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New bounds for roots of polynomials based on Fiedler companion matrices*

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Abstract

Several matrix norms of the classical Frobenius companion matrices of a monic polynomial $p(z)$ have been used in the literature to obtain simple lower and upper bounds on the absolute values of the roots λ of $p(z)$. Recently, M. Fiedler has introduced a new family of companion matrices of $p(z)$ (Lin. Alg. Appl., 372 (2003) 325-331) that has received considerable attention and it is natural to investigate if matrix norms of Fiedler companion matrices may be used to obtain new and sharper lower and upper bounds on $|\lambda|$. The development of such bounds requires first to know simple expressions for some relevant matrix norms of Fiedler matrices and we obtain them in the case of the 1- and ∞- matrix norms. With these expressions at hand, we will show that norms of Fiedler matrices produce many new bounds, but that none of them improves significatively the classical bounds obtained from the Frobenius companion matrices. However, we will prove that if the norms of the inverses of Fiedler matrices are used, then another family of new bounds on $|\lambda|$ is obtained and some of the bounds in this family improve significatively the bounds coming from the Frobenius companion matrices for certain polynomials.

Keywords: bounds, eigenvalues, Fiedler companion matrices, matrix norms, roots of polynomials

AMS subject classification: 15A18, 15A60, 15B99, 65F15, 65F35, 65H04

1 Introduction

This paper is devoted to find bounds for the absolute values of the roots of $p(z)$, a monic polynomial of degree $n \geq 2$ with complex coefficients written as

$$p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k, \qquad a_i \in \mathbb{C}, \quad i = 0, 1, \ldots, n-1. \tag{1}$$

To locate approximately the roots of $p(z)$ through simple operations with its coefficients is a classical problem that has produced a considerable amount of literature (see the comprehensive surveys [15, 17] and the references therein). Simple location rules are used for theoretical purposes, as establishing sufficient conditions guaranteeing that $p(z)$ is stable or that all its roots are inside the unit circle, and they are also used in iterative algorithms for computing the roots of $p(z)$ to find initial guesses of the roots for starting the iteration [2, 3]. Recently, polynomial eigenvalue problems have received much attention and simple criteria for locating approximately the eigenvalues of matrix polynomials have been developed [4, 12], but, to keep the paper concise, matrix polynomials are not covered in this work.

Let us denote by λ any root of $p(z)$. In this paper, we are interested in finding nonnegative numbers $L(p)$ and $U(p)$ depending on the coefficients of $p(z)$, such that

$$L(p) \leq |\lambda| \leq U(p), \tag{2}$$

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by using norms of the Fiedler matrices associated with $p(z)$. The Fiedler matrices [9] of $p(z)$ are a family of 2^{n-1} different matrices whose eigenvalues are precisely the roots of $p(z)$. The family of Fiedler matrices has received considerable attention in the last years, and it includes the well-known *first* and *second (Frobenius) companion forms* of $p(z)$, that is, the matrices

$$C_1(p) = \begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 \\ 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix} \quad \text{and} \quad C_2(p) = \begin{bmatrix} -a_{n-1} & 1 & & \\ \vdots & & \ddots & \\ -a_1 & & & 1 \\ -a_0 & 0 & \cdots & 0 \end{bmatrix}, \tag{3}$$

which have been widely used to obtain classic bounds of type (2) [13, pp. 365–368], as well as other types of location results for roots of polynomials [16]. However, to the best of our knowledge, other Fiedler matrices have not yet been used for these purposes and this is the goal of this paper.

When $a_0 \neq 0$, i.e, when $\lambda = 0$ is not a root of $p(z)$ in (1), the *monic reversal polynomial of* $p(z)$ [13, p. 366] plays an important role in getting bounds for the roots of $p(z)$. It is defined as follows:

$$p^\sharp(z) := \frac{z^n}{a_0} p(z^{-1}) = z^n + \frac{a_1}{a_0} z^{n-1} + \frac{a_2}{a_0} z^{n-2} + \cdots + \frac{a_{n-1}}{a_0} z + \frac{1}{a_0}. \tag{4}$$

Observe that the roots of $p^\sharp(z)$ are the reciprocals of the roots of $p(z)$. Therefore, the eigenvalues of the Frobenius companion forms of $p^\sharp(z)$, i.e., $C_1(p^\sharp)$ and $C_2(p^\sharp)$, are also the reciprocals of the roots of $p(z)$. This can be combined with a well known property of any *submultiplicative matrix norm*, i.e., a matrix norm $\|\cdot\|$ satisfying $\|AB\| \leq \|A\|\|B\|$ for all $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times n}$ [13, Chapter 5]. This property establishes that if $X \in \mathbb{C}^{n\times n}$ and μ is any eigenvalue of X, then $|\mu| \leq \|X\|$ [13, p. 347] and it can be applied to both $C_i(p)$ and $C_i(p^\sharp)$ to prove that

$$\left(\|C_i(p^\sharp)\|\right)^{-1} \leq |\lambda| \leq \|C_i(p)\|, \qquad i = 1, 2, \tag{5}$$

for any root λ of $p(z)$, which allows us to get bounds of type (2). In practice, (5) is used with the 1-, 2-, ∞-, and Frobenius norms. For a matrix $A = (a_{ij}) \in \mathbb{C}^{m\times n}$, these norms are defined as [11, p. 108]

$$\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}|, \quad \|A\|_2 = \sigma_{max}(A), \quad \|A\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|, \quad \|A\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2},$$

where $\sigma_{max}(A)$ denotes the largest singular value of A. Note that $\|A\|_1 = \|A^T\|_\infty$, $\|A\|_2 = \|A^T\|_2$, and $\|A\|_F = \|A^T\|_F$. In [13, pp. 365-368], the inequalities (5) are used with $C_2(p)$ and $C_2(p^\sharp)$ and the ∞-, 1-, 2-, and Frobenius norms to get the following classical bounds.

Theorem 1.1. *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial with complex coefficients and λ be any root of $p(z)$. Then $|\lambda|$ satisfies the following inequalities.*

1. *Cauchy's lower and upper bounds (coming from C_2 and $\|\cdot\|_\infty$):*

$$\frac{|a_0|}{\max\{1, |a_0| + |a_1|, |a_0| + |a_2|, \ldots, |a_0| + |a_{n-1}|\}} \leq |\lambda| \leq \max\{|a_0|, 1 + |a_1|, \ldots, 1 + |a_{n-1}|\}.$$

2. *Montel's lower and upper bounds (coming from C_2 and $\|\cdot\|_1$):*

$$\frac{|a_0|}{\max\{|a_0|, 1 + |a_1| + |a_2| + \cdots + |a_{n-1}|\}} \leq |\lambda| \leq \max\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\}.$$

3. *Carmichael-Mason's lower and upper bounds (coming from C_2 and $\|\cdot\|_2$):*

$$\frac{|a_0|}{\sqrt{1 + |a_0|^2 + |a_1|^2 + \cdots + |a_{n-1}|^2}} \leq |\lambda| \leq \sqrt{1 + |a_0|^2 + |a_1|^2 + \cdots + |a_{n-1}|^2}.$$

4. *Frobenius' lower and upper bounds (coming from C_2 and $\|\cdot\|_F$):*

$$\frac{|a_0|}{\sqrt{1 + (n-1)|a_0|^2 + |a_1|^2 + \cdots + |a_{n-1}|^2}} \leq |\lambda| \leq \sqrt{(n-1) + |a_0|^2 + \cdots + |a_{n-1}|^2}.$$

Note that if $C_1(p)$ and $C_1(p^\sharp)$ are used instead of $C_2(p)$ and $C_2(p^\sharp)$, then the same bounds are obtained, but Cauchy's bounds are obtained for $\|\cdot\|_1$ and Montel's bounds for $\|\cdot\|_\infty$. It is clear that Carmichael-Mason's bounds are always sharper than Frobenius' bounds, but which are the sharpest among the other bounds depend on the particular polynomial that is considered. However, Cauchy's bounds are essentially the sharpest ones in Theorem 1.1. To be precise, if $U_C(p), U_M(p)$, and $U_{CM}(p)$ denote, respectively, the upper Cauchy's, Montel's, and Carmichael-Mason's bounds, then it is easy to prove that $U_C(p) \leq 2\, U_M(p)$ and $U_C(p) \leq \sqrt{2}\, U_{CM}(p)$ for any $p(z)$. Besides, if $L_C(p), L_M(p)$, and $L_{CM}(p)$ denote, respectively, the lower Cauchy's, Montel's, and Carmichael-Mason's bounds, then $L_M(p) \leq 2\, L_C(p)$ and $L_{CM}(p) \leq \sqrt{2}\, L_C(p)$ for any $p(z)$. For this reason, we will use preferably Cauchy's bounds for testing the sharpness of the new bounds obtained in this work.

Apart from their eigenvalues, the rest of Fiedler matrices of $p(z)$ share a key property with the first and second Frobenius companion forms: they contain, in different positions, exactly the same nonzero entries, i.e., $n-1$ entries equal to 1, and n entries equal to $-a_0, -a_1, \ldots, -a_{n-1}$ [8]. In addition, if we denote by $M_\sigma(p)$ a Fiedler matrix of $p(z)$, where σ is a symbol that allows us to distinguish among the 2^{n-1} different Fiedler matrices, then the same argument that we used to get (5) allows us to prove

$$\left(\|M_\sigma(p^\sharp)\|\right)^{-1} \leq |\lambda| \leq \|M_\sigma(p)\|\,, \tag{6}$$

for any root λ of $p(z)$, for any Fiedler matrix of $p(z)$, and for any submultiplicative matrix norm. As a consequence, it is natural to try to use (6) combined with the 1-, 2-, ∞-, and Frobenius norms for obtaining new simple lower and upper bounds on the absolute values of the roots of $p(z)$. Since, there exist $2^{n-1}-2$ Fiedler matrices that are different that the Frobenius companion forms [8], this strategy may expand considerably with respect Theorem 1.1 the arena in which to look for good bounds of type (2). But note that, before applying (6), we need to know which are the expressions for the 1-, 2-, ∞-, and Frobenius norms of Fiedler matrices. The Frobenius norms of all Fiedler matrices associated with $p(z)$ are equal since all of them have the same nonzero entries [8], and therefore new bounds are not obtained from $\|\cdot\|_F$. In addition, it is known [8, Section 6], that except in the case of Frobenius companion matrices, simple expressions for the 2-norm of Fiedler matrices are not available, and it seems very difficult to get them. So, in this context, it only remains to investigate which are the expressions for the ∞- and the 1- norms of any Fiedler matrix and to obtain them is *the first main contribution in this work*. These expressions, together with (6), produce, in fact, many new lower/upper bounds, but none of them improves Cauchy's lower/upper bounds in Theorem 1.1 by a factor larger than two. Moreover, we will see that Cauchy's bounds are the sharpest ones among those obtained for a large subclass of Fiedler matrices. The proof of these facts is *the second main contribution in this work*.

To improve these results, we follow another strategy based on the fact that for any invertible matrix X, the eigenvalues of X^{-1} are the reciprocals of the eigenvalues of X. So, if $a_0 \neq 0$, the eigenvalues of $M_\sigma(p^\sharp)^{-1}$ are the roots of $p(z)$, the eigenvalues of $M_\sigma(p)^{-1}$ are the reciprocals of the roots of $p(z)$, and

$$\left(\|M_\sigma(p)^{-1}\|\right)^{-1} \leq |\lambda| \leq \|M_\sigma(p^\sharp)^{-1}\|, \tag{7}$$

for any root λ of $p(z)$, for any Fiedler matrix of $p(z)$, and for any submultiplicative matrix norm. The practical use of (7) requires to know $\|M_\sigma(p)^{-1}\|$ and $\|M_\sigma(p^\sharp)^{-1}\|$ for the 1-, 2-, ∞-, and Frobenius norms. Expressions for the Frobenius norms are given in [8, Corollary 3.3], such expressions are not available for the 2- norm and it seems very difficult to get them [8, Section 6], and for the 1- and ∞- norms are obtained for first time in this work, and this is *our third main contribution*. Note that for the Frobenius companion matrices $C_i(p)$, $i=1,2$, (7) is exactly the same as (5) for the 1-, 2-, ∞-, and Frobenius norms, since it is easy to see[1] that $\|C_i(p^\sharp)\| = \|C_i(p)^{-1}\|$ and $\|C_i(p)\| = \|C_i(p^\sharp)^{-1}\|$, and new bounds are not obtained. However, we will prove that the use of other Fiedler matrices in (7) gives new bounds for the roots of polynomials and, *more important, that some of these bounds are much sharper than Cauchy's lower/upper bounds in certain cases.* In this setting, the following Fiedler matrix plays a key role:

$$F(p) = \begin{bmatrix} -a_{n-1} & 1 & & & \\ \vdots & & \ddots & & \\ -a_2 & & & 1 & \\ -a_1 & & & & -a_0 \\ 1 & & & & 0 \end{bmatrix} \tag{8}$$

[1] These equalities are proved in Theorem 5.2.

and, for it, (7) with the ∞-norm gives the following lower/upper bounds

$$\frac{|a_0|}{\max\{1+|a_1|, |a_0|(1+|a_2|), \ldots, |a_0|(1+|a_{n-1}|)\}} \leq |\lambda|\,, \tag{9}$$

$$|\lambda| \leq \max\left\{1+\frac{|a_1|}{|a_0|}, 1+\frac{|a_2|}{|a_0|}, \ldots, 1+\frac{|a_{n-2}|}{|a_0|}, |a_0|+|a_{n-1}|\right\}\,. \tag{10}$$

We will prove that if $|a_0| \leq 1$, then, up to a factor 2, (9) is the sharpest bound among all lower bounds in (7) for the 1- and ∞- norms, and it may be much sharper than Cauchy's lower bound under certain conditions. Moreover, we will prove that if $|a_0| \geq 1$, then, up to a factor 2, (10) is the sharpest bound among all upper bounds in (7) for the 1- and ∞- norms, and it may be much sharper than Cauchy's upper bound under certain conditions. These results and the bounds (9)-(10) are *our fourth main contribution*.[2] We will also show that the use of the Frobenius norm in (7) leads to simple new bounds, but that they do not improve Cauchy's lower-upper bounds and the bounds (9)-(10) by factors larger than $\sqrt{2}$.

The bounds in Theorem 1.1 and the ones that can be obtained from Fiedler matrices and their inverses with the 1-, ∞-, and Frobenius norms (see, for instance, (9)-(10)) have an important drawback: the lower bounds are always smaller than 1 and the upper bounds are always larger than 1. This is a consequence of the presence of entries equal to 1 in any Fiedler matrix and its inverse. For $C_1(p)$ and $C_2(p)$ an standard way to overcome this drawback is to use diagonal similarities, which do not change neither the eigenvalues nor the zero pattern, and to use (5). More precisely, let D and $\widetilde{D}$ be nonsingular diagonal matrices, then from (5) we get $\left(\|\widetilde{D}^{-1}C_i(p^\sharp)\widetilde{D}\|\right)^{-1} \leq |\lambda| \leq \|D^{-1}C_i(p)D\|$, for $i=1,2$. Given a polynomial $p(z)$, the selection of a proper D (and/or $\widetilde{D}$) may improve drastically the bounds, but a choice of D that is good for certain polynomials may be bad for others, so the choice of proper diagonal similarities is not immediate. Some specific D's have been used to get the well-know Fujiwara's [10] and Kojima's bounds [14] (see also [13, p. 367]). The use of diagonal similarities is also possible with Fiedler matrices, both combined with (6) and (7), and it is possible to obtain explicit expressions of the involved norms for the 1-, ∞-, and Frobenius norms. However, how to select proper diagonal matrices that improve the known bounds for wide classes of polynomials is not clear. This problem requires further and extensive investigation and in this paper we limit ourselves to give some theoretical results on the optimal bounds that can be obtained with this approach. In this context, it should be noted that the Fiedler matrix $F(p)$ is a very particular diagonal similarity of $C_2(p)$ if $a_0 \neq 0$ (both matrices are also similar if $a_0 = 0$, but then the similarity is not diagonal). In fact, $F(p)$ is the only Fiedler matrix of $p(z)$ that is diagonally similar to $C_2(p)$, because other Fiedler matrices have a different zero pattern.

The paper is organized as follows. In Section 2, we recall the definition of Fiedler companion matrices and some of their properties. Explicit expressions for the ∞- and 1- norms of Fiedler matrices and their inverses are obtained in Section 3. In Sections 4 and 5 we study the bounds that follow from (6) and (7), respectively, for the ∞- and 1- norms. Section 5 includes the most relevant new bounds presented in this manuscript. The bounds obtained from (7) in the Frobenius norm are analyzed in Section 6. Some theoretical results on bounds coming from applying diagonal similarities to Fiedler matrices are introduced in Section 7. Finally, our conclusions and some open problems in this area are presented in Section 8.

2 Definition and basic properties of Fiedler matrices

For the polynomial $p(z)$ in (1), we define the $n \times n$ matrices

$$M_0 := \begin{bmatrix} I_{n-1} & 0 \\ 0 & -a_0 \end{bmatrix} \quad \text{and} \quad M_k := \begin{bmatrix} I_{n-k-1} & & & \\ & -a_k & 1 & \\ & 1 & 0 & \\ & & & I_{k-1} \end{bmatrix}, \quad k = 1, \ldots, n-1, \tag{11}$$

which are the basic factors used to build all Fiedler matrices. Here and in the rest of the paper I_j denotes the $j \times j$ identity matrix for $j > 0$ and I_0 stands for the empty matrix. In [9], Fiedler matrices are constructed as the products

$$M_{i_1} M_{i_2} \cdots M_{i_n}\,,$$

[2] We remark the fact that the use in (7) of other Fiedler matrices different of $F(p)$ may also improve very much the Cauchy bounds is the situations described after equations (9)-(10). However, in these situations, they never improve the bounds coming from $F(p)$ by a factor larger than two and, so, they are not studied in depth in this work.

where $(i_1, i_2, \dots, i_n)$ is any possible permutation of the n-tuple $(0, 1, \dots, n-1)$. In order to better express certain key properties of this permutation and the resulting Fiedler matrix, in [6] the authors index the product of the M_i-factors in a slightly different way, as it is described in Definition 2.1.

Definition 2.1. *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$, with $n \geq 2$, and let M_i, for $i = 0, 1, \dots, n-1$, be the matrices defined in* (11). *Given any bijection $\sigma : \{0, 1, \dots, n-1\} \to \{1, \dots, n\}$, the* Fiedler matrix of $p(z)$ associated with σ *is the $n \times n$ matrix*

$$M_\sigma(p) := M_{\sigma^{-1}(1)} \cdots M_{\sigma^{-1}(n)}. \tag{12}$$

Note that $\sigma(i)$ describes the position of the factor M_i in the product $M_{\sigma^{-1}(1)} \cdots M_{\sigma^{-1}(n)}$, i.e., $\sigma(i) = j$ means that M_i is the jth factor in the product.

$M_\sigma(p)$ in (12) shows explicitly the dependence of a Fiedler matrix on $p(z)$. The building factors (11) of $M_\sigma(p)$ also depend on $p(z)$, but, for simplicity, we do not indicate this dependence explicitly in the factors.

Some important properties of Fiedler matrices are the following. All Fiedler matrices of $p(z)$ are similar, and so all of them have $p(z)$ as characteristic polynomial [9]. The matrices M_i in (11) are symmetric and, therefore, the transpose of any Fiedler matrix is another Fiedler matrix, obtained by reversing the order of the M_i factors in (12). This has a relevant implication in our study: for determining the sharpest bounds (6) and (7) that can be obtained from the set of Fiedler matrices by using the 1- and ∞- norms, we need to analyze only the ∞-norms of all Fiedler matrices and their inverses, because this includes also the bounds coming from the 1-norms, because $\|A\|_1 = \|A^T\|_\infty$ for any matrix A. The first and second Frobenius companion forms of $p(z)$ are particular cases of Fiedler matrices, namely,

$$C_1(p) = M_{n-1} M_{n-2} \cdots M_1 M_0 \quad \text{and} \quad C_2(p) = M_0 M_1 \cdots M_{n-2} M_{n-1}\,.$$

Other relevant matrices included in the set of Fiedler matrices of $p(z)$ are the four pentadiagonal matrices described in [8, Example 2.2] (see also [9]). The low bandwidth of these pentadiagonal matrices makes them very interesting in fast numerical methods for computing roots of polynomials. However, we will see in this work that pentadiagonal Fiedler matrices do not play any special role in getting bounds for the roots of polynomials.

The matrices $\{M_k\}_{k=0}^{n-1}$ in (11) satisfy the following commutativity relations

$$M_i M_j = M_j M_i \quad \text{for } |i - j| \neq 1. \tag{13}$$

These relations imply that some Fiedler matrices associated with different bijections σ are equal. For example, for $n = 3$, the Fiedler matrices $M_0 M_2 M_1$ and $M_2 M_0 M_1$ are equal. These relations suggest that the relative positions of the matrices M_i and M_{i+1} in the product defining $M_\sigma(p)$ are of fundamental interest in studying Fiedler matrices. The fact that this is true has been extensively shown in [6, 7, 8]. This motivates Definition 2.2, that has been introduced in [6, 8].

Definition 2.2. *Let $\sigma : \{0, 1, \dots, n-1\} \to \{1, \dots, n\}$ be a bijection.*

(a) *For $i = 0, \dots, n-2$, we say that σ has a* consecution *at i if $\sigma(i) < \sigma(i+1)$ and that σ has an* inversion *at i if $\sigma(i) > \sigma(i+1)$.*

(b) *The* consecution-inversion structure sequence of σ, *denoted by* $\mathrm{CISS}(\sigma)$, *is the tuple $(\mathfrak{c}_0, \mathfrak{i}_0, \mathfrak{c}_1, \mathfrak{i}_1, \dots, \mathfrak{c}_\ell, \mathfrak{i}_\ell)$, where σ has $\mathfrak{c}_0$ consecutive consecutions at $0, 1, \dots, \mathfrak{c}_0 - 1$; $\mathfrak{i}_0$ consecutive inversions at $\mathfrak{c}_0, \mathfrak{c}_0 + 1, \dots, \mathfrak{c}_0 + \mathfrak{i}_0 - 1$ and so on, up to $\mathfrak{i}_\ell$ inversions at $n - 1 - \mathfrak{i}_\ell, \dots, n-2$.*

(c) *The* number of initial consecutions or inversions of σ, *denoted by t_σ, is*

$$t_\sigma = \begin{cases} \mathfrak{c}_0 & \text{if } \mathfrak{c}_0 \neq 0, \\ \mathfrak{i}_0 & \text{if } \mathfrak{c}_0 = 0. \end{cases}$$

Remark 2.3. The following simple observations on Definition 2.2 will be used freely.

1. σ has a consecution at i if and only if M_i is to the left of M_{i+1} in the Fiedler matrix $M_\sigma(p)$, while σ has an inversion at i if and only if M_i is to the right of M_{i+1} in $M_\sigma(p)$.
2. Note that $\mathfrak{c}_0$ and $\mathfrak{i}_\ell$ in $\mathrm{CISS}(\sigma)$ may be zero (in the first case, σ has an inversion at 0 and in the second one it has a consecution at $n-2$) but $\mathfrak{i}_0, \mathfrak{c}_1, \mathfrak{i}_1, \dots, \mathfrak{i}_{\ell-1}, \mathfrak{c}_\ell$ are all strictly positive. These conditions uniquely determine $\mathrm{CISS}(\sigma)$ and, in particular, the parameter ℓ.

Examples that illustrate the concepts introduced in Definition 2.2 are found in [8, Example 2.5].

For $k = 1, \ldots, n-1$, the matrices M_k defined in (11) are nonsingular for any value of the coefficients a_k, while the matrix M_0 is nonsingular if and only if $a_0 \neq 0$, and the inverses of these matrices are

$$M_0^{-1} = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1/a_0 \end{bmatrix}, \qquad M_k^{-1} = \begin{bmatrix} I_{n-k-1} & & & \\ & 0 & 1 & \\ & 1 & a_k & \\ & & & I_{k-1} \end{bmatrix}, \quad k = 1, 2, \ldots, n-1.$$

Hence, for any bijection σ, the Fiedler matrix $M_\sigma(p)$ in (12) is nonsingular[3] if and only if $a_0 \neq 0$, that is, if $\lambda = 0$ is not a root of $p(z)$, and we can factorize $M_\sigma(p)^{-1}$ as

$$M_\sigma(p)^{-1} = (M_{\sigma^{-1}(1)} \cdots M_{\sigma^{-1}(n)})^{-1} = M_{\sigma^{-1}(n)}^{-1} \cdots M_{\sigma^{-1}(1)}^{-1}.$$

In [8, Theorems 2.6 and 3.1], two algorithms that construct Fiedler matrices and their inverses without multiplying the M_k or M_k^{-1} factors are presented. Both of them are included here in Theorem 2.4, but mixed in just one algorithm for brevity. These algorithms allow us to get expressions for the ∞- and 1-norms of any Fiedler matrix and its inverse in Section 3. In `Algorithm 1`, we use *MATLAB notation for submatrices*, that is, $A(i : j, :)$ indicates the submatrix of A consisting of all rows from i to j and $A(:, k : l)$ indicates the submatrix of A consisting of all columns from k to l.

Theorem 2.4. *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial with degree $n \geq 2$ and $a_0 \neq 0$, let $\sigma : \{0, 1, \ldots, n-1\} \to \{1, \ldots, n\}$ be a bijection, and let $M_\sigma(p)$ be the Fiedler matrix of $p(z)$ associated with σ. Then* `Algorithm 1` *constructs $M_\sigma(p)$ and $M_\sigma(p)^{-1}$.*

`Algorithm 1.` *Given $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ and a bijection σ, the following algorithm constructs $M_\sigma(p)$ and $M_\sigma(p)^{-1}$.*

if σ has a consecution at 0 then

$$W_0 = \begin{bmatrix} -a_1 & 1 \\ -a_0 & 0 \end{bmatrix} \quad \text{and} \quad B_0 = \begin{bmatrix} 0 & -1/a_0 \\ 1 & -a_1/a_0 \end{bmatrix}$$

else

$$W_0 = \begin{bmatrix} -a_1 & -a_0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B_0 = \begin{bmatrix} 0 & 1 \\ -1/a_0 & -a_1/a_0 \end{bmatrix}$$

endif

for $i = 1 : n-2$

if σ has a consecution at i then

$$W_i = \begin{bmatrix} -a_{i+1} & 1 & 0 \\ W_{i-1}(:,1) & 0 & W_{i-1}(:, 2 : i+1) \end{bmatrix} \quad \text{and} \quad B_i = \begin{bmatrix} 0 & B_{i-1}(1,:) \\ 1 & a_{i+1} B_{i-1}(1,:) \\ 0 & B_{i-1}(2 : i+1, :) \end{bmatrix}$$

else

$$W_i = \begin{bmatrix} -a_{i+1} & W_{i-1}(1,:) \\ 1 & 0 \\ 0 & W_{i-1}(2 : i+1, :) \end{bmatrix} \quad \text{and} \quad B_i = \begin{bmatrix} 0 & 1 & 0 \\ B_{i-1}(:,1) & a_{i+1} B_{i-1}(:,1) & B_{i-1}(:, 2 : i+1) \end{bmatrix}$$

endif

endfor

$M_\sigma(p) = W_{n-2}$ and $M_\sigma(p)^{-1} = B_{n-2}$.

In [7, Theorem 3.10] and [8, Theorem 3.2], it is shown that `Algorithm 1` can be used to get information on the entries of $M_\sigma(p)$ and $M_\sigma(p)^{-1}$ in a simple way. Part of this information is stated in Theorem 2.5.

Theorem 2.5. *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial with degree $n \geq 2$, let $\sigma : \{0, 1, \ldots, n-1\} \to \{1, \ldots, n\}$ be a bijection, let $M_\sigma(p)$ be the Fiedler matrix of $p(z)$ associated with σ, and let t_σ be the number of initial consecutions or inversions of σ. Then, the non-identically zero entries of $M_\sigma(p)$ are*

(a) *n entries equal to $-a_0, -a_1, \ldots, -a_{n-1}$, with exactly one copy of each, and*

(b) *$n-1$ entries equal to 1.*

[3] Throughout this paper we will often impose $a_0 \neq 0$, but the reader should note that this assumption is only needed in those results that involve inverses of Fiedler matrices. However, for avoiding cumbersome statements, we often impose $a_0 \neq 0$ also in the results for Fiedler matrices.

Moreover, if $a_0 \neq 0$, the non-identically zero entries of $M_\sigma(p)^{-1}$ are

(a) $t_\sigma + 1$ *entries equal to* $-\dfrac{1}{a_0}, -\dfrac{a_1}{a_0}, \dots, -\dfrac{a_{t_\sigma}}{a_0}$, *with exactly one copy of each,*

(b) $n-1-t_\sigma$ *entries equal to* $a_{t_\sigma+1}, a_{t_\sigma+2}, \dots, a_{n-1}$, *with exactly one copy of each, and*

(c) $n-1$ *entries equal to* 1.

Given a bijection $\sigma : \{0,1,\dots,n-1\} \longrightarrow \{1,\dots,n\}$, the *reversal bijection of* σ, denoted by $\mathrm{rev}(\sigma) : \{0,1,\dots,n-1\} \to \{1,\dots,n\}$, is defined by $\mathrm{rev}(\sigma)(i) = n+1-\sigma(i)$, for $0 \leq i \leq n-1$. In next sections we will use the following result, whose easy proof is omitted.

Theorem 2.6. *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$, with $n \geq 2$, let $\sigma : \{0,1,\dots,n-1\} \to \{1,\dots,n\}$ be a bijection, and let $M_\sigma(p)$ be the Fiedler matrix of $p(z)$ associated with σ. Then,*

$$M_\sigma(p)^T = M_{\mathrm{rev}(\sigma)}(p). \tag{14}$$

3 Formulas for the ∞- and 1-norms of Fiedler matrices and their inverses

Theorem 2.5 describes which are the non-identically zero entries of a Fiedler matrix $M_\sigma(p)$ and its inverse $M_\sigma(p)^{-1}$, but it does not give information on the positions where these entries are placed in. In order to obtain expressions for $\|M_\sigma(p)\|_\infty$ and $\|M_\sigma(p)^{-1}\|_\infty$, it is needed to know how the non-identically zero entries of these two matrices are distributed by rows. This is presented in Lemma 3.1 for $M_\sigma(p)$ and in Lemma 3.2 for $M_\sigma(p)^{-1}$. Once these two lemmas are known, we get easily Theorem 3.3, where the formulas for $\|M_\sigma(p)\|_\infty$ and $\|M_\sigma(p)^{-1}\|_\infty$ are finally stated. As a corollary of Theorem 3.3 and Theorem 2.6 the formulas for $\|M_\sigma(p)\|_1$ and $\|M_\sigma(p)^{-1}\|_1$ are obtained and presented in Theorem 3.4.

The results in this section and in the rest of the paper require the partial sums of the entries of $\mathrm{CISS}(\sigma)$, that were previously used in [6, p. 2193]. We recall now their definitions: let $\sigma : \{0,1,\dots,n-1\} \to \{1,\dots,n\}$ be a bijection and let $\mathrm{CISS}(\sigma) = (\mathfrak{c}_0, \mathfrak{i}_0, \mathfrak{c}_1, \mathfrak{i}_1, \dots, \mathfrak{c}_\ell, \mathfrak{i}_\ell)$ be the consecution-inversion structure sequence of σ introduced in Definition 2.2, then

$$s_k := \sum_{j=0}^{k} (\mathfrak{c}_j + \mathfrak{i}_j), \quad \text{for } k = 0,1,\dots,\ell, \qquad s_{-1} := 0. \tag{15}$$

Observe that $s_\ell = n-1$ is the total number of consecutions and inversions of σ, that if $\mathfrak{c}_0 = 0$ then $s_0 = \mathfrak{i}_0$, and that $s_k = s_{k-1} + \mathfrak{c}_k + \mathfrak{i}_k$, for $k = 0,1,\dots,\ell$.

Lemma 3.1. *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial with degree $n \geq 2$, let $\sigma : \{0,1,\dots,n-1\} \to \{1,\dots,n\}$ be a bijection, let $\mathrm{CISS}(\sigma) = (\mathfrak{c}_0, \mathfrak{i}_0, \mathfrak{c}_1, \mathfrak{i}_1, \dots, \mathfrak{c}_\ell, \mathfrak{i}_\ell)$ be the consecution-inversion structure sequence of σ, and let $\{s_k\}_{k=-1}^{\ell}$ be the partial sums of σ. If $M_\sigma(p)$ is the Fiedler matrix of $p(z)$ associated with σ, then the non-zero entries of $M_\sigma(p)$ are placed as specified in the following statements.*

(a) *Each of the $(n-1)$ entries equal to 1 is in a different row of $M_\sigma(p)$. The only row of $M_\sigma(p)$ which does not contain an entry equal to 1 is*

  (i) *the nth row, if $\mathfrak{c}_0 > 0$;*

  (ii) *the $(n - \mathfrak{i}_0)$th row, if $\mathfrak{c}_0 = 0$.*

(b) *The entries $-a_0, -a_1, \dots, -a_{s_0}$ of $M_\sigma(p)$ satisfy:*

  (i) *If $\mathfrak{c}_0 > 0$, then*

  - *each of the entries $-a_0, -a_1, \dots, -a_{\mathfrak{c}_0 - 1}$ is in a different row of $M_\sigma(p)$, each of these rows does not contain any other entry equal to $-a_i$, for $i = 0,1,\dots,n-1$, and $-a_0$ is in the nth row; and*
  - *the entries $-a_{\mathfrak{c}_0}, -a_{\mathfrak{c}_0+1}, \dots, -a_{\mathfrak{c}_0+\mathfrak{i}_0}$ are all of them in the same row of $M_\sigma(p)$ and this row does not contain any other entry equal to $-a_i$, for $i = 0,1,\dots,n-1$.*

  (ii) *If $\mathfrak{c}_0 = 0$, then the entries $-a_0, -a_1, \dots, -a_{\mathfrak{i}_0}$ are all of them in the $(n - \mathfrak{i}_0)$th row of $M_\sigma(p)$ and these are the only non-zero entries in this row.*

(c) *For each $k = 1,\dots,\ell$, the entries $-a_{s_{k-1}+1}, -a_{s_{k-1}+2}, \dots, -a_{s_k}$ of $M_\sigma(p)$ satisfy:*

  - *each of the entries $-a_{s_{k-1}+1}, \dots, -a_{s_{k-1}+\mathfrak{c}_k-1}$ is in a different row of $M_\sigma(p)$ and each of these rows does not contain any other entry equal to $-a_i$, for $i = 0,1,\dots,n-1$; and*

- *the entries $-a_{s_{k-1}+\mathfrak{c}_k}, -a_{s_{k-1}+\mathfrak{c}_k+1}, \ldots, -a_{s_k}$ are all of them in the same row of $M_\sigma(p)$ and this row does not contain any other entry equal to $-a_i$, for $i = 0, 1, \ldots, n-1$.*

Proof. The formal proof follows an inductive argument based on `Algorithm 1` in Theorem 2.4. We only sketch the main idea and invite the reader to complete the details, which are straightforward but somewhat long. The result is obviously true for the matrix W_0 appearing in `Algorithm 1`, or in other words, is obviously true for polynomials of degree 2. Then the induction hypothesis is that Lemma 3.1 holds for W_{n-3}, or in other words for polynomials of degree $n-1$, and then the way `Algorithm 1` constructs $W_{n-2} = M_\sigma(p)$ from W_{n-2} is used to prove that the entries of $M_\sigma(p)$ satisfy Lemma 3.1. For this purpose, four cases should be considered, depending on whether σ has a consecution or an inversion at $n-3$, and on whether σ has a consecution or an inversion at $n-2$. □

Next, we determine in Lemma 3.2 the distribution by rows of the non-zero entries of $M_\sigma(p)^{-1}$.

Lemma 3.2. *With the same notation and hypotheses that in Lemma 3.1, let us assume in addition that $a_0 \neq 0$ and that t_σ is the number of initial consecutions or inversions of σ. Then the non-zero entries of $M_\sigma(p)^{-1}$ are placed as specified in the following statements.*

(a) *Each of the $(n-1)$ entries equal to 1 is in a different row of $M_\sigma(p)^{-1}$. The only row of $M_\sigma(p)^{-1}$ which does not contain an entry equal to 1 is the $(n-\mathfrak{c}_0)$th row.*

(b) *The entries $-\frac{1}{a_0}, -\frac{a_1}{a_0}, \ldots, -\frac{a_{t_\sigma}}{a_0}, a_{t_\sigma+1}, a_{t_\sigma+2}, \ldots, a_{s_0}$ of $M_\sigma(p)^{-1}$ satisfy*[4]

  (i) *If $\mathfrak{c}_0 > 0$, then*
  - *$-1/a_0$ is the only non-zero entry in the $(n-\mathfrak{c}_0)$th row of $M_\sigma(p)^{-1}$;*
  - *if, in addition, $\mathfrak{c}_0 > 1$, then each of the entries $-a_1/a_0, \ldots, -a_{\mathfrak{c}_0-1}/a_0$ is in a different row of $M_\sigma(p)^{-1}$ and each of these rows does not contain any other entry of the set $\{-1/a_0, -a_1/a_0, \ldots, -a_{\mathfrak{c}_0}/a_0, a_{\mathfrak{c}_0+1}, \ldots, a_{n-1}\}$;*
  - *the entries $-a_{\mathfrak{c}_0}/a_0, a_{\mathfrak{c}_0+1}, \ldots, a_{s_0}$ are all of them in the same row of $M_\sigma(p)^{-1}$ and this row does not contain any other entry of the set $\{-1/a_0, -a_1/a_0, \ldots, -a_{\mathfrak{c}_0}/a_0, a_{\mathfrak{c}_0+1}, \ldots, a_{n-1}\}$.*

  (ii) *If $\mathfrak{c}_0 = 0$, then the entries $-1/a_0, -a_1/a_0, \ldots, -a_{\mathfrak{i}_0}/a_0$ are all of them in the nth row of $M_\sigma(p)^{-1}$ and these are the only non-zero entries in this row.*

(c) *For each $k = 1, \ldots, \ell$, the entries $a_{s_{k-1}+1}, a_{s_{k-1}+2}, \ldots, a_{s_k}$ of $M_\sigma(p)^{-1}$ satisfy:*
  - *each of the entries $a_{s_{k-1}+1}, \ldots, a_{s_{k-1}+\mathfrak{c}_k-1}$ is in a different row of $M_\sigma(p)^{-1}$ and each of these rows does not contain any other entry of the set $\{-1/a_0, -a_1/a_0, \ldots, -a_{t_\sigma}/a_0, a_{t_\sigma+1}, \ldots, a_{n-1}\}$; and*
  - *the entries $a_{s_{k-1}+\mathfrak{c}_k}, a_{s_{k-1}+\mathfrak{c}_k+1}, \ldots, a_{s_k}$ are all of them in the same row of $M_\sigma(p)^{-1}$ and this row does not contain any other entry of the set $\{-1/a_0, -a_1/a_0, \ldots, -a_{t_\sigma}/a_0, a_{t_\sigma+1}, \ldots, a_{n-1}\}$.*

Proof. The proof is similar to the one of Lemma 3.1 but using the matrices B_i appearing in `Algorithm 1` instead of the matrices W_i. We invite the reader to complete the details. □

Lemmas 3.1 and 3.2 allow us to prove easily the main result in this section, that is, Theorem 3.3. The simple proof is omitted.

Theorem 3.3. *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial with degree $n \geq 2$, let $\sigma : \{0, 1, \ldots, n-1\} \to \{1, \ldots, n\}$ be a bijection, let $\mathrm{CISS}(\sigma) = (\mathfrak{c}_0, \mathfrak{i}_0, \mathfrak{c}_1, \mathfrak{i}_1, \ldots, \mathfrak{c}_\ell, \mathfrak{i}_\ell)$ be the consecution-inversion structure sequence of σ, let s_k, for $k = 0, 1, \ldots, \ell$, be the partial sums defined in (15), and let $M_\sigma(p)$ be the Fiedler matrix of $p(z)$ associated with σ. Let us define the quantities*

$$\gamma_{\sigma,0}(p) = \begin{cases} \max\{1+|a_1|, \ldots, 1+|a_{\mathfrak{c}_0-1}|, 1+|a_{\mathfrak{c}_0}|+|a_{\mathfrak{c}_0+1}|+\cdots+|a_{s_0}|\}, & \text{if } \mathfrak{c}_0 > 0, \\ \max\{|a_0|+|a_1|+\cdots+|a_{s_0}|, 1\}, & \text{if } \mathfrak{c}_0 = 0, \end{cases}$$

if $a_0 \neq 0$, also the quantities

$$\delta_\sigma(p) = \begin{cases} \max\left\{1+\frac{|a_1|}{|a_0|}, \ldots, 1+\frac{|a_{\mathfrak{c}_0-1}|}{|a_0|}, 1+\frac{|a_{\mathfrak{c}_0}|}{|a_0|}+|a_{\mathfrak{c}_0+1}|+\cdots+|a_{s_0}|\right\}, & \text{if } \mathfrak{c}_0 > 0, \\ \max\left\{\frac{1}{|a_0|}+\frac{|a_1|}{|a_0|}+\cdots+\frac{|a_{s_0}|}{|a_0|}, 1\right\}, & \text{if } \mathfrak{c}_0 = 0, \end{cases}$$

[4] Observe that, if $\mathfrak{c}_0 = 0$, then there are not entries $a_{t_\sigma+1}, a_{t_\sigma+2}, \ldots, a_{s_0}$ since $s_0 = \mathfrak{i}_0 = t_\sigma$.

and finally, for $k = 1, \ldots, \ell$, the quantities

$$\gamma_{\sigma,k}(p) = \max\{1 + |a_{s_{k-1}+1}|, \ldots, 1 + |a_{s_{k-1}+\mathfrak{c}_k-1}|, 1 + |a_{s_{k-1}+\mathfrak{c}_k}| + \cdots + |a_{s_k}|\},$$

where, if $\mathfrak{c}_k = 1$, for some $k = 0, 1, \ldots, \ell$, then the first $\mathfrak{c}_k - 1$ terms within the maximums defining $\gamma_{\sigma,k}(p)$ or $\delta_\sigma(p)$ do not appear. Then

$$\|M_\sigma(p)\|_\infty = \max\{|a_0|, \gamma_{\sigma,0}(p), \gamma_{\sigma,1}(p), \ldots, \gamma_{\sigma,\ell}(p)\}, \tag{16}$$

and

$$\|M_\sigma(p)^{-1}\|_\infty = \max\left\{\frac{1}{|a_0|}, \delta_\sigma(p), \gamma_{\sigma,1}(p), \ldots, \gamma_{\sigma,\ell}(p)\right\}. \tag{17}$$

As an immediate consequence of Theorems 2.6 and 3.3 we get formulas for $\|M_\sigma(p)\|_1$ and $\|M_\sigma(p)^{-1}\|_1$.

Theorem 3.4. *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial with degree $n \geq 2$, let $\sigma : \{0, 1, \ldots, n-1\} \to \{1, \ldots, n\}$ be a bijection, and let $M_\sigma(p)$ be the Fiedler matrix of $p(z)$ associated with σ. Then,*

$$\|M_\sigma(p)\|_1 = \|M_\sigma(p)^T\|_\infty = \|M_{\mathrm{rev}(\sigma)}(p)\|_\infty,$$

and, if $a_0 \neq 0$,

$$\|M_\sigma(p)^{-1}\|_1 = \|\left(M_\sigma(p)^{-1}\right)^T\|_\infty = \|M_{\mathrm{rev}(\sigma)}(p)^{-1}\|_\infty,$$

where $\mathrm{rev}(\sigma)$ is the reversal bijection of σ.

As it was explained in the Introduction, the main goal of this work is to use (6) and (7) with $\|\cdot\|_\infty$ and $\|\cdot\|_1$ and all Fiedler matrices to get new bounds on the roots of $p(z)$, and to determine which are the sharpest ones among these bounds. An important consequence of Theorem 3.4 is that we only need to study the $\infty-$norms of Fiedler matrices, since we obtain the same bounds from the 1-norm by considering the Fiedler matrices associated with the reversal bijections. *Therefore, in the rest of the paper, 1-norms will no longer appear.*

4 Lower and upper bounds from ∞-norms of Fiedler matrices

As a direct consequence of (6) and the expression in Theorem 3.3 for the ∞-norm of a Fiedler matrix, we obtain in Theorem 4.1 the first family of new lower and upper bounds for the absolute values of the roots of monic polynomials presented in this paper. We use the expression "family of lower/upper bounds" because for each different CISS(σ) we obtain a different couple of lower/upper bounds.

Theorem 4.1. *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ with $n \geq 2$, let $p^\sharp(z)$ be the monic reversal polynomial of $p(z)$, and let $\sigma : \{0, 1, \ldots, n-1\} \to \{1, \ldots, n\}$ be a bijection. If λ is a root of $p(z)$, then*

$$\left(\max\left\{\frac{1}{|a_0|}, \gamma_{\sigma,0}(p^\sharp), \gamma_{\sigma,1}(p^\sharp), \ldots, \gamma_{\sigma,\ell}(p^\sharp)\right\}\right)^{-1} \leq |\lambda| \leq \max\{|a_0|, \gamma_{\sigma,0}(p), \gamma_{\sigma,1}(p), \ldots, \gamma_{\sigma,\ell}(p)\}, \tag{18}$$

where the quantities $\gamma_{\sigma,k}(p)$ and $\gamma_{\sigma,k}(p^\sharp)$, for $k = 0, 1, 2, \ldots, \ell$, are those defined in Theorem 3.3 for $p(z)$ and $p^\sharp(z)$, respectively.

Observe that in the statement of Theorem 4.1 we have not imposed $a_0 \neq 0$, which, strictly speaking, is necessary for obtaining the lower bound in (18). However, if $a_0 = 0$, then the lower bound can be taken to be zero and this is consistent with the fact that $p(z)$ has at least one root equal to zero.

We illustrate in Example 4.2 the upper bound in (18) for a couple of particular Fiedler matrices. These are two of the pentadiagonal Fiedler matrices described in [8, Example 2.2].

Example 4.2. *Consider the Fiedler matrix $M_{\sigma_1}(p)$ of $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ with σ_1 such that*

$$\mathrm{CISS}(\sigma_1) = \begin{cases} (1, 1, 1, 1, \ldots, 1, 1, 1, 1) \in \mathbb{R}^{1\times(n-1)}, & \text{if } n \text{ is odd,} \\ (1, 1, 1, 1, \ldots, 1, 1, 1, 0) \in \mathbb{R}^{1\times n}, & \text{if } n \text{ is even.} \end{cases}$$

For instance, for a monic polynomial $p(z) = z^8 + \sum_{k=0}^{7} a_k z^k$ with degree 8, $M_{\sigma_1}(p)$ is:

$$M_{\sigma_1}(p) = \begin{bmatrix} -a_7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_6 & 0 & -a_5 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_4 & 0 & -a_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_2 & 0 & -a_1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_0 & 0 \end{bmatrix}.$$

Just by looking at $M_{\sigma_1}(p)$ in this particular case, we may realize that the ∞-norm of $M_{\sigma_1}(p)$, for arbitrary degree n, is:

$$\|M_{\sigma_1}(p)\|_\infty = \begin{cases} \max\{|a_0|, 1+|a_1|+|a_2|, \ldots, 1+|a_{n-3}|+|a_{n-2}|, 1+|a_{n-1}|\}, & \text{if } n \text{ is even}, \\ \max\{|a_0|, 1+|a_1|+|a_2|, \ldots, 1+|a_{n-2}|+|a_{n-1}|\}, & \text{if } n \text{ is odd}, \end{cases}$$

which coincides with the upper bound in (18), i.e., with computing the norm via the formula (16).

Next, let $M_{\sigma_2}(p)$ be the Fiedler matrix of $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ with σ_2 such that

$$\mathrm{CISS}(\sigma_2) = \begin{cases} (0, 2, 1, 1, \ldots, 1, 1, 1, 1) \in \mathbb{R}^{1\times(n-1)}, & \text{if } n \text{ is odd}, \\ (0, 2, 1, 1, \ldots, 1, 1, 1, 0) \in \mathbb{R}^{1\times n}, & \text{if } n \text{ is even}. \end{cases}$$

For a monic polynomial $p(z)$ with degree 8, we have

$$M_{\sigma_2}(p) = \begin{bmatrix} -a_7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_6 & 0 & -a_5 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_4 & 0 & -a_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_2 & 0 & -a_1 & -a_0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Again, looking at $M_{\sigma_2}(p)$ in this example leads us to

$$\|M_{\sigma_2}(p)\|_\infty = \begin{cases} \max\{|a_0|+|a_1|+|a_2|, 1+|a_3|+|a_4|, \ldots, 1+|a_{n-3}|+|a_{n-2}|, 1+|a_{n-1}|\}, & \text{if } n \text{ is even}, \\ \max\{|a_0|+|a_1|+|a_2|, 1+|a_3|+|a_4|, \ldots, 1+|a_{n-2}|+|a_{n-1}|\}, & \text{if } n \text{ is odd}, \end{cases}$$

which again coincides with the upper bound in (18).

Although, $M_{\sigma_1}(p)$ and $M_{\sigma_2}(p)$ look almost the same and, as a consequence, $\|M_{\sigma_1}(p)\|_\infty$ and $\|M_{\sigma_2}(p)\|_\infty$ have also the same flavor, there are relevant differences. For instance, it is obvious that $\|C_2(p)\|_\infty \leq \|M_{\sigma_1}(p)\|_\infty$, that is, Cauchy's upper bound is sharper for any $p(z)$ than the upper bound provided by $M_{\sigma_1}(p)$. However, this is not true for $M_{\sigma_2}(p)$. For instance, if $a_0 = 10^{-16}$, $a_2 = 1$, and the rest of the a_i are all equal to zero, then $\|C_2(p)\|_\infty = 2$ and $\|M_{\sigma_2}(p)\|_\infty = 1 + 10^{-16}$. This means that the upper bound provided by $M_{\sigma_2}(p)$ may be sharper than Cauchy's upper bound at least by a factor 2 for certain polynomials. However, this is the maximum improvement over Cauchy's upper bound that can be obtained from the upper bound in (18) for any polynomial, and not only for $M_{\sigma_2}(p)$, but for any other Fiedler matrix, as we prove in Theorem 4.3.

Theorem 4.3 is the main result in this section. It proves that the bounds coming from applying (18) to all Fiedler matrices (i.e., from $\left(\|M_\sigma(p^\sharp)\|_\infty\right)^{-1} \leq |\lambda| \leq \|M_\sigma(p)\|_\infty$) never improve Cauchy's lower (i.e., $\left(\|C_2(p^\sharp)\|_\infty\right)^{-1}$) and Cauchy's upper (i.e., $\|C_2(p)\|_\infty$) bounds by a factor larger than 2. In this sense, the classical Cauchy's bounds in Theorem 1.1 are optimal, up to a factor 2, among those obtained from (18) and, in fact, we will see that they are strictly optimal for a large subclass of Fiedler matrices.

Theorem 4.3. *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial of degree $n \geq 2$ with $a_0 \neq 0$, let $p^\sharp(z)$ be the monic reversal polynomial of $p(z)$, let $\sigma : \{0, 1, \ldots, n-1\} \to \{1, \ldots, n\}$ be a bijection, and let $\mathrm{CISS}(\sigma) = (\mathfrak{c}_0, \mathfrak{i}_0, \mathfrak{c}_1, \mathfrak{i}_1, \ldots, \mathfrak{c}_\ell, \mathfrak{i}_\ell)$ be the consecution-inversion structure sequence of σ. Let $C_2(p)$ be the second Frobenius companion form of $p(z)$ and let $M_\sigma(p)$ be the Fiedler matrix of $p(z)$ associated with σ. Then the following statements hold.*

(a) *If* $\mathfrak{c}_0 > 0$, *then* $\|C_2(p)\|_\infty \leq \|M_\sigma(p)\|_\infty$.
(This means that Cauchy's upper bound is the sharpest upper bound among those in (18) *when* $\mathfrak{c}_0 > 0$.*)*

(b) *If* $\mathfrak{c}_0 = 0$, *then* $\|C_2(p)\|_\infty - 1 \leq \|M_\sigma(p)\|_\infty$.
(This means that Cauchy's upper bound is essentially the sharpest upper bound among those in (18) *when* $\|C_2(p)\|_\infty$ *is large.)*

(c) *If* $\mathfrak{c}_0 = 0$, *then* $\|C_2(p)\|_\infty/2 \leq \|M_\sigma(p)\|_\infty$.
(This means that none of the upper bounds in (18) *improves Cauchy's upper bound by a factor larger than two.)*

(d) *If* $\mathfrak{c}_0 > 0$, *then* $\left(\|M_\sigma(p^\sharp)\|_\infty\right)^{-1} \leq \left(\|C_2(p^\sharp)\|_\infty\right)^{-1}$.
(This means that Cauchy's lower bound is the sharpest lower bound among those in (18) *when* $\mathfrak{c}_0 > 0$.*)*

(e) *If* $\mathfrak{c}_0 = 0$, *then* $\left(\|M_\sigma(p^\sharp)\|_\infty\right)^{-1} \leq \left(\|C_2(p^\sharp)\|_\infty - 1\right)^{-1}$.
(This means that Cauchy's lower bound is essentially the sharpest lower bound among those in (18) *when* $\|C_2(p^\sharp)\|_\infty$ *is large.)*

(f) *If* $\mathfrak{c}_0 = 0$, *then* $\left(\|M_\sigma(p^\sharp)\|_\infty\right)^{-1} \leq 2\,\left(\|C_2(p^\sharp)\|_\infty\right)^{-1}$.
(This means that none of the lower bounds in (18) *improves Cauchy's lower bound by a factor larger than two.)*

Proof. In this proof we use the notation introduced in Theorems 3.3 and 4.1. Parts (a), (b), and (c) are consequences of the following three inequalities:

$$\text{if } \mathfrak{c}_0 > 0, \text{ then } \ \gamma_{\sigma,0}(p) \geq \max\{1+|a_1|, 1+|a_2|, \ldots, 1+|a_{s_0}|\}; \tag{19}$$

$$\text{if } \mathfrak{c}_0 = 0, \text{ then } 1+\gamma_{\sigma,0}(p) \geq \max\{1+|a_1|, 1+|a_2|, \ldots, 1+|a_{s_0}|\}; \tag{20}$$

and, for $k = 1, 2, \ldots, \ell$,

$$\gamma_{\sigma,k}(p) \geq \max\{1+|a_{s_{k-1}+1}|, \ldots, 1+|a_{s_k}|\}. \tag{21}$$

Proof of Part (a). From (16), (19), and (21), we get that if $\mathfrak{c}_0 > 0$, then

$$\|M_\sigma(p)\|_\infty = \max\{|a_0|, \gamma_{\sigma,0}(p), \ldots, \gamma_{\sigma,\ell}(p)\} \geq \max\{|a_0|, 1+|a_1|, \ldots, 1+|a_{n-1}|\} = \|C_2(p)\|_\infty\,.$$

Proof of Part (b). From (16), (20), and (21), we get that if $\mathfrak{c}_0 = 0$, then

$$\begin{aligned}1+\|M_\sigma(p)\|_\infty &= \max\{1+|a_0|, 1+\gamma_{\sigma,0}(p), \ldots, 1+\gamma_{\sigma,\ell}(p)\} \geq \max\{|a_0|, 1+\gamma_{\sigma,0}(p), \gamma_{\sigma,1}(p), \ldots, \gamma_{\sigma,\ell}(p)\} \\ &\geq \max\{|a_0|, 1+|a_1|, \ldots, 1+|a_{n-1}|\} = \|C_2(p)\|_\infty.\end{aligned}$$

Proof of Part (c). From (16), we have that $1 \leq \|M_\sigma(p)\|_\infty$. Therefore, from (b), $\|C_2(p)\|_\infty \leq \|M_\sigma(p)\|_\infty + 1 \leq 2\,\|M_\sigma(p)\|_\infty$, which is part (c).

Proofs of Parts (d), (e), and (f). Parts (a), (b), and (c) have been proved for any monic polynomial $p(z)$. Therefore, they can be applied to $p^\sharp(z)$ for proving parts (d), (e), and (f). □

Observe that there exist polynomials for which the inequalities in parts (b), (c), (e), and (f) of Theorem 4.3 become as close as equalities as desired. In the case of parts (b) and (c) this happens, for instance, for the polynomial considered in the last part of Example 4.2 if we take a_0 as small as we want. Note also that even in the case $\mathfrak{c}_0 = 0$, it is possible to find sufficient conditions on the coefficients of $p(z)$ that guarantee $\|C_2(p)\|_\infty \leq \|M_\sigma(p)\|_\infty$ for wide classes of polynomials and for all Fiedler matrices, and also to find sufficient conditions that guarantee $\left(\|M_\sigma(p^\sharp)\|_\infty\right)^{-1} \leq \left(\|C_2(p^\sharp)\|_\infty\right)^{-1}$ for wide classes of polynomials. We do not pursue this goal here since the inequalities proved in parts (b), (c), (e), and (f) show very clearly that Cauchy's bounds are essentially always the sharpest ones in the family (18).

5 Lower and upper bounds from ∞-norms of inverses of Fiedler matrices

As a direct consequence of (7) and the expression in Theorem 3.3 for the ∞-norm of the inverse of a Fiedler matrix we obtain in Theorem 5.1 the second family of new lower and upper bounds for the absolute values of the roots of monic polynomials presented in this paper. The key difference between Theorem 5.1 and Theorem 4.1 is that some of the bounds presented in Theorem 5.1 improve significantly the classical Cauchy's bounds for wide classes of polynomials. To prove this fact is one of the main goals in this section.

Theorem 5.1. *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ with $n \geq 2$ and $a_0 \neq 0$, let $p^\sharp(z)$ be the monic reversal polynomial of $p(z)$, and let $\sigma : \{0, 1, \ldots, n-1\} \to \{1, \ldots, n\}$ be a bijection. If λ is a root of $p(z)$, then*

$$\left(\max \left\{ \frac{1}{|a_0|}, \delta_\sigma(p), \gamma_{\sigma,1}(p), \ldots, \gamma_{\sigma,\ell}(p) \right\} \right)^{-1} \leq |\lambda| \leq \max\{|a_0|, \delta_\sigma(p^\sharp), \gamma_{\sigma,1}(p^\sharp), \ldots, \gamma_{\sigma,\ell}(p^\sharp)\}, \qquad (22)$$

where the quantities $\delta_\sigma(p)$, $\gamma_{\sigma,k}(p)$, for $k = 1, 2, \ldots, \ell$, and $\delta_\sigma(p^\sharp)$, $\gamma_{\sigma,k}(p^\sharp)$, for $k = 1, 2, \ldots, \ell$, are those defined in Theorem 3.3 for $p(z)$ and $p^\sharp(z)$, respectively.

For making comparisons, a key property that readers should bear in mind is that Cauchy's and Montel's lower and upper bounds are included among the bounds in (22) for certain choices of σ. This is a consequence of the more general result presented in Theorem 5.2.

Theorem 5.2. *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial of degree $n \geq 2$ and $a_0 \neq 0$, and let $p^\sharp(z)$ be the monic reversal polynomial of $p(z)$. Let $C_1(p)$ and $C_2(p)$ be the first and second Frobenius companion forms of $p(z)$. Then*

$$C_1(p)^{-1} = R\, C_1(p^\sharp)\, R, \qquad \text{and} \qquad C_2(p)^{-1} = R\, C_2(p^\sharp)\, R, \qquad (23)$$

where R is the reverse identity matrix, i.e.,

$$R = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

As a consequence,

(a) $\|C_i(p)^{-1}\|_s = \|C_i(p^\sharp)\|_s$, *for $i = 1, 2$ and $s = 1, 2, \infty, F$,*

(b) $\|C_i(p^\sharp)^{-1}\|_s = \|C_i(p)\|_s$, *for $i = 1, 2$ and $s = 1, 2, \infty, F$.*

Proof. The equalities in (23) follow from direct matrix multiplication, from the fact that the inverses of $C_1(p)$ and $C_2(p)$ are given by

$$C_1(p)^{-1} = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -1/a_0 & -a_{n-1}/a_0 & \cdots & -a_1/a_0 \end{bmatrix} \quad \text{and} \quad C_2(p)^{-1} = \begin{bmatrix} 0 & \ldots & 0 & -1/a_0 \\ 1 & & & -a_{n-1}/a_0 \\ & \ddots & & \vdots \\ & & 1 & -a_1/a_0 \end{bmatrix}, \qquad (24)$$

and from the expressions of the coefficients of $p^\sharp(z)$. Then, part (a) follows from (23) and the fact that 1-, 2-, ∞-, and Frobenius-norms are invariant under multiplication by the matrix R. Finally, part (b) follows from applying part (a) to $p^\sharp$ and the fact that $(p^\sharp)^\sharp = p$. □

Recall that Cauchy's and Montel's upper bounds are $\|C_2(p)\|_\infty$ and $\|C_1(p)\|_\infty$, respectively. So part (b) of Theorem 5.2 allows us to express Cauchy's upper bound as $\|C_2(p^\sharp)^{-1}\|_\infty = \|C_2(p)\|_\infty$ and Montel's upper bound as $\|C_1(p^\sharp)^{-1}\|_\infty = \|C_1(p)\|_\infty$. Since the upper bound in (22) is $\|M_\sigma(p^\sharp)^{-1}\|_\infty$, we see that Cauchy's and Montel's upper bounds are included among the upper bounds in (22). Analogously, part (a) of Theorem 5.2 allows us to see that Cauchy's lower bound is $\left(\|C_2(p)^{-1}\|_\infty\right)^{-1} = \left(\|C_2(p^\sharp)\|_\infty\right)^{-1}$, and that Montel's lower bound is $\left(\|C_1(p)^{-1}\|_\infty\right)^{-1} = \left(\|C_1(p^\sharp)\|_\infty\right)^{-1}$. Since the lower bound in (22) is $\left(\|M_\sigma(p)^{-1}\|_\infty\right)^{-1}$, we see that Cauchy's and Montel's lower bounds are two of the lower bounds in (22).

The Fiedler matrix $F(p) := M_1 M_2 \cdots M_{n-1} M_0$ will play a relevant role in determining which are the sharpest bounds among those in (22). The matrix $F(p)$ is associated with any bijection τ such that $\mathrm{CISS}(\tau) = (0, 1, n-2, 0)$ and the explicit expressions of $F(p)$ and $F(p)^{-1}$ are

$$F(p) = \begin{bmatrix} -a_{n-1} & 1 & & & \\ \vdots & & \ddots & & \\ -a_2 & & & 1 & \\ -a_1 & & & & -a_0 \\ 1 & & & & 0 \end{bmatrix} \quad \text{and} \quad F(p)^{-1} = \begin{bmatrix} 0 & & & & 1 \\ 1 & & & & a_{n-1} \\ & \ddots & & & \vdots \\ & & 1 & & a_2 \\ & & & -1/a_0 & -a_1/a_0 \end{bmatrix}. \tag{25}$$

The bounds (22) for $F(p)$ are summarized in Theorem 5.3 for future reference. These bounds are one of the most important contributions in this paper, since as it is explained in Theorems 5.5 and 5.7, they improve significantly Cauchy's upper and lower bounds for certain polynomials. Theorem 5.3 follows immediately from (7), the expression of $F(p)^{-1}$ in (25), and the expression for $F(p^\sharp)^{-1}$ that is obtained from applying the second expression in (25) to $p^\sharp(z)$.

Theorem 5.3. *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial with $n \geq 2$ and $a_0 \neq 0$, let $p^\sharp(z)$ be the monic reversal polynomial of $p(z)$, and let $F(p)$ be the Fiedler matrix in (25). Then*

(a) $\left(\|F(p)^{-1}\|_\infty\right)^{-1} = \min\left\{\frac{|a_0|}{1+|a_1|}, \frac{1}{1+|a_2|}, \ldots, \frac{1}{1+|a_{n-1}|}\right\}$*;*

(b) $\|F(p^\sharp)^{-1}\|_\infty = \max\left\{1 + \frac{|a_1|}{|a_0|}, 1 + \frac{|a_2|}{|a_0|}, \ldots, 1 + \frac{|a_{n-2}|}{|a_0|}, |a_0| + |a_{n-1}|\right\}$*; and,*

(c) $\left(\|F(p)^{-1}\|_\infty\right)^{-1} \leq |\lambda| \leq \|F(p^\sharp)^{-1}\|_\infty$*, that is,*

$$\min\left\{\frac{|a_0|}{1+|a_1|}, \frac{1}{1+|a_2|}, \ldots, \frac{1}{1+|a_{n-1}|}\right\} \leq |\lambda| \leq \max\left\{1 + \frac{|a_1|}{|a_0|}, 1 + \frac{|a_2|}{|a_0|}, \ldots, 1 + \frac{|a_{n-2}|}{|a_0|}, |a_0| + |a_{n-1}|\right\}.$$

Note that the lower bound in Theorem 5.3-(c) is precisely (9) in the Introduction, although written in a different way, and that the upper bound is (10).

Theorem 5.4 is the first important result on comparison of bounds in this section. It proves that either Cauchy's lower/upper bounds or the lower/upper bounds in part (c) of Theorem 5.3 are essentially the sharpest bounds among those coming from applying (22) to all Fiedler matrices. The absolute value of the zero degree coefficient of $p(z)$ is the key to distinguish whether Cauchy's bounds or the ones in Theorem 5.3 are the sharpest. In contrast, $|a_0|$ did not play any role in Theorem 4.3.

Theorem 5.4. *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial of degree $n \geq 2$ with $a_0 \neq 0$, let $p^\sharp(z)$ be the monic reversal polynomial of $p(z)$, let $\sigma : \{0, 1, \ldots, n-1\} \to \{1, \ldots, n\}$ be a bijection, and let $\mathrm{CISS}(\sigma) = (\mathfrak{c}_0, \mathfrak{i}_0, \mathfrak{c}_1, \mathfrak{i}_1, \ldots, \mathfrak{c}_\ell, \mathfrak{i}_\ell)$ be the consecution-inversion structure sequence of σ. Let $C_2(p)$ be the second Frobenius companion form of $p(z)$, let $F(p)$ be the Fiedler matrix of $p(z)$ in (25), and let $M_\sigma(p)$ be the Fiedler matrix of $p(z)$ associated with σ. Then the following statements hold.*

(a) *If $|a_0| \geq 1$ and $\mathfrak{c}_0 = 0$, then $\|F(p^\sharp)^{-1}\|_\infty \leq \|M_\sigma(p^\sharp)^{-1}\|_\infty$.*
(This means that $F(p)$ gives the sharpest upper bound among the upper bounds in (22) when $|a_0| \geq 1$ and $\mathfrak{c}_0 = 0$.)

(b) *If $|a_0| \geq 1$ and $\mathfrak{c}_0 > 0$, then $\|F(p^\sharp)^{-1}\|_\infty / 2 \leq \|M_\sigma(p^\sharp)^{-1}\|_\infty$.*
(This means that, when $|a_0| \geq 1$ and $\mathfrak{c}_0 > 0$, none of the upper bounds in (22) improves the upper bound given by $F(p)$ by a factor larger than two.)

(c) *If $|a_0| < 1$ and $\mathfrak{c}_0 > 0$, then $\|C_2(p^\sharp)^{-1}\|_\infty \leq \|M_\sigma(p^\sharp)^{-1}\|_\infty$.*
(This means that Cauchy's upper bound is the sharpest upper bound among those in (22) when $|a_0| < 1$ and $\mathfrak{c}_0 > 0$.)

(d) *If $|a_0| < 1$ and $\mathfrak{c}_0 = 0$, then $\|C_2(p^\sharp)^{-1}\|_\infty - 1 \leq \|M_\sigma(p^\sharp)^{-1}\|_\infty$.*
(This means that Cauchy's upper bound is essentially the sharpest upper bound among those in (22) when $|a_0| < 1$, $\mathfrak{c}_0 = 0$, and $\|C_2(p^\sharp)^{-1}\|_\infty$ is large.)

(e) *If $|a_0| < 1$ and $\mathfrak{c}_0 = 0$, then $\|C_2(p^\sharp)^{-1}\|_\infty / 2 \leq \|M_\sigma(p^\sharp)^{-1}\|_\infty$.*
(This means that, when $|a_0| < 1$ and $\mathfrak{c}_0 = 0$, none of the upper bounds in (22) improves Cauchy's upper bound by a factor larger than two.)

(f) *If $|a_0| \leq 1$ and $\mathfrak{c}_0 = 0$, then $\left(\|M_\sigma(p)^{-1}\|_\infty\right)^{-1} \leq \left(\|F(p)^{-1}\|_\infty\right)^{-1}$.*
(This means that $F(p)$ gives the sharpest lower bound among the lower bounds in (22) when $|a_0| \leq 1$ and $\mathfrak{c}_0 = 0$.)

(g) *If* $|a_0| \le 1$ *and* $\mathfrak{c}_0 > 0$*, then* $\left(\|M_\sigma(p)^{-1}\|_\infty\right)^{-1} \le 2\,\left(\|F(p)^{-1}\|_\infty\right)^{-1}$.
(This means that, when $|a_0| \le 1$ *and* $\mathfrak{c}_0 > 0$*, none of the lower bounds in* (22) *improves the lower bound given by* $F(p)$ *by a factor larger than two.)*

(h) *If* $|a_0| > 1$ *and* $\mathfrak{c}_0 > 0$*, then* $\left(\|M_\sigma(p)^{-1}\|_\infty\right)^{-1} \le \left(\|C_2(p)^{-1}\|_\infty\right)^{-1}$.
(This means that Cauchy's lower bound is the sharpest lower bound among those in (22) *when* $|a_0| > 1$ *and* $\mathfrak{c}_0 > 0$*.)*

(i) *If* $|a_0| > 1$ *and* $\mathfrak{c}_0 = 0$*, then* $\left(\|M_\sigma(p)^{-1}\|_\infty\right)^{-1} \le \left(\|C_2(p)^{-1}\|_\infty - 1\right)^{-1}$.
(This means that Cauchy's lower bound is essentially the sharpest lower bound among those in (22) *when* $|a_0| > 1$, $\mathfrak{c}_0 = 0$*, and* $\|C_2(p)^{-1}\|_\infty$ *is large.)*

(j) *If* $|a_0| > 1$ *and* $\mathfrak{c}_0 = 0$*, then* $\left(\|M_\sigma(p)^{-1}\|_\infty\right)^{-1} \le 2\,\left(\|C_2(p)^{-1}\|_\infty\right)^{-1}$.
(This means that, when $|a_0| > 1$ *and* $\mathfrak{c}_0 = 0$*, none of the lower bounds in* (22) *improves Cauchy's lower bound by a factor larger than two.)*

Proof. The expression (4) for the monic reversal polynomial of $p(z)$ implies that, $p(0)$, i.e., the zero-degree coefficient of $p(z)$, satisfies $|p(0)| = |a_0| \ge 1$ (resp., $|p(0)| = |a_0| < 1$) if and only if $|p^\sharp(0)| = 1/|a_0| \le 1$ (resp., $|p^\sharp(0)| = 1/|a_0| > 1$). From this, we see: that part (f) applied to $p^\sharp(z)$ implies part (a); that part (g) applied to $p^\sharp(z)$ implies part (b); that part (h) applied to $p^\sharp(z)$ implies part (c); that part (i) applied to $p^\sharp(z)$ implies part (d); and that part (j) applied to $p^\sharp(z)$ implies part (e). Therefore we only need to prove parts (f), (g), (h), (i), and (j). We will use the notation in Theorem 3.3 throughout the proof.

Proof of part (f). If $|a_0| \le 1$ and $\mathfrak{c}_0 = 0$, then

$$\max\left\{\frac{1}{|a_0|}, \delta_\sigma(p)\right\} = \delta_\sigma(p) \ge \max\left\{\frac{1}{|a_0|} + \frac{|a_1|}{|a_0|}, 1 + |a_2|, \ldots, 1 + |a_{s_0}|\right\}.$$

This inequality, together with (21), (17), and (25) imply $\|M_\sigma(p)^{-1}\|_\infty \ge \|F(p)^{-1}\|_\infty$.

Proof of Part (g). If $|a_0| \le 1$ and $\mathfrak{c}_0 > 0$, then

$$\begin{aligned}\max\left\{\frac{1}{|a_0|}, \delta_\sigma(p)\right\} &= \max\left\{\frac{1}{|a_0|}, 1 + \frac{|a_1|}{|a_0|}, \ldots, 1 + \frac{|a_{\mathfrak{c}_0 - 1}|}{|a_0|}, 1 + \frac{|a_{\mathfrak{c}_0}|}{|a_0|} + |a_{\mathfrak{c}_0 + 1}| + \cdots + |a_{s_0}|\right\} \\ &\ge \max\left\{\frac{1}{|a_0|}, 1 + \frac{|a_1|}{|a_0|}, 1 + |a_2|, \ldots, 1 + |a_{s_0}|\right\} \ge \frac{1}{2}\max\left\{\frac{1}{|a_0|} + \frac{|a_1|}{|a_0|}, 1 + |a_2|, \ldots, 1 + |a_{s_0}|\right\},\end{aligned}$$

where in the first inequality we have used that $|a_0| \le 1$. In addition, from (21), for $k = 1, 2, \ldots, \ell$,

$$\gamma_{\sigma,k}(p) \ge \max\left\{1 + |a_{s_{k-1}+1}|, \ldots, 1 + |a_{s_k}|\right\} \ge \frac{1}{2}\max\left\{1 + |a_{s_{k-1}+1}|, \ldots, 1 + |a_{s_k}|\right\}.$$

Combining these results with (17) and (25), we get $\|M_\sigma(p)^{-1}\|_\infty \ge \frac{1}{2}\|F(p)^{-1}\|_\infty$.

Proof of Part (h). If $|a_0| > 1$ and $\mathfrak{c}_0 > 0$, then

$$\begin{aligned}\max\left\{\frac{1}{|a_0|}, \delta_\sigma(p)\right\} &= \max\left\{\frac{1}{|a_0|}, 1 + \frac{|a_1|}{|a_0|}, \ldots, 1 + \frac{|a_{\mathfrak{c}_0 - 1}|}{|a_0|}, 1 + \frac{|a_{\mathfrak{c}_0}|}{|a_0|} + |a_{\mathfrak{c}_0 + 1}| + \cdots + |a_{s_0}|\right\} \\ &\ge \max\left\{\frac{1}{|a_0|}, 1 + \frac{|a_1|}{|a_0|}, 1 + \frac{|a_2|}{|a_0|}, \ldots, 1 + \frac{|a_{s_0}|}{|a_0|}\right\}.\end{aligned}$$

In addition, from (21), for $k = 1, 2, \ldots, \ell$,

$$\gamma_{\sigma,k}(p) \ge \max\left\{1 + |a_{s_{k-1}+1}|, \ldots, 1 + |a_{s_k}|\right\} \ge \max\left\{1 + \frac{|a_{s_{k-1}+1}|}{|a_0|}, \ldots, 1 + \frac{|a_{s_k}|}{|a_0|}\right\}.$$

Combining these results with (17) and (24), we get $\|M_\sigma(p)^{-1}\|_\infty \ge \|C_2(p)^{-1}\|_\infty$.

Proof of Part (i). If $|a_0| > 1$ and $\mathfrak{c}_0 = 0$, then

$$\begin{aligned}\max\left\{\frac{1}{|a_0|}, \delta_\sigma(p)\right\} &\ge \frac{1}{|a_0|} + \frac{|a_1|}{|a_0|} + \cdots + \frac{|a_{s_0}|}{|a_0|} = \left(\frac{1}{|a_0|} + \frac{|a_1|}{|a_0|} + \cdots + \frac{|a_{s_0}|}{|a_0|} + 1\right) - 1 \\ &\ge \max\left\{\frac{1}{|a_0|}, 1 + \frac{|a_1|}{|a_0|}, 1 + \frac{|a_2|}{|a_0|}, \ldots, 1 + \frac{|a_{s_0}|}{|a_0|}\right\} - 1,\end{aligned}$$

and, for $k = 1, 2, \ldots, \ell$,

$$\gamma_{\sigma,k}(p) \geq \max\left\{1 + \frac{|a_{s_{k-1}+1}|}{|a_0|}, \ldots, 1 + \frac{|a_{s_k}|}{|a_0|}\right\} \geq \max\left\{1 + \frac{|a_{s_{k-1}+1}|}{|a_0|}, \ldots, 1 + \frac{|a_{s_k}|}{|a_0|}\right\} - 1.$$

Combining these results with (17) and (24), we get $\|M_\sigma(p)^{-1}\|_\infty \geq \|C_2(p)^{-1}\|_\infty - 1$.

Proof of Part (j). From Part (i) and the fact that $1 \leq \|M_\sigma(p)^{-1}\|_\infty$, it follows that $\|C_2(p)^{-1}\|_\infty \leq \|M_\sigma(p)^{-1}\|_\infty + 1 \leq 2\,\|M_\sigma(p)^{-1}\|_\infty$. □

Although parts (a) and (b) of Theorem 5.4 tell us that $\|F(p^\sharp)^{-1}\|_\infty$ is essentially the sharpest upper bound among those in (22) when $|a_0| \geq 1$, they do not establish whether or not $\|F(p^\sharp)^{-1}\|_\infty$ improves significantly Cauchy's upper bound. Theorem 5.5 shows that it is possible to construct polynomials for which $\|F(p^\sharp)^{-1}\|_\infty$ can be extremely smaller than Cauchy's upper bound.

Theorem 5.5. *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial with degree $n \geq 2$ and $a_0 \neq 0$, let $p^\sharp(z)$ be the monic reverse polynomial of $p(z)$, let $U_C(p)$ be the Cauchy's upper bound for $p(z)$, and let $F(p^\sharp)$ be the Fiedler companion matrix of $p^\sharp(z)$ defined in* (25)*. If the coefficients of $p(z)$ satisfy*

$$\max\{|a_1|, \ldots, |a_{n-2}|\} \geq |a_0|(|a_0| + |a_{n-1}| - 1) \quad \textit{and} \quad |a_0| > 1, \tag{26}$$

then

$$\frac{U_C(p)}{\|F(p^\sharp)^{-1}\|_\infty} \geq \frac{|a_0|}{2}.$$

Proof. If the inequality (26) is satisfied, then

$$\|F(p^\sharp)^{-1}\|_\infty = \max\left\{1 + \frac{|a_1|}{|a_0|}, \ldots, 1 + \frac{|a_{n-2}|}{|a_0|}, |a_0| + |a_{n-1}|\right\} = 1 + \frac{1}{|a_0|}\max\{|a_1|, \ldots, |a_{n-2}|\}$$

and $U_C(p) = \max\{1 + |a_1|, \ldots, 1 + |a_{n-1}|\} \geq 1 + \max\{|a_1|, \ldots, |a_{n-2}|\}$. Therefore,

$$\frac{U_C(p)}{\|F(p^\sharp)^{-1}\|_\infty} \geq \frac{1 + \max\{|a_1|, \ldots, |a_{n-2}|\}}{1 + \frac{1}{|a_0|}\max\{|a_1|, \ldots, |a_{n-2}|\}} \geq \frac{|a_0|}{2},$$

where the last inequality is a particular case of the more general inequality $(1+a)/(1+a/b) \geq b/2$, which is valid for any positive numbers $a > 0$ and $b > 0$ such that $1 + a/2 \geq b/2$. Observe that (26) guarantees that these conditions are satisfied with $a = \max\{|a_1|, \ldots, |a_{n-2}|\}$ and $b = |a_0|$ (it may help to distinguish the cases $|a_0| > 2$ and $2 \geq |a_0| > 1$). □

Theorem 5.5 states that if (26) is satisfied and $|a_0|$ is very large, then Cauchy's upper bound for the absolute values of the roots of a monic polynomial is much larger than the upper bound $\|F(p^\sharp)^{-1}\|_\infty$. Notice that, however, in order for (26) to hold when $|a_0|$ is large, there must be another coefficient of $p(z)$ whose absolute value is larger than approximately $|a_0|^2$. This is the case of the following example that illustrates Theorem 5.5.

Example 5.6. *Consider the monic polynomial $p(z) = z^3 + z^2 + 10^{2m}z + 10^m$, for some integer $m > 0$. For this polynomial we have the following upper bounds*

$$|\lambda| \leq \|F(p^\sharp)^{-1}\|_\infty = 1 + 10^m \approx 10^m,$$
$$|\lambda| \leq U_C(p) = 1 + 10^{2m} \approx 10^{2m}, \qquad (\textit{Cauchy}),$$

and $\max\{|\lambda| : \lambda$ *is a root of* $p(z)\} \approx 10^m$*. We observe that the bound $\|F(p^\sharp)^{-1}\|_\infty$ is essentially optimal, while Cauchy's upper bound is extremely larger than $|\lambda|$ if m is large.*

Although parts (f) and (g) of Theorem 5.4 tell us that $\left(\|F(p)^{-1}\|_\infty\right)^{-1}$ is essentially the sharpest lower bound among those in (22) when $|a_0| \leq 1$, they do not establish whether or not this bound improves significantly Cauchy's lower bound. Theorem 5.7 shows that it is possible to construct polynomials for which $\left(\|F(p)^{-1}\|_\infty\right)^{-1}$ can be extremely larger than Cauchy's lower bound.

Theorem 5.7. *Let* $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ *be a monic polynomial with degree* $n \geq 2$ *and* $a_0 \neq 0$, *let* $L_C(p)$ *be Cauchy's lower bound for* $p(z)$, *and let* $F(p)$ *be the Fiedler matrix defined in* (25). *If the coefficients of* $p(z)$ *satisfy*

$$\max\{|a_2|, \dots, |a_{n-1}|\} \geq \frac{1+|a_1|}{|a_0|}, \quad \textit{and} \quad |a_0| \leq 1, \tag{27}$$

then

$$\frac{\left(\|F(p)^{-1}\|_\infty\right)^{-1}}{L_C(p)} \geq \frac{1}{2|a_0|}. \tag{28}$$

Proof. Conditions (27) and expression (25) imply

$$\|F(p)^{-1}\|_\infty = \max\left\{\frac{1+|a_1|}{|a_0|}, 1+|a_2|, \dots, 1+|a_{n-1}|\right\} = 1 + \max\{|a_2|, \dots, |a_{n-1}|\},$$

and $\max\{|a_2|, \dots, |a_{n-1}|\} \geq 1$. Also, we have that

$$L_C(p)^{-1} = \max\left\{\frac{1}{|a_0|}, 1 + \frac{|a_1|}{|a_0|}, \dots, 1 + \frac{|a_{n-1}|}{|a_0|}\right\} \geq 1 + \frac{\max\{|a_2|, \dots, |a_{n-1}|\}}{|a_0|}.$$

Then

$$\frac{L_C(p)^{-1}}{\|F(p)^{-1}\|_\infty} \geq \frac{1 + \max\{|a_2|, \dots, |a_{n-1}|\}/|a_0|}{1 + \max\{|a_2|, \dots, |a_{n-1}|\}} \geq \frac{1}{2|a_0|}.$$

The last inequality is a particular case of the general inequality $(1 + a/b)/(1+a) \geq 1/(2b)$, which is valid for any numbers such that $b > 0$ and $a \geq 1$. □

Theorem 5.7 states that if (27) is satisfied and $|a_0|$ is very small, then Cauchy's lower bound for the absolute values of the roots of a monic polynomial is much smaller than the lower bound $\left(\|F(p)^{-1}\|_\infty\right)^{-1}$. Note that in order for (27) to hold when $|a_0|$ is small, at least one of the coefficients $a_2, \dots, a_{n-1}$ must have a large absolute value. This is the case in Example 5.8, which illustrates Theorem 5.7.

Example 5.8. *Consider the monic polynomial* $p(z) = z^3 + 2 \cdot 10^m z^2 + z + 10^{-m}$, *with* m *a positive integer. For this polynomial we have the following lower bounds*

$$|\lambda| \geq \left(\|F(p)^{-1}\|_\infty\right)^{-1} = \frac{1}{1 + 2 \cdot 10^m} \approx 0.5 \cdot 10^{-m},$$
$$|\lambda| \geq L_C(p) = \frac{1}{1 + 2 \cdot 10^{2m}} \approx 0.5 \cdot 10^{-2m}, \qquad (\textit{Cauchy}),$$

and $\min\{|\lambda| : \lambda \, \textit{is a root of} \, p(z)\} \approx 0.7 \cdot 10^{-m}$. *We observe that the bound* $\left(\|F(p)^{-1}\|_\infty\right)^{-1}$ *is almost optimal, while Cauchy's bound is extremely smaller than* $|\lambda|$ *if* m *is large.*

6 Lower and upper bounds from Frobenius norms of inverses of Fiedler matrices

As we commented in the Introduction, the use of (6) with the Frobenius norm makes no sense since all Fiedler matrices of a given monic polynomial have the same Frobenius norm [8, Corollary 2.9] and, therefore, we obtain exactly the same bounds as in part 4 of Theorem 1.1 in all cases. However, the use of (7) with the Frobenius norm may produce new bounds, since the inverses of all Fiedler matrices of a given monic polynomial *do not have always* the same Frobenius norm. In fact, given $p(z)$, $\|M_\sigma(p)^{-1}\|_F$ depends only on t_σ, i.e., on the number of initial consecutions or inversions of σ [8, Corollary 3.3]. In this context, the purpose of this section is to study the bounds $\left(\|M_\sigma(p)^{-1}\|_F\right)^{-1} \leq |\lambda| \leq \|M_\sigma(p^\sharp)^{-1}\|_F$ for the absolute values of the roots λ of a monic polynomial $p(z)$ and to compare them with Cauchy's lower/upper bounds and with the bounds in Theorem 5.3-(c). The main conclusion is that, although the new bounds coming from the Frobenius norm may be sharper in certain situations, the improvements are never significative.

Theorem 6.1 is a direct consequence of (7) and [8, Corollary 3.3].

Theorem 6.1. *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial with $n \geq 2$ and $a_0 \neq 0$, let $\sigma : \{0, 1, \dots, n-1\} \to \{1, \dots, n\}$ be a bijection, and let t_σ be the number of initial consecutions or inversions of σ. If λ is a root of $p(z)$, then $\left(\|M_\sigma(p)^{-1}\|_F\right)^{-1} \leq |\lambda| \leq \|M_\sigma(p^\sharp)^{-1}\|_F$, that is,*

$$\frac{1}{\sqrt{(n-1) + \frac{1+|a_1|^2+\cdots+|a_{t_\sigma}|^2}{|a_0|^2} + |a_{t_\sigma+1}|^2 + \cdots + |a_{n-1}|^2}} \leq |\lambda| \qquad \text{and} \tag{29}$$

$$|\lambda| \leq \sqrt{(n-1) + |a_0|^2 + |a_{n-1}|^2 + |a_{n-2}|^2 + \cdots + |a_{n-t_\sigma}|^2 + \frac{|a_{n-t_\sigma-1}|^2 + \cdots + |a_1|^2}{|a_0|^2}}\,. \tag{30}$$

Given $p(z)$, the bounds (29) and (30) depend only on t_σ. On the other hand, the second companion form $C_2(p)$ is a Fiedler matrix that corresponds to the maximum value of t_σ, i.e., $t_\sigma = n-1$, while the matrix $F(p)$ in (25) corresponds to the minimum value $t_\sigma = 1$. This allows us to prove Theorem 6.2 directly from (29)-(30). The reader should recall that the lower and upper bounds of part 4 in Theorem 1.1 are, respectively, $\left(\|C_2(p^\sharp)\|_F\right)^{-1}$ and $\|C_2(p)\|_F$, which are equal, respectively, to $\left(\|C_2(p)^{-1}\|_F\right)^{-1}$ and $\|C_2(p^\sharp)^{-1}\|_F$, by Theorem 5.2.

Theorem 6.2. *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial of degree $n \geq 2$ with $a_0 \neq 0$, let $p^\sharp(z)$ be the monic reversal polynomial of $p(z)$, and let $\sigma : \{0, 1, \dots, n-1\} \to \{1, \dots, n\}$ be a bijection. Let $C_2(p)$ be the second Frobenius companion form of $p(z)$, let $F(p)$ be the Fiedler matrix defined in (25), and let $M_\sigma(p)$ be the Fiedler matrix of $p(z)$ associated with σ. Then the following statements hold.*

(a) *If $|a_0| \geq 1$, then $\|F(p^\sharp)^{-1}\|_F \leq \|M_\sigma(p^\sharp)^{-1}\|_F$.*
(This means that $F(p)$ gives the sharpest upper bound among the upper bounds in (30) when $|a_0| \geq 1$.)

(b) *If $|a_0| < 1$, then $\|C_2(p^\sharp)^{-1}\|_F \leq \|M_\sigma(p^\sharp)^{-1}\|_F$.*
(This means that the upper bound in part 4 of Theorem 1.1 is the sharpest upper bound among the upper bounds in (30) when $|a_0| < 1$.)

(c) *If $|a_0| \leq 1$, then $\left(\|M_\sigma(p)^{-1}\|_F\right)^{-1} \leq \left(\|F(p)^{-1}\|_F\right)^{-1}$.*
(This means that $F(p)$ gives the sharpest lower bound among the lower bounds in (29) when $|a_0| \leq 1$.)

(d) *If $|a_0| > 1$, then $\left(\|M_\sigma(p)^{-1}\|_F\right)^{-1} \leq \left(\|C_2(p)^{-1}\|_F\right)^{-1}$.*
(This means that the lower bound in part 4 of Theorem 1.1 is the sharpest lower bound among the lower bounds in (29) when $|a_0| > 1$.)

Part (b) in Theorem 6.2 shows us that when $|a_0| < 1$, the upper bounds in (30) are of no interest, since all of them are larger than the upper bound in part 4 of Theorem 1.1, which is larger than Carmichael-Mason upper bound, which in turn is larger than Cauchy's upper bound divided by $\sqrt{2}$. Analogously, part (d) in Theorem 6.2 shows us that when $|a_0| > 1$, the lower bounds in (29) are of no interest, since all of them are smaller than the lower bound in part 4 of Theorem 1.1.

However, parts (a) and (c) of Theorem 6.2 suggest that the upper bound $\|F(p^\sharp)^{-1}\|_F$ and/or the lower bound $\left(\|F(p)^{-1}\|_F\right)^{-1}$ might improve in certain situations previously known upper/lower bounds for the absolute values of the roots of monic polynomials. In fact, this is true, but Theorem 6.3 shows that these improvements are never larger than a factor $\sqrt{2}$, that is, the improvements are never really significative. This is shown by comparing these bounds with those established in Theorem 5.3.

Theorem 6.3. *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial with $n \geq 2$ and $a_0 \neq 0$ and let $F(p)$ be the Fiedler matrix of $p(z)$ defined in (25). Then*

(a) $\dfrac{1}{\sqrt{2}}\|F(p^\sharp)^{-1}\|_\infty \leq \|F(p^\sharp)^{-1}\|_F$,

(b) $\dfrac{1}{\sqrt{2}}\left(\|F(p)^{-1}\|_F\right)^{-1} \leq \left(\|F(p)^{-1}\|_\infty\right)^{-1}$.

Proof. Part (a) follows from applying part (b) to $p^\sharp(z)$. Therefore we only prove part (b). We have that

$$\frac{||F^{-1}(p)||_F}{||F^{-1}(p)||_\infty} = \frac{\sqrt{(n-1) + \frac{1}{|a_0|^2} + \frac{|a_1|^2}{|a_0|^2} + |a_2|^2 + \cdots + |a_{n-1}|^2}}{\max\left\{\frac{1}{|a_0|} + \frac{|a_1|}{|a_0|}, 1+|a_2|, 1+|a_3|, \dots, 1+|a_{n-1}|\right\}}.$$

Next, use

$$\frac{1}{|a_0|} + \frac{|a_1|}{|a_0|} \le \sqrt{2}\sqrt{\left(\frac{1}{|a_0|}\right)^2 + \left(\frac{|a_1|}{|a_0|}\right)^2} \quad \text{and} \quad 1 + |a_i| \le \sqrt{2}\sqrt{1+|a_i|^2},\ i = 2, \dots, n-1,$$

and the result follows immediately. □

7 Optimal bounds based on norms of diagonal similarities

All upper bounds presented in this manuscript for the absolute values of the roots λ of $p(z)$, and the majority of the bounds existing in the literature, are functions only of the absolute values of the coefficients of $p(z)$. A well-known bound of this type is the unique positive real root of $u(z) = z^n - \sum_{k=0}^{n-1} |a_k| z^k$, which will be denoted by $R(p)$. The first proof that $|\lambda| \le R(p)$ is attributed to Cauchy [5]. This classical result is also proved in [19] as a corollary of Pellet's theorem and a recent proof can be found in [12, p.14]. Note that the fact that $u(z)$ has a unique positive real root, whenever $a_i \neq 0$ for at least one $i \in \{0, 1, \dots, n-1\}$, follows easily from Descartes's rule of signs. Among all bounds on $|\lambda|$ that depend only on $|a_i|$, $i = 0, 1, \dots, n-1$, the sharpest one is precisely $R(p)$. This was stated in [19] and it is proved in Theorem 7.1 for completeness.

Theorem 7.1. [19, p. 61] *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial with $a_i \neq 0$ for at least one $i \in \{0, 1, \dots, n-1\}$, and let $R(p)$ be the unique positive real root of $u(z) = z^n - \sum_{k=0}^{n-1} |a_k| z^k$. If $B(p)$ is an upper bound on the absolute values of the roots of $p(z)$ that is a function only of $|a_0|, |a_1|, \dots, |a_{n-1}|$, then $R(p) \le B(p)$.*

Proof. Since $B(p)$ depends only on $|a_i|$, $i = 0, 1, \dots, n-1$, we have that $B(p) = B(u)$ and, since $R(p)$ is a root of $u(z)$ and is positive, we have that $R(p) \le B(u) = B(p)$. □

The optimality of $R(p)$ makes it very interesting for the theoretical purpose of testing the quality of other upper bounds for $|\lambda|$ that depend only on the absolute values of the coefficients of the polynomial. However, $R(p)$ has a limited practical interest since its computation requires to compute the root of a polynomial[5]. In the context of this paper, the optimal bound $R(p)$ is used in Theorem 7.4, which establishes that for all Fiedler companion matrices of $p(z)$ the optimal upper bound that can be obtained by using the ∞-norm and diagonal similarities is, in all cases, precisely $R(p)$. However, this result is again mainly of theoretical interest, since there is not an easy way of choosing the optimal diagonal similarity.

The proof of Theorem 7.4 requires to use two lemmas. The first one is Lemma 7.2, which merges Theorem 1, Corollary 1, and Corollary 2 in [18]. The concepts mentioned in the statement of Lemma 7.2 are contained in [13]. Also, note that all the inequalities containing vectors should be understood componentwise.

Lemma 7.2. *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ and let $\rho(|A|)$ be the spectral radius of $|A| = (|a_{ij}|)$. Then:*

(a) $\inf_{D \text{ diagonal}} ||D^{-1}AD||_\infty = \rho(|A|)$.

(b) *There exists a vector $x = (x_i) > 0$ such that $|A|x - \rho(|A|)x \le 0$ if and only if*

$$\min_{D \text{ diagonal}} ||D^{-1}AD||_\infty = \rho(|A|).$$

In this case, the minimum is attained at $D' = \operatorname{diag}(x) := \operatorname{diag}(x_1, x_2, \dots, x_n)$.

(c) *If A is irreducible, then* (b) *holds and the minimum is attained in the right positive eigenvector x of $|A|$ corresponding to $\rho(|A|)$, i.e., in the right Perron vector of $|A|$.*

Lemma 7.3 is the other lemma that we need to prove Theorem 7.4. We use in Lemma 7.3 the concept of *directed graph of a matrix* as is defined in [13, Definition 6.2.11].

Lemma 7.3. *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial, let $\sigma : \{0, 1, \dots, n-1\} \to \{1, \dots, n\}$ be a bijection, and let $M_\sigma(p)$ be the Fiedler matrix of $p(z)$ associated with σ. Then,*

(a) *The directed graph of $M_\sigma(p)$ has a cycle that visits all nodes if and only if $a_0 \neq 0$.*

(b) *$M_\sigma(p)$ is an irreducible matrix if and only if $a_0 \neq 0$.*

[5] Although this root is a very special one, and fast methods for computing it can be easily devised.

Proof. In the proof we will use the fact that $M_\sigma(p)$ has either a row (if σ has a consecution at 0) or a column (if σ has an inversion at 0) whose entries are $-a_0$ together with $n-1$ zeros. This follows easily by induction on the matrices W_i defined in `Algorithm 1` in Theorem 2.4. We will denote by $\Gamma(A)$ the directed graph of a matrix A.

Proof of part (a). If $a_0 = 0$, then $M_\sigma(p)$ has either a row or a column with all its entries equal to zero. In both cases the corresponding vertex cannot be visited by a cycle and, therefore, $\Gamma(M_\sigma(p))$ has not a cycle visiting all nodes.

If $a_0 \neq 0$, then we proceed by induction on the matrices W_i defined in `Algorithm 1`. The result is obviously true for W_0 since the entries $W_0(1,2)$ and $W_0(2,1)$ are both different from zero and, so, $\Gamma(W_0)$ has a cycle visiting all nodes. Let us assume that the result is true for the $(i+1)\times(i+1)$ matrix W_{i-1}. We need to distinguish two cases: σ has a consecution at i or σ has an inversion at i. We only prove the result in the case when σ has a consecution at i, since the other one is similar. The fact that $\Gamma(W_{i-1})$ has a cycle that visits all nodes is equivalent to the fact that there exists a permutation $(j_2, j_3, \ldots, j_{i+1})$ of the indices $(2, 3, \ldots, i+1)$ such that

$$W_{i-1}(1,j_2)\,W_{i-1}(j_2,j_3)\,W_{i-1}(j_3,j_4)\cdots W_{i-1}(j_i,j_{i+1})\,W_{i-1}(j_{i+1},1) \neq 0. \tag{31}$$

The expression of W_i in terms of W_{i-1} given in `Algorithm 1` allows us to write (31) in terms of entries of W_i as follows

$$W_i(2,j_2+1)\,W_i(j_2+1,j_3+1)\,W_i(j_3+1,j_4+1)\cdots W_i(j_i+1,j_{i+1}+1)\,W_i(j_{i+1}+1,1) \neq 0$$

and, since $W_i(1,2) = 1$, we get

$$W_i(1,2)\,W_i(2,j_2+1)\,W_i(j_2+1,j_3+1)\,W_i(j_3+1,j_4+1)\cdots W_i(j_i+1,j_{i+1}+1)\,W_i(j_{i+1}+1,1) \neq 0,$$

which corresponds to a cycle that visits all nodes in $\Gamma(W_i)$.

Proof of part (b). If $a_0 = 0$, then $M_\sigma(p)$ has either a row or a column with all its entries equal to zero. If $M_\sigma(p)$ has a zero row, then select a permutation matrix Π such that $\Pi^T M_\sigma(p)\Pi$ has the nth row equal to zero and we see by definition that $M_\sigma(p)$ is reducible. If $M_\sigma(p)$ has a zero column, then select a permutation matrix Π such that $\Pi^T M_\sigma(p)\Pi$ has the nth column equal to zero and we see by definition that $M_\sigma(p)$ is reducible.

If $a_0 \neq 0$, then, by part (a), $\Gamma(M_\sigma(p))$ is strongly connected [13, Definition 6.2.13] and this equivalent to the fact that $M_\sigma(p)$ is irreducible [13, Theorem 6.2.24]. □

Now, we are in the position of proving the main result of this section.

Theorem 7.4. *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial with $a_i \neq 0$ for at least one $i \in \{0, 1, \ldots, n-1\}$, let $\sigma : \{0, 1, \ldots, n-1\} \to \{1, \ldots, n\}$ be a bijection, let $M_\sigma(p)$ be the Fiedler matrix of $p(z)$ associated with σ, and let $R(p)$ be the unique positive root of $u(z) = z^n - \sum_{k=0}^{n-1} |a_k| z^k$. Then:*

(a) *$R(p)$ is the spectral radius of $|M_\sigma(p)|$.*

(b) $\inf_{D \text{ diagonal}} ||D^{-1}M_\sigma(p)D||_\infty = R(p)$.

(c) *Moreover, if $a_0 \neq 0$ and if we denote by $x_\sigma(p) \in \mathbb{R}^n$ the right Perron vector of $|M_\sigma(p)|$, then*

$$\min_{D \text{ diagonal}} \|D^{-1}M_\sigma(p)D\|_\infty = R(p), \tag{32}$$

and the minimum is attained at $D' = \operatorname{diag}(x_\sigma(p))$.

Proof. By Theorem 2.5, we have that $u(z) = z^n - \sum_{k=0}^{n-1} |a_k| z^k$ is the characteristic polynomial of the nonnegative matrix $|M_\sigma(p)| = M_\sigma(u)$. The discussion at the beginning of this section implies that $R(p) \geq |\mu|$ for any other root μ of $u(z)$, i.e., for any other eigenvalue of $M_\sigma(u)$. This proves (a). Part (b) follows from Lemma 7.2(a). Finally, part (c) follows from Lemma 7.2(c) and Lemma 7.3. □

Theorem 7.4(b) does not guarantee that the infimum is attained and does not explain how to find an optimal diagonal similarity if $a_0 = 0$. However, in the case of the first Frobenius companion form $C_1(p)$ this problem can be easily fixed. This is shown in Proposition 7.5.

Proposition 7.5. *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial with $a_i \neq 0$ for at least one $i \in \{0, 1, \ldots, n-1\}$, let $C_1(p)$ be the first Frobenius companion form of $p(z)$, and let $R(p)$ be the unique positive root of $u(z) = z^n - \sum_{k=0}^{n-1} |a_k| z^k$. If $D = \operatorname{diag}(R(p)^{n-1}, \ldots, R(p), 1)$, then*

$$||D^{-1}C_1(p)D||_\infty = R(p).$$

Proof. If $x^T = \begin{bmatrix} R(p)^{n-1}, & \cdots, & R(p), & 1 \end{bmatrix}$, then it may be checked that $|C_1(p)|x = R(p)x$. Since $x > 0$, Theorem 7.4(a) and Lemma 7.2(b) imply the result. $\square$

It is natural to conjecture that a result similar to Proposition 7.5 also holds for any Fiedler matrix just by replacing $\begin{bmatrix} R(p)^{n-1}, & \cdots, & R(p), & 1 \end{bmatrix}^T$ by the corresponding right Perron vector. However, Example 7.6 shows that this is not true, since the right Perron vectors of the entrywise absolute values of Fiedler matrices different that the first Frobenius companion form may have zero entries when $a_0 = 0$ and, so, we cannot apply Lemma 7.2(b) based on the Perron vectors.

Example 7.6. *Consider the four Fiedler matrices associated with a polynomial $p(z) = z^3 + a_2 z^2 + a_1 z + a_0$ with $a_i \neq 0$ for at least one $i \in \{0, 1, 2\}$, that is,*

$$M_{\sigma_1} = \begin{bmatrix} -a_2 & -a_1 & -a_0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_{\sigma_2} = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix},$$

$$M_{\sigma_3} = \begin{bmatrix} -a_2 & -a_1 & 1 \\ 1 & 0 & 0 \\ 0 & -a_0 & 0 \end{bmatrix}, \quad M_{\sigma_4} = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & -a_0 \\ 1 & 0 & 0 \end{bmatrix},$$

and let $R(p)$ be the unique positive root of $u(z) = z^3 - |a_2|z^2 - |a_1|z - |a_0|$. It can be checked that the eigenvectors of $|M_{\sigma_1}|, |M_{\sigma_2}|, |M_{\sigma_3}|, |M_{\sigma_4}|$ associated with $R(p)$ are, respectively,

$$x_{\sigma_1}(p) = \begin{bmatrix} R(p)^2 \\ R(p) \\ 1 \end{bmatrix}, \quad x_{\sigma_2}(p) = \begin{bmatrix} 1 \\ R(p) - |a_2| \\ R(p)^2 - |a_2|R(p) - |a_1| \end{bmatrix},$$

$$x_{\sigma_3}(p) = \begin{bmatrix} R(p) \\ 1 \\ R(p)^2 - |a_2|R(p) - |a_1| \end{bmatrix}, \quad \text{and} \quad x_{\sigma_4}(p) = \begin{bmatrix} R(p) \\ R(p)^2 - |a_2|R(p) \\ 1 \end{bmatrix},$$

which have nonnegative entries as a consequence of $u(R(p)) = 0$. If we denote by D_1, D_2, D_3, D_4 the diagonal matrices $\mathrm{diag}(x_{\sigma_1}(p)), \mathrm{diag}(x_{\sigma_2}(p)), \mathrm{diag}(x_{\sigma_3}(p)), \mathrm{diag}(x_{\sigma_4}(p))$, respectively, then D_1 is the only one that is nonsingular for any values of a_2, a_1, and a_0. For example, consider the monic polynomial of degree 3 with $a_0 = a_1 = 0$ and $a_2 \neq 0$, then

$$x_{\sigma_1}(p) = \begin{bmatrix} |a_2|^2 \\ |a_2| \\ 1 \end{bmatrix}, \quad x_{\sigma_2}(p) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad x_{\sigma_3}(p) = \begin{bmatrix} |a_2| \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad x_{\sigma_4}(p) = \begin{bmatrix} |a_2| \\ 0 \\ 1 \end{bmatrix}.$$

Explicit formulas for the eigenvectors of Fiedler matrices are available in the literature [6], and this allows us to add further conditions on the coefficients of the polynomial under which Proposition 7.5 can be extended to other Fiedler matrices when $a_0 = 0$. Since the general case is messy, we limit ourselves in Proposition 7.7 to the Fiedler matrix $F(p)$ defined in (8) and that has played a very relevant role in this paper.

Proposition 7.7. *Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial with $a_0 = 0$ and $a_1 \neq 0$, let $F(p)$ be the Fiedler matrix of $p(z)$ defined in (8), and let $R(p)$ be the unique positive root of $u(z) = z^n - \sum_{k=0}^{n-1} |a_k| z^k$. Define the Horner shifts of $u(z)$ as follows*

$$u_0(z) = 1, \quad u_k(z) = z^k - |a_{n-1}|z^{k-1} - |a_{n-2}|z^{k-2} - \cdots - |a_{n-k}|, \quad \text{for } k = 1, \dots, n.$$

If $D = \mathrm{diag}(R(p),\ R(p)u_1(R(p)), \dots, R(p)u_{n-2}(R(p)), 1)$, then

$$||D^{-1}F(p)D||_\infty = R(p)\,.$$

Proof. For brevity, we denote $R = R(p)$ in the proof. Let $x := [R,\ Ru_1(R), \dots, Ru_{n-2}(R), 1]^T$. Then, it is easy to check that $|F(p)|x = Rx$, i.e., x is the Perron right vector of $|F(p)|$. Next, we prove that $x > 0$. To this purpose, observe that the Horner shifts satisfy $u_k(z) = zu_{k-1}(z) - |a_{n-k}|$ for $k = 1, 2, \dots n$, and that $u_n(z) = u(z)$. Since $R > 0$ is a root of $u(z)$, the equation $u(z) = zu_{n-1}(z) - |a_0|$ and $a_0 = 0$ imply $u_{n-1}(R) = 0$. Also, since $a_1 \neq 0$, we have $0 = u_{n-1}(R) = Ru_{n-2}(R) - |a_1|$ which implies $u_{n-2}(R) = |a_1|/R > 0$. With this, the recurrence relation $Ru_{k-1}(R) = u_k(R) + |a_{n-k}|$ implies $u_j(R) > 0$, for $j = n-3, n-4, \dots, 1$. Therefore, the Perron vector x is a positive vector and Theorem 7.4(a) and Lemma 7.2(b) imply the result. $\square$

One point that should be remarked on Theorem 7.4(c) is related to the fact mentioned above that, for a given eigenvalue of any Fiedler matrix, there exists a formula for the corresponding eigenvector [6]. This formula depends, of course, on the eigenvalue and also on the Horner shifts of the polynomial, and is particularly simple in the cases of the Frobenius companion matrices. A potential use of these formulas is to obtain "approximately optimal" diagonal matrices to be used in $\|D^{-1}M_\sigma(p)D\|_\infty$. The idea would be to obtain first an upper bound on the absolute values of the roots of a polynomial by some of the approaches explained in this manuscript, to introduce this bound in the formula for the eigenvector of the corresponding Fiedler matrix $M_\sigma(u)$ for getting a vector y, and to take $D = \mathrm{diag}(y)$. This process can be iterated. This and other approaches for getting good bounds via diagonal similarities will be investigated in near future.

Another interesting point to be commented is that a similar approach to the one explained in this section is possible for the inverses of Fiedler matrices. For brevity, we do not present here all the details, but just the main ideas. Note that by Theorem 2.5, we have $|M_\sigma(p)^{-1}| = M_\sigma(\ell)^{-1}$, where $\ell(z) = z^n + \sum_{k=1}^{n-1} |a_k| z^k - |a_0|$, and, besides, $\ell(z)$ has a unique positive real root [19], that we denote by $r(p)$. In addition, a nonsingular matrix is irreducible if and only if its inverse is irreducible. Therefore, $r(p)^{-1}$ is the Perron eigenvalue of $|M_\sigma(p)^{-1}|$ and $\min_{D \text{ diag}} \|D^{-1}\, M_\sigma(p)^{-1}\, D\|_\infty = r(p)^{-1}$. Finally, note that the developments in this section, and the corresponding ones for inverses of Fiedler matrices, can be applied to the Fiedler matrices of $p^\sharp(z)$ and their inverses and, therefore, the diagonal similarities of all lower and upper bounds in (6) and (7) for the ∞-norm are covered.

8 Conclusions and future work

Explicit expressions and a complete analysis of the bounds on the absolute values of the roots of a monic scalar polynomial that are obtained by using the 1-, ∞-, and Frobenius norms of Fiedler companion matrices and their inverses have been presented in this manuscript. Particular attention has been paid to determine which are the sharpest bounds among those coming from Fiedler matrices and their inverses, and we have found that in many interesting situations the bounds coming from the inverse of the Fiedler matrix $F(p)$ defined in (25) are the sharpest ones and that they improve significatively, for certain polynomials, the classical bounds obtained from the Frobenius companion matrices. We consider that this paper is just a first step in the use of Fiedler matrices for bounding roots of polynomials. Next steps should include: (a) the generalization of the results presented here from scalar to matrix polynomials, since Fiedler companion matrices have been extended, and thoroughly studied, to the context of matrix polynomials [1, 6, 7]; (b) the investigation of concrete diagonal scalings of Fiedler matrices and/or their inverses that can produce sharper bounds for some classes of scalar polynomials; and (c) the use of Fiedler matrices for getting other types of inclusion regions for the roots of scalar polynomials, as it was done in [16] for the classical Frobenius companion matrices.

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