SIAM J. MATRIX ANAL. APPL.
Vol. 35, No. 4, pp. 1303–1328

RELATIVE PERTURBATION THEORY FOR DIAGONALLY DOMINANT MATRICES*

MEGAN DAILEY[†], FROILÁN M. DOPICO[‡], AND QIANG YE[§]

Abstract. In this paper, strong relative perturbation bounds are developed for a number of linear algebra problems involving diagonally dominant matrices. The key point is to parameterize diagonally dominant matrices using their off-diagonal entries and diagonally dominant parts and to consider small relative componentwise perturbations of these parameters. This allows us to obtain new relative perturbation bounds for the inverse, the solution to linear systems, the symmetric indefinite eigenvalue problem, the singular value problem, and the nonsymmetric eigenvalue problem. These bounds are much stronger than traditional perturbation results, since they are independent of either the standard condition number or the magnitude of eigenvalues/singular values. Together with previously derived perturbation bounds for the LDU factorization and the symmetric positive definite eigenvalue problem, this paper presents a complete and detailed account of relative structured perturbation theory for diagonally dominant matrices.

Key words. accurate computations, diagonally dominant matrices, diagonally dominant parts, inverses, linear systems, eigenvalues, singular values, relative perturbation theory

AMS subject classifications. 65F05, 65F15, 65F35, 15A09, 15A12, 15A18

DOI. 10.1137/130943613

1. Introduction.

Diagonally dominant matrices form an important class of matrices that arise in a large number of applications. Finite difference discretizations of elliptic differential operators, Markov chains, and graph Laplacians are some typical examples of this type of matrices. Indeed, diagonal dominance is often a consequence of some natural physical property of a practical problem. Diagonally dominant matrices have some nice numerical and theoretical properties, as explained in [21, 24, 25, 26]. For instance, a strictly diagonally dominant matrix is nonsingular and its LU factorization always exists and can be stably computed without carrying out any pivoting. Furthermore, inverses, and hence condition numbers, of diagonally dominant matrices can be bounded in terms of the minimal diagonal dominance [34, 40, 42].

Recent works have shown that diagonally dominant matrices may enjoy much better numerical properties than those presented in classical texts [21, 24]. The novel idea of these works is to exploit the structure of diagonally dominant matrices using their parametrization in terms of the off-diagonal entries and the diagonally dominant parts [3, 45]. With the new parametrization, stronger perturbation bounds and more accurate algorithms have been obtained for certain linear algebra problems in [10, 14, 45, 46]. Specifically, a relative perturbation theory is presented in [46] for the eigenvalues of a symmetric positive semidefinite diagonally dominant matrix (i.e., a symmetric diagonally dominant matrix with nonnegative diagonals), which simply

*Received by the editors October 31, 2013; accepted for publication (in revised form) by J. L. Barlow July 30, 2014; published electronically October 28, 2014.
http://www.siam.org/journals/simax/35-4/94361.html

[†]Indiana University Kokomo, Kokomo, IN 46904-9003 (medailey@iuk.edu). The research of this author was supported in part by NSF grant DMS-1318633.

[‡]Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM and Departamento de Matemáticas, Universidad Carlos III de Madrid, 28911 Leganés, Spain (dopico@math.uc3m.es). The research of this author was supported in part by Ministerio de Economía y Competitividad of Spain under grant MTM2012-32542.

[§]Department of Mathematics, University of Kentucky, Lexington, KY 40506 (qiang.ye@uky.edu). The research of this author was supported in part by NSF grants DMS-1317424 and DMS-1318633.

bounds the relative variation of the eigenvalues by the relative perturbation of the matrix parameters, without involving any condition number, constant, or amplifying factor. In [14], a structured perturbation theory is presented for the LDU factorization of diagonally dominant matrices that provides simple and strong bounds on the entrywise relative variations for the diagonal matrix D and the normwise relative variations for the factors L and U. This result has been recently improved in an essential way in [10] by allowing the use of a certain pivoting strategy which guarantees that the factors L and U are always well-conditioned. Computationally, a new algorithm is presented in [45] that accurately computes the LDU factorization of diagonally dominant matrices with entrywise accurate factor D and normwise accurate factors L and U, which is a significant improvement over the classical results in [20, 44]. Furthermore, this algorithm can be combined with the algorithms presented in [11] to compute the singular values with relative errors in the order of machine precision. In fact, the algorithm for the LDU factorization in [45] can be combined also with the algorithms in [7, 15, 16] to compute, with high relative accuracy, solutions to linear systems and solutions to least squares problems involving diagonally dominant matrices, and eigenvalues of symmetric diagonally dominant matrices.

We emphasize that some perturbation problems and high relative accuracy algorithms have been considered in [4, 33] for the closely related class of γ-scaled diagonally dominant (γ-s.d.d.) matrices. A matrix H is γ-s.d.d. with respect to a norm $\|\cdot\|$ if it can be expressed as $H = D_1AD_2$, where D_1 and D_2 are nonsingular diagonal matrices, $(A)_{ii} = \pm 1$ for all i, and $\|A - \text{diag}(A)\| \leq \gamma < 1$. The references [4, 33] study the change of the singular values and the eigenvalues, in the symmetric case, of H under perturbations $H + \delta H = D_1(A + \delta A)D_2$, with $\|\delta A\| < 1 - \gamma$. The perturbation bounds obtained in [4, 33] amplify $\|\delta A\|$ by factors $(1 - 2\gamma)^{-1}$ or $(1 - \gamma)^{-1}$, which can be considered as condition numbers of the corresponding problems and are very large if $\gamma \approx 1/2$ or $\gamma \approx 1$. In contrast, the bounds derived in [10, 14, 46] and in this work are free of condition numbers for the class of perturbations we consider.

In this paper, we shall significantly broaden the study of perturbation properties of diagonally dominant matrices in terms of their off-diagonal entries and diagonally dominant parts by establishing strong relative perturbation bounds for a number of linear algebra problems that have not been considered before in this context. First, a perturbation bound is presented for the inverse of a diagonally dominant matrix that is independent of any condition number. As a direct consequence, we shall also establish a perturbation bound for the solution to the linear system $Ax = b$, which is governed by a certain condition number that is of order one for most vectors b and is always smaller than the traditional condition number of A. Then, the relative eigenvalue perturbation bound for a symmetric positive semidefinite diagonally dominant matrix in [46] is generalized to the indefinite case. Next, relative perturbation bounds for the singular values of any diagonally dominant matrix are obtained. These bounds are equal to a dimensional constant times the perturbation of the parameters, without involving any other amplifying factor. Lastly, we shall derive relative perturbation bounds for the eigenvalues of a nonsymmetric diagonally dominant matrix, which are still dependent on the standard eigenvalue condition number as usual [12, 21] but independent of the magnitude of the eigenvalue itself. A remarkable feature of all the bounds presented in this paper is that they are finite rigorous bounds, i.e., they are not asymptotic bounds valid only for infinitesimal perturbations.

We shall rely heavily on the LDU perturbation results from [10, 14]. Indeed, most of the new bounds in this paper are derived starting from the perturbation bounds for the LDU factorization. In addition, some other results included in [17] will also play a relevant role. Our methods can be directly adapted to the structured perturbation problem where a general matrix, i.e., not necessarily diagonally dominant, is perturbed in such a way that a rank-revealing decomposition XDY of this matrix [11] is changed with small entrywise relative variations for the diagonal matrix D and small normwise relative variations for the factors X and Y (see also [16]). We do not insist on this approach, but, as an example, we present one such result in Theorem 6.4. Clearly, other strong perturbation bounds can also be derived for matrices under such a structured perturbation.

This paper can be seen as a contribution to one of the most fruitful lines of research in matrix perturbation theory in the last two decades: the derivation of perturbation bounds much stronger than the traditional ones when structure-preserving perturbations of relevant types of structured matrices are considered (see, for instance, [3, 4, 5, 6, 9, 14, 22, 23, 27, 28, 31, 32, 33, 39, 46] and the references therein). Even more, we can say that this manuscript belongs to a more specific class of recent research works in structured matrix perturbation theory: those that represent certain structured matrices by a proper set of parameters (different from the matrix entries), in such a way that tiny perturbations of these parameters produce tiny variations of some interesting quantities in linear algebra. Apart from the references on diagonally dominant matrices mentioned above, other references dealing with *parameterized perturbations* are [13, 19, 35, 36, 37] for eigenvalues and eigenvectors of tridiagonal matrices parameterized by their bidiagonal LDU factorizations and [29] for eigenvalues and singular values of totally nonnegative matrices parameterized by their bidiagonal decompositions.

The rest of the paper is organized as follows. In section 2, an overview of diagonally dominant matrices and related perturbation results for their LDU factorizations from [10, 14] are presented. In addition, section 2 includes a numerical example that illustrates why the parametrization using off-diagonal entries and diagonally dominant parts is essential to get strong perturbation bounds. We develop relative perturbation bounds for the inverse and solutions to linear systems in section 3, for the symmetric indefinite eigenvalue problem in section 4, for the singular value problem in section 5, and for the nonsymmetric eigenvalue problem in section 6. Finally, we conclude by presenting some remarks in section 7.

Next, we present the notation used in this paper.

Notation. We consider only real matrices and we denote by $\mathbb{R}^{m\times n}$ the set of $m \times n$ real matrices. The entries of a matrix A are a_{ij} or A_{ij}, and $|A|$ is the matrix with entries $|a_{ij}|$. The inequality $A \geq B$ for matrices means $a_{ij} \geq b_{ij}$ for all i, j, and the inequality $v \geq w$ for vectors means $v_i \geq w_i$ for all the entries of the vectors. Analogously, the inequality $v \geq 0$ for the vector v means $v_i \geq 0$ for all its entries. We use MATLAB notation for submatrices. That is, $A(i : j, k : l)$ denotes the submatrix of A formed by rows i through j and columns k through l. We also use $A(i', j')$ to denote the submatrix of A formed by deleting row i and column j from A. Let $\alpha = [i_1, i_2, \ldots, i_p]$, where $1 \leq i_1 < i_2 < \cdots < i_p \leq m$, and $\beta = [j_1, j_2, \ldots, j_q]$, where $1 \leq j_1 < j_2 < \cdots < j_q \leq n$. Then $A(\alpha, \beta)$ denotes the submatrix of A that consists of rows $i_1, i_2, \ldots, i_p$ and columns $j_1, j_2, \ldots, j_q$. We denote by I_s the $s \times s$ identity matrix, by 0_s the $s \times s$ zero matrix, and by $0_{p\times q}$ the $p \times q$ zero matrix. We will simply write I and 0 when their sizes are clear from the context. Five matrix norms will be used: $\|A\|_{\max} = \max_{ij} |a_{ij}|$, $\|A\|_1 = \max_j \sum_i |a_{ij}|$, $\|A\|_\infty = \max_i \sum_j |a_{ij}|$, $\|A\|_F = (\sum_{i,j} |a_{ij}|^2)^{1/2}$, and the spectral norm $\|A\|_2$. The condition numbers of a nonsingular matrix A in any of these norms are denoted as $\kappa_i(A) := \|A\|_i \|A^{-1}\|_i$, for $i = \max, 1, \infty, F, 2$. The sign of $x \in \mathbb{R}$ is $\operatorname{sign}(x)$, where $\operatorname{sign}(0)$ is defined to be 1.

2. Preliminaries and example. In this section, we give an overview of diagonally dominant matrices and present some results proved recently in [10, 14] that will be used in the subsequent sections. More information on diagonally dominant matrices can be found in [10, section 2] and [14, section 2], and the references therein. In addition, at the end of this section, we present and discuss an example which illustrates why the use of a proper parametrization is essential to obtain strong perturbation bounds for diagonally dominant matrices, with special emphasis on a situation that is not covered by the bounds in [4, 33]. We first define diagonally dominant matrices.

Definition 2.1. *A matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is said to be row diagonally dominant if $|a_{ii}| \geq \sum_{j\neq i} |a_{ij}|$ for $i = 1, \dots, n$ and is said to be column diagonally dominant if $|a_{ii}| \geq \sum_{j\neq i} |a_{ji}|$ for $i = 1, \dots, n$.*

For brevity, we will consider only row diagonally dominant matrices, although the results we present hold for column diagonally dominant matrices with obvious modifications or by taking transposes.

An idea that has played an important role in deriving strong perturbation bounds for diagonally dominant matrices is to reparameterize the matrix in terms of its diagonally dominant parts and off-diagonal entries (see [45]).

Definition 2.2. (1) *Given a matrix $M = [m_{ij}] \in \mathbb{R}^{n\times n}$ and a vector $v = [v_i] \in \mathbb{R}^n$, we use $\mathcal{D}(M, v)$ to denote the matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ whose off-diagonal entries are the same as M (i.e., $a_{ij} = m_{ij}$ for $i \neq j$) and whose ith diagonal entry is $a_{ii} = v_i + \sum_{j\neq i} |m_{ij}|$ for $i = 1, \dots, n$.*

(2) *Given a matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, we denote by $A_D \in \mathbb{R}^{n\times n}$ the matrix whose off-diagonal entries are the same as A and whose diagonal entries are zero. Then, letting $v_i = a_{ii} - \sum_{j\neq i} |a_{ij}|$, for $i = 1, \dots, n$, and $v = [v_1, v_2, \dots, v_n]^T \in \mathbb{R}^n$, we have*

$$A = \mathcal{D}(A_D, v)$$

and we call it the representation of A by its diagonally dominant parts v and off-diagonal entries A_D.

We note that the diagonally dominant parts v have been introduced as parameters to represent matrices in [2, 3] for diagonally dominant M-matrices and in [45, 46] for diagonally dominant matrices, but they have also been used previously in the literature for various other purposes (see [1, 40, 42, 43]). Clearly, $v \geq 0$ if and only if A is row diagonally dominant and its diagonal entries are nonnegative. We will frequently use the condition $v \geq 0$ as an assumption without referring explicitly to its meaning. For most problems (i.e., the LDU factorization, inverses, linear systems, and the singular value problem), by considering an equivalent problem for SA with $S = \operatorname{diag}\{\operatorname{sign}(a_{ii})\}$, we can restrict ourselves to diagonally dominant matrices A with nonnegative diagonal without loss of generality. For the eigenvalue problem, however, we need to consider in general diagonally dominant matrices with diagonal entries of any sign. To properly parameterize such matrices, we need the signs of the diagonal entries (i.e., S) as well; we shall leave the details of this more general case to section 4, when we study the symmetric indefinite eigenvalue problem.

For row diagonally dominant matrices parameterized by A_D and v, we shall establish the perturbation theory for several linear algebra problems under small componentwise relative perturbations of these parameters. We note that using A_D and v as parameters is crucial in the development of high relative accuracy algorithms for such matrices in [45]. Study of such parametric perturbations for the LDU factorization is also essential for the error analysis presented in [14] for the algorithm in [45]. Additionally, as mentioned in the introduction, the diagonally dominant parts v and off-diagonal entries often represent physical parameters in applications and it is natural to consider perturbed problems with small variations in these parameters.

Our study of small componentwise relative perturbations assumes that the zero off-diagonal entries incur no perturbation. Although this assumption is key for the error analysis [14] of the algorithm presented in [45], it may be restrictive in some problems. However, in many applications (e.g., discretization of PDEs, Laplacians of graphs, and Markov chains), appearance of zero entries is typically a result of the structure in the underlying problem and will hence be preserved in a perturbed problem. Of course, from the theoretical point of view, it would be interesting to consider nontrivial perturbations to zero parameters, but, probably, the strong bounds derived in this paper cannot be generalized to this situation. For example, as noted in the introduction, the effect of certain small normwise perturbations that allow the perturbation of zero entries in some problems involving γ-s.d.d. matrices have been studied in [4, 33], but the perturbation bounds obtained may become arbitrarily large if γ is close to 1.

Several of our results are based on the perturbation bounds for the LDU factorization recently obtained in [10, 14]. We first recall that if the LU, or LDU, factorization of a nonsingular matrix exists, then it is unique. However, for singular matrices, when an LU, or LDU, factorization exists, it is not unique in general. In this case, in order to study its perturbation properties, we need to consider the following unique form of the LDU factorization (see [14, Definition 1]).

Definition 2.3. *A matrix $A \in \mathbb{R}^{n\times n}$ with rank $r > 0$ is said to have LDU factorization if there exist a unit lower triangular matrix $L_{11} \in \mathbb{R}^{r\times r}$, a unit upper triangular matrix $U_{11} \in \mathbb{R}^{r\times r}$, and a nonsingular diagonal matrix $D_{11} \in \mathbb{R}^{r\times r}$ such that $A = LDU$, where*

$$L = \begin{bmatrix} L_{11} & 0 \\ L_{21} & I_{n-r} \end{bmatrix}, \qquad D = \begin{bmatrix} D_{11} & 0 \\ 0 & 0_{n-r} \end{bmatrix}, \qquad U = \begin{bmatrix} U_{11} & U_{12} \\ 0 & I_{n-r} \end{bmatrix}.$$

Here, if $r = n$, the $(2, 2)$ blocks are considered empty and $L = L_{11}$, $D = D_{11}$, and $U = U_{11}$.

It is easy to see that if this form of LDU factorization exists, then it is unique. For a row diagonally dominant matrix $A \in \mathbb{R}^{n\times n}$, applying any diagonal pivoting strategy (i.e., pivoting with simultaneous and equal row and column permutations) will result in PAP^T that has a unique LDU factorization in the sense of Definition 2.3, where P is the permutation matrix defined by the pivoting strategy. For the purposes of this work, we consider a pivoting strategy suggested in [38] for matrices with other structures and used for first time in [45] for general row diagonally dominant matrices. This strategy is called *column diagonal dominance pivoting*. Let $A^{(1)} = A$ and let $A^{(k)} = [a_{ij}^{(k)}] \in \mathbb{R}^{n\times n}$ denote the matrix obtained after $(k-1)$ stages of Gaussian elimination have been performed on A, i.e., all entries below the diagonal in the first $k-1$ columns of $A^{(k)}$ are zero. It is well known ([14, Theorem 1] or [24]) that the Schur complement $A^{(k)}(k : n, k : n)$ is row diagonally dominant. Then, there is at least one column of this Schur complement that is column diagonally dominant, i.e., $|a_{ii}^{(k)}| - \sum_{j=k,j\neq i}^{n} |a_{ji}^{(k)}| \geq 0$ for some $i \geq k$. So, the column diagonal dominance pivoting scheme permutes into the pivot position (k, k) the maximal diagonal entry that is column diagonally dominant. That is, at step k, after the permutation, we have

$$\left|a_{kk}^{(k)}\right| = \max_{k\le i\le n}\left\{\left|a_{ii}^{(k)}\right| : |a_{ii}^{(k)}| - \sum_{j=k,j\neq i}^{n} |a_{ji}^{(k)}| \ge 0\right\},$$

where we still use $A^{(k)} = [a_{ij}^{(k)}]$ to denote the matrix after the permutation. With this pivoting strategy, at the end, we obtain a row diagonally dominant factor U as usual, but now L is column diagonally dominant. Hence, by [38], L, U, and their inverses can be bounded as

$$\|L\|_{\max} = 1,\ \|L\|_1 \le 2,\ \|L\|_\infty \le n,\ \|L^{-1}\|_{\max} = 1,\ \|L^{-1}\|_1 \le n,\ \|L^{-1}\|_\infty \le n, \tag{2.1}$$
$$\|U\|_{\max} = 1,\ \|U\|_1 \le n,\ \|U\|_\infty \le 2,\ \|U^{-1}\|_{\max} = 1,\ \|U^{-1}\|_1 \le n,\ \|U^{-1}\|_\infty \le n\,. \tag{2.2}$$

The bounds for the inverses in (2.1) and (2.2) follow from Proposition 2.1 in [38], which states that the inverses of triangular diagonally dominant (either by rows or columns) matrices with ones on the diagonal have the absolute values of their entries bounded by one. It is worth observing that this result follows immediately from a classic and more general result to be found in [26, Theorem 2.5.12], where a proof is given for strictly diagonally dominant matrices but can be easily extended to any nonsingular diagonally dominant matrix.

The bounds in (2.1), (2.2) imply that the LDU factorization of a row diagonally dominant matrix A obtained by column diagonal dominance pivoting is always a rank-revealing decomposition [11], which is of fundamental interest for performing accurate computations. For all the linear algebra problems we consider here, since the permuted matrix PAP^T coming from any diagonal pivoting strategy results in trivially equivalent problems, we can assume that the row diagonally dominant matrix we consider is arranged for column diagonal dominance pivoting, i.e., A has the permutation P applied already. More importantly, the unique LDU factorization obtained under this pivoting scheme is stable under componentwise perturbations of the diagonally dominant parts and off-diagonal entries. Indeed, the following perturbation bounds are obtained in [10, 14].

THEOREM 2.4 (see [10, Theorem 3.2], [14, Theorem 3]). *Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ be such that $v \ge 0$. Suppose that A has LDU factorization $A = LDU$, where $L = [l_{ij}]$, $D = \operatorname{diag}(d_1, \dots, d_n)$, and $U = [u_{ij}]$. Let $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v}) \in \mathbb{R}^{n\times n}$ be a matrix that satisfies*

$$|\tilde{v} - v| \le \epsilon v \quad \textit{and} \quad |\tilde{A}_D - A_D| \le \epsilon |A_D| \tag{2.3}$$

for some positive ϵ with $(12n+1)\epsilon < 1$. Then, we have

(1) *$\tilde{A}$ is row diagonally dominant with nonnegative diagonal entries,* $\operatorname{rank}(\tilde{A}) = \operatorname{rank}(A)$*, and $\tilde{A}$ has LDU factorization $\tilde{A} = \tilde{L}\tilde{D}\tilde{U}$, where $\tilde{L} = [\tilde{l}_{ij}]$, $\tilde{D} = \operatorname{diag}(\tilde{d}_1, \dots, \tilde{d}_n)$, and $\tilde{U} = [\tilde{u}_{ij}]$;*

(2) *for $i = 1, \dots, n$, $\tilde{d}_i = d_i(1 + w_i)$ with*

$$\left(\frac{1-\epsilon}{1+\epsilon}\right)^n - 1 \le w_i \le \left(\frac{1+\epsilon}{1-\epsilon}\right)^n - 1, \tag{2.4}$$

and, in particular, $|\tilde{d}_i - d_i| \le \frac{2n\epsilon}{1-2n\epsilon}|d_i|$, for $i = 1, \dots, n$;

(3) *$|\tilde{u}_{ij} - u_{ij}| \le 3n\epsilon$, for $1 \le i, j \le n$, and $\frac{\|\tilde{U}-U\|_\infty}{\|U\|_\infty} \le 3n^2\epsilon$;*

(4) *and, if A is arranged for column diagonal dominance pivoting, then*

$$\|\tilde{L} - L\|_1 \le \frac{n(8n-2)\epsilon}{1-(12n+1)\epsilon} \quad \textit{and} \quad \frac{\|\tilde{L} - L\|_1}{\|L\|_1} \le \frac{n(8n-2)\epsilon}{1-(12n+1)\epsilon}.$$

The main remark on the relative bounds presented in Theorem 2.4 is that they do not depend on any condition number, either of the matrix A or of its factors, and so they imply that for *any* row diagonally dominant matrix, small componentwise perturbations as in (2.3) always produce small relative changes in the LDU factors. Observe also that $v \geq 0$ and the fact that $0 \leq \epsilon < 1$ in (2.3) imply immediately that $\tilde{v} \geq 0$, which is why the perturbations in (2.3) preserve the diagonally dominant structure and the nonnegativity of the diagonal entries.

We shall also use in the rest of the paper the following lemma, which combines Lemmas 3, 4, and 7 of [14] and studies the perturbation of the determinant and certain minors of diagonally dominant matrices with nonnegative diagonals under structured perturbations of type (2.3).

LEMMA 2.5. *Let* $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ *be such that* $v \geq 0$ *and let* $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v}) \in \mathbb{R}^{n\times n}$ *satisfy* (2.3) *for some* ϵ *with* $0 \leq \epsilon < 1$. *Then*

(a) $\det \tilde{A} = (\det A)(1+\eta_1)\cdots(1+\eta_n)$, *where* $|\eta_j| \leq \epsilon$ *for* $1 \leq j \leq n$;

(b) *if* $\alpha = [i_1, i_2, \ldots, i_t]$, *where* $1 \leq i_1 < i_2 < \cdots < i_t \leq n$, *then*

$$\det \tilde{A}(\alpha,\alpha) = (\det A(\alpha,\alpha))(1+\beta_1)\cdots(1+\beta_t), \quad \text{where } |\beta_j| \leq \epsilon, \quad \text{for } 1 \leq j \leq t;$$

(c) *if* $k+1 \leq p, q \leq n$ *and* $p \neq q$, *then*

$$\begin{aligned}\Big|\det \tilde{A}([1:k,p],[1:k,q]) - \det A([1:k,p],[1:k,q])\Big| \\ \leq 2\left((1+\epsilon)^{k+1}-1\right)\det A([1:k,p],[1:k,p]).\end{aligned}$$

To finish this section, we present an example to illustrate why parameterized perturbations of type (2.3) may be expected to lead to stronger bounds than general perturbations, perturbations that only preserve the diagonally dominant property, or perturbations of the type considered in [4, 33]. For brevity, Example 2.6 focuses only on singular values, but similar examples can be devised for the other linear algebra problems considered in this paper.

Example 2.6. Let us consider the following row diagonally dominant matrix A, whose vector of diagonally dominant parts is denoted by v_A:

$$A = \begin{bmatrix} 3 & -1.5 & 1.5 \\ -1 & 2.002 & 1 \\ 2 & 0.5 & 2.5 \end{bmatrix}, \qquad v_A = \begin{bmatrix} 0 \\ 0.002 \\ 0 \end{bmatrix}.$$

The two row diagonally dominant matrices

$$B = \begin{bmatrix} 3 & -1.5 & 1.5 \\ -1 & 2.001 & 1 \\ 2 & 0.5 & 2.5 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3.0015 & -1.5015 & 1.5 \\ -1 & 2.002002 & 1 \\ 2 & 0.5 & 2.5 \end{bmatrix}$$

are very close to A in an standard entrywise sense, since they satisfy

$$|A-B| \leq 5\cdot 10^{-4}\,|A| \quad \text{and} \quad |A-C| \leq 10^{-3}\cdot |A|.$$

However, their vectors of diagonally dominant parts are

$$v_B = \begin{bmatrix} 0 & 0.001 & 0 \end{bmatrix}^T \quad \text{and} \quad v_C = \begin{bmatrix} 0 & 0.002002 & 0 \end{bmatrix}^T,$$

and we see that v_B is very different from v_A, while v_C is very close to v_A. More precisely,

$$|v_A - v_B| = 0.5\, v_A \quad \text{and} \quad |v_A - v_C| = 10^{-3}\, v_A. \tag{2.5}$$

Thus, in plain words, we can say that C is a nearby perturbation of A in the sense of diagonally dominant parts and off-diagonal entries, while B is not. That is, B is a nearby perturbation of A only in the traditional entrywise sense.

In the following table, we show the singular values, $\sigma_1 \geq \sigma_2 \geq \sigma_3$, of A, B, and C in the usual decreasing order (all numbers in the table are given with four significant digits):

	A	B	C
σ_1	4.641	4.640	4.642
σ_2	2.910	2.909	2.910
σ_3	$6.663 \cdot 10^{-4}$	$3.332 \cdot 10^{-4}$	$6.673 \cdot 10^{-4}$

The first key remark on this table is that the smallest singular values of A and B do *not* agree in a single digit, despite the fact that A and B are very close to each other and both are row diagonally dominant. Therefore, just preserving the diagonally dominant property may not be enough to get good perturbation properties. In contrast, the smallest singular values of A and C agree in the first two digits. More precisely, for $i = 1, 2, 3$,

$$\max_i \frac{|\sigma_i(A) - \sigma_i(B)|}{\sigma_i(A)} = 0.49989 \quad \text{and} \quad \max_i \frac{|\sigma_i(A) - \sigma_i(C)|}{\sigma_i(A)} = 1.4444 \cdot 10^{-3}.$$

The behavior we observe in this example is not by chance, since in section 5 we will show that for row diagonally dominant matrices, tiny relative perturbations of diagonally dominant parts and off-diagonal entries always result in tiny relative variations of the singular values, independently of their magnitudes.

We finish by looking at this example from the perspective of the γ-s.d.d. matrices studied in [4, 33]. Perturbation bounds for singular values are not presented in [4] but are presented in [33, Theorem 4.1]. There symmetric diagonal scalings and the spectral norm are considered. In our example, it amounts to writing A as $A = DXD$ with D diagonal and $\mathrm{diag}(X) = I_3$. So, the parameter γ is $\gamma_2 = \|X - \mathrm{diag}(X)\|_2 = 0.9997$. This means that the set of perturbations E of A allowed in [33, Theorem 4.1] is empty since they must satisfy $\|D^{-1}ED^{-1}\|_2 < 1 - 2\,\gamma_2 < 0$. Even in the case that the allowable perturbations were defined by $\|D^{-1}ED^{-1}\|_2 < 1 - \gamma_2 = 2.6644 \cdot 10^{-4}$, as it happens in some results stated in [4] for eigenvalues of symmetric matrices, the matrix C cannot be considered as an allowable perturbation of A in the γ-s.d.d. sense, since $\|D^{-1}(C - A)D^{-1}\|_2 = 7.9033 \cdot 10^{-4} > 1 - \gamma_2$. Even more, the amplifying factor (eigenvalue condition number) in this case would be large: $(1 - \gamma_2)^{-1} = 3.75 \cdot 10^3$. The situation would be worse if we consider the 1- and ∞-norms, since the gamma parameters in these cases would be $\gamma_1 = 1.1383$ and $\gamma_\infty = 1.1598$, respectively, both larger than one.

Two other natural options are to scale the matrix A on the left or on the right. On the left, one writes $A = D_1 W$, with D_1 diagonal and $\mathrm{diag}(W) = I_3$. In this case the γ-parameters in the 1-, 2-, and ∞-norms are $\gamma_1 = 1.2995, \gamma_2 = 1.0461$, and $\gamma_\infty = 1$, respectively. None of them are smaller than 1. On the right, one writes $A = YD_1$, with $\mathrm{diag}(Y) = I_3$, and the parameters are $\gamma_1 = 1, \gamma_2 = 1.0520, \gamma_\infty = 1.3493$; again, none of them are smaller than 1.

The bottom line of the discussion above is that the perturbations considered in this paper are not a subset of the perturbations considered in [4, 33] and vice versa and that there are diagonally dominant matrices ill-conditioned in the γ-s.d.d. sense which are well-conditioned under parameterized perturbations (2.3). Big differences between both approaches may appear, for instance, when γ is very close to one.

3. Bounds for inverses and solutions to linear systems. The perturbation theory for the inverse of a matrix A and for the solution to linear systems $Ax = b$ is well established and can be found in many books on numerical linear algebra [12, 21, 24, 41]. The classical perturbation bounds of a general matrix depend on the traditional condition number $\kappa(A) := \|A\|\,\|A^{-1}\|$ for normwise perturbations, while for entrywise perturbations, they are governed by the Bauer–Skeel condition number $\|\,|A^{-1}|\,|A|\|$. Both these condition numbers may be unbounded in general. In this section, we show that row diagonally dominant matrices are well-conditioned with respect to inversion under perturbations in the parameterization given by Definition 2.2. Similarly, for the solution to linear systems, this structured perturbation allows us to present normwise bounds that are dependent on a smaller condition number that is almost always a modest value.

The main idea in this section is simple: noting that the entries of A^{-1} can be expressed in terms of minors of A [25], we utilize the perturbation results for determinants presented in Lemma 2.5 to obtain the following entrywise perturbation bounds for the inverse of a row diagonally dominant matrix.

THEOREM 3.1. *Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ be such that $v \geq 0$ and suppose that A is nonsingular. Let $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v}) \in \mathbb{R}^{n\times n}$ be such that*

$$|\tilde{v} - v| \leq \epsilon v \quad \textit{and} \quad |\tilde{A}_D - A_D| \leq \epsilon |A_D| \quad \textit{for some} \quad 0 \leq \epsilon < 1.$$

Then $\tilde{A}$ is nonsingular and if $2n\epsilon < 1$, we have for $1 \leq i, j \leq n$,

$$\left|(\tilde{A}^{-1})_{ij} - (A^{-1})_{ij}\right| \leq \frac{(3n-2)\epsilon}{1-2n\epsilon}\left|(A^{-1})_{jj}\right|. \tag{3.1}$$

Proof. By Lemma 2.5(a), we have $\det \tilde{A} = (\det A)(1+\eta_1)\cdots(1+\eta_n) \neq 0$, since $|\eta_j| \leq \epsilon < 1$ for all j. Therefore, $\tilde{A}$ is nonsingular.

To prove (3.1), we consider $j \neq i$ first. Without loss of generality, we assume that $i = n-1$ and $j = n$, since this can always be obtained using proper simultaneous *row and column* permutations of A^{-1}, corresponding, respectively, to the same simultaneous *row and column* permutations in A. Using Lemma 2.5(c) with $k = n-2$, $p = n-1$, and $q = n$, we have

$$|\det \tilde{A}(j', i') - \det A(j', i')| \leq 2\left((1+\epsilon)^{n-1} - 1\right)\det A(j', j'). \tag{3.2}$$

It follows from this and Lemma 2.5(a)–(b) that

$$\begin{aligned}(\tilde{A}^{-1})_{ij} - (A^{-1})_{ij} &= \frac{(-1)^{i+j}\det \tilde{A}(j',i')}{\det \tilde{A}} - \frac{(-1)^{i+j}\det A(j',i')}{\det A}\\ &= \frac{(-1)^{i+j}\det \tilde{A}(j',i')}{(\det A)(1+\eta_1)\cdots(1+\eta_n)} - \frac{(-1)^{i+j}\det A(j',i')}{\det A}\\ &= \frac{(-1)^{i+j}\chi(\det \tilde{A}(j',i') - \det A(j',i'))}{\det A} + (\chi - 1)\frac{(-1)^{i+j}\det A(j',i')}{\det A},\end{aligned}$$

where $\chi := \frac{1}{(1+\eta_1)\cdots(1+\eta_n)}$. Noting that $|\chi - 1| \le \frac{1}{(1-\epsilon)^n} - 1$ and using (3.2), we have

$$\begin{aligned}
\left|(\tilde{A}^{-1})_{ij} - (A^{-1})_{ij}\right| &\le \frac{|\chi||\det \tilde{A}(j',i') - \det A(j',i')|}{|\det A|} + |\chi - 1| \frac{|\det A(j',i')|}{|\det A|} \\
&\le \frac{2\left((1+\epsilon)^{n-1} - 1\right)|\chi||\det A(j',j')|}{|\det A|} + |\chi - 1| \frac{|\det A(j',i')|}{|\det A|} \\
&= 2\left((1+\epsilon)^{n-1} - 1\right)|\chi|\left|(A^{-1})_{jj}\right| + |\chi - 1|\left|(A^{-1})_{ij}\right| \\
&\le \frac{2\left((1+\epsilon)^{n-1} - 1\right)}{(1-\epsilon)^n}\left|(A^{-1})_{jj}\right| + \left[\frac{1}{(1-\epsilon)^n} - 1\right]\left|(A^{-1})_{ij}\right| \\
&\le \frac{2(n-1)\epsilon}{1-2n\epsilon}\left|(A^{-1})_{jj}\right| + \frac{n\epsilon}{1-n\epsilon}\left|(A^{-1})_{ij}\right|,
\end{aligned}$$

where we have used (see [24, Chapter 3]) that

$$\frac{(1+\epsilon)^{n-1} - 1}{(1-\epsilon)^n} \le \frac{(n-1)\epsilon/(1-(n-1)\epsilon)}{1 - n\epsilon/(1-n\epsilon)} \le \frac{(n-1)\epsilon}{1-2n\epsilon}. \tag{3.3}$$

From [14, Theorem 1(e)], we have $|(A^{-1})_{ij}| \le |(A^{-1})_{jj}|$, which leads to (3.1) for $i \ne j$.

Finally, we prove (3.1) for $i = j$. Again, we use Lemma 2.5(a)–(b) to prove that

$$\begin{aligned}
(\tilde{A}^{-1})_{ii} = \frac{\det \tilde{A}(i',i')}{\det \tilde{A}} &= \frac{(\det A(i',i'))(1+\beta_1)\cdots(1+\beta_{n-1})}{(\det A)(1+\eta_1)\cdots(1+\eta_n)} \\
&= (A^{-1})_{ii}\frac{(1+\beta_1)\cdots(1+\beta_{n-1})}{(1+\eta_1)\cdots(1+\eta_n)},
\end{aligned}$$

where $|\eta_j| \le \epsilon < 1$ and $|\beta_j| \le \epsilon < 1$. According to [24, Lemma 3.1], this equality can be written as

$$(\tilde{A}^{-1})_{ii} = (A^{-1})_{ii}\,(1+\theta_{2n-1}), \quad \text{where} \quad |\theta_{2n-1}| \le \frac{(2n-1)\epsilon}{1-(2n-1)\epsilon}.$$

Therefore, $\left|(\tilde{A}^{-1})_{ii} - (A^{-1})_{ii}\right| = |\theta_{2n-1}|\left|(A^{-1})_{ii}\right|$, and

$$\left|(\tilde{A}^{-1})_{ii} - (A^{-1})_{ii}\right| \le \frac{(2n-1)\epsilon}{1-(2n-1)\epsilon}\left|(A^{-1})_{ii}\right| \le \frac{(3n-2)\epsilon}{1-2n\epsilon}\left|(A^{-1})_{ii}\right|,$$

which completes the proof. ∎

We note that the assumption $2n\epsilon < 1$ in Theorem 3.1 is not essential and is only made to simplify bounds as those in (3.3), which allows us to obtain the simple bound in (3.1). Namely, a bound similar to, but more complicated to express than, (3.1) can be proved if only the assumption $0 \le \epsilon < 1$ is satisfied. Note also that Theorem 3.1 gives that small relative componentwise perturbations in the data $\mathcal{D}(A_D, v)$ result in small relative perturbations in the diagonal entries of the inverse. However, the perturbation of an off-diagonal entry can only be guaranteed to be small relative to the diagonal entry in the corresponding column of the inverse, rather than relative to the off-diagonal entry itself. This might seem unsatisfactory at first glance, but the perturbation to an off-diagonal entry may not be small relative to itself. This is illustrated in Example 3.2 below, which also shows that the bound (3.1) cannot be essentially improved. In addition, note that for a row diagonally dominant matrix, its inverse has the property that the entry with largest absolute value in each column is precisely the diagonal one (see [14, Theorem 1(e)], [26, p. 125]). Therefore, (3.1) is essentially a columnwise perturbation bound, weaker than a componentwise bound, but stronger than a general normwise perturbation bound, and in fact much stronger in certain situations. This is also illustrated in Example 3.2.

Example 3.2. Let us consider the real matrices

$$A = \begin{bmatrix} d_1 & d_1 s & d_1 t \\ & d_2 & d_2 \\ & & d_3 \end{bmatrix} \quad \text{and} \quad \tilde{A} = \begin{bmatrix} d_1 & d_1 s & d_1 \tilde{t} \\ & d_2 & d_2 \\ & & d_3 \end{bmatrix},$$

where $d_1 > 0$, $d_2 > 0$, $d_3 > 0$, $|s| + |t| < 1/2$, $\tilde{t} = t(1+\epsilon)$, and $0 \le \epsilon < 1/6$. Then A and $\tilde{A}$ are both row diagonally dominant and it can be checked that they satisfy the assumptions of Theorem 3.1. The inverses of A and $\tilde{A}$ are

$$A^{-1} = \begin{bmatrix} d_1^{-1} & -d_2^{-1}s & d_3^{-1}(s-t) \\ & d_2^{-1} & -d_3^{-1} \\ & & d_3^{-1} \end{bmatrix} \quad \text{and} \quad \tilde{A}^{-1} = \begin{bmatrix} d_1^{-1} & -d_2^{-1}s & d_3^{-1}(s-\tilde{t}) \\ & d_2^{-1} & -d_3^{-1} \\ & & d_3^{-1} \end{bmatrix}$$

and the absolute perturbation in the $(1,3)$ entry of the inverse is

$$\left|(\tilde{A}^{-1})_{13} - (A^{-1})_{13}\right| = \epsilon \frac{|t|}{d_3} = \epsilon |t| \left|(A^{-1})_{33}\right|. \tag{3.4}$$

Observe that if $|t|$ is not too small (for instance, $t \approx 1/4$), then (3.4) shows that the bound (3.1) is essentially attained in this case. However, the relative perturbation

$$\frac{\left|(\tilde{A}^{-1})_{13} - (A^{-1})_{13}\right|}{|(A^{-1})_{13}|} = \epsilon \frac{|t|}{|s-t|}$$

is arbitrarily large if $s - t$ is close to 0, for instance, if $s \approx t \approx 1/4$. Finally, observe also that if $d_1 \ll d_3$, then $d_3^{-1} \ll d_1^{-1} \le \left\|A^{-1}\right\|_i$, with $i = 1, 2, \infty$, and any normwise perturbation bound for the variation of the inverse cannot give reliable information on the variation of the $(1,3)$ entry, while the bound (3.1) is sharp in this example (unless $|t|$ is very small).

With the bound (3.1), we prove in Corollary 3.3 a very satisfactory relative normwise bound for the inverse that is completely independent of any condition number.

COROLLARY 3.3. *Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ be such that $v \ge 0$ and suppose that A is nonsingular. Let $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v}) \in \mathbb{R}^{n\times n}$ be such that*

$$|\tilde{v} - v| \le \epsilon v \quad \textit{and} \quad |\tilde{A}_D - A_D| \le \epsilon |A_D| \qquad \textit{for some } 0 \le \epsilon < 1/(2n).$$

Let $\|\cdot\|$ be the 1-norm, the 2-norm, the ∞-norm, or the Frobenius norm. Then

$$\frac{\|\tilde{A}^{-1} - A^{-1}\|}{\|A^{-1}\|} \le \frac{n(3n-2)\epsilon}{1-2n\epsilon}.$$

Proof. Theorem 1(e) in [14] implies $|(A^{-1})_{ij}| \le |(A^{-1})_{jj}|$ for all i, j. Thus, it follows from Theorem 3.1 that for $1 \le i, j \le n$

$$\left|(\tilde{A}^{-1})_{ij} - (A^{-1})_{ij}\right| \le \frac{(3n-2)\epsilon}{1-2n\epsilon} \max_{k,l} |(A^{-1})_{k,l}|.$$

Then,

$$\|\tilde{A}^{-1} - A^{-1}\| \le n\frac{(3n-2)\epsilon}{1-2n\epsilon} \max_{k,l} |(A^{-1})_{kl}| \le n\frac{(3n-2)\epsilon}{1-2n\epsilon}\|A^{-1}\|. \qquad \square$$

With the results of Corollary 3.3, we can now present perturbation bounds for the solution to linear systems, whose coefficient matrices are row diagonally dominant.

THEOREM 3.4. *Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ be such that $v \ge 0$ and suppose that A is nonsingular. Let $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v}) \in \mathbb{R}^{n\times n}$ be such that*

$$|\tilde{v} - v| \le \epsilon v \quad \text{and} \quad |\tilde{A}_D - A_D| \le \epsilon |A_D| \qquad \text{for some } 0 \le \epsilon < 1/(2n). \tag{3.5}$$

Let $\|\cdot\|$ be the 1-norm, the 2-norm, or the ∞-norm. Let $b, \delta b \in \mathbb{R}^{n\times 1}$ be vectors such that $\|\delta b\| \le \epsilon\|b\|$ and consider the following two linear systems:

$$Ax = b \quad \text{and} \quad \tilde{A}\tilde{x} = b + \delta b.$$

Then,

$$\frac{\|\tilde{x} - x\|}{\|x\|} \le \left[\frac{(3n^2 - 2n + 1)\epsilon + (3n^2 - 4n)\epsilon^2}{1 - 2n\epsilon}\right] \frac{\|A^{-1}\| \, \|b\|}{\|x\|}. \tag{3.6}$$

Proof. Observe that $\tilde{x} - x = (\tilde{A}^{-1} - A^{-1})b + \tilde{A}^{-1}\delta b$. Then, applying Corollary 3.3 gives

$$\begin{aligned}
\|\tilde{x} - x\| &\le \|\tilde{A}^{-1} - A^{-1}\| \, \|b\| + \|\tilde{A}^{-1}\| \, \|\delta b\| \\
&\le \|\tilde{A}^{-1} - A^{-1}\| \, \|b\| + \left[\|\tilde{A}^{-1} - A^{-1}\| + \|A^{-1}\|\right] \epsilon\|b\| \\
&\le \frac{n(3n-2)\epsilon}{1-2n\epsilon}\|A^{-1}\| \, \|b\| + \left[\frac{n(3n-2)\epsilon}{1-2n\epsilon}\|A^{-1}\| + \|A^{-1}\|\right] \epsilon\|b\| \\
&\le \left[\frac{n(3n-2)\epsilon}{1-2n\epsilon} + \epsilon\left(\frac{n(3n-2)\epsilon}{1-2n\epsilon} + 1\right)\right] \|A^{-1}\| \, \|b\|.
\end{aligned}$$

Simplifying, this bound leads to (3.6). $\square$

Theorem 3.4 shows that the sensitivity of the linear system $Ax = b$ to parameterized perturbations of type (3.5) is mainly determined by $\|A^{-1}\| \, \|b\|/\|x\|$. For general unstructured matrices, the condition number $\kappa(A, b) := \|A^{-1}\| \, \|b\|/\|x\|$ measures the normwise sensitivity of the solution x when only b is perturbed and A remains unchanged. It is immediate to see that $\kappa(A, b) \le \kappa(A)$ always holds, but much more important is to note that if $\kappa(A) \gg 1$, then $\kappa(A, b) \ll \kappa(A)$ for most vectors b (i.e., randomly chosen b), that is, the condition number $\kappa(A, b)$ is usually a moderate number compared to $\kappa(A)$. This fact is well known in numerical linear algebra and it was noted for first time in [8]. Some additional discussions on this point can be found in [16, section 3.2].

We note that a bound like (3.6) can also be derived from the perturbation bound for the solution to $Ax = b$ under a *structured* perturbation of a rank-revealing decomposition of A presented in [16] (see Remark 3.3 and Theorem 3.2 in [16]) as follows: for the perturbation considered in Theorem 3.4, we can first apply the results in [10, 14] (i.e., Theorem 2.4) to derive a perturbation bound for the *LDU* factorization of A with the column diagonal dominance pivoting, and then use the bound in [16] to obtain a perturbation bound similar to (3.6) for x. However, the bound so obtained will be weaker since it involves a dimensional constant of order n^4 instead of n^2 as in (3.6). In addition, this other approach requires considering perturbations of size $c\,n^4\epsilon < 1$, where c is a moderate constant, and so much smaller than those in (3.5).

4. Bounds for eigenvalues of symmetric matrices. In this section, we present perturbation bounds for eigenvalues of symmetric diagonally dominant matrices under parameterized perturbations of type (2.3). A first point to keep in mind is that if a matrix A enjoys, simultaneously, the properties of symmetry and row diagonal dominance, then A must be both *row and column* diagonally dominant. These properties give us two additional properties which are essential in this section: (1) the LDU decomposition of A inherits the symmetry, i.e., $A = LDL^T$, and (2) since $L = U^T$, the L factor satisfies the entrywise perturbation bounds in Theorem 2.4(3), instead of only the normwise bounds in Theorem 2.4(4). Note also that, in this case, column diagonal dominance pivoting coincides with complete diagonal pivoting.

A second point to be noted is that in [46], a strong relative perturbation bound has already been obtained for the eigenvalues of symmetric diagonally dominant matrices with nonnegative diagonals (hence positive semidefinite). More precisely, it is shown in [46] that if a symmetric diagonally dominant matrix $A = \mathcal{D}(A_D, v)$ with $v \geq 0$ and a perturbed symmetric matrix $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v})$ satisfy $|\tilde{v} - v| \leq \epsilon v$ and $|\tilde{A}_D - A_D| \leq \epsilon |A_D|$ for some ϵ with $0 \leq \epsilon < 1$, then the relative perturbation in the eigenvalues is bounded precisely by ϵ. That is, if $\lambda_1 \geq \cdots \geq \lambda_n$ are the eigenvalues of A and $\tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_n$ are the eigenvalues of $\tilde{A}$, then

$$|\tilde{\lambda}_i - \lambda_i| \leq \epsilon \lambda_i \quad \text{for } i = 1, \ldots, n. \tag{4.1}$$

This result is certainly strong and simple, but the techniques used in [46] for proving (4.1) rely heavily on the positive semidefinite character of the matrix A and we do not see how to generalize them to cover symmetric indefinite diagonally dominant matrices. In this section, we will use an approach completely different to the one in [46] to show that a relative perturbation bound similar to (4.1) holds for the eigenvalues of symmetric indefinite diagonally dominant matrices. This approach is inspired in the algorithm presented in [15] for computing, with high relative accuracy, the eigenvalues of *any* symmetric matrix expressed as a rank-revealing decomposition, and by its error analysis. The key point is that in the case of symmetric diagonally dominant matrices the strong perturbation bounds given in Theorem 2.4 for the LDL^T factorization can be expressed as a small multiplicative perturbation of the original matrix, which allows us to apply the eigenvalue relative perturbation results developed in [17].

If A is symmetric indefinite diagonally dominant, then A has both negative and positive diagonal entries and the parametrization introduced in Definition 2.2 is no longer useful. In this case, it is more appropriate to define the diagonally dominant parts using the absolute values of the diagonal entries, i.e., $v_i := |a_{ii}| - \sum_{j \neq i} |a_{ij}|$. However, a_{ii} cannot be obtained from a_{ij} $(j \neq i)$ and v_i defined this way. We need to take the signs of the diagonal entries as additional parameters to define the diagonal entries and, hence, the whole matrix. Thus, we generalize Definition 2.2 to include these additional parameters as follows.

DEFINITION 4.1. *For any $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, let A_D be the matrix whose off-diagonal entries are the same as A and whose diagonal entries are zero. Let*

$$v_i = |a_{ii}| - \sum_{j \neq i} |a_{ij}| \text{ for } i = 1, \ldots, n,$$
$$S = \operatorname{diag}(\operatorname{sign}(a_{11}), \ldots, \operatorname{sign}(a_{nn})).$$

Noting that A is uniquely determined from the parameters A_D, $v = [v_i] \in \mathbb{R}^n$, and S, we write $A = \mathcal{D}(A_D, v, S)$ to indicate that A is defined by these parameters.

With this parametrization, note that row diagonal dominance is equivalent to $v \geq 0$. Next, we introduce Lemmas 4.2 and 4.3, which are simple auxiliary results needed in the proof of the main result in this section, i.e., Theorem 4.4.

LEMMA 4.2. *Let $y \geq 0$ and $0 \leq \epsilon < 1$ be real numbers. Then,*

$$\left(\frac{1+\epsilon}{1-\epsilon}\right)^y - 1 \geq 1 - \left(\frac{1-\epsilon}{1+\epsilon}\right)^y.$$

Proof. Let $x = \left(\frac{1+\epsilon}{1-\epsilon}\right)^y$ and observe $x > 0$. Thus, $x + \frac{1}{x} \geq 2$ and, hence, $x - 1 \geq 1 - \frac{1}{x}$. ∎

LEMMA 4.3. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. If A has LDU factorization $A = LDU$ in the sense of Definition* 2.3*, then $U = L^T$ and $A = LDL^T$.*

Proof. Let $r = \operatorname{rank}(A)$ and let

$$L = \begin{bmatrix} L_{11} & 0 \\ L_{21} & I_{n-r} \end{bmatrix}, \qquad D = \begin{bmatrix} D_{11} & 0 \\ 0 & 0_{n-r} \end{bmatrix}, \qquad U = \begin{bmatrix} U_{11} & U_{12} \\ 0 & I_{n-r} \end{bmatrix},$$

with $L_{11}, D_{11}, U_{11} \in \mathbb{R}^{r\times r}$, be the LDU factorization of A in the sense of Definition 2.3. Partition A accordingly as

$$A = \begin{bmatrix} A_{11} & A_{21}^T \\ A_{21} & A_{22} \end{bmatrix}.$$

Then $A_{11} = L_{11}D_{11}U_{11}$ is the unique LDU factorization of the nonsingular matrix A_{11}. Since A_{11} is symmetric, we have $U_{11} = L_{11}^T$. Furthermore, it follows from $A_{21} = L_{21}D_{11}U_{11}$ and $A_{21}^T = L_{11}D_{11}U_{12}$ that $U_{12} = L_{21}^T$. Therefore $U = L^T$ and $A = LDL^T$. ∎

We now present the main theorem of this section, in which we consider a perturbation of $A = \mathcal{D}(A_D, v, S)$ that has small relative errors in each component of A_D, v, and S. Since S is a diagonal matrix of ± 1, this necessarily implies that S is unperturbed, which means that the signs of the diagonal entries of the matrix are preserved under the perturbation.

THEOREM 4.4. *Let $A = \mathcal{D}(A_D, v, S) \in \mathbb{R}^{n\times n}$ be a symmetric matrix such that $v \geq 0$. Let $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v}, S)$ be another symmetric matrix that satisfies*

$$|\tilde{v} - v| \leq \epsilon v \quad \text{and} \quad |\tilde{A}_D - A_D| \leq \epsilon |A_D| \quad \text{for some } 0 \leq \epsilon < 1.$$

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots \geq \tilde{\lambda}_n$ be the eigenvalues of A and $\tilde{A}$, respectively. If $2n^2(n+2)\epsilon < 1$, then

$$|\tilde{\lambda}_i - \lambda_i| \leq (2\nu + \nu^2)|\lambda_i| \quad \text{for } i = 1, \ldots, n, \tag{4.2}$$

where $\nu = \frac{2n^2(n+1)\epsilon}{1-n\epsilon}$.

Proof. First, if P is the permutation matrix defined by any diagonal pivoting scheme for A that ensures existence of the LDU factorization in the sense of Definition 2.3 (e.g., the complete pivoting), we can consider PAP^T and $P\tilde{A}P^T$, which do not change the eigenvalues or the perturbation assumptions. Therefore, we can assume without loss of generality that A is already arranged such that its LDU factorization exists. Observe that S is assumed to be unperturbed. Multiplying S on the left with the matrices A and $\tilde{A}$, we get two diagonally dominant matrices with nonnegative diagonals $C = \mathcal{D}(C_D, v_C)$ and $\tilde{C} = \mathcal{D}(\tilde{C}_D, \tilde{v}_C)$, where

$$C = SA, \quad C_D = SA_D, \quad v_C = v,$$

$$\tilde{C} = S\tilde{A}, \quad \tilde{C}_D = S\tilde{A}_D, \quad \tilde{v}_C = \tilde{v}.$$

Then,

$$|\tilde{C}_D - C_D| \leq \epsilon |C_D| \qquad \text{and} \qquad |\tilde{v}_C - v_C| \leq \epsilon v_C.$$

Since A is symmetric, it has the LDU factorization with structure $A = LDL^T$ by Lemma 4.3. In addition, C has LDU factorization, $C = L_C D_C U_C$, which satisfies $L_C = SLS$, $D_C = SD$, and $U_C = L^T$, because

$$C = SA = SLDL^T = (SLS)(SD)L^T. \tag{4.3}$$

Now, apply Theorem 2.4(1) to C and $\tilde{C}$ to obtain that $\tilde{C}$ has LDU factorization, which is denoted as $\tilde{C} = \tilde{L}_C \tilde{D}_C \tilde{U}_C$. This implies that $\tilde{A}$ also has LDU factorization, which by Lemma 4.3 has the structure $\tilde{A} = \tilde{L}\tilde{D}\tilde{L}^T$. The same argument we used in (4.3) leads to $\tilde{L}_C = S\tilde{L}S$, $\tilde{D}_C = S\tilde{D}$, and $\tilde{U}_C = \tilde{L}^T$. Next, we apply Theorem 2.4(2)–(3) to C and $\tilde{C}$ by taking into account $D_C = SD$, $U_C = L^T$, $\tilde{D}_C = S\tilde{D}$, and $\tilde{U}_C = \tilde{L}^T$, and, with the notation $L = [l_{ij}]$, $D = \operatorname{diag}[d_i]$, $\tilde{L} = [\tilde{l}_{ij}]$, and $\tilde{D} = \operatorname{diag}[\tilde{d}_i]$, we get

$$\tilde{d}_i = d_i(1 + w_i), \quad \text{with} \left(\frac{1-\epsilon}{1+\epsilon}\right)^n - 1 \leq w_i \leq \left(\frac{1+\epsilon}{1-\epsilon}\right)^n - 1 \tag{4.4}$$

for $i = 1, \dots, n$ and

$$|\tilde{l}_{ij} - l_{ij}| \leq 3n\epsilon \quad \text{for } 1 \leq i, j \leq n, \text{ i.e., } \|\tilde{L} - L\|_{\max} \leq 3n\epsilon. \tag{4.5}$$

Set $\gamma_i = \sqrt{1 + w_i} - 1$ and observe that $\tilde{d}_i = d_i(1 + \gamma_i)^2$. From (4.4), we get

$$\left(\frac{1-\epsilon}{1+\epsilon}\right)^{n/2} - 1 \leq \gamma_i \leq \left(\frac{1+\epsilon}{1-\epsilon}\right)^{n/2} - 1$$

and, from Lemma 4.2, we have

$$|\gamma_i| \leq \left(\frac{1+\epsilon}{1-\epsilon}\right)^{n/2} - 1 \leq \frac{1}{(1-\epsilon)^n} - 1 \leq \frac{n\epsilon}{1 - n\epsilon}.$$

Now, set $W = \operatorname{diag}(\gamma_1, \gamma_2, \dots, \gamma_n)$. Then, we can write $\tilde{D}$ as

$$\tilde{D} = (I + W)D(I + W) \quad \text{with} \ \|W\|_2 \leq \frac{n\epsilon}{1 - n\epsilon}. \tag{4.6}$$

Thus, letting $\Delta_L = \tilde{L} - L$, we have

$$\begin{aligned}
\tilde{A} = \tilde{L}\tilde{D}\tilde{L}^T &= (L + \Delta_L)(I + W)D(I + W)(L + \Delta_L)^T \\
&= [L + \Delta_L + LW + \Delta_L W]\, D\, [L + \Delta_L + LW + \Delta_L W]^T \\
&= (I + F)LDL^T(I + F)^T \\
&= (I + F)\, A\, (I + F)^T, \qquad (4.7)
\end{aligned}$$

where $F = \Delta_L L^{-1} + LWL^{-1} + \Delta_L WL^{-1}$. Since L is column diagonally dominant, then $\|L\|_{\max} = 1$ and $\|L^{-1}\|_{\max} = 1$ by (2.1). These bounds, combined with (4.5) and (4.6), yield

$$\begin{aligned}
\|F\|_2 &\leq \|\Delta_L\|_2 \|L^{-1}\|_2 + \|L\|_2 \|W\|_2 \|L^{-1}\|_2 + \|\Delta_L\|_2 \|W\|_2 \|L^{-1}\|_2 \\
&\leq \|\Delta_L\|_F \|L^{-1}\|_F + \|L\|_F \|W\|_2 \|L^{-1}\|_F + \|\Delta_L\|_F \|W\|_2 \|L^{-1}\|_F \\
&\leq \frac{n(n+1)}{2} [\|\Delta_L\|_{\max} \|L^{-1}\|_{\max} + \|L\|_{\max} \|W\|_2 \|L^{-1}\|_{\max} \\
&\qquad\qquad + \|\Delta_L\|_{\max} \|W\|_2 \|L^{-1}\|_{\max}] \\
&\leq \frac{n(n+1)}{2} \left[3n\epsilon + \frac{n\epsilon}{1-n\epsilon} + 3n\epsilon \left(\frac{n\epsilon}{1-n\epsilon} \right) \right] = \frac{2n^2(n+1)\epsilon}{1-n\epsilon} =: \nu,
\end{aligned} \tag{4.8}$$

where we have used that $\|M\|_F \leq \sqrt{n(n+1)/2}\, \|M\|_{\max}$ for any lower triangular matrix M. Since $2n^2(n+2)\epsilon < 1$, we have $\|F\|_2 < 1$, which implies $I + F$ is nonsingular. Hence, we can apply [17, Theorem 2.1], which states that if $\tilde{A} = (I + F)A(I + F)^T$ for a nonsingular matrix $(I + F)$, then

$$|\tilde{\lambda}_i - \lambda_i| \leq |\lambda_i|\, \|(I + F)(I + F)^T - I\|_2 \quad \text{for } i = 1, \ldots, n. \tag{4.9}$$

Note that

$$\|(I + F)(I + F)^T - I\|_2 = \|F + F^T + FF^T\|_2 \leq 2\|F\|_2 + \|F\|_2^2 \leq 2\nu + \nu^2.$$

The theorem is proved by combining this inequality with (4.9). ◻

Theorem 4.4 demonstrates that the relative perturbations of the eigenvalues are proportional to ϵ and are independent of any condition number. However, the bound in (4.2) is weaker than the one in (4.1) proved in [46] for positive semidefinite matrices, since (4.2) contains the dimensional factor n^3. This is mostly the result of bounding the 2-norm of various matrices and vectors from the max norm. It is obviously pessimistic but it is not clear how it can be improved with our current approach. For sparse matrices, the bound may be improved slightly. For example, if A is a sparse matrix such that the sparsity pattern of L as determined from that of A has at most n_L nonzeros per row, then $\tilde{L}$ and hence Δ_L have the same sparsity pattern as L. Then, using, for example,

$$\|\Delta_L L^{-1}\|_2 \leq \|\Delta_L\|_2 \|L^{-1}\|_2 \leq \|\Delta_L\|_F \|L^{-1}\|_F \leq \sqrt{n_L n} \|\Delta_L\|_{\max}\, n \|L^{-1}\|_{\max}$$

in the bound (4.8), the $n(n+1)/2$ factor there can be improved to $\sqrt{n_L} n^{3/2}$.

5. Bounds for singular values. We consider in this section perturbation bounds for singular values of nonsymmetric row diagonally dominant matrices with nonnegative diagonals. Classic perturbation bounds for the singular values of a general matrix A are obtained as by-products of the eigenvalue perturbation theory of symmetric matrices [41] just by applying this theory to

$$B = \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix} \quad \text{or} \quad A^T A.$$

However, this approach cannot be followed here, owing to the fact that A being row diagonally dominant does not imply that B or $A^T A$ is diagonally dominant. So, we need to develop a different approach and for this purpose we follow a three-step procedure similar to the one used in the proof of Theorem 4.4: in a first step the perturbation of the LDU factorization is considered using Theorem 2.4, in a second step the bounds for the LDU factors are used to express $\tilde{A}$ as a multiplicative perturbation of A (see (4.7)), and the final step uses on this expression the multiplicative perturbation results from [17]. This allows us to prove Theorem 5.1. Note that in Theorem 5.1 the matrix A is not symmetric and, so, A is only row diagonally dominant, instead of being simultaneously row and column diagonally dominant as in Theorem 4.4. This partially explains why the bound presented in Theorem 5.1 is weaker than the one in Theorem 4.4 if the matrix is symmetric.

THEOREM 5.1. *Let* $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ *be such that* $v \geq 0$ *and let* $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v}) \in \mathbb{R}^{n\times n}$ *be such that*

$$|\tilde{v} - v| \leq \epsilon v \quad \textit{and} \quad |\tilde{A}_D - A_D| \leq \epsilon |A_D| \quad \textit{for some } 0 \leq \epsilon < 1/(12n+1).$$

Let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ *and* $\tilde{\sigma}_1 \geq \tilde{\sigma}_2 \geq \cdots \geq \tilde{\sigma}_n$ *be the singular values of* A *and* $\tilde{A}$*, respectively, and let*

$$\nu := \frac{n^2(4n-1)\sqrt{2(n+1)}}{1-(12n+1)\epsilon}\epsilon.$$

If $0 \leq \nu < 1$*, then*

$$|\tilde{\sigma}_i - \sigma_i| \leq (2\nu + \nu^2)\,\sigma_i \quad \textit{for} \quad i = 1, \ldots, n. \tag{5.1}$$

Proof. As in the proof of Theorem 4.4, we can assume without loss of generality that A is arranged for column diagonal dominance pivoting. So, A has LDU factorization and, by Theorem 2.4, $\tilde{A}$ has also LDU factorization. Let $A = LDU$ and $\tilde{A} = \tilde{L}\tilde{D}\tilde{U}$ be these factorizations, and use the notation $D = \operatorname{diag}(d_1, \ldots, d_n)$ and $\tilde{D} = \operatorname{diag}(\tilde{d}_1, \ldots, \tilde{d}_n)$. Then, Theorem 2.4 implies

$$\tilde{d}_i = d_i(1 + w_i) \quad \text{with } |w_i| \leq \frac{2n\epsilon}{1-2n\epsilon} \text{ for } i = 1, \ldots, n, \tag{5.2}$$

$$\|\Delta_U\|_{\max} \leq 3n\epsilon \quad \text{with } \Delta_U := \tilde{U} - U, \tag{5.3}$$

and

$$\|\Delta_L\|_2 \leq \sqrt{n}\|\Delta_L\|_1 \leq \frac{n^{3/2}(8n-2)\epsilon}{1-(12n+1)\epsilon} \quad \text{with } \Delta_L := \tilde{L} - L. \tag{5.4}$$

If we write

$$\tilde{D} = D(I + W) \quad \text{with } W = \operatorname{diag}(w_1, w_2, \ldots, w_n),$$

then

$$\begin{aligned} \tilde{A} = \tilde{L}\tilde{D}\tilde{U} &= (L + \Delta_L)D(I + W)(U + \Delta_U) \\ &= (I + \Delta_L L^{-1})LD(U + \Delta_U + WU + W\Delta_U) \\ &= (I + E)A(I + F), \end{aligned}$$

where

$$E := \Delta_L L^{-1} \text{ and} \tag{5.5}$$

$$F := U^{-1}\Delta_U + U^{-1}WU + U^{-1}W\Delta_U. \tag{5.6}$$

Since U is row diagonally dominant, we have $\|U\|_{\max} = \|U^{-1}\|_{\max} = 1$ by (2.2). Then, from (5.2), (5.3), and (5.6), we get

$$\begin{aligned}
\|F\|_2 &\leq \|U^{-1}\|_2\|\Delta_U\|_2 + \|U^{-1}\|_2\|W\|_2\|U\|_2 + \|U^{-1}\|_2\|W\|_2\|\Delta_U\|_2 \\
&\leq \|U^{-1}\|_F\|\Delta_U\|_F + \|U^{-1}\|_F\|W\|_2\|U\|_F + \|U^{-1}\|_F\|W\|_2\|\Delta_U\|_F \\
&\leq \frac{n(n+1)}{2}[\|U^{-1}\|_{\max}\|\Delta_U\|_{\max} + \|U^{-1}\|_{\max}\|W\|_2\|U\|_{\max} \\
&\qquad\qquad + \|U^{-1}\|_{\max}\|W\|_2\|\Delta_U\|_{\max}] \\
&\leq \frac{n(n+1)}{2}\left[3n\epsilon + \frac{2n\epsilon}{1-2n\epsilon} + \frac{2n\epsilon}{1-2n\epsilon}(3n\epsilon)\right] \\
&= \frac{5n^2(n+1)\epsilon}{2(1-2n\epsilon)} \leq \nu.
\end{aligned}$$

Since L is column diagonally dominant, then $\|L^{-1}\|_{\max} = 1$, by (2.1), and hence $\|L^{-1}\|_2 \leq \|L^{-1}\|_F \leq \sqrt{n(n+1)/2}$. From (5.5) and (5.4), we obtain

$$\|E\|_2 \leq \|\Delta_L\|_2\|L^{-1}\|_2 \leq \frac{n^2(4n-1)\sqrt{2(n+1)}}{1-(12n+1)\epsilon}\epsilon = \nu.$$

Since $0 \leq \nu < 1$, both $I+E$ and $I+F$ are nonsingular. Therefore, we can apply [17, Theorem 3.3] to obtain

$$|\tilde{\sigma}_i - \sigma_i| \leq \gamma\sigma_i \quad \text{for } i = 1, 2, \ldots, n, \tag{5.7}$$

where $\gamma = \max\{\|(I+E)(I+E)^T - I\|_2, \|(I+F)^T(I+F) - I\|_2\}$. Note that

$$\begin{aligned}
\|(I+E)(I+E)^T - I\|_2 &= \|I + E + E^T + EE^T - I\|_2 = \|E + E^T + EE^T\|_2 \\
&\leq \|E\|_2 + \|E^T\|_2 + \|E\|_2\|E^T\|_2 \leq 2\|E\|_2 + \|E\|_2^2 \\
&\leq 2\nu + \nu^2.
\end{aligned}$$

Similarly, $\|(I+F)^T(I+F) - I\|_2 \leq 2\nu + \nu^2$. The theorem is proved by combining the last two inequalities with (5.7). ∎

While the significant part of Theorem 5.1 is that the relative changes of the singular values are proportional to ϵ and are independent of any condition number, the provided bound is pessimistic since it contains the dimensional factor $n^{7/2}$. This is partly inherited from the perturbation bound for L, but it is also the result of bounding the 2-norm of various matrices and vectors from the max norm. As in the case of Theorem 4.4, we do not see how this pessimistic dimensional constant can be improved with the current approach, although, as we also commented after Theorem 4.4, if A is sparse, then some slight improvement is possible.

A bound like (5.1) can also be derived by directly applying the singular value perturbation bound in [11, Theorem 2.1] for a matrix represented as a rank-revealing decomposition to the perturbed LDU factors. However, because the perturbation bounds for the L factor and the U factor given in Theorem 2.4 differ quite significantly while the approach in [11] treats the two factors equally, and because the bound in [11] involves condition numbers in the 2-norm, the bound so obtained will be weaker with a larger dimensional constant of order n^4.

Finally, note that the bound (4.1) proved in [46] for symmetric positive semidefinite diagonally dominant matrices makes it natural to conjecture that the singular values satisfy the same perturbation bound, i.e., $|\tilde{\sigma}_i - \sigma_i| \leq \epsilon\sigma_i$, since the singular value problem is essentially a symmetric positive semidefinite eigenvalue problem. However, the matrices A and C in Example 2.6 show that this conjecture is false.

6. Bounds for eigenvalues of nonsymmetric matrices. The perturbation theory for the nonsymmetric eigenvalue problem is generally much more complex than for the symmetric eigenvalue problem. For example, general normwise perturbations of a nonsymmetric matrix A produce an absolute variation of the eigenvalues of A that may be much larger than the norm of the perturbation. The reason is that the absolute variation of each simple eigenvalue of A is governed by its standard condition number [12, p. 149]. This condition number is eigenvalue dependent and is determined by the acute angle made by the left and right eigenvectors of the eigenvalue, which is related to the departure from normality of A since for normal matrices the standard condition number is always equal to one. If A is a nonnormal row diagonally dominant matrix and we consider parameterized perturbations using diagonally dominant parts and off-diagonal entries, then the dependence of the eigenvalue variation on the standard condition number can still be expected; however, we will show in this section that the relative variation is independent of the magnitude of the eigenvalue itself.

Consider the classical analytic perturbation theory for the nonsymmetric eigenvalue problem (see [12, p. 149]). Let λ be a simple eigenvalue of a general matrix $A \in \mathbb{R}^{n\times n}$ with a right eigenvector x and a left eigenvector y. The matrix $\tilde{A} = A + E$ has an eigenvalue $\tilde{\lambda}$ such that

$$\tilde{\lambda} - \lambda = \frac{y^* E x}{y^* x} + \mathcal{O}\left(\|E\|_2^2\right) \tag{6.1}$$

and

$$|\tilde{\lambda} - \lambda| \leq \sec\theta(y,x)\|E\|_2 + \mathcal{O}\left(\|E\|_2^2\right), \tag{6.2}$$

where $\theta(y,x)$ is the acute angle between x and y, and $\sec\theta(y,x) = \frac{\|y\|_2\|x\|_2}{|y^*x|}$ is the standard condition number of the eigenvalue λ. The perturbation bound (6.2) concerns the absolute variation of the eigenvalue. The corresponding relative perturbation bound depends also on the magnitude of the eigenvalue itself as follows:

$$\frac{|\tilde{\lambda} - \lambda|}{|\lambda|} \leq \left(\sec\theta(y,x)\,\frac{\|A\|_2}{|\lambda|}\right)\frac{\|E\|_2}{\|A\|_2} + \mathcal{O}\left(\|E\|_2^2\right). \tag{6.3}$$

Observe that (6.3) shows that the relative variation of λ can be large compared to the relative size of the perturbation $\|E\|_2/\|A\|_2$ as a consequence of two facts: $\|A\|_2/|\lambda|$ can be large and/or $\sec\theta(y,x)$ can be large. For parameterized perturbations of row diagonally dominant matrices, we present in this section a new perturbation bound that removes the dependence on the magnitude of the eigenvalue, i.e., it removes the factor $\|A\|_2/|\lambda|$. We first present a modified version of (6.1) by using the left eigenvector $\tilde{y}$ of $\tilde{A}$.

LEMMA 6.1. *Let λ be an eigenvalue of $A \in \mathbb{R}^{n\times n}$ with a right eigenvector x and let $\tilde{\lambda}$ be an eigenvalue of $\tilde{A} = A + E$ with a left eigenvector $\tilde{y}$ such that $\tilde{y}^* x \neq 0$. Then,*

$$\tilde{\lambda} - \lambda = \frac{\tilde{y}^* E x}{\tilde{y}^* x} \tag{6.4}$$

and

$$|\tilde{\lambda} - \lambda| \leq \sec\theta(\tilde{y},x)\|E\|_2. \tag{6.5}$$

Proof. Since $E = \tilde{A} - A$, we have

$$\tilde{y}^* E x = \tilde{y}^* \tilde{A} x - \tilde{y}^* A x = \left(\tilde{y}^* \tilde{\lambda}\right) x - \tilde{y}^* (\lambda x) = \left(\tilde{\lambda} - \lambda\right) \tilde{y}^* x,$$

from which (6.4) and hence (6.5) follow. ◻

Notice that (6.5) is very similar to (6.2); however, one advantage of (6.5) is that it is a straightforward inequality not containing asymptotically higher order error terms. On the other hand, (6.5) depends on the left eigenvector $\tilde{y}$ of $\tilde{A}$, which is not assumed to be known in a general setting. Interestingly, this turns out to be advantageous for our purpose, as will become evident in the proof of Theorem 6.2. We also note that in Lemma 6.1 neither λ nor $\tilde{\lambda}$ need be simple eigenvalues and that $\tilde{\lambda}$ can be any eigenvalue of $\tilde{A}$, not necessarily the closest one to λ. However, for small perturbations E, if λ is not simple, then $y^* x = 0$ and $\tilde{y}^* x \approx 0$. The fact that $\tilde{y}^* x \approx 0$ may happen even if λ is simple when A is close to a matrix with a multiple eigenvalue near λ. Also, if $\tilde{\lambda}$ approximates some eigenvalue λ_1 of A that is different from λ with $\tilde{y}$ approximating the left eigenvector y_1 of A corresponding to λ_1, then $\tilde{y}^* x \approx y_1^* x = 0$. In all these cases, $\sec\theta(\tilde{y}, x)$ is expected to be extremely large and, even though the bound (6.5) is valid, it is not meaningful.

We now present in Theorem 6.2 a relative perturbation bound for eigenvalues of nonsymmetric row diagonally dominant matrices. We consider the general case of matrices with possibly both positive and negative diagonal entries and, therefore, the parametrization $A = \mathcal{D}(A_D, v, S)$ introduced in Definition 4.1 is used. Note that the perturbations considered in Theorem 6.2 preserve the signs of the diagonal entries. See the remarks before Theorem 4.4 concerning this assumption.

THEOREM 6.2. *Let $A = \mathcal{D}(A_D, v, S) \in \mathbb{R}^{n\times n}$ be such that $v \geq 0$ and let λ be an eigenvalue of A with a right eigenvector x. Let $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v}, S) \in \mathbb{R}^{n\times n}$ be such that*

$$|\tilde{v} - v| \leq \epsilon v \quad \text{and} \quad |\tilde{A}_D - A_D| \leq \epsilon |A_D| \quad \text{for some } 0 \leq \epsilon < 1,$$

and let $\tilde{\lambda}$ be an eigenvalue of $\tilde{A}$ with a left eigenvector $\tilde{y}$ such that $\tilde{y}^* x \neq 0$. If $(13n + 7n^3 \sec\theta(\tilde{y}, x))\,\epsilon < 1$, then

$$|\tilde{\lambda} - \lambda| \leq \frac{8n^{7/2} + 7n^3}{1 - (13n + 7n^3 \sec\theta(\tilde{y}, x))\,\epsilon} \sec\theta(\tilde{y}, x)\epsilon\,|\lambda|, \tag{6.6}$$

where $\sec\theta(\tilde{y}, x) = \frac{\|\tilde{y}\|_2 \|x\|_2}{|\tilde{y}^* x|}$.

Proof. Clearly SA and $S\tilde{A}$ are row diagonally dominant with nonnegative diagonal entries and they satisfy condition (2.3) of Theorem 2.4. Without loss of generality, we assume that SA is arranged for column diagonal dominance pivoting. Then SA has LDU factorization $SA = LDU$ with L being column diagonally dominant and U being row diagonally dominant. By (2.1)–(2.2), we have $\|L^{-1}\|_2 \leq n\|L^{-1}\|_{\max} \leq n$, $\|U\|_2 \leq \sqrt{n}\|U\|_\infty \leq 2\sqrt{n}$, and $\|U^{-1}\|_2 \leq n\|U^{-1}\|_{\max} \leq n$.

It follows from Theorem 2.4 that $S\tilde{A}$ has LDU factorization $S\tilde{A} = \tilde{L}\tilde{D}\tilde{U}$ and

$$|\Delta_D| \leq \frac{2n\epsilon}{1 - 2n\epsilon} D \ \text{ with } \Delta_D := \tilde{D} - D, \tag{6.7}$$

$$\|\Delta_U\|_2 \leq n\|\Delta_U\|_{\max} \leq 3n^2\epsilon \ \text{ with } \Delta_U := \tilde{U} - U, \tag{6.8}$$

and

$$\|\Delta_L\|_2 \leq \sqrt{n}\|\Delta_L\|_1 \leq \frac{n^{3/2}(8n-2)}{1 - (12n+1)\epsilon}\epsilon \ \text{ with } \Delta_L := \tilde{L} - L. \tag{6.9}$$

We write $E := \tilde{A} - A = S(\tilde{L}\tilde{D}\tilde{U} - LDU)$ as

$$E = S\Delta_L DU + S\tilde{L}\Delta_D U + S\tilde{L}\tilde{D}\Delta_U.$$

Combining this expression for E with Lemma 6.1, we obtain

$$\begin{aligned}(\tilde{\lambda} - \lambda)(\tilde{y}^* x) &= \tilde{y}^* S\Delta_L DUx + \tilde{y}^* S\tilde{L}\Delta_D Ux + \tilde{y}^* S\tilde{L}\tilde{D}\Delta_U x \\ &= \lambda\tilde{y}^* S\Delta_L L^{-1}Sx + \tilde{\lambda}\tilde{y}^*\tilde{U}^{-1}\tilde{D}^\dagger\Delta_D Ux + \tilde{\lambda}\tilde{y}^*\tilde{U}^{-1}\Delta_U x,\end{aligned} \tag{6.10}$$

where $\tilde{D}^\dagger$ is the Moore–Penrose pseudoinverse of $\tilde{D}$ and we have used $DUx = L^{-1}SAx = \lambda L^{-1}Sx$, $\tilde{y}^* S\tilde{L}\tilde{D} = \tilde{y}^*\tilde{A}\tilde{U}^{-1} = \tilde{\lambda}\tilde{y}^*\tilde{U}^{-1}$, and $\tilde{y}^* S\tilde{L}\Delta_D = \tilde{y}^* S\tilde{L}(\tilde{D}\tilde{D}^\dagger)\Delta_D = \tilde{\lambda}\tilde{y}^*\tilde{U}^{-1}\tilde{D}^\dagger\Delta_D$. In addition, note that $S\tilde{A}$ is also row diagonally dominant, since $\tilde{v} \geq 0$, and, so, $\tilde{U}$ is row diagonally dominant. This implies, by (2.2), that $\|\tilde{U}^{-1}\|_2 \leq n\|\tilde{U}^{-1}\|_{\max} \leq n$. With this bound and (6.7), (6.8), and (6.9), we get

$$\|\Delta_L L^{-1}\|_2 \leq \|\Delta_L\|_2\|L^{-1}\|_2 \leq \frac{n^{5/2}(8n-2)}{1-(12n+1)\epsilon}\epsilon,$$

$$\|\tilde{D}^\dagger\Delta_D\|_2 = \|(I + D^\dagger\Delta_D)^{-1}D^\dagger\Delta_D\|_2 \leq \|(I + D^\dagger\Delta_D)^{-1}\|_2\,\|D^\dagger\Delta_D\|_2 \leq \frac{2n\epsilon}{1-4n\epsilon},$$

and

$$\|\tilde{U}^{-1}\Delta_U\|_2 \leq \|\tilde{U}^{-1}\|_2\,\|\Delta_U\|_2 \leq 3n^3\epsilon.$$

Substituting these into (6.10), we obtain

$$\begin{aligned}|\tilde{\lambda} - \lambda||\tilde{y}^* x| &\leq \|\tilde{y}\|_2\|x\|_2\left(|\lambda|\frac{n^{5/2}(8n-2)\epsilon}{1-(12n+1)\epsilon} + |\tilde{\lambda}|\,n\,\frac{2n\epsilon}{1-4n\epsilon}\,2\sqrt{n} + |\tilde{\lambda}|\,3n^3\epsilon\right) \\ &\leq \|\tilde{y}\|_2\|x\|_2\left(|\lambda|\frac{8n^{7/2}\epsilon}{1-13n\epsilon} + |\tilde{\lambda}|\frac{7n^3\epsilon}{1-4n\epsilon}\right)\end{aligned}$$

and thus

$$|\tilde{\lambda} - \lambda| \leq \sec\theta(\tilde{y}, x)\left(|\lambda|\frac{8n^{7/2}\epsilon}{1-13n\epsilon} + |\tilde{\lambda}|\frac{7n^3\epsilon}{1-13n\epsilon}\right).$$

Finally, use $|\tilde{\lambda}| \leq |\tilde{\lambda} - \lambda| + |\lambda|$ and rearrange the inequality above to produce the desired result. ∎

Theorem 6.2 improves the classical bound (6.3) in that the relative perturbation $|\tilde{\lambda} - \lambda|/|\lambda|$ in the eigenvalue is proportional to $\sec\theta(\tilde{y}, x)\epsilon$ but independent of the eigenvalue itself. In addition, we have a rigorous inequality independent of any high order term. A drawback of Theorem 6.2 is that the condition number $\sec\theta(\tilde{y}, x)$ is defined from the left eigenvector of $\tilde{A}$. However, if $\tilde{\lambda}$ approximates a simple eigenvalue λ and A is not close to a matrix with a multiple eigenvalue near λ, then $\tilde{y} \approx y$ and $\sec\theta(\tilde{y}, x) \approx \sec\theta(y, x)$ in an asymptotic sense. In addition, as discussed just after Lemma 6.1, this is the only situation in which Theorem 6.2 is really useful and meaningful, since otherwise $\tilde{y}^* x \approx 0$ and it renders a meaningless bound. Therefore, Theorem 6.2 implicitly requires that $\tilde{\lambda}$ is an eigenvalue of $\tilde{A}$ that approximates a simple eigenvalue λ of A and that A is not very close to a matrix with a multiple eigenvalue near λ.

Example 6.3 illustrates the key feature of Theorem 6.2, i.e., that tiny eigenvalues of row diagonally dominant nonsymmetric matrices with small or moderate standard eigenvalue condition number are robust under small componentwise perturbations of their diagonally dominant parts and off-diagonal entries, while this is not the case under traditional entrywise perturbations of the matrix. In addition, Example 6.3 illustrates that $\sec\theta(y,x)$ can be reliably used instead of $\sec\theta(\tilde{y},x)$ in (6.6).

Example 6.3. Let us consider the following three row diagonally dominant matrices A, B, C with vectors of diagonally dominant parts denoted by v_A, v_B, v_C:

$$A = \begin{bmatrix} 3 & -1.5 & 1.5 \\ -1 & 2.002 & 1 \\ 3 & 1.5 & 4.5 \end{bmatrix}, \qquad v_A = \begin{bmatrix} 0 \\ 0.002 \\ 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 3 & -1.5 & 1.5 \\ -1 & 2.001 & 1 \\ 3 & 1.5 & 4.5 \end{bmatrix}, \qquad C = \begin{bmatrix} 3.0015 & -1.5015 & 1.5 \\ -1 & 2.002002 & 1 \\ 3 & 1.5 & 4.5 \end{bmatrix},$$

$$v_B = \begin{bmatrix} 0 & 0.001 & 0 \end{bmatrix}^T, \quad \text{and} \quad v_C = \begin{bmatrix} 0 & 0.002002 & 0 \end{bmatrix}^T.$$

Observe that $|A-B| \le 5\cdot 10^{-4}\,|A|$, $|A-C| \le 10^{-3}\cdot|A|$, $|v_A - v_B| = 0.5\,v_A$, and $|v_A - v_C| = 10^{-3}\,v_A$. Thus, C is a nearby perturbation of A in the sense of diagonally dominant parts and off-diagonal entries, while B is a nearby perturbation of A only in the traditional entrywise sense. The eigenvalues of A, B, and C are shown in the table below, together with the corresponding standard eigenvalue condition numbers $\sec\theta(y,x)$ for A and the effective condition numbers $\sec\theta(\tilde{y},x)$ appearing in the bound (6.6) when considering $\tilde{A} = C$ (all numbers in the table are given with four significant digits):

	A	B	C	$\sec\theta(y,x)$	$\sec\theta(\tilde{y},x)$
λ_1	6.000	6.000	6.000	1.086	1.086
λ_2	3.501	3.501	3.502	1.080	1.080
λ_3	$8.569\cdot 10^{-4}$	$4.285\cdot 10^{-4}$	$8.580\cdot 10^{-4}$	1.035	1.035

Observe that the tiniest eigenvalues of A and C agree in the first two digits, while that of B is completely different. Observe also that $\sec\theta(y,x)$ and $\sec\theta(\tilde{y},x)$ coincide in all digits shown in the table.

Theorem 6.2 can be generalized to a structured perturbation problem where a general matrix, i.e., not necessarily diagonally dominant, is perturbed with small changes in the factors of a rank-revealing decomposition of the matrix. Recall that given a matrix $A \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(A) = r$, we say that $A = XDY \in \mathbb{R}^{m\times n}$ is a rank-revealing decomposition of A [11] if $D \in \mathbb{R}^{r\times r}$ is diagonal and nonsingular, and $X \in \mathbb{R}^{m\times r}$ and $Y \in \mathbb{R}^{r\times n}$ are well-conditioned matrices. Since X and Y may be rectangular matrices, their condition numbers are defined using their Moore–Penrose pseudoinverses, denoted by $X^\dagger$ and $Y^\dagger$, as $\kappa_2(X) = \|X\|_2\,\|X^\dagger\|_2$ and $\kappa_2(Y) = \|Y\|_2\,\|Y^\dagger\|_2$. Next, we consider in Theorem 6.4 perturbations of A obtained with small entrywise relative perturbations in the diagonal factor D and small normwise relative perturbations in the factors X and Y. The bound in Theorem 6.4 may be applicable to some special matrices for which a rank-revealing decomposition can be accurately computed [11, 15, 16].

THEOREM 6.4. *Let $A = XDY \in \mathbb{R}^{n\times n}$ be a rank-revealing decomposition and let $\tilde{A} = \tilde{X}\tilde{D}\tilde{Y} \in \mathbb{R}^{n\times n}$ be such that*

$$\tilde{X} = X + \Delta_X, \quad \tilde{D} = D + \Delta_D, \quad \tilde{Y} = Y + \Delta_Y$$

with

$$|\Delta_D| \leq \epsilon |D|, \quad \|\Delta_X\|_2 \leq \epsilon \|X\|_2, \text{ and } \|\Delta_Y\|_2 \leq \epsilon \|Y\|_2 \tag{6.11}$$

for some $0 \leq \epsilon < 1$. *Let* λ *be an eigenvalue of* A *with right eigenvector* x *and let* $\tilde{\lambda}$ *be an eigenvalue of* $\tilde{A}$ *with left eigenvector* $\tilde{y}$ *such that* $\tilde{y}^* x \neq 0$. *Let* $\kappa = \max\{\kappa_2(X), \kappa_2(Y)\}$. *If* $\epsilon\, \kappa\, (1 + \sec\theta(\tilde{y}, x)) < 1$, *then*

$$|\tilde{\lambda} - \lambda| \leq \epsilon\, \kappa \sec\theta(\tilde{y}, x)\, \frac{3 + \epsilon}{1 - \epsilon\, \kappa\, (1 + \sec\theta(\tilde{y}, x))}\, |\lambda|. \tag{6.12}$$

Proof. Let $r = \operatorname{rank}(A)$. So $X^\dagger X = I_r$ and $YY^\dagger = I_r$, since XDY is a rank-revealing decomposition. In addition, note that[1] $\operatorname{rank}(X) = \operatorname{rank}(\tilde{X}) = r$ and $\operatorname{rank}(Y) = \operatorname{rank}(\tilde{Y}) = r$. So $\tilde{X}^\dagger \tilde{X} = I_r$ and $\tilde{Y}\tilde{Y}^\dagger = I_r$ also hold.

Observe that

$$\tilde{A} - A = \Delta_X DY + \tilde{X}\Delta_D Y + \tilde{X}\tilde{D}\Delta_Y.$$

Applying (6.4) yields

$$\begin{aligned}(\tilde{y}^* x)\left(\tilde{\lambda} - \lambda\right) &= \tilde{y}^* \Delta_X DYx + \tilde{y}^* \tilde{X}\Delta_D Yx + \tilde{y}^* \tilde{X}\tilde{D}\Delta_Y x \\ &= \lambda \tilde{y}^* \Delta_X X^\dagger x + \lambda \tilde{y}^* \tilde{X}\Delta_D D^{-1} X^\dagger x + \tilde{\lambda}\tilde{y}^* \tilde{Y}^\dagger \Delta_Y x,\end{aligned} \tag{6.13}$$

since $DYx = X^\dagger Ax = \lambda X^\dagger x$, $\Delta_D Yx = \Delta_D(D^{-1}D)Yx = \lambda \Delta_D D^{-1} X^\dagger x$, and $\tilde{y}^* \tilde{X}\tilde{D} = \tilde{y}^* \tilde{A}\tilde{Y}^\dagger = \tilde{\lambda}\tilde{y}^* \tilde{Y}^\dagger$. From the assumption (6.11), we get

$$\|\Delta_X X^\dagger\|_2 \leq \|\Delta_X\|_2\, \|X^\dagger\|_2 \leq \epsilon\, \kappa_2(X), \tag{6.14}$$

$$\|\tilde{X}\Delta_D D^{-1} X^\dagger\|_2 \leq \|X + \Delta_X\|_2\, \|\Delta_D D^{-1}\|_2 \|X^\dagger\|_2 \leq \epsilon (1 + \epsilon)\, \kappa_2(X), \tag{6.15}$$

and if $\tilde{\sigma}_r$ and σ_r are, respectively, the smallest singular values of $\tilde{Y}$ and Y,

$$\|\tilde{Y}^\dagger \Delta_Y\|_2 \leq \frac{\|\Delta_Y\|_2}{\tilde{\sigma}_r} \leq \frac{\epsilon\, \|Y\|_2}{\sigma_r - \|\Delta_Y\|_2} \leq \frac{\epsilon\, \|Y\|_2}{\sigma_r - \epsilon\, \|Y\|_2} \leq \frac{\epsilon\, \kappa_2(Y)}{1 - \epsilon\, \kappa_2(Y)}. \tag{6.16}$$

Combining (6.14)–(6.15)–(6.16) with (6.13), we have

$$|\tilde{y}^* x|\, |\tilde{\lambda} - \lambda| \leq \|\tilde{y}\|_2 \|x\|_2 \left(|\lambda|\, \epsilon\, (2 + \epsilon)\, \kappa_2(X) + |\tilde{\lambda}|\, \frac{\epsilon\, \kappa_2(Y)}{1 - \epsilon\, \kappa_2(Y)} \right).$$

Finally, use $|\tilde{\lambda}| \leq |\tilde{\lambda} - \lambda| + |\lambda|$ and rearrange the inequality above to obtain (6.12). $\square$

We finish this section with some remarks on other possible strategies for obtaining relative perturbation bounds for eigenvalues of nonsymmetric matrices when they

[1] These two equalities follow from (6.11). Let us prove it only for X, since it is similar for Y. According to the Weyl perturbation theorem [41] for singular values, we have $|\sigma_i(\tilde{X}) - \sigma_i(X)| \leq \|\Delta_X\|_2 \leq \epsilon \|X\|_2$ for $i = 1, \ldots, r$, where $\sigma_i(X)$ and $\sigma_i(\tilde{X})$ are the singular values of X and $\tilde{X}$ respectively arranged in decreasing order. So $|\sigma_i(\tilde{X}) - \sigma_i(X)|/\sigma_i(X) \leq \epsilon\kappa_2(X) < 1$ for $i = 1, \ldots, r$. This and $\sigma_i(X) \neq 0$ imply that $\sigma_i(\tilde{X}) \neq 0$ for all i.

are perturbed using a rank-reveling decomposition as in (6.11). This type of perturbation of rank-revealing decompositions can always be written as a *multiplicative* perturbation of the original matrix. This has been used before in [7, 11, 15, 16]. Then, it is possible to use relative bounds for eigenvalues of nonsymmetric matrices under multiplicative perturbations which are already available in the literature [27, section 5] (see also the original references [18] and [30]). Essentially, two types of relative bounds can be found: Bauer–Fike and Hoffman–Wielandt bounds. The Bauer–Fike bounds require A to be diagonalizable and depend on the condition number of the whole eigenvector matrix of A, that is, the square matrix whose columns are all the eigenvectors of A. The Hoffman–Wielandt bounds still require stronger assumptions, since they require both A and $\tilde{A}$ to be diagonalizable and they depend on the product of the condition numbers of *both* the whole eigenvector matrices of A and $\tilde{A}$. The main drawback of these bounds is that the condition number of the whole eigenvector matrix is larger than the largest condition number of all the individual eigenvalues [12, Theorem 4.7]. Thus, the relative Bauer–Fike and Hoffman–Wielandt bounds may be very pessimistic in situations where only some eigenvalues have large standard condition numbers, but the condition numbers of other eigenvalues are moderate. Theorem 6.4 presented here has the obvious advantage of depending essentially only on the condition number of each individual eigenvalue and, in addition, it does not require that the matrix A be diagonalizable.

7. Concluding remarks. We have systematically studied the relative perturbation theory for row diagonally dominant matrices under small componentwise perturbations of their diagonally dominant parts and off-diagonal entries. The use of this parameterized perturbation has been the key to deriving strong relative perturbation bounds for inverses, solutions to linear systems, the symmetric indefinite eigenvalue problem, the singular value problem, and the nonsymmetric eigenvalue problem. These bounds demonstrate that potentially much more accurate algorithms than the traditional ones are possible for solving all these problems from the use of diagonally dominant parts and off-diagonal entries. Indeed, such high relative accuracy algorithms have already been obtained for the LDU factorization and the singular value problem in [45], and the results in the present paper show that highly accurate algorithms for other problems also can be obtained by combining the LDU algorithm in [45] with the algorithms in [7, 15, 16]. One challenging open problem in this area is to develop algorithms to compute the eigenvalues of nonsymmetric diagonally dominant matrices with the relative accuracy determined by Theorem 6.2. This will be the subject of future research.

Acknowledgments. The authors are indebted to the associate editor handling this paper for proposing many technical improvements and suggestions on the presentation that have contributed significantly to improving this manuscript. The authors also thank two anonymous referees for many other very useful comments.

REFERENCES

[1] J. Ahlberg and E. Nilson, *Convergence properties of the spline fit*, J. SIAM, 11 (1963), pp. 95–104.
[2] A. S. Alfa, J. Xue, and Q. Ye, *Accurate computation of the smallest eigenvalue of a diagonally dominant M-matrix*, Math. Comp., 71 (2002), pp. 217–236.
[3] A. S. Alfa, J. Xue, and Q. Ye, *Entrywise perturbation theory for diagonally dominant M-matrices with applications*, Numer. Math., 90 (2002), pp. 401–414.

[4] J. Barlow and J. Demmel, *Computing accurate eigensystems of scaled diagonally dominant matrices*, SIAM J. Numer. Anal., 27 (1990), pp. 762–791.
[5] S. Bora, *Structured eigenvalue condition number and backward error of a class of polynomial eigenvalue problems*, SIAM J. Matrix Anal. Appl., 31 (2009), pp. 900–917.
[6] S. Bora and V. Mehrmann, *Linear perturbation theory for structured matrix pencils arising in control theory*, SIAM J. Matrix Anal. Appl., 28 (2006), pp. 148–169.
[7] N. Castro-González, J. Ceballos, F. M. Dopico, and J. M. Molera, *Accurate solution of structured least squares problems via rank-revealing decompositions*, SIAM J. Matrix Anal. Appl., 34 (2013), pp. 1112–1128.
[8] T. F. Chan and D. E. Foulser, *Effectively well-conditioned linear systems*, SIAM J. Sci. Statist. Comput., 9 (1988), pp. 963–969.
[9] X.-W. Chang and R.-C. Li, *Multiplicative perturbation analysis for QR factorizations*, Numer. Algebra Control Optim., 1 (2011), pp. 301–316.
[10] M. Dailey, F. M. Dopico, and Q. Ye, A new perturbation bound for the LDU factorization of diagonally dominant matrices, SIAM J. Matrix Anal. Appl., 35 (2014), pp. 904–930.
[11] J. Demmel, M. Gu, S. Eisenstat, I. Slapničar, K. Veselić, and Z. Drmač, *Computing the singular value decomposition with high relative accuracy*, Linear Algebra Appl., 299 (1999), pp. 21–80.
[12] J. W. Demmel, *Applied Numerical Linear Algebra*, SIAM, Philadelphia, 1997.
[13] I. S. Dhillon and B. N. Parlett, *Orthogonal eigenvectors and relative gaps*, SIAM J. Matrix Anal. Appl., 25 (2003), pp. 858–899.
[14] F. M. Dopico and P. Koev, *Perturbation theory for the LDU factorization and accurate computations for diagonally dominant matrices*, Numer. Math., 119 (2011), pp. 337–371.
[15] F. M. Dopico, P. Koev, and J. M. Molera, *Implicit standard Jacobi gives high relative accuracy*, Numer. Math., 113 (2009), pp. 519–553.
[16] F. M. Dopico and J. M. Molera, *Accurate solution of structured linear systems via rank-revealing decompositions*, IMA J. Numer. Anal., 32 (2012), pp. 1096–1116.
[17] S. C. Eisenstat and I. C. F. Ipsen, *Relative perturbation techniques for singular value problems*, SIAM J. Numer. Anal., 32 (1995), pp. 1972–1988.
[18] S. C. Eisenstat and I. C. F. Ipsen, *Relative perturbation results for eigenvalues and eigenvectors of diagonalisable matrices*, BIT, 38 (1998), pp. 502–509.
[19] C. Ferreira, B. Parlett, and F. M. Dopico, *Sensitivity of eigenvalues of an unsymmetric tridiagonal matrix*, Numer. Math., 122 (2012), pp. 527–555.
[20] R. Funderlic, M. Neumann, and R. Plemmons, *LU decompositions of generalized diagonally dominant matrices*, Numer. Math., 40 (1982), pp. 57–69.
[21] G. Golub and C. Van Loan, *Matrix Computations*, 3rd ed., Johns Hopkins University Press, Baltimore, MD, 1996.
[22] D. J. Higham and N. J. Higham, *Backward error and condition of structured linear systems*, SIAM J. Matrix Anal. Appl., 13 (1992), pp. 162–175.
[23] D. J. Higham and N. J. Higham, *Structured backward error and condition of generalized eigenvalue problems*, SIAM J. Matrix Anal. Appl., 20 (1999), pp. 493–512.
[24] N. J. Higham, *Accuracy and Stability of Numerical Algorithms*, 2nd ed., SIAM, Philadelphia, PA, 2002.
[25] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1985.
[26] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1994.
[27] I. C. F. Ipsen, *Relative perturbation results for matrix eigenvalues and singular values*, Acta Numer., 7 (1998), pp. 151–201.
[28] M. Karow, D. Kressner, and F. Tisseur, *Structured eigenvalue condition numbers*, SIAM J. Matrix Anal. Appl., 28 (2006), pp. 1052–1068.
[29] P. Koev, *Accurate eigenvalues and SVDs of totally nonnegative matrices*, SIAM J. Matrix Anal. Appl., 27 (2005), pp. 1–23.
[30] R.-C. Li, *Relative perturbation theory.* III. *More bounds on eigenvalue variation*, Linear Algebra Appl., 266 (1997), pp. 337–345.
[31] R.-C. Li, *A bound on the solution to a structured Sylvester equation with an application to relative perturbation theory*, SIAM J. Matrix Anal. Appl., 21 (1999), pp. 440–445.
[32] R.-C. Li, Y. Nakatsukasa, N. Truhar, and S. Xu, *Perturbation of partitioned Hermitian definite generalized eigenvalue problems*, SIAM J. Matrix Anal. Appl., 32 (2011), pp. 642–663.
[33] J. Matejaš and V. Hari, *Relative eigenvalue and singular value perturbations of scaled diagonally dominant matrices*, BIT, 48 (2008), pp. 769–781.

[34] A. M. Ostrowski, *Note on bounds for determinants with dominant principal diagonal*, Proc. Amer. Math. Soc., 3 (1952), pp. 26–30.
[35] B. N. Parlett, *Spectral sensitivity of products of bidiagonals*, Linear Algebra Appl., 275/276 (1998), pp. 417–431.
[36] B. N. Parlett, *Perturbation of eigenpairs of factored symmetric tridiagonal matrices*, Found. Comput. Math., 3 (2003), pp. 207–223.
[37] B. N. Parlett, *A bidiagonal matrix determines its hyperbolic SVD to varied relative accuracy*, SIAM J. Matrix Anal. Appl., 26 (2005), pp. 1022–1057.
[38] J. M. Peña, *LDU decompositions with L and U well conditioned*, Electron. Trans. Numer. Anal., 18 (2004), pp. 198–208.
[39] S. M. Rump, *Eigenvalues, pseudospectrum and structured perturbations*, Linear Alg. Appl., 413 (2006), pp. 567–593.
[40] P. N. Shivakumar, J. J. Williams, Q. Ye, and C. A. Marinov, *On two-sided bounds related to weakly diagonally dominant M-matrices with application to digital circuit dynamics*, SIAM J. Matrix Anal. Appl., 17 (1996), pp. 298–312.
[41] G. W. Stewart and J.-G. Sun, *Matrix Perturbation Theory*, Academic Press, New York, 1990.
[42] J. M. Varah, *A lower bound for the smallest singular value of a matrix*, Linear Algebra Appl., 11 (1975), pp. 3–5.
[43] R. Varga, *On diagonal dominance arguments for bounding* $\|A^{-1}\|_\infty$, Linear Algebra Appl., 14 (1976), pp. 211–217.
[44] R. Varga and D.-Y. Cai, *On the LU factorization of M-matrices*, Numer. Math., 38 (1981), pp. 179–192.
[45] Q. Ye, *Computing singular values of diagonally dominant matrices to high relative accuracy*, Math. Comp., 77 (2008), pp. 2195–2230.
[46] Q. Ye, *Relative perturbation bounds for eigenvalues of symmetric positive definite diagonally dominant matrices*, SIAM J. Matrix Anal. Appl., 31 (2009), pp. 11–17.