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NON-ORIENTABLE LAGRANGIAN COBORDISMS BETWEEN LEGENDRIAN KNOTS

ORSOLA CAPOVILLA-SEARLE AND LISA TRAYNOR

Abstract. In the symplectization of standard contact 3-space, $\mathbb{R} \times \mathbb{R}^3$, it is known that an orientable Lagrangian cobordism between a Legendrian knot and itself, also known as an orientable Lagrangian endocobordism for the Legendrian knot, must have genus 0. We show that any Legendrian knot has a non-orientable Lagrangian endocobordism, and that the cross-cap genus of such a non-orientable Lagrangian endocobordism must be a positive multiple of 4. The more restrictive exact, non-orientable Lagrangian endocobordisms do not exist for any exactly fillable Legendrian knot but do exist for any stabilized Legendrian knot. Moreover, the relation defined by exact, non-orientable Lagrangian cobordism on the set of stabilized Legendrian knots is symmetric and defines an equivalence relation, a contrast to the non-symmetric relation defined by orientable Lagrangian cobordisms.

1. Introduction

Smooth cobordisms are a common object of study in topology. Motivated by ideas in symplectic field theory, [21], Lagrangian cobordisms that are cylindrical over Legendrian submanifolds outside a compact set have been an active area of research interest. Throughout this paper, we will study Lagrangian cobordisms in the symplectization of the standard contact $\mathbb{R}^3$, namely the symplectic manifold $(\mathbb{R}\times\mathbb{R}^3, d(e^t\alpha))$ where $\alpha = dz - y dx$, that coincide with the cylinders $\mathbb{R}\times\Lambda_+$ (respectively, $\mathbb{R}\times\Lambda_-$) when the $\mathbb{R}$-coordinate is sufficiently positive (respectively, negative). Our focus will be on non-orientable Lagrangian cobordisms between Legendrian knots Λ_+ and Λ_- and non-orientable Lagrangian endocobordisms, which are non-orientable Lagrangian cobordisms with $\Lambda_+ = \Lambda_-$.

Smooth endocobordisms in $\mathbb{R} \times \mathbb{R}^3$ without the Lagrangian condition are abundant: for any smooth knot $K \subset \mathbb{R}^3$, and an arbitrary $j \geq 0$, there is a smooth 2-dimensional orientable submanifold M of genus j such that M agrees with the cylinder $\mathbb{R}\times K$ when the $\mathbb{R}$ coordinate lies outside an interval $[T_-, T_+]$; the analogous statement holds for non-orientable M and cross-cap genus[1] when $j > 0$. For any Legendrian knot Λ, it is easy to construct an orientable Lagrangian endocobordism of genus 0, namely the trivial Lagrangian cylinder $\mathbb{R} \times \Lambda$. In fact, with the added Lagrangian condition, *orientable* Lagrangian endocobordisms must be concordances:

[1] the number of real projective planes in a connected sum decomposition

Theorem (Chantraine, [8]). *For any Legendrian knot Λ, any orientable, Lagrangian endocobordism for Λ must have genus 0.*

Non-orientable Lagrangian endocobordisms also exist and have topological restrictions:

Theorem 1.1. *For an arbitrary Legendrian knot Λ, there exists a non-orientable Lagrangian endocobordism for Λ of cross-cap genus g if and only if $g \in 4\mathbb{Z}^+$.*

Theorem 1.1 is proved in Theorem 3.2 and Theorem 3.3. The fact that the cross-cap genus of a non-orientable Lagrangian endocobordism must be a positive multiple of 4 follows from a result of Audin about the obstruction to the Euler characteristic for closed, Lagrangian submanifolds in $\mathbb{R}^4$, [1]. It is easy to construct *immersed* Lagrangian endocobordisms; the existence of the desired embedded endocobordisms follows from Lagrangian surgery, as developed, for example, by Polterovich in [38].

Of special interest are Lagrangian cobordisms that satisfy an additional "exactness" condition. Exactness is known to be quite restrictive: by a foundational result of Gromov, [30], there are no closed, exact Lagrangian submanifolds in $\mathbb{R}^{2n}$ with its standard symplectic structure. The non-closed trivial Lagrangian cylinder $\mathbb{R} \times \Lambda$ is exact, and Section 2 describes some general methods to construct exact Lagrangian cobordisms. In contrast to Theorem 1.1, there are some Legendrians that do not admit *exact*, non-orientable Lagrangian endocobordisms:

Theorem 1.2. *There does not exist an exact, non-orientable Lagrangian endocobordism for any Legendrian knot Λ that is exactly orientably or non-orientably fillable.*

A Legendrian knot Λ is exactly fillable if there exists an exact Lagrangian cobordism that is cylindrical over Λ at the positive end and does not intersect $\{T_-\} \times \mathbb{R}^3$, for $T_- \ll 0$; precise definitions can be found in Section 2. Theorem 1.2 is proved in Section 4; it follows from the Seidel Isomorphism (Theorem 4.1), which relates the topology of a filling to the linearized contact cohomology of the Legendrian at the positive end. Theorem 1.2 implies that on the set of Legendrian knots in $\mathbb{R}^3$ that are exactly fillable, orientably or not, the relation defined by exact, non-orientable Lagrangian cobordism is *anti-reflexive* and *anti-symmetric*; see Corollary 4.2. Figure 6 gives some particular examples of Legendrians that are exactly fillable and thus do not admit exact, non-orientable Lagrangian endocobordisms. Many of these examples are maximal *tb* Legendrian representatives of twist or torus knots. In fact, using the classification results of Etnyre and Honda, [25], and of Etnyre, Ng, and Vértesi, [26], we show:

Corollary 1.3. *Let K be the smooth knot type of either a twist knot or a positive torus knot or a negative torus knot of the form $T(-p, 2k)$, for p odd and $p > 2k > 0$. Then any maximal tb Legendrian representative of K does not have an exact, non-orientable Lagrangian endocobordism.*

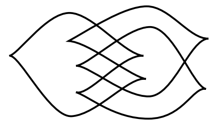

FIGURE 1. The max *tb* Legendrian representative of $m(8_{19})$.

However, stabilized Legendrian knots do admit exact, non-orientable Lagrangian endocobordisms: a Legendrian knot is said to be stabilized if, after Legendrian isotopy, a strand contains a zig-zag as shown in Figure 4.

Theorem 1.4. *For any stabilized Legendrian knot* Λ *and any* $k \in \mathbb{Z}^+$*, there exists an exact, non-orientable Lagrangian endocobordism for* Λ *of cross-cap genus* $4k$*.*

Some Legendrian knots are neither exactly fillable nor stabilized. Thus, a natural question is:

Question 1.5. *If a Legendrian knot is not exactly fillable and is not stabilized, does it have an exact, non-orientable Lagrangian endocobordism? In particular, does the unique Legendrian representative of* $m(8_{19}) = T(-4, 3)$ *with maximal tb whose front projection is shown in Figure 1 have an exact, non-orientable Lagrangian endocobordism?*

A description of how the Legendrian knot can be recovered from the front projection is given in Subsection 2.1. The max *tb* version of $m(8_{19})$ is not exactly fillable since the upper bound on the *tb* invariant for all Legendrian representatives of $m(8_{19})$ given by the Kauffman polynomial is not sharp; see Section 6 for more details. In response to Question 1.5, Chantraine, Dimitroglou Rizell, Ghiggini, and Golovko, [10, Corollary 12.3], proved an extension of Theorem 1.2 that shows an exact, non-orientable, Lagrangian endocobordism does not exist for an orientable Legendrian that admits an augmentation or, more generally, for an orientable Legendrian whose characteristic algebra admits a finite-dimensional representation. The max *tb* Legendrian representative of $m(8_{19}) = T(-4, 3)$ does not have an augmentation, but by results of Sivek, [46, Corollary 3.5], the characteristic algebra of this Legendrian does have a 2-dimensional representation. Thus the answer to Question 1.5 is no; see Section 6 for additional questions.

Given the existence of exact, non-orientable Lagrangian endocobordisms for a stabilized Legendrian, it is natural to ask: What Legendrian knots can appear as a "slice" of such an endocobordism? The parallel question for orientable Lagrangian endocobordisms has been studied in [9, 4, 13]. The non-orientable version of this question is closely tied to the question of whether or not non-orientable Lagrangian cobordisms define an equivalence relation on the set of Legendrian knots. By a result of Chantraine, [8], it is known that the relation defined on the set of Legendrian knots by *orientable* Lagrangian cobordism is *not* an equivalence relation since symmetry fails. In fact, the relation defined on the set of *stabilized* Legendrian knots by exact, non-orientable Lagrangian cobordism is symmetric: see Theorem 5.2. In addition, this relation is transitive by “stacking” (Lemma 2.2) and reflexive by Theorem 1.4. Thus we get:

Theorem 1.6. *On the set of stabilized Legendrian knots, the relation defined by exact, non-orientable Lagrangian cobordism is an equivalence relation. Moreover, all stabilized Legendrian knots are equivalent with respect to this relation.*

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2. Background

In this section, we give some basic background on Legendrian and Lagrangian submanifolds.

2.1. Contact Manifolds and Legendrian Submanifolds.

Below is some basic background on contact manifolds and Legendrian knots. More information can be found, for example, in [23] and [24].

A **contact manifold** (Y, ξ) is an odd-dimensional manifold together with a contact structure, which consists of a maximally non-integrable field of tangent hyperplanes. The **standard contact structure** on $\mathbb{R}^3$ is the field $\xi_p = \ker \alpha_0(p)$, for $\alpha_0(x, y, z) = dz - y dx$. A **Legendrian link** Λ is a submanifold of $\mathbb{R}^3$ diffeomorphic to a disjoint union of circles such that for all $p \in \Lambda$, we have $T_p\Lambda \subset \xi_p$; if, in addition, Λ is connected, Λ is a **Legendrian knot**. It is common to examine Legendrian links from their xz-projections, known as their **front projections**. A Legendrian link will generically have an immersed front projection with semi-cubical cusps, no vertical tangents, and no self-tangencies; any such projection can be uniquely lifted to a Legendrian link using $y = dz/dx$.

Two Legendrian links Λ_0 and Λ_1 are **equivalent Legendrian links** if there exists a 1-parameter family of Legendrian links Λ_t, $t \in [0, 1]$, joining Λ_0 and Λ_1. In fact, Legendrian links Λ_0, Λ_1 are equivalent if and only if their front projections are equivalent by planar isotopies that do not introduce vertical tangents and the **Legendrian Reidemeister moves** as shown in Figure 2.

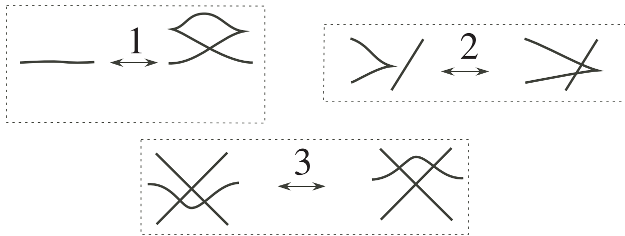

Figure 2. The three Legendrian Reidemeister moves. There is another type 1 move obtained by flipping the planar figure about a horizontal line, and there are three additional type 2 moves obtained by flipping the planar figure about a vertical, a horizontal, and both a vertical and horizontal line.

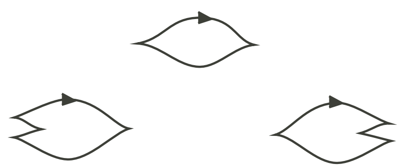

Figure 3. Three different Legendrian unknots; the one with maximal tb invariant of -1 and two others obtained by $\pm$-stabilizations.

Every knot has a Legendrian representative. In fact, every knot has an infinite number of different Legendrian representatives. For example, Figure 3 shows three different oriented Legendrians that are all topologically the unknot. These unknots can be distinguished by classical Legendrian invariants: the Thurston-Bennequin number, tb, and the rotation number, r. These invariants can easily be computed from a front projection; see, for example, [5].

The two unknots in the second line of Figure 3 are obtained from the one at the top by stabilization. In general, from an oriented Legendrian Λ, one can obtain oriented Legendrians $S_\pm(\Lambda)$: the **positive (negative) stabilization, S_+ (S_-),** is obtained by replacing a portion of a strand with a strand that contains a down (up) **zig-zag**, as shown in Figure 4. This stabilization procedure will not change the underlying smooth knot type but will decrease the Thurston-Bennequin number by 1; adding an up (down) zig-zag will decrease (increase) the rotation number by 1. It is possible to move a zig-zag to any strand of a Legendrian knot, [28]. Bennequin and Slice-Bennequin Inequalities (see, for example, [24]) show that for any Legendrian representative Λ of a fixed smooth knot type K, $tb(\Lambda) + |r(\Lambda)|$ is bounded above. Because of such bounds, the set of oriented Legendrian representatives of a fixed smooth knot type can be visualized by a "mountain range" in the plane where each Legendrian representative Λ is recorded by a vertex at coordinates $(r(\Lambda), tb(\Lambda))$; two vertices are connected by an edge if

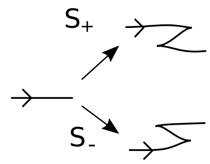

FIGURE 4. The positive (negative) stabilization of an oriented knot is obtained by introducing a down (an up) zig-zag.

the corresponding Legendrians are related by stabilization. Many examples of known and conjectured mountain ranges can be found in the Legendrian knot atlas of Chongchitmate and Ng, [12].

2.2. **Symplectic Manifolds, Lagrangian Submanifolds, and Lagrangian Cobordisms.** We will now discuss some basic concepts in symplectic geometry. Additional background can be found, for example, in [34].

A **symplectic manifold** (M, ω) is an even-dimensional manifold together with a 2-form ω that is closed and non-degenerate; when ω is an exact 2-form, $(M, \omega = d\beta)$ is said to be an **exact symplectic manifold**. A basic example of an exact symplectic manifold is $(\mathbb{R}^4, \omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$. The cobordisms constructed in this paper live inside the symplectic manifold that is constructed as the symplectization of $(\mathbb{R}^3, \xi_0 = \ker \alpha_0)$, namely, $\mathbb{R} \times \mathbb{R}^3$ with symplectic form given by $\omega = d(e^t \alpha_0)$. In fact, there is an exact symplectic diffeomorphism between the symplectization $(\mathbb{R} \times \mathbb{R}^3, \omega)$ and the standard $(\mathbb{R}^4, \omega_0)$, see for example [6].

A **Lagrangian submanifold** L of a 4-dimensional symplectic manifold (M, ω) is a 2-dimensional submanifold so that $\omega|_L = 0$. When M is an exact symplectic manifold, $\omega = d\beta$, $\beta|_L$ is necessarily a closed 1-form; when, in addition, $\beta|_L$ is an exact 1-form, $\beta|_L = df$, then L is said to be an **exact Lagrangian submanifold**. It is easy to verify that the exactness of the Lagrangian does not depend on the choice of β.

Remark 2.1. There is a (non-exact) Lagrangian torus in the standard symplectic $\mathbb{R}^4$: this can be seen as the product of two embedded circles in each of the (x_1, y_1) and (x_2, y_2) planes. By classical algebraic topology, it follows that the torus is the only compact, orientable surface that admits a Lagrangian embedding into $\mathbb{R}^4$: a result of Whitney equates a signed count of double points of an immersion to the Euler number of the normal bundle, but for a Lagrangian submanifold, the normal and tangent bundles are isomorphic, [3].

We turn our focus to non-compact Lagrangians that are cylindrical over Legendrians.

Definition 2.1. Let Λ_-, Λ_+ be Legendrian links in $\mathbb{R}^3$.

(1) A Lagrangian submanifold without boundary $L \subset \mathbb{R} \times \mathbb{R}^3$ is a **Lagrangian cobordism from Λ_+ to Λ_-** if it is of the form

$$L = ((-\infty, T_-] \times \Lambda_-) \cup \overline{L} \cup ([T_+, +\infty) \times \Lambda_+),$$

for some $T_- < T_+$, where $\overline{L} \subset [T_-, T_+] \times \mathbb{R}^3$ is compact with boundary $\partial\overline{L} = (\{T_-\} \times \Lambda_-) \cup (\{T_+\} \times \Lambda_+)$.

(2) A Lagrangian cobordism from Λ_+ to Λ_- is **orientable (resp., non-orientable)** if L is orientable (resp., non-orientable).

(3) A Lagrangian cobordism from Λ_+ to Λ_- is **exact** if L is exact, namely $e^t\alpha_0|_L = df|_L$, and the primitive, f, is constant on the cylindrical ends: there exist constants $C_\pm$ such that

$$f|_{L\cap((-\infty,T_-)\times\mathbb{R}^3)} = C_-, \qquad f|_{L\cap((T_+,+\infty)\times\mathbb{R}^3)} = C_+.$$

A Legendrian knot Λ is **(exactly) fillable** if there exists an (exact) Lagrangian cobordism from $\Lambda_+ = \Lambda$ to $\Lambda_- = \emptyset$.

An important property of Lagrangian cobordisms is that they can be stacked/composed:

Lemma 2.2 (Stacking Cobordisms, [19])**.** *If L_{12} is a Lagrangian cobordism from $\Lambda_+ = \Lambda_1$ to $\Lambda_- = \Lambda_2$, and L_{23} is a Lagrangian cobordism from $\Lambda_+ = \Lambda_2$ to $\Lambda_- = \Lambda_3$, then there exists a Lagrangian cobordism L_{13} from $\Lambda_+ = \Lambda_1$ to $\Lambda_- = \Lambda_3$. Furthermore, if L_{12} and L_{23} are exact, then there exists an exact L_{13}.*

Constructions of exact Lagrangian cobordisms are an active area of research. In this paper, we will use the fact that there exist exact Lagrangian cobordisms between Legendrians related by isotopy and certain surgeries. The existence of exact Lagrangian cobordisms from isotopy is well-known, see, for example, [22], [8], [19], and [6].

Lemma 2.3 (Exact Cobordisms from Isotopy)**.** *Suppose that Λ and Λ' are isotopic Legendrian links. Then there exists an exact, orientable Lagrangian cobordism, in fact concordance, from $\Lambda_+ = \Lambda$ to $\Lambda_- = \Lambda'$.*

Remark 2.2. In general, the trace of a Legendrian isotopy is not a Lagrangian cobordism. However it is possible to add a "correction term" so that it will be Lagrangian. More precisely, let $\lambda_t(u) = (x(t,u), y(t,u), z(t,u))$, $t \in \mathbb{R}$, be a Legendrian isotopy such that $\frac{\partial\lambda}{\partial t}(t,u)$ has compact support with $\operatorname{Im}\lambda_t(u) = \Lambda_-$ for $t \le -T$ and $\operatorname{Im}\lambda_t(u) = \Lambda_+$ for $t \ge T$, and let

$$\eta(t,u) = \alpha_0\left(\frac{\partial\lambda}{\partial t}(t,u)\right).$$

Then $\Gamma(t,u) = (t, x(t,u), y(t,u), z(t,u) + \eta(t,u))$ is an exact Lagrangian immersion. If $\eta(t,u)$ is sufficiently small, which can be guaranteed by "slowing down" the isotopy via a t-reparameterization, then $\Gamma(t,u)$ is an exact Lagrangian embedding.

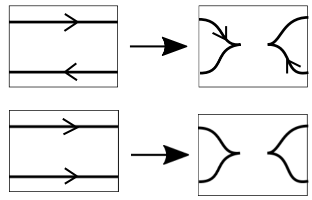

FIGURE 5. Orientable and Non-Orientable Legendrian surgeries.

In addition, Legendrians Λ and Λ' that differ by "surgery" can be connected by an exact Lagrangian cobordism. In one of these surgery operations, a Legendrian 0-tangle, consisting of two strands with no crossings and no cusp points, is replaced with a Legendrian ∞-tangle, consisting of two strands that each have 1 cusp and no crossings; see Figure 5. When the strands of the 0-tangle are oppositely oriented, this is an **orientable surgery**; otherwise this is a **non-orientable surgery**. In addition, there is another surgery operation that shows the maximal *tb* Legendrian representative of the unknot, shown at the top of Figure 3, can be filled.

Lemma 2.4 (Exact Cobordisms from Surgery, [19, 15, 6])**.**

(1) *Suppose that Λ_+ and Λ_- are Legendrian knots where Λ_- is obtained from Λ_+ by orientable (non-orientable) surgery, as shown in Figure 5. Then there exists an exact, orientable (non-orientable) Lagrangian cobordism from Λ_+ to Λ_-.*

(2) *Suppose Λ_+ is the Legendrian unknot with tb equal to the maximum value of -1. Then there exists an exact, orientable Lagrangian filling of Λ_+ by a disk.*

Remark 2.3. By Lemmas 2.2, 2.3 and 2.4, to show there exists an exact Lagrangian cobordism from Λ_+ to Λ_-, it suffices to show that there is a string of Legendrian links $(\Lambda_+ = \Lambda_0, \Lambda_1, \dots, \Lambda_n = \Lambda_-)$, where each Λ_{i+1} is obtained from Λ_i by a single surgery, as shown in Figure 5, and Legendrian isotopy. In the case where each surgery is orientable, the exact Lagrangian cobordism will be orientable. If all surgeries are orientable and $\Lambda_\pm$ are both knots, then the length, n, of this string must be even and will agree with twice the genus of the Lagrangian cobordism; for more details, see [5]. If there is at least one non-orientable surgery, the exact Lagrangian cobordism will be non-orientable and the length of the string agrees with the cross-cap genus of the Lagrangian cobordism. To construct an exact Lagrangian filling of Λ_+, it suffices to construct such a string to $\Lambda_- = U$, where U is a trivial link of maximal *tb* Legendrian unknots.

3. Constructions of Non-orientable Lagrangian Endocobordisms

In this section, we show that *any* Legendrian knot has a non-orientable Lagrangian endocobordism with cross-cap genus an arbitrary multiple of 4. We then show that it is not possible to get any other cross-cap genera.

The strategy to show existence is to first construct an immersed orientable Lagrangian cobordism, and then apply "Lagrangian surgery" to modify it so that it is embedded. The following description of Lagrangian surgery follows Polterovich's construction, [38]; see also work of Lalonde and Sikorav, [33].

To describe Lagrangian surgery precisely, we first need to explain the "sign" of a double point. Suppose that x is a point of self-intersection of a generic, immersed, oriented 2-dimensional submanifold L of $\mathbb{R}^4$. Then $\operatorname{sgn}(x) \in \{\pm 1\}$ will denote the **sign of self-intersection of** L at x: let (v_1, v_2) and (w_1, w_2) be positively oriented bases of the transverse tangent spaces at x, then

$$\operatorname{sgn}(x) = +1 \iff (v_1, v_2, w_1, w_2) \text{ is a positively oriented basis of } \mathbb{R}^4,$$

and otherwise $\operatorname{sgn}(x) = -1$.

By constructing a Lagrangian handle in a Darboux chart, it is possible to remove double points of a Lagrangian immersion:

Lemma 3.1 (Lagrangian Surgery, [38])**.** *Let Σ be a 2-dimensional manifold. Suppose $\phi : \Sigma \to \mathbb{R}^4$ is a Lagrangian immersion, and $U \subset \mathbb{R}^4$ contains a single transversal double point x of ϕ. Then there exists a 2-dimensional manifold Σ' and a Lagrangian immersion $\phi' : \Sigma' \to \mathbb{R}^4$ such that*

(1) $\operatorname{Im}\phi = \operatorname{Im}\phi'$ *on* $\mathbb{R}^4 - U$*;*
(2) ϕ' *has no double points in* U*.*

Furthermore, let $\phi^{-1}(\{x\}) = \{p_1, p_2\} \subset \Sigma$.

(1) *If p_1, p_2 are in disjoint components of Σ, then Σ' is obtained from Σ by a connect sum operation.*
(2) *If p_1, p_2 are in the same component of Σ, then*
    (a) *if Σ is not orientable, $\Sigma' = \Sigma \# K (= \Sigma \# T)$,*
    (b) *if Σ is orientable, then $\Sigma' = \Sigma \# T$, when $\operatorname{sgn}(x) = +1$, and $\Sigma' = \Sigma \# K$, when $\operatorname{sgn}(x) = -1$,*

    where K denotes the Klein bottle and T denotes the torus.

We now have the necessary background to show the existence of a non-orientable Lagrangian endocobordism for any Legendrian knot:

Theorem 3.2. *For any Legendrian knot Λ and any $k \in \mathbb{Z}^+$, there exists a non-orientable Lagrangian endocobordism for Λ of cross-cap genus $4k$.*

Proof. For an arbitrary Legendrian knot Λ, begin with a cylindrical Lagrangian cobordism, $L = \mathbb{R} \times \Lambda$ in $\mathbb{R} \times \mathbb{R}^3$, which is a space that is symplectically equivalent to the standard $\mathbb{R}^4$. As explained in Remark 2.1, there exists an embedded Lagrangian torus, T, such that $T \cap L = \emptyset$. After a suitable shift and perturbation, we can assume that L and T intersect at exactly two points, x_1 and x_2 where $\mathrm{sgn}(x_1) = +1$ and $\mathrm{sgn}(x_2) = -1$. By Lemma 2.4, Lagrangian surgery at x_1 results in the connected, oriented, immersed Lagrangian diffeomorphic to $(\mathbb{R} \times S^1)\#T$ with a double point at x_2 of index -1; a second Lagrangian surgery at x_2 results in an embedded, non-orientable Lagrangian cobordism diffeomorphic to $(\mathbb{R}\times S^1)\#T\#K$, and thus of cross-cap genus 4. Stacking these endocobordisms, using Lemma 2.2, produces an embedded, non-orientable Lagrangian cobordism of cross-cap genus $4k$, for any $k \in \mathbb{Z}^+$. □

In fact, the possible cross-cap genera that appear in Theorem 3.2 are all that can exist:

Theorem 3.3. *Any non-orientable Lagrangian endocobordism in $\mathbb{R} \times \mathbb{R}^3$ must have cross-cap genus $4k$, for some $k \in \mathbb{Z}^+$.*

This cross-cap genus restriction is closely tied to Euler characteristic obstructions for *compact*, non-orientable submanifolds that admit Lagrangian embeddings in $(\mathbb{R}^4, \omega_0)$, or equivalently in $(\mathbb{R} \times \mathbb{R}^3, d(e^t\alpha))$:

Lemma 3.4 (Audin, [1])**.** *Any compact, non-orientable Lagrangian submanifold of $\mathbb{R} \times \mathbb{R}^3$ has an Euler characteristic divisible by 4.*

This result can be seen as an extension, to the non-orientable setting, of a formula of Whitney that relates the number of double points of a smooth immersion to the Euler number of the normal bundle of the immersion; see [1, 3].

Remark 3.1. Lemma 3.4 implies that any compact, non-orientable, Lagrangian submanifold L in $\mathbb{R}\times\mathbb{R}^3$ has cross-cap genus $2+4j$, for some $j \geq 0$. There are explicit constructions of compact, non-orientable Lagrangian submanifolds of cross-cap genus $2 + 4j$, for all $j > 0$, [29, 2]. It has been shown that there is no embedded, Lagrangian Klein bottle ($j = 0$), [35, 44].

To utilize the cross-cap genus restrictions for compact Lagrangians, we will employ the following lemma, which shows that for any Lagrangian endocobordism, it is possible to construct a compact, non-orientable Lagrangian submanifold into which we can glue the compact portion of a Lagrangian endocobordism.

Lemma 3.5. *For any Legendrian knot $\Lambda \subset \mathbb{R}^3$, any open set $D \subset \mathbb{R}^3$ containing Λ, and any $T \in \mathbb{R}^+$, there exists a compact, non-orientable Lagrangian submanifold L in $\mathbb{R} \times \mathbb{R}^3$ such that*

$$L \cap ([-T, T] \times D) = [-T, T] \times \Lambda.$$

Proof. The strategy will be to construct a Lagrangian torus with double points, thought of as two finite cylinders with top and bottom circles identified, and then apply Lagrangian surgery to remove the double points. As a first step, we construct (non-disjoint) Lagrangian embeddings of two cylinders via Legendrian isotopies, Lemma 2.3. Namely, start with two disjoint copies of Λ: Λ in D and a translated version $\Lambda' \subset \mathbb{R}^3 - D$. Now, for $t \in [0, U]$, consider Legendrian isotopies Λ_t of Λ and Λ'_t of Λ' that satisfy the following conditions: $\Lambda_t = \Lambda$, for $t \in [0, U]$; $\Lambda'_t = \Lambda'$, for $t \in [0, T]$, and for $t \in [T, U]$, Λ'_t is a Legendrian isotopy of Λ' such that $\Lambda'_t = \Lambda$, for t near U. By repeating an analogous procedure for $t \in [-U, 0]$, we can obtain a smooth immersion of the torus into $[-U, U] \times \mathbb{R}^3$. The arguments used to prove Lemma 2.3 (see Remark 2.2) show that for $U - T$ sufficiently large, the trace of these isotopies can be perturbed to two non-disjoint embedded Lagrangian cylinders that do not have any intersection points in $[-T, T] \times \mathbb{R}^3$, and a direct calculation shows that each double point with $t \in [T, U]$ can be paired up with a double point with $t \in [-U, -T]$ of opposite sign. Then by applying Lagrangian surgery, Lemma 3.1, at each double point we get a compact, non-orientable Lagrangian submanifold L in $\mathbb{R} \times \mathbb{R}^3$ with the desired properties. □

We are now ready to prove the cross-cap genus restriction for arbitrary non-orientable, Lagrangian endocobordisms:

Proof of Theorem 3.3. Let C be a non-orientable Lagrangian endocobordism. Suppose $C \subset \mathbb{R} \times D$ and C agrees with a standard cylinder outside $[-T, T] \times \mathbb{R}^3$. By Lemma 3.5, there is a compact, non-orientable Lagrangian submanifold L in $\mathbb{R} \times \mathbb{R}^3$ so that

$$L \cap ([-T, T] \times D) = [-T, T] \times \Lambda.$$

Let L' be the Lagrangian submanifold obtained by removing the standard cylindrical portion of L in $[-T, T] \times D$ and replacing it with $C \cap ([-T, T] \times \mathbb{R}^3)$. Then L' will be a compact, non-orientable Lagrangian submanifold whose cross-cap genus, $k(L')$, differs from the cross-cap genus of L, $k(L)$, by the cross-cap genus of C, $k(C)$: $k(L') = k(L) + k(C)$. By Lemma 3.4, there exist $j, j' \in \mathbb{Z}^+$ so that $k(L) = 2 + 4j$ and $k(L') = 2 + 4j'$. Thus we find that the cross-cap genus of C, $k(C)$, must be divisible by 4. □

Remark 3.2. For exact Lagrangian cobordisms that are constructed from isotopy and surgery, Lemmas 2.3 and 2.4, it is possible to show that the cross-cap genus must be a multiple of 4 by an alternate argument that relies on a careful analysis of the possible changes to $tb(\Lambda)$ under surgery; [7].

4. Obstructions to Exact Non-Orientable Lagrangian Endocobordisms

We will now begin to focus on *exact*, non-orientable Lagrangian cobordisms. In this section, we will prove Theorem 1.2, which states that any Legendrian knot that is exactly fillable does not have an exact non-orientable Lagrangian endocobordism. The proof of this theorem will involve applying the Seidel Isomorphism, which relates the topology of a filling to the linearized Legendrian contact cohomology of the Legendrian at the positive end. We will then apply Theorem 1.2 and give examples of maximal *tb*

Legendrian knots that do not have exact, non-orientable Lagrangian endocobordisms.

We begin with a brief description of Legendrian contact homology; additional background information can be found, for example, in [24]. Legendrian contact homology is a Floer-type invariant of a Legendrian submanifold that lies within Eliashberg, Givental, and Hofer's Symplectic Field Theory framework; [20, 21, 11]. It is possible to associate to a Legendrian submanifold $\Lambda \subset \mathbb{R}^3$ the stable, tame isomorphism class of an associative differential graded algebra (DGA), $(\mathcal{A}(\Lambda), \partial)$. The algebra is freely generated by the so-called Reeb chords of Λ, and is graded using a Maslov index. The differential comes from counting pseudo-holomorphic curves in the symplectization of $\mathbb{R}^3$; for our interests, we will always use $\mathbb{Z}/2$ coefficients. **Legendrian contact homology**, namely the homology of $(\mathcal{A}(\Lambda), \partial)$, is a Legendrian isotopy invariant. Legendrian contact homology has been defined for Legendrian in contact manifolds other than $\mathbb{R}^3$; see, for example, [18], [40].

In general, it is difficult to extract information directly from the Legendrian contact homology. An important computational technique arises from the existence of augmentations of the DGA. An **augmentation** ε of $\mathcal{A}(\Lambda)$ is a differential algebra homomorphism $\varepsilon : (\mathcal{A}(\Lambda), \partial) \to (\mathbb{Z}_2, 0)$; a **graded augmentation** is an augmentation such that ε is supported on elements of degree 0. Not all Legendrians have an augmentation; for any Legendrian Λ, there are only a finite number of augmentations. Given a graded augmentation ε, one can linearize $(\mathcal{A}(\Lambda), \partial)$ to a finite dimensional differential graded complex $(A(\Lambda), \partial^\varepsilon)$ and obtain **linearized contact homology**, denoted $LCH_*(\Lambda, \varepsilon; \mathbb{Z}/2)$, and its dual **linearized contact cohomology**, $LCH^*(\Lambda, \varepsilon; \mathbb{Z}/2)$. The set of all linearized (co)homology groups with respect to all possible graded augmentations is an invariant of Λ. If the augmentation is ungraded, one can still examine the rank of the non-graded linearized (co)homology, $\dim LCH(\Lambda, \varepsilon; \mathbb{Z}/2)$, and obtain as an invariant of Λ the set of ranks of this total linearized (co)homology for all possible augmentations. Ungraded linearized (co)homology is not an effective invariant: of the many examples of Legendrian knots in the Legendrian knot atlas of Chongchitmate and Ng, [12], that have the same classical invariants yet can be distinguished through graded linearized homology, none can be distinguished by examining ungraded homology. However, ungraded (co)homology will be useful in arguments below.

Ekholm, [16], has shown that an exact Lagrangian filling, F, of a Legendrian submanifold $\Lambda \subset \mathbb{R}^3$ induces an augmentation ε_F of $(\mathcal{A}(\Lambda), \partial)$. When this filling has Maslov class 0, the augmentation will be graded. Informally, Maslov 0 means that along each loop in the filling, the corresponding loop of Lagrangian tangent planes is trivial in the Lagrangian Grassmannian.

The following result of Seidel will play a central role in showing obstructions to exact, non-orientable Lagrangian endocobordisms. A proof of this result was sketched by Ekholm in [17] and given in detail in Dimitroglou-Rizell, [14]; a parallel result using generating family homology is given in [42].

Theorem 4.1 (Seidel Isomorphism, [17], [14], [19])**.** *Let $\Lambda \subset \mathbb{R}^{2n+1}$ be an n-dimensional Legendrian submanifold with an exact Lagrangian filling F; let ε_F denote the augmentation induced by the filling. Then*

$$\dim H(F; \mathbb{Z}/2) = \dim LCH(\Lambda, \varepsilon_F; \mathbb{Z}/2).$$

If the filling F of the n-dimensional Legendrian has Maslov class 0, then a graded version of the above equality holds:

$$\dim H_{n-*}(F; \mathbb{Z}/2) = \dim LCH^*(\Lambda, \varepsilon_F; \mathbb{Z}/2).$$

The ungraded version of the Seidel Isomorphism will be used to prove that any Legendrian Λ that is exactly fillable does not have an exact, non-orientable Lagrangian endocobordism:

Proof of Theorem 1.2. For a contradiction, suppose that there is a Legendrian knot Λ that has an exact Lagrangian filling and an exact non-orientable Lagrangian endocobordism. Then by stacking the endocobordisms, Lemma 2.2, it follows that Λ has an infinite number of topologically distinct exact, non-orientable Lagrangian fillings. Each of these exact Lagrangian fillings induces an augmentation. Since there are only a finite number of possible augmentations, there must exist two topologically distinct fillings that induce the same augmentation. However, this gives a contradiction to the Seidel Isomorphism, Theorem 4.1. □

Theorem 1.2 implies that on the set of Legendrian knots in $\mathbb{R}^3$ that are exactly fillable, orientably or not, the relation defined by exact, non-orientable Lagrangian cobordism is anti-reflexive. Thus, by stacking, Lemma 2.2, we immediately also see:

Corollary 4.2. *On the set of Legendrian knots in $\mathbb{R}^3$ that are exactly fillable, orientably or not, the relation $\sim$ defined by exact, non-orientable Lagrangian cobordism is anti-symmetric: $\Lambda_1 \sim \Lambda_2 \implies \Lambda_2 \not\sim \Lambda_1$.*

We now apply Theorem 1.2 to give examples of Legendrians that do not have exact, non-orientable Lagrangian endocobordisms. Hayden and Sabloff, [31], showed that every positive knot type has a Legendrian representative that has an exact, orientable Lagrangian filling. Combining this with Theorem 1.2, immediately gives:

Corollary 4.3 ([31])**.** *Each positive knot has a Legendrian representative that does not have an exact, non-orientable Lagrangian endocobordism.*

There is work in progress to show that every +-adequate knot has a Legendrian representative with an exact filling, [43].

Many maximal *tb* representatives of low crossing knots have fillings, orientable or not. Figure 6 illustrates some Legendrians that can be verified to

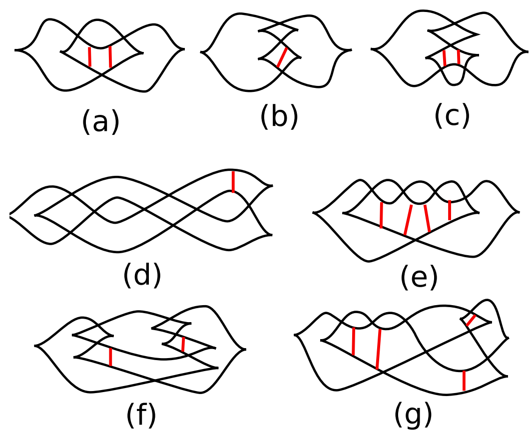

FIGURE 6. Examples of Legendrians that do not have exact, non-orientable Lagrangian endocobordisms: maximal tb representatives of (a) $m(3_1) = T(3,2) = K_{-2}$, (b) $3_1 = T(-3,2) = K_1$, (c) $4_1 = K_2 = K_{-3}$, (d) $5_1 = T(-5,2)$, (e) $m(5_1) = T(5,2)$, (f) 6_2, and (g) $m(6_2)$. The red lines indicate points for surgeries.

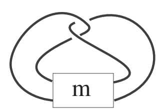

FIGURE 7. The smooth twist knot K_m; the box contains m right-handed half twists if $m \geq 0$, and $|m|$ left-handed twists if $m < 0$. Notice that K_0 and K_{-1} are unknots.

have exact, Lagrangian fillings: see Remark 2.3. Many of the examples in Figure 6 are Legendrian representatives of twist knots, K_m, or torus knots, $T(p,q)$. Using Theorem 1.2 together with classification results of Etnyre and Honda, [25], and of Etnyre, Ng, and Vértesi, [26], we show that *all* maximal tb representatives of twist knots, positive torus knots, and negative torus knots of the form $T(-p, 2k)$, $p > 2k > 0$, do not have exact, non-orientable Lagrangian endocobordisms:

Proof of Corollary 1.3. By Theorem 1.2, to show the non-existence of an exact, non-orientable Lagrangian endocobordism, it suffices to show the existence of an exact Lagrangian filling.

First consider the case where Λ is a maximal tb representative of a twist knot, whose form is shown in Figure 7. Etnyre, Ng, and Vértesi, have classified all Legendrian twist knots, [26]: every maximal tb Legendrian representative of K_m, for $m \leq -2$, is Legendrian isotopic to one of the form in Figure 8, and every maximal tb Legendrian representative of K_m, for $m \geq 1$, is Legendrian isotopic to one of the form in Figure 9. For a

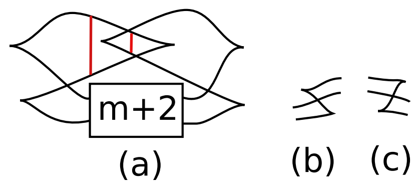

Figure 8. Any maximal tb Legendrian representative of a negative twist knot, K_m with $m \leq -2$, is Legendrian isotopic to a Legendrian of the form (a) with the box containing $|m+2|$ half twists, where each half twist has form S as shown in (b) or form Z as shown in (c). Two surgeries produces a max tb Legendrian unknot.

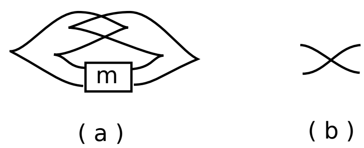

Figure 9. Any maximal tb Legendrian representative of a positive twist knot, K_m with $m \geq 1$, is Legendrian isotopic to a Legendrian in the form of (a), where the box contains m half twists of form X as shown in (b).

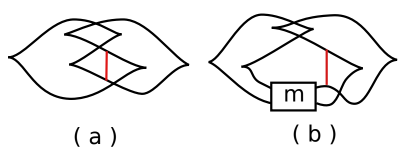

Figure 10. An inductive argument shows that every max tb representative of a positive twist knot has an exact Lagrangian filling.

max tb representative of a negative twist knot, Figure 8 illustrates the two surgeries that show the existence of an exact Lagrangian filling. For a max tb Legendrian representative of a positive twist knot, the existence of an exact filling can be shown by an induction argument: Figure 10 (a), indicates the surgery point when $m = 1$; for all $m \geq 1$, a maximal tb representative of K_{m+1} can be reduced to a maximal tb representative of K_m by one surgery as indicated in Figure 10 (b).

Next consider maximal tb Legendrian representatives of a torus knot, a knot that can be smoothly isotoped so that it lies on the surface of an unknotted torus in $\mathbb{R}^3$. Every torus knot can be specified by a pair (p, q) of coprime integers: we will use the convention that the (p, q)-torus knot, $T(p, q)$, winds p times around a meridional curve of the torus and q times in the longitudinal direction. In fact, $T(p, q)$ is equivalent to $T(q, p)$ and to

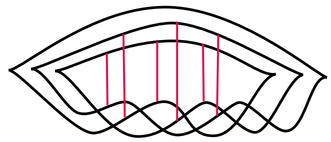

FIGURE 11. Surgeries that result in an exact filling of the maximal tb representative of the positive torus knot $T(5,3)$.

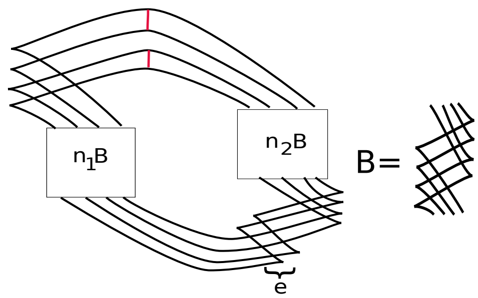

FIGURE 12. The general form of a maximal tb representative of a negative torus knot $T(-p, 2k)$, with $p > 2k > 0$, with $k = 2$ and $|p| = (1+n_1+n_2)2k+e$; the indicated k surgeries produce a Legendrian trivial link of maximal tb unknots.

$T(-p,-q)$. We will always assume that $|p| > q \geq 2$, since we are interested in non-trivial torus knots.

Etnyre and Honda, [25], showed there is a unique maximal tb representative of a positive torus knot, $T(p,q)$ with $p > 0$. The surgeries used in [5, Theorem 4.2] show that this maximal representative is exactly fillable. Figure 11 illustrates the orientable surgeries for the $(5,3)$-torus knot; in this sequence of surgeries, one begins with surgeries on the innermost strands, and then performs a Legendrian isotopy so that it is possible to do a surgery on the next set of innermost strands.

Lastly consider the case where Λ is topologically a negative torus knot, $T(-p, 2k)$ with $p > 2k > 0$. In this case, Etnyre and Honda have shown that the number of different maximal tb Legendrian representations depends on the divisibility of p by $2k$: if $|p| = m2k + e$, $0 < e < 2k$, there are m non-oriented Legendrian representatives of $T(-p, 2k)$ with maximal tb. These different representatives with maximal tb are obtained by writing $m = 1 + n_1 + n_2$, where $n_1, n_2 \geq 0$, and then $\Lambda_{(n_1,n_2)}$ is constructed using the form shown in Figure 12 with n_1 and n_2 copies of the tangle B inserted as indicated; this figure also shows k surgeries that guarantee the existence of an exact Lagrangian filling. □

Some comments on obstructions to exact fillings are discussed in Section 6.

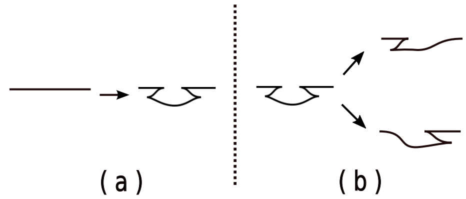

FIGURE 13. It is possible to construct exact non-orientable Lagrangian cobordisms between (a) $\Lambda_+ = \Lambda$ and $S_-S_+(\Lambda)$, (b) $\Lambda_+ = S_-S_+(\Lambda)$ and $\Lambda_- = S_+(\Lambda)$ or $\Lambda_- = S_-(\Lambda)$.

5. Constructions of Exact, Non-orientable Lagrangian Cobordisms

In this section, we will construct an exact, non-orientable Lagrangian endocobordism of cross-cap genus 4 for any stabilized Legendrian knot, and a non-orientable Lagrangian cobordism between any two stabilized Legendrian knots. All these exact Lagrangian cobordisms are constructed through isotopy and surgery, see Remark 2.3.

Central to these constructions will be the following lemma, which says that one can always introduce a pair of "oppositely oriented" zig-zags to Λ_+, and if one has a pair of oppositely oriented zig-zags in Λ_+, then one can remove either element of this pair; see Figure 13. One needs to be careful when discussing orientations for the ends of a non-orientable Lagrangian cobordism: given an orientation on Λ_+, there is no canonical orientation for Λ_-. However, although an orientation is needed on Λ to distinguish between $S_+(\Lambda)$ and $S_-(\Lambda)$, $S_-S_+(\Lambda)$ is a well-defined operation on unoriented knots.

Lemma 5.1. *Let Λ be a Legendrian knot. Then there exists an exact, non-orientable Lagrangian cobordism:*

(1) *of cross-cap genus* 2 *between $\Lambda_+ = \Lambda$ and $\Lambda_- = S_-S_+(\Lambda)$;*
(2) *of cross-cap genus* 1 *between $\Lambda_+ = S_-S_+(\Lambda)$ and $\Lambda_- = S_+(\Lambda)$ or $\Lambda_- = S_-(\Lambda)$.*

Proof. The strategy will be to construct the desired exact, non-orientable Lagrangian cobordism via Legendrian isotopy and surgeries that are performed locally, near the site of the stabilizations. Figure 14 illustrates the isotopy and surgeries, the second of which is non-orientable, that imply the existence of a cross-cap genus 2 Lagrangian cobordism between $\Lambda_+ = \Lambda$ and $\Lambda_- = S_-S_+(\Lambda)$. Figure 15 illustrates the isotopy and surgery that imply the existence of a cross-cap genus 1 Lagrangian cobordism between $\Lambda_+ = S_-S_+(\Lambda)$ and $\Lambda_- = S_+(\Lambda)$, when the original strand is oriented from right to left, or $\Lambda_- = S_-(\Lambda)$, when the original strand is oriented from left to right. $\square$

5.1. **Exact, Non-Orientable Lagrangian Endocobordisms.** In Theorem 1.2, it was shown that Legendrians that are exactly fillable do not

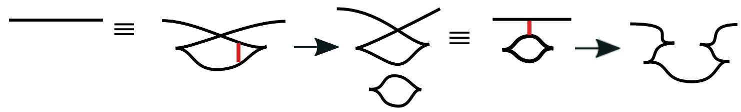

Figure 14. By applying an orientable and a non-orientable surgery, any strand can have a pair of oppositely oriented zig-zags introduced.

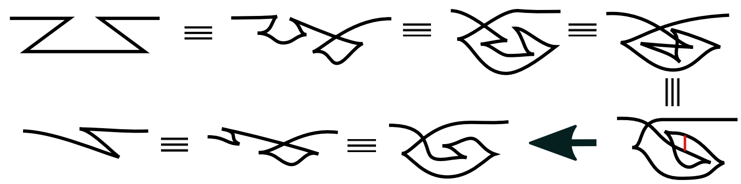

Figure 15. In the presence of oppositely oriented zig-zags, via one non-orientable surgery, one of the zig-zags can be removed.

have exact, non-orientable Lagrangian endocobordisms. However exact, non-orientable Lagrangian endocobordisms do exist for stabilized knots:

Proof of Theorem 1.4. First consider the case where Λ is the negative stabilization of a Legendrian: $\Lambda = S_-(\widehat{\Lambda})$. Then by Lemma 5.1, there exists an exact, non-orientable Lagrangian cobordism:

(1) of cross-cap genus 2 between Λ and $S_-S_+(\Lambda)$;
(2) of cross-cap genus 1 between $S_-S_+(\Lambda)$ and $S_+(\Lambda)$;
(3) of cross-cap genus 1 between $S_+(\Lambda) = S_+(S_-(\widehat{\Lambda}))$ and $S_-(\widehat{\Lambda}) = \Lambda$.

Stacking these cobordisms results in an exact, non-orientable Lagrangian endocobordism of cross-cap genus 4. Additional stacking results in arbitrary multiples of cross-cap genus 4.

An analogous argument proves the case where Λ is the positive stabilization of a Legendrian: $\Lambda = S_+(\widehat{\Lambda})$. $\square$

5.2. **Exact, Non-Orientable Lagrangian Cobordisms between Stabilized Legendrians.** Given that every stabilized Legendrian knot has a non-orientable Lagrangian endocobordism, a natural question is: What Legendrian knots can appear as a "slice" of such an endocobordism? In this section, we show that *any* stabilized Legendrian knot can appear as such a slice.

Theorem 5.2. *For smooth knot types K, K', let Λ be any Legendrian representative of K and let Λ' be a stabilized Legendrian representative of K'. Then there exists an exact, non-orientable Lagrangian cobordism between $\Lambda_+ = \Lambda$ and $\Lambda_- = \Lambda'$.*

Before moving to the proof of Theorem 5.2, we show that non-orientable Lagrangian cobordisms define an equivalence relation on the set of stabilized Legendrian knots:

Proof of Theorem 1.6. Let $\mathcal{L}^s$ denote the set of all stabilized Legendrian knots of any smooth knot type. Define the relation $\sim$ on $\mathcal{L}^s$ by $\Lambda_1 \sim \Lambda_2$ if there exists an exact, non-orientable Lagrangian cobordism from $\Lambda_+ = \Lambda_1$ to $\Lambda_- = \Lambda_2$. Reflexivity of $\sim$ follows from Theorem 1.4. Symmetry of $\sim$ follows from Theorem 5.2. Transitivity of $\sim$ follows from Lemma 2.2. Thus $\sim$ defines an equivalence relation. Moreover, by Theorem 5.2, we see that with respect to this equivalence relation, there is only one equivalence class. □

To prove Theorem 5.2, it will be useful to first show that there is an exact, non-orientable Lagrangian cobordism between any two stabilized Legendrians of a fixed knot type:

Proposition 5.3. *Let K be any smooth knot type, and let Λ, Λ' be Legendrian representatives of K, where Λ' is stabilized. Then there exists an exact, non-orientable Lagrangian cobordism between $\Lambda_+ = \Lambda$ and $\Lambda_- = \Lambda'$.*

Proof. Fix a smooth knot type K, and let Λ_1, Λ_2 be Legendrian representatives, where Λ_2 is stabilized. By results of Fuchs and Tabachnikov, [28], we know that there exist r_1, ℓ_1, r_2, ℓ_2 such that $S_-^{\ell_1} S_+^{r_1}(\Lambda_1) = S_-^{\ell_2} S_+^{r_2}(\Lambda_2)$. By applying additional positive stabilizations, if needed, we can assume $r_1 > \ell_1$.

Consider the case where Λ_2 is the negative stabilization of some Legendrian: $\Lambda_2 = S_-(\hat{\Lambda}_2)$. By applications of Lemma 5.1, there exists an exact, non-orientable Lagrangian cobordism between:

(1) Λ_1 and $S_-^{r_1} S_+^{r_1}(\Lambda_1)$;
(2) $S_-^{r_1} S_+^{r_1}(\Lambda_1)$ and $S_-^{\ell_1} S_+^{r_1}(\Lambda_1)$, and thus between $S_-^{r_1} S_+^{r_1}(\Lambda_1)$ and $S_-^{\ell_2} S_+^{r_2}(\Lambda_2)$;
(3) $S_-^{\ell_2} S_+^{r_2}(\Lambda_2)$ and $S_+^{r_2}(\Lambda_2)$;
(4) $S_+^{r_2}(\Lambda_2) = S_+^{r_2}(S_-(\hat{\Lambda}_2))$ and $S_-(\hat{\Lambda}_2) = \Lambda_2$.

By stacking these cobordisms (Lemma 2.2), we have our desired exact, non-orientable Lagrangian cobordism between Λ_1 and Λ_2. An analogous argument proves the case where Λ_2 is the positive stabilization of some Legendrian. □

Proof of Theorem 5.2. The strategy here is to first show that one can construct an exact, non-orientable Lagrangian cobordism between Λ and a stabilized Legendrian unknot Λ_0. Similarly, it is possible to construct an exact, non-orientable Lagrangian cobordism between Λ' and a stabilized Legendrian unknot Λ_0'; we will show it is possible to "reverse" this sequence of surgeries and construct an exact, non-orientable Lagrangian cobordism between Λ_0' and $\widetilde{\Lambda}'$, which is a stabilization of Λ'. By Proposition 5.3, there exists an exact, non-orientable Lagrangian cobordism between Λ_0 and Λ_0' and between $\widetilde{\Lambda}'$ and Λ'. Thus we will have the desired exact, nonorientable

Lagrangian cobordism between Λ and Λ_0 by stacking the cobordisms between Λ and Λ_0, between Λ_0 and Λ_0', between Λ_0? and $\tilde{\Lambda}'$, and between $\tilde{\Lambda}$? and Λ'.

We first show how it is possible to construct an exact, non-orientable Lagrangian cobordism from Λ to a Legendrian unknot; cf., [5]. Let Λ be an arbitrary Legendrian knot. We can assume that Λ has at least one positive crossing by, if necessary, applying a Legendrian Reidemeister 1 move. As shown in Figure 16, performing an orientable or non-orientable surgery near a crossing produces a crossing that can be removed through Legendrian Reidemeister moves. Perform such a surgery on every crossing in Λ until you have obtained k disjoint stabilized Legendrian unknots; since Λ has at least one positive crossing, we have performed at least one non-orientable surgery. Align the k Legendrian unknots vertically and perform surgeries so that we obtain a single stabilized Legendrian unknot Λ_0. In this way, we have constructed an exact, non-orientable Lagrangian cobordism between Λ and Λ_0.

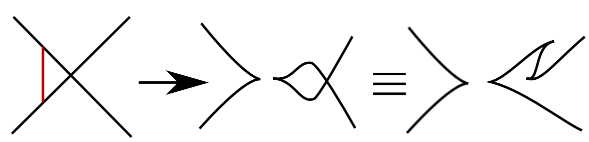

FIGURE 16. For any Legendrian knot Λ, perform a surgery near each crossing in order to get a set of disjoint Legendrian unknots.

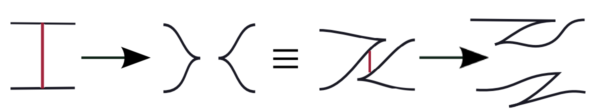

FIGURE 17. Surgeries used to convert to a link of Legendrian unknots can be "undone", at the cost of additional stabilizations.

A similar procedure can be used to construct a sequence of surgeries from Λ' to another Legendrian unknot Λ_0'; now we show it is possible to "reverse" this procedure and construct a sequence of surgeries from Λ_0' to $\widetilde{\Lambda}'$, a Legendrian obtained by applying stabilizations to Λ'. Figure 17 illustrates how every surgery that was used to get to a Legendrian unknot can be undone at the cost of adding additional zig-zags into the original strands. Figure 18 illustrates this procedure with an example.

As outlined at the beginning of the proof, these constructions prove the existence of an exact Lagrangian cobordism from $\Lambda_+ = \Lambda$ to $\Lambda_- = \Lambda'$. $\square$

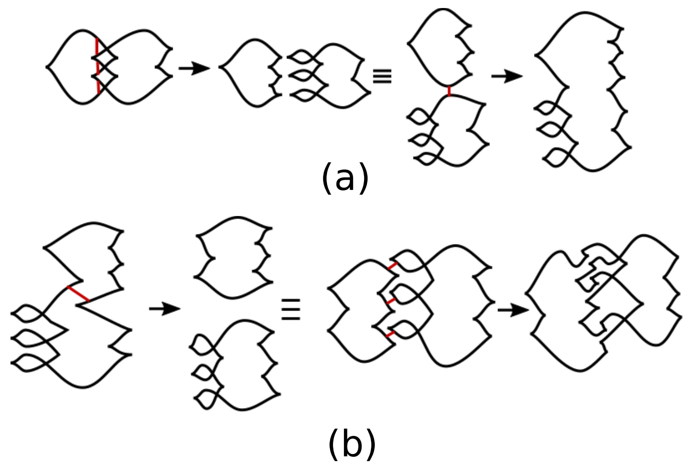

FIGURE 18. (a) Surgeries that give rise to an exact non-orientable Lagrangian cobordism from the max *tb* version of 3_1 to a stabilized unknot. (b) Surgeries that give rise to an exact non-orientable Lagrangian cobordism from the stabilized unknot to a stabilized representative of 3_1.

6. Additional Questions

We end with a brief discussion of some additional questions.

From results above, we know that exactly fillable Legendrian knots do not admit exact, non-orientable Lagrangian endocobordisms while stabilized Legendrian knots do. There are examples of Legendrian knots that are neither exactly fillable nor stabilized. As mentioned above, Ekholm, [16], has shown that if Λ is exactly fillable, then there exists an ungraded augmentation of $\mathcal{A}(\Lambda)$. By work of Sabloff, [41], and independently, Fuchs and Ishkhanov, [27], we then know that there exists an ungraded ruling of Λ. (Definitions of graded/ungraded rulings can be found, for example, in [32].) Then it follows by work of Rutherford, that the Kauffman bound on the maximal *tb* value for all Legendrian representatives of the smooth knot type of Λ is sharp, [39]. Thus, if the Kauffman bound is not sharp for the smooth knot type K, then no Legendrian representative of K is exactly fillable. So a natural question is:

Question 6.1. *If Λ is a maximal tb representative of a knot type K for which the upper bound on tb for all Legendrian representatives given by the Kauffman polynomial is not sharp, does Λ have an exact, non-orientable Lagrangian endocobordism?*

The Legendrian representative of $m(8_{19})$ shown in Figure 1 satisfies the hypotheses in Question 6.1; the Kauffman bound is known to be sharp for all knots with 10 or fewer crossings except $m(8_{19})$, $m(9_{42})$, $m(10_{124})$, $m(10_{128})$, $m(10_{132})$, and $m(10_{136})$; [36, 37, 39]. As mentioned in the Introduction, the paper [10] contains results that address this question which imply that the answer to Question 6.1 is no. However, this now spawns new questions. For example, consider the the max *tb* Legendrian representative of $m(10_{132})$ given as K_2 in Sivek's paper, [46, Figure 2]. The Legendrian K_2 is not stabilized, does not have an augmentation (and thus is not exactly fillable), and does not have a finite-dimensional representation. Does K_2 have an exact, non-orientable, Lagrangian endocobordism?

There are also examples of Legendrians with non-maximal *tb* that are not stabilized. For example, $m(10_{161})$ is a knot type where the unique maximal *tb* representative has a filling. However, there is a Legendrian representative with non-maximal *tb* that does not arise as a stabilization. As shown in [45, Figure 1], this non-maximal *tb*, non-stabilized Legendrian does have an ungraded ruling, and the characteristic algebra of K_2 does not have a finite-dimensional representation, [46].

Question 6.2. *Does the non-stabilized, non-maximal tb Legendrian representative of* $m(10_{161})$ *have an exact, non-orientable Lagrangian endocobordism?*

Additional examples of non-stabilized and non-maximal *tb* representatives can be found in the Legendrian knot atlas of Chongchitmate and Ng, [12].

There are additional questions that arise from the constructions of fillings. For example, it is known by results of Chantraine, [8], that orientable fillings realize the smooth 4-ball genus. In Figure 6, examples are given of non-orientable Lagrangian fillings of maximal *tb* representatives of 6_2 and $m(6_2)$ of cross-cap genus 2 and 4, respectively: the smooth 4-dimensional cross-cap number of both 6_2 and $m(6_2)$ is 1.

Question 6.3. *Does there exist a non-orientable Lagrangian filling of these Legendrian representatives of* 6_2 *and* $m(6_2)$ *of cross-cap genus* 1*?*

References

1. M. Audin, *Fibrés normal d'immersions en dimension double, points double d'immersions lagrangiennes et plongements totalement réels*, Comment. Math. Helvetici **63** (1988), 593–623.
2. ______, *Quelques remarques sur les surfaces lagrangiennes de Givental*, J. Geom. Physics **7** (1990), no. 4, 593–623.
3. M. Audin, F. Lalonde, and L. Polterovich, *Symplectic rigidity: Lagrangian submanifolds*, Holomorphic curves in symplectic geometry (M. Audin and J. Lafontaine, eds.), Progress in Mathematics, Birkhauser Verlag, 1994, pp. 271–321.
4. J. Baldwin and S. Sivek, *Invariants of Legendrian and transverse knots in monopole knot homology*, Preprint available as arXiv:1405.3275, 2014.
5. B. Boranda, L. Traynor, and S. Yan, *The surgery unknotting number of Legendrian links*, Involve **6** (2013), no. 3, 273–299. MR 3101761
6. F. Bourgeois, J. Sabloff, and L. Traynor, *Lagrangian cobordisms via generating families: construction and geography*, Algebr. Geom. Topol. **15**, 2439–2477.
7. O. Capovilla-Searle, *Surgery strings and loops of Legendrian knots*, Bachelor's Thesis, Bryn Mawr College, 2015.

8. B. Chantraine, *On Lagrangian concordance of Legendrian knots*, Algebr. Geom. Topol. **10** (2010), 63–85.
9. ———, *Lagrangian concordance is not a symmetric relation*, Quantum Topol. (2013), To Appear, Preprint available as arXiv:1301.3767.
10. B. Chantraine, G. Dimitroglou Rizell, P. Ghiggini, and R. Golovko, *Floer theory for Lagrangian cobordisms*, Preprint available on arXiv as arXiv:1511.0947v1, 2015.
11. Yu. Chekanov, *Differential algebra of Legendrian links*, Invent. Math. **150** (2002), 441–483.
12. W. Chongchitmate and L. Ng, *An atlas of Legendrian knots*, Experimental Mathematics (2013), 26–37.
13. C. Cornwell, L. Ng, and S. Sivek, *Obstructions to Lagrangian concordance*, Algebr. Geom. Topol., To appear.
14. G. Dimitroglou Rizell, *Lifting pseudo-holomorphic polygons to the symplectisation of $P \times R$ and applications*, Quantum Topol., To Appear, Preprint available as arXiv:1305.1572.
15. ———, *Legendrian ambient surgery and Legendrian contact homology*, J. Symplectic Geom. (2015), To appear, Preprint available as arXiv:1205.5544v5.
16. T. Ekholm, *Rational symplectic field theory over $\mathbb{Z}_2$ for exact Lagrangian cobordisms*, J. Eur. Math. Soc. (JEMS) **10** (2008), no. 3, 641–704.
17. ———, *Rational SFT, linearized Legendrian contact homology, and Lagrangian Floer cohomology*, Perspectives in analysis, geometry, and topology, Progr. Math., vol. 296, Birkhäuser/Springer, New York, 2012, pp. 109–145.
18. T. Ekholm, J. Etnyre, and M. Sullivan, *Legendrian contact homology in $P \times \mathbb{R}$*, Trans. Amer. Math. Soc. **359** (2007), no. 7, 3301–3335 (electronic).
19. T. Ekholm, K. Honda, and T. Kálmán, *Legendrian knots and exact Lagrangian cobordisms*, J. Eur. Math. Soc. (JEMS) (2012), To Appear, Preprint available as arXiv:1212.1519.
20. Ya. Eliashberg, *Invariants in contact topology*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), no. Extra Vol. II, 1998, pp. 327–338 (electronic).
21. Ya. Eliashberg, A. Givental, and H. Hofer, *Introduction to symplectic field theory*, Geom. Funct. Anal. (2000), no. Special Volume, Part II, 560–673.
22. Ya. Eliashberg and M. Gromov, *Lagrangian intersection theory: finite-dimensional approach*, Geometry of differential equations, Amer. Math. Soc. Transl. Ser. 2, vol. 186, Amer. Math. Soc., Providence, RI, 1998, pp. 27–118.
23. J. Etnyre, *Introductory lectures on contact geometry*, Topology and geometry of manifolds (Athens, GA, 2001), Proc. Sympos. Pure Math., vol. 71, Amer. Math. Soc., Providence, RI, 2003, pp. 81–107.
24. ———, *Legendrian and transversal knots*, Handbook of knot theory, Elsevier B. V., Amsterdam, 2005, pp. 105–185.
25. J. Etnyre and K. Honda, *Knots and contact geometry I: Torus knots and the figure eight knot*, J. Symplectic Geom. **1** (2001), no. 1, 63–120.
26. J. Etnyre, L. Ng, and V. Vértesi, *Legendrian and transverse twist knots*, J. Eur. Math. Soc. (JEMS) **15** (2013), no. 3, 451–512.
27. D. Fuchs and T. Ishkhanov, *Invariants of Legendrian knots and decompositions of front diagrams*, Mosc. Math. J. **4** (2004), no. 3, 707–717.
28. D. Fuchs and S. Tabachnikov, *Invariants of Legendrian and transverse knots in the standard contact space*, Topology **36** (1997), 1025–1053.
29. A. Givental, *Lagrange imbeddings of surfaces and the open Whitney umbrella*, Funktsional. Anal. i Prilozhen. **20** (1986), no. 3, 35–41,96.
30. M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), 307–347.

31. K. Hayden and J. Sabloff, *Positive knots and Lagrangian fillability*, Proc. Amer. Math. Soc. **143** (2015), 1813–1821.
32. T. Kálmán, *Rulings of Legendrian knots as spanning surfaces*, Pacific J. Math. **237** (2008), no. 2, 287–297.
33. F. Lalonde and J.C. Sikorav, *Sous-variétés lagrangiennes et lagrangiennes exactes des fibrés cotangents*, Comment. Math. Helvetici **66** (1991), 18–33.
34. D. McDuff and D. Salamon, *Introduction to symplectic topology*, second ed., Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1998.
35. S. Yu. Nemirovski, *Homology class of a Lagrangian Klein bottle*, Investiya Math. **73** (2009), no. 4, 689–698.
36. L. Ng, *Maximal Thurston-Bennequin number of two-bridge links*, Algebr. Geom. Topol. **1** (2001), 427–434 (electronic).
37. ———, *A Legendrian Thurston-Bennequin bound from Khovanov homology*, Algebr. Geom. Topol. **5** (2005), 1637–1653.
38. L. Polterovich, *The surgery of Lagrange submanifolds*, Geom. Funct. Anal. **1** (1991), no. 2, 198–210.
39. D. Rutherford, *Thurston-Bennequin number, Kauffman polynomial, and ruling invariants of a Legendrian link: the Fuchs conjecture and beyond*, Int. Math. Res. Not. (2006), Art. ID 78591, 15.
40. J. Sabloff, *Invariants of Legendrian knots in circle bundles*, Comm. Contemp. Math. **5** (2003), no. 4, 569–627.
41. ———, *Augmentations and rulings of Legendrian knots*, Int. Math. Res. Not. (2005), no. 19, 1157–1180.
42. J. Sabloff and L. Traynor, *Obstructions to Lagrangian cobordisms between Legendrian submanifolds*, Algebr. Geom. Topol. **13** (2013), 2733–2797.
43. J.M. Sabloff, Private Communication.
44. V.V. Shevchishin, *Lagrangian embeddings of the Klein bottle and combinatorial properties of mapping class groups*, Investiya Math. **73** (2009), no. 4, 797–859.
45. C. Shonkwiler and D. Vela-Vick, *Legendrian contact homology and nondestabilizability*, J. Symplectic Geom. **9** (2011), no. 1, 33–44.
46. S. Sivek, *The contact homology of Legendrian knots with maximal Thurston-Bennequin invariant*, J. Symplectic Geom. **11** (2013), no. 2, 167–178.

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