$M|G|\infty$ Queue Busy Period Length with PME Distribution Analysis through Laplace Transform

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ABSTRACT

In this article it is shown that if the busy period of a $M|G|\infty$ queue system is PME distributed, the respective service time is a random variable with a long-tail distribution. The result is obtained through Laplace transforms analysis.

***Keywords**: $M|G|\infty$, busy period, PME distribution, long-tail distribution, Laplace transform.*

INTRODUCTION

In a $M|G|\infty$ queue system, λ is the Poisson process arrivals rate, α is the mean service time, $G(.)$ represents the service time distribution function and so $\alpha = \int_0^\infty [1 - G(t)]\, dt$ since $G(.)$ is the distribution function of a positive random variable. The traffic intensity is $\rho = \lambda\alpha$ and B is the busy period length.

Note the busy period study importance, for this queuing system, because in its operation any customer, when it arrives, finds immediately an available server. So the problem is "for how long the servers – and how many servers? – must be available? That is: how long is the busy period length?"

When looking for a family of positive distribution functions, $F_r(t)$, with tail behavior:

$$F_r^c(t) \sim \alpha_r t^{-r}, \text{as } t \to \infty,$$

with mean 1 and manageable Laplace transform, to serve as test in their queues, with long-tail service-time distributions, waiting-time tail probabilities study (Abate, Choudhury and Whitt, 1994) created the PME-**P**areto **M**ixture of **E**xponentials distributions family.

Being mandatory a finite mean it must be $r > 1$. So the Pareto distribution could be the adequate choice. For this distribution $F_r^c(t) = \left(\frac{r-1}{r}\right)^r t^{-r}, t \geq \frac{r-1}{r}$, and its density is $f_r(t) = r\left(\frac{r-1}{r}\right)^r t^{-(r+1)}, t \geq \frac{r-1}{r}$. As their moments are $m_n' = \frac{r}{r-n}\left(\frac{r-1}{r}\right)^r, 1 \leq n < r$, the squared coefficient of variation is $c = \frac{1}{r(r-2)}$.

As this Pareto family does not allow small values and its modifications, that would do so, Laplace transforms are not expressible in terms of elementary functions, (Abate, Choudhury, Whitt, 1994) proposed a new modification, the PME-**P**areto **M**ixture of **E**xponentials distributions family, with

$$g_r(t) = \int_{\frac{r-1}{r}}^{\infty} f_r(y) y^{-1} e^{-\frac{t}{y}} dy, r > 1 \qquad (1.1)$$

where $f_r(y) = r\left(\frac{r-1}{r}\right)^r x^{-(r+1)}, x \geq \frac{r-1}{r}$, is a Pareto distribution probability density function. It is long-tail type distribution. The g_r moments are

$$m_n = n! \frac{r}{r-n}\left[\frac{r-1}{r}\right]^n, n = 1,2, \ldots \qquad (1.2).$$

It will be supposed that the *B* probability density function is given by (1.1). And through Laplace transform analysis it will be emphasized that in these circumstances the service time is a random variable with a long-tail distribution.

THE PME LAPLACE TRANSFORM

Calling $\hat{g}_r(s)$ the Laplace transform of a PME with parameter *r*, see again (Abate, Choudhury, Whitt, 1994),

$$\hat{g}_r(s) = r\left(\frac{r-1}{r}\right)^r \int_0^{\frac{r}{r-1}} \frac{x^r}{s+x} dx, r > 1 \qquad (2.1).$$

But $\hat{g}_r(s) = r\left(\frac{r-1}{r}\right)^r \left(\int_0^{\frac{r}{r-1}} \left(x^{r-1} - \frac{sx^{r-1}}{s+x}\right) dx\right) = r\left(\frac{r-1}{r}\right)^r \left(\int_0^{\frac{r}{r-1}} x^{r-1} dx - s\int_0^{\frac{r}{r-1}} \frac{x^{r-1}}{s+x} dx\right) = r\left(\frac{r-1}{r}\right)^r \left(\left[\frac{x^r}{r}\right]_0^{\frac{r}{r-1}} - s\int_0^{\frac{r}{r-1}} \frac{x^{r-1}}{s+x} dx\right) = r\left(\frac{r-1}{r}\right)^r \left(\frac{1}{r}\left(\frac{r}{r-1}\right)^r - s\int_0^{\frac{r}{r-1}} \frac{x^{r-1}}{s+x} dx\right) = 1 - sr\left(\frac{r-1}{r}\right)^r \int_0^{\frac{r}{r-1}} \frac{x^{r-1}}{s+x} dx$.

If $\hat{h}_r(s)$ is the PME with parameter *r* tail Laplace transform, as $\hat{h}_r(s) = \frac{1}{s} - \frac{1}{s}\hat{g}_r(s)$,

$$\hat{h}_r(s) = r\left(\frac{r-1}{r}\right)^r \int_0^{\frac{r}{r-1}} \frac{x^{r-1}}{s+x} dx, r > 1 \quad (2.2).$$

So, after (2.2),

$$\hat{h}_r^{(n)}(s) = (-1)^n n!\, r\left(\frac{r-1}{r}\right)^r \int_0^{\frac{r}{r-1}} \frac{x^{r-1}}{(s+x)^{n+1}}\, dx, n = 0,1,2,\ldots \qquad (2.3)$$

where $\hat{h}_r^{(n)}$ is the n^th^ order derivative of $\hat{h}_r$.

Then $\hat{h}_r^{(n)}(0) = (-1)^n n!\, r\left(\frac{r-1}{r}\right)^r \int_0^{\frac{r}{r-1}} x^{r-n-2}\, dx, n = 0,1,2,\ldots$. But

$$\int_0^{\frac{r}{r-1}} x^{r-n-2}\, dx = \begin{cases} \left[\frac{x^{r-n-1}}{r-n-1}\right]_0^{\frac{r}{r-1}}, n \neq r-1 \\ [\log|x|]_0^{\frac{r}{r-1}}, n = r-1 \end{cases} = \begin{cases} \frac{\left(\frac{r}{r-1}\right)^{r-(n+1)}}{r-(n+1)}, r > n+1 \\ -\infty, r \leq n+1 \end{cases}.$$

So $\hat{h}_r^{(n)}(0) = \begin{cases} (-1)^n n!\, r\left(\frac{r-1}{r}\right)^r \frac{\left(\frac{r}{r-1}\right)^{r-(n+1)}}{r-(n+1)}, r > n+1 \\ (-1)^n(-\infty), 1 < r \leq n+1 \end{cases}$ or, equivalently,

$$\hat{h}_r^{(n)}(0) = \begin{cases} (-1)^n n!\, r\left(\frac{r-1}{r}\right)^r \frac{\left(\frac{r}{r-1}\right)^{r-(n+1)}}{r-(n+1)}, n < r+1 \\ (-1)^n(-\infty), n \geq r-1 \end{cases}, r > 1 \quad (2.4).$$

$M|G|\infty$ QUEUE BUSY PERIOD TAIL LAPLACE TRANSFORM

Call $U(t)$ the $M|G|\infty$ busy period tail and $u(s)$ the respective Laplace transform so, see (Ferreira, Andrade, 2010a),

$$\frac{d}{dt}\left(\frac{1 - e^{-\lambda \int_0^t [1-G(v)]dv}}{1 - e^{-\rho}}\right) = TL^{-1}\left(\frac{1}{1-e^{-\rho}} \frac{\lambda u(s)}{\lambda u(s) + 1}\right) \qquad (3.1)$$

and

$$\frac{e^{-\lambda \int_0^t [1-G(v)]dv} \lambda(1 - G(t))}{1 - e^{-\rho}} = TL^{-1}\left(\frac{1}{1-e^{-\rho}} \frac{\lambda u(s)}{\lambda u(s) + 1}\right) \quad (3.2)$$

being $TL^{-1}(.)$ the inverse Laplace transform.

So, $\int_0^\infty t^n \frac{e^{-\lambda\int_0^t[1-G(v)]dv}\lambda(1-G(t))}{1-e^{-\rho}}dt = (-1)^n \frac{1}{1-e^{-\rho}}\left(\frac{\lambda u(s)}{\lambda u(s)+1}\right)_{s=0}^{(n)} \Leftrightarrow \int_0^\infty t^n e^{-\lambda\int_0^t[1-G(v)]dv}\lambda(1-G(t))dt = (-1)^n\left(\frac{\lambda u(s)}{\lambda u(s)+1}\right)_{s=0}^{(n)}$.

As $e^{-\lambda\int_0^t[1-G(v)]dv} \leq 1$ the consequence is that $\int_0^\infty t^n\lambda(1-G(t))dt \geq (-1)^n\left(\frac{\lambda u(s)}{\lambda u(s)+1}\right)_{s=0}^{(n)} \Leftrightarrow \int_0^\infty \frac{1-G(t)}{\alpha}\,dt \geq \frac{(-1)^n}{\rho}\left(\frac{\lambda u(s)}{\lambda u(s)+1}\right)_{s=0}^{(n)}$ this entire happening if $\frac{1}{1-e^{-\rho}}\frac{\lambda u(s)}{\lambda u(s)+1}$ is in fact a probability density function Laplace transform, being enough that $(-1)^n\left(\frac{\lambda u(s)}{\lambda u(s)+1}\right)_{s=0}^{(n)} \geq 0, n = 0,1,2,\ldots$.

$M|G|\infty$ QUEUE BUSY PERIOD WITH PME DISTRIBUTION

Note that in the $\frac{\lambda u(s)}{\lambda u(s)+1}$ n^th order derivative, $u^{(n)}(s)$ always appears with a positive sign in the numerator and, for $s = 0$, if $u(s)$ is given by (2.1), in that order n derivative the denominator is $(\lambda+1)^{n+1}$. It is enough to be attentive to the quotient derivative expression and note that $\hat{h}_r(0) = 1$.

So, after (2.4), it is concluded:

-If there is a service distribution such that the $M|G|\infty$ queue busy period is distributed as a PME distribution with parameter *r,* the service equilibrium distribution moments of order greater than $r-1$, centered in the origin, are infinite.

Note that

-The service equilibrium distribution, with these moments, is a long-tail distribution, see again (Abate, Choudhury, Whitt, 1994),

-As $\frac{u(s)}{\frac{e^{\rho}-1}{\lambda}}$ is the $M|G|\infty$ queue busy period equilibrium distribution Laplace transform it is also concluded that if this has moments of order greater than $n, . n \in \mathbb{N}$, infinite, the same happens with the service time equilibrium distribution, which is: they are both long-tail distributions.

CONCLUDING REMARKS

As it is stated in (Abate, Choudhury, Whitt, 1994) the PME are long-tail distributions. So it is checked in this work that there is an uncontested association between long-tail service distributions and long-tail busy period distributions for the $M|G|\infty$ queue, as it was shown in (Ferreira, Andrade, 2012).

The PMEs were introduced in (Abate, Choudhury, Whitt, 1994). There they were a tool to investigate properties of waiting times tail probabilities in queues with long-tail service-time distributions. For this investigation the authors developed algorithms for computing the waiting time distribution by Laplace transform inversion when the Laplace transforms of the inter-arrival time and service time distributions are known. The procedure here trailed is similar.

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