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LARGE VECTOR SPACES OF BLOCK-SYMMETRIC STRONG LINEARIZATIONS OF MATRIX POLYNOMIALS

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Abstract. Given a matrix polynomial $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ of degree k, where A_i are $n \times n$ matrices with entries in a field $\mathbb{F}$, the development of linearizations of $P(\lambda)$ that preserve whatever structure $P(\lambda)$ might posses has been a very active area of research in the last decade. Most of the structure-preserving linearizations of $P(\lambda)$ discovered so far are based on certain modifications of block-symmetric linearizations. The block-symmetric linearizations of $P(\lambda)$ available in the literature fall essentially into two classes: linearizations based on the so-called Fiedler pencils with repetition, which form a finite family, and a vector space of dimension k of block-symmetric pencils, called $\mathbb{DL}(P)$, such that most of its pencils are linearizations. One drawback of the pencils in $\mathbb{DL}(P)$ is that none of them is a linearization when $P(\lambda)$ is singular. In this paper we introduce new vector spaces of block-symmetric pencils, most of which are strong linearizations of $P(\lambda)$. The dimensions of these spaces are $O(n^2)$, which, for $n \geq \sqrt{k}$, are much larger than the dimension of $\mathbb{DL}(P)$. When k is odd, many of these vector spaces contain linearizations also when $P(\lambda)$ is singular. The coefficients of the block-symmetric pencils in these new spaces can be easily constructed as $k \times k$ block-matrices whose $n \times n$ blocks are of the form 0, $\pm\alpha I_n$, $\pm\alpha A_i$, or arbitrary $n \times n$ matrices, where α is an arbitrary nonzero scalar.

Key words. block-symmetric linearizations, Fiedler pencils with repetition, generalized Fiedler pencils with repetition, matrix polynomials, strong linearizations, structured matrix polynomials, vector space $\mathbb{DL}(P)$

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1. Introduction.

Let $\mathbb{F}$ be an arbitrary field and $M_n(\mathbb{F})$ be the set of $n \pm n$ matrices with entries in $\mathbb{F}$. Throughout this paper we consider the matrix polynomial of degree k and size $n \pm n$

$$P(\lambda) = \bigcup_{i=0}^{k} \lambda^i A_i, \quad \text{where } A_0, \ldots, A_k \,/\, M_n(\mathbb{F}),\; A_k \neq 0. \tag{1.1}$$

The matrix polynomial $P(\lambda)$ is said to be singular if $\det P(\lambda)$ is identically zero, and it is said to be regular otherwise. Matrix polynomials arise in many applications, are receiving considerable attention in the literature in the last years, and some general references on this topic are [13, 19, 29].

The most extended way to deal in theory and applications with matrix polynomials is via *linearizations* [13]. A linearization of the matrix polynomial $P(\lambda)$ is a pencil $L(\lambda) = \lambda L_1 \quad L_0$ of size $(nk) \pm (nk)$ such that there exist two unimodular matrix polynomials, i.e., matrix polynomials with constant nonzero determinant, $U(\lambda)$ and $V(\lambda)$, which satisfy

$$U(\lambda)\, L(\lambda)\, V(\lambda) = \begin{bmatrix} P(\lambda) & 0 \\ 0 & I_{n(k-1)} \end{bmatrix},$$

where $I_{n(k-1)}$ is the identity matrix of size $n(k \quad 1) \pm n(k \quad 1)$. Even more interesting are the *strong linearizations* of $P(\lambda)$ [12], whose definition requires to introduce first the *reversal*

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of $P(\lambda)$ as the polynomial $\text{rev}\, P(\lambda) := \lambda^k P(1/\lambda)$. Then a linearization $L(\lambda)$ of $P(\lambda)$ is said to be *strong* if $\text{rev}\, L(\lambda)$ is also a linearization of $\text{rev}\, P(\lambda)$. The key property of strong linearizations is that they have the same finite and infinite regular spectral structures as $P(\lambda)$ [11]. We emphasize that we are using the classical definitions and sizes of linearizations and strong linearizations. Extensions of these concepts allowing other sizes have been considered recently by several authors (see [11] and the references therein). In this paper, given a pencil $L(\lambda) = \lambda L_1 \quad L_0$, we will call the constant matrices L_0 and L_1 the coefficients of $L(\lambda)$ of degree 0 and 1, respectively.

The matrix polynomials that arise in applications have very often particular structures. For example, they can be symmetric, Hermitian, palindromic, or alternating, among many other possible structures (see [20, 21, 23, 27] and the references therein for more details). Most structured matrix polynomials have spectra with particular symmetries or constrains. Therefore, it is important to have strong linearizations of such structured matrix polynomials that share the same structure, so that they preserve the properties of the spectrum under the effect of rounding errors if structure-preserving numerical algorithms for the corresponding generalized eigenvalue problem are used on the linearizations. Additionally, it is desirable to have a large class of structured strong linearizations that can be constructed easily, from which one can select a linearization with the most favorable properties in terms, for instance, of conditioning and backward errors of eigenvalues [28], sparsity patterns, or the recovery of eigenvectors. This has motivated an intense research activity in the last years on the development of structure-preserving strong linearizations of matrix polynomials. The following list of references is an incomplete sample of recent papers on this topic [2, 3, 4, 6, 7, 10, 17, 18, 22, 23, 24, 25, 26, 32].

Most structure-preserving strong linearizations of matrix polynomials appearing in the literature are obtained from *block-symmetric* strong linearizations via some manipulations. The notion of block-symmetry is defined as follows (see for instance [17]): let $H = (H_{ij})_{i,j=1}^k$, with $H_{ij} \ / \ M_n(\mathbb{F})$, be a $k \pm k$ block-matrix and define the block-transpose $H^{\mathcal{B}}$ of H as the $k \pm k$ block-matrix having the block H_{ji} in the block position (i, j), for $1 \geq i, j \geq k$. Then, we say that H is block-symmetric if $H = H^{\mathcal{B}}$ and [1] we say that a pencil $L(\lambda) = \lambda L_1 \quad L_0$ is block-symmetric if both matrices L_1 and L_0 are block-symmetric. Therefore, the development of block-symmetric strong linearizations of matrix polynomials is the key step towards obtaining structure-preserving strong linearizations and has received considerable attention in the literature. Essentially, two main classes of block-symmetric strong linearizations are available nowadays. They are described in the next two paragraphs.

Given a matrix polynomial $P(\lambda)$ as in (1.1), the associated vector space of block-symmetric pencils $\mathbb{DL}(P)$ was introduced and studied in [17, 22]. Recently, an insightful functional interpretation for $\mathbb{DL}(P)$ has been presented in [30]. The space $\mathbb{DL}(P)$ is a k-dimensional vector space over $\mathbb{F}$ of block-symmetric pencils most of which are strong linearizations of $P(\lambda)$ when $P(\lambda)$ is regular. The pencils in this space are easily constructible from the matrix coefficients of $P(\lambda)$. However, a drawback of $\mathbb{DL}(P)$ is that it does not contain any linearization of $P(\lambda)$ if $P(\lambda)$ is singular [9, Theorem 6.1], which questions the utility of such space also when $P(\lambda)$ is regular but very close to be singular, even in the case a staircase algorithm valid for "numerically singular pencils" is used [31]. To the best of our knowledge, no vector spaces of block-symmetric strong linearizations for singular matrix polynomials have appeared so far in the literature. To introduce spaces of this type for polynomials with odd degree is one of the main contributions of this paper.

The other main class of block-symmetric strong linearizations of $P(\lambda)$ as in (1.1) are

[1] Observe that if $H_{ij} = H_{ij}^T$ for all (i, j), then H is block-symmetric if and only if it is symmetric.

based on the so-called Fiedler pencils with repetition (FPR) introduced in [32], and whose definition is reminded in Definition 5.1. The family of FPR is a *finite* family of pencils that are strong linearizations of $P(\lambda)$ under certain conditions. It was shown in [32] that some FPR are in fact block-symmetric, and the complete characterization of *all* FPR that are block-symmetric has been recently obtained in [4]. It is important to remark that some block-symmetric FPR are strong linearizations when $P(\lambda)$ is singular and its degree is odd. It is also worth noting that the pencils in the standard basis of $\mathbb{DL}(P)$ [17, Section 3.3] are block-symmetric FPR [32, Corollary 2].

In this paper, we expand in a significative way the classes of block-symmetric strong linearizations of $P(\lambda)$ discussed previously. In a certain sense, we proceed by combining both approaches since, starting from the *finite family* of block-symmetric FPR, we are able to construct many vector spaces, consisting mostly of block-symmetric strong linearizations of $P(\lambda)$, with dimension much larger than $\mathbb{DL}(P)$, that is, we get large *infinite families* of block-symmetric strong linearizations.

We start our trip by extending the family of FPR to a new much larger family of pencils that we call *generalized Fiedler pencils with repetition* (GFPR). The pencils in this family are easily constructible from the coefficients of the k-degree matrix polynomial $P(\lambda)$ in (1.1). More precisely, their coefficient matrices can be seen as block-matrices whose blocks are of the form 0, I_n, $\oplus A_i$, or arbitrary $n \pm n$ matrices. We give sufficient conditions for a GFPR to be a strong linearization of $P(\lambda)$. Then, we consider a subset of block-symmetric pencils inside the family of GFPR, which contains the block-symmetric FPR, and, therefore, the standard basis for $\mathbb{DL}(P)$. We define the vector space generated by these block-symmetric GFPR and call it the extended $\mathbb{DL}(P)$, denoted by $\mathbb{EDL}(P)$. Thus, the space $\mathbb{DL}(P)$ is a subspace of $\mathbb{EDL}(P)$. When $k \sim 3$, we construct many other subspaces of $\mathbb{EDL}(P)$ which consist mostly of strong linearizations of $P(\lambda)$ and give an easily constructible basis for each of them. Although some of these spaces contain linearizations only when $P(\lambda)$ is regular (satisfying some extra additional nonsingularity conditions), we also obtain many subspaces consisting mostly of block-symmetric strong linearizations when $P(\lambda)$ is a singular matrix polynomial with odd degree. The largest dimension of the new subspaces we introduce is $\left\lfloor \frac{(k-1)^2}{4} \right. \; n^2+1$, which is much larger than the dimension k of $\mathbb{DL}(P)$ if $n \sim \;\; \overline{k}$. Note that the dimension depends not only on the degree of $P(\lambda)$ but also on its size. An interesting property of the pencils in these large subspaces is that all of them can be obtained from a block-symmetric FPR by choosing carefully some of its blocks, replacing them by $n \pm n$ arbitrary matrices, and, finally, multiplying the remaining blocks by an arbitrary scalar. This simple replacement procedure is described in Section 2 in the particular case of the FPR in the standard basis of $\mathbb{DL}(P)$.

Another remarkable property of many of the new block-symmetric strong linearizations that we introduce is that they preserve the sign characteristic [13, 14] of $P(\lambda)$, when $P(\lambda)$ is Hermitian and has real eigenvalues. At present, this property has been proved only for one linearization: the last pencil in the standard basis of $\mathbb{DL}(P)$ [1, Lemma 2.8]. Note that the proof in [1] can be easily extended to prove that many other pencils in $\mathbb{DL}(P)$ preserve the sign characteristic, but that it remains valid only for matrix polynomials with semisimple eigenvalues. For brevity, we postpone the study of this interesting question to [5].

The paper is organized as follows. In Section 2, we present simple examples that illustrate how to construct some large new spaces of block-symmetric strong linearizations of a matrix polynomial $P(\lambda)$ as in (1.1). The complete formal construction of all the spaces that we introduce in this paper is rather technical, as well as the proof that they mostly contain strong linearizations, and is deferred to Section 7. In Section 3, we present some definitions and results introduced in previous papers relative to tuples of integers. The elementary matrices used throughout the paper to construct different classes of pencils associated with $P(\lambda)$ are considered in Section 4 and, based on them, we introduce in Section 5 the family of GFPR and give conditions for a pencil in this family to be a strong linearization of $P(\lambda)$. In Section 6, we study the index tuples that define a block-symmetric GFPR and give a canonical form for these tuples, which allows us to obtain a canonical expression for each of the block-symmetric GFPR. In this section, we also define the extended $\mathbb{DL}(P)$ space, $\mathbb{EDL}(P)$, as the subspace generated by the block-symmetric GFPR and show that $\mathbb{DL}(P)$ is one of its subspaces. In Section 7, we introduce the key spaces in this paper, the so-called Single-$\mathbb{EDL}(P)$ subspaces, give a basis for each of them, and show that their elements are almost all strong linearizations of $P(\lambda)$ when $P(\lambda)$ is regular (satisfying some nonsingularity conditions) or when $P(\lambda)$ is singular and has odd degree. We also provide in Section 7 some concrete examples of Single-$\mathbb{EDL}(P)$ subspaces. Finally, in Section 8, the conclusions and some lines of future research are summarized.

2. Large spaces of block-symmetric strong linearizations obtained from pencils in the standard basis of $\mathbb{DL}(P)$. In this section we show how to construct some special cases of the new large dimensional vector spaces of block-symmetric pencils associated with a matrix polynomial $P(\lambda)$ as in (1.1) introduced in this work. Moreover, we present *without proofs* sufficient conditions under which pencils in these subspaces are strong linearizations of $P(\lambda)$. These conditions clearly indicate that almost all pencils in these subspaces are strong linearizations of $P(\lambda)$ when the field $\mathbb{F}$ is infinite. The main purpose of this section is to convince the reader that, once a given block-symmetric Fiedler pencil with repetition (FPR) [4, 32] is given, to construct the new space of potential strong linearizations based on it is very simple, as well as it is very simple to state sufficient conditions under which the pencils in this space are strong linearizations of $P(\lambda)$. However, we will see in subsequent sections that, both, proving rigorously that these conditions are indeed sufficient and describing all the families of new spaces of block-symmetric linearizations require many technical details that may hide this simplicity. These technicalities are mainly a consequence of the fact that block-symmetric FPR are naturally defined as long products of certain elementary matrices [4, 32], instead of being defined in terms of its entries. Before starting, we warn the reader that although this section includes several forward references to results in next sections, these references should be skipped in a first reading, since they are not needed for understanding this section and, in some sense, they are envisioned as a guide to some of the most important results proved in this work.

The new subspaces of block-symmetric pencils associated with $P(\lambda)$ in (1.1) that we introduce are presented in Definition 7.1 and are called *Single-*$\mathbb{EDL}(P)$ *subspaces*. We define one such subspace for each block-symmetric FPR introduced in [4] (the concepts of FPR and block-symmetric FPR are refreshed in Definition 5.1 and in the second paragraph in Section 6, respectively). In this section, we define in a very simple way some Single-$\mathbb{EDL}(P)$ subspaces of large dimension. More precisely, we consider here the Single-$\mathbb{EDL}(P)$ subspaces associated with those block-symmetric FPR in the standard basis for the space of block-symmetric pencils $\mathbb{DL}(P)$ [17, 22]. The pencils in this basis were proved to be block-symmetric FPR recently in [32, Corollary 2], but they are very well-known since long time ago [20] and have the fundamental advantage over other block-symmetric FPR that their descriptions in terms of block entries is extremely simple [17, Section 3.3].

Let us describe the standard basis of $\mathbb{DL}(P)$. It was shown in [17] that if $P(\lambda)$ is a matrix polynomial of degree k and size $n \pm n$ as in (1.1), then for every $L(\lambda) \ / \ \mathbb{DL}(P)$ there exists a unique vector $v \ / \ \mathbb{F}^k$ such that

$$L(\lambda) \times (\Lambda \circ \ I_n) = v \circ \ P(\lambda) \quad \text{and} \quad (\Lambda^T \circ \ I_n) \times L(\lambda) = v^T \circ \ P(\lambda),$$

where $\Lambda = [\lambda^{k-1} \ \lambda^{k-2} \cdots \lambda \ 1]^T$ and $\otimes$ denotes the Kronecker product. This vector v is called the *ansatz vector* of $L(\lambda)$. As a consequence of this result $\mathbb{DL}(P)$ is isomorphic to $\mathbb{F}^k$ and, therefore, $\mathbb{DL}(P)$ has dimension k. In [17], the so-called *standard basis for* $\mathbb{DL}(P)$ was obtained by considering the pencils whose ansatz vector is one of the vectors in the standard basis for $\mathbb{F}^k$. The description of the pencils in the standard basis for $\mathbb{DL}(P)$ in terms of the coefficients of $P(\lambda)$ can be obtained as follows [17, Theorem 3.5]: for $j = 1 : k$, define

$$L_j(P) := \begin{bmatrix} 0 & \cdots & \cdots & A_k \\ \vdots & & \iddots & A_{k-1} \\ \vdots & \iddots & \iddots & \vdots \\ A_k & A_{k-1} & \dots & A_{k-j+1} \end{bmatrix} \quad \text{and} \quad U_j(P) := \begin{bmatrix} A_{j-1} & \dots & A_1 & A_0 \\ \vdots & \iddots & \iddots & \vdots \\ A_1 & \iddots & & \vdots \\ A_0 & \cdots & \cdots & 0 \end{bmatrix}$$

and the $k \times k$ block-symmetric matrix

$$X_m := \begin{bmatrix} L_m(P) & 0 \\ 0 & U_{k-m}(P) \end{bmatrix}, \quad \text{for } m = 0 : k,$$

where the matrices $L_0(P)$ and $U_0(P)$ are taken to be void. Then, the standard basis for $\mathbb{DL}(P)$ is $\{D_1(\lambda, P), \dots, D_k(\lambda, P)\}$ with

$$D_m(\lambda, P) = \lambda X_m - X_{m-1}, \quad \text{for } m = 1 : k. \tag{2.1}$$

Each of the Single-$\mathbb{EDL}(P)$ subspaces discussed in this section is obtained from a pencil $D_m(\lambda, P)$ as follows:

- Multiply the pencil $D_m(\lambda, P)$ by an arbitrary scalar $\alpha \in \mathbb{F}$.
- Replace some (not arbitrary) blocks of the form $\pm\alpha A_i$ in αX_m and αX_{m-1} by arbitrary $n \times n$ matrices.

We illustrate in Examples 2.1 and 2.2 which blocks are replaced by arbitrary matrices in two concrete cases and, after that, we provide the general replacement rule.

EXAMPLE 2.1. *Let $P(\lambda)$ be the matrix polynomial in* (1.1) *with $k = 5$ and consider the pencil $D_1(\lambda, P)$ in* (2.1)*, that is,*

$$\lambda \left[\begin{array}{c|cccc} A_5 & 0 & 0 & 0 & 0 \\ \hline 0 & A_3 & A_2 & A_1 & A_0 \\ 0 & A_2 & A_1 & A_0 & 0 \\ 0 & A_1 & A_0 & 0 & 0 \\ 0 & A_0 & 0 & 0 & 0 \end{array}\right] - \begin{bmatrix} A_4 & A_3 & A_2 & A_1 & A_0 \\ A_3 & A_2 & A_1 & A_0 & 0 \\ A_2 & A_1 & A_0 & 0 & 0 \\ A_1 & A_0 & 0 & 0 & 0 \\ A_0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and, based on it, define the block-symmetric pencils of the form

$$\lambda \left[\begin{array}{c|cccc} \alpha A_5 & 0 & 0 & 0 & 0 \\ \hline 0 & \alpha A_3 & B_{2,1} & B_{3,1} & B_{4,1} \\ 0 & B_{2,1} & \alpha A_1 & B_{3,2} & 0 \\ 0 & B_{3,1} & B_{3,2} & 0 & 0 \\ 0 & B_{4,1} & 0 & 0 & 0 \end{array}\right] - \begin{bmatrix} \alpha A_4 & \alpha A_3 & B_{2,1} & B_{3,1} & B_{4,1} \\ \alpha A_3 & \alpha A_2 & \alpha A_1 & B_{3,2} & 0 \\ B_{2,1} & \alpha A_1 & \alpha A_0 & 0 & 0 \\ B_{3,1} & B_{3,2} & 0 & 0 & 0 \\ B_{4,1} & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\alpha \in \mathbb{F}$ is an arbitrary scalar and $B_{2,1}$, $B_{3,1}$, $B_{3,2}$, and $B_{4,1}$ are arbitrary $n \times n$ matrices. It is immediate to see that this family of pencils forms a vector space of dimension $4n^2 + 1$ over $\mathbb{F}$. This family is an instance of a Single-$\mathbb{EDL}(P)$ subspace that we call $F_1(P)$. A basis for $F_1(P)$ is obtained by taking $\alpha = 1$ and considering the pencil in which all the blocks $B_{i,j}$ are 0, as well as all the pencils in which 3 of the blocks B_{ij} are 0 and the *remaining one has exactly one entry equal to* 1 *and all the other entries equal to* 0*. It can be shown that any pencil in* $F_1(P)$ *such that* $\alpha \neq 0$ *and* $B_{4,1}$ *and* $B_{3,2}$ *are nonsingular is a strong linearization of* $P(\lambda)$*. Observe that this implies that even when* $P(\lambda)$ *is singular almost all pencils in* $F_1(P)$ *are strong linearizations of* $P(\lambda)$*.*

EXAMPLE 2.2. *In our second example, we consider* $P(\lambda)$ *in* (1.1) *with degree* $k = 7$ *and start the construction from the pencil* $D_4(\lambda, P) = \lambda X_4 - X_3$ *in* (2.1)*:*

$$X_4 = \left[\begin{array}{cccc|ccc} 0 & 0 & 0 & A_7 & 0 & 0 & 0 \\ 0 & 0 & A_7 & A_6 & 0 & 0 & 0 \\ 0 & A_7 & A_6 & A_5 & 0 & 0 & 0 \\ A_7 & A_6 & A_5 & A_4 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & A_2 & A_1 & A_0 \\ 0 & 0 & 0 & 0 & A_1 & A_0 & 0 \\ 0 & 0 & 0 & 0 & A_0 & 0 & 0 \end{array}\right]$$

$$X_3 = \left[\begin{array}{ccc|cccc} 0 & 0 & A_7 & 0 & 0 & 0 & 0 \\ 0 & A_7 & A_6 & 0 & 0 & 0 & 0 \\ A_7 & A_6 & A_5 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & A_3 & A_2 & A_1 & A_0 \\ 0 & 0 & 0 & A_2 & A_1 & A_0 & 0 \\ 0 & 0 & 0 & A_1 & A_0 & 0 & 0 \\ 0 & 0 & 0 & A_0 & 0 & 0 & 0 \end{array}\right]$$

Based on $D_4(\lambda, P)$ *define the family of block-symmetric pencils of the form* $\lambda \widetilde{X}_4 - \widetilde{X}_3$ *with*

$$\widetilde{X}_4 = \left[\begin{array}{cccc|ccc} 0 & 0 & 0 & C_{1,3} & 0 & 0 & 0 \\ 0 & 0 & \alpha A_7 & C_{2,3} & 0 & 0 & 0 \\ 0 & \alpha A_7 & \alpha A_6 & \alpha A_5 & 0 & 0 & 0 \\ C_{1,3} & C_{2,3} & \alpha A_5 & \alpha A_4 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \alpha A_2 & B_{2,1} & B_{3,1} \\ 0 & 0 & 0 & 0 & B_{2,1} & \alpha A_0 & 0 \\ 0 & 0 & 0 & 0 & B_{3,1} & 0 & 0 \end{array}\right]$$

$$\widetilde{X}_3 = \left[\begin{array}{ccc|cccc} 0 & 0 & C_{1,3} & 0 & 0 & 0 & 0 \\ 0 & \alpha A_7 & C_{2,3} & 0 & 0 & 0 & 0 \\ C_{1,3} & C_{2,3} & \alpha A_5 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \alpha A_3 & \alpha A_2 & B_{2,1} & B_{3,1} \\ 0 & 0 & 0 & \alpha A_2 & \alpha A_1 & \alpha A_0 & 0 \\ 0 & 0 & 0 & B_{2,1} & \alpha A_0 & 0 & 0 \\ 0 & 0 & 0 & B_{3,1} & 0 & 0 & 0 \end{array}\right]$$

where $\alpha \in \mathbb{F}$ *is an arbitrary scalar and* $B_{2,1}$*,* $B_{3,1}$*,* $C_{1,3}$*, and* $C_{2,3}$ *are arbitrary* $n \times n$ *matrices. Again, it is immediate to see that this family of pencils forms a vector space of dimension* $4n^2 + 1$ *over* $\mathbb{F}$*. It is also a particular case of a Single-*$\mathbb{EDL}(P)$ *subspace that we call* $F_4(P)$*. A basis for* $F_4(P)$ *is obtained by taking* $\alpha = 1$ *and considering the pencil in which all the blocks* $B_{i,j}$ *and* $C_{p,q}$ *are* 0*, as well as all the pencils in which* 3 *of the blocks* B_{ij} *and* $C_{p,q}$ *are* 0 *and the remaining one has exactly one entry equal to* 1 *and all the other entries equal to* 0*.* If A_0 and A_7 are nonsingular*, then it can be shown that any pencil in* $F_4(P)$ *such that* $\alpha \neq 0$ *and* $B_{3,1}$ *and* $C_{1,3}$ *are nonsingular is a strong linearization of* $P(\lambda)$*.*

In contrast with the situation in Example 2.1, we cannot ensure that a pencil in $F_4(P)$ for $k = 7$ is a strong linearization of $P(\lambda)$ if $P(\lambda)$ is singular.

The replacement rule that we have used in Examples 2.1 and 2.2 to construct the spaces $F_m(P)$ starting from $\alpha\, D_m(\lambda, P) = \lambda\,(\alpha X_m) \quad (\alpha X_{m-1})$ is very simple and is as follows:

(a) Replace in αX_m all non-identically zero off-diagonal blocks in $\quad \alpha\, U_{k-m}(P)$ by arbitrary $n \pm n$ matrices preserving block-symmetry;

(b) The arbitrary matrices introduced in step (a) located below (resp. above) the block-diagonal of $\quad \alpha\, U_{k-m}(P)$ replace the blocks located one column leftwards (resp. one row upwards) in $\alpha\, X_{m-1}$;

(c) Replace in αX_{m-1} all non-identically zero off-diagonal blocks in $\alpha\, L_{m-1}(P)$ by arbitrary $n \pm n$ matrices preserving block-symmetry;

(d) The arbitrary matrices introduced in step (c) located below (resp. above) the block-diagonal of $\alpha\, L_{m-1}(P)$ replace the blocks located one row downwards (resp. one column rightwards) in $\alpha\, X_m$.

This replacement rule is valid for defining the Single-$\mathbb{EDL}(P)$ subspace $F_m(P)$ associated with the FPR $D_m(\lambda, P)$ for any polynomial $P(\lambda)$ of arbitrary degree $k \sim 3$ and for any value of $m\ /\ \}1, \ldots, k|$. This rule can be stated as the formal Definition 2.1 of the spaces of block-symmetric pencils $F_m(P)$ in terms of block indices.

DEFINITION 2.1. *Let $P(\lambda)$ be an $n \pm n$ matrix polynomial of degree k as in* (1.1)*. Let $D_m(\lambda, P)$, $m = 1 : k$, be the m*th *pencil in the standard basis of $\mathbb{DL}(P)$ defined in* (2.1)*. The family of block-symmetric pencils $F_m(P)$ is the set of those pencils obtained from $D_m(\lambda, P)$ as follows: first, for any $\alpha\ /\ \mathbb{F}$ get $\alpha D_m(\lambda, P) = \lambda(\alpha X_m) \quad (\alpha X_{m-1})$ and then*

≤ replace the blocks in positions $(m + i, m + j)$ and $(m + j, m + i)$ in αX_m, and the blocks in positions $(m + i, m + j \quad 1)$ and $(m + j \quad 1, m + i)$ in αX_{m-1}, by an arbitrary $n \pm n$ matrix $B_{i,j}$, for $i = 2 : k \quad m$ and $j = 1 : min\}i \quad 1, k \quad m \quad i+1|$; and

≤ replace the blocks in positions (i, j) and (j, i) in αX_{m-1} and the blocks in positions $(i, j+1)$ and $(j+1, i)$ in αX_m, by an arbitrary $n \pm n$ matrix $C_{i,j}$, for $i = 1 : m \quad 2$ and $j = \max\}m \quad i, i + 1| \ : m \quad 1$.

Let us go back to our guiding Examples 2.1 and 2.2 and observe that the coefficients of the pencils in $F_m(P)$ have the same block diagonal structures as X_m and X_{m-1}. These structures allow us to describe very easily sufficient conditions under which the pencils in $F_m(P)$ are strong linearizations of $P(\lambda)$. For this purpose, we only need to consider the block-antidiagonals of the diagonal blocks of the coefficients of the pencils in $F_m(P)$. Then the mentioned sufficient conditions are that the arbitrary matrices in these block-antidiagonals are nonsingular and that if αA_k or $\quad \alpha A_0$ appears more than once in the same block-antidiagonal, then A_k or A_0 must be nonsingular. These sufficient conditions are stated rigorously in Theorem 2.2, together with other basic properties of the family of pencils $F_m(P)$. The complete proof of this theorem will be presented in Subsection 7.2 taking into account the representation of generalized Fiedler pencils with repetition in terms of products of certain elementary matrices. However, we observe that the fact that $F_m(P)$ is a vector subspace is a straightforward consequence of Definition 2.1 and that the value of the dimension of $F_m(P)$ follows from counting the number of arbitrary matrices mentioned in Definition 2.1.

THEOREM 2.2. *Let $P(\lambda)$ be an $n \pm n$ matrix polynomial of degree k as in (1.1) and let $F_m(P)$ be the family of block-symmetric pencils introduced in Definition 2.1. Then, $F_m(P)$ is a vector space over $\mathbb{F}$ of dimension*

$$\Big)\Big\rfloor \frac{(k \quad m)^2}{4} \Big\lceil + \Big\rfloor \frac{(m \quad 1)^2}{4} \Big\lceil\Big\lceil\, n^2 + 1.$$

Moreover, if the notation in Definition 2.1 is used, then a pencil $L(\lambda)$ in $F_m(P)$ is a strong linearization of $P(\lambda)$ when the following conditions hold:

(a) $\alpha \neq 0$*;*
(b) *the matrices* $B_{i,k-m-i+1}$*, for* $i = \lfloor \frac{k-m}{2} \rfloor + 1 : k-m$*, and* $C_{i,m-i}$*, for* $i = 1 : \lfloor \frac{m-1}{2} \rfloor$*, are nonsingular matrices;*
(c) A_0 *is nonsingular if* $k-m$ *is odd;*
(d) A_k *is nonsingular if* m *is even.*

A basis for the space $F_m(P)$ can be obtained as described in Examples 2.1 and 2.2. A general procedure for obtaining a basis of any Single-$\mathbb{EDL}(P)$ subspace will be presented in Theorem 7.2. Observe that, for each m, the intersection of the spaces $F_m(P)$ and $\mathbb{DL}(P)$ is the vector space of dimension 1 generated by $D_m(\lambda, P)$.

Concerning conditions (c) and (d) in Theorem 2.2 note that, in each pencil in $F_m(P)$ corresponding to some $\alpha \neq 0$, we have at least one block equal to $-\alpha A_0$ and one block equal to αA_k in the matrix coefficient of the term of degree 0 and 1, respectively. The block $-\alpha A_0$ or αA_k appears more than once in the respective matrix coefficient when $k-m$ is odd or m is even, respectively, and if it is singular, then Theorem 2.2 does not ensure that the corresponding pencil is a strong linearization of $P(\lambda)$. In particular, if $P(\lambda)$ is a singular matrix polynomial, for Theorem 2.2 to ensure that $F_m(P)$ contains strong linearizations of $P(\lambda)$ we need that m and the degree of $P(\lambda)$ are odd as, in this case, the blocks $-A_0$ and A_k are both singular. A simple consequence of the general Theorem 5.5 is that, if $P(\lambda)$ is regular, then the sufficient conditions (b), (c), and (d) given in Theorem 2.2 for a pencil in $F_m(P)$ with $\alpha \neq 0$ to be a strong linearization of $P(\lambda)$ are also necessary.

REMARK 2.1. *From Definition 2.1 (see also Examples 2.1 and 2.2), it is clear that if* $P(\lambda)$ *is symmetric the pencils in* $F_m(P)$ *are symmetric if and only if all the arbitrary* $n \times n$ *matrices* $B_{i,j}$ *and* $C_{i,j}$ *are symmetric. This extra condition defines a subspace Sym*$(F_m(P))$ *of* $F_m(P)$ *whose dimension is*

$$\left(\left\lfloor \frac{(k-m)^2}{4} \right\rfloor + \left\lfloor \frac{(m-1)^2}{4} \right\rfloor \right) \left(\frac{n^2+n}{2} \right) + 1.$$

A procedure in terms of block-entries similar to the one given in this section could be applied to produce a space of pencils such that almost all pencils are strong linearizations of a matrix polynomial $P(\lambda)$ as in (1.1) starting with any of the block-symmetric FPR associated with $P(\lambda)$ introduced in [4], not necessarily with a pencil in the standard basis of $\mathbb{DL}(P)$. However, for those Single-$\mathbb{EDL}(P)$ subspaces obtained from an FPR not in the standard basis of $\mathbb{DL}(P)$, a description in terms of products of elementary matrices is more convenient and natural, since the Fiedler pencils and the FPR are originally described in this way (as opposed to $\mathbb{DL}(P)$). Therefore, a block-entry description of general Single-$\mathbb{EDL}(P)$ subspaces is complicated and will not be pursued in this paper. Instead, we present a unified description of all the cases in terms of products of elementary matrices. This description requires the use of some tools that will be introduced in the next sections.

3. Equivalence of index tuples. In this section we include some definitions and results on index tuples that will be crucial in the paper, in particular, in the construction of the Fiedler pencils with repetition (FPR) and the generalized Fiedler pencils with repetition (GFPR).

We will use the following notation. If a and b are two integers, we denote

$$a : b := \begin{cases} a, a+1, \ldots, b, & \text{if } a \leq b, \\ \emptyset & \text{if } a > b. \end{cases}$$

DEFINITION 3.1. *We call an index tuple to a finite ordered sequence of integer numbers.*

Each of these integers is called an index. The number of indices in an index tuple $\mathbf{t}$ *is called its length and is denoted by* $|\mathbf{t}|$.

Given an index tuple $\mathbf{t}$ and an integer a, we denote by $a+\mathbf{t}$ the index tuple obtained from $\mathbf{t}$ by adding a to each of its indices. Given index tuples $\mathbf{t}_1,\ldots,\mathbf{t}_s$, we denote by $(\mathbf{t}_1,\ldots,\mathbf{t}_s)$ the index tuple obtained by concatenating the indices in the index tuples $\mathbf{t}_1,\ldots,\mathbf{t}_s$ in the indicated order.

DEFINITION 3.2. *We say that two nonnegative indices* i,j *in an index tuple* commute *if* $|i-j|\neq 1$.

DEFINITION 3.3. *Let* $\mathbf{t}$ *and* $\mathbf{t}'$ *be two index tuples of nonnegative integers. We say that* $\mathbf{t}'$ *is obtained from* $\mathbf{t}$ *by a* transposition *if* $\mathbf{t}'$ *is obtained from* $\mathbf{t}$ *by interchanging two distinct commuting indices in adjacent positions. If* i *and* $i+1$ *are the positions of the interchanged indices in* $\mathbf{t}$, *we call* the transposition *that produces* $\mathbf{t}'$ *from* $\mathbf{t}$ *to the permutation of* $\{1:|\mathbf{t}|\}$ *obtained by interchanging* i *and* $i+1$ *in the identity permutation* $(1:|\mathbf{t}|)$.

We will use the standard notation in abstract algebra for transpositions [16]. That is, by $(i,i+1)$ we denote the permutation of $\{1:|\mathbf{t}|\}$ obtained by interchanging i and $i+1$ in $(1:|\mathbf{t}|)$.

DEFINITION 3.4. *Given two index tuples* $\mathbf{t}$ *and* $\mathbf{t}'$ *of nonnegative integers, we say that* $\mathbf{t}$ *is* equivalent *to* $\mathbf{t}'$ *if* $\mathbf{t}=\mathbf{t}'$ *or if* $\mathbf{t}'$ *can be obtained from* $\mathbf{t}$ *by a sequence of transpositions. If* $\mathbf{t}$ *and* $\mathbf{t}'$ *are index tuples of negative integers and* a *is the minimum index among the indices in* $\mathbf{t}$ *and* $\mathbf{t}'$, *we say that* $\mathbf{t}$ *is equivalent to* $\mathbf{t}'$ *if* $-a+\mathbf{t}$ *is equivalent to* $-a+\mathbf{t}'$. *If* $\mathbf{t}$ *and* $\mathbf{t}'$ *are equivalent index tuples, we write* $\mathbf{t}\sim\mathbf{t}'$.

Note that the relation $\sim$ is an equivalence relation.

We observe that, if $\mathbf{t}=(i_1,\ldots,i_r)$ and $\mathbf{t}'$ are two equivalent index tuples of nonnegative (resp. negative) integers, and $\mathbf{t}'$ (resp. $-a+\mathbf{t}'$, where a is the minimum index in $\mathbf{t}$) can be obtained from $\mathbf{t}$ (resp. from $-a+\mathbf{t}$) by an ordered sequence of transpositions $\sigma_1,\ldots,\sigma_s$, then $\mathbf{t}'=(i_{\sigma(1)},\ldots,i_{\sigma(r)})$, where $\sigma=\sigma_1\circ\cdots\circ\sigma_s$. Since no transposition σ_j corresponds to positions where the same index lies, it follows that, though such sequence of transpositions is not unique in general, the composition $\sigma=\sigma_1\circ\cdots\circ\sigma_s$ of any such sequence of transpositions is unique. We then have the following definition.

DEFINITION 3.5. *Let* $\mathbf{t}$ *and* $\mathbf{t}'$ *be two equivalent index tuples of nonnegative (resp. negative) integers. Suppose that* $\mathbf{t}'$ *(resp.* $-a+\mathbf{t}'$, *where* a *is the minimum index in* $\mathbf{t}$*) is obtained from* $\mathbf{t}$ *(resp. from* $-a+\mathbf{t}$*) by an ordered sequence of transpositions* $\sigma_1,\ldots,\sigma_s$. *Then we say that the composition* $\sigma=\sigma_1\circ\cdots\circ\sigma_s$ *is the* allowed permutation that transforms $\mathbf{t}$ into $\mathbf{t}'$.

Observe that, if $\mathbf{t}'$ and $\mathbf{t}=(i_1,\ldots,i_r)$ are equivalent we may have $\mathbf{t}'=(i_{\tau(1)},\ldots,i_{\tau(r)})$, where τ is not the allowed permutation that transforms $\mathbf{t}$ into $\mathbf{t}'$.

EXAMPLE 3.1. *Let* $\mathbf{t}=(1:3,0:1,0)=(i_1,i_2,i_3,i_4,i_5,i_6)$. *The tuple* $\mathbf{t}$ *is equivalent to the tuple* $\mathbf{t}'=(1,0,2,1,0,3)$ *which is obtained from* $\mathbf{t}$ *by the following sequence of transpositions*

$$\begin{aligned}\mathbf{t}=(i_1,i_2,i_3,i_4,i_5,i_6)&\sim(i_1,i_2,i_4,i_3,i_5,i_6)\sim(i_1,i_4,i_2,i_3,i_5,i_6)\\ &\sim(i_1,i_4,i_2,i_5,i_3,i_6)\sim(i_1,i_4,i_2,i_5,i_6,i_3)=\mathbf{t}'.\end{aligned}$$

The transpositions associated with the previous interchanges of indices, given in the same order in which they were implemented, are: $(3,4),(2,3),(4,5),(5,6)$. *Therefore,* $\sigma=(3,4)\circ(2,3)\circ(4,5)\circ(5,6)=(1,4,2,5,6,3)$ *is the allowed permutation that transforms* $\mathbf{t}$ *into* $\mathbf{t}'$.

Note that we also have $\mathbf{t}'=(i_5,i_4,i_2,i_1,i_6,i_3)$. *However the permutation* $\tau=(5,4,2,1,6,3)$ *is not the allowed permutation that transforms* $\mathbf{t}$ *into* $\mathbf{t}'$.

In the construction of the FPR and GFPR we will require some index tuples satisfying the following property.

DEFINITION 3.6. *[32, Definition 7] Let* $\mathbf{t} = (i_1, i_2, \ldots, i_r)$ *be an index tuple of either nonnegative integers or negative integers. Then,* $\mathbf{t}$ *is said to satisfy the* Successor Infix Property (SIP) *if for every pair of indices* i_a, i_b / $\mathbf{t}$*, with* $1 \geq a < b \geq r$*, satisfying* $i_a = i_b$*, there exists at least one index* $i_c = i_a + 1$ *with* $a < c < b$*.*

Note that any subtuple of consecutive indices of a tuple satisfying the SIP also satisfies the SIP. Also, the SIP is invariant under equivalence.

Next we give a canonical form for tuples satisfying the SIP under the equivalence relation given in Definition 3.4.

DEFINITION 3.7. *[32, Theorem 1] Let* $\mathbf{t}$ *be an index tuple with indices from* $\}0 : h|$*,* $h \sim 0$*. Then* $\mathbf{t}$ *is said to be in* column standard form*, or, analogously,* $\mathbf{t}$ *is said to be a* csf-tuple*, if*

$$\mathbf{t} = (a_s : b_s, a_{s-1} : b_{s-1}, \ldots, a_2 : b_2, a_1 : b_1),$$

with $h \sim b_s > b_{s-1} > \times\times\times > b_2 > b_1 \sim 0$ *and* $0 \geq a_j \geq b_j$*, for all* $j = 1 : s$*. We call each subtuple of consecutive indices* $(a_i : b_i)$ *a* string *in* $\mathbf{t}$*.*

EXAMPLE 3.2. *The tuples* $\mathbf{t}_1 = (3 : 4, 0 : 2)$ *and* $\mathbf{t}_2 = (1 : 4, 0 : 3, 0 : 2, 1, 0)$ *are csf-tuples with indices from* $\}0 : 4|$*.*

The connection between the column standard form of an index tuple and the SIP is shown in the following result.

LEMMA 3.8. *[32, Theorem 2] Let* $\mathbf{t}$ *be an index tuple.*

≤ *If the indices of* $\mathbf{t}$ *are all nonnegative integers, then* $\mathbf{t}$ *satisfies the SIP if and only if* $\mathbf{t}$ *is equivalent to a tuple in column standard form.*

≤ *If the indices of* $\mathbf{t}$ *are all negative integers and* a *is the minimum index in* $\mathbf{t}$*, then* $\mathbf{t}$ *satisfies the SIP if and only if* $a + \mathbf{t}$ *is equivalent to a tuple in column standard form.*

Taking into account Proposition 2.12 in [4], it follows that two tuples in column standard form are equivalent if and only if they coincide. We then have the following definition.

DEFINITION 3.9. *The unique index tuple in column standard form equivalent to an index tuple* $\mathbf{t}$ *of nonnegative integers satisfying the SIP is called the* column standard form of $\mathbf{t}$ *and is denoted by* $csf(\mathbf{t})$*.*

Note that, in particular, if $\mathbf{t}$ is simple (that is, $\mathbf{t}$ has no repeated indices), then $\mathbf{t}$ satisfies the SIP and, therefore, $\mathbf{t}$ is equivalent to a unique tuple in column standard form.

In the construction of our linearizations the symmetric tuples will play a crucial role.

DEFINITION 3.10. *An index tuple* $\mathbf{t} = (t_1, \ldots, t_r)$ *is symmetric if* $rev(\mathbf{t}) \subset \mathbf{t}$*, where* $rev(\mathbf{t}) := (t_r, \ldots, t_1)$*.*

4. Elementary matrices and matrix assignments. We start this section by defining some matrices, denoted by $M_i(B)$, that will be used in the definition of the Fiedler pencils with repetition (FPR) and the generalized Fiedler pencils with repetition (GFPR) presented in the next section. Note that these matrices, or a permutation similarity of them, were considered in the literature before [2, 3, 4], but associated with a matrix polynomial $P(\lambda)$, i.e., with the block B being a matrix coefficient of $P(\lambda)$ or obtained from a matrix coefficient of $P(\lambda)$ by inversion or/and multiplication by 1.

Let $k \sim 2$ be an integer and B an arbitrary $n \pm n$ matrix. Then we define the following elementary block-matrices partitioned into $k \pm k$ blocks of size $n \pm n$:

$$M_0(B) := \left]\begin{array}{c|c} I_{(k-1)n} & 0 \\ \hline 0 & B \end{array}\right\lfloor, \quad M_{-k}(B) := \left]\begin{array}{c|c} B & 0 \\ \hline 0 & I_{(k-1)n} \end{array}\right\lfloor,$$

$$M_i(B) := \left[\begin{array}{c|cc|c} I_{(k-i-1)n} & 0 & 0 & 0 \\ \hline 0 & B & I_n & 0 \\ 0 & I_n & 0 & 0 \\ \hline 0 & 0 & 0 & I_{(i-1)n} \end{array}\right], \quad i = 1 : k-1, \tag{4.1}$$

and

$$M_{-i}(B) := \left[\begin{array}{c|cc|c} I_{(k-i-1)n} & 0 & 0 & 0 \\ \hline 0 & 0 & I_n & 0 \\ 0 & I_n & B & 0 \\ \hline 0 & 0 & 0 & I_{(i-1)n} \end{array}\right], \quad i = 1 : k-1.$$

Note that, for $i = 1 : k-1$, $M_i(B)$ and $M_{-i}(B)$ are invertible for any B. Moreover, $(M_i(B))^{-1} = M_{-i}(-B)$. On the other hand, the matrices $M_0(B)$ and $M_{-k}(B)$ are invertible if and only if B is invertible.

REMARK 4.1. *It is easy to check that the commutativity relations*

$$M_i(B_1)M_j(B_2) = M_j(B_2)M_i(B_1) \tag{4.2}$$

hold for any $n \times n$ matrices B_1 and B_2 if $||i| - |j|| \neq 1$ and $|i| \neq |j|$.

DEFINITION 4.1. *Let $\mathbf{t} = (i_1, i_2, \ldots, i_r)$ be an index tuple with indices from $\{-k : k-1\}$. Then, we call a* matrix assignment for $\mathbf{t}$ *to an ordered collection $X := (X_1, \ldots, X_r)$ of arbitrary $n \times n$ matrices. We say that the matrix X_j is assigned to the position j in $\mathbf{t}$. The matrix assignment X for $\mathbf{t}$ is said to be nonsingular if the matrices assigned by X to the positions in $\mathbf{t}$ occupied by the 0 and $-k$ indices are nonsingular.*

Note that, if 0 and $-k$ are not indices in $\mathbf{t}$, then any matrix assignment for $\mathbf{t}$ is nonsingular.

Given a permutation σ of $\{1 : r\}$ and a matrix assignment $X = (X_1, \ldots, X_r)$ for an index tuple $\mathbf{t}$, we denote by $\sigma(X)$ the matrix assignment for $\mathbf{t}$ given by $(X_{\sigma(1)}, \ldots, X_{\sigma(r)})$. In particular, we denote by $rev(X)$ the matrix assignment for $\mathbf{t}$ obtained from X by reversing the order of the matrices, that is, $rev(X) = \sigma(X)$, where $\sigma = (r, r-1, \ldots, 1)$.

DEFINITION 4.2. *Let $\mathbf{t} = (i_1, i_2, \ldots, i_r)$ be a symmetric index tuple with indices from either $\{-k : -1\}$ or $\{0 : k-1\}$, X be a matrix assignment for $\mathbf{t}$, and σ be the allowed permutation that transforms $\mathbf{t}$ into $rev(\mathbf{t})$. Then, X is said to be a symmetric matrix assignment for $\mathbf{t}$ if $rev(X) = \sigma(X)$.*

If $X = (X_1, \ldots, X_r)$ is a matrix assignment for a nonempty index tuple $\mathbf{t} = (i_1, \ldots, i_r)$ with indices from $\{-k : k-1\}$, we denote

$$M_{\mathbf{t}}(X) := M_{i_1}(X_1)M_{i_2}(X_2)\cdots M_{i_r}(X_r).$$

If $\mathbf{t}$ and X are empty, then $M_{\mathbf{t}}(X) := I_{kn}$.

Note that, in particular, if $X^1, \ldots X^s$ are matrix assignments for the index tuples $\mathbf{t}_1, \ldots, \mathbf{t}_s$, respectively, then

$$M_{\mathbf{t}_1,\ldots,\mathbf{t}_s}(X^1, \ldots, X^s) := M_{(\mathbf{t}_1,\ldots,\mathbf{t}_s)}(X^1, \ldots, X^s) = M_{\mathbf{t}_1}(X^1)\cdots M_{\mathbf{t}_s}(X^s).$$

If $P(\lambda)$ is an $n \times n$ matrix polynomial of degree k as in (1.1) and $i \in \{-k : k-1\}$, we consider the following abbreviated notation:

$$M_i^P := \begin{cases} M_i(-A_i), & \text{if } i \geq 0 \\ M_i(A_{-i}), & \text{if } i < 0 \end{cases}.$$

If $\mathbf{t} = (i_1, ..., i_r)$ is a nonempty index tuple, we denote $M_{\mathbf{t}}^P := M_{i_1}^P M_{i_2}^P \cdots M_{i_r}^P$. If $\mathbf{t}$ is the empty tuple, $M_{\mathbf{t}}^P := I_{kn}$. Note that, in particular, if $\mathbf{t}_1, \ldots, \mathbf{t}_s$ are index tuples, then $M_{\mathbf{t}_1,\ldots,\mathbf{t}_s}^P = M_{\mathbf{t}_1}^P \cdots M_{\mathbf{t}_s}^P$.

DEFINITION 4.3. *We say that $X = (X_1, \ldots, X_r)$ is the* trivial matrix assignment *for the index tuple $\mathbf{t} = (i_1, \ldots, i_r)$, with indices from $\{-k : -1\}$, associated with the matrix polynomial $P(\lambda)$ as in (1.1), if $M_{i_j}(X_j) = M_{i_j}^P$ for $j = 1 : r$.*

Note that, if X is the trivial matrix assignment for $\mathbf{t}$ associated with $P(\lambda)$, then $M_{\mathbf{t}}(X) = M_{\mathbf{t}}^P$. Moreover, if $\mathbf{t}$ is a symmetric index tuple, then the trivial matrix assignment for $\mathbf{t}$ associated with $P(\lambda)$ is a symmetric matrix assignment for $\mathbf{t}$ since it assigns equal matrices to positions occupied by equal indices.

The next lemma shows that, when computing $M_{\mathbf{t}}(X)$, the tuple $\mathbf{t}$ can be replaced by any equivalent tuple, as long as some appropriate reordering of the matrices in the matrix assignment X for $\mathbf{t}$ is done. This result generalizes one part of Lemma 4.3 in [4].

LEMMA 4.4. *Let $\mathbf{t}$ be an index tuple with indices from either $\{0 : k-1\}$ or $\{-k : -1\}$ and let X be a matrix assignment for $\mathbf{t}$. If $\mathbf{t}$ is equivalent to $\mathbf{t}'$ and σ is the allowed permutation that transforms $\mathbf{t}$ into $\mathbf{t}'$, then $M_{\mathbf{t}}(X) = M_{\mathbf{t}'}(\sigma(X))$.*

Proof. Assume $\mathbf{t} = (i_1, \ldots, i_r)$ is equivalent to $\mathbf{t}'$. Then, $\mathbf{t}'$ can be obtained from $\mathbf{t}$ by a sequence of interchanges of distinct indices which are not consecutive integers. By Remark 4.1, if the indices i_j and i_l, with $||i_j| - |i_l|| \neq 1$, are interchanged, then $M_{i_j}(X_j)$ and $M_{i_l}(X_l)$ commute, where X_j and X_l are the matrices assigned by X to the positions j and l in $\mathbf{t}$, respectively. Thus the result follows. □

The next example illustrates the previous lemma.

EXAMPLE 4.1. *Let $\mathbf{t} = (1 : 3, 0 : 1, 0) = (i_1, i_2, i_3, i_4, i_5, i_6)$ and $\mathbf{t}' = (1, 0, 2, 1, 0, 3)$ be the equivalent tuples given in Example 3.1. It was shown there that the permutation $\sigma = (1, 4, 2, 5, 6, 3)$ is the allowed permutation that transforms $\mathbf{t}$ into $\mathbf{t}'$. Let $X = (X_1, \ldots, X_6)$ be a matrix assignment for $\mathbf{t}$. Then, taking into account Lemma 4.4,*

$$
\begin{aligned}
M_{\mathbf{t}}(X) &= M_{(1,2,3,0,1,0)}(X_1, X_2, X_3, X_4, X_5, X_6) \\
&= M_{(i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}, i_{\sigma(4)}, i_{\sigma(5)}, i_{\sigma(6)})}(X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)}, X_{\sigma(4)}, X_{\sigma(5)}, X_{\sigma(6)}) \\
&= M_{(1,0,2,1,0,3)}(X_1, X_4, X_2, X_5, X_6, X_3) \\
&= M_{\mathbf{t}'}(\sigma(X)).
\end{aligned}
$$

As will be seen in the next section the matrix coefficients of a generalized Fiedler pencil with repetition associated with a matrix polynomial $P(\lambda)$ as in (1.1) can be viewed as block-matrices whose blocks are of the form 0_n, I_n, $\pm A_i$, or arbitrary $n \times n$ matrices. This fact motivates the following definition.

DEFINITION 4.5. *Let $\mathbf{t} = (i_1, \ldots, i_r)$ be a tuple with indices from either $\{0 : k-1\}$ or $\{-k : -1\}$. We say that $\mathbf{t}$ is operation free if, for any matrix assignment $X = (X_1, \ldots, X_r)$ for $\mathbf{t}$, the $n \times n$ block-entries of the product $M_{\mathbf{t}}(X)$ are either 0_n, I_n, or X_j, with $j \in \{1 : r\}$, and the positions where each of these blocks lies do not depend on the particular matrix assignment X, that is, those positions only depend on $\mathbf{t}$.*

THEOREM 4.6. *Let $\mathbf{t}$ be a tuple with indices from either $\{0 : k-1\}$ or $\{-k : -1\}$. The tuple $\mathbf{t}$ is operation free if and only if $\mathbf{t}$ satisfies the SIP.*

Proof. Suppose that $\mathbf{t} = (i_1, \ldots, i_r)$, as in the statement, is an operation free tuple. Then, for any matrix polynomial $P(\lambda)$ as in (1.1), the blocks of $M_{\mathbf{t}}(X)$ are 0_n, I_n, or $-A_{i_j}$ (resp. A_{i_j}), for $j = 1 : r$, if $\mathbf{t}$ has nonnegative (resp. negative) indices, when X is the trivial matrix assignment associated with $P(\lambda)$, that is, $M_{\mathbf{t}}^P$ is operation free according to [32, Definition 6]. By Theorem 2 in [32], if $\mathbf{t}$ has nonnegative indices, and by Theorem 3 in [32], if $\mathbf{t}$ has negative indices, it follows that $\mathbf{t}$ satisfies the SIP.

Now suppose that $\mathbf{t} = (i_1, \ldots, i_r)$ satisfies the SIP. We prove the result when $\mathbf{t}$ has nonnegative indices. By Theorem 2 in [32], for any matrix polynomial $P(\lambda)$ as in (1.1), all $n \pm n$ blocks of $M_{\mathbf{t}}^P$ are either 0_n, I_n or $\quad A_{i_j}$, for $j = 1 : r$. Since each block-entry of $M_{\mathbf{t}}^P$ can be viewed as a sum of products of r factors, each factor being an entry of each of the matrices M_{i_j} (which is either 0_n, I_n, or a block $\quad A_{i_j}$ for $j = 1 : r$), we conclude that no such nonzero summands cancel for arbitrary A_{i_j}. Therefore, the operation free property on $M_{\mathbf{t}}^P$ implies that at most one nonzero such summand may appear in each block-entry of $M_{\mathbf{t}}^P$ and it is equal to either I_n or $\quad A_{i_j}$. Thus, all the factors in this summand are I_n except, maybe, one factor of the form $\quad A_{i_j}$. Also, all the zero summands are associated with products having a factor 0. As a consequence, when we consider replacements of the matrices $\quad A_{i_j}$ by arbitrary matrices, more precisely, when we consider the assignment $X = (X_1, \ldots, X_r)$ for $\mathbf{t}$, we conclude that all the $n \pm n$ blocks of $M_{\mathbf{t}}(X)$ are identically either 0_n, I_n, or a block X_j, $j = 1 : r$, and the position of these blocks is independent of X. Thus, $\mathbf{t}$ is operation free. □

Taking into account [32, Theorem 2], a consequence of Theorem 4.6 is that an index tuple $\mathbf{t}$ is operation free if and only if $M_{\mathbf{t}}^P$ is operation free according to [32, Definition 6].

The following lemma provides an insight on the structure of products of elementary matrices associated with index tuples satisfying the SIP. Based on it, in Theorem 5.3 we describe the structure of the matrix coefficients of a GFPR.

LEMMA 4.7. *Let $\mathbf{t} = (i_1, \ldots, i_r)$ be a tuple satisfying the SIP with indices from either $\}0 : k \quad 1|$ or $\} \quad k : \quad 1|$ and let $X = (X_1, \ldots, X_r)$ be a generic matrix assignment for $\mathbf{t}$. Then, each block-entry of $M_{\mathbf{t}}(X)$ is either 0_n, I_n or X_j; for each X_j there is exactly one position in the block-matrix $M_{\mathbf{t}}(X)$ whose entry is identically equal to X_j; $M_{\mathbf{t}}(X)$ has no block-rows and no block-columns identically zero; and in each block-row and in each block-column of $M_{\mathbf{t}}(X)$ at most one block identically equal to I_n appears.*

Proof. We prove the result when $\mathbf{t}$ has nonnegative indices. The claim that each block-entry of $M_{\mathbf{t}}(X)$ is either 0_n, I_n, or X_j is an immediate consequence of Theorem 4.6.

The fact that $M_{\mathbf{t}}(X)$ has no block-rows and no block-columns identically zero follows from Theorem 4.6 and the nonsingularity of $M_{\mathbf{t}}(X)$ for nonsingular matrix assignments. Note that this fact was observed in [7, Section 2.2].

Now we prove the rest of the result by induction on r. If $r = 1$, by the definition of the elementary matrices $M_i(B)$ introduced at the beginning of this section, the block X_1 appears exactly once in $M_{i_1}(X_1)$ and there is at most one block I_n in each block-row and block-column of $M_{i_1}(X_1)$. Now suppose that $r > 1$ and the claim is true for $r \quad 1$. By Theorem 4.6, $\mathbf{t}' := (i_1, \ldots, i_{r-1})$ and $\mathbf{t}$ are operation free. Thus, by the inductive hypothesis, for $X' = (X_1, \ldots, X_{r-1})$, each X_j appears exactly once in the matrix $M_{\mathbf{t}'}(X')$, and at most one block I_n appears in each block-row and in each block-column of $M_{\mathbf{t}'}(X')$. If $i_r = 0$, this implies that the last block-column of $M_{\mathbf{t}'}(X')$ has no blocks X_j and, therefore, has exactly one block I_n, since there are no zero block-rows or block-columns. In this case, the claim follows. Now suppose that $i_r > 0$. We consider the product of the submatrix of $M_{\mathbf{t}'}(X')$ corresponding to the $k \quad i_r, k \quad i_r + 1$ block-columns, say $Q_{\mathbf{t}'}$, by the matrix

$$R_{i_r} := \begin{bmatrix} X_r & I_n \\ I_n & 0 \end{bmatrix}.$$

Observe that the remaining block-columns of $M_{\mathbf{t}'}(X')$ stay unchanged in $M_{\mathbf{t}}(X)$. Denote by $\begin{bmatrix} Y_1 & Y_2 \end{bmatrix}$, where Y_1, Y_2 are $n \pm n$ blocks, an arbitrary block-row of $Q_{\mathbf{t}'}$. We have

$$\begin{bmatrix} Y_1 & Y_2 \end{bmatrix} \begin{bmatrix} X_r & I_n \\ I_n & 0 \end{bmatrix} = \begin{bmatrix} Y_1 X_r + Y_2 & Y_1 \end{bmatrix}, \tag{4.3}$$

which implies that either $Y_1 = I_n$ and $Y_2 = 0$, or $Y_1 = 0$. Thus, we deduce that the $(k - i_r)$th block-column of $M_{\mathbf{t}'}(X')$ has no blocks X_j. Since $M_{\mathbf{t}'}(X')$ has no zero block-rows and no zero block-columns, by the inductive hypothesis, the $(k - i_r)$th block-column of $M_{\mathbf{t}'}(X')$ must have exactly one block I_n. Then, it follows that the $(k - i_r + 1)$th block-column of $M_{\mathbf{t}}(X)$ has a block I_n and all the other entries are 0.

Also, the $(k - i_r)$th block-column of $M_{\mathbf{t}}(X)$ has exactly one block X_r, among some other possible blocks X_j with $j \neq r$, and at most one block I_n.

The number of blocks equal to I_n in each block-row of $M_{\mathbf{t}}(X)$ is the same as in the corresponding block-row of $M_{\mathbf{t}'}(X')$, as the number of blocks equal to I_n in the second member of (4.3) is the same as in $[Y_1 \; Y_2]$. Thus, the claim follows. □

The next lemma, which generalizes Lemma 5 in [32], shows that the block-transpose of a product of elementary matrices associated with an operation-free index tuple behaves as the regular transpose.

LEMMA 4.8. *Let* $\mathbf{t}$ *be an index tuple satisfying the SIP with indices from either* $\{0 : k-1\}$ *or* $\{-k : -1\}$ *and let* X *be a matrix assignment for* $\mathbf{t}$. *Then,*

$$M_{\mathbf{t}}^{\mathcal{B}}(X) = M_{rev(\mathbf{t})}(rev(X)).$$

Proof. Let $\mathbf{t} = (i_1, \ldots, i_r)$ and $X = (X_{i_1}, \ldots, X_{i_r})$, and suppose that $\mathbf{t}$ has indices from $\{0 : k-1\}$. The proof is similar if $\mathbf{t}$ has indices from $\{-k : -1\}$.

We prove the result by induction on the length r of $\mathbf{t}$. The result is obvious for $r = 1$ since the elementary matrix $M_{i_1}(X_1)$ is block-symmetric. Now suppose that $r > 1$ and the claim is true for $r - 1$. Let $\mathbf{t}' = (i_1, \ldots, i_{r-1})$ and $X' = (X_1, \ldots, X_{r-1})$. By the inductive hypothesis,

$$M_{i_r}(X_r)M_{\mathbf{t}'}^{\mathcal{B}}(X') = M_{rev(\mathbf{t})}(rev(X)).$$

Let us view $M_{\mathbf{t}}(X)$ as a $k \times k$ block-matrix and let $p, q \in \{1 : k\}$. We show that the blocks in position (p, q) in $H_1 := M_{i_r}(X_r)M_{\mathbf{t}'}^{\mathcal{B}}(X')$ and in $H_2^{\mathcal{B}} = [M_{\mathbf{t}'}(X')M_{i_r}(X_r)]^{\mathcal{B}}$ are the same.

Assume that $p \neq k - i_r, k - i_r + 1$. Then, the block in position (p, q) in H_1 is the block in position (p, q) in $M_{\mathbf{t}'}^{\mathcal{B}}(X')$. On the other hand, the block in position (p, q) in $H_2^{\mathcal{B}}$ is the block in position (q, p) in H_2, which is the block in position (q, p) in $M_{\mathbf{t}'}(X')$ and the result follows.

Assume now that $p = k - i_r + 1$. Note that $i_r \neq 0$ in this case. Then, the block in position (p, q) in H_1 is the block in position $(k - i_r, q)$ in $M_{\mathbf{t}'}^{\mathcal{B}}(X')$. On the other hand, the block in position (p, q) in $H_2^{\mathcal{B}}$ is the block in position (q, p) in H_2, which is the block in position $(q, k - i_r)$ of $M_{\mathbf{t}'}(X')$ and the result follows.

Finally, let $p = k - i_r$. Suppose that $i_r \neq 0$. Let Y_1 and Y_2 be the blocks of $M_{\mathbf{t}'}^{\mathcal{B}}(X')$ in positions $(k - i_r, q)$ and $(k - i_r + 1, q)$, respectively. Then, the element in position (p, q) of H_1 is given by $X_rY_1 + Y_2$. Since the qth block-row of $M_{\mathbf{t}'}(X')$ is the qth block-column of its block-transpose, the element in position (p, q) in $H_2^{\mathcal{B}}$, which is the element in position (q, p) in H_2, is $Y_1X_r + Y_2$. Since an index tuple satisfying the SIP is operation free, Y_1 must be either 0 or I_n. Thus, $X_rY_1 + Y_2 = Y_1X_r + Y_2$ and the result follows. If $i_r = 0$, the proof can be done using similar arguments. □

The last lemma in this section gives conditions that ensure that a product of elementary matrices is block-symmetric.

LEMMA 4.9. *Let* $\mathbf{t}$ *be a symmetric index tuple satisfying the SIP with indices from either* $\{0 : k-1\}$ *or* $\{-k : -1\}$, *and let* X *be a symmetric matrix assignment for* $\mathbf{t}$. *Then,* $M_{\mathbf{t}}(X)$ *is block-symmetric.*

Proof. Let $\mathbf{t}$ and X be as in the statement. Since $\mathbf{t}$ is symmetric, $\mathbf{t}$ is equivalent to $rev(\mathbf{t})$. Let σ be the allowed permutation that transforms $\mathbf{t}$ into $rev(\mathbf{t})$. Then, $rev(X) = \sigma(X)$, which implies that

$$M_{\mathbf{t}}(X) = M_{rev(\mathbf{t})}(\sigma(X)) = M_{rev(\mathbf{t})}(rev(X)) = M^{\mathcal{B}}_{\mathbf{t}}(X),$$

where the first and last equalities follow from Lemmas 4.4 and 4.8, respectively. □

5. Generalized Fiedler pencils with repetition. Let $P(\lambda)$ be an $n \times n$ matrix polynomial of degree k as in (1.1). Next we introduce the family of Fiedler pencils with repetition associated with $P(\lambda)$, defined in [32]. The matrix coefficients of these pencils are products of elementary matrices $M_i(B)$, where the blocks B are coefficients of $P(\lambda)$.

DEFINITION 5.1. *(FPR) Let $P(\lambda)$ be a matrix polynomial of degree k as in (1.1). Let $h \in \{0 : k-1\}$. Let $\mathbf{q}$ and $\mathbf{z}$ be permutations of $\{0 : h\}$ and $\{-k : -h-1\}$, respectively. Let $\mathbf{l}_q$ and $\mathbf{r}_q$ be tuples with indices from $\{0 : h-1\}$ such that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ satisfies the SIP. Let $\mathbf{l}_z$ and $\mathbf{r}_z$ be tuples with indices from $\{-k : -h-2\}$ such that $(\mathbf{l}_z, \mathbf{z}, \mathbf{r}_z)$ satisfies the SIP. Then, the pencil*

$$M^P_{\mathbf{l}_q,\mathbf{l}_z}(\lambda M^P_{\mathbf{z}} - M^P_{\mathbf{q}})M^P_{\mathbf{r}_z,\mathbf{r}_q} \tag{5.1}$$

is called a Fiedler pencil with repetition (FPR) *associated with* $P(\lambda)$.

An important property of an FPR is that its matrix coefficients can be seen as $k \times k$ block-matrices whose blocks are of the form 0_n, I_n, or $\pm A_i$ [32].

The pencils in the standard basis of $\mathbb{DL}(P)$ (introduced in Section 2) are examples of FPR ([4, Lemma 5.7] and [32, Corollary 2]).

EXAMPLE 5.1. *Let $P(\lambda)$ be a matrix polynomial of degree 6 as in (1.1). The FPR given by $\mathbf{q} = (3:4, 1:2, 0)$, $\mathbf{l}_q = (0:2, 0)$, $\mathbf{r}_q = (3, 1:2, 0:1, 0)$, $\mathbf{z} = (-6:-5)$, $\mathbf{l}_z = \emptyset$ and $\mathbf{r}_z = (-6)$ is the pencil $D_2(\lambda, P)$ defined in Section 2, that is,*

$$\lambda \begin{bmatrix} 0 & A_6 & 0 & 0 & 0 & 0 \\ A_6 & A_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & A_2 & A_1 & A_0 \\ 0 & 0 & A_2 & A_1 & A_0 & 0 \\ 0 & 0 & A_1 & A_0 & 0 & 0 \\ 0 & 0 & A_0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} A_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_4 & A_3 & A_2 & A_1 & A_0 \\ 0 & A_3 & A_2 & A_1 & A_0 & 0 \\ 0 & A_2 & A_1 & A_0 & 0 & 0 \\ 0 & A_1 & A_0 & 0 & 0 & 0 \\ 0 & A_0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here we extend the family of FPR to a larger family of pencils which we call *generalized Fiedler pencils with repetition*.

DEFINITION 5.2. *(GFPR) Let $P(\lambda)$ be a matrix polynomial of degree k as in (1.1). Let $h \in \{0 : k-1\}$. Let $\mathbf{q}$ and $\mathbf{z}$ be permutations of $\{0 : h\}$ and $\{-k : -h-1\}$, respectively. Let $\mathbf{l}_q$ and $\mathbf{r}_q$ be tuples with indices from $\{0 : h-1\}$ such that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ satisfies the SIP. Let $\mathbf{l}_z$ and $\mathbf{r}_z$ be tuples with indices from $\{-k : -h-2\}$ such that $(\mathbf{l}_z, \mathbf{z}, \mathbf{r}_z)$ satisfies the SIP. Let X, Y, Z and W be matrix assignments for $\mathbf{l}_q$, $\mathbf{r}_q$, $\mathbf{l}_z$ and $\mathbf{r}_z$, respectively. Then, the pencil*

$$L_P(\lambda) = M_{\mathbf{l}_q,\mathbf{l}_z}(X,Z)(\lambda M^P_{\mathbf{z}} - M^P_{\mathbf{q}})M_{\mathbf{r}_z,\mathbf{r}_q}(W,Y) \tag{5.2}$$

is called a generalized Fiedler pencil with repetition (GFPR) *associated with* $P(\lambda)$.

Note that, if X, Y, Z, and W are, respectively, the trivial matrix assignments for $\mathbf{l}_q$, $\mathbf{r}_q$, $\mathbf{l}_z$ and $\mathbf{r}_z$, then $L_P(\lambda)$ is a FPR associated with $P(\lambda)$. Thus, the family of FPR is contained in the family of GFPR.

EXAMPLE 5.2. *The pencil $D_2(\lambda, P)$, given in Example 5.1, can also be seen as the GFPR defined by the tuples $\mathbf{q}$, $\mathbf{r}_q$, $\mathbf{l}_q$, $\mathbf{z}$, $\mathbf{r}_z$, and $\mathbf{l}_z$ given in that example, and the trivial matrix assignments for these tuples.*

The GFPR given by the tuples $\mathbf{q}$, $\mathbf{r}_q$, $\mathbf{l}_q$, $\mathbf{z}$, $\mathbf{r}_z$, and $\mathbf{l}_z$ in Example 5.1 and the matrix assignments $X = (X_1, X_2, X_3, X_4)$ for $\mathbf{l}_q$, $Y = (Y_1, Y_2, Y_3, Y_4, Y_5, Y_6)$ for $\mathbf{r}_q$, $Z = A$ for $\mathbf{l}_z$, and $W = (W_1)$ for $\mathbf{r}_z$ is

$$L_P(\lambda) = \lambda \begin{bmatrix} 0 & A_6 & 0 & 0 & 0 & 0 \\ W_1 & A_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & Y_1 & Y_3 & Y_5 & Y_6 \\ 0 & 0 & X_3 & Y_2 & Y_4 & 0 \\ 0 & 0 & X_2 & X_4 & 0 & 0 \\ 0 & 0 & X_1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} W_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_4 & Y_1 & Y_3 & Y_5 & Y_6 \\ 0 & A_3 & A_2 & Y_2 & Y_4 & 0 \\ 0 & X_3 & A_1 & A_0 & 0 & 0 \\ 0 & X_2 & X_4 & 0 & 0 & 0 \\ 0 & X_1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REMARK 5.1. *If $\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q$ are as in Definition 5.2, then, since $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ satisfies the SIP, both $\mathbf{l}_q$ and $\mathbf{r}_q$ satisfy the SIP. Moreover, because $\mathbf{q}$ is a permutation of $\}0 : h|$, it can be seen that $(\mathbf{l}_q, \mathbf{r}_q)$ satisfies the SIP, as stated in Theorem 4 in [32]. Similarly, if $\mathbf{l}_z$, $\mathbf{z}$, $\mathbf{r}_z$ are as in Definition 5.2, $(\mathbf{l}_z, \mathbf{r}_z)$ satisfies the SIP, as stated in Theorem 5 in [32].*

REMARK 5.2. *Using the commutativity relations in Remark 4.1, the matrix coefficient of the term of degree 1 of a GFPR as in (5.2) can be written in the form*

$$[M_{\mathbf{l}_z}(Z) M_{\mathbf{z}}^P M_{\mathbf{r}_z}(W)] M_{\mathbf{l}_q, \mathbf{r}_q}(X, Y).$$

Similarly, the matrix coefficient of the term of degree 0 can be written in the form

$$[M_{\mathbf{l}_q}(X) M_{\mathbf{q}}^P M_{\mathbf{r}_q}(Y)] M_{\mathbf{l}_z, \mathbf{r}_z}(Z, W).$$

We now describe the structure of the matrix coefficients of a GFPR associated with a matrix polynomial $P(\lambda)$ as in (1.1). Each of these matrix coefficients is the product of some elementary matrices determined by two index tuples (one of negative indices and one of nonnegative indices, at least one of which is nonempty) satisfying the SIP. Among other properties, we prove that these matrix coefficients are operation-free, which implies that each of them can be seen as a block-matrix whose blocks are of the form 0_n, I_n, $\oplus A_i$, or arbitrary $n \pm n$ matrices.

THEOREM 5.3. *(Structure of a GFPR) Let $P(\lambda)$ be a matrix polynomial of degree k as in (1.1) and let*

$$L_P(\lambda) := \lambda L_1 \quad L_0 = M_{\mathbf{l}_q, \mathbf{l}_z}(X, Z)(\lambda M_{\mathbf{z}}^P \quad M_{\mathbf{q}}^P) M_{\mathbf{r}_z, \mathbf{r}_q}(W, Y)$$

be the GFPR associated with $P(\lambda)$ as in Definition 5.2. Then:

- *(a) L_1 (resp. L_0) is a $k \pm k$ block matrix in which each of the $n \pm n$ blocks is either 0_n, I_n, $A_k, A_{k-1}, \ldots, A_{h+1}$ (resp. A_h, $A_{h-1}, \ldots,$ A_0) or a matrix in the matrix assignments for the tuples $\mathbf{l}_q$, $\mathbf{r}_q$, $\mathbf{l}_z$, and $\mathbf{r}_z$.*
- *(b) For each block A_k, $A_{k-1}, \ldots, A_{h+1}$ (resp. A_h, $A_{h-1}, \ldots,$ A_0) and each of the matrices in the matrix assignments for $\mathbf{l}_q$, $\mathbf{r}_q$, $\mathbf{l}_z$, and $\mathbf{r}_z$, there is exactly one position in the block-matrix L_1 (resp. L_0) whose entry is identically equal to that block; L_1 (resp. L_0) has neither identically zero block-rows nor identically zero block-columns; and in each block-row and in each block-column of L_1 (resp. L_0) at most one block identically equal to I_n appears.*

(c) $L_1 = diag(C_{11}, C_{22})$, *where* C_{11} *is a* $(k \quad h) \pm (k \quad h)$ *block-matrix which contains the blocks in the matrix assignments for the tuples* $\mathbf{z}$, $\mathbf{l}_z$, *and* $\mathbf{r}_z$, *and* C_{22} *is an* $h \pm h$ *block-matrix which contains the blocks in the matrix assignments for the tuples* $\mathbf{l}_q$ *and* $\mathbf{r}_q$.

(d) $L_0 = diag(D_{11}, D_{22})$, *where* D_{11} *is a* $(k \quad h \quad 1) \pm (k \quad h \quad 1)$ *block-matrix which contains the blocks in the matrix assignments for the tuples* $\mathbf{l}_z$ *and* $\mathbf{r}_z$, *and* D_{22} *is a* $(h+1) \pm (h+1)$ *block-matrix which contains the blocks in the matrix assignments for the tuples* $\mathbf{q}$, $\mathbf{l}_q$, *and* $\mathbf{r}_q$.

Proof. We only prove the result for $L_1 = M_{\mathbf{l}_q,\mathbf{l}_z}(X,Z) M^P_{\mathbf{z}} M_{\mathbf{r}_z,\mathbf{r}_q}(W,Y)$. From Remark 5.2, we get $L_1 = \rfloor M_{\mathbf{l}_z}(Z) M^P_{\mathbf{z}} M_{\mathbf{r}_z}(W) \lfloor \; M_{\mathbf{l}_q,\mathbf{r}_q}(X,Y)$. In addition, $(\mathbf{l}_z, \mathbf{z}, \mathbf{r}_z)$ satisfies the SIP by definition of GFPR and $(\mathbf{l}_q, \mathbf{r}_q)$ satisfies the SIP by Remark 5.1. Therefore, Lemma 4.7 can be applied separately to $M_{\mathbf{l}_z}(Z) M^P_{\mathbf{z}} M_{\mathbf{r}_z}(W)$ and to $M_{\mathbf{l}_q,\mathbf{r}_q}(X,Y)$. Finally, note that from the definition of the elementary matrices, we get that $M_{\mathbf{l}_z}(Z) M^P_{\mathbf{z}} M_{\mathbf{r}_z}(W) = diag(C_{11}, I_{hn})$ and $M_{\mathbf{l}_q,\mathbf{r}_q}(X,Y) = diag(I_{(k-h)n}, C_{22})$, where C_{11} and C_{22} are as in the statement. □

From the definitions of FPR and GFPR and taking into account Theorem 5.3, it follows that the family of GFPR strictly contains the family of FPR.

Next we study conditions for a GFPR to be a strong linearization of a matrix polynomial $P(\lambda)$ as in (1.1). For this purpose, we introduce first the following definition.

DEFINITION 5.4. *Let* $P(\lambda)$ *be a matrix polynomial as in (1.1) and* $L_P(\lambda)$ *be a GFPR associated with* $P(\lambda)$ *as in (5.2). We say that* $L_P(\lambda)$ *satisfies the* nonsingularity conditions *if the matrix assignments* X *for* $\mathbf{l}_q$, Y *for* $\mathbf{r}_q$, Z *for* $\mathbf{l}_z$, *and* W *for* $\mathbf{r}_z$ *are all nonsingular.*

Note that, if $L_P(\lambda)$ is a FPR as in (5.1), $L_P(\lambda)$ satisfies the nonsingularity conditions when none of the following happens:

1. 0 is an index in $\mathbf{l}_q$ or $\mathbf{r}_q$ and A_0 is singular;
2. k is an index in $\mathbf{l}_z$ or $\mathbf{r}_z$ and A_k is singular.

The fact that $\lambda M^P_{\mathbf{z}} \quad M^P_{\mathbf{q}}$ in (5.2) is a generalized Fiedler pencil and that these pencils are strong linearizations of $P(\lambda)$ [8] allow us to present in Theorem 5.5 sufficient conditions for a GFPR to be a strong linearization.

THEOREM 5.5. *Let* $P(\lambda)$ *be a matrix polynomial as in (1.1) and* $L_P(\lambda)$ *be a GFPR associated with* $P(\lambda)$ *as in (5.2). Then:*

- ≤ *If* $L_P(\lambda)$ *satisfies the nonsingularity conditions,* $L_P(\lambda)$ *is a strong linearization for* $P(\lambda)$.
- ≤ *If* $P(\lambda)$ *is regular and* $L_P(\lambda)$ *is a strong linearization for* $P(\lambda)$, $L_P(\lambda)$ *satisfies the nonsingularity conditions.*

Proof. Let $L_P(\lambda)$ be a pencil of the form (5.2), where X, Y, Z, and W are matrix assignments as described in Definition 5.2. If $L_P(\lambda)$ satisfies the nonsingularity conditions, the matrices $M_{\mathbf{l}_q,\mathbf{l}_z}(X,Z)$ and $M_{\mathbf{r}_z,\mathbf{r}_q}(W,Y)$ are nonsingular matrices. Thus, the pencils $L_P(\lambda)$ and $\lambda M^P_{\mathbf{z}} \quad M^P_{\mathbf{q}}$ are strictly equivalent. Since $\lambda M^P_{\mathbf{z}} \quad M^P_{\mathbf{q}}$ is a strong linearization of $P(\lambda)$ (when $P(\lambda)$ is regular or singular), the first claim follows.

To prove the second claim, note that, if $L_P(\lambda)$ does not satisfy the nonsingularity conditions, then $L_P(\lambda)$ would be a singular pencil and, therefore, it would not be a linearization of a regular $P(\lambda)$, as any linearization of a regular matrix polynomial is also regular. □

6. Block-symmetric GFPR. In this section, given a matrix polynomial $P(\lambda)$ as in (1.1), we characterize a family of block-symmetric GFPR associated with $P(\lambda)$ from which we construct, in Section 7, vector spaces most of whose pencils are strong linearizations of $P(\lambda)$.

In [4, Corollary 5.6 and Theorem 3.17] it was proven that an FPR of the form (5.1) is block-symmetric for any matrix polynomial $P(\lambda)$ if and only if $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$, $(\mathbf{l}_q, \mathbf{r}_q)$, $(\mathbf{l}_z, \mathbf{z}, \mathbf{r}_z)$, and $(\mathbf{l}_z, \mathbf{r}_z)$ are symmetric index tuples. We are interested in constructing a family of block-symmetric GFPR which, in particular, contains the block-symmetric FPR. Taking into account these facts and motivated by Lemma 4.9, we will use these symmetric tuples in the construction of our block-symmetric GFPR, which we characterize in this section. For this purpose, we need to refresh some concepts and results about symmetric index tuples that were introduced in [4].

DEFINITION 6.1. *Let $h \sim 0$ be an integer. We say that* $\mathbf{w}$ *is an* h-admissible tuple *if* $\mathbf{w}$ *is a permutation of* $\}0 : h|$ *and*

$$csf(\mathbf{w}) = (h \quad 1 : h, h \quad 3 : h \quad 2, \ldots, p+1 : p+2, 0 : p)$$

for some $0 \geq p \geq h$*. We call* p *the index of* $\mathbf{w}$.

Observe that, if $\mathbf{w}$ has index p, then the sequence of lengths of the strings in $csf(\mathbf{w})$ is of the form $(2, \ldots, 2, p+1)$. This implies that h and p have the same parity.

DEFINITION 6.2. *Let $h \sim 0$ and $\mathbf{w}$ be an h-admissible tuple with index p. We call the* symmetric complement of $\mathbf{w}$ *to the tuple* $\mathbf{c}_w$ *defined as follows:*

$\leq \mathbf{c}_w = (h \quad 1, h \quad 3, \ldots, p+3, p+1, (0 : p)_{rev_c})$, *if* $p \sim 1$,

$\leq \mathbf{c}_w = (h \quad 1, h \quad 3, \ldots, 1)$, *if* $p = 0$ *and* $h > 0$,

$\leq \mathbf{c}_w$ *is empty if* $h = 0$,

where $(0 : p)_{rev_c} := (0 : p \quad 1, 0 : p \quad 2, \ldots, 0 : 1, 0)$.

In [4, Lemma 3.11] it was shown that, if $\mathbf{w}$ and $\mathbf{c}_w$ are as in Definitions 6.1 and 6.2, respectively, then $(\mathbf{w}, \mathbf{c}_w)$ and $\mathbf{c}_w$ are symmetric and satisfy the SIP. Moreover, by [4, Lemma 3.15], if a tuple $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ as in Definition 5.2 is symmetric, with $(\mathbf{l}_q, \mathbf{r}_q)$ symmetric, then

$$(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) \subset (\mathbf{t}_w, \mathbf{w}, \mathbf{c}_w, rev(\mathbf{t}_w))$$

and

$$(\mathbf{l}_q, \mathbf{r}_q) \subset (\mathbf{t}_w, \mathbf{c}_w, rev(\mathbf{t}_w))$$

for some h-admissible tuple $\mathbf{w}$ and some index tuple $\mathbf{t}_w$ with indices from $\}0 : h \quad 1|$.

Next we provide a family of block-symmetric GFPR that extends the family of block-symmetric FPR introduced in [4].

THEOREM 6.3. *Let $P(\lambda)$ be a matrix polynomial of degree k as in (1.1) and let $h \,/\, \}0 : k \quad 1|$. Let $\mathbf{w}$ and $k+\mathbf{v}$ be h-admissible and $(k \quad h \quad 1)$-admissible index tuples, respectively, and let $\mathbf{t}_w$ and $k + \mathbf{t}_v$ be index tuples with indices from $\}0 : h \quad 1|$ and $\}0 : k \quad h \quad 2|$, respectively, such that $(\mathbf{t}_w, \mathbf{w}, \mathbf{c}_w, rev(\mathbf{t}_w))$ and $(\mathbf{t}_v, \mathbf{v}, \mathbf{c}_v, rev(\mathbf{t}_v))$ satisfy the SIP ($\mathbf{c}_w$ and $k + \mathbf{c}_v$ are the symmetric complements of $\mathbf{w}$ and $k + \mathbf{v}$, respectively). Let X and Y be matrix assignments for $\mathbf{t}_w$ and $\mathbf{t}_v$, respectively. Then, the GFPR*

$$L_P(\lambda) = M_{\mathbf{t}_w,\mathbf{t}_v}(X, Y)(\lambda M^P_{\mathbf{v}} \quad M^P_{\mathbf{w}})M^P_{\mathbf{c}_w,\mathbf{c}_v} M_{rev(\mathbf{t}_w),rev(\mathbf{t}_v)}(rev(X), rev(Y)). \quad (6.1)$$

is block-symmetric.

Proof. Let $L_P(\lambda) := \lambda L_1 \quad L_0$. We need to see that $L^{\mathcal{B}}_P(\lambda) = L_P(\lambda)$, or equivalently, $L^{\mathcal{B}}_1 = L_1$ and $L^{\mathcal{B}}_0 = L_0$. We prove the result for L_0. The result for L_1 can be proven similarly. By Remark 5.2, L_0 can be written in the form

$$H_1 H_2 := [M_{\mathbf{t}_w}(X) M^P_{\mathbf{w},\mathbf{c}_w} M_{rev(\mathbf{t}_w)}(rev(X))][M_{\mathbf{t}_v}(Y) M^P_{\mathbf{c}_v} M_{rev(\mathbf{t}_v)}(rev(Y))].$$

Note that $H_1 = I_{(k-h-1)n} \otimes H'_1$ and $H_2 = H'_2 \otimes I_{(h+1)n}$, for some $(h+1) \pm (h+1)$ and $(k \quad h \quad 1) \pm (k \quad h \quad 1)$ matrices H'_1 and H'_2, respectively. Thus, we have $L^{\mathcal{B}}_0 = H^{\mathcal{B}}_1 H^{\mathcal{B}}_2$. Now the result follows from Lemma 4.9 taking into account Remark 5.1 and the fact that $(X, \mathcal{Y}, rev(X))$ and $(Y, \mathcal{Z}, rev(Y))$ are symmetric matrix assignments for the symmetric index tuples $(\mathbf{t}_w, \mathbf{w}, \mathbf{c}_w, rev(\mathbf{t}_w))$ and $(\mathbf{t}_v, \mathbf{c}_v, rev(\mathbf{t}_v))$, respectively, where $\mathcal{Y}$ and $\mathcal{Z}$ are the trivial matrix assignments for $(\mathbf{w}, \mathbf{c}_w)$ and $(\mathbf{c}_v)$, respectively, associated with $P(\lambda)$.□

Note that any pencil of the form (6.1) is a strong linearization of a matrix polynomial $P(\lambda)$ if the matrices $M_{\mathbf{t}_w,\mathbf{t}_v}(X, Y)$ and $M^P_{\mathbf{c}_w,\mathbf{c}_v}$ are nonsingular.

Observe that the block-symmetric FPR introduced in [4] form a subset of the set of block-symmetric GFPR introduced above. To see that, take X and Y to be the trivial matrix assignments associated with $P(\lambda)$ for $\mathbf{t}_w$ and $\mathbf{t}_v$, respectively.

It is interesting to remark that, for $k > 2$, when the field $\mathbb{F}$ is infinite, while the family of block-symmetric FPR associated with a matrix polynomial $P(\lambda)$ as in (1.1) is finite, the family of block-symmetric GFPR contains infinitely many pencils, since the matrices in the assignments X and Y are arbitrary.

We now consider the smallest vector space containing the block-symmetric GFPR associated with a matrix polynomial $P(\lambda)$ introduced above.

DEFINITION 6.4. *Let $P(\lambda)$ be a matrix polynomial as in (1.1). We call* the extended $\mathbb{DL}(P)$ *to the vector space generated by the family of block-symmetric GFPR associated with $P(\lambda)$ given in Theorem 6.3, and denote it by $\mathbb{EDL}(P)$.*

We observe that the space $\mathbb{EDL}(P)$ contains the space $\mathbb{DL}(P)$ as will follow from Remark 6.2.

REMARK 6.1. *Though any block-symmetric FPR associated with a matrix polynomial $P(\lambda)$ is symmetric when $P(\lambda)$ is [4, Theorem 5.2 and Corollary 5.6], this is not necessarily true when the pencil is a block-symmetric GFPR. However, if the matrices in the matrix assignments X and Y are all symmetric, then a block-symmetric GFPR associated with $P(\lambda)$ as in Theorem 6.3 is symmetric when $P(\lambda)$ is. Moreover, the subspace of $\mathbb{EDL}(P)$ generated by the family of block-symmetric GFPR associated with $P(\lambda)$ in which X and Y are matrix assignments formed by symmetric matrices is such that all the pencils in it are symmetric when $P(\lambda)$ is.*

6.1. Canonical expression for the block-symmetric GFPR. Every index tuple can, in general, be expressed in several equivalent ways. Next we give a canonical form under equivalence for symmetric index tuples of the form $(\mathbf{t}_w, \mathbf{w}, \mathbf{c}_w, rev(\mathbf{t}_w))$ which allows us to express a block-symmetric GFPR of the form (6.1) associated to a generic $P(\lambda)$ in a standard way. More precisely, the tuples $\mathbf{w}$ and $\mathbf{v}$ used in the construction of a block-symmetric GFPR can be chosen to have minimum index. This canonical expression of a GFPR also facilitates the computation of the dimension of the subspaces of block-symmetric strong linearizations that we obtain in Section 7, which will be given in terms of the lengths of the corresponding tuples $\mathbf{t}_w$ and $\mathbf{t}_v$. We observe that among equivalent tuples of the form $(\mathbf{t}_w, \mathbf{w}, \mathbf{c}_w, rev(\mathbf{t}_w))$, the tuple in canonical form has $\mathbf{t}_w$ maximal, that is, of largest length. This can be easily seen taking into account that equivalent tuples of the form $(\mathbf{t}_w, \mathbf{w}, \mathbf{c}_w, rev(\mathbf{t}_w))$ have the same indices and the corresponding subtuples $\mathbf{w}$ are all permutations of $\}1 : h|$. Moreover, $\mathbf{c}_w$ has minimum number of indices when the index of $\mathbf{w}$ is the smallest.

Clearly, given an integer $h \sim 0$, there exists a unique h-admissible tuple, up to equivalence, of index 0 or 1. This index is 0 if h is even and is 1 if h is odd.

DEFINITION 6.5. *We call* the admissible tuple associated with $h \sim 0$ *to the unique h-admissible tuple of index 0 or 1 in column standard form. We denote this tuple by $\mathbf{w}_h$.*

EXAMPLE 6.1. *The admissible tuple associated with $h = 4$ is* $(3 : 4, 1 : 2, 0)$. *The admissible tuple associated with $h = 5$ is* $(4 : 5, 2 : 3, 0 : 1)$.

Next we characterize the index tuples $\mathbf{t}_{w_h}$ such that $(\mathbf{t}_{w_h}, \mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h}))$ satisfy the SIP, where $\mathbf{w}_h$ is the admissible tuple associated with h and $\mathbf{c}_{w_h}$ is the corresponding symmetric complement.

DEFINITION 6.6. *Given $h \sim 0$, we say that an index tuple $\mathbf{t}$ is in* canonical form for *h if*

$\mathbf{t}$ *is of the form*

$$\Big) a_1 : h - 2, a_2 : h - 4, \ldots, a_{\lfloor \frac{h}{2} \rfloor} : h - 2 \Big\rfloor \frac{h}{2} \Big\lceil \Big[\tag{6.2}$$

with $0 \geq a_i \geq h - 2i + 1$ *(if* $a_i = h - 2i + 1$*, then the string* $a_i : h - 2i$ *is empty). We say that such a tuple is* maximal *if* $a_i = 0$ *for all* i*. In this case, we denote the tuple by* $\mathbf{t}_h$ *and we say that* $(0 : h - 2i)$ *is its* i*-th string.*

Note that, if $h = 0, 1$, an index tuple in canonical form for h is necessarily empty. Moreover, for $h > 1$, a tuple in canonical form for h is, by definition, in column standard form.

LEMMA 6.7. *Let* $h \sim 0$ *and* $\mathbf{w}_h$ *be the admissible tuple associated with* h*. Let* $\mathbf{c}_{w_h}$ *be the symmetric complement of* $\mathbf{w}_h$ *and let* $\mathbf{t}_{w_h}$ *be an index tuple with indices from* $\}0 : h - 1|$*. Then,* $(\mathbf{t}_{w_h}, \mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h}))$ *satisfies the SIP if and only if* $\mathbf{t}_{w_h}$ *is equivalent to a tuple in canonical form for* h*.*

Proof. Assume that $(\mathbf{t}_{w_h}, \mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h}))$ satisfies the SIP. Since $\mathbf{t}_{w_h}$ is a subtuple of indices in consecutive positions of a tuple satisfying the SIP, it also satisfies the SIP and, by Lemma 3.8, it is equivalent to a tuple in column standard form. Thus, the tuple $(\mathbf{t}_{w_h}, \mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h}))$ is equivalent to $(csf(\mathbf{t}_{w_h}), csf(\mathbf{w}_h), \mathbf{c}_{w_h}, rev(csf(\mathbf{t}_{w_h})))$, which also satisfies the SIP. Therefore, for any b_i that is the last index of a string in $csf(\mathbf{t}_{w_h})$, the index $b_i + 1$ must appear to the left of b_i in $csf(\mathbf{w}_h)$. Since $\mathbf{w}_h$ is the admissible tuple associated with h,

$$csf(\mathbf{w}) = (h - 1 : h, h - 3 : h - 2, \ldots, p + 1 : p + 2, 0 : p), \quad p \,/\, \}0, 1|. \tag{6.3}$$

Thus, $b_i + 1$ occurs to the left of b_i in $csf(\mathbf{w}_h)$ only if b_i is the last index of a string in $csf(\mathbf{w}_h)$ different from h, that is, $b_i = h - 2j$ for some $j \,/\, \}1 : \rfloor \frac{h}{2} \lceil |$.

Assume now that $\mathbf{t}_{w_h}$ is equivalent to a tuple of the form (6.2). It is easy to verify that $(csf(\mathbf{t}_{w_h}), csf(\mathbf{w}_h), \mathbf{c}_{w_h}, rev(csf(\mathbf{t}_{w_h})))$ satisfies the SIP and, therefore, so does $(\mathbf{t}_{w_h}, \mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h}))$. □

DEFINITION 6.8. *Let* $h \sim 0$ *and* $\mathbf{w}$ *be an* h*-admissible tuple. Let* $\mathbf{c}_w$ *be the symmetric complement of* $\mathbf{w}$ *and let* $\mathbf{t}_w$ *be an index tuple with indices from* $\}0 : h - 1|$*. We say that* $(\mathbf{t}_w, \mathbf{w}, \mathbf{c}_w, rev(\mathbf{t}_w))$ *is in* reduced index form *if*

1. $\mathbf{w} = \mathbf{w}_h$*, that is,* $\mathbf{w}$ *is the admissible tuple associated with* h*; and*
2. $\mathbf{t}_w$ *is in canonical form for* h*.*

Note that, by Lemma 6.7, a tuple $(\mathbf{t}_w, \mathbf{w}, \mathbf{c}_w, rev(\mathbf{t}_w))$ in reduced index form satisfies the SIP.

LEMMA 6.9. *Let* $h \sim 0$ *and* $\mathbf{w}$ *be an* h*-admissible tuple of index* p*. Let* $\mathbf{c}_w$ *be the symmetric complement of* $\mathbf{w}$*. Then,* $(\mathbf{w}, \mathbf{c}_w)$ *is equivalent to the index tuple* $(\mathbf{t}_p, \mathbf{w}_h, \mathbf{c}_{\mathbf{w}_h}, rev(\mathbf{t}_p))$ *and* $\mathbf{c}_w$ *is equivalent to* $(\mathbf{t}_p, \mathbf{c}_{\mathbf{w}_h}, rev(\mathbf{t}_p))$*, where* $\mathbf{w}_h$ *is the admissible tuple associated with* h*,* $\mathbf{c}_{\mathbf{w}_h}$ *is the symmetric complement of* $\mathbf{w}_h$*, and* $\mathbf{t}_p$ *is the maximal tuple in canonical form for* p*, that is,* $\mathbf{t}_p = (0 : p - 2, 0 : p - 4, \ldots, 0 : p - 2 \rfloor \frac{p}{2} \lceil)$ *for* $p \sim 2$ *(*$\mathbf{t}_p$ *is empty if* $p < 2$*).*

Proof. Let $(\mathbf{w}, \mathbf{c}_w)$ be as in the statement of the theorem. We prove the result by induction on the index p of $\mathbf{w}$. If $p \,/\, \}0, 1|$, the result follows trivially as $\mathbf{w}_h = \mathbf{w}$ and $\mathbf{t}_p$ is empty.

Now suppose that $p > 1$ and that the result holds for $p' < p$. Recall the notation rev_c introduced in Definition 6.2. We have

$$\begin{aligned} \mathbf{w} &\subset (h - 1 : h, h - 3 : h - 2, \ldots, p + 1 : p + 2, 0 : p) \\ &\subset ((0 : p - 2), (h - 1 : h, h - 3 : h - 2, \ldots, p - 1 : p)) \end{aligned}$$

and

$$\mathbf{c}_w = (h \quad 1, h \quad 3, \ldots, p+1, (0:p)_{rev_c})$$

$$\subset ((0:p \quad 2), (h \quad 1, h \quad 3, \ldots, p \quad 1, (0:p \quad 2)_{rev_c}), rev(0:p \quad 2)).$$

Let $\mathbf{w}^* = (h \quad 1:h, \ldots, p \quad 1:p, 0:p \quad 2)$. Note that $\mathbf{w}^*$ is an h-admissible tuple of index $p \quad 2$. Moreover, the symmetric complement of $\mathbf{w}^*$ is $\mathbf{c}_{w^*} = (h \quad 1, h \quad 3, \ldots, p \quad 1, (0 : p \quad 2)_{rev_c})$. As a consequence,

$$(\mathbf{w}, \mathbf{c}_w) \subset (0:p \quad 2, \mathbf{w}^*, \mathbf{c}_{w^*}, rev(0:p \quad 2)) \tag{6.4}$$

and

$$\mathbf{c}_w \subset (0:p \quad 2, \mathbf{c}_{w^*}, rev(0:p \quad 2)). \tag{6.5}$$

By the inductive hypothesis, $(\mathbf{w}^*, \mathbf{c}_{w^*})$ is equivalent to $(\mathbf{t}_{p-2}, \mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_{p-2}))$ and $\mathbf{c}_{w^*}$ is equivalent to $(\mathbf{t}_{p-2}, \mathbf{c}_{w_h}, rev(\mathbf{t}_{p-2}))$. Now the result follows from (6.4) and (6.5). Note that $p \quad 2 \quad 2\rfloor \frac{p-2}{2} \lceil = p \quad 2\rfloor \frac{p}{2} \lceil$. □

The next lemma shows that the reduced index form is canonical under equivalence in the set of symmetric index tuples of the form $(\mathbf{t}_w, \mathbf{w}, \mathbf{c}_w, rev(\mathbf{t}_w))$ that satisfy the SIP.

LEMMA 6.10. *Let $h \sim 0$ and $\mathbf{w}$ be an h-admissible tuple of index p. Let $\mathbf{c}_w$ be the symmetric complement of $\mathbf{w}$ and let $\mathbf{t}_w$ be an index tuple with indices from $\}0 : h \quad 1|$ such that $(\mathbf{t}_w, \mathbf{w}, \mathbf{c}_w, rev(\mathbf{t}_w))$ satisfies the SIP. Then, $(\mathbf{t}_w, \mathbf{w}, \mathbf{c}_w, rev(\mathbf{t}_w))$ is equivalent to a unique tuple $(\mathbf{t}_{w_h}, \mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h}))$ in reduced index form. Moreover, the tuple $\mathbf{t}_{w_h}$ is equivalent to $(\mathbf{t}_w, \mathbf{t}_p)$ and $(\mathbf{t}_{w_h}, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h}))$ is equivalent to $(\mathbf{t}_w, \mathbf{c}_w, rev(\mathbf{t}_w))$, where $\mathbf{t}_p$ is the maximal tuple in canonical form for p.*

Proof. First we show the existence part of the first claim. By Lemma 6.9, $(\mathbf{t}_w, \mathbf{w}, \mathbf{c}_w, rev(\mathbf{t}_w))$ is equivalent to $(\mathbf{t}_w, \mathbf{t}_p, \mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_p), rev(\mathbf{t}_w))$. By Lemma 6.7, $(\mathbf{t}_w, \mathbf{t}_p)$ is equivalent to a tuple $\mathbf{t}_{w_h}$ in canonical form for h. Thus, $(\mathbf{t}_w, \mathbf{w}, \mathbf{c}_w, rev(\mathbf{t}_w))$ is equivalent to the tuple $(\mathbf{t}_{w_h}, \mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h}))$, which is in reduced index form.

Now we prove uniqueness. Suppose that $(\mathbf{t}_w, \mathbf{w}, \mathbf{c}_w, rev(\mathbf{t}_w))$, as in the statement of the theorem, is equivalent to the index tuples in reduced index form $(\mathbf{t}_{w_h}, \mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h}))$ and $(\mathbf{t}'_{w_h}, \mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}'_{w_h}))$. By Lemma 2.14 in [4], we have $\mathbf{t}_{w_h} \subset \mathbf{t}'_{w_h}$, which implies $csf(\mathbf{t}_{w_h}) = csf(\mathbf{t}'_{w_h})$. Since $\mathbf{t}_{w_h} = csf(\mathbf{t}_{w_h})$ and $\mathbf{t}'_{w_h} = csf(\mathbf{t}'_{w_h})$, we deduce that $\mathbf{t}_{w_h} = \mathbf{t}'_{w_h}$.

The first part of the second claim follows from the proof of the existence claim and the uniqueness of $\mathbf{t}_{w_h}$.

The second part of the second claim follows from Lemma 6.9 taking into account that $\mathbf{t}_{w_h}$ is equivalent to $(\mathbf{t}_w, \mathbf{t}_p)$. □

Next we use Lemma 6.10 to obtain a canonical expression for the block-symmetric GFPR given in (6.1).

THEOREM 6.11. *Let $0 \geq h < k$. There exist index tuples $\mathbf{t}_{w_h}$ and $k + \mathbf{t}_{v_h}$ in canonical form for h and $k \quad h \quad 1$, respectively, such that, for any matrix polynomial $P(\lambda)$ of degree k as in (1.1), the block-symmetric GFPR given in Theorem 6.3 can be expressed as*

$$M_{\mathbf{t}_{w_h}, \mathbf{t}_{v_h}}(\mathcal{Y}, \mathcal{Z})(\lambda M^P_{\mathbf{v}_h} \quad M^P_{\mathbf{w}_h}) M^P_{\mathbf{c}_{w_h}, \mathbf{c}_{v_h}} M_{rev(\mathbf{t}_{w_h}), rev(\mathbf{t}_{v_h})}(rev(\mathcal{Y}), rev(\mathcal{Z})), \tag{6.6}$$

where $\mathbf{w}_h$ is the admissible tuple associated with h, $\mathbf{c}_{w_h}$ is the symmetric complement of $\mathbf{w}_h$, $\mathbf{w}_{k-h-1}$ is the admissible tuple associated with $k \quad h \quad 1$, $\mathbf{v}_h = \quad k + \mathbf{w}_{k-h-1}$, $k + \mathbf{c}_{v_h}$

is the symmetric complement of $\mathbf{w}_{k-h-1}$, *and* $\mathcal{Y}$ *and* $\mathcal{Z}$ *are matrix assignments for* $\mathbf{t}_{w_h}$ *and* $\mathbf{t}_{v_h}$, *respectively. Moreover, any pencil of the form (6.6) is a block-symmetric GFPR.*

Proof. The fact that any pencil of the form (6.6) is a block-symmetric GFPR is an immediate consequence of Theorem 6.3. Now we prove the first claim. Consider a pencil of the form (6.1). By Remark 5.2, this pencil can be written as $\lambda L_1 - L_0$, with

$$L_0 = [M_{\mathbf{t}_w}(X) M^P_{\mathbf{w},\mathbf{c}_w} M_{rev(\mathbf{t}_w)}(rev(X))][M_{\mathbf{t}_v}(Y) M^P_{\mathbf{c}_v} M_{rev(\mathbf{t}_v)}(rev(Y))] \tag{6.7}$$

and

$$L_1 = [M_{\mathbf{t}_v}(Y) M^P_{\mathbf{v},\mathbf{c}_v} M_{rev(\mathbf{t}_v)}(rev(Y))][M_{\mathbf{t}_w}(X) M^P_{\mathbf{c}_w} M_{rev(\mathbf{t}_w)}(rev(X))]. \tag{6.8}$$

Let $(\mathbf{t}_{w_h}, \mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h}))$ and $k + (\mathbf{t}_{v_h}, v_h, \mathbf{c}_{v_h}, rev(\mathbf{t}_{v_h}))$ be the unique tuples in reduced index form equivalent to $(\mathbf{t}_w, \mathbf{w}, \mathbf{c}_w, rev(\mathbf{t}_w))$ and $k + (\mathbf{t}_v, \mathbf{v}, \mathbf{c}_v, rev(\mathbf{t}_v))$, respectively, given by Lemma 6.10. We then have that $(\mathbf{t}_{w_h}, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h}))$ is equivalent to $(\mathbf{t}_w, \mathbf{c}_w, rev(\mathbf{t}_w))$, and $k + (\mathbf{t}_{v_h}, \mathbf{c}_{v_h}, rev(\mathbf{t}_{v_h}))$ is equivalent to $k + (\mathbf{t}_v, \mathbf{c}_v, rev(\mathbf{t}_v))$. Also,

$$\mathbf{t}_{w_h} \subset \mathbf{t}^*_w := (\mathbf{t}_w, \mathbf{t}_p) \quad \text{and} \quad \mathbf{t}_{v_h} \subset \mathbf{t}^*_v := (\mathbf{t}_v, -k + \mathbf{t}_{p'}),$$

where p and p' are the index of $\mathbf{w}$ and $k + \mathbf{v}$, respectively.

Because of Lemma 6.9, the tuple $(\mathbf{t}^*_w, \mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}^*_w))$ (resp. $(\mathbf{t}^*_w, \mathbf{c}_{w_h}, rev(\mathbf{t}^*_w))$) can be obtained from the tuple $(\mathbf{t}_w, \mathbf{w}, \mathbf{c}_w, rev(\mathbf{t}_w))$ (resp. $(\mathbf{t}_w, \mathbf{c}_w, rev(\mathbf{t}_w))$) by a sequence of transpositions not involving the indices of $\mathbf{t}_w$. Analogously, the tuple $k+(\mathbf{t}^*_v, v_h, \mathbf{c}_{v_h}, rev(\mathbf{t}^*_v))$ (resp. $k + (\mathbf{t}^*_v, \mathbf{c}_{v_h}, rev(\mathbf{t}^*_v))$) can be obtained from the tuple $k + (\mathbf{t}_v, \mathbf{v}, \mathbf{c}_v, rev(\mathbf{t}_v))$ (resp. $k + (\mathbf{t}_v, \mathbf{c}_v, rev(\mathbf{t}_v))$ by a sequence of transpositions not involving the indices of $k + \mathbf{t}_v$.

Let σ (resp. τ) be the allowed permutation that transforms $\mathbf{t}^*_w$ (resp. $\mathbf{t}^*_v$) into $\mathbf{t}_{w_h}$ (resp. $\mathbf{t}_{v_h}$). Let $\mathcal{Y}'$ be the trivial matrix assignment for $\mathbf{t}_p$ and $\mathcal{Z}'$ be the trivial matrix assignment for $-k + \mathbf{t}_{p'}$. Using Lemma 4.4, the result follows for the matrix assignments $\mathcal{Y} = \sigma(X, \mathcal{Y}')$ and $\mathcal{Z} = \tau(Y, \mathcal{Z}')$ for $\mathbf{t}_{w_h}$ and $\mathbf{t}_{v_h}$, respectively. □

A pencil of the form (6.6) associated with $P(\lambda)$ is uniquely determined by h, $\mathbf{t}_{w_h}$, $\mathbf{t}_{v_h}$, $\mathcal{Y}$, and $\mathcal{Z}$. We then have the following notation.

DEFINITION 6.12. *We denote by* $L_P(h, \mathbf{t}_{w_h}, \mathbf{t}_{v_h}, \mathcal{Y}, \mathcal{Z})$ *a block-symmetric GFPR as in Theorem 6.11.*

Observe that, if $h = k - 1$, $\mathbf{t}_{v_h}$ is necessarily empty. Analogously, if $h = 0$, $\mathbf{t}_{w_h}$ is necessarily empty.

According to Theorem 6.11, every block-symmetric GFPR as in Theorem 6.3 is of the form $L_P(h, \mathbf{t}_{w_h}, \mathbf{t}_{v_h}, \mathcal{Y}, \mathcal{Z})$, for some h, $\mathbf{t}_{w_h}$, $\mathbf{t}_{v_h}$, $\mathcal{Y}$, $\mathcal{Z}$.

We denote $L_P(h, \mathbf{t}_{w_h}, \mathbf{t}_{v_h}) := L_P(h, \mathbf{t}_{w_h}, \mathbf{t}_{v_h}, \mathcal{Y}, \mathcal{Z})$, when $\mathcal{Y}$ and $\mathcal{Z}$ are the trivial matrix assignments for $\mathbf{t}_{w_h}$ and $\mathbf{t}_{v_h}$, respectively. Note that $L_P(h, \mathbf{t}_{w_h}, \mathbf{t}_{v_h})$ is a FPR. Moreover, every block-symmetric FPR associated with $P(\lambda)$ is of the form $L_P(h, \mathbf{t}_{w_h}, \mathbf{t}_{v_h})$ for some h, $\mathbf{t}_{w_h}$, $\mathbf{t}_{v_h}$ [4, Corollary 5.6].

REMARK 6.2. *The space* $\mathbb{EDL}(P)$ *introduced in Definition 6.4 contains the space* $\mathbb{DL}(P)$ *as, for* $m = 1 : k$, *the pencil* $D_m(\lambda, P)$ *in the standard basis for* $\mathbb{DL}(P)$ *(see Section 2) is the FPR* $L_P(k - m, \mathbf{t}_{k-m}, -k + \mathbf{t}_{m-1})$, *where* $\mathbf{t}_{k-m}$ *and* $\mathbf{t}_{m-1}$ *are the maximal index tuples in canonical form for* $k - m$ *and* $m - 1$, *respectively ([32, Corollaries 1 and 2] and [4, Lemma 5.7]).*

The next example illustrates the previous remark.

EXAMPLE 6.2. *Let* $P(\lambda)$ *be a matrix polynomial of degree 6 as in (1.1). Let* $h = 4$, $\mathbf{t}_{w_h} = (0 : 2, 0)$, $\mathbf{t}_{v_h} = \mathcal{A}$ $\mathcal{Y} = (X_1, X_2, X_3, X_4)$ *be a matrix assignment for* $\mathbf{t}_{w_h}$, *and* $\mathcal{Z}$ *be* *the empty matrix assignment. Then, the pencil $L_P(h, \mathbf{t}_{w_h}, \mathbf{t}_{v_h}, \mathcal{Y}, \mathcal{Z})$ is given by*

$$\lambda \begin{bmatrix} 0 & A_6 & 0 & 0 & 0 & 0 \\ A_6 & A_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & X_3 & X_2 & X_1 \\ 0 & 0 & X_3 & A_1 & X_4 & 0 \\ 0 & 0 & X_2 & X_4 & 0 & 0 \\ 0 & 0 & X_1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_4 & A_3 & X_3 & X_2 & X_1 \\ 0 & A_3 & A_2 & A_1 & X_4 & 0 \\ 0 & X_3 & A_1 & A_0 & 0 & 0 \\ 0 & X_2 & X_4 & 0 & 0 & 0 \\ 0 & X_1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that if $\mathcal{Y}$ is the trivial matrix assignment for $\mathbf{t}_{w_h}$, then $L_P(h, \mathbf{t}_{w_h}, \mathbf{t}_{v_h}, \mathcal{Y}, \mathcal{Z}) = L_P(h, \mathbf{t}_{w_h}, \mathbf{t}_{v_h})$ is the FPR $D_2(\lambda, P)$ of the standard basis of $\mathbb{DL}(P)$, that is, the pencil

$$\lambda \begin{bmatrix} 0 & A_6 & 0 & 0 & 0 & 0 \\ A_6 & A_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & A_2 & A_1 & A_0 \\ 0 & 0 & A_2 & A_1 & A_0 & 0 \\ 0 & 0 & A_1 & A_0 & 0 & 0 \\ 0 & 0 & A_0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_4 & A_3 & A_2 & A_1 & A_0 \\ 0 & A_3 & A_2 & A_1 & A_0 & 0 \\ 0 & A_2 & A_1 & A_0 & 0 & 0 \\ 0 & A_1 & A_0 & 0 & 0 & 0 \\ 0 & A_0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7. Single-$\mathbb{EDL}(P)$ subspaces. It is well-known that if $P(\lambda)$ is regular, almost all pencils in $\mathbb{DL}(P)$ are strong linearizations of $P(\lambda)$ [17, Section 7]. However, no pencil in $\mathbb{DL}(P)$ is a linearization of a singular $P(\lambda)$ [9, Theorem 6.1]. In this section, we introduce subspaces of the space $\mathbb{EDL}(P)$ presented in Definition 6.4, other than $\mathbb{DL}(P)$, that we call Single-$\mathbb{EDL}(P)$ subspaces, in which almost all pencils are strong linearizations of $P(\lambda)$ when $\mathbb{F}$ is infinite. These subspaces are not contained in $\mathbb{DL}(P)$, many of them have trivial intersection with this space, and have dimensions much larger than $\mathbb{DL}(P)$. Moreover, when $P(\lambda)$ is singular and its degree k is odd, we also provide subspaces in which almost all pencils are strong linearizations of $P(\lambda)$, which is in stark contrast with the situation for $\mathbb{DL}(P)$. As a consequence, the subspaces that we introduce are, as far as we know, the first vector spaces of strong block-symmetric linearizations for singular matrix polynomials available in the literature.

In Section 2 we discussed particular Single-$\mathbb{EDL}(P)$ subspaces, which we called $F_m(P)$. Each subspace $F_m(P)$ can be obtained from the pencil $D_m(\lambda, P)$ in the standard basis of $\mathbb{DL}(P)$ by multiplying some blocks by an arbitrary fixed scalar and replacing the remaining blocks, identified in Definition 2.1, by arbitrary $n \pm n$ matrices. It turns out that our general Single-$\mathbb{EDL}(P)$ subspaces can be obtained in a similar way, by starting with an arbitrary block-symmetric FPR, not necessarily a pencil in the standard basis of $\mathbb{DL}(P)$, as illustrated in the following example. However, the explicit description of the positions of the blocks to be replaced is in general very technical and involved and, thus, we do not give it here.

EXAMPLE 7.1. *Let $P(\lambda)$ be a matrix polynomial of degree $k = 5$ as in (1.1). Let $h = 4$, $\mathbf{t}_w = (2, 0)$, and $\mathbf{t}_v = \mathcal{A}$ The FPR $L_P(4, \mathbf{t}_w, \mathbf{t}_v)$ is*

$$\lambda L_1 \quad L_0 := \lambda \left[\begin{array}{c|cccc} A_5 & 0 & 0 & 0 & 0 \\ \hline 0 & A_3 & A_2 & I_n & 0 \\ 0 & A_2 & A_1 & 0 & A_0 \\ 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & A_0 & 0 & 0 \end{array}\right] \begin{bmatrix} A_4 & A_3 & A_2 & I_n & 0 \\ A_3 & A_2 & A_1 & 0 & A_0 \\ A_2 & A_1 & A_0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 \\ 0 & A_0 & 0 & 0 & 0 \end{bmatrix}$$

Note that each matrix coefficient L_1 and L_0 can be viewed as the direct sum of two matrices (one of them being empty in the case of L_0). The matrix in the upper left corner corresponds to the blocks assigned to the indices in the tuple $(\mathbf{t}_v, \mathbf{c}_v, rev(\mathbf{t}_v))$ (in L_0) and *in the tuple* $(\mathbf{t}_v, \mathbf{v}, \mathbf{c}_v, rev(\mathbf{t}_v))$ *(in* L_1*); the matrix in the lower right corner corresponds to the blocks assigned to the indices in the tuple* $(\mathbf{t}_w, \mathbf{c}_w, rev(\mathbf{t}_w))$ *(in* L_1*) and the tuple* $(\mathbf{t}_w, \mathbf{w}, \mathbf{c}_w, rev(\mathbf{t}_w))$ *(in* L_0*).*

We will replace some selected blocks in L_1 *and* L_0 *by arbitrary matrices and multiply the remaining blocks by an arbitrary constant* γ *in order to obtain the pencils in a subspace of* $\mathbb{EDL}(P)$*. We first make the replacements in the block associated with the tuple* $(\mathbf{t}_w, \mathbf{c}_w, rev(\mathbf{t}_w))$*, that is, the lower right corner of* L_1*, say* Q*. Note that* $\mathbf{c}_w$ *has no repeated indices. Also, the GFPR* $L_P(4, \mathbf{t}_w, \mathbf{t}_v, X, Y)$ *is block-symmetric for any matrix assignments* X *and* Y *for* $\mathbf{t}_w$ *and* $\mathbf{t}_v$*. Moreover, the positions in the matrix coefficients of* $L_P(4, \mathbf{t}_w, \mathbf{t}_v, X, Y)$ *of the block assigned to an index* i *in* $\mathbf{t}_w$ *or* $\mathbf{t}_v$ *is independent of the matrix assignment. Thus, we can conclude that the block assigned to an index in* $\mathbf{t}_w$ *(and the same index in* $rev(\mathbf{t}_w)$*) occurs in symmetric positions in* Q*, and, conversely, if the same block occurs in symmetric positions in* Q*, then this block is assigned to an index in* $\mathbf{t}_w$ *(and the same index in* $rev(\mathbf{t}_w)$*). There are two pairs of symmetric positions in* Q *occupied by the same block* A_i*, namely, the block* A_2 *(associated with the index 2 in* $\mathbf{t}_w$ *and* $rev(\mathbf{t}_w)$*) occurs in positions (3,2) and (2,3) in* L_1*, and the block* A_0 *(associated with the index 0 in* $\mathbf{t}_w$ *and* $rev(\mathbf{t}_w)$*) occurs in positions (5,3) and (3,5) in* L_1*. We replace the two occurrences of the block* A_2 *by an arbitrary matrix* X_2 *and the two occurrences of the block* A_0 *by an arbitrary matrix* X_1*. We then multiply the remaining blocks in* Q *by* γ *and get the pencil*

$$\lambda \begin{bmatrix} A_5 & 0 & 0 & 0 & 0 \\ 0 & \gamma A_3 & X_2 & \gamma I_n & 0 \\ 0 & X_2 & \gamma A_1 & 0 & X_1 \\ 0 & \gamma I_n & 0 & 0 & 0 \\ 0 & 0 & X_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_4 & A_3 & A_2 & I_n & 0 \\ A_3 & A_2 & A_1 & 0 & A_0 \\ A_2 & A_1 & A_0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 \\ 0 & A_0 & 0 & 0 & 0 \end{bmatrix}$$

Next we focus on the block associated with the tuple $(\mathbf{t}_w, \mathbf{w}, \mathbf{c}_w, rev(\mathbf{t}_w))$*, that is, the lower right corner of* L_0*, say* Q'*. We will replace the blocks corresponding to the same indices involved in the replacements in* Q *by the respective matrices* X_i*. The block corresponding to the index 2 lies in* L_0 *in positions* $(3,1)$ *and* $(1,3)$*, and we replace it by* X_2*; the block corresponding to the index 0 lies in* L_0 *in positions* $(5,2)$ *and* $(2,5)$*, and we replace it by* X_1*. Note that the positions of these replaced blocks are obtained from the positions in* L_1 *by decreasing the column (resp. the row) by 1 if the position is below (resp. above) the main diagonal of* L_1*. Also note that, in general, the blocks* A_i *associated with the index* i *in the subtuple* $(\mathbf{w}, \mathbf{c}_w)$ *could occur in symmetric positions in* Q'*, though it is not the case here. Finally, we multiply the rest of nonzero blocks in* Q' *by* γ*. In this way we obtain the set of pencils of the form*

$$\lambda \begin{bmatrix} A_5 & 0 & 0 & 0 & 0 \\ 0 & \gamma A_3 & X_2 & \gamma I_n & 0 \\ 0 & X_2 & \gamma A_1 & 0 & X_1 \\ 0 & \gamma I_n & 0 & 0 & 0 \\ 0 & 0 & X_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma A_4 & \gamma A_3 & X_2 & \gamma I_n & 0 \\ \gamma A_3 & \gamma A_2 & \gamma A_1 & 0 & X_1 \\ X_2 & \gamma A_1 & \gamma A_0 & 0 & 0 \\ \gamma I_n & 0 & 0 & 0 & 0 \\ 0 & X_1 & 0 & 0 & 0 \end{bmatrix}$$

In a similar way, we could make the replacements in the left upper corner matrices of L_0 *and* L_1*, starting first with the matrix* L_0*. However, in this case, there are no symmetric positions occupied by blocks of the form* A_i *and, thus, there are no replacements to make. The only* *thing left to do is to multiply the blocks in Q' by γ. We get the pencil*

$$\lambda \begin{bmatrix} \gamma A_5 & 0 & 0 & 0 & 0 \\ 0 & \gamma A_3 & X_2 & \gamma I_n & 0 \\ 0 & X_2 & \gamma A_1 & 0 & X_1 \\ 0 & \gamma I_n & 0 & 0 & 0 \\ 0 & 0 & X_1 & 0 & 0 \end{bmatrix}^{\mathcal{B}} \begin{bmatrix} \gamma A_4 & \gamma A_3 & X_2 & \gamma I_n & 0 \\ \gamma A_3 & \gamma A_2 & \gamma A_1 & 0 & X_1 \\ X_2 & \gamma A_1 & \gamma A_0 & 0 & 0 \\ \gamma I_n & 0 & 0 & 0 & 0 \\ 0 & X_1 & 0 & 0 & 0 \end{bmatrix}^{\mathcal{B}} \tag{7.1}$$

The set of pencils of the previous form, with γ running over $\mathbb{F}$ and X_1, X_2 running over $M_n(\mathbb{F})$, forms a subspace of $\mathbb{EDL}(P)$, which we will call a Single-$\mathbb{EDL}(P)$.

From now on we assume that $P(\lambda)$ is a matrix polynomial of degree $k \sim 3$.

In this section we use very often the notation $L_P(h, \mathbf{t}_w, \mathbf{t}_v, \mathcal{Y}, \mathcal{Z})$, introduced in Definition 6.12, to denote a block-symmetric GFPR as in Theorem 6.11.

7.1. General approach to Single-$\mathbb{EDL}(P)$ subspaces. In this section we construct subspaces of $\mathbb{EDL}(P)$ obtained from each block-symmetric FPR associated with a matrix polynomial $P(\lambda)$ (not necessarily obtained from a pencil in the standard basis of $\mathbb{DL}(P)$, as those subspaces considered in Section 2) and we show that most of the pencils in these subspaces are strong linearizations of $P(\lambda)$, when $P(\lambda)$ is regular satisfying some nonsingularity conditions or when $P(\lambda)$ is singular of odd degree. The subspaces described in Theorem 2.2 are particular cases of these general subspaces that we call the Single-$\mathbb{EDL}(P)$ subspaces.

Let $\mathbf{t}$ be an index tuple. We denote by $S(n, \mathbf{t})$ the set of all matrix assignments for $\mathbf{t}$ whose matrices are of size $n \pm n$ and have entries in $\mathbb{F}$.

DEFINITION 7.1. *Let $P(\lambda)$ be an $n \pm n$ matrix polynomial of degree k as in (1.1) and $0 \geq h < k$ be an integer. Let $\mathbf{t}_w$ (resp. $k + \mathbf{t}_v$) be a tuple in canonical form for h (resp. $k \quad h \quad 1$). Define*

$$S_1^P(h, \mathbf{t}_w, \mathbf{t}_v) :=$$
$$\}L_P(h, \mathbf{t}_w, \mathbf{t}_v, \mathcal{Y}, \mathcal{Z}) + \alpha L_P(h, \mathbf{t}_w, \mathbf{t}_v, 0, 0) : \alpha \ / \ \mathbb{F}, \mathcal{Y} \ / \ S(n, \mathbf{t}_w),\ \mathcal{Z} \ / \ S(n, \mathbf{t}_v)| \ .$$

We call $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$ a Single-$\mathbb{EDL}(P)$ subspace.

It is easy to see that $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$ is a vector space over $\mathbb{F}$.

Note that each Single-$\mathbb{EDL}(P)$ subspace $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$ is associated with a single FPR, namely, with $L_P(h, \mathbf{t}_w, \mathbf{t}_v)$, since the pencils in $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$ are obtained from the multiples of $L_P(h, \mathbf{t}_w, \mathbf{t}_v)$ by replacing the blocks corresponding to the indices in $\mathbf{t}_w$ and $\mathbf{t}_v$ by arbitrary matrices. This observation is the motivation for the name *Single*-$\mathbb{EDL}(P)$ subspace. We also observe that, by Theorem 6.11, any block-symmetric GFPR associated with $P(\lambda)$ as in Theorem 6.3, belongs to a Single-$\mathbb{EDL}(P)$ subspace associated with $P(\lambda)$.

EXAMPLE 7.2. *Let $k = 5$, $h = 4$, $\mathbf{t}_w = (0 : 2, 0)$, $\mathbf{t}_v = \mathcal{A}$ Then, $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$ is given by the family of pencils*

$$\lambda \begin{bmatrix} \gamma A_5 & 0 & 0 & 0 & 0 \\ 0 & \gamma A_3 & X_3 & X_2 & X_1 \\ 0 & X_3 & \gamma A_1 & X_4 & 0 \\ 0 & X_2 & X_4 & 0 & 0 \\ 0 & X_1 & 0 & 0 & 0 \end{bmatrix}^{\mathcal{B}} \begin{bmatrix} \gamma A_4 & \gamma A_3 & X_3 & X_2 & X_1 \\ \gamma A_3 & \gamma A_2 & \gamma A_1 & X_4 & 0 \\ X_3 & \gamma A_1 & \gamma A_0 & 0 & 0 \\ X_2 & X_4 & 0 & 0 & 0 \\ X_1 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathcal{B}}$$

where $\gamma \ / \ \mathbb{F}$ and $X_i \ / \ M_n(\mathbb{F})$ for $i = 1 : 4$. Notice that $\gamma = \alpha + 1$, where α is as in Definition 7.1. Moreover, this Single-$\mathbb{EDL}(P)$ subspace coincides with the subspace $F_1(\lambda)$ introduced in Definition 2.1, that is, it is built on $D_1(\lambda, P)$, the first pencil in the standard basis of $\mathbb{DL}(P)$.

Next we give a basis and the dimension of the vector space $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$. We denote by E_{ij} the $n \pm n$ matrix with 1 in position (i, j) and 0 in all other positions.

THEOREM 7.2. *Let $P(\lambda)$ be an $n \pm n$ matrix polynomial of degree k as in (1.1) and $0 \geq h < k$. Let $\mathbf{t}_w$ (resp. $k + \mathbf{t}_v$) be a tuple in canonical form for h (resp. $k \quad h \quad 1$) with length l (resp. l'). Then, $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$ is a subspace of $\mathbb{EDL}(P)$ of dimension $(l + l')n^2 + 1$. A basis of $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$ is given by*

$$\beta := \{L_P(h, \mathbf{t}_w, \mathbf{t}_v, 0, 0)\} \cup \bigcup_{\substack{s=1:l \\ t_1,t_2=1:n}} L_P(h, \mathbf{t}_w, \mathbf{t}_v, X_{s,t_1,t_2}, 0) \cup \bigcup_{\substack{p=1:l' \\ q_1,q_2=1:n}} L_P(h, \mathbf{t}_w, \mathbf{t}_v, 0, Y_{p,q_1,q_2}),$$

where 0 denotes a matrix assignment in which all matrices are zero, X_{s,t_1,t_2} is the matrix assignment for $\mathbf{t}_w$ in which all matrices are 0 except the one in position s which is E_{t_1,t_2}, and Y_{p,q_1,q_2} is the matrix assignment for $\mathbf{t}_v$ in which all matrices are 0 except the one in position p which is E_{q_1,q_2}.

Proof. It follows easily from Lemma 4.7 that $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$ is a vector space generated by the pencils in β. Note that the cardinality of β is $r := (l + l')n^2 + 1$. Now we prove that the pencils in β are linearly independent. Let us denote these pencils by $b_1, \ldots, b_r$, with $b_1 := L_P(h, \mathbf{t}_w, \mathbf{t}_v, 0, 0)|$, and consider the equation

$$\alpha_1 b_1 + \times\times\times + \alpha_r b_r = 0, \quad \alpha_i \ / \ \mathbb{F}. \tag{7.2}$$

We want to show that $\alpha_i = 0$, $i = 1 : r$. Since $P(\lambda)$ is a polynomial of degree k, the matrix A_k is not 0. By Lemma 4.7, the block A_k appears in the same position in the matrix coefficient of the term of degree 1 of each pencil b_i. This implies that

$$\bigcup_{i=1}^{r} \alpha_i = 0. \tag{7.3}$$

On the other hand, taking into account Lemma 4.7 and the definition of the matrix assignments X_{s,t_1,t_2} and Y_{p,q_1,q_2}, it follows that each pencil b_i, $i = 2 : r$, has an entry equal to 1 in a position in which all other pencils have a 0. Thus, (7.2) implies $\alpha_i = 0$ for $i = 2 : r$. From (7.3), we get $\alpha_1 = 0$ and the result follows. □

EXAMPLE 7.3. *Let $P(\lambda)$ be a 2 ± 2 matrix polynomial of degree 3 as in (1.1). According to Theorem 7.2, $S_1^P(0, \mathcal{A}(\;\;3))$ has dimension 5 and a basis for this vector space is $\}L_1(\lambda), L_2(\lambda), L_3(\lambda), L_4(\lambda), L_5(\lambda)|$, where*

$$L_1(\lambda) = \lambda \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_3 & A_2 \\ 0 & A_2 & A_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_0 \end{bmatrix},$$

$$L_2(\lambda) = \lambda \begin{bmatrix} 0 & 0 & E_{11} \\ 0 & A_3 & A_2 \\ E_{11} & A_2 & A_1 \end{bmatrix} + \begin{bmatrix} 0 & E_{11} & 0 \\ E_{11} & A_2 & 0 \\ 0 & 0 & A_0 \end{bmatrix},$$

$$L_3(\lambda) = \lambda \begin{bmatrix} 0 & 0 & E_{12} \\ 0 & A_3 & A_2 \\ E_{12} & A_2 & A_1 \end{bmatrix} + \begin{bmatrix} 0 & E_{12} & 0 \\ E_{12} & A_2 & 0 \\ 0 & 0 & A_0 \end{bmatrix},$$

$$L_4(\lambda) = \lambda \begin{bmatrix} 0 & 0 & E_{21} \\ 0 & A_3 & A_2 \\ E_{21} & A_2 & A_1 \end{bmatrix} + \begin{bmatrix} 0 & E_{21} & 0 \\ E_{21} & A_2 & 0 \\ 0 & 0 & A_0 \end{bmatrix},$$

$$L_5(\lambda) = \lambda \begin{bmatrix} 0 & 0 & E_{22} \\ 0 & A_3 & A_2 \\ E_{22} & A_2 & A_1 \end{bmatrix} - \begin{bmatrix} 0 & E_{22} & 0 \\ E_{22} & A_2 & 0 \\ 0 & 0 & -A_0 \end{bmatrix},$$

with

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \textit{and} \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Observe that $S_1^P(0, \mathcal{A}(-3))$ coincides with the subspace $F_3(P)$ introduced in Definition 2.1.

Next, the largest possible dimension that a Single-$\mathbb{EDL}(P)$ subspace $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$ may have is computed when h, $\mathbf{t}_w$ and $\mathbf{t}_v$ vary. Note that, for each h, taking into account Theorem 7.2, the maximal dimension is attained when the tuples $\mathbf{t}_w$ and $\mathbf{t}_v$ have largest lengths, that is, when $\mathbf{t}_w$ and $k + \mathbf{t}_v$ are maximal tuples according to Definition 6.6. It can easily be seen that the length of a maximal tuple $\mathbf{t}_w$ in canonical form for h is $\lfloor \frac{h^2}{4} \rfloor$. Thus, we have the following result, which follows from the previous observations and Theorem 7.2.

COROLLARY 7.3. *Let $P(\lambda)$ be an $n \times n$ matrix polynomial of degree k as in (1.1). The maximum dimension attained by the vector spaces $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$, when $0 \leq h < k$ and $\mathbf{t}_w$, $k + \mathbf{t}_v$ are canonical tuples for h and $k - h - 1$, respectively, is*

$$\max_{0 \leq h < k} \left\{ \left(\left\lfloor \frac{h^2}{4} \right\rfloor + \left\lfloor \frac{(k-h-1)^2}{4} \right\rfloor \right) n^2 + 1 \right\} = \left\lfloor \frac{(k-1)^2}{4} \right\rfloor n^2 + 1.$$

REMARK 7.1. (Symmetric pencils in $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$ when $P(\lambda)$ is symmetric). *According to Remark 6.1, in order to guarantee that a pencil $T(\lambda) \in S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$ is symmetric when $P(\lambda)$ is, $T(\lambda)$ must be generated by matrix assignments $\mathcal{Y}$ and $\mathcal{Z}$ whose matrices are all symmetric. These symmetric pencils span a subspace* Sym $\left(S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)\right)$ *of $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$. A basis of* Sym $\left(S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)\right)$ *can be obtained in a similar way to the basis of $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$ in Theorem 7.2 with the following obvious modifications for guaranteeing symmetry: impose $t_1 \geq t_2$, $q_1 \geq q_2$, and replace E_{t_1,t_2} (resp. E_{q_1,q_2}) by a matrix with 1 in positions (t_1, t_2) and (t_2, t_1) (resp. (q_1, q_2) and (q_2, q_1)) and 0 in all other positions. So, with the notation of Theorem 7.2, the dimension of* Sym $\left(S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)\right)$ *is $(l + l')\left(\frac{n^2+n}{2}\right) + 1$. Also, the maximum dimension attained by* Sym $\left(S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)\right)$*, when h, $\mathbf{t}_w$, and $\mathbf{t}_v$ vary, can be obtained with a similar argument to that used to deduce Corollary 7.3. This maximum dimension is equal to*

$$\left\lfloor \frac{(k-1)^2}{4} \right\rfloor \left(\frac{n^2+n}{2} \right) + 1.$$

Note that $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$ is isomorphic to the space $\mathbb{F}^{n^2(l+l')+1}$, where l and l' are, as in Theorem 7.2, the lengths of $\mathbf{t}_w$ and $\mathbf{t}_v$, respectively. Next we prove that, when some restrictions on $P(\lambda)$ are considered, almost all the pencils in the vector space $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$ are strong linearizations of the matrix polynomial $P(\lambda)$. To make the notion of "almost all" rigorous we need to assume that the field $\mathbb{F}$ is infinite. With this assumption in mind, in Theorem 7.5, we will mean by "almost all" that the isomorphic images in $\mathbb{F}^{n^2(l+l')+1}$ of the pencils in $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$ which are not strong linearizations of $P(\lambda)$ form a closed nowhere dense set in the Zariski topology (see [15] for details). Note that this condition holds if the set is contained in an (affine) algebraic set on which a non constant multivariable polynomial vanishes. In the particular case when $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, any closed nowhere dense subset of $\mathbb{F}^{n^2(l+l')+1}$ in the Zariski sense is closed nowhere dense when the standard topology on $\mathbb{F}^{n^2(l+l')+1}$ is considered, and, also, has zero Lebesgue measure.

LEMMA 7.4. *Let $P(\lambda)$ be a matrix polynomial of degree k as in (1.1). Let $0 \geq h < k$ and let $\mathbf{t}_w$ and $k+\mathbf{t}_v$ be tuples in canonical form for h and $k \quad h \quad 1$, respectively. Suppose that h is even if A_0 is singular and $k \quad h$ is odd if A_k is singular. Let $L(\lambda) = L_P(h, \mathbf{t}_w, \mathbf{t}_v, X, Y) + \alpha L_P(h, \mathbf{t}_w, \mathbf{t}_v, 0, 0)$ be a pencil in $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$. If $\alpha \not\Leftarrow \quad 1$ and X and Y are nonsingular matrix assignments for $\mathbf{t}_w$ and $\mathbf{t}_v$, respectively, then $L(\lambda)$ is a strong linearization of $P(\lambda)$.*

Proof. Suppose that $\alpha \not\Leftarrow \quad 1$. The pencil $L(\lambda)$ can be written as

$$(\alpha+1)M_{\mathbf{t}_w,\mathbf{t}_v}(X',Y') \quad \lambda M^P_{\mathbf{v}_h} \quad M^P_{\mathbf{w}_h} \left[M^P_{\mathbf{c}_{w_h},\mathbf{c}_{v_h}} M_{rev(\mathbf{t}_w),rev(\mathbf{t}_v)}(rev(X'), rev(Y')) \right) \tag{7.4}$$

where $\mathbf{v}_h = \quad k + \mathbf{w}_{k-h-1}$, and X' and Y' are the matrix assignments obtained from X and Y, respectively, by multiplying each matrix by $(\alpha+1)^{-1}$. By Theorem 5.5, if $M^P_{\mathbf{c}_{w_h}}$, $M^P_{\mathbf{c}_{v_h}}$ and $M_{\mathbf{t}_w,\mathbf{t}_v}(X',Y')$ are nonsingular, then (7.4) is a strong linearization of $P(\lambda)$. Note that $M_{\mathbf{t}_w,\mathbf{t}_v}(X',Y')$ is nonsingular if and only if X' and Y' are nonsingular matrix assignments if and only if X and Y are nonsingular matrix assignments. Also, $M^P_{\mathbf{c}_{w_h}}$ (resp. $M^P_{\mathbf{c}_{v_h}}$) is nonsingular if h (resp. $k \quad h \quad 1$) is even or A_0 (resp. A_k) is nonsingular. □

THEOREM 7.5. *Let $P(\lambda)$ be a matrix polynomial of degree k as in (1.1). Let $0 \geq h < k$ and let $\mathbf{t}_w$ and $k+\mathbf{t}_v$ be tuples in canonical form for h and $k \quad h \quad 1$, respectively. Suppose that h is even if A_0 is singular and $k \quad h$ is odd if A_k is singular. Then, almost all the pencils in the vector space $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$ are strong linearizations of $P(\lambda)$.*

Proof. Let $L(\lambda) = L_P(h, \mathbf{t}_w, \mathbf{t}_v, X, Y) + \alpha L_P(h, \mathbf{t}_w, \mathbf{t}_v, 0, 0)$ be a pencil in $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$. It follows from Lemma 7.4 that $L(\lambda)$ is a strong linearization of $P(\lambda)$, unless $\alpha = \quad 1$ or any of the matrix assignments X and Y is singular. Thus, letting l and l' be the lengths of $\mathbf{t}_w$ and $\mathbf{t}_v$, respectively, the set containing the pencils in $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$ that are not strong linearizations of $P(\lambda)$ is isomorphic to a subset of

$$\mathcal{X} := \Big)\} \quad 1| \pm \mathbb{F}^{n^2(l+l')} \Big(\lfloor \Big]_{i=1}^{r} \mathbb{F}^{1+n^2(l+l'-i)} \pm S \pm \mathbb{F}^{n^2(i-1)} \Big\lceil,$$

where $S \in \mathbb{F}^{n^2}$ is the isomorphic image of the set of singular matrices in $M_n(\mathbb{F})$ and r is the sum of the number of occurrences of the indices 0 and $\quad k$ in $\mathbf{t}_w$ and $\mathbf{t}_v$, respectively. The set $\mathcal{X}$ is an (affine) algebraic set in $\mathbb{F}^{n^2(l+l')+1}$ that coincides with the set of roots of the non constant polynomial $(x_0+1)det(X_{l+l'-r+1}) \times\!\!\times\!\!\times det(X_{l+l'}) \,/\, \mathbb{F}[x_0, x_1^1, \ldots, x_{n^2}^1, \ldots, x_1^{l+l'}, \ldots, x_{n^2}^{l+l'}]$, where X_i, $i = l+l' \quad r+1, \ldots, l+l'$, is an $n \pm n$ matrix whose entries are the variables x_j^i, $j = 1 : n^2$. The set $\mathcal{X}$ is closed nowhere dense in $\mathbb{F}^{n^2(l+l')+1}$ in the Zariski topology, which implies the claim. □

Note that, from Theorem 7.5, we can conclude that, when $P(\lambda)$ is singular of odd degree, by choosing h even, we get a subspace $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$, for every possible choice of $\mathbf{t}_w$ and $\mathbf{t}_v$, in which almost all the pencils are strong linearizations of $P(\lambda)$.

We close this section with some remarks. First we note that two distinct pairs of tuples $\mathbf{t}_w$ and $\mathbf{t}_v$ associated with a fixed h do not necessarily produce two Single-$\mathbb{EDL}(P)$ subspaces with trivial intersection. Consider $0 \geq h < k$ and tuples $\mathbf{t}_w$ and $\mathbf{t}_v$, where at least one of $\mathbf{t}_w$ or $k+\mathbf{t}_v$ contains the zero index. If $\mathbf{t}_w^1$ and $k+\mathbf{t}_v^1$ are the tuples obtained from $\mathbf{t}_w$ and $k+\mathbf{t}_v$ by deleting one or more indices equal to zero in at least one of them (note that deleting a zero index in $k+\mathbf{t}_v$ corresponds to the deletion of a $\quad k$ index in $\mathbf{t}_v$), then $L_P(h, \mathbf{t}_w^1, \mathbf{t}_v^1, X^1, Y^1)$ coincides with $L_P(h, \mathbf{t}_w, \mathbf{t}_v, X, Y)$, when X and Y are matrix assignments for $\mathbf{t}_w$ and $\mathbf{t}_v$, respectively, with the property that the matrices corresponding to the positions of the deleted indices in $\mathbf{t}_w$ and $\mathbf{t}_v$, respectively, are I_n, and X^1 and Y^1 are matrix assignments for $\mathbf{t}_w^1$ and

$\mathbf{t}_v^1$ obtained from the matrix assignments X and Y by deleting the matrices in the positions of the deleted indices in $\mathbf{t}_w$ and $\mathbf{t}_v$. Thus, any pencil in $S_1^P(h, \mathbf{t}_w^1, \mathbf{t}_v^1)$ is in $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$ and, therefore, we can conclude that $S_1^P(h, \mathbf{t}_w^1, \mathbf{t}_v^1)$ is a subspace of $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$. These ideas are illustrated by the following example.

EXAMPLE 7.4. *Let $P(\lambda)$ be a matrix polynomial of degree $k = 5$ as in (1.1). Let $h = 4$, $\mathbf{t}_w = (0 : 2, 0)$, and $\mathbf{t}_v = \mathcal{A}$ The subspace $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$ is given in Example 7.2. Now consider $\mathbf{t}_w^1 = (1 : 2)$ and $\mathbf{t}_v^1 = \mathcal{A}$ Note that we have deleted both zeros in $\mathbf{t}_w$. Then, in order to obtain $S_1^P(h, \mathbf{t}_w^1, \mathbf{t}_v^1)$ we replace in the pencils in $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$ the blocks X_1 and X_4, corresponding to the zero indices in $\mathbf{t}_w$, by γI_n and get the pencils of the form*

$$\lambda \begin{bmatrix} \gamma A_5 & 0 & 0 & 0 & 0 \\ 0 & \gamma A_3 & X_3 & X_2 & \gamma I_n \\ 0 & X_3 & \gamma A_1 & \gamma I_n & 0 \\ 0 & X_2 & \gamma I_n & 0 & 0 \\ 0 & \gamma I_n & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma A_4 & \gamma A_3 & X_3 & X_2 & \gamma I_n \\ \gamma A_3 & \gamma A_2 & \gamma A_1 & \gamma I_n & 0 \\ X_3 & \gamma A_1 & \gamma A_0 & 0 & 0 \\ X_2 & \gamma I_n & 0 & 0 & 0 \\ \gamma I_n & 0 & 0 & 0 & 0 \end{bmatrix}$$

where X_i / $M_n(\mathbb{F})$ and γ / $\mathbb{F}$. It is clear that $S_1^P(h, \mathbf{t}_w^1, \mathbf{t}_v^1) \in S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$ and that the dimension of $S_1^P(h, \mathbf{t}_w^1, \mathbf{t}_v^1)$ is strictly less than the dimension of $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$ since the matrix coefficients of the general pencil in $S_1^P(h, \mathbf{t}_w^1, \mathbf{t}_v^1)$ contain fewer free variables than those in $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$.

Note that the space $S_1^P(h, \mathbf{t}_w^1, \mathbf{t}_v^1)$ presented in the previous example corresponds to a Single-$\mathbb{EDL}(P)$ subspace constructed from a block-symmetric FPR which is not in the standard basis of $\mathbb{DL}(P)$. However, $S_1^P(h, \mathbf{t}_w^1, \mathbf{t}_v^1)$ is a subspace of $F_1(P)$ introduced in Definition 2.1. Next we include an example of a Single-$\mathbb{EDL}(P)$ subspace constructed from a block-symmetric FPR not in the standard basis of $\mathbb{DL}(P)$ and which is not a subspace of $F_m(P)$ for any m.

EXAMPLE 7.5. *If $P(\lambda)$ is a matrix polynomial of degree $k = 5$ as in (1.1), $h = 4$, $\mathbf{t}_w = (2, 0)$, and $\mathbf{t}_v = \mathcal{A}$(as in Example 7.1), then the subspace $S_1^P(h, \mathbf{t}_w, \mathbf{t}_v)$ is given by the set of pencils of the form (7.1) with X_i / $M_n(\mathbb{F})$ and γ / $\mathbb{F}$.*

7.2. Proof of Theorem 2.2. Here we pay special attention to the subspaces discussed in Section 2. We show that they are particular cases of Single-$\mathbb{EDL}(P)$ subspaces and prove Theorem 2.2.

PROPOSITION 7.6. *Let $P(\lambda)$ be a matrix polynomial of degree k as in (1.1) and $1 \geq m \geq k$. The family $F_m(P)$ of pencils, introduced in Definition 2.1, is the Single-$\mathbb{EDL}(P)$ subspace $S_1^P(k \quad h, \mathbf{t}_{k-m}, \quad k + \mathbf{t}_{m-1})$.*

As mentioned in Remark 6.2, the pencils in the standard basis for $\mathbb{DL}(P)$, where $P(\lambda)$ is a matrix polynomial of degree k, are those block-symmetric FPR of the form $L_P(k \quad m, \mathbf{t}_{k-m}, \quad k + \mathbf{t}_{m-1})$, $m = 1 : k$, where t_{k-m} and t_{m-1} are the maximal index tuples in canonical form for $k \quad m$ and $m \quad 1$, respectively (see Definition 6.6). Namely, the mth pencil $D_m(\lambda, P)$ in the standard basis of $\mathbb{DL}(P)$ can be expressed as follows:

$$M^P_{\mathbf{t}_{k-m}, -k+\mathbf{t}_{m-1}}, (\lambda M^P_{-k+\mathbf{w}_{m-1}} \quad M^P_{\mathbf{w}_{k-m}}) M^P_{\mathbf{c}_{w_{k-m}}, -k+\mathbf{c}_{w_{m-1}}} M^P_{rev(\mathbf{t}_{k-m}), rev(-k+\mathbf{t}_{m-1})}.$$

Before we prove the proposition, we give a technical lemma regarding the positions in the matrix coefficients of $D_m(\lambda, P)$ of the blocks associated with the index tuples that characterize $D_m(\lambda, P)$ as an FPR.

Given an index tuple $\mathbf{t}$ of the form $(\mathbf{t}_{w_h}, \mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h}))$ as in Theorem 6.11, where $0 \geq h < k$, and a matrix assignment X for $\mathbf{t}$, we have $M_{\mathbf{t}}(X) = I_{n(k-h-1)} \otimes B$, where B / $M_{n(h+1)}(\mathbb{F})$. Similarly, if $\mathbf{t}' := (\mathbf{t}_{w_h}, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h}))$ and X' is a matrix assignment for $\mathbf{t}'$, then $M_{\mathbf{t}'}(X') = I_{n(k-h)} \otimes C$, where C / $M_{nh}(\mathbb{F})$. Here we study the positions in

B and C of the blocks in the matrix assignments for $\mathbf{t}$ and $\mathbf{t}'$, respectively. This justifies the following notation.

Let $\mathbf{t}$ be an index tuple of nonnegative integers and let X be a matrix assignment for $\mathbf{t}$. We denote by $M^s_{\mathbf{t}}(X)$, or simply $M^{P,s}_{\mathbf{t}}$ if X is the trivial matrix assignment for $\mathbf{t}$ associated with a matrix polynomial $P(\lambda)$ of degree k as in (1.1), the submatrix of $M_{\mathbf{t}}(X)$ corresponding to the block-rows and block-columns indexed by $\}s : k|$.

LEMMA 7.7. *Let $P(\lambda)$ be a matrix polynomial of degree k as in (1.1) and $0 \geq h < k$. Let $\mathbf{w}_h$ be the admissible tuple associated with h, $\mathbf{c}_{w_h}$ be the symmetric complement of $\mathbf{w}_h$, and $\mathbf{t}_h$ be the maximal tuple in canonical form for h. Let X be a matrix assignment for $\mathbf{t}_h$. The following conditions hold:*

1. *The block-positions in $M^{k-h}_{\mathbf{t}_h}(X)M^{P,k-h}_{\mathbf{w}_h,\mathbf{c}_{w_h}}M^{k-h}_{rev(\mathbf{t}_h)}(rev(X))$ corresponding to the indices in $(\mathbf{w}_h, \mathbf{c}_{w_h})$ are (i, i), with i / $\}1 : \rceil\frac{h+1}{2}\{|$, and $(i, i+1)$, $(i+1, i)$, with i / $\}1 : \rceil\frac{h}{2}\{|$.*
2. *The block-positions in $M^{k-h+1}_{\mathbf{t}_h}(X)M^{P,k-h+1}_{\mathbf{c}_{w_h}}M^{k-h+1}_{rev(\mathbf{t}_h)}(rev(X))$ corresponding to the indices in $\mathbf{c}_{w_h}$ are (i, i), with i / $\}1 : \rceil\frac{h}{2}\{|$.*
3. *If the block-position in $M^{k-h}_{\mathbf{t}_h}(X)M^{P,k-h}_{\mathbf{w}_h,\mathbf{c}_{w_h}}M^{k-h}_{rev(\mathbf{t}_h)}(rev(X))$ or in $M^{k-h+1}_{\mathbf{t}_h}(X)M^{P,k-h+1}_{\mathbf{c}_{w_h}}M^{k-h+1}_{rev(\mathbf{t}_h)}(rev(X))$ corresponding to an index in $\mathbf{t}_h$ is (i, j), then $i > j$ and the block-position corresponding to the same index in $rev(\mathbf{t}_h)$ is (j, i).*
4. *If the block-position in $M^{k-h}_{\mathbf{t}_h}(X)M^{P,k-h}_{\mathbf{w}_h,\mathbf{c}_{w_h}}M^{k-h}_{rev(\mathbf{t}_h)}(rev(X))$ corresponding to an index in $\mathbf{t}_h$ is (i, j), then the position in $M^{k-h+1}_{\mathbf{t}_h}(X)M^{P,k-h+1}_{\mathbf{c}_{w_h}}M^{k-h+1}_{rev(\mathbf{t}_h)}(rev(X))$ corresponding to the same index is $(i\ \ 1, j)$.*

Proof. Here we prove the case when h is even, that is, $\mathbf{w}_h$ is an admissible tuple of index 0. The case h odd can be proven similarly. We will denote by $\cup$ the trivial matrix assignment for $(\mathbf{w}, \mathbf{c}_w)$ associated with $P(\lambda)$ and by Z any $n\pm n$ block in $\cup$. Denote by Y_l, $l = 0 : h\ \ 2$, the block in X corresponding to the position $l+1$ in the string $(0 : h\ \ 2)$ of $\mathbf{t}_h$ (and therefore, the block assigned to the position in the subtuple $(rev(0 : h\ \ 2))$ of $rev(\mathbf{t}_h)$ where the same index occurs). Recall that $\mathbf{t}_h$ is of the form (6.2) with $a_i = 0$ for all i. Consider the following matrix assignments for subtuples of $\mathbf{t}_h$: $Y = (Y_0, \ldots, Y_{h-2})$ and $\bar{X}$ such that $X = (Y, \bar{X})$. Also consider the following block-matrices: $Y' = \rceil\ Y_{h-2}\ \ \times\times\times\ \ Y_1\ \lfloor$ and $Y'' = \rceil\ Y_{h-3}\ \ \times\times\times\ \ Y_1\ \lfloor$.

The proof is by induction on h. If $h = 0$, we have $\mathbf{w}_h = (0)$ and $\mathbf{c}_{w_h}$ and $\mathbf{t}_h$ empty. In this case the claim holds trivially. Let $h = 2$. Then $\mathbf{w}_h = (1 : 2, 0)$, $\mathbf{c}_{w_h} = (1)$ and $\mathbf{t}_h = (0)$. By direct multiplication, we get

$$M^{k-2}_{(0)}(Y_0)M^{P,k-2}_{(1:2,0),1)}M^{k-2}_{(0)}(Y_0) = \left\rceil \begin{array}{ccc} Z & Z & Y_0 \\ Z & Z & 0 \\ Y_0 & 0 & 0 \end{array} \right\lfloor,$$

$$M^{k-1}_{(0)}(Y_0)M^{P,k-1}_{(1)}M^{k-1}_{(0)}(Y_0) = \left\rceil \begin{array}{cc} Z & Y_0 \\ Y_0 & 0 \end{array} \right\lfloor$$

which implies the result. Assume that $h \sim 4$. Let $\mathbf{s}_h := (\mathbf{t}_h, \mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_h))$. Note that

$$\mathbf{s}_h \subset \mathbf{s}'_h := ((0 : h\ \ 2, h\ \ 1 : h), \mathbf{s}_{h-2}, (h\ \ 1, rev(0 : h\ \ 2))).$$

Moreover, if σ is the allowed permutation transforms $\mathbf{s}_h$ into $\mathbf{s}'_h$, then $\sigma(X, \cup, rev(X))$ has the form

$$(Y, Z, Z, \bar{X}, Z, \ldots, Z, rev(\bar{X}), Z, rev(Y)),$$

where each block Z is the trivial matrix assignment, associated with $P(\lambda)$, for the index in $\mathbf{s}'_h$ in the corresponding position. Let

$$M^{k-h+2}_{\mathbf{s}_{h-2}} := M^{k-h+2}_{\mathbf{t}_{h-2}}(\bar{X})M^{P,k-h+2}_{\mathbf{w}_{h-2},\mathbf{c}_{w_{h-2}}}M^{k-h+2}_{rev(\mathbf{t}_{h-2})}(rev(\bar{X})) = \begin{bmatrix} A & B \\ C & M^{k-h+3}_{\mathbf{s}_{h-2}} \end{bmatrix}$$

where A, B, C are $n \pm n$ blocks. Then, the product $M^{P,k-h}_{h-1:h}(I_{2n} \otimes M^{k-h+2}_{\mathbf{s}_{h-2}})M^{P,k-h}_{h-1}$ has the form

$$\begin{bmatrix} Z & I_n & 0 & 0 \\ Z & 0 & I_n & 0 \\ I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n(h-2)} \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & A & B \\ 0 & 0 & C & M^{k-h+3}_{\mathbf{s}_{h-2}} \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & Z & I_n & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_{n(h-2)} \end{bmatrix}$$
$$= \begin{bmatrix} Z & Z & I_n & 0 \\ Z & A & 0 & B \\ I_n & 0 & 0 & 0 \\ 0 & C & 0 & M^{k-h+3}_{\mathbf{s}_{h-2}} \end{bmatrix}.$$

This implies that

$$\begin{aligned} M^{k-h}_{\mathbf{s}_h} &:= M^{k-h}_{\mathbf{t}_h}(X)M^{P,k-h}_{\mathbf{w}_h,\mathbf{c}_{w_h}}M^{k-h}_{rev(\mathbf{t}_h)}(rev(X)) \\ &= M^{k-h}_{0:h-2}(Y)M^{P,k-h}_{h-1:h}(I_{2n} \otimes M^{k-h+2}_{\mathbf{s}_{h-2}})M^{P,k-h}_{h-1}M^{k-h}_{rev(0:h-2)}(rev(Y)). \end{aligned}$$

has the form

$$\begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & (Y')^{\mathcal{B}} & I_{n(h-2)} \\ 0 & 0 & Y_0 & 0 \end{bmatrix} \begin{bmatrix} Z & Z & I_n & 0 \\ Z & A & 0 & B \\ I_n & 0 & 0 & 0 \\ 0 & C & 0 & M^{k-h+3}_{\mathbf{s}_{h-2}} \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & Y' & Y_0 \\ 0 & 0 & I_{n(h-2)} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} Z & Z & Y' & Y_0 \\ Z & A & B & 0 \\ (Y')^{\mathcal{B}} & C & M^{k-h+3}_{\mathbf{s}_{h-2}} & 0 \\ Y_0 & 0 & 0 & 0 \end{bmatrix}.$$

By the inductive hypothesis, the first claim follows.

Let $\mathbf{p}_h := (\mathbf{t}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_h))$. Note that

$$\mathbf{p}_h \subset \mathbf{p}'_h := ((0 : h-2, h-1), \mathbf{p}_{h-2}, rev(0 : h-2)).$$

Moreover, if μ is the allowed permutation that transforms $\mathbf{p}_h$ into $\mathbf{p}'_h$ then $\mu(X, \bar{\cup}, rev(X))$, where $\bar{\cup}$ is the trivial matrix assignment for $\mathbf{c}_{w_h}$ associated with $P(\lambda)$, has the form

$$(Y, Z, \bar{X}, Z, \ldots, Z, rev(\bar{X}), rev(Y)),$$

where each block Z is the trivial matrix assignment, associated with $P(\lambda)$, for the index in $\mathbf{p}'_h$ in the corresponding position. Let

$$M^{k-h+3}_{\mathbf{p}_{h-2}} := M^{k-h+3}_{\mathbf{t}_{h-2}}(\bar{X})M^{P,k-h+3}_{\mathbf{c}_{w_{h-2}}}M^{k-h+3}_{rev(\mathbf{t}_{h-2})}(rev(\bar{X})) = \begin{bmatrix} D & E \\ F & M^{k-h+4}_{\mathbf{p}_{h-2}} \end{bmatrix},$$

where D, E, F are $n \times n$ blocks. The product $M_{h-1}^{P,k-h+1}(I_{2n} \otimes M_{\mathbf{p}_{h-2}}^{k-h+3})$ is a matrix of the form

$$\begin{bmatrix} Z & I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_{n(h-3)} \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & D & E \\ 0 & 0 & F & M_{\mathbf{p}_{h-2}}^{k-h+4} \end{bmatrix} = \begin{bmatrix} Z & I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \\ 0 & 0 & D & E \\ 0 & 0 & F & M_{\mathbf{p}_{h-2}}^{k-h+4} \end{bmatrix}.$$

Thus, the product

$$\begin{aligned} M_{\mathbf{p}_h}^{k-h+1} &:= M_{\mathbf{t}_h}^{k-h+1}(X) M_{\mathbf{c}_{w_h}}^{P,k-h+1} M_{rev(\mathbf{t}_h)}^{k-h+1}(rev(X)) \\ &= M_{0:h-2}^{k-h+1}(Y) M_{h-1}^{P,k-h+1}(I_{2n} \otimes M_{\mathbf{p}_{h-2}}^{k-h+3}) M_{rev(0:h-2)}^{k-h+1}(rev(Y)) \end{aligned}$$

is a matrix of the form

$$\begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & Y_{h-2} & I_n & 0 \\ 0 & (Y'')^{\mathcal{B}} & 0 & I_{n(h-3)} \\ 0 & Y_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Z & I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \\ 0 & 0 & D & E \\ 0 & 0 & F & M_{\mathbf{p}_{h-2}}^{k-h+4} \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & Y_{h-2} & Y'' & Y_0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & I_{n(h-3)} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} Z & Y_{h-2} & Y'' & Y_0 \\ Y_{h-2} & D & E & 0 \\ (Y'')^{\mathcal{B}} & F & M_{\mathbf{p}_{h-2}}^{k-h+4} & 0 \\ Y_0 & 0 & 0 & 0 \end{bmatrix}.$$

By the inductive hypothesis, the second claim follows.

The last two claims also follow by applying the induction hypothesis and by taking into account that the blocks in the matrices $M_{\mathbf{s}_h}^{k-h}$ and $M_{\mathbf{p}_h}^{k-h+1}$ corresponding to the indices in the subtuple $(0 : h-2)$ of $\mathbf{t}_w$ appear in the first block-column. □

From Lemma 7.7 a corresponding result can be obtained for negative tuples of the form $-k+(\mathbf{t}_h, \mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_h))$ by noting that $M_{-k+\mathbf{s}_h}(X) = RM_{\mathbf{s}_h}(X)R$ and $M_{-k+\mathbf{p}_h}(X') = RM_{\mathbf{p}_h}(X')R$, where $\mathbf{s}_h$ and $\mathbf{p}_h$ are as in the proof of Lemma 7.7, X and X' are matrix assignments for $\mathbf{s}_h$ and $\mathbf{p}_h$, respectively, and

$$R := \begin{bmatrix} 0 & \dots & I_n \\ \vdots & \iddots & \vdots \\ I_n & \dots & 0 \end{bmatrix}. \tag{7.5}$$

Proof of Proposition 2.2: Next we show that, for a fixed $m \in \{1 : k\}$, the pencils in the vector space $F_m(P)$, obtained from $\alpha D_m(\lambda, P)$, are of the form

$$L_P(k-m, \mathbf{t}_{k-m}, -k+\mathbf{t}_{m-1}, \mathcal{Y}, \mathcal{Z}) + (\alpha-1) L_P(k-m, \mathbf{t}_{k-m}, -k+\mathbf{t}_{m-1}, 0, 0), \tag{7.6}$$

where $\alpha \in \mathbb{F}$ and $\mathcal{Y}$ and $\mathcal{Z}$ denote arbitrary matrix assignments for $\mathbf{t}_{k-m}$ and $-k+\mathbf{t}_{m-1}$, respectively, proving the claim. Note that it is enough to show that the blocks in the coefficient matrices of $D_m(\lambda, P)$ that are replaced by arbitrary matrices (as described in Definition 2.1) correspond to the indices in $\mathbf{t}_{k-m}$ and $-k+\mathbf{t}_{m-1}$, and that the blocks corresponding to the same index in $\mathbf{t}_{k-m}$ and $rev(\mathbf{t}_{k-m})$ (resp. $-k+\mathbf{t}_{m-1}$ and $-k+rev(\mathbf{t}_{m-1})$) are the same.

Using the notation in Section 2, we have $D_m(\lambda, P) = \lambda X_m - X_{m-1}$, where $X_m = diag(L_m(P), -U_{k-m}(P))$ and $X_{m-1} = diag(L_{m-1}(P), -U_{k-m+1}(P))$. Note that $C_{11} :=$

$L_m(P)$ and $C_{22} := \quad U_{k-m}(P)$ are $m \pm m$ and $(k \quad m) \pm (k \quad m)$ block-matrices, respectively; also, $D_{11} := L_{m-1}(P)$ and $D_{22} := \quad U_{k-m+1}(P)$ are $(m \quad 1) \pm (m \quad 1)$ and $(k \quad m+1) \pm (k \quad m+1)$ block-matrices, respectively. By Theorem 5.3, D_{11} contains the blocks in the matrix assignment for the tuple $\quad k + (\mathbf{t}_{m-1}, \mathbf{c}_{w_{m-1}}, rev(\mathbf{t}_{m-1}))$ and D_{22} contains the blocks in the matrix assignment for the tuple $(\mathbf{t}_{k-m}, \mathbf{w}_{k-m}, \mathbf{c}_{w_{k-m}}, rev(\mathbf{t}_{k-m}))$. Similarly, C_{11} contains the blocks in the matrix assignment for the tuple $\quad k + (\mathbf{t}_{m-1}, \mathbf{w}_{m-1}, \mathbf{c}_{w_{m-1}}, rev(\mathbf{t}_{m-1}))$ and C_{22} contains the blocks in the matrix assignment for the tuple $(\mathbf{t}_{k-m}, \mathbf{c}_{w_{k-m}}, rev(\mathbf{t}_{k-m}))$.

Taking into account the form of the matrices X_m and X_{m-1}, it follows that the not identically zero blocks in D_{22} are located in X_{m-1} in the positions given by the set

$$S_1 = \}(m \quad 1+i, m \quad 1+j),\ i = 1 : k \quad m+1,\ j = 1 : k \quad m+2 \quad i| \,,$$

while the not identically zero blocks in C_{22} are located in X_m in the positions given by the set

$$S_2 = \}(m+i, m+j),\ i = 1 : k \quad m,\ j = 1 : k \quad m+1 \quad i| \,.$$

Note that S_1 can also be expressed as the union of the two sets

$$\Big\}(m \quad 1+i, m \quad 1+i),\ i = 1 : \Big\rceil \frac{k \quad m+1}{2} \Big\{\Big[$$

and

$$\}(m \quad 1+i, m \quad 1+j), (m \quad 1+j, m \quad 1+i),\ i = 2 : k \quad m+1,\ j = 1 : \min\}i \quad 1, k \quad m+2 \quad i| \,| \,.$$

In item 1 of Lemma 7.7 we claim that the nonzero blocks in D_{22} corresponding to the indices in the tuple $(\mathbf{w}_{k-m}, \mathbf{c}_{w_{k-m}})$ are located in X_{m-1} in the positions given by the set

$$\begin{aligned} S'_1 = & \Big\}(m \quad 1+i, m+i),\ (m+i, m \quad 1+i),\ i = 1 : \Big\rceil \frac{k \quad m}{2} \Big\{\Big[\\ & \lfloor \Big\}(m \quad 1+i, m \quad 1+i),\ i = 1 : \Big\rceil \frac{k \quad m+1}{2} \Big\{\Big[\,. \end{aligned}$$

Thus, the positions in X_{m-1} occupied by the blocks corresponding to the indices in $(\mathbf{t}_{k-m}, rev(\mathbf{t}_{k-m}))$ are given by $S_1 \quad S'_1$. Bearing in mind the item 2 of Lemma 7.7, similar observations can be done for C_{22}. Taking now into account items 3 and 4 of Lemma 7.7, we get that the matrices obtained from C_{22} and D_{22} by replacing some blocks by arbitrary matrices, as indicated in Definition 2.1, coincide with the corresponding matrices in the pencil (7.6), for some assignment $\mathcal{Y}$. Because of the comment after Lemma 7.7, similar results can be obtained for D_{11} and C_{11}.

Proof of Theorem 2.2: Since by Proposition 7.6, $F_m(P)$ is a Single-$\mathbb{EDL}(P)$ subspace, taking into account Theorem 7.2, $F_m(P)$ is a vector space of the claimed dimension, proving the first claim in the theorem. Note that the tuples $\mathbf{t}_{k-m}$ and $\mathbf{t}_{m-1}$ have lengths $\Big\rfloor \frac{(k-m)^2}{4}$ and $\Big\rfloor \frac{(m-1)^2}{4}$, respectively.

Now we show the second claim, that is, a pencil in $F_m(P)$ satisfying the four conditions in Theorem 2.2 is a strong linearization of $P(\lambda)$. We first observe that the blocks corresponding to the indices 0 in the tuple $\mathbf{t}_{k-m}$ must lie among the positions occupied by the matrix

A_0 in the matrix coefficients X_m and X_{m-1} of $D_m(\lambda, P)$. Lemma 7.7 below gives the positions occupied by the block $-A_0$ corresponding to the 0 indices in $(\mathbf{w}_{k-m}, \mathbf{c}_{w_{k-m}})$. Items 3 and 4 of this lemma tell us that the index 0 in $\mathbf{t}_{k-m}$ and the same index in $rev(\mathbf{t}_{k-m})$ occupy symmetric positions. Thus, we conclude that the blocks corresponding to the indices 0 in $\mathbf{t}_{k-m}$ lie in X_m in positions $(m+i, k-i+1), (k-i+1, m+i,), i = \lceil \frac{k-m}{2} \rceil + 1 : k-m$, and lie in X_{m-1} in positions $(m+i, k-i), (k-i, m+i), i = \lceil \frac{k-m}{2} \rceil + 1 : k-m$. Similarly, it can be seen that the positions in X_{m-1} and X_m of the blocks corresponding to the indices $-k$ in $-k + \mathbf{t}_{m-1}$ are those to which the blocks $C_{i,m-i}$, $i = 1 : \lfloor \frac{m-1}{2} \rfloor$, are assigned according to Definition 2.1. Now, because of Theorem 5.5, the second claim in Theorem 2.2 follows. Note that, if $\alpha \neq 0$, the pencil in (7.6) is a nonzero multiple of a GFPR.

8. Conclusions and future work. In this paper, we extend the family of Fiedler pencils with repetition (FPR) associated with a matrix polynomial $P(\lambda)$ as in (1.1), which was introduced in [32], to a much larger family of pencils that we call the generalized Fiedler pencils with repetition (GFPR) and we give sufficient conditions for a GFPR to be a strong linearization of $P(\lambda)$. Additionally, we identify a subfamily of block-symmetric GFPR which generates a vector space $\mathbb{EDL}(P)$ that we call the extended $\mathbb{DL}(P)$ since the k-dimensional $\mathbb{DL}(P)$, introduced and studied in [17, 22], is one of its subspaces. We construct numerous subspaces of $\mathbb{EDL}(P)$, called Single-$\mathbb{EDL}(P)$ subspaces, with the property that almost all of their elements are strong linearizations of $P(\lambda)$. In particular, the largest dimension of such subspaces is $d(k,n) = \lfloor \frac{(k-1)^2}{4} \rfloor n^2 + 1$, which is much larger than the dimension of $\mathbb{DL}(P)$. If $P(\lambda)$ is regular or singular with odd degree, there always exist Single-$\mathbb{EDL}(P)$ spaces with the property that almost all of its pencils are strong linearizations of $P(\lambda)$, in stark contrast with the situation for $\mathbb{DL}(P)$ in which no pencil is a linearization of $P(\lambda)$ when $P(\lambda)$ is singular. We are presently studying in [5] strong linearizations contained in the Single-$\mathbb{EDL}(P)$ subspaces that preserve the sign characteristic of $P(\lambda)$ when it is Hermitian and has real eigenvalues [14]. Another line of future research is to study how to construct from the pencils in the Single-$\mathbb{EDL}(P)$ subspaces of strong linearizations that are palindromic, alternating, or skew-symmetric when $P(\lambda)$ has any of these structures. Moreover, it is still an open question to compute the dimension of $\mathbb{EDL}(P)$, as well as to establish if it contains subspaces of (almost all) strong linearizations with dimension larger than $d(k,n)$.

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