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AUTOMATIC REALIZABILITY OF GALOIS GROUPS OF ORDER 16

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ABSTRACT. This article examines the realizability of small groups of order $2^k, k \leq 4$, as Galois groups over arbitrary fields of characteristic not 2. In particular we consider automatic realizability of certain groups given the realizability of others.

1. INTRODUCTION

In this article we are concerned with automatic realizability results for small Galois 2-groups, particularly those of order 16. We ask when it is true that the realizability of a particular group G as a Galois group over a field K implies the realizability of another group H as a Galois group over K. In some cases, these results hold over all fields. In others, the automatic realizability depends on specific properties of K. We are also interested in field-theoretic criteria which guarantee the realizability of certain groups. By a famous result of Witt [Wi:1936], Satz, p.237, the realizability of a 2-group G over a field of characteristic 2 depends only on the minimal number of generators of G. For this reason we will always assume our fields to be of characteristic not 2.

We obtain our results primarily by considering the obstructions to the realizability of the groups, expressed in terms of products of quaternion algebras in the Brauer group $Br(K)$ of the base field K. By manipulating these expressions, one can often show that triviality of the obstruction for one group implies triviality of the obstruction for another. This is particularly true if additional information on the field, such as its level, is taken into consideration. The obstructions for these groups have appeared in various references. A complete survey of known results is provided in [GSS:1995]. We quote these results below and use them heavily in this article. Realizability questions have been considered previously in the two articles by C. U. Jensen, [Je:1989] and [Je:1992]. Jensen considers the structure of the groups explicitly in most instances, and often describes precisely how to construct fields realizing the groups, whereas our approach is less constructive. He also works

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with finite 2-groups in general, whereas we work just with groups of order at most 16. We obtain a number of new realizability results by our methods.

Our goal is to systematically examine the groups of order 16, and what can be said about the realizability of each group as a Galois group over fields of characteristic not 2. There are fourteen different groups of order 16, five of which are abelian. We let C_n denote the cyclic group of order n. The five abelian groups are $C_{16}, C_8 \times C_2, C_4^2, C_4 \times C_2^2$, and C_2^4. By known results ([Wh:1957], [KuLe:1975], [GSS:1995]) realizability for these groups is determined solely by realizability of C_4 and the size of the square class group $|\dot{K}/\dot{K}^2|$. Note that the criterion for the realizability of C_4^2 over K is simply the existence of two C_4-extensions containing distinct quadratic subfields. There are also the two decomposable groups obtained by taking the direct products of each of the two nonabelian groups of order 8, the dihedral group and the quaternion group, with C_2; we denote these two groups as $D_8 \times C_2$ and $Q_8 \times C_2$ respectively. Again, it is easy to see that $D_8 \times C_2$ (respectively $Q_8 \times C_2$) is realizable if and only if D_8 (respecitively Q_8) is realizable, and $|\dot{K}/\dot{K}^2|$ is large enough; see [GSS:1995]. (We write D_{2n} to denote the dihedral group of order $2n$.)

We introduce notation and presentations for each of the remaining seven indecomposable nonabelian groups below. We then give a list of the obstructions for each of the ten groups in which we will be interested. In §2 we give group-by-group descriptions of the results we have obtained for realizability, providing proofs for those results that are not widely known. In §3 we present results in terms of conditions on the level of the base field. Our results agree with or extend those obtained in [Je:1989], [Je:1992] for groups of order 16. Similar results have also recently been obtained in [L:1995].

The seven indecomposable nonabelian groups of order 16 are: DC, the central product of D_8 with C_4 amalgamating the central elements of order 2; $Q \curlywedge C$, the pullback of the system $(Q_8 \to C_2, C_4 \to C_2)$ and also a semidirect product $C_4 \rtimes C_4$; $D \curlywedge C$, the pullback of the system $(D_8 \to C_2, C_4 \to C_2)$ and also a semidirect product $(C_4 \times C_2) \rtimes C_2)$; D_{16}, the dihedral group of order 16; Q_{16}, the quaternion group of order 16; SD_{16}, the semidihedral or quasidihedral group of order 16; and M_{16}, the modular group of order 16. Their presentations are as follows:

$$\begin{gathered}
DC \cong \langle x, y, z | x^2 = y^2 = z^4 = 1, y^{-1}xy = xz^2, [x, z] = [y, z] = 1\rangle, \\
Q \curlywedge C \cong \langle x, y | x^4 = y^4 = 1, y^{-1}xy = x^{-1}\rangle, \\
D \curlywedge C \cong \langle x, y, z | x^4 = y^2 = z^2 = 1, y^{-1}xy = xz, [x, z] = [y, z] = 1\rangle, \\
D_{16} \cong \langle x, y | x^8 = y^2 = 1, y^{-1}xy = x^{-1}\rangle, \\
Q_{16} \cong \langle x, y | x^8 = y^4 = 1, y^{-1}xy = x^{-1}, y^2 = x^4\rangle, \\
SD_{16} \cong \langle x, y | x^8 = y^2 = 1, y^{-1}xy = x^3\rangle, \\
M_{16} \cong \langle x, y | x^8 = y^2 = 1, y^{-1}xy = x^5\rangle.
\end{gathered}$$

The following proposition is a compilation of all of the obstructions for the realizability of the ten groups we are interested in, taken from [GSS:1995]. Details and additional references can be found in that article.

Proposition 1. (1) *A necessary and sufficient condition for the realizability of the group C_4 as a Galois group over the field K is the existence of an element*

$a \in \dot{K} - \dot{K}^2$ such that $(a,-1) = 1 \in Br_2(K)$. In this case, the group C_4 can be realized by an extension of K which is quadratic over $K(\sqrt{a})$.

(2) *A necessary and sufficient condition for the realizability of the dihedral group D_8 of order 8 as a Galois group over the field K is the existence of independent elements $a, b \in \dot{K}/\dot{K}^2$ such that $(a,b) = 1 \in Br_2(K)$. In this case, the group D_8 is realized by an extension of K which is cyclic over $K(\sqrt{ab})$.*

(3) *A necessary and sufficient condition for the realizability of the group Q_8 as a Galois group over the field K is the existence of independent elements $a, b \in \dot{K}/\dot{K}^2$ such that $(a,b)(ab,-1) = 1 \in Br_2(K)$.*

(4) *A necessary and sufficient condition for the realizability of the group DC as a Galois group over the field K is the existence of independent elements $a, b, c \in \dot{K}/\dot{K}^2$ such that $(a,b)(c,-1) = 1 \in Br_2(K)$.*

(5) *A necessary and sufficient condition for the realizability of the group $Q \curlywedge C$ as a Galois group over the field K is the existence of independent elements $a, b \in \dot{K}/\dot{K}^2$ such that $(a,-1) = (a,b)(b,-1) = 1 \in Br_2(K)$.*

(6) *A necessary and sufficient condition for the realizability of the group $D \curlywedge C$ as a Galois group over the field K is the existence of independent elements $a, b \in \dot{K}/\dot{K}^2$ such that $(a,-1) = (a,b) = 1 \in Br_2(K)$.*

(7) *A necessary and sufficient condition for the realizability of the group D_{16} as a Galois group over the field K is the existence of independent elements $a, b \in \dot{K}/\dot{K}^2$ and an element $x \in \dot{K}$ such that $(a,b) = (a,2)(x,-ab) = 1 \in Br_2(K)$.*

(8) *A necessary and sufficient condition for the realizability of the group Q_{16} as a Galois group over the field K is the existence of independent elements $a, b \in \dot{K}/\dot{K}^2$ and an element $x \in \dot{K}$ such that $(a,b) = (a,2)(x,-ab)(ab,-1) = 1 \in Br_2(K)$.*

(9) *A necessary and sufficient condition for the realizability of the group SD_{16} as a Galois group over the field K is the existence of independent elements $a, b \in \dot{K}/\dot{K}^2$ and an element $x \in \dot{K}$ such that $(a,b) = (a,-2)(x,-ab) = 1 \in Br_2(K)$.*

(10) *A necessary and sufficient condition for the realizability of the group M_{16} as a Galois group over the field K is the existence of independent elements $a, b \in \dot{K}/\dot{K}^2$ and an element $x \in \dot{K}$ such that $(a,-1) = (a,2b)(x,-1) = 1 \in Br_2(K)$.*

2. Realizability results for specific groups

We begin this section with some definitions, and then proceed to examine results on realizability of each of the ten groups $C_4, D_8, Q_8, DC, Q \curlywedge C, D \curlywedge C, D_{16}, Q_{16}, SD_{16}$, and M_{16}. We use the obstructions provided in Proposition 1 as our criteria for realizability.

Definition. The *level* $s(K)$ of the field K is the least positive integer n such that -1 can be expressed as a sum of n squares; $s(K) = \infty$ if no such integer exists. If $s(K)$ is finite, then $s(K)$ is a power of 2. If $s(K) = \infty$, K is *formally real.*

Definition. The field K is *pythagorean* if every element which is a sum of squares is in fact a square. In this case either $s(K) = 1$ and K is quadratically closed, or $s(K) = \infty$.

Definition. An element $a \in \dot{K} - \dot{K}^2$ is *rigid* if the set of elements in K represented by the quadratic form $\langle 1, a \rangle$ is precisely $\dot{K}^2 \cup a\dot{K}^2$. The field K is said to be *rigid* if every $a \in \dot{K} - \pm\dot{K}^2$ is rigid. Notice that if K is rigid and $s(K) = 2$, then $2 \in \pm\dot{K}^2$, and if K is rigid and $s(K) = \infty$, then $2 \in \dot{K}^2$.

Proposition 2.1. *The group C_4 is realizable as a Galois group over K if and only if K is not pythagorean.*

Proof. The obstruction to the realizability of C_4 is equivalent to the existence of an element $a \in K$ which is a sum of two squares but not a square. □

Proposition 2.2. *Assume $|\dot{K}/\dot{K}^2| \geq 4$. The dihedral group D_8 of order 8 is always realizable over K unless $s(K) = 1$ and K is rigid.*

Proof. If $s(K) \geq 2$ and $|\dot{K}/\dot{K}^2| \geq 4$, then there exists $a \in \dot{K}$ such that $a, -a$ are independent mod $\dot{K}^2$. Since $(a, -a) = 1 \in Br_2(K)$ always, we see that D_8 is realizable. If $s(K) = 1$ and there do not exist $a, b \in \dot{K}$, independent mod $\dot{K}^2$, such that $(a, b) = 1 \in Br_2(K)$, then for all $a \in \dot{K} - \dot{K}^2$ we must have that the form $\langle 1, a\rangle$ only represents elements in $\dot{K}^2 \cup a\dot{K}^2$. Thus K is rigid. □

Proposition 2.3. *Assume $|\dot{K}/\dot{K}^2| \geq 4$. The quaternion group Q_8 of order 8 is always realizable over K unless either $s(K) = \infty$ and sums of squares are rigid in K, or $s(K) = 1$ and K is rigid. In particular, if K is not formally real, then Q_8 is realizable if and only if D_8 is realizable. If Q_8 is realizable, then C_4 is realizable.*

Proof. The proof of all but the last statement is contained in [Wa:1990] and can be derived from Proposition 1 using the equality $(a, b)(ab, -1) = (-a, -b)(-1, -1)$ in $Br_2(F)$. The last statement follows by noting that the given conditions for realizability of Q_8 imply K cannot be pythagorean. □

Proposition 2.4. *Assume $|\dot{K}/\dot{K}^2| \geq 8$. The group DC is realizable over K unless K is rigid, in which case $s(K) = 1, 2,$ or ∞.*

Proof. The result is derived by manipulating the obstruction $(a, b)(c, -1)$ (see [MiSm:1991]). □

We now consider realizability of $Q \curlywedge C$. Notice first that if $s(K) = 1$, the obstruction for realizability of $Q \curlywedge C$ reduces to that for Q_8 (and for D_8). If $s(K) = 2$, let $a = -1$, and choose b independent of -1. Then $(-1, -1) = 1 \in Br_2(K)$ and $(-1, b)(b, -1) = 1$, so there is a $Q \curlywedge C$-extension of K. The following proposition considers realizability of $Q \curlywedge C$ in general, giving an automatic realizability result that holds over any field K.

Proposition 2.5. *The group $Q \curlywedge C$ is realizable over K if and only if Q_8 is realizable over K.*

Proof. If $Q \curlywedge C$ is realizable, then clearly so is Q_8, as well as C_4, since they are quotients. Conversely, assume Q_8 is realizable. Then we have independent a, b such that $(a, b)(a, -1)(b, -1) = 1 \in Br_2(K)$. By manipulation of quaternion algebras we have $(-a, -b)(-1, -1) = 1$, and working with known criteria for the splitting of a tensor product of two quaternion algebras, we have $\langle a, b, ab\rangle \simeq \langle 1, 1, 1\rangle$, so a is a sum of three squares, but not a square. If a is a sum of two squares, then $(a, -1) = 1$, giving the desired $Q \curlywedge C$-extension. If not, then there exists $c \in K$ which is a sum of two squares but not a square, such that a is represented by the form $\langle 1, c\rangle$, so $\langle 1, c\rangle \simeq \langle a, ac\rangle$. Then $(a, -c) = 1 \in Br_2(K)$, so $(a, c)(a, -1) = 1$, and since c is a sum of two squares, $(c, -1) = 1$. Now a, c are independent since c is a sum of two squares but a is not, so we have a $Q \curlywedge C$-extension of K. □

Proposition 2.6. *Assume $|\dot{K}/\dot{K}^2| \geq 4$. The group $D \curlywedge C$ is realizable over K unless* (a) *$s(K) = 1$ and K is rigid,* (b) *$s(K) = \infty$ and K is pythagorean, or* (c) *$s(K) = 2$ and $C_4 \times C_4$ is not realizable as a Galois group over K.*

Proof. If $s(K) = 1$, the obstruction for $D \curlywedge C$ reduces to that for D_8, and the group is not realizable precisely when K is rigid (condition (a)). If $s(K) \geq 2$ and there exists $a \neq -1 \in \dot{K}$ which is a sum of two squares but not a square, then $(a, -1) = (a, -a) = 1 \in Br_2(K)$, satisfying the obstruction to the realizability of $D \curlywedge C$ as a Galois group over K. If there is no such a, then either all sums of squares are in fact squares, in which case K is pythagorean and formally real (condition (b)), or the only element $a \in \dot{K} - \dot{K}^2$ for which $(a, -1) = 1$ is $a = -1$, in which case $s(K) = 2$ and $C_4 \times C_4$ cannot be realized as a Galois group over K (condition (c)). Note that in condition (c), the only C_4-extensions of K will be those lying over $K(\sqrt{-1})$. □

When $s(K) = 1$, all the nonabelian groups of order 16 and exponent 4 are realizable if any of them are, provided $|\dot{K}/\dot{K}^2|$ is large enough. This is clear from the obstructions given in Proposition 1, since the obstructions for all of these groups become identical when -1 is a square in K. We have the following result.

Corollary 2.7. *If $s(K) = 1$, then any one of the groups $D_8, Q_8, Q \curlywedge C, D \curlywedge C$ is realizable over K if and only if the other three groups are realizable over K. If $|\dot{K}/\dot{K}^2| \geq 8$, the realizability of these groups is also equivalent to the realizability of DC.*

The following proposition links realizability of $D \curlywedge C$ with that of DC when $s(K) \geq 2$. Combining this with Corollary 2.7 shows that $D \curlywedge C$ automatically realizes DC if K has enough square classes.

Proposition 2.8. *Assume $s(K) \geq 2$ and $|\dot{K}/\dot{K}^2| \geq 8$. If $2 \in -\dot{K}^2$, then realizability of $D \curlywedge C$ implies D_{16} is realizable, which implies DC is realizable. If $2 \in \dot{K}^2$, then realizability of $D \curlywedge C$ implies SD_{16} is realizable, which implies DC is realizable. If $2 \notin \pm\dot{K}^2$, DC is realizable.*

Proof. First observe that if $2 \notin \pm\dot{K}^2$, then since the form $\langle 1, -2 \rangle$ represents -1, K cannot be rigid and DC is realizable. Now assume $D \curlywedge C$ is realizable. We may fix a, b independent mod $\dot{K}^2$ such that $(a, -1) = (a, b) = 1 \in Br_2(K)$. First assume $2 \equiv -1 \bmod \dot{K}^2$. In the obstruction for the realizability of D_{16}, choose the same a, b and let $x = 1$. Then the obstruction for D_{16} becomes (a, b) and $(a, 2)(1, -ab) = (a, -1)$, both of which we have assumed to be trivial. Thus D_{16} is realizable. Now assume $2 \equiv -1 \bmod \dot{K}^2$ and D_{16} is realizable. Then there exist a, b independent and $x \in \dot{K}$ such that $(a, b) = (a, -1)(x, -ab) = 1$. If $a \neq -b \in \dot{K}/\dot{K}^2$, then $a, b, -1$ are independent; and since $s(K) = 2$ (because $2 \equiv -1$), we have $(a, b)(-1, -1) = 1$, realizing DC. If $a = -b$, then we have $(a, -1) = 1$. Choose $c \in \dot{K}$ such that $a, c, -1$ are independent. Then $(c, -c)(a, -1) = 1$, realizing DC.

For the remainder of the proof, assume $2 \in \dot{K}^2$. Then the criterion for the realizability of SD_{16} is the existence of independent $c, d \in \dot{K}$ and $x \in \dot{K}$ such that $(c, d) = (c, -1)(x, -cd) = 1$. Choose $c = a, d = b$, where $(a, -1) = (a, b) = 1$ is given by the realizability of $D \curlywedge C$, and take $x = 1$.

We show below (Proposition 2.15) that, in general, realizability of SD_{16} implies realizability of C_4. If $s(K) \geq 4$, C_4 automatically realizes DC. If $s(K) = 2$, realizability of SD_{16} gives us independent a, b such that $(a, b) = (a, -1)(x, -ab) = 1$

for some $x \in \dot{K}$. If $a = -b$, then $(a, -1) = 1$. Choose $c \in \dot{K}$ such that $a, c, -1$ are independent. Then $(c, -c)(a, -1) = 1$, realizing DC. If $a \neq -b$, then $a, b, -1$ are independent, and $(a, b)(-1, -1) = 1$, since $s(K) = 2$, so we again obtain the realizability of DC. □

Proposition 2.9. *Assume $|\dot{K}/\dot{K}^2| \geq 4$. The group D_{16} is always realizable over K if $s(K) \geq 4$. If $s(K) = 2$, then D_{16} is realizable unless $2 \in -\dot{K}^2$, in which case realizability of $D \curlywedge C$ implies realizability of D_{16}. If $s(K) = 1$, any one of the groups D_{16}, Q_{16}, and SD_{16} is realizable if and only if the other two are, and realizability of any of these groups implies the realizability of D_8. This in turn implies the realizability of $M_{16}, Q_8, Q \curlywedge C$, and $D \curlywedge C$, and if $|\dot{K}/\dot{K}^2| \geq 8$, of DC as well. Thus if $s(K) = 1$ and $|\dot{K}/\dot{K}^2|$ is large enough, D_{16} automatically realizes all other groups of order 16.*

Proof. The criterion for the realizability of D_{16} is the existence of independent $a, b \in \dot{K}$, and $x \in \dot{K}$, such that $(a, b) = (a, 2)(x, -ab) = 1 \in Br_2(K)$. Assume first $s(K) \geq 2$. If $2 \notin \pm\dot{K}^2$, choose $a = -2, b = 2$; then D_{16} is realizable. If $2 \in \dot{K}^2$, choose $a = -b$ for some $a \notin \pm\dot{K}^2$. If $2 \in -\dot{K}^2$, then necessarily $s(K) = 2$, and $D \curlywedge C$ realizes D_{16} as shown in Proposition 2.8.

Assume now that $s(K) = 1$. The realizability criteria for all three groups D_{16}, Q_{16}, SD_{16} reduce to the existence of independent $a, b \in \dot{K}$ and $x \in \dot{K}$ such that $(a, b) = (a, 2)(x, ab) = 1 \in Br_2(K)$. Thus they are all equivalent and clearly imply the realizability of D_8. Realizability of D_8 implies the realizability of the other groups listed, as shown in Corollary 2.7 above for all groups except M_{16}, and in Corollary 2.14 below for M_{16}. When $s(K) = 1$, $(a, a) = 1$ for all $a \in \dot{K}$, realizing C_4, so realizability of D_{16} is sufficient to guarantee the realizability of all groups of order 16 over K when $|\dot{K}/\dot{K}^2|$ is large enough. □

Corollary 2.10. *If $s(K) = 1$ and $2 \in \dot{K}^2$, then the realizability of any one of the groups $D_8, Q_8, D \curlywedge C, Q \curlywedge C, D_{16}, Q_{16}, SD_{16}, M_{16}$ implies the realizability of all of the others.*

Proof. Under the given assumptions on K, we have already seen the equivalence of the realizability criteria for the groups D_{16}, Q_{16}, and SD_{16} (Proposition 2.9), and for the groups $D_8, Q_8, Q \curlywedge C, D \curlywedge C$, (Corollary 2.7), and the result for M_{16} follows from Corollary 2.14 below. Also D_8 is a quotient of D_{16}, so is trivially realized by it. Thus, what remains to show is that realizability of D_8 implies realizability of D_{16}. If $-1, 2 \in \dot{K}^2$, the criterion for D_{16} becomes $(a, b) = (x, -ab) = 1$ for some independent $a, b \in \dot{K}$ and some $x \in \dot{K}$. Choosing $x = 1$ makes D_{16} realizable whenever D_8 is. □

Just as realizability of Q_8 implies realizability of D_8, so does realizability of Q_{16} imply realizability of D_{16}, as the following proposition, together with Proposition 2.9, shows.

Proposition 2.11. *If $s(K) \leq 2$, Q_{16} is realizable if and only if D_{16} is. If $s(K) \geq 4$, realizability of Q_{16} implies realizability of $C_4, D_8, D \curlywedge C, D_{16}, SD_{16}$, and M_{16}, and also of DC if $|\dot{K}/\dot{K}^2| \geq 8$. If $2 \in \dot{K}^2$, realizability of Q_{16} is equivalent to realizability of Q_8.*

Proof. We have already seen in Proposition 2.9 that for $s(K) = 1$, Q_{16} is realizable over K if and only if D_{16} is. If $s(K) = 2$ and either group is realizable, then we have independent $a, b \in \dot{K}/\dot{K}^2$ with $(a, b) = 1$. If $a = -b$, then $(a, 2) = (a, 2)(-1, -1)$, and so realizability of either group implies that of the other. If $a \neq -b$, then $(a, 2)(x, -ab) = (a, 2)(x, -ab)(-1, -1) = (a, 2)(x, -ab)(-1, -ab)(-1, ab) = (a, 2)(-x, -ab)(-1, ab)$, so the obstruction for D_{16} becomes the obstruction for Q_{16} by replacing x with $-x$.

For the remainder of the proof, we assume $s(K) \geq 4$. The groups D_8 and D_{16} are always realizable if $|\dot{K}/\dot{K}^2| \geq 4$. If $2 \notin \dot{K}^2$, then we have seen that the groups $C_4, D \curlywedge C$, and, if $|\dot{K}/\dot{K}^2| \geq 8$, also DC are always realizable. To see that SD_{16} and M_{16} are realizable, note that $s(K) \geq 4$ and $2 \notin \dot{K}^2$ imply that $2, -2$ are independent. So taking $a = 2$, $b = -2$, and $x = 1$, the obstructions for SD_{16} and M_{16} are each trivial. Thus we now assume $2 \in \dot{K}^2$.

First we show realizability of Q_{16} is equivalent to realizability of Q_8. Given a, b realizing Q_{16}, we cannot have $a = -b$, since then $(-1, -1) = 1$, implying $s(K) \leq 2$. Thus we have $(a, b) = (x, -ab)(ab, -1)$ for some $x \in \dot{K}$. If $x \in \dot{K}^2$, then $(ab, -1) = 1$, so $(a, b)(ab, -1) = 1$, realizing Q_8. If $xab \in \dot{K}^2$, then $(x, -ab) = (x, -x) = 1$, so $(ab, -1) = 1$, and we again have Q_8 as above. Finally, if neither x nor xab is a square, then x, ab are independent. Further, $(x, ab)(xab, -1) = (x, ab)(x, -1)(ab, -1) = (x, -ab)(ab, -1) = 1$, realizing Q_8. Conversely, given Q_8 as a K-Galois group, we know we also have $Q \curlywedge C$ as a K-Galois group. Thus, there exist a, b independent such that $(a, -1) = (a, b)(b, -1) = 1$. Choose $c = ab, d = b, x = a$. Then $(c, d) = (ab, b) = (a, b)(b, -1) = 1$ and $(c, 2)(x, -cd)(cd, -1) = (a, -1) = 1$, and so we have a Q_{16}-extension of K.

It remains to show that when $2 \in \dot{K}^2$, realizability of Q_{16} implies that of C_4, $D \curlywedge C, SD_{16}$, and M_{16}. We have seen that Q_{16} realizes $Q \curlywedge C$, so we get C_4 because it is a quotient of $Q \curlywedge C$. Since $s(K) \geq 4$, C_4 is realizable if and only if $D \curlywedge C$ is realizable (Propositions 2.1 and 2.6). Suppose $(a, -1) = 1$ realizes C_4. Necessarily a and -1 are independent, since if $a = -1$, $s(K) \leq 2$. Let $b = -a$, $x = 1$ in the obstruction for M_{16}. Then we have $(a, -1) = (a, -a)(1, -1) = 1$, so M_{16} is realizable. To see realizability of C_4 implies realizability of SD_{16}, again let $b = -a$. Then under the assumption $2 \in \dot{K}^2$, $(a, -a) = (a, -1) = 1$, and so SD_{16} is realizable. □

Proposition 2.12. *If $s(K) = 1$, then SD_{16} is realizable if and only if D_{16} and Q_{16} are. If $s(K) \geq 2$ and $|\dot{K}/\dot{K}^2| \geq 4$, then SD_{16} is always realizable unless $2 \in \dot{K}^2$, in which case realizability of SD_{16} is implied by that of $D \curlywedge C$.*

Proof. We have already seen in Proposition 2.9 that if $s(K) = 1$, then SD_{16} is realizable if and only if D_{16} and Q_{16} are. Also we have seen in Proposition 2.8 that if $s(K) \geq 2$, $2 \in \dot{K}^2$, and $D \curlywedge C$ is realizable, then so is SD_{16}. So it remains to show that if $2 \notin \dot{K}^2$ and $s(K) \geq 2$, we can always realize SD_{16}. If $2 \notin -\dot{K}^2$, then $2, -2$ are independent, and we can let $a = 2, b = -2$; then $(2, -2) = 1$, so the obstruction for SD_{16} is trivial. If $2 \in -\dot{K}^2$, then necessarily $s(K) = 2$. Choose $a \notin \pm\dot{K}^2$, and let $b = -a$. Then $(a, -2) = 1$, so again SD_{16} is realizable. □

Proposition 2.13. *Assume $|\dot{K}/\dot{K}^2| \geq 4$. The group M_{16} is always realizable over K unless either $s(K) = 1$ and K is rigid, or $s(K) = \infty$ and K is pythagorean. Thus realizability of either $D \curlywedge C$ or Q_8 implies that of M_{16}.*

Proof. If $s(K) = 1$, the obstruction for M_{16} becomes simply $(a, 2b)$. If $2 \notin \dot{K}^2$, letting $b = 2$ makes the obstruction trivial. If $2 \in \dot{K}^2$, then the obstruction becomes (a, b), which is the same as the obstruction for D_8. This group is realizable if and only if K is not rigid. Now assume $s(K) = 2$. Let $a = -1, b \notin \pm\dot{K}^2$, and $x = 2b$. Then $(-1,-1) = (-1,2b)(-1,2b) = 1$, so M_{16} is realizable. Finally assume $s(K) \geq 4$. If $2 \notin \dot{K}^2$, then $2, -2$ are independent. Let $a = 2, b = -2, x = 1$. Then $(2,-1) = (2,-1)(1,-1) = 1$, so M_{16} is realizable. If $2 \in \dot{K}^2$, we have shown in the proof of Proposition 2.11 that realizability of C_4 implies the realizability of M_{16}, and C_4 is realizable unless $s(K) = \infty$ and K is pythagorean. □

Corollary 2.14. *Assume $s(K) = 1$. If $2 \in \dot{K}^2$, realizability of any of $D_8, Q_8, Q \curlywedge C, D \curlywedge C$ is equivalent to realizability of M_{16}. If $2 \notin \dot{K}^2$ and $|\dot{K}/\dot{K}^2| \geq 4$, then M_{16} is always realizable.*

Proof. This follows directly from the preceding proof and Corollary 2.7. □

Proposition 2.15. *Realizability of any of M_{16}, Q_{16}, or SD_{16} implies realizability of C_4.*

Proof. First, C_4 is a quotient of M_{16}, so this is a case of trivial realizability. It is not a quotient of either of the other two groups, however, so these results are of more interest. We may assume $s(K) = \infty$ and K is pythagorean, since C_4 otherwise is always realizable. Thus $2 \in \dot{K}^2$. We have observed in this case that Q_{16} is equivalent to Q_8 (Proposition 2.11) which is equivalent to $Q \curlywedge C$ (Proposition 2.5), which has C_4 as a quotient, so realizability of Q_{16} implies realizability of C_4.

Finally, suppose SD_{16} is realizable and $2 \in \dot{K}^2$. Then there exist independent $a, b \in \dot{K}$ such that $(a,b) = 1$ and $(a,-1)(x,-ab) = 1$ for some $x \in \dot{K}$. By the common slot, or linkage, property for quaternion algebras, there exists $y \in \dot{K}$ such that $(a,-1) = (y,-1)$, $(x,-ab) = (y,-ab)$, and $(y,ab) = 1$. Thus $(ay,-1) = (a,-1)(y,-1) = 1$. If $y \neq a$, then $(ay,-1) = 1$ implies the existence of a C_4-extension containing $K(\sqrt{ay})$. If $y = a$, then $(a,ab) = 1 = (a,b)$ implies $(a,a) = 1$, so there exists a C_4-extension containing $K(\sqrt{a})$. □

3. Realizability results by conditions on the field K

In this section we organize the realizability results according to the level of K. For groups of order 16, our results yield or improve the results obtained in [Je:1989], [Je:1992] and [L:1995]. Jensen's results are obtained using a variety of methods, and many hold for larger classes of 2-groups, not just those of order 16. When considering just groups of order 16, however, most of these results can be obtained and even strengthened working solely with the obstructions to the realizability of the groups.

Theorem 3.1. *Assume $s(K) = 1$.*

(1) *The group $(C_4)^n$ is realizable if and only if $|\dot{K}/\dot{K}^2| \geq 2^n$.*

(2) *If any one of the groups $D_8, Q_8, D \curlywedge C, Q \curlywedge C$ is realizable, so are the rest. If $|\dot{K}/\dot{K}^2| \geq 8$, then realizablility of these groups is equivalent to realizability of DC.*

(3) *If $2 \notin \dot{K}^2$ and $|\dot{K}/\dot{K}^2| \geq 4$, then M_{16} is always realizable.*

(4) *If any one of the groups D_{16}, Q_{16}, SD_{16} is realizable, then so are the other two. Realizability of any of these groups trivially realizes D_8. Thus, if $|\dot{K}/\dot{K}^2| \geq 16$, realizability of any of these groups implies realizability of all groups of order 16.*

(5) *If $2 \in \dot{K}^2$, i.e. if K contains a primitive 8th root of unity, then realizability of any of the groups D_8, D_{16}, M_{16} implies realizability of the others. Thus, if $|\dot{K}/\dot{K}^2| \geq 16$ and K contains a primitive 8th root of unity, realizability of any nonabelian group of order 16 implies realizability of all groups of order 16.*

Proof. Most of the results follow directly from the propositions and corollaries in §2. For (1), observe that $(a,-1)=1$ for all $a \in \dot{K}$. Thus the number of independent C_4-extensions that exist over K is precisely the number of independent square classes in K. (2) follows directly from Corollary 2.7 and (3) from Corollary 2.14. The first statement of (4) follows from Proposition 2.9. The first statement of (5) comes from Corollary 2.10. The final statements of (4) and (5) now follow from the rest of the theorem. Notice that $|\dot{K}/\dot{K}^2| \geq 8$ is sufficient for realizability of all of these groups except C_2^4. □

Theorem 3.2. *Assume $s(K)=2$.*

(1) *The group C_4 is always realizable.*

(2) *If $|\dot{K}/\dot{K}^2| \geq 4$, the groups $D_8, Q_8, Q \curlywedge C$, and M_{16} are always realizable.*

(3) *The group D_{16} is realizable if and only if Q_{16} is.*

(4) *The group $D \curlywedge C$ is realizable if and only if $C_4 \times C_4$ is.*

(5) *If $|\dot{K}/\dot{K}^2| \geq 4$ and $2 \in \dot{K}^2$, then D_{16} is always realizable, and realizability of $D \curlywedge C$ implies realizability of SD_{16}. If $|\dot{K}/\dot{K}^2| \geq 8$, realizability of SD_{16} implies realizability of DC.*

(6) *If $|\dot{K}/\dot{K}^2| \geq 4$ and $-2 \in \dot{K}^2$, then SD_{16} is always realizable, and realizability of $D \curlywedge C$ implies realizability of D_{16}. If $|\dot{K}/\dot{K}^2| \geq 8$, realizability of D_{16} implies realizability of DC.*

(7) *If $|\dot{K}/\dot{K}^2| \geq 4$ and $2 \notin \pm\dot{K}^2$, then the groups $D_{16}, Q_{16}, D \curlywedge C, SD_{16}$, and $C_4 \times C_4$ are always realizable. If $|\dot{K}/\dot{K}^2| \geq 8$, DC is also realizable. If $|\dot{K}/\dot{K}^2| \geq 16$, every group of order 16 is realizable.*

Proof. (1) follows from Proposition 2.1; note in particular that $(-1,-1)=1 \in Br_2(K)$, realizing C_4. (2) comes from Propositions 2.2, 2.3, 2.5, and 2.13. Proposition 2.11 gives (3). Proposition 2.6 shows the realizability of $D \curlywedge C$ is implied by that of $C_4 \times C_4$. Conversely, if $D \curlywedge C$ is realizable, there exists independent $a, b \in \dot{K}$ with $(a,-1)=(a,b)=1$. If $a \neq -1$, there exist C_4-extensions containing $K(\sqrt{a})$ and $K(\sqrt{-1})$. If $a=-1$, there exist C_4-extensions containing $K(\sqrt{b})$ and $K(\sqrt{-1})$. This gives (4). Propositions 2.8 and 2.9 give both (5) and (6).

Finally, for (7), Proposition 2.9 gives realizability of D_{16}; this together with Proposition 2.11 gives realizability of Q_{16}. The group $C_4 \times C_4$ is realizable because $(-1,-1)=(-1,2)=1$, so there exist independent C_4-extensions of K containing $K(\sqrt{-1})$ and $K(\sqrt{2})$ respectively. Proposition 2.12 gives the realizability of SD_{16}. The realizability of DC follows from Proposition 2.8. Finally, all of these combined with (2) give the last statement. □

Theorem 3.3. *Assume $s(K) \geq 4$ and $|\dot{K}/\dot{K}^2| \geq 4$.*

(1) *The groups D_8 and D_{16} are always realizable.*

(2) *The group Q_8 is realizable if and only if $Q \curlywedge C$ is.*

(3) *If $2 \notin \dot{K}^2$, the groups $C_4, D \curlywedge C, SD_{16}$, and M_{16} are always realizable. If $|\dot{K}/\dot{K}^2| \geq 8$, then DC is also realizable.*

(4) *If $2 \in \dot{K}^2$, then Q_{16} is realizable if and only if Q_8 is. If any one of the groups $C_4, D \curlywedge C, SD_{16}, M_{16}$ is realizable, so are the rest. All of these groups are automatically realized by Q_8. If $|\dot{K}/\dot{K}^2| \geq 8$, realizability of any of these groups implies realizability of DC. Thus, in this case, realizability of all nonabelian groups of order 16 is implied by realizability of Q_8.*

Proof. Propositions 2.2 and 2.9 give (1), and (2) is exactly Proposition 2.5. Since $(-1,2) = 1 \in Br_2(K)$ always, if $2 \notin \dot{K}^2$, K cannot be pythagorean, and since $-2 \notin \dot{K}^2$, this also implies that K cannot be rigid. Thus Propositions 2.1, 2.6, 2.12, and 2.13 give (3). The first statement of (4) follows from Proposition 2.11. The second statment can be derived as follows: Propositions 2.12 and 2.13 show realizability of $D \curlywedge C$ imply realizability of M_{16} and SD_{16}. Proposition 2.15 shows realizability of either of these groups implies realizability of C_4. And Proposition 2.6 together with Proposition 2.1 gives the fact that C_4 realizes $D \curlywedge C$ when $s(K) \geq 4$. The third statement of (4) follows from Proposition 2.3, and the fourth from Proposition 2.4. The final statement summarizes the results above. □

We have the following proposition, improving on results of Jensen which state that realizability of Q_8 implies realizability of $Q \curlywedge C$, implying realizability of D_8, and realizability of D_8 implies realizability of one of D_{16}, SD_{16}, or M_{16}.

Proposition 3.4. *Realizability of Q_8 is equivalent to that of $Q \curlywedge C$ and implies that of D_8. Realizability of D_8 implies realizability of at least one of D_{16} or M_{16}.*

The first result in Proposition 3.4 follows from Proposition 2.5 and Theorems 3.1(2), 3.2(2), and 3.3(2). The second result can be derived as follows. Theorem 3.1(3),(5) shows that if $s(K) = 1$, D_8 automatically realizes M_{16}. The realizability of D_8 implies $|\dot{K}/\dot{K}^2| \geq 4$, so if $s(K) = 2$, M_{16} is realizable by Theorem 3.2(2), and if $s(K) \geq 4$, D_{16} is realizable by Theorem 3.3(1).

Jensen also derives a number of results which depend on square class considerations and existence of certain roots of unity in the field. Again, if one considers just groups of order 16, these results can be obtained from Theorems 3.1–3.3. The roots of unity needed by Jensen in this case are at most 8th roots of 1, so considering the behavior of -1 and 2 in $\dot{K}/\dot{K}^2$ suffices.

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