ISCTE IUL
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ISCTE-Lisbon University Institute

Business School

Department of Finance

Three Essays on the Valuation of American-style Options

A Thesis presented in partial fulfillment of the Requirements for the Degree of Doctor in Finance

by

João Pedro Bento Ruas

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July, 2013

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Resume

This thesis addresses the valuation of American-style standard and barrier options in three separate and self-contained papers:

A. **Pricing and Static Hedging of American-style Options under the Jump to Default Extended CEV Model**

This paper prices (and hedges) American-style options through the *static hedge approach* (SHP) proposed by Chung and Shih (2009) and extends the literature in two directions. First, the SHP approach is adapted to the *jump to default extended CEV* (JDCEV) model of Carr and Linetsky (2006), and plain-vanilla American-style options on defaultable equity are priced. The robustness and efficiency of the proposed pricing solutions are compared with the *optimal stopping approach* offered by Nunes (2009), under both the JDCEV framework and the nested *constant elasticity of variance* (CEV) model of Cox (1975), using different elasticity parameter values. Second, both the SHP and the optimal stopping approaches are extended to the valuation of American-style capped options.

B. **General Put-Call Symmetry for American-style Barrier Options**

This paper derives put-call symmetries for American-style single and double barrier options. Using the change of numeraire technique proposed by Geman et al. (1995) and Schroder (1999) we are able to derive these symmetries without im-

posing previous assumptions on the process followed by the underlying asset. Our results are tested through an extensive numerical analysis run under the *constant elasticity of variance* model.

C. **In-Out Parity Relations and Early Exercise Boundaries for American-style Barrier Options**

This paper derives new in-out parity relations for American-style puts with a down barrier and American-style calls with an up barrier. More importantly, we also propose a novel representation for the early exercise boundary of American-style double knock-out options in terms of the simpler optimal stopping boundary for a nested single barrier American-style option. Therefore, we are able to extend the *static hedge portfolio* approach to the valuation of American-style double barrier knock-out options. Our results are tested through an extensive numerical analysis run under the *geometric Brownian motion* (GBM) and the *constant elasticity of variance* models.

JEL Classification: G13.

Keywords: American-style options; Barrier options; GBM model; CEV model; JDCEV model.

Resumo

Esta tese aborda a avaliação de opções de estilo Americano, com e sem barreira, em três artigos distintos:

A. **Pricing and Static Hedging of American-style Options under the Jump to Default Extended CEV Model**

   Este artigo avalia (e faz o hedging) de opções de estilo Americano através do *static hedge approach* (SHP) proposto por Chung and Shih (2009) e estende a literatura em duas direções. Primeiramente, o SHP é adaptado ao modelo *jump to default extended CEV* (JDCEV) de Carr and Linetsky (2006), e são avaliadas opções de estilo Americano sem barreira sobre activos com possibilidade de falência. A robustez e a eficiência das soluções de avaliação propostas, são comparadas com o *optimal stopping approach* de Nunes (2009), no âmbito dos modelos JDCEV e *constant elasticity of variance* (CEV) de Cox (1975), considerando diferentes valores para o parâmetro de elasticidade. Em segundo lugar, tanto o SHP como o *optimal stopping approach* são estendidos para a avaliação de opções de estilo Americano com um *cap*.

B. **General Put-Call Symmetry for American-style Barrier Options**

   Este artigo deriva relações de simetria *put-call* para opções de estilo Americano com uma e duas barreiras. Usando a técnica de mudança de numerário proposta

por Geman et al. (1995) e Schroder (1999) estas simetrias são derivadas sem impor restrições prévias sobre o processo estocástico seguido pelo activo subjacente. Os resultados são testados através de uma extensa análise numérica sob o modelo *constant elasticity of variance*.

C. **In-Out Parity Relations and Early Exercise Boundaries for American-style Barrier Options**

   Este artigo deriva novas relações de paridade *in-out* para puts de estilo Americano com uma barreira inferior e calls de estilo Americano com uma barreira superior. Mais importante, é proposta uma nova representação da fronteira de exercício antecipado para opções de estilo Americano com dupla barreira *knock-out*, em termos da fronteira de exercício óptimo de uma opção de estilo Americano com uma só barreira. Assim sendo, o método *static hedge portfolio* é estendido para a avaliação de opções de estilo Americano com dupla barreira *knock-out*. Os resultados são testados através de uma extensa análise numérica sob os modelos *geometric Brownian motion* e *constant elasticity of variance*.

JEL Classification: G13.

Keywords: American-style options; Barrier options; GBM model; CEV model; JDCEV model.

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Contents

1. Introduction

This thesis addresses the valuation of American-style standard and barrier options in three separate and self-contained papers.

The valuation of American-style option contracts remains as one of the most challenging problems in the financial economics literature, given the difficulty, if not impossibility, of achieving elegant analytical pricing solutions as those offered in the prominent work of Black and Scholes (1973) and Merton (1973). This problem is even enhanced when we allow the possibility of a single or double barrier feature for the American-style option.

In this thesis, singles and doubles, knock-outs and knock-ins, American-style barrier options, are considered. An American-style knock-out option (without rebate) becomes worthless if the barrier is touched by the underlying asset price; otherwise, it can be exercised at or before the expiry date. Additionally, and as pointed by Gao et al. (2000, Footnote 15), for the knock-out event and the exercise date to be well defined, the option contract must be specified in a way such that when the asset price first touches the barrier, the option holder has the option to either exercise or let the option contract be knocked out. An American-style knock-in option becomes an American-style standard option if and only if the barrier is touched by the underlying spot price, before or at, the option's expiry date.

The first paper offers three contributions for the existent option pricing literature. First, and most importantly, the *static hedge portfolio* (SHP) approach of Chung and Shih (2009) for pricing American-style standard options under the *geometric Brownian motion* (GBM) and *constant elasticity of variance* (CEV) model of Cox (1975) is extended to American-style standard and capped options under the *jump to default extended CEV* (JDCEV) model of Carr and Linetsky (2006). Such extension should prove useful to researchers and practitioners in corporate debt and equity derivatives markets, because the JDCEV model is consistent with three well-known facts that have found empirical support in the literature, namely: the existence of a negative correlation between stock returns and realized volatility (*leverage effect*), as observed, for instance, by Bekaert and Wu (2000); the inverse relation between the implied volatility and the strike price of an option contract (*implied volatility skew*), as documented, for example, in Dennis and Mayhew (2002); and the empirical evidence of a positive relationship between default probabilities and equity volatility, documented, for instance, in Campbell and Taksler (2003).

Second, the optimal stopping approach of Nunes (2009) is extended for the pricing of American-style capped options, assuming that the recovery value associated to the put can be paid at the default time—as in Nunes (2009, Section VII)—or at the maturity date of the option.

Third, the SHP approach is implemented to price American-style options under the CEV model of Cox (1975) for other values of the elasticity parameter (beta) besides the $4/3$ benchmark used by Chung and Shih (2009), thus accommodating both direct and indirect leverage effects observed across a wide variety of options markets.

Additionally, analytical solutions are offered to efficiently compute the hedge ratios of the European-style pricing solutions proposed by Carr and Linetsky (2006), which contain an embedded credit derivative (i.e. a European-style default claim) in the case of the put contracts. Given the recent market practitioners' concerns of linking equity derivatives markets and credit markets, such closed-form solutions should be a viable alternative for implementing efficient schemes to jointly hedge equity and credit derivatives under this class of hybrid credit-equity models.

The second paper focus on the put-call symmetry. The put-call symmetry holds whenever the price of a put option can be recovered from the price of a call option (and vice versa) through a suitable change in its function arguments.

Using the same change of numéraire technique as Schroder (1999), and considering a market structure in which the underlying asset price follows a single factor but a Markovian diffusion process, Detemple (2001) derives the put-call symmetry for American-style single knock-out options. Using this same change of numéraire technique presented in Schroder (1999) and based on two mild assumptions we are able to extend the put-call symmetry to other types of American-style barrier options.

Our contribution to the existent literature is threefold: First, we extend the put-call symmetry to American-style single knock-in barrier options; Second, and more importantly, the symmetry is extended to American-style double knock-out and knock-in options; Third, we provide numerical results for all the put-call symmetries derived under the CEV model of Cox (1975).

The third paper provides three important results concerning the valuation of American-style barrier options. First, we prove a new *in-out parity* relation between American-style barrier options. Second, we derive the early exercise boundary of American-style double knock-out options in terms of the simpler exercise boundary of single barrier American-style options. Finally, we use the previous finding to extend the SHP approach of Chung and Shih (2009) to the valuation of American-style double barrier knock-out options.

Concerning our first result, it is well known that for European-style contracts, the sum of the prices of knock-in and knock-out barrier options is equal to the price of a standard European-style option. However, and as argued by Dai and Kwok (2004, Page 187) or Chung et al. (2013, Page 191), such in-out barrier parity relation cannot be applied to American-style barrier options. In this article, we extend the in-out barrier parity relation for American-style puts with a down barrier and calls with an up barrier.

Our second theoretical contribution to the existent literature on option pricing consists in obtaining the early exercise boundary of any American-style double barrier option in terms of the barrier levels and as a function of the optimal stopping boundary of a nested single barrier American-style option. The rational behind our main result is similar to the one used, for instance, by Broadie and Detemple (1995, Theorem 1) or Gao et al. (2000, Theorem 6) to relate the early exercise boundaries of American-style standard and single barrier option contracts. Therefore, we are able to reduce the valuation of American-style double barrier options to the same complexity level as the one faced to price simpler single barrier contracts. Bearing in mind that only a few numerical methods have been proposed for pricing American-style double barrier options—as, for instance, the trinomial method of Ritchken (1995) or the PDE schemes of Zvan et al. (2000)—and that most of them are confined to the simpler GBM assumption of Black and Scholes (1973) and Merton (1973), our result should be used for all option pricing models that generate viable pricing solutions for single barrier American-style options.

To illustrate the potential of the previous finding, we easily extend the SHP methodology of Chung and Shih (2009) to the valuation of American-style double barrier knock-out options.

All the new results proposed in the third paper are only based on two mild assumptions: the existence of a risk neutral measure (i.e. on no arbitrage, in the Harrison and Pliska

(1981) sense); and the existence of a unique, continuous and monotone early exercise boundary that separates the *continuation* and *exercise* regions of any American-style option contract. Therefore, our results can be applied to the whole class of single diffusion processes discussed in Detemple and Tian (2002, Propostion 1), and even to jump-diffusion models—but subject to the technical conditions stated in Pham (1997, Theorem 3.2).

This thesis proceeds as follows. Chapter 2 presents the first paper. Chapter 3 presents the second paper. Chapter 4 presents the third paper. Finally, Chapter 5 concludes.

2. Pricing and Static Hedging of American-style Options under the Jump to Default Extended CEV Model*

Abstract: This paper prices (and hedges) American-style options through the *static hedge approach* (SHP) proposed by Chung and Shih (2009) and extends the literature in two directions. First, the SHP approach is adapted to the *jump to default extended CEV* (JDCEV) model of Carr and Linetsky (2006), and plain-vanilla American-style options on defaultable equity are priced. The robustness and efficiency of the proposed pricing solutions are compared with the *optimal stopping approach* offered by Nunes (2009), under both the JDCEV framework and the nested *constant elasticity of variance* (CEV) model of Cox (1975), using different elasticity parameter values. Second, both the SHP and the optimal stopping approaches are extended to the valuation of American-style capped options.

JEL Classification: G13.

Keywords: American-style options; Static hedging; CEV model; JDCEV model.

*This paper is a joint work with José Carlos Dias and João Pedro Nunes and was submitted to the Journal of Banking and Finance.

2.1 Introduction

The valuation (and hedging) of American-style option contracts remains as one of the most challenging problems in the financial economics literature, given the difficulty, if not impossibility, of achieving elegant analytical pricing solutions as those offered in the prominent work of Black and Scholes (1973) and Merton (1973) (hereafter, BSM). The absence of an exact and closed-form pricing solution for the American-style put (or call, but on a dividend-paying asset) stems from the fact that the option price and the early exercise boundary must be determined simultaneously as the solution of the same free boundary problem that has been set up by McKean (1965). These difficulties have thus lead to the development of several alternative valuation methodologies, ranging from numerical solution methods to analytical approximations, all attempting to efficiently price a variety of financial products with early exercise features.[2.1]

The numerical methods include, for instance, the finite difference schemes introduced by Brennan and Schwartz (1977), the binomial models of Cox et al. (1979) and Rendleman and Bartter (1979), the trinomial lattice schemes of Boyle and Tian (1999) and Tian (1993), and the least-squares Monte Carlo scheme of Longstaff and Schwartz (2001). Even though these numerical methods are flexible, simple to implement, and generally convergent, they are also too time consuming and do not provide the comparative statics attached to an analytical representation of the option pricing solution.

One of the first analytical approximations is offered by Barone-Adesi and Whaley (1987), using the quadratic method of MacMillan (1986), but its convergence properties are still weak, especially for long maturity options. Johnson (1983) and Broadie and Detemple (1996) provide lower and upper bounds for American options, but these are based on regression coefficients that are estimated through a time-demanding calibration to a

[2.1] See Barone-Adesi (2005) for a general overview of the literature.

large set of options contracts.

Carr (1998) proposes a fast and accurate randomization approach that uses Richardson extrapolation. Geske and Johnson (1984) approximate the American option price through an infinite series of Bermudan-style options exercisable at a finite number of exercise points, and use also Richardson extrapolation. Several extensions of the original Geske-Johnson methodology have been proposed in the literature to overcome its non-uniform convergence feature. For instance, Bunch and Johnson (1992) implement a modified two-point Geske-Johnson scheme, Chang et al. (2007) propose a *repeated-Richardson extrapolation* procedure, and Chung and Shackleton (2007) generalize the Geske and Johnson (1984) method through a two-point scheme based not only on the inter-exercise time dimension, but also on the time to maturity of the option contract. However, one of the main disadvantages of all these extrapolation schemes is the indetermination of the sign for the approximation error.

Kim (1990), Jacka (1991), Carr et al. (1992), and Jamshidian (1992) initiated another stream of the option pricing literature: The so-called *integral representation method*. However, the numerical efficiency of this approach depends on the specification that is adopted for the unknown early exercise boundary. For example, Huang et al. (1996) adopt a time consuming step function approximation, while Ju (1998) proposes a multipiece exponential representation of the early exercise boundary.

All the aforementioned studies are based on the usual lognormal assumption of BSM, and most of them differ only in the specification adopted for the early exercise boundary. Kim and Yu (1996), Detemple and Tian (2002), and Nunes (2009) constitute three notable exceptions. The former two studies extend the integral representation method to alternative diffusion processes. However, and in opposition to the standard geometric Brownian motion case, such an extension does not offer an analytic representation for the integral equation representing the early exercise premium, which undermines its computational efficiency. Based on the *optimal stopping approach* of Bensoussan (1984) and Karatzas (1988), Nunes (2009) proposes an alternative characterization of the standard American-style option price that is valid for any continuous representation of the exercise boundary and for any Markovian price process describing the dynamics of the underlying asset price, including the *jump to default constant elasticity of variance* (JDCEV) model of Carr and Linetsky (2006).

Chung and Shih (2009) tackle the American-style option pricing problem through the *static hedge approach* (hereafter, SHP) initially developed by Bowie and Carr (1994), Derman et al. (1995), and Carr et al. (1998) for hedging European-style exotic options (in which case the boundary is known ex-ante). Bowie and Carr (1994) and Carr et al. (1998) hedge via static positions of European-style options for a continuum of strikes (but with the same maturity date as the exotic option), while Derman et al. (1995) use a continuum of standard European-style options with subsequent maturities and strikes equaling the (known) boundary until the maturity of the exotic option. The pricing methodology proposed by Chung and Shih (2009) for valuing American-style options combines both methods: It uses standard European-style options with multiple strikes and multiple maturities, because the optimal exercise boundary is not known ex-ante. This approach creates a static portfolio of European-style options whose values match the payoff of the American-style option being hedged at expiration and along the boundary, by applying the value-matching and smooth-pasting conditions on the early exercise boundary.

This paper offers three contributions for the existent option pricing literature. First, and most importantly, we generalize the SHP approach for pricing American-style standard and capped options under the JDCEV model. Such extension should prove useful to researchers and practitioners in corporate debt and equity derivatives markets, because the JDCEV model is consistent with three well-known facts that have found empirical support in the literature, namely: the existence of a negative correlation between stock returns and realized volatility (*leverage effect*), as observed, for instance, by Bekaert and Wu (2000); the inverse relation between the implied volatility and the strike price of an option contract (*implied volatility skew*), as documented, for example, in Dennis and Mayhew (2002); and the empirical evidence of a positive relationship between default probabilities and equity volatility, documented, for instance, in Campbell and Taksler (2003).

Second, we extend the optimal stopping approach of Nunes (2009) for the pricing of American-style capped options, assuming that the recovery value associated to the put can be paid at the default time—as in Nunes (2009, Section VII)—or at the maturity date of the option. Our numerical results show that the SHP methodology is more efficient (and as accurate as) the optimal stopping approach of Nunes (2009).

Third, we implement the SHP approach to price American-style options under the *constant elasticity of variance* (CEV) model of Cox (1975) for other values of the elasticity parameter (beta) besides the $4/3$ benchmark used by Chung and Shih (2009), thus accommodating both direct and indirect leverage effects observed across a wide variety of options markets. As argued in Nunes (2009, page 1250), the optimal stopping approach offers a better speed-accuracy trade-off than the pricing methodology of Detemple and Tian (2002)—which is based on the (very time consuming) full recursive method of Huang et al. (1996)—and the accelerated recursive scheme of Kim and Yu (1996), for valuing option contracts under the CEV assumption. Therefore, the accuracy and efficiency of the SHP approach for valuing American-style options under the CEV model will be compared against the option pricing framework proposed by Nunes (2009).

Additionally, we offer analytical solutions to efficiently compute the hedge ratios of the European-style pricing solutions proposed by Carr and Linetsky (2006), which contain an embedded credit derivative (i.e. a European-style default claim) in the case of the put contracts. Given the recent market practitioners' concerns of linking equity derivatives markets and credit markets, such closed-form solutions should be a viable alternative for implementing efficient schemes to jointly hedge equity and credit derivatives under this class of hybrid credit-equity models.

The remainder of this article is organized as follows. Section 2.2 presents a brief summary of the JDCEV framework. Section 2.3 extends the optimal stopping and SHP approaches for the valuation of American-style standard and capped options under the JDCEV model. Both valuation methods are numerically tested in Section 2.4, under both the CEV and JDCEV setups. Finally, Section 2.5 summarizes the results and contains concluding remarks. All accessory results are relegated to the Appendix.

2.2 JDCEV model

For the analysis to remain self-contained, the next three subsections provide, respectively, a brief summary of the building blocks for the general JDCEV setup, the closed-form solutions for pricing European-style options under the time-homogeneous JDCEV model with constant parameters, and a specialization of the JDCEV modeling architecture to the classic CEV model.

2.2.1 Model setup

Carr and Linetsky (2006) construct a unified framework for the valuation of corporate liabilities, credit derivatives, and equity derivatives as contingent claims written on a defaultable stock. The price of the defaultable stock is modeled as a time-inhomogeneous diffusion process solving the stochastic differential equation

$$\frac{dS_t}{S_t} = \left[r_t - q_t + \lambda\left(t, S\right)\right] dt + \sigma\left(t, S\right) dW_t^{\mathbb{Q}}, \tag{2.1}$$

with $S_{t_0} > 0$, and where the risk-free interest rate r_t and the dividend yield q_t are deterministic functions of time, while the instantaneous volatility of equity returns $\sigma\left(t, S\right)$ and the default intensity $\lambda\left(t, S\right)$ can also be state-dependent. $W_t^{\mathbb{Q}} \in \mathbb{R}$ is a standard Wiener process generating the filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq t_0\}$, and the martingale probability measure $\mathbb{Q}$, associated to the "money market account" numéraire, is taken as given.[2.2]

The pricing model proposed by Carr and Linetsky (2006) can either diffuse or jump to default. In the first case, bankruptcy occurs at the first passage time of the stock price to 0:

$$\tau_0 := \inf\left\{t > t_0 : S_t = 0\right\}. \tag{2.2}$$

Alternatively, the stock price can also jump to default at the first jump time

$$\tilde{\zeta} := \inf\left\{t > t_0 : \frac{1}{\mathbf{1}_{\{t<\tau_0\}}} \int_{t_0}^{t} \lambda\left(u, S\right) du \geq \Theta\right\}, \tag{2.3}$$

of the integrated hazard process to the level drawn from an exponential random variable Θ independent of $W_t^{\mathbb{Q}}$ and with unit mean. Therefore, the time of default is simply given by[2.3]

$$\zeta = \tau_0 \wedge \tilde{\zeta}, \tag{2.4}$$

and $\mathbb{D} = \{\mathcal{D}_t, t \geq t_0\}$ is the filtration generated by the default indicator process $\mathcal{D}_t = \mathbf{1}_{\{t>\zeta\}}$.

[2.2]Note that the inclusion of the hazard rate $\lambda(t, S)$ in the drift of equation (2.1) compensates the stockholders for default (with zero recovery) and insures, under the risk-neutral measure $\mathbb{Q}$, an expected rate of return equal to the risk-free interest rate. Nevertheless, such an equivalent martingale measure will not be unique because the arbitrage-free market considered by Carr and Linetsky (2006) is incomplete in the sense that the jump to default will not be modeled as a stopping time of $\mathbb{F}$.

[2.3]For any two real numbers x and y, we denote by $x \vee y$ and $x \wedge y$, respectively, their maximum and minimum.

As in the classical CEV model of Cox (1975), Carr and Linetsky (2006) accommodate the leverage effect and the implied volatility skew by specifying the instantaneous stock volatility as a power function:

$$\sigma(t, S) = a_t S_t^{\bar{\beta}}, \tag{2.5}$$

where $\bar{\beta} < 0$ is the volatility elasticity parameter and $a_t > 0$, $\forall t$, is a deterministic volatility scale function. Yet, to be consistent with the empirical evidence of a positive relationship between default probabilities and equity volatility, Carr and Linetsky (2006) further assume that the default intensity is an increasing affine function of the instantaneous stock variance:

$$\lambda(t, S) = b_t + c\,\sigma(t, S)^2, \tag{2.6}$$

where $c \geq 0$, and $b_t \geq 0$, $\forall t$, is a deterministic function of time.

Following the hybrid credit-equity modeling framework of Carr and Linetsky (2006), taking $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$, and assuming no default by time t_0 (i.e. $\zeta > t_0$), the time-t_0 value of a European-style call (if $\phi = -1$) or put (if $\phi = 1$) on the stock price S, with strike K, recovery value R (i.e. the amount that the owner of a defaulted claim receives upon default), and maturity date T ($\geq t_0$), can be represented by the following building blocks:

$$v_{t_0}(S_{t_0}, K, T, R; \phi, \eta) = v_{t_0}^0(S_{t_0}, K, T; \phi) + v_{t_0}^D(S_{t_0}, R, T; \phi, \eta), \tag{2.7}$$

where

$$v_{t_0}^0(S_{t_0}, K, T; \phi) := \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^T r_l dl}(\phi K - \phi S_T)^+ \mathbf{1}_{\{\zeta > T\}} \middle| \mathcal{G}_{t_0}\right], \tag{2.8}$$

is the option value but conditional on no default by time T, and

$$v_{t_0}^D(S_{t_0}, R, T; \phi, \eta) := \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\eta} r_l dl}(\phi R)^+ \mathbf{1}_{\{\zeta \leq T\}} \middle| \mathcal{G}_{t_0}\right], \tag{2.9}$$

for $\eta \in \{\zeta, T\}$. In the case of a European call, there is no recovery if the firm defaults. However, for the European put, equation (2.9) corresponds to a recovery payment equal to the strike (i.e. $R = K$), that can be paid at the default time ζ or at the maturity date T, depending on the recovery assumption.[2.4] In the latter case, equation (2.9) can be rewritten as

$$v_{t_0}^{D}\left(S_{t_0}, R, T; \phi, T\right) = (\phi R)^{+} e^{-\int_{t_0}^{T} r_l dl} \left[1 - SP\left(S_{t_0}, t_0; T\right)\right], \tag{2.10}$$

where

$$SP\left(S_{t_0}, t_0; T\right) := \mathbb{E}_{\mathbb{Q}}\left(\mathbf{1}_{\{\zeta > T\}} \middle| \mathcal{G}_{t_0}\right) = \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_{t_0}^{T} \lambda(l,S) dl} \mathbf{1}_{\{\tau_0 > T\}} \middle| \mathcal{F}_{t_0}\right), \tag{2.11}$$

is understood as the risk-neutral probability of surviving beyond time $T > t_0$, and is defined in Carr and Linetsky (2006, Equation 3.1).

2.2.2 Pricing solutions for European-style options

For constant r, q, a, b, and c, and assuming that $\zeta > t_0$, Carr and Linetsky (2006, Proposition 5.5) show that the t_0-price of a European-style call option with strike price K and expiry date at time T $(\geq t_0)$ is given by[2.5]

$$\begin{aligned} v_{t_0}\left(S_{t_0}, K, T, 0; -1, \eta\right) &= e^{-q(T-t_0)} S_{t_0} \Phi_{+1}\left(0, \frac{k^2}{\rho}; \delta_+, \frac{x^2}{\rho}\right) \\ &\quad - e^{-(r+b)(T-t_0)} K \left(\frac{x^2}{\rho}\right)^{\frac{1}{2|\bar{\beta}|}} \Phi_{+1}\left(-\frac{1}{2|\bar{\beta}|}, \frac{k^2}{\rho}; \delta_+, \frac{x^2}{\rho}\right), \end{aligned} \tag{2.12}$$

[2.4]Note that recovery claims with $\eta = T$ and $\eta = \zeta$ correspond to defaultable zero-coupon bonds under *fractional recovery of treasury* and *fractional recovery of face value*, respectively—see, for instance, Schönbucher (2003, Section 6.1), Bélanger et al. (2004, Section 3), and Lando (2004, Section 5.7).

[2.5]Note that the recovery component of the European-style call, $v_{t_0}^{D}\left(S_{t_0}, R, T; -1, \eta\right)$, is zero, and, therefore, $v_{t_0}\left(S_{t_0}, K, T, 0; -1, \eta\right) = v_{t_0}^{0}\left(S_{t_0}, K, T; -1\right)$.

whereas the t_0-price of the corresponding European-style put, but conditional on no default by time T, is given by

$$
\begin{aligned}
v_{t_0}^0\left(S_{t_0}, K, T; 1\right) &= e^{-(r+b)(T-t_0)}\, K \left(\frac{x^2}{\rho}\right)^{\frac{1}{2|\bar{\beta}|}} \Phi_{-1}\left(-\frac{1}{2|\bar{\beta}|}, \frac{k^2}{\rho}; \delta_+, \frac{x^2}{\rho}\right) \\
&\quad - e^{-q(T-t_0)}\, S_{t_0}\, \Phi_{-1}\left(0, \frac{k^2}{\rho}; \delta_+, \frac{x^2}{\rho}\right),
\end{aligned}
\tag{2.13}
$$

where

$$x := \frac{1}{|\bar{\beta}|} S_{t_0}^{|\bar{\beta}|}, \tag{2.14}$$

$$k := \frac{1}{|\bar{\beta}|} K^{|\bar{\beta}|} e^{-|\bar{\beta}|(r-q+b)(T-t_0)}, \tag{2.15}$$

$$\delta_+ := \frac{2c+1}{|\bar{\beta}|} + 2, \tag{2.16}$$

and

$$\rho \equiv \rho(t_0, T) := \begin{cases} a^2\left(T - t_0\right) & \Leftarrow \quad r - q + b = 0 \\ \frac{a^2}{2|\bar{\beta}|(r-q+b)}\left(1 - e^{-2|\bar{\beta}|(r-q+b)(T-t_0)}\right) & \Leftarrow \quad r - q + b \neq 0 \end{cases}. \tag{2.17}$$

The functions $\Phi_\theta\left(p, y; v, \lambda\right) := \mathbb{E}^{\chi^2(v,\lambda)}\left(X^p \mathbf{1}_{\{\theta X \geq \theta y\}}\right)$ represent, for $\theta \in \{-1, 1\}$, the truncated p-th moments of a noncentral chi-square random variable X with v degrees of freedom and noncentrality parameter λ, as defined in Carr and Linetsky (2006, Equations 5.11 and 5.12).

The time-t_0 value of the recovery part of the European-style put option, to be paid at the maturity date T, is given by equation (2.10) with

$$SP\left(S_{t_0}, t_0; T\right) = e^{-b(T-t_0)} \left(\frac{x^2}{\rho}\right)^{\frac{1}{2|\bar{\beta}|}} M\left(-\frac{1}{2|\bar{\beta}|}; \delta_+, \frac{x^2}{\rho}\right), \tag{2.18}$$

and where $M\left(p; v, \lambda\right) := \mathbb{E}^{\chi^2(v,\lambda)}\left(X^p\right)$ is the p-th raw moment of a noncentral chi-square random variable X with v degrees of freedom and noncentrality parameter λ, as defined in Carr and Linetsky (2006, Equation 5.10).

There may be, however, put option contracts paying also the fixed recovery value R, but at the default time ζ (i.e. considering the fractional recovery of face value assumption). Following Carr and Linetsky (2006, Equation 5.15), the value of a claim that pays R dollars at the default time ζ is given by

$$\begin{aligned} & v_{t_0}^D\left(S_{t_0}, R, T; 1, \zeta\right) \\ = \quad & R \int_{t_0}^T e^{-(r+b)(u-t_0)} \left[b \left(\frac{x^2}{\rho\left(t_0, u\right)} \right)^{\frac{1}{2|\bar{\beta}|}} M\left(-\frac{1}{2|\bar{\beta}|}; \delta_+, \frac{x^2}{\rho\left(t_0, u\right)} \right) \right. \\ & \left. + c\, a^2\, S_{t_0}^{2\bar{\beta}}\, e^{-2|\bar{\beta}|\,(r-q+b)(u-t_0)} \left(\frac{x^2}{\rho\left(t_0, u\right)} \right)^{\frac{1}{2|\bar{\beta}|}+1} M\left(-\frac{1}{2|\bar{\beta}|} - 1; \delta_+, \frac{x^2}{\rho\left(t_0, u\right)} \right) \right] du. \end{aligned} \tag{2.19}$$

Remark 2.1 *In all numerical computations presented in this paper, and to enhance the efficient computation of the pricing solutions* (2.12) *and* (2.13)*, we use the algorithm recently offered by Dias and Nunes (2012) for valuing the truncated* p*-th moments* $\Phi_\theta\left(p, y; v, \lambda\right)$*, with* $\theta \in \{-1, 1\}$*. The raw moments* $M\left(p; v, \lambda\right)$ *contained in the right-hand side of equations* (2.18) *and* (2.19) *are computed also using the same algorithm via the identity provided by Carr and Linetsky (2006, Equation 5.13).*

2.2.3 CEV model

As shown by Carr and Linetsky (2006, Remark 5.2), the (no bankruptcy and *local volatility*) standard time-homogeneous CEV model of Cox (1975) can be nested into the aforementioned modeling framework.[2.6]

[2.6] For additional background on the CEV process, see, for instance, Cox (1975), Emanuel and MacBeth (1982), Schroder (1989), Davydov and Linetsky (2001, 2003), Nunes (2009), Dias and Nunes (2011), and Larguinho et al. (2013).

Definition 2.1 *The classic CEV model of Cox (1975) can be nested into the general framework described by equations* (2.1) *to* (2.6) *through the following restrictions:* $r_t = r$, $q_t = q$, $\lambda(t, S) = 0$, $\sigma(t, S) = \delta S_t^{\frac{\beta}{2}-1}$, *and* $\tau_0 = \infty$, *with* $\delta, \beta \in \mathbb{R}$.

Remark 2.2 *For all numerical experiments under the CEV assumption we adopt the Schroder (1989) pricing solutions by expressing the time-*t_0 *value of a European-style call option on the asset price* S*, with strike* K*, and maturity at time* T *(*$\geq t_0$*) in terms of the complementary distribution function of a noncentral chi-square law. As usual, the corresponding time-*t_0 *value of a European-style put arises immediately if one applies the put-call parity.*

Remark 2.3 *The implementation of the SHP approach for valuing American-style options under the CEV model requires the knowledge of the analytical solutions for the hedge ratios of the corresponding European-style plain-vanilla options. Fortunately, the necessary delta measures can be computed in closed-form and are given in Larguinho et al. (2013).*

Remark 2.4 *The valuation of option prices and deltas under the CEV model requires the computation of the noncentral chi-square distribution function. There is an extensive literature devoted to the efficient computation of this cumulative distribution function (cdf). For all numerical computations of option prices and hedge ratios under the CEV assumption we use Benton and Krishnamoorthy (2003, Algorithm 7.3) for computing the cdf of a noncentral probability law.*[2.7]

[2.7] See Larguinho et al. (2013) who have shown that this algorithm clearly offers the best speed-accuracy trade-off for computing the cdf of a noncentral probability law in the context of the CEV model.

2.3 Valuation of American-style options

To the authors knowledge, the valuation of American-style standard options under the JDCEV framework is only pursued by Nunes (2009, Section VII) under an optimal stopping approach, and assuming the recovery payment at the default time (i.e. $\eta = \zeta$). In this section, we extend the optimal stopping approach of Nunes (2009) for the payment of the recovery value at the maturity date T (i.e. $\eta = T$), and for the valuation of American-style capped options. More importantly, we also generalize the SHP approach proposed by Chung and Shih (2009) for the pricing of both standard and capped American-style options under the JDCEV model.

2.3.1 Standard American-style contracts

Following Nunes (2009, Equation 53), and assuming that $\zeta > t_0$, the time-t_0 value of an American-style standard option under the JDCEV model, on the stock price S, with strike K, recovery value R, and maturity date T ($\geq t_0$), can be represented by the following Snell envelope:

$$\begin{aligned} V_{t_0}\left(S_{t_0}, K, T, R; \phi, \eta\right) &= \sup_{\tau \in \mathcal{T}} \left\{ \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{T\wedge\tau} r_l dl} \left(\phi K - \phi S_{T\wedge\tau}\right)^{+} \mathbf{1}_{\{\zeta > T\wedge\tau\}} \middle| \mathcal{G}_{t_0} \right] \right. \qquad (2.20) \\ &\quad \left. + \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\eta} r_l dl} (\phi R)^{+} \mathbf{1}_{\{\zeta \leq T\wedge\tau\}} \middle| \mathcal{G}_{t_0} \right] \right\}, \end{aligned}$$

where $\phi \in \{-1, 1\}$, $\eta \in \{\zeta, T\}$, and $\mathcal{T}$ is the set of all stopping times (taking values in $[t_0, \infty]$) for the enlarged filtration $\mathbb{G} = \{\mathcal{G}_t, t \geq t_0\}$. In the case of an American call ($\phi = -1$), there is no recovery if the firm defaults. However, for the American put ($\phi = 1$), the second expectation on the right-hand side of equation (2.20) corresponds to a recovery payment equal to the strike (i.e. $R = K$) at the default time ζ or at the maturity date T (as long as the default event precedes both expiry and early exercise dates).

Given that the random variable Θ is independent of $\mathbb{F}$, Carr and Linetsky (2006, Equations 3.2 and 3.4) or Schönbucher (2003, Proposition 5.3 and Equation 5.32) imply that equation (2.20) can be rewritten in terms of the restricted filtration $\mathbb{F}$:

$$\begin{aligned} & V_{t_0}\left(S_{t_0}, K, T, R; \phi, \eta\right) \qquad (2.21) \\ = \; & \sup_{\tau \in \mathcal{T}} \left\{ \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_{t_0}^{T\wedge\tau}(r_l+\lambda(l,S))dl} \left(\phi K - \phi S_{T\wedge\tau}\right)^{+} \mathbf{1}_{\{\tau_0 > T\wedge\tau\}} \middle| \mathcal{F}_{t_0} \right] \right. \\ & + \mathbf{1}_{\{\eta = T\}} (\phi R)^{+} e^{-\int_{t_0}^{T} r_l dl} \left[1 - \mathbb{E}_{\mathbb{Q}} \left(e^{-\int_{t_0}^{T\wedge\tau} \lambda(l,S)dl} \mathbf{1}_{\{\tau_0 > T\wedge\tau\}} \middle| \mathcal{F}_{t_0} \right) \right] \\ & \left. + \mathbf{1}_{\{\eta = \zeta\}} (\phi R)^{+} \mathbb{E}_{\mathbb{Q}} \left[\int_{t_0}^{T\wedge\tau} e^{-\int_{t_0}^{v}(r_l+\lambda(l,S))dl} \lambda(v,S) \mathbf{1}_{\{\tau_0 > v\}} dv \middle| \mathcal{F}_{t_0} \right] \right\}. \end{aligned}$$

Moreover, since S behaves as a pure diffusion process with respect to the filtration $\mathbb{F}$, Detemple and Tian (2002, Propositions 1 and 2) show that there exists (at each time $t \in [t_0, T]$) a critical asset price E_t below (above) which the American-style put (call) price equals its intrinsic value and, therefore, early exercise should occur. Consequently, the optimal policy should be to exercise the American-style option when the underlying asset price first touches its critical level. Representing the first passage time of the underlying asset price S to its early exercise boundary $\{E_t, t_0 \leq t \leq T\}$ by

$$\tau_e := \inf\left\{t \geq t_0 : S_t = E_t\right\}, \qquad (2.22)$$

equation (2.21) can be restated as:

$$V_{t_0}\left(S_{t_0}, K, T, R; \phi, \eta\right) = V_{t_0}^{0}\left(S_{t_0}, K, T; \phi\right) + V_{t_0}^{D}\left(S_{t_0}, R, T; \phi, \eta\right), \qquad (2.23)$$

where

$$V_{t_0}^{0}\left(S_{t_0},K,T;\phi\right)=\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{T\wedge\tau_e}(r_l+\lambda(l,S))dl}\left(\phi K-\phi S_{T\wedge\tau_e}\right)^{+}\mathbf{1}_{\{\tau_0>T\wedge\tau_e\}}\middle|\mathcal{F}_{t_0}\right],\qquad(2.24)$$

corresponds to Nunes (2009, Equation 55), i.e. to the American-style option price conditional on no default (before the expiry and early exercise dates), and

$$\begin{aligned}&V_{t_0}^{D}\left(S_{t_0},R,T;\phi,\eta\right)\qquad(2.25)\\=\;&\mathbf{1}_{\{\eta=T\}}(\phi R)^{+}e^{-\int_{t_0}^{T}r_l dl}\left[1-\mathbb{E}_{\mathbb{Q}}\left(e^{-\int_{t_0}^{T\wedge\tau_e}\lambda(l,S)dl}\mathbf{1}_{\{\tau_0>T\wedge\tau_e\}}\middle|\mathcal{F}_{t_0}\right)\right]\\&+\mathbf{1}_{\{\eta=\zeta\}}(\phi R)^{+}\mathbb{E}_{\mathbb{Q}}\left[\int_{t_0}^{T\wedge\tau_e}e^{-\int_{t_0}^{v}(r_l+\lambda(l,S))dl}\lambda(v,S)\mathbf{1}_{\{\tau_0>v\}}dv\middle|\mathcal{F}_{t_0}\right],\end{aligned}$$

represents the present value of the recovery payment made at the maturity date or at the default time.

Next proposition decomposes the American-style option price (2.23) into its European-style counterpart and an early exercise premium, and generalizes Nunes (2009, Proposition 7) to different recovery assumptions.

Proposition 2.1 *Under the JDCEV model described by equations (2.1) to (2.4), and assuming that $\zeta > t_0$, the time-t_0 value of an American-style standard option on the stock price S, with strike K, recovery value R, and with maturity date T ($\geq t_0$) is equal to*

$$V_{t_0}\left(S_{t_0},K,T,R;\phi,\eta\right)=v_{t_0}\left(S_{t_0},K,T,R;\phi,\eta\right)+EEP_{t_0}\left(S_{t_0},K,T,R;\phi,\eta\right),\qquad(2.26)$$

where the corresponding European-style option price $v_{t_0}\left(S_{t_0},K,T,R;\phi,\eta\right)$ is given by *equation (2.7),*

$$
\begin{aligned}
& EEP_{t_0}\left(S_{t_0}, K, T, R; \phi, \eta\right) && (2.27) \\
= & \int_{t_0}^{T} \Big\{ e^{-\int_{t_0}^{u} r_l dl} \left[(\phi K - \phi E_u)^+ - v_u^0 \left(E_u, K, T; \phi\right) \right] SP\left(S_{t_0}, t_0; u\right) \\
& -\mathbf{1}_{\{\eta = T\}} e^{-\int_{t_0}^{u} r_l dl} v_u^D \left(E_u, R, T; \phi, T\right) - \mathbf{1}_{\{\eta = \zeta\}} v_u^D \left(E_u, R, T; \phi, \zeta\right) \Big\} \mathbb{Q}\left(\tau_e \in du \middle| \mathcal{F}_{t_0}\right),
\end{aligned}
$$

is the early exercise premium of the American-style put ($\phi = 1$) or call ($\phi = -1$) option, $\{E_u, t_0 \leq u \leq T\}$ is the (unknown) early exercise boundary, and functions $v_u^0(\cdot)$, $v_u^D(\cdot)$ and $SP(\cdot)$ are defined by equations (2.8), (2.9) and (2.11), respectively.

Proof. Please see Appendix A. ■

As usual, the time path $\{E_u, t_0 \leq u \leq T\}$ of critical asset prices is not known ex ante. To implement Proposition 2.1, we must first parameterize such early exercise boundary, and maximize (with respect to those parameters) the early exercise premium (2.27). For this purpose, the density of the first passage time τ_e can be easily recovered by solving the non-linear integral equation of Nunes (2009, Equation 35) through the standard partition method proposed by Park and Schuurmann (1976).

2.3.2 SHP approach

This subsection provides an alternative pricing method to Proposition 2.1 as well as the main theoretical contribution of this paper: The extension of the SHP approach of Chung and Shih (2009) to the JDCEV model. Such extension is based on the fact that the process S behaves as a pure diffusion process with respect to the filtration $\mathbb{F}$. Therefore, the usual value-matching and smooth-pasting conditions can be imposed to equations (2.24) and (2.25), by including in the SHP portfolio the European-style contracts (2.8) and (2.9) with different maturities and different strikes.

As in Chung and Shih (2009), we start at the maturity date of the American-style option and proceed backwards until the valuation date. At time T, we start our static hedge portfolio with one unit of the European-style option (2.7) with strike K, and expiry date at time T. Note that such long position now includes two components: One long position on the European-style contract (2.8) that assumes no default, as in Chung and Shih (2009); but also a new long position on the recovery component (2.9), that will ensure that the portfolio is worth the recovery value R if default occurs.

Similarly to Chung and Shih (2009), we divide the time to maturity of the option contract into n evenly-spaced time points such that $\delta t := (T - t_0)/n$. At each time $t_i := t_0 + i\delta t$ (for $i = n-1, \ldots, 1, 0$), the unkown early exercise boundary E_i is matched by adding w_i units of only the no-default component (2.8) with strike equal to E_i, and maturity at time t_{i+1}. For each time step, the unkowns E_i and w_i are found by solving simultaneously the following two recurrence conditions:

$$\phi K - \phi E_{n-i} = v_{t_{n-i}}\left(E_{n-i}, K, T, R; \phi, \eta\right) + \sum_{j=1}^{i} w_{n-j} \times v^0_{t_{n-i}}\left(E_{n-i}, E_{n-j}, t_{n-j+1}; \phi\right), \quad (2.28)$$

and

$$-\phi = \Delta_{v_{t_{n-i}}(E_{n-i},K,T,R;\phi,\eta)} + \sum_{j=1}^{i} w_{n-j} \times \Delta_{v^0_{t_{n-i}}(E_{n-i},E_{n-j},t_{n-j+1};\phi)}, \quad (2.29)$$

for $i = 1, 2, ..., n$, and where Δ represents the delta (or hedge ratio) of the option.

After solving for all the unknowns E_i and w_i (for $i = n-1, \ldots, 1, 0$), the time-t_0 SHP price of the American-style option, under the JDCEV model, is finally given by:

$$V^{shp}_{t_0}\left(S_{t_0}, K, T, R; \phi, \eta\right) = \begin{cases} V^{shpu}_{t_0}\left(S_{t_0}, K, T, R; \phi, \eta\right) & \Leftarrow \quad \phi S_{t_0} > \phi E_{t_0} \\ \phi K - \phi S_{t_0} & \Leftarrow \quad \phi S_{t_0} \leq \phi E_{t_0} \end{cases}, \quad (2.30)$$

where

$$V_{t_0}^{shpu}\left(S_{t_0},K,T,R;\phi,\eta\right):=v_{t_0}\left(S_{t_0},K,T,R;\phi,\eta\right)+\sum_{j=1}^{n}w_{n-j}\times v_{t_0}^{0}\left(S_{t_0},E_{n-j},t_{n-j+1};\phi\right). \tag{2.31}$$

Remark 2.5 *Note that equation (2.31) constitutes only an upper bound for the true American-style option price, and the true SHP price must be found through equation (2.30) whenever the early exercise boundary has been crossed by the valuation date. Under such scenario, Appendix B shows that equation (2.31) would overvalue the option contract.*

Remark 2.6 *To simultaneously solve the two recurrence conditions* (2.28) *and* (2.29)*, we must provide an initial guess for* E_{n-1}*. Following, for instance, Huang et al. (1996, Footnote 5) and Kim and Yu (1996, Page 67), we initialize the early exercise boundary at*

$$E_n=\phi\left(\phi K\wedge\phi\frac{r_T K}{q_T}\right), \tag{2.32}$$

and take $E_{n-1}=E_n$ *as an initial guess.*

2.3.3 Hedge ratios

As usual, the implementation of the SHP approach for valuing American-style options under the JDCEV model requires the knowledge of the analytical solutions for the hedge ratios of the corresponding European-style plain-vanilla options. The next proposition offers the closed-form solutions for the hedge ratios of the pricing solutions proposed by Carr and Linetsky (2006), which represent a novel contribution to the credit and equity derivatives literature.[2.8]

Proposition 2.2 *Let x, k, δ_+, and ρ be defined as in equations* (2.14), (2.15), (2.16), *and* (2.17), *respectively. Assume that default has not occurred by time $t_0 \geq 0$, that is $\zeta > t_0$, and take $S_{t_0} > 0$.*

i. The delta of the call option (2.12) *is given by*

$$\begin{aligned}\Delta_{v_{t_0}\left(S_{t_0},K,T,0;-1,\eta\right)} \quad = \quad & e^{-q(T-t_0)}\left[\Phi_{+1}\left(0,\frac{k^2}{\rho};\delta_+,\frac{x^2}{\rho}\right)+2\,|\bar{\beta}|\,\frac{x^2}{\rho}\,p\left(\frac{k^2}{\rho};2+\delta_+,\frac{x^2}{\rho}\right)\right]\\ & -\frac{K}{S_{t_0}}\,e^{-(r+b)(T-t_0)}\left(\frac{x^2}{\rho}\right)^{\frac{1}{2|\bar{\beta}|}}\left[\Phi_{+1}\left(-\frac{1}{2|\bar{\beta}|},\frac{k^2}{\rho};\delta_+,\frac{x^2}{\rho}\right)\right.\\ & \left.\left(1-|\bar{\beta}|\,\frac{x^2}{\rho}\right)+2\,|\bar{\beta}|\,\widetilde{\Phi}_{+1}\left(-\frac{1}{2|\bar{\beta}|},\frac{k^2}{\rho};\delta_+,\frac{x^2}{\rho}\right)\right], \qquad (2.33)\end{aligned}$$

where $p(.;v,\lambda)$ is the probability density function of a noncentral chi-square distribution with v degrees of freedom and noncentrality parameter λ, as given in Johnson et al. (1995, Equation 29.4), and

$$\widetilde{\Phi}_{+1}\left(p,w;v,\lambda\right):=2^p\sum_{i=0}^{\infty}\frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^i}{(i-1)!}\,\frac{\Gamma\left(p+\frac{v}{2}+i,\frac{w}{2}\right)}{\Gamma\left(\frac{v}{2}+i\right)}, \qquad (2.34)$$

with $\Gamma\left(a,z\right)$ and $\Gamma\left(a\right)$ representing the upper incomplete gamma function and the Euler gamma function given in Abramowitz and Stegun (1972, Equations 6.5.3 and 6.1.1), respectively, for $a,z\in\mathbb{R}_+$.

[2.8] Even though we are concentrating our analysis on the time-homogeneous JDCEV model with constant parameters, it is straightforward to extend the analytical solutions of the hedge ratios proposed in Proposition 2.2 for the time-dependent JDCEV model.

ii. The delta of the put option conditional on no default (2.13) *is given by*

$$
\begin{aligned}
\Delta_{v^0_{t_0}\left(S_{t_0},K,T;1\right)} &= \frac{K}{S_{t_0}}\, e^{-(r+b)(T-t_0)} \left(\frac{x^2}{\rho}\right)^{\frac{1}{2|\bar{\beta}|}} \left[\Phi_{-1}\left(-\frac{1}{2|\bar{\beta}|}, \frac{k^2}{\rho}; \delta_+, \frac{x^2}{\rho}\right)\right. \qquad (2.35)\\
&\quad \left(1-|\bar{\beta}|\,\frac{x^2}{\rho}\right) + 2\,|\bar{\beta}|\,\widetilde{\Phi}_{-1}\left(-\frac{1}{2|\bar{\beta}|}, \frac{k^2}{\rho}; \delta_+, \frac{x^2}{\rho}\right)\Bigg]\\
&\quad -e^{-q(T-t_0)}\left[\Phi_{-1}\left(0, \frac{k^2}{\rho}; \delta_+, \frac{x^2}{\rho}\right) - 2\,|\bar{\beta}|\,\frac{x^2}{\rho}\,p\left(\frac{k^2}{\rho}; 2+\delta_+, \frac{x^2}{\rho}\right)\right],
\end{aligned}
$$

where

$$
\widetilde{\Phi}_{-1}\left(p, w; v, \lambda\right) := 2^p \sum_{i=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^i}{(i-1)!}\,\frac{\gamma\left(p+\frac{v}{2}+i, \frac{w}{2}\right)}{\Gamma\left(\frac{v}{2}+i\right)}, \qquad (2.36)
$$

with $\gamma\left(a, z\right)$ being the lower incomplete gamma function given in Abramowitz and Stegun (1972, Equation 6.5.2), for $a, z \in \mathbb{R}_+$.

iii. The delta of the recovery part of the put (2.10), under the fractional recovery of treasury assumption, is given by

$$
\begin{aligned}
\Delta_{v^D_{t_0}\left(S_{t_0},R,T;1,T\right)} &= -\frac{R}{S_{t_0}}\, e^{-r(T-t_0)}\, SP\left(S_{t_0}, t_0; T\right) \qquad (2.37)\\
&\quad \left[1+|\bar{\beta}|\,\frac{x^2}{\rho}\left(\left(1-\frac{1}{|\bar{\beta}|\,\delta_+}\right)\frac{{}_1F_1\left(\frac{\delta_+}{2}+p+1, \frac{\delta_+}{2}+1, \frac{x^2}{2\rho}\right)}{{}_1F_1\left(\frac{\delta_+}{2}+p, \frac{\delta_+}{2}, \frac{x^2}{2\rho}\right)} - 1\right)\right],
\end{aligned}
$$

where

$$
{}_1F_1\left(a, b, z\right) := \sum_{i=0}^{\infty} \frac{(a)_i}{(b)_i}\frac{z^i}{i!}, \qquad (2.38)
$$

is the Kummer confluent hypergeometric function of the first kind as given, for instance, by Slater (1960, Equation 1.1.8), Abramowitz and Stegun (1972, Equation 13.1.2), or Lebedev (1972, Equation 9.9.1), and $(a)_i$ is the Pochhammer function defined, for example, in Abramowitz and Stegun (1972, Equation 6.1.22).

iv. The delta of the recovery part of the put (2.19)*, under the fractional recovery of face* *value assumption, is given by*

$$
\begin{aligned}
&\Delta_{v^D_{t_0}\left(S_{t_0},R,T;1,\zeta\right)} \qquad (2.39)\\
= \;& R\int_{t_0}^{T} e^{-(r+b)(u-t_0)}\left[\frac{b\,A}{S_{t_0}}\left(\frac{x^2}{\rho\left(t_0,u\right)}\right)^{\frac{1}{2|\bar{\beta}|}} M\left(-\frac{1}{2|\bar{\beta}|};\delta_+,\frac{x^2}{\rho\left(t_0,u\right)}\right)\right.\\
&\left.+ca^2BS_{t_0}^{2\beta-1}e^{-2|\bar{\beta}|(r-q+b)(u-t_0)}\left(\frac{x^2}{\rho\left(t_0,u\right)}\right)^{\frac{1}{2|\bar{\beta}|}+1} M\left(-\frac{1}{2|\bar{\beta}|}-1;\delta_+,\frac{x^2}{\rho\left(t_0,u\right)}\right)\right]du,
\end{aligned}
$$

where

$$
A := 1+|\bar{\beta}|\,\frac{x^2}{\rho\left(t_0,u\right)}\,\left[\left(1-\frac{1}{|\bar{\beta}|\,\delta_+}\right)\frac{{}_1F_1\left(\frac{\delta_+}{2}+p+1,\frac{\delta_+}{2}+1,\frac{x^2}{2\rho(t_0,u)}\right)}{{}_1F_1\left(\frac{\delta_+}{2}+p,\frac{\delta_+}{2},\frac{x^2}{2\rho(t_0,u)}\right)}-1\right], \qquad (2.40)
$$

and

$$
B := 1+|\bar{\beta}|\,\frac{x^2}{\rho\left(t_0,u\right)}\,\left[\left(1-\frac{1}{|\bar{\beta}|\,\delta_+}-\frac{2}{\delta_+}\right)\frac{{}_1F_1\left(\frac{\delta_+}{2}+p,\frac{\delta_+}{2}+1,\frac{x^2}{2\rho(t_0,u)}\right)}{{}_1F_1\left(\frac{\delta_+}{2}+p-1,\frac{\delta_+}{2},\frac{x^2}{2\rho(t_0,u)}\right)}-1\right]. \qquad (2.41)
$$

Proof. The delta of the recovery parts of the put (2.10) and (2.19) involve the derivative of the Kummer confluent hypergeometric function (2.38) with respect to z, given, for instance, in Slater (1960, Equation 2.1.1), Abramowitz and Stegun (1972, Equation 13.4.8), or Lebedev (1972, Equation 9.9.4). The proof of the hedge ratios (2.33) and (2.35) involves straightforward calculus and is omitted. ■

Remark 2.7 *Note that, under the fractional recovery of treasury assumption, the delta of the European-style put option* (2.7) *is given by the sum of equations* (2.35) *and* (2.37). *Alternatively, such hedge ratio can be easily obtained through the put-call parity*

$$
\Delta_{v_{t_0}\left(S_{t_0},K,T,R;1,T\right)} = \Delta_{v^0_{t_0}\left(S_{t_0},K,T;-1\right)} - e^{-q(T-t_0)}, \qquad (2.42)
$$

which avoids the need of computing the Kummer confluent hypergeometric function (2.38).

Remark 2.8 *One of the ingredients for efficiently compute the hedge ratios offered in Proposition 2.2 is the computation of the value function* $\widetilde{\Phi}_{\theta}\left(p, w; v, \lambda\right)$*, with* $\theta \in \{-1, 1\}$*. To enhance the efficiency of the analytical formulas* (2.33) *and* (2.35)*, we have adapted the algorithm offered by Dias and Nunes (2012) for computing the series solutions* (2.34) *and* (2.36)*. Details are available upon request.*

2.3.4 American-style capped contracts

Equation (2.23) can be interpreted as an American-style down-and-out option with the down-and-out barrier set at zero (with the short-term interest rate replaced by an intensity-adjusted short-rate, and with possible recovery at default). Therefore, the extension of the SHP approach to the valuation of equation (2.23) highlights that the SHP method can also be easily applied to the pricing of American-style barrier options under single factor diffusion processes. Such task has been successfully undertaken by Chung et al. (2013) for American knock-in put options under the CEV model (but only with $\beta = \frac{4}{3}$). To illustrate the extension of the SHP approach to the pricing of American-style barrier options under the JDCEV framework, we now consider the valuation of American-style capped call and put options.

Upon exercise, the payoff of a capped option on the asset price S, with strike K, and constant cap H, is equal to $(S \wedge H - K)^{+}$, for a capped call, and to $(K - S \vee H)^{+}$, for a capped put. Therefore, and as argued, for instance, by Detemple (2006, Page 89), a capped option is equivalent to "a knock-out barrier option with rebate equal to the option payoff at the trigger date".

Under the JDCEV model and assuming the automatic exercise at the constant cap level H, the time-t_0 value of an American-style capped option on the stock price S, with strike K, and with expiry date at time T ($\geq t_0$) can be obtained through the following augmentation of the Snell envelope (2.20):[2.9]

$$\begin{aligned} & \bar{V}_{t_0}\left(S_{t_0}, K, H, T; \phi, \eta\right) \\ = & \sup_{\tau \in \mathcal{T}} \left\{ \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_{t_0}^{T \wedge \tau \wedge \tau_H} r_l dl} \left(\phi K - \phi S_{T \wedge \tau \wedge \tau_H}\right)^+ \mathbf{1}_{\{\zeta > T \wedge \tau \wedge \tau_H\}} \middle| \mathcal{G}_{t_0} \right] \right. \\ & \left. + \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_{t_0}^{\eta} r_l dl} \left(\phi \left(K - H\right)^+\right)^+ \mathbf{1}_{\{\zeta \leq T \wedge \tau \wedge \tau_H\}} \middle| \mathcal{G}_{t_0} \right] \right\}, \end{aligned} \tag{2.43}$$

where

$$\tau_H := \inf\left\{t > t_0 : S_t = H\right\}, \tag{2.44}$$

is the first passage time of the underlying asset price to the cap level H, which is such that

$$\phi H < \phi S_{t_0}, \tag{2.45}$$

with $\phi = 1$ for a put option, $\phi = -1$ for a call option, and $\eta \in \{\zeta, T\}$. Since the random variable Θ is independent of $\mathbb{F}$, because S behaves as a pure diffusion process with respect to the filtration $\mathbb{F}$, and following the same steps as in Subsection 2.3.1, equation (2.43) can be simply restated as

$$\bar{V}_{t_0}\left(S_{t_0}, K, H, T; \phi, \eta\right) = \bar{V}_{t_0}^0\left(S_{t_0}, K, H, T; \phi\right) + \bar{V}_{t_0}^D\left(S_{t_0}, K, H, T; \phi, \eta\right), \tag{2.46}$$

where

$$\begin{aligned} & \bar{V}_{t_0}^0\left(S_{t_0}, K, H, T; \phi\right) \\ = & \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_{t_0}^{T \wedge \bar{\tau}_e \wedge \tau_H} (r_l + \lambda(l,S)) dl} \left(\phi K - \phi S_{T \wedge \bar{\tau}_e \wedge \tau_H}\right)^+ \mathbf{1}_{\{\tau_0 > T \wedge \bar{\tau}_e \wedge \tau_H\}} \middle| \mathcal{F}_{t_0} \right], \end{aligned} \tag{2.47}$$

[2.9] As usual, for call options the recovery value upon default is zero. For American-style capped puts, such recovery value is assumed to be equal to $(K - 0 \vee H)^+ = (K - H)^+$.

is the American-style capped option value conditional on no default, and

$$
\begin{aligned}
& \bar{V}_{t_0}^{D}\left(S_{t_0}, K, H, T; \phi, \eta\right) && (2.48)\\
= \quad & \mathbf{1}_{\{\eta=T\}}\left(\phi\left(K-H\right)^{+}\right)^{+} e^{-\int_{t_0}^{T} r_l dl}\left[1-\mathbb{E}_{\mathbb{Q}}\left(e^{-\int_{t_0}^{T\wedge\bar{\tau}_e\wedge\tau_H}\lambda(l,S)dl}\mathbf{1}_{\{\tau_0>T\wedge\bar{\tau}_e\wedge\tau_H\}}\middle|\mathcal{F}_{t_0}\right)\right]\\
& +\mathbf{1}_{\{\eta=\zeta\}}\left(\phi\left(K-H\right)^{+}\right)^{+}\mathbb{E}_{\mathbb{Q}}\left[\int_{t_0}^{T\wedge\bar{\tau}_e\wedge\tau_H} e^{-\int_{t_0}^{v}(r_l+\lambda(l,S))dl}\lambda(v,S)\mathbf{1}_{\{\tau_0>v\}}dv\middle|\mathcal{F}_{t_0}\right],
\end{aligned}
$$

is the present value of the recovery payment (made at the maturity date or at the default time), and

$$\bar{\tau}_e := \inf\left\{t > t_0 : S_t = \bar{E}_t^{\phi}\right\}, \tag{2.49}$$

is the optimal stopping time through the early exercise boundary $\left\{\bar{E}_t^{\phi}, t_0 \leq t \leq T\right\}$ of the capped put (if $\phi = 1$) or call (if $\phi = -1$).

Moreover, the early exercise boundary of both capped options can be easily recovered from the boundary of the corresponding uncapped option, since Gao et al. (2000, Theorem 6) and Detemple and Tian (2002, Proposition 8) have shown that

$$\bar{E}_t^{1} = E_t \vee H, \tag{2.50}$$

and

$$\bar{E}_t^{-1} = E_t \wedge H, \tag{2.51}$$

for all $t \in [t_0, T]$, and where $\{E_t, t_0 \leq t \leq T\}$ is the early exercise boundary of the corresponding standard American-style option. Consequently, equation (2.46) can be simply rewritten as equation (2.23) but with τ_e and R replaced by $\bar{\tau}_e \wedge \tau_H$ and $(K-H)^{+}$, respectively.

Under the optimal stopping approach of Nunes (2009), the evaluation of equation (2.46) is straightforward. First, the early exercise boundary $\{E_t, t_0 \leq t \leq T\}$ is found via Proposition 2.1, by maximizing the early exercise premium (2.27) for $R = (K - H)^{+}$ and with respect to a polynomial parameterization of the boundary. Then, the capped boundary $\left\{ \bar{E}_t^{\phi}, t_0 \leq t \leq T \right\}$ is obtained through equations (2.50) or (2.51), and fed into equation (2.26) with $R = (K - H)^{+}$.

Under the SHP approach, the evaluation of equation (2.46) is based on two steps that combine the Derman et al. (1995) and the Chung and Shih (2009) approaches. Again, we divide the time to maturity of the option contract into n evenly-spaced time points such that $\delta t := (T - t_0)/n$, and $t_i := t_0 + i\delta t$ (for $i = n-1, ..., 1, 0$). But now we only need to find the unknown early exercise boundary $\bar{E}_i^{\phi}$ until $\bar{E}_i^{\phi} = H$, since equations (2.50) and (2.51) imply that the remaining boundary is simply given by H.

The recurrence conditions (2.28) and (2.29) are easily adapted to the valuation of American-style capped options:

$$\begin{aligned} \phi K - \phi \bar{E}_{n-i}^{\phi} &= v_{t_{n-i}} \left(\bar{E}_{n-i}^{\phi}, K, T, (K-H)^{+}; \phi, \eta \right) \\ &\quad + \sum_{j=1}^{i} w_{n-j} \times v_{t_{n-i}}^{0} \left(\bar{E}_{n-i}^{\phi}, \bar{E}_{n-j}^{\phi}, t_{n-j+1}; \phi \right), \end{aligned} \tag{2.52}$$

and

$$-\phi \mu = \left[\Delta_{v_{t_{n-i}}\left(\bar{E}_{n-i}^{\phi}, K, T, (K-H)^{+}; \phi, \eta\right)} + \sum_{j=1}^{i} w_{n-j} \times \Delta_{v_{t_{n-i}}^{0}\left(\bar{E}_{n-i}^{\phi}, \bar{E}_{n-j}^{\phi}, t_{n-j+1}; \phi\right)} \right] \mu, \tag{2.53}$$

with $\mu = 1$ while $\bar{E}_{n-i}^{\phi} = E_{n-i}^{\phi}$ and $\mu = 0$ when $\bar{E}_{n-i}^{\phi} = H$. Finally, the time-t_0 SHP price of the American-style capped option, under the JDCEV model, is given by:

$$\bar{V}_{t_0}^{shp}\left(S_{t_0}, K, H, T; \phi, \eta\right) = \begin{cases} \bar{V}_{t_0}^{shpu}\left(S_{t_0}, K, H, T; \phi, \eta\right) & \Leftarrow \quad \phi S_{t_0} > \phi \bar{E}_{t_0}^{\phi} \\ \phi K - \phi S_{t_0} & \Leftarrow \quad \phi S_{t_0} \leq \phi \bar{E}_{t_0}^{\phi} \end{cases}, \tag{2.54}$$

where

$$\bar{V}_{t_0}^{shpu}(S_{t_0}, K, H, T; \phi, \eta) := v_{t_0}\left(S_{t_0}, K, T, (K-H)^+; \phi, \eta\right) + \sum_{j=1}^{n} w_{n-j} \times v_{t_0}^0\left(S_{t_0}, \bar{E}_{n-j}^{\phi}, t_{n-j+1}; \phi\right). \tag{2.55}$$

Note that the procedure of Chung and Shih (2009) is only used until $\bar{E}_{n-i}^{\phi} = H$; then, the procedure of Derman et al. (1995) is applied to the remaining early exercise boundary, since the boundary is known and we only have to find the weights of the hedging portfolio. Using this two-step procedure, the valuation of American-style capped options is always faster than the valuation of standard American-style options (except for the case when both contracts share the same exercise boundary and, hence, the CPU time is the same).

2.4 Numerical experiments

To test the efficiency of the SHP and optimal stopping approaches under the CEV and JDCEV frameworks, we divide the analysis in two parts: First, we consider the valuation of standard American-style options; then we deal with the pricing of American-style capped options.

2.4.1 American-style standard options

It is noteworthy to emphasize that Chung and Shih (2009) have successfully applied the SHP approach to price (and hedge) American-style options under the lognormal assumption of BSM and for the CEV diffusion model of Cox (1975). For the latter model, however, they consider only the case where the elasticity parameter of the CEV process (β) is equal to $4/3$. As observed by Schroder (1989), the prices of plain-vanilla calls and puts under the CEV assumption with $\beta = 4/3$ are easy to compute since the corresponding complementary noncentral chi-square distribution functions $Q(.;1,\lambda)$, $Q(.;3,\lambda)$, and $Q(.;5,\lambda)$ can be determined using only the standard normal density and distribution functions.

CEV model

The main goal of our first numerical experiments is to further test the accuracy and efficiency of the SHP method for pricing American-style options under the CEV model described in Subsection 2.2.3 by extending the analysis for any β parameter, thus accommodating both direct and indirect leverage effects commonly observed across a variety of options markets.

For this purpose, the pricing solutions of the SHP procedure will be compared against the optimal stopping approach offered by Nunes (2009) for the parameter constellations considered in Nunes (2009, Tables 2, 3, and 4).[2.10] All numerical results in this paper are obtained through *Matlab* (R2010a) running on an Intel Xeon X5680 3.33GHz processor.

Table 2.1 reports American-style put prices with a time to maturity of six months and assuming an elasticity parameter $\beta = 3$, while Table 2.2 values American-style call options with an expiry date of one year, and under a CEV square root process with $\beta = 1$.[2.11] The proxy of the *exact* American option price (4th column) is computed through the Crank-Nicolson finite-difference scheme with $15,000$ time intervals and $10,000$ space steps. The optimal stopping approach of Nunes (2009) is implemented

[2.10]The use of the Nunes (2009) valuation methodology will also allow us to compare the SHP results to be obtained under the JDCEV model proposed in Carr and Linetsky (2006).

[2.11]Note that the value of an American-style call option under the CEV model can be obtained also through the put-call symmetry offered by Schroder (1999, Corollary 1).

with conditional minimization through the *Matlab* "fmincon" algorithm, and using a four and five degree polynomial specification for the early exercise boundary (5th and 6th columns, respectively).[2.12] As suggested by Nunes (2009, page 1246), the parameters defining the exercise policy are first estimated by discretizing both Nunes (2009, Propositions 1 and 5) using $N = 2^4$. Then, and based on this approximation for the optimal exercise boundary, the early exercise premium is computed via Nunes (2009, Proposition 6) using $N = 2^8$ time steps. The last four columns of Tables 2.1 and 2.2 contain the American-style option prices generated by the SHP method, which is implemented using the *Matlab* "fsolve" algorithm for solving the recurrence conditions (2.28) and (2.29), subject to the restrictions described in Definition 2.1, and with $n \in \{4, 12, 24, 52\}$. Accuracy is measured by the mean average absolute percentage error (over the 20 contracts considered) of each valuation approach and with respect to the exact American option price. Efficiency is evaluated by the total CPU time (expressed in seconds) spent to value the whole set of contracts considered.

There are four points that are noteworthy to highlight in these two tables. First, both methods are accurate: the mean average absolute percentage errors (MAPE) in all tested cases are well below the typical bid-ask spread observed in the market. Second, the SHP method with $n = 12$ gives similar results in terms of accuracy to the Nunes (2009) approach with a four degree polynomial specification for the early exercise boundary, but with less than half of the computational burden. Third, the SHP method with $n = 24$ gives more accurate results than the Nunes (2009) approach with a five degree polynomial specification for the early exercise boundary, though using a similar CPU time. Fourth, the computational expenses contained in Table 2.2 are about half of those presented in Table 2.1 essentially due to the fact that more than half of the American-style options evaluated in Table 2.2 are equal to their European-style

[2.12] The early exercise premium (2.27) is maximized subject to the terminal condition (2.32), and imposing that the optimal exercise boundary is non-negative and non-decreasing.

counterparts, and in these cases both methods become faster.

Tables 2.1 and 2.2 compile 40 option prices which obviously do not represent a large enough sample to take more robust conclusions, thus giving only a preliminary flavor of the results. Hence, to better assess the speed-accuracy trade-off between the two methods tested we follow the guidelines of Broadie and Detemple (1996) by conducting a careful large sample evaluation of 1,250 randomly generated American-style put option prices.

Table 2.3 reproduces the pricing errors of the optimal stopping approach of Nunes (2009) with a five and six polynomial specification for the early exercise boundary (2nd and 3rd columns, respectively) and the pricing errors of the SHP approach for four different evenly-spaced time grids (the last four columns), where all the option parameters, with the exception of β and δ, are extracted from the same uniform distributions as in Ju (1998, Table 3). As before, the *true* American-style put option price is computed through the Crank-Nicolson finite-difference scheme with $15,000$ time intervals and $10,000$ space steps.[2.13] Even though both methods produce extremely accurate results, the SHP approach offers the best speed-accuracy trade-off for pricing American-style standard options under the CEV model.

JDCEV model

Armed with the closed-form solutions of European-style options and the corresponding hedge ratios, the implementation of the SHP approach for valuing standard American-style options under the JDCEV model (under both recovery assumptions) follows in a

[2.13] It is well-known—see, for instance, Schroder (1989) or Larguinho et al. (2013)—that option pricing under the CEV assumption is computationally expensive especially when β is close to two (the lognormal case), volatility is low, or the time to maturity is small. For this reason, we have excluded from the original sample option contracts with an elasticity parameter $\beta \in [1.75, 2.25]$, thus leaving 1,085 contracts to be evaluated. Additional results, not reported here but available upon request, show that it takes almost the same computational time to value the 165 contracts removed from the initial sample and the remaining 1,085 contracts.

straightforward way.

To illustrate the robustness and efficiency of the proposed pricing methodology, we focus on the valuation of standard American-style puts under the JDCEV model, though the procedure is easily adapted for valuing their counterpart calls. To the best of our knowledge, the optimal stopping approach of Nunes (2009) is the only available methodology, until now, for pricing plain-vanilla American-style options under the JDCEV model—now extended for the payment of the recovery value at the maturity date T—, and hence it will be used as our benchmark.

Tables 2.4 and 2.5 test the efficiency of the SHP algorithm for valuing standard American put options under the time-homogeneous JDCEV model for different parameter configurations borrowed from Carr and Linetsky (2006, Table 1), and assuming, respectively, recovery at maturity (i.e. $\eta = T$) and recovery at default time (i.e. $\eta = \zeta$). The 4th, 5th, and 6th columns of both tables show the European-style put option price components obtained via equations (2.13), (2.9), and (2.7), respectively. As expected, put options under the fractional recovery of treasury assumption are worth less than the corresponding options under the fractional recovery of face value assumption, due to the different discount effects over the recovery value (equal to K). The optimal stopping approach of Nunes (2009) is implemented with conditional minimization, using the *Matlab* “fmincon” algorithm, considering a six degree polynomial specification for the early exercise boundary. The parameters defining the exercise policy are first estimated using $N = 2^4$ time steps. Then, and based on this approximation for the optimal exercise boundary, the early exercise premium (7th column of both tables), the American put option price conditional on no default before the expiry and early exercise dates (8th column of both tables), and the present value of the recovery payment made at the maturity date (9th column of Table 2.4) or at the default time (9th column of Table 2.5) are computed via equations (2.27), (2.24), and (2.25), respectively, using $N = 2^8$

time steps. The standard American-style put option price contained in the 10th column of both tables is recovered through equations (2.23) or (2.26).[2.14] Finally, columns 11 to 13 of Tables 2.4 and 2.5 report the values of the American-style put given in equation (2.30), which are obtained through the SHP procedure using again the "fsolve" algorithm available in *Matlab* for solving the recurrence conditions (2.28) and (2.29), with $n \in \{12, 24, 52\}$.

To sum up, the results computed via the SHP approach are shown to be robust along both tables. For instance, it is possible to obtain extremely accurate option prices (for both recovery assumptions) even using the SHP pricing procedure with only 12 evenly-spaced time points n, but with much less computational burden. As expected, both pricing frameworks require higher CPU times to compute standard American-style put option contracts under the fractional recovery of face value assumption, because, in this case, one has to use a numerical integration scheme for computing equations (2.19) and (2.25) under the optimal stopping approach, and equations (2.19) and (2.39) under the SHP procedure.[2.15]

2.4.2 American-style capped options

This subsection aims to compare the optimal stopping and the SHP approaches for pricing American-style capped put options under both the CEV and JDCEV models. The valuation of the corresponding calls can be treated similarly.

Table 2.6 prices American-style capped put options under the CEV model using the pa-

[2.14] Both tables highlight that when $K = 120$ the put is sufficiently in-the-money, so that the time-t_0 spot price (S_{t_0}) is already below the critical asset price E_{t_0}, and, therefore, the standard American-style put price equals its intrinsic value.

[2.15] At the expense of a higher computational burden, both valuation methodologies have been implemented through the Gauss-Kronrod integration method, using the "quadgk" algorithm available in *Matlab*. Even though there are other numerical integration schemes that may be also applied to diminish the computational effort, we do not pursue such numerical issues here.

rameter constellations considered in Table 2.1, but now augmented with a barrier level $H = \$75$. The 3rd and 4th columns of the table highlight the values obtained under the optimal stopping approach of Nunes (2009), using equations (2.26), (2.27), and (2.50), with $R = (K - H)^+$, and considering a polynomial boundary specification with four and five degrees of freedom, respectively. Columns 5 to 8 report the results obtained via the SHP method with $n \in \{4, 12, 24, 52\}$, and computed through the recurrence conditions (2.52) and (2.53), and equation (2.54). Again, both valuation methodologies have been implemented subject to the restrictions described in Definition 2.1.

Table 2.7 prices American-style capped put options under the JDCEV model (with recovery at maturity) using the parameter constellations considered in Table 2.4 and a barrier $H = \$75$. The valuation of the corresponding put with recovery at the default time (and calls with both recovery assumptions) can be treated similarly. Columns 4 to 6 value American-style capped put options using the optimal stopping approach of Nunes (2009). The no default component (column 4) is computed using equation (2.47), and the recovery component (column 5) is computed through equation (2.48). Finally, column 6 gives the sum of these two components obtained via equation (2.46).[2.16] The last three columns report the results obtained via the SHP method with $n \in \{12, 24, 52\}$, and computed through the recurrence conditions (2.52) and (2.53), and equation (2.54).

In summary, the tables reveal that both pricing methodologies produce results that are almost indistinguishable, though the SHP procedure seems to be more efficient in terms of computational burden. As expected, and as explained in Subsection 2.3.4, the CPU time required to value these two sets of capped American-style put options contracts under the SHP approach is smaller than the computational effort for valuing

[2.16] Similarly to the standard American-style put option case, when $K = 120$ the capped put is sufficiently in-the-money, so that the time-t_0 spot price (S_{t_0}) is already below the critical asset price $E^1_{t_0}$, and, therefore, the capped American-style put price equals its intrinsic value.

the corresponding standard American-style put options of Tables 2.1 and 2.4. On the contrary, the optimal stopping approach of Nunes (2009) requires further CPU time, since we still need to find the whole early exercise boundary of the standard American-style put option.

2.5 Conclusions

The most important theoretical contribution of this paper is the generalization of the SHP procedure for valuing American-style standard and capped options under the JDCEV model of Carr and Linetsky (2006). To accomplish this purpose, novel analytical representations were obtained for the hedge ratios of the corresponding European-style standard options, which can be used to jointly price equity and credit derivatives under this general and flexible modeling framework. The SHP approach is also implemented to price American-style standard and capped options under the unrestricted CEV model, thus accommodating both direct and indirect leverage effects typically observed by market practitioners. Furthermore, we extend the optimal stopping approach of Nunes (2009) for the pricing of American-style capped options, assuming that the recovery value associated to the put can be paid at the default time or at the maturity date of the option.

Overall, the numerical experiments run have shown that the SHP pricing methodology is as accurate as but (generally) faster than the optimal stopping approach, thus offering a better speed-accuracy trade-off for pricing American-style standard and capped options under both the (single-factor) CEV and JDCEV models. Nevertheless, and as shown by Nunes (2011, Theorem 1), the optimal stopping approach should be easier to extend for multifactor models (incorporating, for instance, stochastic volatility) because it only requires a numerically tractable solution for both the corresponding European-style option and for the transition density function of the underlying asset process.

Appendix A

This appendix proves Proposition 2.1.

Nunes (2009, Equations 57 and 60) has already shown that

$$V_{t_0}^0\left(S_{t_0}, K, T; \phi\right) = v_{t_0}^0\left(S_{t_0}, K, T; \phi\right) + EEP_{t_0}^0\left(S_{t_0}, K, T; \phi\right), \tag{A-1}$$

where the first term on the right-hand side of equation (A-1) is given by equation (2.8), and

$$\begin{aligned} & EEP_{t_0}^0\left(S_{t_0}, K, T; \phi\right) \qquad\qquad \text{(A-2)} \\ = & \int_{t_0}^{T} e^{-\int_{t_0}^{u} r_l dl}\left[\left(\phi K - \phi E_u\right)^+ - v_u^0\left(E_u, K, T; \phi\right)\right] SP\left(S_{t_0}, t_0; u\right) \mathbb{Q}\left(\tau_e \in du \middle| \mathcal{F}_{t_0}\right). \end{aligned}$$

A similar decomposition must also be made to the recovery component (2.25), and will be illustrated in the next lines for the case of recovery at the maturity date.[2.17]

Assuming that $\eta = T$ and $\zeta > t_0$, equation (2.25) can be rewritten as

$$\begin{aligned} V_{t_0}^D\left(S_{t_0}, R, T; \phi, T\right) &= (\phi R)^+ e^{-\int_{t_0}^{T} r_l dl}\left[1 - \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_{t_0}^{T}\lambda(l,S)dl}\mathbf{1}_{\{\tau_0 > T, \tau_e \geq T\}}\middle| \mathcal{F}_{t_0}\right)\right. \\ &\quad \left. - \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_{t_0}^{\tau_e}\lambda(l,S)dl}\mathbf{1}_{\{\tau_0 > \tau_e, \tau_e < T\}}\middle| \mathcal{F}_{t_0}\right)\right], \end{aligned}$$

[2.17] The recovery at default time case can be treated similarly, and the proof is available upon request.

i.e.

$$
\begin{aligned}
& V_{t_0}^{D}\left(S_{t_0}, R, T; \phi, T\right) && \text{(A-3)} \\
= \; & (\phi R)^{+} e^{-\int_{t_0}^{T} r_l dl} \left[1 - \mathbb{E}_{\mathbb{Q}} \left(e^{-\int_{t_0}^{T} \lambda(l,S) dl} \mathbf{1}_{\{\tau_0 > T\}} \middle| \mathcal{F}_{t_0} \right) \right. \\
& \left. + \mathbb{E}_{\mathbb{Q}} \left(e^{-\int_{t_0}^{T} \lambda(l,S) dl} \mathbf{1}_{\{\tau_0 > T, \tau_e < T\}} \middle| \mathcal{F}_{t_0} \right) - \mathbb{E}_{\mathbb{Q}} \left(e^{-\int_{t_0}^{\tau_e} \lambda(l,S) dl} \mathbf{1}_{\{\tau_0 > \tau_e, \tau_e < T\}} \middle| \mathcal{F}_{t_0} \right) \right],
\end{aligned}
$$

since $\mathbf{1}_{\{\tau_e \geq T\}} = 1 - \mathbf{1}_{\{\tau_e < T\}}$.

Equations (2.10) and (2.11), and the law of iterative expectations, imply that equation (A-3) can be further simplified into

$$
\begin{aligned}
& V_{t_0}^{D}\left(S_{t_0}, R, T; \phi, T\right) && \text{(A-4)} \\
= \; & (\phi R)^{+} e^{-\int_{t_0}^{T} r_l dl} \left[1 - \mathbb{E}_{\mathbb{Q}} \left(\mathbf{1}_{\{\zeta > T\}} \middle| \mathcal{G}_{t_0} \right) \right. \\
& \left. + \mathbb{E}_{\mathbb{Q}} \left(\mathbf{1}_{\{\zeta > T, \tau_e < T\}} \middle| \mathcal{G}_{t_0} \right) - \mathbb{E}_{\mathbb{Q}} \left(\mathbf{1}_{\{\zeta > \tau_e, \tau_e < T\}} \middle| \mathcal{G}_{t_0} \right) \right] \\
= \; & (\phi R)^{+} e^{-\int_{t_0}^{T} r_l dl} \left\{ 1 - SP\left(S_{t_0}, t_0; T\right) - \mathbb{E}_{\mathbb{Q}} \left[\left(\mathbf{1}_{\{\zeta > \tau_e\}} - \mathbf{1}_{\{\zeta > T\}} \right) \mathbf{1}_{\{\tau_e < T\}} \middle| \mathcal{G}_{t_0} \right] \right\} \\
= \; & v_{t_0}^{D}\left(S_{t_0}, R, T; \phi, T\right) - (\phi R)^{+} e^{-\int_{t_0}^{T} r_l dl} \mathbb{E}_{\mathbb{Q}} \left(\mathbf{1}_{\{\tau_e < \zeta < T\}} \mathbf{1}_{\{\tau_e < T\}} \middle| \mathcal{G}_{t_0} \right) \\
= \; & v_{t_0}^{D}\left(S_{t_0}, R, T; \phi, T\right) - (\phi R)^{+} e^{-\int_{t_0}^{T} r_l dl} \mathbb{E}_{\mathbb{Q}} \left[\mathbb{E}_{\mathbb{Q}} \left(\mathbf{1}_{\{\tau_e < \zeta < T\}} \middle| \mathcal{G}_{\tau_e} \right) \mathbf{1}_{\{\tau_e < T\}} \middle| \mathcal{G}_{t_0} \right].
\end{aligned}
$$

Using again definition (2.11), equation (A-4) can be restated in terms of the restricted filtration $\mathbb{F}$, with respect to which the asset price process S behaves as a pure diffusion process:

$$
\begin{aligned}
& V_{t_0}^{D}\left(S_{t_0}, R, T; \phi, T\right) && \text{(A-5)} \\
= \; & v_{t_0}^{D}\left(S_{t_0}, R, T; \phi, T\right) \\
& -(\phi R)^{+} e^{-\int_{t_0}^{T} r_l dl} \mathbb{E}_{\mathbb{Q}} \left\{ \left[1 - \mathbb{E}_{\mathbb{Q}} \left(e^{-\int_{\tau_e}^{T} \lambda(l,S) dl} \mathbf{1}_{\{\tau_0 > T\}} \middle| \mathcal{F}_{\tau_e} \right) \right] \mathbf{1}_{\{\tau_e < T\}} \middle| \mathcal{F}_{t_0} \right\}.
\end{aligned}
$$

Taking advantage of the Markovian nature of the underlying price process S, the outer expectation on the right-hand side of equation (A-5) can be written as a convolution against the density of the first passage time τ_e, yielding

$$
\begin{aligned}
& V_{t_0}^D\left(S_{t_0}, R, T; \phi, T\right) && \text{(A-6)} \\
= \; & v_{t_0}^D\left(S_{t_0}, R, T; \phi, T\right) \\
& -(\phi R)^+ e^{-\int_{t_0}^T r_l dl} \int_{t_0}^T \left[1 - \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_u^T \lambda(l,S)dl} \mathbf{1}_{\{\tau_0 > T\}} \middle| S_u = E_u\right)\right] \mathbb{Q}\left(\tau_e \in du \middle| \mathcal{F}_{t_0}\right) \\
= \; & v_{t_0}^D\left(S_{t_0}, R, T; \phi, T\right) - (\phi R)^+ e^{-\int_{t_0}^T r_l dl} \int_{t_0}^T \left[1 - SP\left(E_u, u; T\right)\right] \mathbb{Q}\left(\tau_e \in du \middle| \mathcal{F}_{t_0}\right) \\
= \; & v_{t_0}^D\left(S_{t_0}, R, T; \phi, T\right) - \int_{t_0}^T e^{-\int_{t_0}^u r_l dl} v_u^D\left(E_u, R, T; \phi, T\right) \mathbb{Q}\left(\tau_e \in du \middle| \mathcal{F}_{t_0}\right),
\end{aligned}
$$

where the last two lines follow from equations (2.11) and (2.10), respectively.

Combining equations (2.7), (2.23), (A-1), (A-2), and (A-6), equations (2.26) and (2.27) arise for $\eta = T$. ■

Appendix B

This appendix shows that whenever $\phi S_{t_0} \leq \phi E_{t_0}$ (for $\phi \in \{-1, 1\}$), the standard SHP pricing equation (2.31) would overvalue the American-style option. This is explained by the fact that all European-style options (conditional on no default) in the replicating portfolio (with a strike price equal to the value of the exercise boundary) would end in-the-money until the American-style option is exercised, i.e. until the spot price touches the early exercise boundary. On the contrary, when $\phi S_{t_0} > \phi E_{t_0}$, all European-style options included in the replicating portfolio end out-of-the-money until the American-style option is exercised. Next proposition formalizes the aforementioned economic rationale.

Proposition B.1 *When $\phi S_{t_0} \leq \phi E_{t_0}$, then*

$$V_{t_0}^{shpu}\left(S_{t_0}, K, T, R; \phi, \eta\right) > V_{t_0}\left(S_{t_0}, K, T, R; \phi, \eta\right), \tag{B-1}$$

for $\phi \in \{-1, 1\}$.

Proof. Given that $\phi S_{t_0} \leq \phi E_{t_0}$, let $t^* \leq T$ denote the next passage time of the asset price S through the early exercise boundary, i.e. $S_{t^*} = E_{t^*}$. This date corresponds to the space time point $n - i^*$. Replacing t_0 and S_{t_0} by t^* (or $n - i^*$) and $S_{t^*} \equiv E_{n-i^*}$, respectively, in equation (2.31), the value of the SHP portfolio at time t^* is equal to

$$\begin{aligned} V_{t^*}^{shpu}\left(E_{t^*}, K, T, R; \phi, \eta\right) &= v_{t_{n-i^*}}\left(E_{n-i^*}, K, T, R; \phi, \eta\right) \\ &+ \sum_{j=1}^{n} w_{n-j} \times v_{t_{n-i^*}}^{0}\left(E_{n-i^*}, E_{n-j}, t_{n-j+1}; \phi\right). \end{aligned} \tag{B-2}$$

Note, however, that all the European-style options with expiry date at times t_{n-j+1}, for $j = i^* + 1, ..., n$, that is all the options with a time to maturity equal to $(i^* - j + 1) \times \delta t$, have already expired by time t^*, and, therefore, their terminal payoff has been previously reinvested until time t^*. Hence,

$$v_{t_{n-i^*}}^{0}\left(E_{n-i^*}, E_{n-j}, t_{n-j+1}; \phi\right) = \left(\phi E_{n-j} - \phi S_{n-j+1}\right)^{+} \prod_{k=n-j+2}^{n-i^*} e^{r_k \times \delta t}, \tag{B-3}$$

for $j = i^* + 1, ..., n$, and equation (B-2) can be rewritten as

$$\begin{aligned} V_{t^*}^{shpu}\left(E_{t^*}, K, T, R; \phi, \eta\right) &= v_{t_{n-i^*}}\left(E_{n-i^*}, K, T, R; \phi, \eta\right) \\ &+ \sum_{j=1}^{i^*} w_{n-j} \times v_{t_{n-i^*}}^{0}\left(E_{n-i^*}, E_{n-j}, t_{n-j+1}; \phi\right) \\ &+ \sum_{j=i^*+1}^{n} w_{n-j} \times \left(\phi E_{n-j} - \phi S_{n-j+1}\right)^{+} \prod_{k=n-j+2}^{n-i^*} e^{r_k \times \delta t}. \end{aligned} \tag{B-4}$$

Using equation (2.28), equation (B-4) can be further rewritten as

$$
\begin{aligned}
V_{t^*}^{shpu}\left(E_{t^*}, K, T, R; \phi, \eta\right) &= \phi K - \phi E_{n-i^*} \\
&+ \sum_{j=i^*+1}^{n} w_{n-j} \times \left(\phi E_{n-j} - \phi S_{n-j+1}\right)^{+} \prod_{k=n-j+2}^{n-i^*} e^{r_k \times \delta t}.
\end{aligned} \tag{B-5}
$$

Finally, and since

$$\phi K - \phi E_{n-i^*} = V_{t^*}\left(E_{t^*}, K, T, R; \phi, \eta\right)$$

corresponds to the well known value-matching condition, then equation (B-5) becomes

$$
\begin{aligned}
V_{t^*}^{shpu}\left(E_{t^*}, K, T, R; \phi, \eta\right) &= V_{t^*}\left(E_{t^*}, K, T, R; \phi, \eta\right) \\
&+ \sum_{j=i^*+1}^{n} w_{n-j} \times \left(\phi E_{n-j} - \phi S_{n-j+1}\right)^{+} \prod_{k=n-j+2}^{n-i^*} e^{r_k \times \delta t}.
\end{aligned} \tag{B-6}
$$

Given that $\phi S_{t_0} \leq \phi E_{t_0}$, we have $\phi S_t < \phi E_t$ for all $t < t^*$, which makes the last term on the right-hand side of equation (B-6) almost surely strictly positive. Therefore,

$$V_{t^*}^{shpu}\left(E_{t^*}, K, T, R; \phi, \eta\right) > V_{t^*}\left(E_{t^*}, K, T, R; \phi, \eta\right). \tag{B-7}$$

Since equation (B-7) holds at any t^* along the early exercise boundary, equation (B-1) follows immediately.

If $t^* > T$, then all the European-style options in the replicating portfolio would end up in-the-money, thus augmenting even further the positive difference between the option values $V_{t_0}^{shpu}\left(S_{t_0}, K, T, R; \phi, \eta\right)$ and $V_{t_0}\left(S_{t_0}, K, T, R; \phi, \eta\right)$. ■

Table 2.1: Prices of standard American-style put options under the CEV model ($\beta = 3$, $S_{t_0} = \$100$, and $T - t_0 = 0.5$ years)

Parameters	Strike	Europ.	Exact	4d Pol.	5d Pol.	shp4	shp12	shp24	shp52
	80	0.159	0.162	0.162	0.162	0.161	0.162	0.162	0.162
$r = 7\%$	90	1.255	1.297	1.295	1.295	1.292	1.296	1.296	1.297
$q = 3\%$	100	4.579	4.792	4.788	4.791	4.786	4.791	4.792	4.792
$\delta = 0.02$	110	10.542	11.215	11.214	11.214	11.217	11.216	11.215	11.215
	120	18.452	20.025	20.023	20.023	20.031	20.028	20.026	20.026
	80	2.293	2.331	2.331	2.331	2.326	2.330	2.331	2.331
$r = 7\%$	90	5.385	5.491	5.489	5.490	5.482	5.489	5.490	5.491
$q = 3\%$	100	10.030	10.262	10.261	10.261	10.252	10.261	10.262	10.262
$\delta = 0.04$	110	16.043	16.474	16.472	16.473	16.465	16.473	16.474	16.474
	120	23.132	23.843	23.842	23.842	23.838	23.843	23.843	23.843
	80	0.822	0.852	0.851	0.851	0.848	0.851	0.851	0.851
$r = 7\%$	90	2.843	2.969	2.968	2.968	2.961	2.967	2.968	2.969
$q = 0\%$	100	6.698	7.060	7.059	7.059	7.052	7.059	7.060	7.060
$\delta = 0.03$	110	12.371	13.175	13.173	13.173	13.173	13.176	13.176	13.176
	120	19.493	20.992	20.991	20.991	21.001	20.995	20.993	20.992
	80	1.419	1.419	1.419	1.419	1.419	1.419	1.419	1.419
$r = 3\%$	90	4.311	4.311	4.311	4.311	4.311	4.311	4.311	4.311
$q = 7\%$	100	9.254	9.254	9.254	9.254	9.254	9.254	9.254	9.254
$\delta = 0.03$	110	15.980	15.980	15.980	15.980	15.980	15.980	15.980	15.980
	120	23.978	23.978	23.978	23.978	23.978	23.978	23.978	23.978
Mean absolute percentage error				0.0253%	0.0218%	0.1279%	0.0287%	0.0105%	0.0031%
CPU (seconds)			468.11	82.37	90.42	10.80	38.38	90.66	247.64

Table 2.1 values standard American-style put options under the CEV model, adopting the parameter configurations of Nunes (2009, Table 2). The 3rd column contains the European-style put prices, while the exact American-style put values (4th column) are based on the Crank-Nicolson method with $15,000$ time intervals and $10,000$ space steps. The 5th and 6th columns show the American-style put prices obtained through the optimal stopping approach of Nunes (2009), using a polynomial boundary—see Nunes (2009, Equation 47)—with 4 and 5 degrees of freedom, respectively. The last four columns report the American-style put prices under the SHP method, using time steps $n \in \{4, 12, 24, 52\}$.

Table 2.2: Prices of standard American-style call options under the CEV model ($\beta = 1$, $S_{t_0} = \$100$, and $T - t_0 = 1$ year)

Parameters	Strike	Europ.	Exact	4d Pol.	5d Pol.	shp4	shp12	shp24	shp52
	80	23.370	23.370	23.370	23.370	23.370	23.370	23.370	23.370
$r = 7\%$	90	15.735	15.735	15.735	15.735	15.735	15.735	15.735	15.735
$q = 3\%$	100	9.635	9.635	9.635	9.635	9.635	9.635	9.635	9.635
$\delta = 2$	110	5.315	5.315	5.315	5.315	5.315	5.315	5.315	5.315
	120	2.630	2.630	2.630	2.630	2.630	2.630	2.630	2.630
	80	28.249	28.254	28.253	28.253	28.254	28.254	28.254	28.254
$r = 7\%$	90	22.203	22.205	22.205	22.205	22.205	22.205	22.205	22.205
$q = 3\%$	100	17.083	17.083	17.083	17.083	17.083	17.083	17.083	17.083
$\delta = 4$	110	12.870	12.870	12.870	12.870	12.870	12.870	12.870	12.870
	120	9.499	9.499	9.499	9.499	9.499	9.499	9.499	9.499
	80	28.022	28.022	28.022	28.022	28.022	28.022	28.022	28.022
$r = 7\%$	90	21.061	21.061	21.061	21.061	21.061	21.061	21.061	21.061
$q = 0\%$	100	15.221	15.221	15.221	15.221	15.221	15.221	15.221	15.221
$\delta = 3$	110	10.567	10.567	10.567	10.567	10.567	10.567	10.567	10.567
	120	7.047	7.047	7.047	7.047	7.047	7.047	7.047	7.047
	80	20.301	21.882	21.881	21.882	21.895	21.886	21.884	21.883
$r = 3\%$	90	14.257	15.187	15.186	15.187	15.187	15.189	15.188	15.188
$q = 7\%$	100	9.552	10.084	10.083	10.083	10.076	10.083	10.084	10.084
$\delta = 3$	110	6.106	6.401	6.400	6.400	6.391	6.400	6.401	6.401
	120	3.728	3.886	3.885	3.885	3.876	3.884	3.886	3.886
Mean absolute percentage error				0.0031%	0.0026%	0.0285%	0.0057%	0.0020%	0.0008%
CPU (seconds)			468.94	54.69	57.47	5.65	20.51	46.07	135.98

Table 2.2 values standard American-style call options under the CEV model, adopting the constellation parameters of Nunes (2009, Table 3). The 3rd column contains the European-style call prices, while the exact American-style call values (4th column) are based on the Crank-Nicolson method with $15,000$ time intervals and $10,000$ space steps. The 5th and 6th columns report the American-style call prices obtained through the optimal stopping approach of Nunes (2009), using a polynomial boundary—see Nunes (2009, Equation 47)—with 4 and 5 degrees of freedom, respectively. The last four columns report the American-style call prices under the SHP method, using time steps $n \in \{4, 12, 24, 52\}$.

Table 2.3: Accuracy of the SHP approach for a large sample of randomly generated standard American-style puts

	5d Pol.	6d Pol.	shp4	shp12	shp24	shp52
Mean	0.0075%	0.0067%	0.0314%	0.0095%	0.0047%	0.0025%
Maximum	0.1598%	0.0818%	0.4698%	0.1182%	0.0451%	0.0149%
Minimum	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%
99th percentile	0.0328%	0.0268%	0.2372%	0.0660%	0.0263%	0.0096%
CPU (seconds)	4,592.45	4,900.69	484.65	1,777.68	4,170.60	11,367.63

Table 2.3 reports the absolute percentage pricing errors associated with the valuation of 1,250 randomly generated standard American-style put options, under the CEV model. The strike price is always set at $\$100$ while the other option features were generated from uniform distributions and within the following intervals: instantaneous volatility between 10% and 60%; interest rate and dividend yield between 0% and 10%; underlying spot price between $\$70$ an $\$130$; β between 0 and 4.0; and time-to-maturity ranging from 0 to 3.0 years. The pricing errors produced by the SHP and Nunes (2009) methods were computed against the Crank-Nicolson scheme with 15,000 time intervals and 10,000 space steps. The 2nd and 3rd columns report the pricing errors obtained through the optimal stopping approach of Nunes (2009), using a polynomial boundary—see Nunes (2009, Equation 47)—with 5 and 6 degrees of freedom, respectively. The last four columns highlight the pricing errors under the SHP method, using time steps $n \in \{4, 12, 24, 52\}$.

Table 2.4: Prices of standard American-style put options under the JDCEV model with recovery at maturity ($\bar{\beta} = -1$, $a = 20$, $S_{t_0} = \$100$, $T - t_0 = 0.5$ years, $r = 5\%$, and $q = 0\%$)

			Nunes (2009)							SHP approach		
			European put			American put						
b	c	K	$v_{t_0}^0$	$v_{t_0}^D$	v_{t_0}	EEP_{t_0}	$V_{t_0}^0$	$V_{t_0}^D$	V_{t_0}	shp12	shp24	shp52
0.00	0.5	80	0.280	0.773	1.053	0.005	0.291	0.767	1.058	1.058	1.058	1.058
0.00	0.5	90	1.254	0.869	2.123	0.035	1.324	0.834	2.159	2.159	2.160	2.160
0.00	0.5	100	3.941	0.966	4.907	0.185	4.279	0.812	5.091	5.092	5.093	5.093
0.00	0.5	110	9.152	1.063	10.215	0.716	10.391	0.540	10.931	10.934	10.934	10.934
0.00	0.5	120	16.727	1.159	17.886	-	-	-	20.000	20.000	20.000	20.000
0.00	1	80	0.226	1.522	1.748	0.004	0.239	1.513	1.752	1.752	1.752	1.752
0.00	1	90	1.067	1.712	2.780	0.029	1.154	1.654	2.808	2.809	2.810	2.810
0.00	1	100	3.500	1.903	5.402	0.156	3.921	1.637	5.558	5.562	5.563	5.564
0.00	1	110	8.397	2.093	10.490	0.636	9.975	1.152	11.127	11.137	11.137	11.137
0.00	1	120	15.709	2.283	17.992	-	-	-	20.000	20.000	20.000	20.000
0.02	0.5	80	0.239	1.534	1.773	0.004	0.250	1.526	1.777	1.777	1.777	1.777
0.02	0.5	90	1.099	1.726	2.824	0.030	1.177	1.678	2.854	2.855	2.856	2.856
0.02	0.5	100	3.546	1.917	5.463	0.160	3.940	1.683	5.623	5.626	5.627	5.627
0.02	0.5	110	8.436	2.109	10.545	0.633	9.947	1.231	11.178	11.188	11.188	11.188
0.02	0.5	120	15.722	2.301	18.023	-	-	-	20.000	20.000	20.000	20.000
0.02	1	80	0.192	2.268	2.460	0.003	0.205	2.259	2.464	2.464	2.464	2.464
0.02	1	90	0.932	2.552	3.484	0.024	1.019	2.488	3.508	3.509	3.509	3.510
0.02	1	100	3.140	2.836	5.975	0.135	3.592	2.518	6.110	6.115	6.116	6.116
0.02	1	110	7.721	3.119	10.841	0.563	9.493	1.911	11.404	11.417	11.417	11.416
0.02	1	120	14.739	3.403	18.142	-	-	-	20.000	20.000	20.000	20.000
CPU (seconds)									145.63	31.39	90.95	367.61

Table 2.4 values standard American-style put options under a time-homogeneous JDCEV model with recovery at maturity. The 4th, 5th, and 6th columns show the European-style put option price components recovered via equations (2.13), (2.10), and (2.7), respectively. Columns 7 to 10 contain the values offered by the approach of Nunes (2009) for the American-style put option components given in equations (2.27), (2.24), (2.25), and (2.26), respectively, using a 6 degree polynomial specification for the early exercise boundary. The last three columns of the table report the values of the standard American-style put provided by the SHP approach with $n \in \{12, 24, 52\}$.

Table 2.5: Prices of standard American-style put options under the JDCEV model with recovery at default time ($\bar{\beta} = -1$, $a = 20$, $S_{t_0} = \$100$, $T - t_0 = 0.5$ years, $r = 5\%$, and $q = 0\%$)

			Nunes (2009)							SHP approach		
			European put			American put						
b	c	K	$v^0_{t_0}$	$v^D_{t_0}$	v_{t_0}	EEP_{t_0}	$V^0_{t_0}$	$V^D_{t_0}$	V_{t_0}	shp12	shp24	shp52
0.00	0.5	80	0.280	0.783	1.063	0.005	0.291	0.776	1.068	1.058	1.068	1.068
0.00	0.5	90	1.254	0.880	2.134	0.035	1.326	0.844	2.170	2.159	2.171	2.171
0.00	0.5	100	3.941	0.978	4.919	0.182	4.279	0.822	5.101	5.092	5.104	5.104
0.00	0.5	110	9.152	1.076	10.228	0.709	10.390	0.547	10.937	10.934	10.943	10.942
0.00	0.5	120	16.727	1.174	17.901	-	-	-	20.000	20.000	20.000	20.000
0.00	1	80	0.226	1.541	1.768	0.004	0.239	1.532	1.771	1.752	1.771	1.772
0.00	1	90	1.067	1.734	2.801	0.028	1.154	1.675	2.829	2.809	2.831	2.831
0.00	1	100	3.500	1.927	5.426	0.153	3.922	1.657	5.580	5.562	5.586	5.586
0.00	1	110	8.397	2.120	10.517	0.625	9.974	1.168	11.142	11.137	11.155	11.155
0.00	1	120	15.709	2.312	18.021	-	-	-	20.000	20.000	20.000	20.000
0.02	0.5	80	0.239	1.553	1.792	0.004	0.250	1.546	1.796	1.777	1.796	1.796
0.02	0.5	90	1.099	1.747	2.846	0.030	1.178	1.698	2.876	2.855	2.877	2.878
0.02	0.5	100	3.546	1.942	5.488	0.156	3.939	1.705	5.643	5.626	5.649	5.650
0.02	0.5	110	8.436	2.136	10.572	0.622	9.945	1.250	11.194	11.188	11.207	11.207
0.02	0.5	120	15.722	2.330	18.052	-	-	-	20.000	20.000	20.000	20.000
0.02	1	80	0.192	2.297	2.489	0.003	0.205	2.288	2.492	2.464	2.493	2.493
0.02	1	90	0.932	2.585	3.516	0.023	1.020	2.519	3.539	3.509	3.542	3.542
0.02	1	100	3.140	2.872	6.011	0.129	3.588	2.552	6.140	6.115	6.150	6.150
0.02	1	110	7.721	3.159	10.880	0.544	9.480	1.944	11.424	11.417	11.446	11.446
0.02	1	120	14.739	3.446	18.185	-	-	-	20.000	20.000	20.000	20.000
CPU (seconds)									25,559.28	428.57	801.61	1,222.80

Table 2.5 values standard American-style put options under a time-homogeneous JDCEV model with recovery at default time. The 4th, 5th, and 6th columns show the European-style put option price components recovered via equations (2.13), (2.19), and (2.7), respectively. Columns 7 to 10 contain the values offered by the approach of Nunes (2009) for the American-style put option components given in equations (2.27), (2.24), (2.25), and (2.26), respectively, using a 6 degree polynomial specification for the early exercise boundary. The last three columns of the table report the values of the standard American-style put provided by the SHP approach with $n \in \{12, 24, 52\}$.

Table 2.6: Prices of capped American-style put options under the CEV model ($\beta = 3$, $S_{t_0} =$ \$100, $H =$ \$75, and $T - t_0 = 0.5$ years)

Parameters	Strike	4d Pol.	5d Pol.	shp4	shp12	shp24	shp52
	80	0.158	0.158	0.158	0.158	0.158	0.158
$r = 7\%$	90	1.290	1.292	1.296	1.296	1.296	1.297
$q = 3\%$	100	4.791	4.790	4.792	4.792	4.792	4.792
$\delta = 0.02$	110	11.214	11.214	11.215	11.215	11.215	11.215
	120	20.021	20.021	20.026	20.026	20.026	20.026
	80	1.573	1.573	1.570	1.570	1.570	1.572
$r = 7\%$	90	5.296	5.296	5.294	5.294	5.294	5.295
$q = 3\%$	100	10.237	10.237	10.237	10.237	10.237	10.238
$\delta = 0.04$	110	16.473	16.473	16.473	16.473	16.473	16.474
	120	23.842	23.842	23.843	23.843	23.843	23.843
	80	0.723	0.723	0.722	0.722	0.722	0.723
$r = 7\%$	90	2.964	2.966	2.966	2.966	2.966	2.967
$q = 0\%$	100	7.059	7.059	7.060	7.060	7.060	7.060
$\delta = 0.03$	110	13.173	13.174	13.176	13.176	13.176	13.176
	120	20.991	20.991	20.993	20.993	20.993	20.992
	80	1.066	1.066	1.064	1.064	1.064	1.065
$r = 3\%$	90	4.172	4.172	4.170	4.170	4.170	4.171
$q = 7\%$	100	9.165	9.165	9.165	9.165	9.165	9.165
$\delta = 0.03$	110	15.909	15.909	15.909	15.909	15.909	15.909
	120	23.920	23.920	23.920	23.920	23.920	23.920
CPU (seconds)		99.90	117.80	7.05	23.92	53.53	145.28

Table 2.6 values capped American-style put options under the CEV model, adopting the parameter configurations of Table 2.1. The 3rd and 4th columns show the capped American-style put prices obtained through the optimal stopping approach, using a polynomial boundary with 4 and 5 degrees of freedom, respectively. The last four columns report the capped American-style put prices under the SHP method, using time steps $n \in \{4, 12, 24, 52\}$.

Table 2.7: Prices of capped American-style put options under the JDCEV model with recovery at maturity ($\bar{\beta} = -1$, $a = 20$, $S_{t_0} = \$100$, $H = \$75$, $T - t_0 = 0.5$ years, $r = 5\%$, and $q = 0\%$)

			Nunes (2009)			SHP approach		
b	c	K	$\bar{V}^0_{t_0}$	$\bar{V}^D_{t_0}$	$\bar{V}_{t_0}$	shp12	shp24	shp52
0.00	0.5	80	0.261	0.047	0.308	0.307	0.308	0.308
0.00	0.5	90	1.331	0.137	1.468	1.468	1.469	1.469
0.00	0.5	100	4.301	0.194	4.495	4.496	4.496	4.497
0.00	0.5	110	10.457	0.140	10.597	10.601	10.599	10.598
0.00	0.5	120	-	-	20.000	20.000	20.000	20.000
0.00	1	80	0.222	0.093	0.315	0.314	0.315	0.315
0.00	1	90	1.172	0.270	1.443	1.444	1.444	1.444
0.00	1	100	4.000	0.378	4.378	4.381	4.381	4.380
0.00	1	110	10.225	0.247	10.472	10.479	10.476	10.475
0.00	1	120	-	-	20.000	20.000	20.000	20.000
0.02	0.5	80	0.229	0.094	0.323	0.322	0.323	0.323
0.02	0.5	90	1.192	0.276	1.467	1.467	1.468	1.468
0.02	0.5	100	4.006	0.394	4.400	4.402	4.402	4.402
0.02	0.5	110	10.206	0.267	10.473	10.479	10.476	10.475
0.02	0.5	120	-	-	20.000	20.000	20.000	20.000
0.02	1	80	0.194	0.139	0.333	0.333	0.333	0.333
0.02	1	90	1.051	0.406	1.457	1.457	1.457	1.457
0.02	1	100	3.733	0.572	4.305	4.309	4.308	4.308
0.02	1	110	10.022	0.349	10.371	10.381	10.377	10.376
0.02	1	120	-	-	20.000	20.000	20.000	20.000
CPU (seconds)					190.04	21.97	71.22	288.61

Table 2.7 values capped American-style put options under a time-homogeneous JDCEV model with recovery at maturity, adopting the parameter configurations of Table 2.4. Columns 4 to 6 contain the values offered by the approach of Nunes (2009) for the American-style capped put option components given in equations (2.47), (2.48), and (2.46), respectively, using a 6 degree polynomial specification for the early exercise boundary. The last three columns of the table report the values of the capped American-style put provided by the SHP approach with $n \in \{12, 24, 52\}$.

3. General Put-Call Symmetry for American-style Barrier Options*

Abstract: This paper derives put-call symmetries for American-style single and double barrier options. Using the change of numéraire technique proposed by Geman et al. (1995) and Schroder (1999) we are able to derive these symmetries without imposing previous assumptions on the stochastic process followed by the underlying asset. Our results are tested through an extensive numerical analysis run under the *constant elasticity of variance* model of Cox (1975).

JEL Classification: G13.

Keywords: American-style options; Barrier options; Put-call symmetry; CEV model.

3.1 Introduction

The put-call symmetry holds whenever the price of a put option can be recovered from the price of a call option (and vice versa) through a suitable change in its function arguments. This relation is important for practitioners since they can value put and call options using the same numerical algorithm.

*This paper is a joint work with João Pedro Nunes and José Carlos Dias.

In the case of standard American-style options, the put-call symmetry was demonstrated by McDonald and Schroder (1998) as well as Bjerksund and Stensland (1993) under the geometric Brownian motion (hereafter, GBM) assumption. In the context of diffusion models in which the drift is an arbitrary function of the asset price but the volatility is a symmetric function of time, Carr and Chesney (1996) show that the symmetry also holds. Based on the change of numéraire technique proposed in Geman et al. (1995), Schroder (1999) uses the reinvested asset price as the numéraire and shows that the put-call symmetry holds for a very general class of models, including models with stochastic coefficients and jumps.

The extension of the put-call symmetry for American-style single knock-out options was achieved by Gao et al. (2000), but under the GBM assumption. Using the same change of numéraire technique as Schroder (1999), and considering a market structure in which the underlying asset price follows a single factor but a Markovian diffusion process, Detemple (2001) derives the put-call symmetry for American-style single knock-out options.

Using the change of numéraire technique presented in Schroder (1999) and based on two mild assumptions we are able to extend the put-call symmetry to other types of American-style barrier options. Our two basic assumptions are: The existence of a risk neutral measure (i.e. no arbitrage, in the Harrison and Pliska (1981) sense), and the existence of a unique early exercise boundary that separates the *continuation* and *exercise* regions of any American-style option contract.

Our contribution to the existent literature is threefold: First, we extend the put-call symmetry to American-style single knock-in barrier options; Second, and more importantly, the symmetry is extended to American-style double knock-out and knock-in options; Third, we provide numerical results for all the put-call symmetries derived in this paper, under the CEV model of Cox (1975).

This paper proceeds as follows. Section 3.2 describes the change of numéraire technique, the contract specifications and the main assumptions presented in Schroder (1999), which allow us to prove, in Section 3.3, the put-call symmetry for American-style barrier options. Section 3.4 applies the previous results to the CEV model of Cox (1975), and offers numerical examples. Finally, Section 3.5 concludes.

3.2 Modelling assumptions and contract specifications

3.2.1 Modelling assumptions

Schroder (1999) uses a change of numéraire argument to derive a general parity relation between plain-vanilla American-style call and put option prices. Schroder (1999) shows that any American-style call pricing solution can be converted, after a change of numéraire, into an American-style put option valuation formula. His argument is valid for a large class of diffusion and jump-diffusion pricing models. Below we state the main assumptions and results that we borrow from Schroder (1999).

We assume throughout that the financial market is arbitrage-free and frictionless, and that trading takes place continuously on the time-interval $\mathcal{T} := [t_0, T]$, for some initial date t_0 and fixed time $T > t_0$. Uncertainty is represented by a complete probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, and the information accruing to all the agents in the economy is described by the complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \in \mathcal{T}\}$ that satisfies the usual conditions.

Furthermore,

Assumption 3.1 *There exists a risk-neutral measure $\mathbb{Q}$ such that the relative reinvested price of every asset, with respect to the reinvested short-rate process (or "money-market account" numéraire), are $\mathbb{Q}$-martingales.*

Following Schroder (1999, Equations 1 and 2), if we change the numéraire from the "money-market account" to a reinvested asset price process $S_t \exp\left(\int_{t_0}^{t} q_u du\right)$, the probability measure $\bar{\mathbb{Q}}$ that corresponds to such new numéraire is such that

$$\bar{\mathbb{Q}}\left(A \middle| \mathcal{F}_{t_0}\right) = \mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{A\}} Z_T \middle| \mathcal{F}_{t_0}\right), \tag{3.1}$$

for $A \in \mathcal{F}$, and where Z_T is the Radon-Nikodym derivative $\frac{d\bar{\mathbb{Q}}}{d\mathbb{Q}}$, i.e.

$$Z_t := \frac{\exp\left(\int_{t_0}^{t} \left(q_u - r_u\right) du\right) S_t}{S_{t_0}},\ t \in \mathcal{T}, \tag{3.2}$$

with S representing the asset price process, r the short rate process, and q the dividend yield.

Based on equations (3.1) and (3.2) it is now possible to obtain Schroder (1999, Proposition 1 and Corollary 1) that we simply summarize.

Proposition 3.1 *Define measure $\bar{\mathbb{Q}}$ by equation (3.1), and*

$$\bar{S}_t := \frac{K S_{t_0}}{S_t}, \tag{3.3}$$

with K being the strike price of the option contract. Then, the time-t_0 price on an asset with the $\mathcal{F}_\tau$-measurable payoff P_τ at the stopping time $\tau \in \mathcal{T}$ is

$$\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{t_0}^{\tau} r_u du\right) P_\tau \middle| \mathcal{F}_{t_0}\right] = \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{\tau} q_u du\right) P_\tau \frac{\bar{S}_\tau}{K} \middle| \mathcal{F}_{t_0}\right]. \tag{3.4}$$

Proof. See Schroder (1999, Proposition 1). ■

Next corollary follows from Proposition 3.1 by taking P_τ to be the intrinsic value $(S_\tau - K)^+$ of a call option.

Corollary 1 *Define measure $\bar{\mathbb{Q}}$ by equation (3.1), and the process $\bar{S}_t$ through equation (3.3). Then, the time-t_0 value of a call option on S, and with strike K, is the same, after a change of numéraire, as the value of a put option on $\bar{S}$, and with strike S_{t_0}, i.e.:*

$$\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{t_0}^{\tau} r_u du\right)(S_\tau - K)^+ \middle| \mathcal{F}_{t_0}\right] = \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{\tau} q_u du\right)\left(S_{t_0} - \bar{S}_\tau\right)^+ \middle| \mathcal{F}_{t_0}\right], \tag{3.5}$$

for any stopping time $\tau \leq T$.

Proof. See Schroder (1999, Corollary 1). ■

The left-hand side of equation (3.5) can be understood as the time-t_0 price of a European call on S, with strike K, and maturity at time τ, whereas the right-hand side corresponds to a European-style put on $\bar{S}$, with strike S_{t_0}, and maturity at the same time τ. Nevertheless, Schroder (1999, Page 1149) shows that Proposition 3.1 and Corollary 1 also hold for American-style options as long as we further adopt Schroder (1999, Assumption 2), i.e.

Assumption 3.2 *Let V_{t_0} be the time-t_0 price of an American-style option allowing the holder to exercise and receive, at any stopping time $\tau \in \mathcal{T}$, the payoff P_τ, where P is an adapted process. Then*

$$V_{t_0} = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{t_0}^{\tau} r_u du\right) P_\tau \middle| \mathcal{F}_{t_0}\right]. \tag{3.6}$$

The *Snell envelope* (3.6) is proved by Karatzas (1988, Theorem 5.4) in a complete market setting, and adapted by Pham (1997, Equation 1.5) to a jump-diffusion framework (although conditional upon the specification of the market price of jump risk).

However, our main results will be further based on the existence and uniqueness of an early exercise boundary $\{B_t : t \in \mathcal{T}\}$ that separates the *continuation* and *exercise* (or stopping) regions of the American-style option contract. This should constitute a mild assumption that is satisfied by most of the option pricing models already proposed in the literature. Under a single factor diffusion model, Detemple and Tian (2002, Propostion 1) prove the existence of the exercise boundary $\{B_t : t \in \mathcal{T}\}$, which is only a function of time (and a continuous function as long as the risk-free interest rate is deterministic). Pham (1997, Theorems 3.2 and 4.1) extends the previous result to a jump-diffusion model (conditional on the positiveness of the riskless interest rate corrected by the jump risk). More importantly, under a very general multifactor and diffusion framework (that accommodates stochastic interest rates, volatilities, and dividend yields), Detemple and Tian (2002, Propostion 2) show that there still exists a single exercise boundary $\{B_t : t \in \mathcal{T}\}$, which now depends on time and on all the other state variables besides the underlying asset price.[3.1]

Therefore, we finally assume that

Assumption 3.3 *There exists a unique early exercise boundary $\{B_t : t \in \mathcal{T}\}$ such that equation (3.6) can be rewritten as*

$$V_{t_0} = \mathbb{E}_{\mathbb{Q}} \left[\exp\left(-\int_{t_0}^{T \wedge \tau_B} r_u du \right) P_{T \wedge \tau_B} \middle| \mathcal{F}_{t_0} \right], \tag{3.7}$$

where τ_B is the first passage time of the underlying asset price process through the

[3.1] This result is conditional on Detemple and Tian (2002, Assumption M), which basically imposes monotonicity restrictions on the trajectories of the asset price, the interest rate, and the dividend yield.

early exercise boundary $\{B_t : t \in \mathcal{T}\}$.

Assumption 3.3 simply states that, for each time $t \in \mathcal{T}$ there exists a *critical asset price* B_t below (above) which the American put (call) price equals its intrinsic value and early exercise should occur. Consequently, the optimal policy should be to exercise the American option when the underlying asset price first touches its critical level.

3.2.2 Contract specifications

In this section we state the contract specifications for all the array of American-style options used in the remaining of this article.

American-style standard options

Using Assumption 3.3, the time-t_0 value of a standard American-style put (if $\phi = 1$) or call (if $\phi = -1$) option on the asset price S, with strike K, and maturity at time T ($\geq t_0$), is equal to

$$\begin{aligned} & AS_{t_0}\left(S_{t_0}, K, T, r, q, B^{s(\phi)}; \phi\right) \\ = \; & \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{t_0}^{T\wedge\tau_{B^{s(\phi)}}} r_u du\right)\left(\phi K - \phi S_{T\wedge\tau_{B^{s(\phi)}}}\right)^+ \middle| \mathcal{F}_{t_0}\right], \end{aligned} \tag{3.8}$$

where

$$\tau_{B^{s(\phi)}} := \inf\left\{t \geq t_0 : \phi S_t \leq \phi B_t^{s(\phi)}\right\}, \tag{3.9}$$

and $B^{s(\phi)}$ denotes the exercise boundary of the standard American-style option.

American-style knock-out options

An American-style knock-out option (without rebate) becomes worthless if the single barrier is touched by the underlying asset price; otherwise, it can be exercised at or before the expiry date. Additionally, and as pointed by Gao et al. (2000, Footnote 15), for the knock-out event and the exercise date to be well defined, the option contract must be specified in a way such that when the asset price first touches the barrier, the option holder has the option to either exercise or let the option be knocked out.

Therefore, using Assumption 3.3, the time-t_0 value of an American-style down-and-out put (if $\phi = 1$) or call (if $\phi = -1$) option on the asset price S, with strike K, lower barrier level L, and maturity at time T ($\geq t_0$), is equal to

$$\begin{aligned} & ADO_{t_0}\left(S_{t_0}, K, L, T, r, q, B^{do(\phi)}; \phi\right) \\ = \;& \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{t_0}^{T\wedge\tau_{B^{do(\phi)}}\wedge\tau_L} r_u du\right)\left(\phi K - \phi S_{T\wedge\tau_{B^{do(\phi)}}\wedge\tau_L}\right)^+ \middle| \mathcal{F}_{t_0}\right], \end{aligned} \tag{3.10}$$

with

$$\tau_L := \inf\left\{t \geq t_0 : S_t \leq L\right\}, \tag{3.11}$$

$$\tau_{B^{do(\phi)}} := \inf\left\{t \geq t_0 : \phi S_t \leq \phi B_t^{do(\phi)}\right\}, \tag{3.12}$$

and where $B^{do(\phi)}$ denotes the exercise boundary of the American-style down-and-out option.

Using Assumption 3.3, the time-t_0 value of an American-style up-and-out put (if $\phi = 1$) or call (if $\phi = -1$) option on the asset price S, with strike K, upper barrier level U, and maturity at time T ($\geq t_0$) is equal to

$$\begin{aligned}
& AUO_{t_0}\left(S_{t_0}, K, U, T, r, q, B^{uo(\phi)}; \phi\right) \qquad (3.13) \\
= \; & \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{t_0}^{T\wedge\tau_{B^{uo(\phi)}}\wedge\tau_U} r_u du\right)\left(\phi K - \phi S_{T\wedge\tau_{B^{uo(\phi)}}\wedge\tau_U}\right)^{+}\middle|\mathcal{F}_{t_0}\right],
\end{aligned}$$

with

$$\tau_U := \inf\left\{t \geq t_0 : S_t \geq U\right\}, \qquad (3.14)$$

$$\tau_{B^{uo(\phi)}} := \inf\left\{t \geq t_0 : \phi S_t \leq \phi B_t^{uo(\phi)}\right\}, \qquad (3.15)$$

and where $B^{uo(\phi)}$ denotes the exercise boundary of the American-style up-and-out option.

Using Assumption 3.3, the time-t_0 value of an American-style double knock-out put (if $\phi = 1$) or call (if $\phi = -1$) option on the asset price S, with strike K, lower barrier level L, upper barrier level U, and maturity at time T ($\geq t_0$), is equal to

$$\begin{aligned}
& AKODB_{t_0}\left(S_{t_0}, K, L, U, T, r, q, B^{ko(\phi)}; \phi\right) \qquad (3.16) \\
= \; & \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{t_0}^{T\wedge\tau_{B^{ko(\phi)}}\wedge\tau_{LU}} r_u du\right)\left(\phi K - \phi S_{T\wedge\tau_{B^{ko(\phi)}}\wedge\tau_{LU}}\right)^{+}\middle|\mathcal{F}_{t_0}\right],
\end{aligned}$$

with

$$\tau_{B^{ko(\phi)}} := \inf\left\{t \geq t_0 : \phi S_t \leq \phi B_t^{ko(\phi)}\right\}, \qquad (3.17)$$

$$\tau_{LU} := \inf\left\{t > t_0 : S_t \leq L \quad or \quad S_t \geq U\right\}, \qquad (3.18)$$

and where $B^{ko(\phi)}$ denotes the exercise boundary of the American-style double knock-out option.

American-style knock-in options

An American-style knock-in option becomes an American-style standard option if and only if the barrier is touched by the underlying spot price, before or at, the option's expiry date. Therefore, using Assumption 3.3, the time-t_0 value of an American-style down-and-in put (if $\phi = 1$) or call (if $\phi = -1$) option on the asset price S, with strike K, lower barrier level L, and maturity at time T ($\geq t_0$), is equal to

$$\begin{aligned} & ADI_{t_0}\left(S_{t_0}, K, L, T, r, q, B^{di(\phi)}; \phi\right) \\ = \quad & \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{t_0}^{\tau_L} r_u du\right) AS_{\tau_L}\left(L, K, T, r, q, B^{s(\phi)}\right) \mathbb{1}_{\{\tau_L \leq T\}} \middle| \mathcal{F}_{t_0}\right], \end{aligned} \tag{3.19}$$

where the first hitting time τ_L is defined by equation (3.11), and $B^{di(\phi)}$ denotes the exercise boundary of the American-style down-and-in option.

Alternatively, and using equations (3.8) and (3.9), equation (3.19) can be rewritten as

$$\begin{aligned} & ADI_{t_0}\left(S_{t_0}, K, L, T, r, q, B^{di(\phi)}; \phi\right) \\ = \quad & \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{t_0}^{T \wedge \gamma^L_{B^{s(\phi)}}} r_u du\right) \left(\phi K - \phi S_{T \wedge \gamma^L_{B^{s(\phi)}}}\right)^+ \mathbb{1}_{\{\tau_L \leq T\}} \middle| \mathcal{F}_{t_0}\right], \end{aligned} \tag{3.20}$$

where

$$\gamma^L_{B^{s(\phi)}} := \inf\left\{t \geq \tau_L : \phi S_t \leq \phi B^{s(\phi)}_t\right\}. \tag{3.21}$$

Using Assumption 3.3, the time-t_0 value of an American-style up-and-in put (if $\phi = 1$) or call (if $\phi = -1$) option on the asset price S, with strike K, upper barrier level U, and maturity at time T ($\geq t_0$), is equal to

$$
\begin{aligned}
& AUI_{t_0}\left(S_{t_0}, K, U, T, r, q, B^{ui(\phi)}; \phi\right) && (3.22)\\
= \;& \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{t_0}^{\tau_U} r_u du\right) AS_{\tau_U}\left(U, K, T, r, q, B^{s(\phi)}; \phi\right) \mathbb{1}_{\{\tau_U \leq T\}} \middle| \mathcal{F}_{t_0}\right]\\
= \;& \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{t_0}^{T\wedge\gamma^U_{B^{s(\phi)}}} r_u du\right)\left(\phi K - \phi S_{T\wedge\gamma^U_{B^{s(\phi)}}}\right)^+ \mathbb{1}_{\{\tau_U \leq T\}} \middle| \mathcal{F}_{t_0}\right],
\end{aligned}
$$

where τ_U is defined by equation (3.14),

$$
\gamma^U_{B^{s(\phi)}} := \inf\left\{t \geq \tau_U : \phi S_t \leq \phi B^{s(\phi)}_t\right\}, \tag{3.23}
$$

and $B^{ui(\phi)}$ denotes the exercise boundary of the American-style up-and-in option.

Finally, using Assumption 3.3, the time-t_0 value of an American-style double knock-in put (if $\phi = 1$) or call (if $\phi = -1$) option on the asset price S, with strike K, lower barrier level L, upper barrier level U, and maturity at time T ($\geq t_0$), is equal to

$$
\begin{aligned}
& AKIDB_{t_0}\left(S_{t_0}, K, L, U, T, r, q, B^{ki(\phi)}; \phi\right) && (3.24)\\
= \;& \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{t_0}^{\tau_{LU}} r_u du\right) AS_{\tau_{LU}}\left(S_{\tau_{LU}}, K, T, r, q, B^{s(\phi)}; \phi\right) \mathbb{1}_{\{\tau_{LU} \leq T\}} \middle| \mathcal{F}_{t_0}\right]\\
= \;& \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{t_0}^{T\wedge\gamma^{LU}_{B^{s(\phi)}}} r_u du\right)\left(\phi K - \phi S_{T\wedge\gamma^{LU}_{B^{s(\phi)}}}\right)^+ \mathbb{1}_{\{\tau_{LU} \leq T\}} \middle| \mathcal{F}_{t_0}\right],
\end{aligned}
$$

where τ_{LU} is defined by equation (3.18),

$$
\gamma^{LU}_{B^{s(\phi)}} := \inf\left\{t \geq \tau_L \wedge \tau_U : \phi S_t \leq \phi B^{s(\phi)}_t\right\}, \tag{3.25}
$$

and $B^{ki(\phi)}$ denotes the exercise boundary of the American-style double knock-in option.

3.3 American-style barrier options put-call symmetry

Using the results from the previous section, i.e. without imposing any specific asset price dynamics, we will obtain put-call symmetries for American-style single, double, knock-in and knock-out barrier options.

3.3.1 Knock-out options

Following Detemple (2001), next two propositions deal with single barrier contracts.

Proposition 3.2 *Define $\bar{S}_t$ through equation (3.3). Under Assumptions 3.1 and 3.3, the time-t_0 value of an American-style down-and-out call option on the asset price S, with strike K, lower barrier level L, maturity at time T $(\geq t_0)$, and exercise boundary $B^{do(-1)}$, is the same, after a change of numéraire, as the time-t_0 value of an American-style up-and-out put option on the asset price $\bar{S}$, with strike S_{t_0}, upper barrier level KS_{t_0}/L, maturity at time T $(\geq t_0)$, and exercise boundary $S_{t_0}K/B^{do(-1)}$, i.e.*

$$\begin{aligned} & ADO_{t_0}\left(S_{t_0}, K, L, T, r, q, B^{do(-1)}; -1\right) \\ = \; & AUO_{t_0}\left(K, S_{t_0}, KS_{t_0}/L, T, q, r, S_{t_0}K/B^{do(-1)}; 1\right). \end{aligned} \tag{3.26}$$

Remark 3.1 *Under the geometric Brownian motion assumption, equation (3.26) was already derived, for instance, by Gao et al. (2000, Theorem 3) or Detemple (2006, Proposition 50).*

Proof. Applying Proposition 3.1 to equation (3.10) and using definition (3.3), then

$$\begin{aligned}
& ADO_{t_0}\left(S_{t_0}, K, L, T, r, q, B^{do(-1)}; -1\right) && (3.27)\\
= \; & \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{T\wedge\tau_{B^{do(-1)}}\wedge\tau_L} q_u du\right)\left(S_{T\wedge\tau_{B^{do(-1)}}\wedge\tau_L} - K\right)^+ \frac{\bar{S}_{T\wedge\tau_{B^{do(-1)}}\wedge\tau_L}}{K}\right|\left.\mathcal{F}_{t_0}\right] \\
= \; & \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{T\wedge\tau_{B^{do(-1)}}\wedge\tau_L} q_u du\right)\left(S_{T\wedge\tau_{B^{do(-1)}}\wedge\tau_L} - K\right)^+ \frac{S_{t_0}}{S_{T\wedge\tau_{B^{do(-1)}}\wedge\tau_L}}\right|\left.\mathcal{F}_{t_0}\right] \\
= \; & \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{T\wedge\tau_{B^{do(-1)}}\wedge\tau_L} q_u du\right)\left(S_{t_0} - \bar{S}_{T\wedge\tau_{B^{do(-1)}}\wedge\tau_L}\right)^+\right|\left.\mathcal{F}_{t_0}\right].
\end{aligned}$$

Based on Corollary 1, the last line of equation (3.27) corresponds to the time-t_0 price of an American-style put option on the asset price $\bar{S}$, with strike S_{t_0}, and maturity at time T. However, all first passage times contained in the last line of equation (3.27) are still expressed in terms of S (and not $\bar{S}$). Nevertheless, equations (3.3) and (3.11) can be combined into

$$\begin{aligned}
\tau_L &= \inf\left\{t \geq t_0 : S_t\left(KS_{t_0}/S_t\right) \leq L\left(KS_{t_0}/S_t\right)\right\} && (3.28)\\
&= \inf\left\{t \geq t_0 : \bar{S}_t \geq KS_{t_0}/L\right\} \\
&= \bar{\tau}_L,
\end{aligned}$$

i.e. τ_L can be also understood as the first passage time of the process $\bar{S}$ through the up-barrier KS_{t_0}/L, which is represented by $\bar{\tau}_L$.

Similarly, equations (3.3) and (3.12) can also yield a new interpretation to the stopping time $\tau_{B^{do(-1)}}$:

$$\begin{aligned}
\tau_{B^{do(-1)}} &= \inf\left\{t \geq t_0 : S_t\left(KS_{t_0}/S_t\right) \geq B_t^{do(-1)}\left(KS_{t_0}/S_t\right)\right\} && (3.29)\\
&= \inf\left\{t \geq t_0 : \bar{S}_t \leq KS_{t_0}/B_t^{do(-1)}\right\} \\
&= \bar{\tau}_{B^{do(-1)}}.
\end{aligned}$$

Hence, $\tau_{B^{do(-1)}}$ can be rewritten as the first passage time of the process $\bar{S}$ through the exercise boundary $KS_{t_0}/B^{do(-1)}$, which is represented by $\bar{\tau}_{B^{do(-1)}}$.

Finally, equations (3.27), (3.28) and (3.29) can be combined into

$$\begin{aligned} & ADO_{t_0}\left(S_{t_0}, K, L, T, r, q, B^{do(-1)}; -1\right) \\ = \ & \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{T\wedge\bar{\tau}_{B^{do(-1)}}\wedge\bar{\tau}_L} q_u du\right)\left(S_{t_0} - \bar{S}_{T\wedge\bar{\tau}_{B^{do(-1)}}\wedge\bar{\tau}_L}\right)^+ \middle| \mathcal{F}_{t_0}\right] \\ = \ & AUO_{t_0}\left(\bar{S}_{t_0}, S_{t_0}, KS_{t_0}/L, T, q, r, KS_{t_0}/B^{do(-1)}; 1\right), \end{aligned} \tag{3.30}$$

where the last line follows from equation (3.13). Equation (3.26) arises immediately after noting that $\bar{S}_{t_0} = K$. ■

Proposition 3.3 *Let $\bar{S}_t$ be defined by equation (3.3). Under Assumptions 3.1 and 3.3, the time-t_0 value of an American-style up-and-out call option on the asset price S, with strike K, upper barrier level U, maturity at time T ($\geq t_0$), and exercise boundary $B^{uo(-1)}$, is the same, after a change of numéraire, as the time-t_0 value of an American-style down-and-out put option on the asset price $\bar{S}$, with strike S_{t_0}, lower barrier level KS_{t_0}/U, maturity at time T ($\geq t_0$), and exercise boundary $S_{t_0}K/B^{uo(-1)}$, i.e.*

$$\begin{aligned} & AUO_{t_0}\left(S_{t_0}, K, U, T, r, q, B^{uo(-1)}; -1\right) \\ = \ & ADO_{t_0}\left(K, S_{t_0}, KS_{t_0}/U, T, q, r, S_{t_0}K/B^{uo(-1)}; 1\right). \end{aligned} \tag{3.31}$$

Proof. Applying Proposition 3.1 to equation (3.13) and using definition (3.3), then

$$
\begin{aligned}
& AUO_{t_0}\left(S_{t_0}, K, U, T, r, q, B^{uo(-1)}; -1\right) && (3.32)\\
= \; & \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{T\wedge\tau_{B^{uo(-1)}}\wedge\tau_U} q_u du\right)\left(S_{T\wedge\tau_{B^{uo(-1)}}\wedge\tau_U} - K\right)^+ \frac{\bar{S}_{T\wedge\tau_{B^{uo(-1)}}\wedge\tau_U}}{K}\middle|\mathcal{F}_{t_0}\right] \\
= \; & \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{T\wedge\tau_{B^{uo(-1)}}\wedge\tau_U} q_u du\right)\left(S_{T\wedge\tau_{B^{uo(-1)}}\wedge\tau_U} - K\right)^+ \frac{S_{t_0}}{S_{T\wedge\tau_{B^{uo(-1)}}\wedge\tau_U}}\middle|\mathcal{F}_{t_0}\right] \\
= \; & \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{T\wedge\tau_{B^{uo(-1)}}\wedge\tau_U} q_u du\right)\left(S_{t_0} - \bar{S}_{T\wedge\tau_{B^{uo(-1)}}\wedge\tau_U}\right)^+\middle|\mathcal{F}_{t_0}\right].
\end{aligned}
$$

Based on Corollary 1, the last line of equation (3.32) corresponds to the time-t_0 price of an American-style put option on the asset price $\bar{S}$, with strike S_{t_0}, and maturity at time T. However, all first passage times contained in the last line of equation (3.32) are still expressed in terms of S (and not $\bar{S}$). Nevertheless, equations (3.3) and (3.14) can be combined into

$$
\begin{aligned}
\tau_U &= \inf\left\{t \geq t_0 : S_t\left(KS_{t_0}/S_t\right) \geq U\left(KS_{t_0}/S_t\right)\right\} && (3.33)\\
&= \inf\left\{t \geq t_0 : \bar{S}_t \leq KS_{t_0}/U\right\} \\
&= \bar{\tau}_U,
\end{aligned}
$$

i.e. τ_U can be also understood as the first passage time of the process $\bar{S}$ through the low-barrier KS_{t_0}/U, which will be denoted by $\bar{\tau}_U$.

Similarly, equations (3.3) and (3.15) can also yield a new interpretation to the stopping time $\tau_{B^{uo(-1)}}$:

$$
\begin{aligned}
\tau_{B^{uo(-1)}} &= \inf\left\{t \geq t_0 : S_t\left(KS_{t_0}/S_t\right) \geq B_t^{uo(-1)}\left(KS_{t_0}/S_t\right)\right\} && (3.34)\\
&= \inf\left\{t \geq t_0 : \bar{S}_t \leq KS_{t_0}/B_t^{uo(-1)}\right\} \\
&= \bar{\tau}_{B^{uo(-1)}}.
\end{aligned}
$$

Hence, $\tau_{B^{uo(-1)}}$ can be rewritten as the first passage time of the process $\bar{S}$ through the exercise boundary $KS_{t_0}/B_t^{uo(-1)}$, which is represented by $\bar{\tau}_{B^{uo(-1)}}$.

Finally, equations (3.32), (3.33) and (3.34) can be combined into

$$\begin{aligned} & AUO_{t_0}\left(S_{t_0}, K, U, T, r, q, B^{uo(-1)}; -1\right) \\ = \; & \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{T\wedge\bar{\tau}_{B^{uo(-1)}}\wedge\bar{\tau}_U} q_u du \right) \left(S_{t_0} - \bar{S}_{T\wedge\bar{\tau}_{B^{uo(-1)}}\wedge\bar{\tau}_U} \right)^+ \middle| \mathcal{F}_{t_0} \right] \\ = \; & ADO_{t_0}\left(\bar{S}_{t_0}, S_{t_0}, KS_{t_0}/U, T, q, r, KS_{t_0}/B^{uo(-1)}; 1\right), \end{aligned} \tag{3.35}$$

where the last line follows from equation (3.10). Equation (3.31) arises immediately after noting that $\bar{S}_{t_0} = K$. ■

Next proposition extends Detemple (2001) to American-style double knock-out options.

Proposition 3.4 *Let $\bar{S}_t$ be defined by equation (3.3). Under Assumptions 3.1 and 3.3, the time-t_0 value of an American-style double knock-out call option on the asset price S, with strike K, lower barrier level L, upper barrier level U, maturity at time T ($\geq t_0$), and exercise boundary $B^{ko(-1)}$, is the same, after a change of numéraire, as the time-t_0 value of an American-style double knock-out put option on the asset price $\bar{S}$, with strike S_{t_0}, lower barrier level KS_{t_0}/U, upper barrier level KS_{t_0}/L , maturity at time T ($\geq t_0$), and exercise boundary $S_{t_0}K/B^{ko(-1)}$, i.e.*

$$\begin{aligned} & AKODB_{t_0}\left(S_{t_0}, K, L, U, T, r, q, B^{ko(-1)}; -1\right) \\ = \; & AKODB_{t_0}\left(K, S_{t_0}, KS_{t_0}/U, KS_{t_0}/L, T, q, r, S_{t_0}K/B^{ko(-1)}; 1\right). \end{aligned} \tag{3.36}$$

Proof. Applying Proposition 3.1 to equation (3.16) and using definition (3.3), then

$$\begin{aligned}
& AKODB_{t_0}\left(S_{t_0}, K, L, U, T, r, q, B^{ko(-1)}; -1\right) \qquad (3.37)\\
= \;& \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{T\wedge\tau_{B^{ko(-1)}}\wedge\tau_{LU}} q_u du\right)\left(S_{T\wedge\tau_{B^{ko(-1)}}\wedge\tau_{LU}} - K\right)^+ \frac{\bar{S}_{T\wedge\tau_{B^{ko(-1)}}\wedge\tau_{LU}}}{K}\right|\left.\mathcal{F}_{t_0}\right]\\
= \;& \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{T\wedge\tau_{B^{ko(-1)}}\wedge\tau_{LU}} q_u du\right)\left(S_{T\wedge\tau_{B^{ko(-1)}}\wedge\tau_{LU}} - K\right)^+ \frac{S_{t_0}}{S_{T\wedge\tau_{B^{ko(-1)}}\wedge\tau_{LU}}}\right|\left.\mathcal{F}_{t_0}\right]\\
= \;& \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{T\wedge\tau_{B^{ko(-1)}}\wedge\tau_{LU}} q_u du\right)\left(S_{t_0} - \bar{S}_{T\wedge\tau_{B^{ko(-1)}}\wedge\tau_{LU}}\right)^+\right|\left.\mathcal{F}_{t_0}\right].
\end{aligned}$$

Based on Corollary 1, the last line of equation (3.37) corresponds to the time-t_0 price of an American-style put option on the asset price $\bar{S}$ with strike S_{t_0}, and maturity at time T, but all first passage times contained in thelast line of equation (3.37) are still expressed in terms of S (and not $\bar{S}$). Nevertheless, equations (3.3) and (3.18) can be combined into

$$\begin{aligned}
\tau_{LU} &= \inf\left\{t \geq t_0 : S_t\left(KS_{t_0}/S_t\right) \leq L\left(KS_{t_0}/S_t\right) \; or \; S_t\left(KS_{t_0}/S_t\right) \geq U\left(KS_{t_0}/S_t\right)\right\}\\
&= \inf\left\{t \geq t_0 : \bar{S}_t \leq KS_{t_0}/U \quad or \quad \bar{S}_t \geq KS_{t_0}/L\right\}\\
&= \bar{\tau}_{LU}, \qquad (3.38)
\end{aligned}$$

i.e. τ_{LU} can be also understood as the first passage time of the process $\bar{S}$ through the lower barrier KS_{t_0}/U or the upper barrier KS_{t_0}/L, which will be denoted by $\bar{\tau}_{LU}$.

Similarly, equations (3.3) and (3.17) can also yield a new interpretation to the stopping time $\tau_{B^{ko(-1)}}$:

$$\begin{aligned}
\tau_{B^{ko(-1)}} &= \inf\left\{t \geq t_0 : S_t\left(KS_{t_0}/S_t\right) \geq B_t^{ko(-1)}\left(KS_{t_0}/S_t\right)\right\} \qquad (3.39)\\
&= \inf\left\{t \geq t_0 : \bar{S}_t \leq KS_{t_0}/B_t^{ko(-1)}\right\}\\
&= \bar{\tau}_{B^{ko(-1)}}.
\end{aligned}$$

Hence, $\tau_{B^{ko(-1)}}$ can be rewritten as the first passage time of the process $\bar{S}$ through the exercise boundary $KS_{t_0}/B_t^{ko(-1)}$, which is represented by $\bar{\tau}_{B^{ko(-1)}}$.

Finally, equations (3.37), (3.38) and (3.39) can be combined into

$$\begin{aligned} & AKODB_{t_0}\left(S_{t_0}, K, L, U, T, r, q, B^{ko(-1)}; -1\right) \\ = \quad & \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{T\wedge\bar{\tau}_{B^{ko(-1)}}\wedge\bar{\tau}_{LU}} q_u du\right)\left(S_{t_0} - \bar{S}_{T\wedge\bar{\tau}_{B^{ko(-1)}}\wedge\bar{\tau}_{LU}}\right)^+ \middle| \mathcal{F}_{t_0}\right] \\ = \quad & AKODB_{t_0}\left(\bar{S}_{t_0}, S_{t_0}, KS_{t_0}/U, KS_{t_0}/L, T, q, r, KS_{t_0}/B^{ko(-1)}; 1\right), \end{aligned} \tag{3.40}$$

where the last line follows from equation (3.16). Equation (3.36) arises immediately after noting that $\bar{S}_{t_0} = K$. ■

3.3.2 Knock-in options

Next three propositions extend the previous analysis for both single and double knock-in options.

Proposition 3.5 *Define $\bar{S}_t$ through equation (3.3). Under Assumptions 3.1 and 3.3, the time-t_0 value of an American-style down-and-in call option on the asset price S, with strike K, lower barrier level L, maturity at time T $(\geq t_0)$, and exercise boundary $B^{di(-1)}$, is the same, after a change of numéraire, as the time-t_0 value of an American-style up-and-in put option on the asset price $\bar{S}$, with strike S_{t_0}, upper barrier level KS_{t_0}/L, maturity at time T $(\geq t_0)$, and exercise boundary $S_{t_0}K/B^{s(-1)}$, i.e.*

$$ADI_{t_0}\left(S_{t_0}, K, L, T, r, q, B^{di(-1)}; -1\right) = AUI_{t_0}\left(K, S_{t_0}, KS_{t_0}/L, T, q, r, S_{t_0}K/B^{s(-1)}, 1\right). \tag{3.41}$$

Proof. Applying Proposition 3.1 to equation (3.20) and using definition (3.3), then

$$
\begin{aligned}
& ADI_{t_0}\left(S_{t_0}, K, L, T, r, q, B^{di(-1)}; -1\right) \qquad (3.42)\\
= \; & \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{T\wedge\gamma^L_{B^{s(-1)}}} q_u du\right)\left(S_{T\wedge\gamma^L_{B^{s(-1)}}} - K\right)^+ \mathbb{1}_{\{\tau_L \le T\}} \frac{\bar{S}_{T\wedge\gamma^L_{B^{s(-1)}}}}{K}\middle| \mathcal{F}_{t_0}\right] \\
= \; & \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{T\wedge\gamma^L_{B^{s(-1)}}} q_u du\right)\left(S_{T\wedge\gamma^L_{B^{s(-1)}}} - K\right)^+ \mathbb{1}_{\{\tau_L \le T\}} \frac{S_{t_0}}{S_{T\wedge\gamma^L_{B^{s(-1)}}}}\middle| \mathcal{F}_{t_0}\right] \\
= \; & \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{T\wedge\gamma^L_{B^{s(-1)}}} q_u du\right)\left(S_{t_0} - S_{T\wedge\gamma^L_{B^{s(-1)}}}\right)^+ \mathbb{1}_{\{\tau_L \le T\}}\middle| \mathcal{F}_{t_0}\right].
\end{aligned}
$$

Based on Corollary 1, the last line of equation (3.42) corresponds to the time-t_0 price of an American-style put option on the asset price $\bar{S}$, with strike S_{t_0}, and maturity at time T. However, all first passage times contained in the last line of equation (3.42) are still expressed in terms of S (and not $\bar{S}$). Nevertheless, equations (3.3), (3.21) and (3.28) can be combined and yield a new interpretation to the stopping time $\gamma_{B^{s(-1)}}$:

$$
\begin{aligned}
\gamma^L_{B^{s(-1)}} &= \inf\left\{t \ge \tau_L : S_t\left(KS_{t_0}/S_t\right) \ge B_t^{s(-1)}\left(KS_{t_0}/S_t\right)\right\} \qquad (3.43)\\
&= \inf\left\{t \ge \bar{\tau}_L : \bar{S}_t \le KS_{t_0}/B_t^{s(-1)}\right\} \\
&= \bar{\gamma}^L_{B^{s(-1)}}.
\end{aligned}
$$

Finally, equations (3.28), (3.42) and (3.43) can be combined into

$$
\begin{aligned}
& ADI_{t_0}\left(S_{t_0}, K, L, T, r, q, B^{di(-1)}; -1\right) \qquad (3.44)\\
= \; & \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{T\wedge\bar{\gamma}^L_{B^{s(-1)}}} q_u du\right)\left(S_{t_0} - S_{T\wedge\bar{\gamma}^L_{B^{s(-1)}}}\right)^+ \mathbb{1}_{\{\bar{\tau}_L \le T\}}\middle| \mathcal{F}_{t_0}\right] \\
= \; & AUI_{t_0}\left(\bar{S}_{t_0}, S_{t_0}, KS_{t_0}/L, T, q, r, KS_{t_0}/B^{s(-1)}\right),
\end{aligned}
$$

where the last line follows from equation (3.22). Equation (3.41) arises immediately after noting that $\bar{S}_{t_0} = K$. ■

Proposition 3.6 *Let $\bar{S}_t$ be defined by equation (3.3). Under Assumptions 3.1 and 3.3, the time-t_0 value of an American-style up-and-in call option on the asset price S, with strike K, upper barrier level U, maturity at time T ($\geq t_0$), and exercise boundary $B^{ui(-1)}$, is the same, after a change of numéraire, as the time-t_0 value of an American-style down-and-in put option on the asset price $\bar{S}$, with strike S_{t_0}, lower barrier level KS_{t_0}/U, maturity at time T ($\geq t_0$), and exercise boundary $S_{t_0}K/B^{s(-1)}$, i.e.*

$$AUI_{t_0}\left(S_{t_0}, K, U, T, r, q, B^{ui(-1)}; -1\right) = ADI_{t_0}\left(K, S_{t_0}, KS_{t_0}/U, T, q, r, S_{t_0}K/B^{s(-1)}; 1\right). \tag{3.45}$$

Proof. Applying Proposition 3.1 to equation (3.22) and using definition (3.3), then

$$\begin{aligned}
& AUI_{t_0}\left(S_{t_0}, K, U, T, r, q, B^{ui(-1)}; -1\right) \qquad (3.46)\\
= \;& \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{T\wedge\gamma^U_{B^{s(-1)}}} q_u du\right)\left(S_{T\wedge\gamma^U_{B^{s(-1)}}} - K\right)^+ \mathbb{1}_{\{\tau_U \leq T\}} \frac{\bar{S}_{T\wedge\gamma^U_{B^{s(-1)}}}}{K}\middle|\mathcal{F}_{t_0}\right]\\
= \;& \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{T\wedge\gamma^U_{B^{s(-1)}}} q_u du\right)\left(S_{T\wedge\gamma^U_{B^{s(-1)}}} - K\right)^+ \mathbb{1}_{\{\tau_U \leq T\}} \frac{S_{t_0}}{S_{T\wedge\gamma^U_{B^{s(-1)}}}}\middle|\mathcal{F}_{t_0}\right]\\
= \;& \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{T\wedge\gamma^U_{B^{s(-1)}}} q_u du\right)\left(S_{t_0} - S_{T\wedge\gamma^U_{B^{s(-1)}}}\right)^+ \mathbb{1}_{\{\tau_U \leq T\}}\middle|\mathcal{F}_{t_0}\right].
\end{aligned}$$

Based on Corollary 1, the last line of equation (3.46) corresponds to the time-t_0 price of an American-style put option on the asset price $\bar{S}$, with strike S_{t_0}, and maturity at time T. However, all first passage times contained in the last line of equation (3.46) are still expressed in terms of S (and not $\bar{S}$). Nevertheless, equations (3.3), (3.23) and (3.33) yield a new interpretation to the stopping time $\gamma^U_{B^{s(-1)}}$:

$$\begin{aligned}\gamma^U_{B^{s(\phi)}} &= \inf\left\{t \geq \tau_U : S_t\left(KS_{t_0}/S_t\right) \geq B_t^{s(-1)}\left(KS_{t_0}/S_t\right)\right\} \\ &= \inf\left\{t \geq \bar{\tau}_U : \bar{S}_t \leq KS_{t_0}/B_t^{s(-1)}\right\} \\ &= \bar{\gamma}^U_{B^{s(-1)}}.\end{aligned} \tag{3.47}$$

Finally, equations (3.33), (3.46), and (3.47) can be combined into

$$\begin{aligned}& AUI_{t_0}\left(S_{t_0}, K, U, T, r, q, B^{ui(-1)}; -1\right) \\ = \; & \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{T\wedge\bar{\gamma}^U_{B^{s(-1)}}} q_u du\right)\left(S_{t_0} - S_{\bar{\gamma}^U_{B^{s(-1)}}}\right)^+ \mathbb{1}_{\{\bar{\tau}_U < T\}} \middle| \mathcal{F}_{t_0}\right] \\ = \; & ADI_{t_0}\left(\bar{S}_{t_0}, S_{t_0}, KS_{t_0}/U, T, q, r, KS_{t_0}/B^{s(-1)}; 1\right),\end{aligned} \tag{3.48}$$

where the last line follows from equation and (3.20). Equation (3.45) arises immediately after noting that $\bar{S}_{t_0} = K$. ■

Proposition 3.7 *Let $\bar{S}_t$ be defined by equation (3.3). Under Assumptions 3.1 and 3.3, the time-t_0 value of an American-style knock-in double barrier call option on the asset price S, with strike K, lower barrier L , upper barrier level U, maturity at time T ($\geq t_0$), and exercise boundary $B^{ki(-1)}$, is the same, after a change of numéraire, as the time-t_0 value of an American-style double knock-in put option on the asset price $\bar{S}$, with strike S_{t_0}, lower barrier level KS_{t_0}/U, upper barrier level KS_{t_0}/L, maturity at time T ($\geq t_0$), and exercise boundary $S_{t_0}K/B^{s(-1)}$, i.e.*

$$\begin{aligned}& AKIDB_{t_0}\left(S_{t_0}, K, L, U, T, r, q, B^{ki(-1)}; -1\right) \\ = \; & AKIDB_{t_0}\left(K, S_{t_0}, KS_{t_0}/U, KS_{t_0}/L, T, q, r, S_{t_0}K/B^{s(-1)}; 1\right).\end{aligned} \tag{3.49}$$

Proof. Applying Proposition 3.1 to equation (3.24) and using definition (3.3), then

$$\begin{aligned}
& AKIDB_{t_0}\left(S_{t_0}, K, L, U, T, r, q, B^{ki(-1)}; -1\right) && (3.50)\\
= \; & \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{T\wedge\gamma^{LU}_{B^{s(-1)}}} q_u du\right)\left(S_{T\wedge\gamma^{LU}_{B^{s(-1)}}} - K\right)^+ \mathbb{1}_{\{\tau_{LU}\leq T\}} \frac{\bar{S}_{T\wedge\gamma^{LU}_{B^{s(-1)}}}}{K}\middle| \mathcal{F}_{t_0}\right]\\
= \; & \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{T\wedge\gamma^{LU}_{B^{s(-1)}}} q_u du\right)\left(S_{T\wedge\gamma^{LU}_{B^{s(-1)}}} - K\right)^+ \mathbb{1}_{\{\tau_{LU}\leq T\}} \frac{S_{t_0}}{S_{T\wedge\gamma^{LU}_{B^{s(-1)}}}}\middle| \mathcal{F}_{t_0}\right]\\
= \; & \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{T\wedge\gamma^{LU}_{B^{s(-1)}}} q_u du\right)\left(S_{t_0} - S_{T\wedge\gamma^{LU}_{B^{s(-1)}}}\right)^+ \mathbb{1}_{\{\tau_{LU}\leq T\}}\middle| \mathcal{F}_{t_0}\right].
\end{aligned}$$

Based on Corollary 1, the last line of equation (3.50) corresponds to the time-t_0 price of an American-style double knock-in barrier put option on the asset price $\bar{S}$, with strike S_{t_0}, and maturity at time T. However, all first passage times contained in the last line of equation (3.50) are still expressed in terms of S (and not $\bar{S}$). Nevertheless, equations (3.3), (3.25) and (3.38) can be combined to yield a new interpretation to the stopping time $\gamma^{LU}_{B^{s(-1)}}$:

$$\begin{aligned}
\gamma^{LU}_{B^{s(-1)}} &= \inf\left\{t \geq \tau_{LU} : S_t\left(KS_{t_0}/S_t\right) \geq B^{s(-1)}_t\left(KS_{t_0}/S_t\right)\right\} && (3.51)\\
&= \inf\left\{t \geq \bar{\tau}_{LU} : \bar{S}_t \leq KS_{t_0}/B^{s(-1)}_t\right\}\\
&= \bar{\gamma}^{LU}_{B^{s(-1)}}.
\end{aligned}$$

Finally, equations (3.50), and (3.51) can be combined into

$$\begin{aligned}
& AKIDB_{t_0}\left(S_{t_0}, K, L, U, T, r, q, B^{ki(-1)}; -1\right) && (3.52)\\
= \; & \mathbb{E}_{\bar{\mathbb{Q}}}\left[\exp\left(-\int_{t_0}^{T\wedge\bar{\gamma}^{LU}_{B^{s(-1)}}} q_u du\right)\left(S_{t_0} - S_{T\wedge\bar{\gamma}^{LU}_{B^{s(-1)}}}\right)^+ \mathbb{1}_{\{\bar{\tau}_{LU}<T\}}\middle| \mathcal{F}_{t_0}\right]\\
= \; & AKIDB_{t_0}\left(\bar{S}_{t_0}, S_{t_0}, KS_{t_0}/U, T, q, r, KS_{t_0}/B^{s(-1)}; 1\right),
\end{aligned}$$

where the last line follows from equation (3.24). Equation (3.49) arises immediately after noting that $\bar{S}_{t_0} = K$. ■

3.4 CEV Process

This section applies the previous put-call symmetries, to the constant elasticity of variance (CEV) process of Cox (1975).

The CEV process is consistent with two well known facts that have found empirical support in the literature: the existence of a negative correlation between stock returns and realized volatility (leverage effect), as observed, for instance, in Bekaert and Wu (2000); and the inverse relation between the implied volatility and the strike price of an option contract (implied volatility skew) — see, for example, Dennis and Mayhew (2002). More importantly, being a local volatility model, the CEV diffusion is consistent with a complete market setup and, therefore, allows the hedging of short option positions only through the underlying asset.

Under the risk-neutral probability measure $\mathbb{Q}$, the CEV process assumes that the asset price $(S_t, t \geq 0)$ is described by the following stochastic differential equation:

$$\frac{dS_t}{S_t} = (r - q)\, dt + \sigma(t, S)\, dW_t^{\mathbb{Q}}, \tag{3.53}$$

with

$$\sigma(t, S) := \delta S_t^{\frac{\beta}{2} - 1}, \tag{3.54}$$

for $\delta, \beta \in \mathbb{R}$ and where $r \geq 0$ denotes the instantaneous riskless interest rate, which is assumed to be constant, $q \geq 0$ represents the dividend yield for the underlying asset price, and $W_t^{\mathbb{Q}} \in \mathbb{R}$ is a standard Brownian motion under $\mathbb{Q}$, initialized at zero and generating the augmented, right continuous and complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq t_0\}$.

To apply the change of numéraire technique in order to obtain put-call symmetries, we first need to transform the CEV process given by equation (3.53), under the probability measure $\mathbb{Q}$, into an equivalent CEV process but under the new probability measure $\bar{\mathbb{Q}}$.

Using equation (3.3), and applying the Itô lemma and Girsanov theorem to equation (3.53), we obtain

$$\frac{d\bar{S}_t}{\bar{S}_t} = (q - r)\, dt + \bar{\delta} \bar{S}_t^{-\left(\frac{\beta}{2}-1\right)} dW_t^{\bar{\mathbb{Q}}}, \tag{3.55}$$

with

$$\bar{\delta} := \delta \left(K S_0\right)^{\left(\frac{\beta}{2}-1\right)}, \tag{3.56}$$

and the following Corollaries follow easily from Propositions 3.2 to 3.7:

i. Under the CEV process the put-call option symmetry between an American-style down-and-out call and an American-style up-and-out put is given by

$$ADO_{t_0}\left(S_0, K, \beta, \delta, L, T, r, q; -1\right) = AUO_{t_0}\left(K, S_0, 4 - \beta, \delta (K S_0)^{\frac{\beta}{2}-1}, K S_0 / L, T, q, r; 1\right). \tag{3.57}$$

ii. Under the CEV process the put-call option symmetry between an American-style up-and-out call and an American-style down-and-out put is given by

$$AUO_{t_0}\left(S_0, K, \beta, \delta, U, T, r, q; -1\right) = ADO_{t_0}\left(K, S_0, 4 - \beta, \delta (K S_0)^{\frac{\beta}{2}-1}, K S_0 / U, T, q, r; 1\right). \tag{3.58}$$

iii. Under the CEV process the put-call option symmetry between an American-style double knock-out call and put option is given by

$$\begin{aligned}&AKODB_{t_0}\left(S_0,K,\beta,\delta,L,U,T,r,q;-1\right) \\ =\;& AKODB_{t_0}\left(K,S_0,4-\beta,\delta(KS_0)^{\frac{\beta}{2}-1},KS_0/U,KS_0/L,T,q,r;1\right).\end{aligned} \tag{3.59}$$

iv. Under the CEV process the put-call option symmetry between an American-style down-and-in call and an American-style up-and-in put is given by

$$ADI_{t_0}\left(S_0,K,\beta,\delta,L,T,r,q;-1\right)=AUI_{t_0}\left(K,S_0,4-\beta,\delta(KS_0)^{\frac{\beta}{2}-1},KS_0/L,T,q,r;1\right). \tag{3.60}$$

v. Under the CEV process the put-call option symmetry between an American-style up-and-in call and an American-style down-and-in put is given by

$$AUI_{t_0}\left(S_0,K,\beta,\delta,U,T,r,q;-1\right)=ADI_{t_0}\left(K,S_0,4-\beta,\delta(KS_0)^{\frac{\beta}{2}-1},KS_0/U,T,q,r;1\right). \tag{3.61}$$

vi. Under the CEV process the put-call option symmetry between an American-style double knock-in call and put option is given by

$$\begin{aligned}&AKIDB_{t_0}\left(S_0,K,\beta,\delta,L,U,T,r,q;-1\right) \\ =\;& AKIDB_{t_0}\left(K,S_0,4-\beta,\delta(KS_0)^{\frac{\beta}{2}-1},KS_0/U,KS_0/L,T,q,r;1\right).\end{aligned} \tag{3.62}$$

3.4.1 Valuation procedures

To test empirically the symmetry relations i to vi, we need a viable valuation method to price the array of American-style contracts involved. The static hedging approach of Chung and Shih (2009) (hereafter, SHP), allows us to value all types of American-style contracts needed.

Under the CEV model assumption, standard American-style options are valued through the Chung and Shih (2009) procedure—enhanced by the Ruas et al. (2012a) extension to other values of the elasticity parameter (beta) besides $\frac{4}{3}$—and using closed-form solutions for the hedge ratios of the corresponding European-style options. American-style down-and-out puts and up-and-out calls are priced using the SHP extension (for American-style capped options) offered in Ruas et al. (2012a). American-style up-and-out puts and down-and-out calls are priced using the Chung et al. (2009) procedure. American-style knock-in options are valued using the algorithm offered by Chung et al. (2013), but with greeks (deltas and thetas) generalized for $\beta \neq \frac{4}{3}$. American-style double knock-out options are valued using the Ruas et al. (2012b) procedure.

The SHP method is implemented with 52 time steps, and all computations are obtained via Matlab (2009a) running on an Intel Core 2 Duo E8500 3.16GHz processor.

3.4.2 Numerical results

To run our numerical experiments under the CEV assumption, we borrow parameters values from Davydov and Linetsky (2001), namely: the time-t_0 underlying asset price is $S_{t_0} = 100$; different moneyness levels are considered by letting the strike price $K \in \{95, 100, 105\}$; the lower barrier is $L = 90$, and the upper barrier is set at $U = 120$; $\beta \in \{2, 1, 0, -2, -4, -6\}$, and δ is selected so that the initial instantaneous volatility $\sigma(t_0, S) = 25\%$ is the same across the different CEV processes; the time to maturity is set at 6 months, i.e. $T - t_0 = 0.5$ years. We change the riskless interest rate in Davydov and Linetsky (2001) from $r = 10\%$ to $r = 5\%$ and the dividend yield from $q = 0\%$ to $q = 7\%$, since the original values for these parameters would lead the American-style options to degenerate into European-style options.

Tables 3.1 to 3.4 apply the put-call symmetry for single barrier knock-out and knock-in options. In all these tables, columns 1 to 5 contain the parameters values considered for each call contract, and column 6 shows the results for the American-style call under analysis. Columns 7 to 11 contain the parameters values for the corresponding put contract that follows from the put-call symmetry, and column 12 shows the American-style put prices obtained through the put-call symmetry.

Table 3.5 implements the put-call symmetry between double knock-out call / put options. Columns 1 to 6 contain the parameters values considered for each call contract, and column 7 presents the American-style call prices. Columns 8 to 13 contain the parameters values for the corresponding put contract that follows from the put-call symmetry, and column 14 shows the result for the American-style put prices, obtained via the put-call symmetry.

Tables 3.1 to 3.5 show that the prices for American-style barrier options are very sensitive to changes in the value of the beta parameter (specially for the knock-in barriers). In several cases, the difference in option values when moving from $\beta = 2$ (i.e when the CEV model degenerates into the GBM model) to $\beta = -6$ is higher than 50%.

3.5 Conclusions

Using the change of numéraire technique proposed by Geman et al. (1995) and used by Schroder (1999), we are able to derive the put-call symmetry for American-style single and double barrier options without imposing a previous assumption on the stochastic process followed by the underlying asset price. Our results are validated by applying the proposed symmetries to the CEV model.

Table 3.1: Put-call symmetry between a down-and-out call and an up-and-out put under the CEV model

S_{t_0}	K	β	δ	L	ADO	S_{t_0}	K	β	δ	U	AUO
100	105	2	2.50E-01	90	4.015	105	100	2	2.50E-01	116.67	4.015
100	105	1	2.50E+00	90	3.911	105	100	3	2.44E-02	116.67	3.911
100	105	0	2.50E+01	90	3.812	105	100	4	2.38E-03	116.67	3.812
100	105	-2	2.50E+03	90	3.623	105	100	6	2.27E-05	116.67	3.623
100	105	-4	2.50E+05	90	3.446	105	100	8	2.16E-07	116.67	3.446
100	105	-6	2.50E+07	90	3.279	105	100	10	2.06E-09	116.67	3.279
100	100	2	2.50E-01	90	5.544	100	100	2	2.50E-01	111.11	5.544
100	100	1	2.50E+00	90	5.487	100	100	3	2.50E-02	111.11	5.487
100	100	0	2.50E+01	90	5.432	100	100	4	2.50E-03	111.11	5.432
100	100	-2	2.50E+03	90	5.325	100	100	6	2.50E-05	111.11	5.325
100	100	-4	2.50E+05	90	5.223	100	100	8	2.50E-07	111.11	5.223
100	100	-6	2.50E+07	90	5.126	100	100	10	2.50E-09	111.11	5.126
100	95	2	2.50E-01	90	7.376	95	100	2	2.50E-01	105.56	7.376
100	95	1	2.50E+00	90	7.357	95	100	3	2.56E-02	105.56	7.357
100	95	0	2.50E+01	90	7.338	95	100	4	2.63E-03	105.56	7.338
100	95	-2	2.50E+03	90	7.300	95	100	6	2.77E-05	105.56	7.300
100	95	-4	2.50E+05	90	7.262	95	100	8	2.92E-07	105.56	7.262
100	95	-6	2.50E+07	90	7.224	95	100	10	3.07E-09	105.56	7.224

Table 3.1 values American-style down-and-out call options and the corresponding American-style up-and-out put options through the put-call symmetry (3.26) under the CEV model. The time to maturity is $T - t_0 = 0.5$ years, the riskless interest rate is $r = 5\%$, and the dividend yield is $q = 7\%$. L denotes the lower barrier for the American-style down-and-out call options and U the upper barrier for the American-style up-and-out put options. Columns 1 to 5 contain the parameters values considered for each call contract, and column 6 shows the American-style call prices. Columns 7 to 11 contain the parameters values considered for the corresponding put contract, and column 12 shows the American-style put prices obtained via the put-call symmetry.

Table 3.2: Put-call symmetry between a up-and-out call and an down-and-out put under the CEV model

S_{t_0}	K	β	δ	U	AUO	S_{t_0}	K	β	δ	L	ADO
100	105	2	2.50E-01	120	4.388	105	100	2	2.50E-01	87.50	4.388
100	105	1	2.50E+00	120	4.357	105	100	3	2.44E-02	87.50	4.357
100	105	0	2.50E+01	120	4.321	105	100	4	2.38E-03	87.50	4.321
100	105	-2	2.50E+03	120	4.237	105	100	6	2.27E-05	87.50	4.237
100	105	-4	2.50E+05	120	4.143	105	100	8	2.16E-07	87.50	4.143
100	105	-6	2.50E+07	120	4.054	105	100	10	2.06E-09	87.50	4.054
100	100	2	2.50E-01	120	6.444	100	100	2	2.50E-01	83.33	6.444
100	100	1	2.50E+00	120	6.467	100	100	3	2.50E-02	83.33	6.467
100	100	0	2.50E+01	120	6.489	100	100	4	2.50E-03	83.33	6.489
100	100	-2	2.50E+03	120	6.536	100	100	6	2.50E-05	83.33	6.536
100	100	-4	2.50E+05	120	6.599	100	100	8	2.50E-07	83.33	6.599
100	100	-6	2.50E+07	120	6.694	100	100	10	2.50E-09	83.33	6.694
100	95	2	2.50E-01	120	8.995	95	100	2	2.50E-01	79.17	8.995
100	95	1	2.50E+00	120	9.087	95	100	3	2.56E-02	79.17	9.087
100	95	0	2.50E+01	120	9.184	95	100	4	2.63E-03	79.17	9.184
100	95	-2	2.50E+03	120	9.398	95	100	6	2.77E-05	79.17	9.398
100	95	-4	2.50E+05	120	9.650	95	100	8	2.92E-07	79.17	9.650
100	95	-6	2.50E+07	120	9.962	95	100	10	3.07E-09	79.17	9.962

Table 3.2 values American-style up-and-out call options and the corresponding American-style down-and-out put options through the put-call symmetry (3.31) under the CEV model. The time to maturity is $T - t_0 = 0.5$ years, the riskless interest rate is $r = 5\%$, and the dividend yield is $q = 7\%$. U denotes the upper barrier for the American-style up-and-out call options and L the lower barrier for the American-style down-and-out put options. Columns 1 to 5 contain the parameters values considered for each call contract, and column 6 shows the American-style call prices. Columns 7 to 11 contain the parameters values considered for the corresponding put contract, and column 12 shows the American-style put prices obtained via the put-call symmetry.

Table 3.3: Put-call symmetry between a down-and-in call and an up-and-in put under the CEV model

S_{t_0}	K	β	δ	L	ADI	S_{t_0}	K	β	δ	U	AUI
100	105	2	2.50E-01	90	0.525	105	100	2	2.50E-01	116.67	0.525
100	105	1	2.50E+00	90	0.553	105	100	3	2.44E-02	116.67	0.553
100	105	0	2.50E+01	90	0.581	105	100	4	2.38E-03	116.67	0.581
100	105	-2	2.50E+03	90	0.640	105	100	6	2.27E-05	116.67	0.640
100	105	-4	2.50E+05	90	0.703	105	100	8	2.16E-07	116.67	0.703
100	105	-6	2.50E+07	90	0.775	105	100	10	2.06E-09	116.67	0.775
100	100	2	2.50E-01	90	0.939	100	100	2	2.50E-01	111.11	0.939
100	100	1	2.50E+00	90	1.003	100	100	3	2.50E-02	111.11	1.003
100	100	0	2.50E+01	90	1.070	100	100	4	2.50E-03	111.11	1.070
100	100	-2	2.50E+03	90	1.213	100	100	6	2.50E-05	111.11	1.213
100	100	-4	2.50E+05	90	1.376	100	100	8	2.50E-07	111.11	1.376
100	100	-6	2.50E+07	90	1.569	100	100	10	2.50E-09	111.11	1.569
100	95	2	2.50E-01	90	1.628	95	100	2	2.50E-01	105.56	1.628
100	95	1	2.50E+00	90	1.738	95	100	3	2.56E-02	105.56	1.738
100	95	0	2.50E+01	90	1.855	95	100	4	2.63E-03	105.56	1.855
100	95	-2	2.50E+03	90	2.108	95	100	6	2.77E-05	105.56	2.108
100	95	-4	2.50E+05	90	2.401	95	100	8	2.92E-07	105.56	2.401
100	95	-6	2.50E+07	90	2.753	95	100	10	3.07E-09	105.56	2.753

Table 3.3 values American-style down-and-in call options and the corresponding American-style up-and-in put options through the put-call symmetry (3.41) under the CEV model. The time to maturity is $T - t_0 = 0.5$ years, the riskless interest rate is $r = 5\%$, and the dividend yield is $q = 7\%$. L denotes the lower barrier for the American-style down-and-in call options and U the upper barrier for the American-style up-and-in put options. Columns 1 to 5 contain the parameters values considered for each call contract, and column 6 shows the American-style call prices. Columns 7 to 11 contain the parameters values considered for the corresponding put contract, and column 12 shows the American-style put prices obtained via the put-call symmetry.

Table 3.4: Put-call symmetry between an up-and-in call and a down-and-in put under the CEV model

S_{t_0}	K	β	δ	U	AUI	S_{t_0}	K	β	δ	L	ADI
100	105	2	2.50E-01	120	3.984	105	100	2	2.50E-01	87.50	3.984
100	105	1	2.50E+00	120	3.813	105	100	3	2.44E-02	87.50	3.813
100	105	0	2.50E+01	120	3.633	105	100	4	2.38E-03	87.50	3.633
100	105	-2	2.50E+03	120	3.238	105	100	6	2.27E-05	87.50	3.238
100	105	-4	2.50E+05	120	2.790	105	100	8	2.16E-07	87.50	2.790
100	105	-6	2.50E+07	120	2.294	105	100	10	2.06E-09	87.50	2.294
100	100	2	2.50E-01	120	5.148	100	100	2	2.50E-01	83.33	5.148
100	100	1	2.50E+00	120	4.965	100	100	3	2.50E-02	83.33	4.965
100	100	0	2.50E+01	120	4.761	100	100	4	2.50E-03	83.33	4.761
100	100	-2	2.50E+03	120	4.284	100	100	6	2.50E-05	83.33	4.284
100	100	-4	2.50E+05	120	3.712	100	100	8	2.50E-07	83.33	3.712
100	100	-6	2.50E+07	120	3.058	100	100	10	2.50E-09	83.33	3.058
100	95	2	2.50E-01	120	6.390	95	100	2	2.50E-01	79.17	6.390
100	95	1	2.50E+00	120	6.178	95	100	3	2.56E-02	79.17	6.178
100	95	0	2.50E+01	120	5.936	95	100	4	2.63E-03	79.17	5.936
100	95	-2	2.50E+03	120	5.352	95	100	6	2.77E-05	79.17	5.352
100	95	-4	2.50E+05	120	4.640	95	100	8	2.92E-07	79.17	4.640
100	95	-6	2.50E+07	120	3.823	95	100	10	3.07E-09	79.17	3.823

Table 3.4 values American-style up-and-in call options and the corresponding American-style down-and-in put options through the put-call symmetry (3.45) under the CEV model. The time to maturity is $T - t_0 = 0.5$ years, the riskless interest rate is $r = 5\%$, and the dividend yield is $q = 7\%$. U denotes the upper barrier for the American-style up-and-in call options and L the lower barrier for the American-style down-and-in put options. Columns 1 to 5 contain the parameters values considered for each call contract, and column 6 shows the American-style call prices. Columns 7 to 11 contain the parameters values considered for the corresponding put contract, and column 12 shows the American-style put prices obtained via the put-call symmetry.

Table 3.5: Put-call symmetry between double knock-out calls and puts under the CEV model

S_{t_0}	K	β	δ	L	U	AKODB	S_{t_0}	K	β	δ	L	U	AKODB
100	105	2	2.50E-01	90	120	3.8680	105	100	2	2.50E-01	87.50	116.67	3.8680
100	105	1	2.50E+00	90	120	3.8076	105	100	3	2.44E-02	87.50	116.67	3.8076
100	105	0	2.50E+01	90	120	3.7420	105	100	4	2.38E-03	87.50	116.67	3.7420
100	105	-2	2.50E+03	90	120	3.5974	105	100	6	2.27E-05	87.50	116.67	3.5974
100	105	-4	2.50E+05	90	120	3.4403	105	100	8	2.16E-07	87.50	116.67	3.4403
100	105	-6	2.50E+07	90	120	3.2790	105	100	10	2.06E-09	87.50	116.67	3.2790
100	100	2	2.50E-01	90	120	5.5086	100	100	2	2.50E-01	83.33	111.11	5.5086
100	100	1	2.50E+00	90	120	5.4661	100	100	3	2.50E-02	83.33	111.11	5.4661
100	100	0	2.50E+01	90	120	5.4206	100	100	4	2.50E-03	83.33	111.11	5.4206
100	100	-2	2.50E+03	90	120	5.3232	100	100	6	2.50E-05	83.33	111.11	5.3232
100	100	-4	2.50E+05	90	120	5.2230	100	100	8	2.50E-07	83.33	111.11	5.2230
100	100	-6	2.50E+07	90	120	5.1255	100	100	10	2.50E-09	83.33	111.11	5.1255
100	95	2	2.50E-01	90	120	7.3754	95	100	2	2.50E-01	79.17	105.56	7.3754
100	95	1	2.50E+00	90	120	7.3567	95	100	3	2.56E-02	79.17	105.56	7.3567
100	95	0	2.50E+01	90	120	7.3378	95	100	4	2.63E-03	79.17	105.56	7.3378
100	95	-2	2.50E+03	90	120	7.2997	95	100	6	2.77E-05	79.17	105.56	7.2997
100	95	-4	2.50E+05	90	120	7.2618	95	100	8	2.92E-07	79.17	105.56	7.2618
100	95	-6	2.50E+07	90	120	7.2241	95	100	10	3.07E-09	79.17	105.56	7.2241

Table 3.5 values American-style double knock-out call options and the corresponding American-style double knock-out put options through the put-call symmetry (3.36) under the CEV model. The time to maturity is $T-t_0 = 0.5$ years, the riskless interest rate is $r = 5\%$, and the dividend yield is $q = 7\%$. L denotes the lower barrier level and U the upper barrier level. Columns 1 to 6 contain the parameter values considered for each call contract, and column 7 shows the American-style call prices. Columns 8 to 13 contain the parameters values considered for the corresponding put contract, and column 14 shows the American-style put prices obtained via the put-call symmetry.

4. In-Out Parity Relations and Early Exercise Boundaries for American-style Barrier Options*

Abstract: This paper derives new in-out parity relations for American-style puts with a down barrier and American-style calls with an up barrier. More importantly, we also propose a novel representation for the early exercise boundary of American-style double knock-out options in terms of the simpler optimal stopping boundary for a nested single barrier American-style option. Therefore, we are able to extend the *static hedge portfolio* approach to the valuation of American-style double barrier knock-out options. Our results are tested through an extensive numerical analysis run under the *geometric Brownian motion* and the *constant elasticity of variance* models.

JEL Classification: G13.

Keywords: American-style options; Barrier options; In-out parity; Early exercise boundary.

*This paper is a joint work with João Pedro Nunes and José Carlos Dias.

4.1 Introduction

This paper provides three important results concerning the valuation of American-style barrier options. First, we prove a new *in-out parity* relation between American-style barrier options. Second, we derive the early exercise boundary of American-style double knock-out options in terms of the simpler exercise boundary of single barrier American-style options. Finally, we use the previous finding to extend the static hedge portfolio (hereafter, SHP) approach of Chung and Shih (2009) to the valuation of American-style double barrier knock-out options.

Concerning our first result, it is well known that for European-style contracts, the sum of the prices of knock-in and knock-out barrier options is equal to the price of a standard European-style option. However, and as argued by Dai and Kwok (2004, Page 187) or Chung et al. (2013, Page 191), such in-out barrier parity relation cannot be applied to American-style barrier options. In this article, we extend the in-out barrier parity relation for American-style puts with a down barrier and calls with an up barrier, and show that the missing link between knock-in, knock-out, and standard options is simply the value of a non-deferrable rebate.

Our second theoretical contribution to the existent literature on option pricing consists in obtaining the early exercise boundary of any American-style double barrier option in terms of the barrier levels and as a function of the optimal stopping boundary of a nested single barrier American-style option. The rational behind our main result is similar to the one used, for instance, by Broadie and Detemple (1995, Theorem 1) or Gao et al. (2000, Theorem 6) to relate the early exercise boundaries of American-style standard and single barrier option contracts. Therefore, we are able to reduce the valuation of American-style double barrier options to the same complexity level as the one faced to price simpler single barrier contracts. Bearing in mind that only a few numerical methods have been proposed for pricing American-style double barrier options—as, for instance, the trinomial method of Ritchken (1995) or the PDE schemes of Zvan et al. (2000)—and that most of them are confined to the simpler geometric Brownian motion assumption of Black and Scholes (1973) and Merton (1973) (hereafter, GBM), our result should be used for all option pricing models that generate viable pricing solutions for single barrier American-style options.

To illustrate the potential of the previous finding, we easily extend the SHP methodology of Chung and Shih (2009) to the valuation of American-style double barrier knock-out options. The SHP approach was initially developed by Bowie and Carr (1994), Derman et al. (1995), and Carr et al. (1998) for hedging European-style exotic options, and then adapted by Chung and Shih (2009), and Chung et al. (2013) to the pricing of standard and single barrier knock-in American-style options, respectively. For this purpose, a static portfolio of European-style options (with multiple strikes and multiple maturities) is created with weights that ensure its value matches the payoff of the option being hedged at expiration and along the (unknown) early exercise boundary as well as along the (known) knock-in boundary. However, for double knock-out barrier American-style options, three different boundaries would have to be met until the expiry date of the contract. Hence, our results greatly simplify the pricing of double knock-out barrier options since we simply rely on the valuation of a single barrier contract.

All the new results proposed in this paper are only based on two mild assumptions: the existence of a risk neutral measure (i.e. on no arbitrage, in the Harrison and Pliska (1981) sense); and the existence of a unique, continuous and monotone early exercise boundary that separates the *continuation* and *exercise* regions of any American-style option contract. Therefore, our results can be applied to the whole class of single diffusion processes discussed in Detemple and Tian (2002, Propostion 1), and even to jump-diffusion models—but subject to the technical conditions stated in Pham (1997, Theorem 3.2).

This paper proceeds as follows. Section 4.2 presents the main assumptions adopted as well as the notation used throughout the paper. Section 4.3 derives the in-out parity for American-style barriers options, and Section 4.4 offers a simpler representation for the early exercise boundary of American-style double knock-out barrier options. Section 4.5 implements the previous results under both the GBM assumption and the *constant elasticity of variance* (CEV) model of Cox (1975), and through the SHP approach. Finally, Section 4.6 concludes.

4.2 Modelling assumptions

Our main modelling assumptions are borrowed from Schroder (1999). We assume throughout that the financial market is frictionless, and that trading takes place continuously on the time-interval $\mathcal{T} := [t_0, T]$, for some initial date t_0 and fixed time $T > t_0$. Uncertainty is represented by a complete probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, where the information accruing to all the agents in the economy is described by the complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \in \mathcal{T}\}$ that satisfies the usual conditions, and

Assumption 4.1 *There exists a risk-neutral measure $\mathbb{Q}$ such that the relative reinvested price of every asset, with respect to the reinvested short-rate process (or "money-market account"numeraire), is a $\mathbb{Q}$-martingale.*

Furthermore, and since this paper deals with the valuation of American-style options, we further adopt Schroder (1999, Assumption 2), i.e.

Assumption 4.2 *Let V_{t_0} be the time-t_0 price of an American-style option allowing the holder to exercise and receive, at any stopping time $\tau \in \mathcal{T}$, the payoff V_τ, where V is an adapted process. Then*

$$V_{t_0} = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} \left[\exp\left(-\int_{t_0}^{\tau} r_u du \right) V_\tau \middle| \mathcal{F}_{t_0} \right], \tag{4.1}$$

where r is the risk-free (and possibly time-dependent) short rate process.

The *Snell envelope* (4.1) is proved by Karatzas (1988, Theorem 5.4) in a complete market setting, and adapted by Pham (1997, Equation 1.5) to a jump-diffusion framework (although conditional upon the specification of the market price of jump risk).

However, our main results will be further based on the existence and uniqueness of an early exercise boundary $\{B_t : t \in \mathcal{T}\}$ that separates the *continuation* and *exercise* (or stopping) regions of the American-style option contract. This should constitute a mild assumption that is satisfied by most of the option pricing models already proposed in the literature. Under a single factor diffusion model, Detemple and Tian (2002, Propostion 1) prove the existence of the exercise boundary $\{B_t : t \in \mathcal{T}\}$, which is only a function of time (and a continuous function, as long as the risk-free interest rate is deterministic). Pham (1997, Theorems 3.2 and 4.1) extends the previous result to a jump-diffusion model (conditional on the positiveness of the riskless interest rate corrected by the jump risk).

Therefore, we finally assume that[4.1]

Assumption 4.3 *There exists a unique and continuous exercise boundary $\{B_t : t \in \mathcal{T}\}$*

[4.1] For $x, y \in \mathbb{R}$, we denote by $x \vee y$ and $x \wedge y$, respectively, their maximum and minimum.

such that equation (4.1) can be rewritten as

$$V_{t_0} = \mathbb{E}_{\mathbb{Q}} \left[\exp\left(-\int_{t_0}^{T\wedge\tau_B} r_u du \right) V_{T\wedge\tau_B} \middle| \mathcal{F}_{t_0} \right], \tag{4.2}$$

where τ_B is the first passage time of the underlying asset price process through the early exercise boundary.

Assumption 4.3 simply states that, for each time $t \in \mathcal{T}$ there exists a *critical asset price* B_t below (above) which the American-style put (call) price equals its intrinsic value and early exercise should occur. Consequently, the optimal policy should be to exercise the American-style option when the underlying asset price first touches its critical level. Therefore, Assumption 4.3 yields the following representation for the time-t_0 value of a standard American-style put (if $\phi = 1$) or call (if $\phi = -1$) option on the asset price S, with strike K, and maturity at time T ($\geq t_0$):

$$AS_{t_0}\left(S_{t_0}, K, T, B^{s(\phi)}; \phi\right) = \mathbb{E}_{\mathbb{Q}} \left[\exp\left(-\int_{t_0}^{T\wedge\tau_{B^{s(\phi)}}} r_u du \right) \left(\phi K - \phi S_{T\wedge\tau_{B^{s(\phi)}}} \right)^+ \middle| \mathcal{F}_{t_0} \right], \tag{4.3}$$

where

$$\tau_{B^{s(\phi)}} := \inf\left\{ t \geq t_0 : \phi S_t \leq \phi B_t^{s(\phi)} \right\}, \tag{4.4}$$

and $B^{s(\phi)}$ denotes the early exercise boundary of the standard American-style put (if $\phi = 1$) or call (if $\phi = -1$), with $\phi S_{t_0} > \phi B_{t_0}^{s(\phi)}$.

The valuation of the standard American-style contracts described by equations (4.3) and (4.4) is already well established in the literature for several single factor and multifactor option pricing models. Under the usual lognormal assumption underlying the GBM process, several accurate analytical approximations have been proposed, such as the randomization approach of Carr (1998), or the integral representation method of Kim (1990), Jacka (1991), Carr et al. (1992), and Jamshidian (1992). For alternative, but still single factor, diffusion processes, Kim and Yu (1996) extend the integral representation method, Nunes (2009) proposes an optimal stopping approach based on the density of the first passage time $\tau_{B^{s(\phi)}}$, and Chung and Shih (2009) adopted the SHP approach offered by Bowie and Carr (1994), Derman et al. (1995), and Carr et al. (1998). For multifactor option pricing models, the literature is more sparse but, nevertheless, Detemple and Tian (2002) adapt the integral representation method, Medvedev and Scaillet (2010) implement an extremely efficient asymptotic expansion that accommodates stochastic interest rates and stochastic volatility, and Nunes (2011) extends the optimal stopping approach to deal with stochastic interest rates.

The purpose of the present paper is essentially to reduce the valuation of American-style single and double barrier options, under any single factor pricing model, to the same complexity level as the one faced with the already well-established solution of equation (4.3).

4.3 In-out parity for American-style barrier options

For European-style barrier options, it is well known that the sum of the prices of down-and-in and down-and-out barrier options is equal to the price of the corresponding standard European-style option. However, and as argued, for instance, by Dai and Kwok (2004, Page 187) or Chung et al. (2013, Page 191), such in-out barrier parity relation cannot be simply transposed to American-style barrier options. For instance, a portfolio consisting of a down-and-in and a down-and-out American-style put always dominates the corresponding American-style but plain-vanilla put option. Along the barrier, the American-style knock-in becomes equal to the standard American-style put. Since the holder of the portfolio has the right to exercise the American-style knock-out, the portfolio value is greater than the standard American-style put price exactly in the amount of its intrinsic value. For European-style options this is not true, since the holder of the portfolio cannot exercise the European-style knock-out along the barrier.

In this section we extend the in-out barrier parity relation for American-style puts with a down barrier and calls with an up barrier. For these contracts, our results follow easily because we are able to relate their early exercise boundaries to the optimal stopping boundaries of the corresponding American-style standard contracts.[4.2]

4.3.1 Puts with a down barrier

The goal now is to relate the prices of American-style down-and-out and down-and-in puts.

Starting with the knock-out contract, an American-style knock-out option (without rebate) becomes worthless if the single barrier is touched by the underlying asset price; otherwise, it can be exercised at or before the expiry date.[4.3] Additionally, and as pointed by Gao et al. (2000, Footnote 15), for the knock-out event and the exercise date to be well defined, the option contract must be specified in a way such that when the asset price first touches the barrier, the option holder has the option to either exercise or let the option be knocked out. Therefore, and using Assumption 4.3, the time-t_0 value of an American-style down-and-out put (if $\phi = 1$) or call (if $\phi = -1$) option on the asset price S, with strike K, lower barrier level L, and maturity at time T ($\geq t_0$), is equal

[4.2]For the American-style put (call) with an up (down) barrier this is not possible because its early exercise boundary never matches the optimal stopping boundary of the corresponding standard American-style option (except at the maturity date). The early exercise boundary shifts upwards (downwards) for the American-style up-and-out (down-and-out) put (call) option, when compared to the corresponding American-style standard put (call).

[4.3]Unless stated otherwise, and following Gao et al. (2000, Page 1788), it is assumed in the remainder of this article that there is no rebate. Moreover, for valuation purposes, rebates can always be detached from the barrier option contract and dealt with via equations (4.11) or (4.28).

to

$$\begin{aligned} & ADO_{t_0}\left(S_{t_0}, K, L, T, B^{do(\phi)}; \phi\right) \\ = \quad & \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{t_0}^{T\wedge\tau_{B^{do(\phi)}}\wedge\tau_L} r_u du\right)\left(\phi K - \phi S_{T\wedge\tau_{B^{do(\phi)}}\wedge\tau_L}\right)^+ \middle| \mathcal{F}_{t_0}\right], \end{aligned} \tag{4.5}$$

where

$$\tau_L := \inf\left\{t \geq t_0 : S_t \leq L\right\}, \tag{4.6}$$

$$\tau_{B^{do(\phi)}} := \inf\left\{t \geq t_0 : \phi S_t \leq \phi B_t^{do(\phi)}\right\}, \tag{4.7}$$

$B^{do(\phi)}$ represents the exercise boundary of the American-style down-and-out put (if $\phi = 1$) or call (if $\phi = -1$), and $\phi S_{t_0} > \phi B_{t_0}^{do(\phi)}$.

Concerning the knock-in contract, an American-style knock-in option becomes a standard American-style option if and only if the barrier is touched by the underlying spot price before the option' expiry date. Therefore, and using again Assumption 4.3, the time-t_0 value of an American-style down-and-in put (if $\phi = 1$) or call (if $\phi = -1$) option on the asset price S, with strike K, lower barrier level L, and maturity at time T ($\geq t_0$) is given by

$$\begin{aligned} & ADI_{t_0}\left(S_{t_0}, K, L, T, B^{di(\phi)}; \phi\right) \\ = \quad & \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{t_0}^{\tau_L} r_u du\right) AS_{\tau_L}\left(L, K, T, B^{s(\phi)}; \phi\right) \mathbb{1}_{\{\tau_L \leq T\}} \middle| \mathcal{F}_{t_0}\right], \end{aligned} \tag{4.8}$$

where the first hitting time τ_L is defined by equation (4.6), $B^{di(\phi)}$ represents the exercise boundary of the American-style down-and-in put (if $\phi = 1$) or call (if $\phi = -1$), and $\phi S_{t_0} > \phi B_{t_0}^{di(\phi)}$. Based on equation (4.3), it is possible to obtain the following decomposition for the price of an American-style down-and-in put.

Proposition 4.1 *Under Assumptions 4.1 and 4.3, and if*

$$\frac{\partial B_t^{s(1)}}{\partial t} \geq 0, \tag{4.9}$$

then the time-t_0 value of an American-style down-and-in put option on the asset price S, with strike K, lower barrier level L, and maturity at time T ($\geq t_0$) is equal to

$$\begin{aligned} & ADI_{t_0}\left(S_{t_0}, K, L, T, B^{di(1)}; 1\right) \\ = \; & \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{t_0}^{T\wedge\tau_{B^{s(1)}}} r_u du\right)\left(K - S_{T\wedge\tau_{B^{s(1)}}}\right)^+ \mathbb{1}_{\left\{\tau_L \leq \tau_{B^{s(1)}},\, \tau_L \leq T\right\}}\middle|\, \mathcal{F}_{t_0}\right] \\ & + \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{t_0}^{\tau_L} r_u du\right)(K - L)^+ \mathbb{1}_{\left\{\tau_L > \tau_{B^{s(1)}},\, \tau_L \leq T\right\}}\middle|\, \mathcal{F}_{t_0}\right], \end{aligned} \tag{4.10}$$

where the first passage times τ_L and $\tau_{B^{s(1)}}$ are defined by equations (4.6) and (4.4), respectively, while $B^{di(1)}$ represents the unknown exercise boundary of the American-style down-and-in put.

Proof. See Appendix C. ■

Remark 4.1 *Equation (4.10) requires the early exercise boundary of the American-style put to be a nondecreasing function of time. Under the simpler GBM assumption, Jacka (1991, Proposition 2.2.2) has already proved equation (4.9). Additionally, Gao et al. (2000, Appendix B.3) have shown that equation (4.9) only requires the standard American-style option price to satisfy a value matching condition. Therefore, equation (4.9) does not constitute a too restrictive assumption.*

Remark 4.2 *Equation (4.10) decomposes the American-style down-and-in put price into two components: A pseudo-standard American-style put (but conditional on the knock-in event to occur before the first early exercise and expiry dates), and a rebate* *of the option's intrinsic value at the knock-in date (if this date is between the first early exercise and expiry dates).*

To relate equations (4.5) and (4.10) through equation (4.3), we still need two more ingredients. First, the time-t_0 value of a down rebate $(K - L)^+$ paid at the first passage time through the barrier L, will be given by

$$RD_{t_0}(S_{t_0}, K, L, T) = \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{t_0}^{\tau_L} r_u du\right)(K - L)^+ \mathbb{1}_{\{\tau_L < T\}} \middle| \mathcal{F}_{t_0}\right], \tag{4.11}$$

where τ_L is still defined by equation (4.6). Note that equation (4.11) simply requires the knowledge of the first passage time density through the barrier (and not through the exercise boundary), which—as shown, for instance, by Kuan and Webber (2003, Equation 4)—can be recovered from the probability distribution of the underlying asset price.

Second, the following relationship between the early exercise boundaries of American-style down-and-out and standard puts will be used.

Proposition 4.2 *Under Assumptions 4.1 and 4.3, and if equation (4.9) is satisfied, then the early exercise boundary of an American-style down-and-out put option with strike K, lower barrier level L ($< K$), and maturity at time T $(\geq t_0)$, is equal to*

$$B_t^{do(1)} = L \vee B_t^{s(1)}, \tag{4.12}$$

for $t \in \mathcal{T}$, and where $B^{s(1)}$ represents the early exercise boundary of the corresponding standard American-style put (on the same underlying asset, with the same strike and identical maturity).

Remark 4.3 *Equation (4.12) is already proved by Broadie and Detemple (1995, The-*

orem 1) or Gao et al. (2000, Theorem 6) but under the GBM assumption. Proposition 4.2 simply extends Gao et al. (2000, Theorem 6) to other asset price processes for which the exercise boundaries $B^{do(1)}$ and $B^{s(1)}$ exist, and the latter is a nondecreasing function of calendar time.

Proof. See Appendix D. ■

Next proposition contains the new in-out parity for American-style puts with a down barrier.

Proposition 4.3 *Under Assumptions 4.1 and 4.3, for $L < K$, and if equation (4.9) is satisfied, then*

$$\begin{aligned} & ADO_{t_0}\left(S_{t_0}, K, L, T, B^{do(1)}; 1\right) \\ = \; & AS_{t_0}\left(S_{t_0}, K, T, B^{s(1)}; 1\right) - ADI_{t_0}\left(S_{t_0}, K, L, T, B^{di(1)}; 1\right) \\ & + RD_{t_0}\left(S_{t_0}, K, L, T\right), \end{aligned} \tag{4.13}$$

where the left-hand side of equation (4.13) is given by equation (4.5), while the first, second and third terms on the right-hand side are defined by equations (4.3), (4.8) and (4.11), respectively.

Proof. We prove the above result by showing that the value of the portfolio on the right hand-side of equation (4.13) always matches the value of the American-style down-and-out put at any point of its optimal exercise boundary $\left\{B_t^{do(1)} : t \in \mathcal{T}\right\}$. For this purpose, we divide the analysis into three mutually exclusive cases. In the first case (Case A), $B_t^{s(1)} > L$ and $B_t^{do(1)} = B_t^{s(1)}$, for any $t \in \mathcal{T}$, turning the American-style down-and-out put into a standard American-style put option. In the second case (Case B), the early exercise boundary $\left\{B_t^{s(1)} : t \in \mathcal{T}\right\}$ is intersected by the lower barrier level L.

Finally, in the third case (Case C), $B_t^{s(1)} < L$, $\forall t \in \mathcal{T}$, and the American-style down-and-out put is exercised at the first passage time through the barrier level L.

Case A If the exercise boundary of the standard American-style put $B^{s(1)}$ is always above the lower barrier L, then Proposition 4.2 implies that $B_t^{do(1)} = B_t^{s(1)}$, for all $t \in \mathcal{T}$. Consequently, and since an American-style option is worth its intrinsic value on its early exercise boundary, the following two value matching conditions are obtained for all $t \in \mathcal{T}$:

$$ADO_t\left(B_t^{do(1)}, K, L, T, B^{do(1)}; 1\right) = \left(K - B_t^{do(1)}\right)^+, \tag{4.14}$$

$$AS_t\left(B_t^{do(1)}, K, T, B^{s(1)}; 1\right) = \left(K - B_t^{do(1)}\right)^+. \tag{4.15}$$

Hence, we just need to prove that

$$ADI_t\left(B_t^{do(1)}, K, L, T, B^{di(1)}; 1\right) = RD_t\left(B_t^{do(1)}, K, L, T\right) \tag{4.16}$$

for equation (4.13) to hold. However, since the lower barrier L is always below $B_t^{s(1)}$, the stopping time τ_L can only occur after $\tau_{B^{s(1)}}$, meaning that the American-style down-and-in put will be exercised immediately at τ_L; i.e., for $\tau_L > \tau_{B^{s(1)}}$, equation (4.10) can be rewritten as

$$\begin{aligned} & ADI_t\left(S_t, K, L, T, B^{di(1)}; 1\right) \\ = \; & \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{t_0}^{\tau_L} r_u du\right)(K-L)^+ \mathbb{1}_{\{\tau_L \leq T\}}\middle| \mathcal{F}_{t_0}\right], \end{aligned} \tag{4.17}$$

which is exactly the same as equation (4.11). Since equation (4.17) holds for any S_t, it also holds for $S_t = B_t^{do(1)}$, and equation (4.16) follows.

Case B If $L \in \left[B_{t_0}^{s(1)}, B_T^{s(1)}\right]$, and since $B_t^{s(1)}$ is a nondecreasing function of t, then there exists a unique $t^* := \left\{t \in \mathcal{T} : B_t^{s(1)} = L\right\}$. For any time $t \leq t^*$, $L > B_t^{s(1)}$, and Proposition 4.2 implies that $B_t^{do(1)} = L \vee B_t^{s(1)} = L$, yielding the following value matching condition:

$$ADO_t\left(L, K, L, T, B^{do(1)}; 1\right) = K - L. \tag{4.18}$$

Moreover, equation (4.11) can be restated as

$$RD_t\left(L, K, L, T\right) = K - L, \tag{4.19}$$

and we just need to prove that

$$AS_t\left(L, K, T, B^{s(1)}; 1\right) = ADI_t\left(L, K, L, T, B^{di(1)}; 1\right), \tag{4.20}$$

for equation (4.13) to hold. But equation (4.20) arises immediately because the American-style down-and-in option is knocked-in at the lower barrier L, becoming a standard American-style put option. For any $t > t^*$, we have $B_t^{s(1)} > L$, and equation (4.13) follows by using the same arguments as presented in Case A.

Case C Finally, if $B_t^{s(1)} < L$, $\forall t \in \mathcal{T}$, then Proposition 4.2 ensures that $B_t^{do(1)} = L \vee B_t^{s(1)} = L$. Therefore, we can use the same arguments as in Case B, for $t \leq t^*$, to show that equations (4.18), (4.19), and (4.20) yield equation (4.13).

4.3.2 Calls with an up barrier

As in the previous subsection, it is also possible to relate the prices of American-style up-and-out and up-and-in calls. Starting with the former contract, and using again Assumption 4.3, the time-t_0 value of an American-style up-and-out put (if $\phi = 1$) or call (if $\phi = -1$) on the asset price S, with strike K, upper barrier level U, and maturity at time T ($\geq t_0$) will be represented by

$$\begin{aligned}
& AUO_{t_0}\left(S_{t_0}, K, U, T, B^{uo(\phi)}; \phi\right) \\
= \; & \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{t_0}^{T\wedge\tau_{B^{uo(\phi)}}\wedge\tau_U} r_u du\right)\left(\phi K - \phi S_{T\wedge\tau_{B^{uo(\phi)}}\wedge\tau_U}\right)^+ \middle| \mathcal{F}_{t_0}\right],
\end{aligned} \tag{4.21}$$

where

$$\tau_U := \inf\left\{t \geq t_0 : S_t \geq U\right\}, \tag{4.22}$$

$$\tau_{B^{uo(\phi)}} := \inf\left\{t \geq t_0 : \phi S_t \leq \phi B_t^{uo(\phi)}\right\}, \tag{4.23}$$

$B^{uo(\phi)}$ denotes the exercise boundary of the American-style up-and-out put (if $\phi = 1$) or call (if $\phi = -1$), and it is assumed that $\phi S_{t_0} > \phi B_{t_0}^{uo(\phi)}$. Similarly, the time-$t_0$ value of an American-style up-and-in put (if $\phi = 1$) or call (if $\phi = -1$) on the asset price S, with strike K, upper barrier U, maturity at time T ($\geq t_0$), and early exercise boundary $B^{ui(\phi)}$, will be written as

$$\begin{aligned}
& AUI_{t_0}\left(S_{t_0}, K, U, T, B^{ui(\phi)}; \phi\right) \\
= \; & \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{t_0}^{\tau_U} r_u du\right) AS_{\tau_U}\left(U, K, T, B^{s(\phi)}; \phi\right) \mathbb{1}_{\{\tau_U \leq T\}} \middle| \mathcal{F}_{t_0}\right],
\end{aligned} \tag{4.24}$$

assuming that $\phi S_{t_0} > \phi B_{t_0}^{ui(\phi)}$.

To relate equations (4.21) and (4.24) for $\phi = -1$, we take advantage of the following relationship—proved, for instance, by Gao et al. (2000, Theorem 5) under the GBM process, although it simply requires $B^{s(-1)}$ to be a nonincreasing function of t—between the optimal stopping boundaries of up-and-out and plain-vanilla American-style calls:

$$B_t^{uo(-1)} = U \wedge B_t^{s(-1)}, \tag{4.25}$$

for $t \in \mathcal{T}$. Using the same reasoning as in the previous subsection, next proposition offers an in-out parity for American-style calls with an up barrier.

Proposition 4.4 *Under Assumptions 4.1 and 4.3, for $U > K$, and if*

$$\frac{\partial B_t^{s(-1)}}{\partial t} \leq 0, \tag{4.26}$$

then

$$\begin{aligned} & AUO_{t_0}\left(S_{t_0}, K, U, T, B^{uo(-1)}; -1\right) && (4.27)\\ = \quad & AS_{t_0}\left(S_{t_0}, K, T, B^{s(-1)}; -1\right) - AUI_{t_0}\left(S_{t_0}, K, U, T, B^{ui(-1)}; -1\right) \\ & + RU_{t_0}\left(S_{t_0}, K, U, T\right), \end{aligned}$$

where the left-hand side of equation (4.27) is given by equation (4.21), the first two terms on the right-hand side of equation (4.27) are given by equations (4.3) and (4.24), respectively, while

$$RU_{t_0}\left(S_{t_0}, K, U, T\right) := \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{t_0}^{\tau_U} r_u du\right)(U-K)^+ \mathbb{1}_{\{\tau_U \leq T\}}\middle| \mathcal{F}_{t_0}\right] \tag{4.28}$$

is the time-t_0 value of the up rebate $(U-K)^+$ paid at the first passage time through the barrier U.

Remark 4.4 *Again, note that equation (4.28) simply requires the knowledge of the first passage time density through the barrier U (and not through the exercise boundary).*

Proof. The proof of Proposition 4.4 is similar to the one presented for Proposition 4.3, and is available upon request.

4.4 Exercise boundaries for American-style double knock-out options

The literature on the valuation of American-style double barrier options is even more sparse than for single barrier contracts. To the authors knowledge, only a few numerical methods have been proposed for pricing American-style double barrier options—as, for instance, the trinomial method of Ritchken (1995) or the PDE schemes of Zvan et al. (2000)—and most of them are confined to the simpler GBM assumption.

In this section, we derive the early exercise boundary of American-style double knock-out options from the optimal stopping boundaries of American-style but single barrier options. Therefore, the novel results presented in Propositions 4.5 and 4.6 should reduce the valuation of American-style double knock-out options to the same complexity level as the one faced when pricing single barrier American-style option contracts.

An American-style double knock-out option becomes worthless if one of the two barriers is touched by the underlying asset price; otherwise, it can be exercised at or before the expiry date. Additionally, and following again Gao et al. (2000, Footnote 15), for the knock-out event and the exercise date to be well defined, the option contract must be specified in a way such that when the asset price first touches any of the barriers, the option holder has the option to either exercise or let the option be knocked out. Therefore, and using Assumption 4.3, the time-t_0 value of an American-style double knock-out option on the asset price S, with strike K, lower barrier level L $(< K)$, upper barrier level U $(> K)$, and maturity at time T $(\geq t_0)$, is equal to

$$\begin{aligned} & AKODB_{t_0}\left(S_{t_0}, K, L, U, T, B^{ko(\phi)}; \phi\right) \\ = \; & \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{t_0}^{T\wedge\tau_{B^{ko(\phi)}}\wedge\tau_{LU}} r_u du\right)\left(\phi K - \phi S_{T\wedge\tau_{B^{ko(\phi)}}\wedge\tau_{LU}}\right)^+ \middle| \mathcal{F}_{t_0}\right], \end{aligned} \tag{4.29}$$

where $\phi = 1$ for a put option, $\phi = -1$ for a call option, $B^{ko(\phi)}$ is the (unknown) early exercise boundary of the American-style double knock-out option,

$$\tau_{B^{ko(\phi)}} := \inf\left\{t \geq t_0 : \phi S_t \leq \phi B_t^{ko(\phi)}\right\}, \tag{4.30}$$

and

$$\tau_{LU} := \inf\left\{t > t_0 : S_t \leq L \quad or \quad S_t \geq U\right\}, \tag{4.31}$$

is the first passage time of the asset price to one of the two barriers.

Next proposition recovers the put early exercise boundary $B^{ko(1)}$ from the one associated to the simpler American-style up-and-out put option.

Proposition 4.5 *Under Assumptions 4.1 and 4.3, and if*

$$\frac{\partial B_t^{uo(\phi)}}{\partial t} \geq 0, \tag{4.32}$$

then the early exercise boundary $B^{ko(1)}$ of an American-style double knock-out put option on the asset price S, with strike K, lower barrier level L, upper barrier level U, and maturity at time T, is equal to

$$B_t^{ko(1)} = L \vee B_t^{uo(1)}, \tag{4.33}$$

for all $t \in \mathcal{T}$, where $B_t^{uo(\phi)}$ denotes the exercise boundary of the American-style up-and-out put option defined in equation (4.21), and $L < K < U$.

Proof. We prove Proposition 4.5 by contradiction, showing that arbitrage is possible if equation (4.33) does not hold. For this purpose, we divide the analysis into three mutually exclusive cases.

Case A First, we assume that $B_t^{uo(1)} > L$, for all $t \in \mathcal{T}$, i.e. the exercise boundary of the American-style up-and-out put is always above the lower barrier of the double knock-out put option. Furthermore, suppose that $B_t^{ko(1)} > B_t^{uo(1)}$. Then, at $S_t = B_t^{ko(1)}$, equation (4.29) implies that

$$AKODB_t\left(B_t^{ko(1)}, K, L, U, T, B^{ko(1)}; 1\right) = K - B_t^{ko(1)}, \tag{4.34}$$

and

$$AKODB_t\left(B_t^{ko(1)}, K, L, U, T, B^{ko(1)}; 1\right) < AUO_t\left(B_t^{ko(1)}, K, U, T, B^{uo(1)}; 1\right), \tag{4.35}$$

since the point $B_t^{ko(1)}$ is a possible, but not optimal, exercise policy for the American up-and-out put defined in equation (4.21).[4.4]

To show that the scenario $B_t^{ko(1)} > B_t^{uo(1)}$ cannot prevail as it would generate arbitrage opportunities, we form a self-financed portfolio V by going long ω units of the double barrier option, while financing this position by going short one unit of the up-and-out put. Hence, ω must be such that the time-t value of the portfolio is zero, i.e.

$$\begin{aligned} V_t &= \omega \times AKODB_t\left(B_t^{ko(1)}, K, L, U, T, B^{ko(1)}; 1\right) \\ &\quad - AUO_t\left(B_t^{ko(1)}, K, U, T, B^{uo(1)}; 1\right) \\ &= 0. \end{aligned} \tag{4.36}$$

Note that inequality (4.35) implies that the solution of equation (4.36) must be such that $\omega > 1$.

We maintain this portfolio until our short position, i.e. the one unit of the up-and-

[4.4] Note that $B_t^{ko(1)} < U$ because the upper barrier level U is shared by both the single barrier and the double barrier puts.

out put, is exercised at its optimal boundary, is knocked-out, or expires; that is until time

$$\tau_V := \tau_{B^{uo(1)}} \wedge \tau_U \wedge T, \tag{4.37}$$

where the first passage times τ_U and $\tau_{B^{uo(1)}}$ are defined by equations (4.22) and (4.23), respectively. Consequently, we have three possible final payouts for our self-financed portfolio. First, if $\tau_V = \tau_U$, both put options are knocked-out and the terminal payoff of the portfolio equals

$$V_{\tau_U} = \omega \times 0 - 0 = 0. \tag{4.38}$$

Second, if $\tau_V = T$, then

$$V_T = \omega \times (K - S_T)^+ - (K - S_T)^+ \geq 0, \tag{4.39}$$

since $\omega > 1$, and an arbitrage opportunity exists for $S_T < K$. Third, if $\tau_V = \tau_{B^{uo(1)}}$, then both options are exercised,

$$V_{\tau_{B^{uo(1)}}} = \omega \times \left(K - S_{\tau_{B^{uo(1)}}}\right) - \left(K - S_{\tau_{B^{uo(1)}}}\right) > 0, \tag{4.40}$$

because $\omega > 1$, and again a riskless profit is available.

In summary, if $B_t^{ko(1)} > B_t^{uo(1)}$, arbitrage is possible, and since $B_t^{ko(1)} < B_t^{uo(1)}$ cannot happen also—as it would mean that the double knock-out put would be more expensive than the up-and-out put—then we must have that $B_t^{ko(1)} = B_t^{uo(1)} = L \vee B_t^{uo(1)}$, for all $t \in \mathcal{T}$ where $B_t^{uo(1)} > L$.

Case B If $L \in \left[B_{t_0}^{uo(1)}, B_T^{uo(1)}\right]$, and since $B_t^{uo(1)}$ is a nondecreasing function of t, then there exists a unique $t^* := \left\{t \in \mathcal{T} : B_t^{uo(1)} = L\right\}$.

For any time $t \geq t^*$, we face the same situation as in Case A (i.e. $B_t^{uo(1)} > L$), and hence it follows that we must have $B_t^{ko(1)} = B_t^{uo(1)} = L \vee B_t^{uo(1)}$.

For $t < t^*$, and given that $B_{t^*}^{ko(1)} = L$, we can only have $B_t^{ko(1)} > L$, if the optimal exercise boundary $B_t^{ko(1)}$ is decreasing in t. However, and using the same argument as Gao et al. (2000, Appendix B.3), i.e. differentiating (with respect to time) the value matching condition (4.34), then it follows that

$$\frac{\partial B_t^{ko(1)}}{\partial t} \geq 0, \tag{4.41}$$

as long as $\frac{\partial AKODB_t}{\partial t} \leq 0$. Therefore, $B_t^{ko(1)} > L$ cannot happen, and since we must have $B_t^{ko(1)} \geq L$ for all $t \in \mathcal{T}$, then $B_t^{ko(1)} = L = L \vee B_t^{uo(1)}$ for $t < t^*$.

Case C Finally, we just have to consider the scenario where $B_t^{uo(1)} < L$, for all $t \in \mathcal{T}$, and hence $B_T^{uo(1)} < L$. Since we must have $B_t^{ko(1)} \geq B_t^{uo(1)}$ but also $B_t^{ko(1)} \geq L$, for all $t \in \mathcal{T}$, then it easy to see that $B_T^{ko(1)} = L$. For $t < T$, and given that $B_T^{ko(1)} = L$, we can only have $B_t^{ko(1)} > L$ if the optimal exercise boundary $B_t^{ko(1)}$ is decreasing in t. However, equation (4.41) rules out such possibility, and, consequently, we must have $B_t^{ko(1)} = L = L \vee B_t^{uo(1)}$ for all $t \in \mathcal{T}$.

Finally, Proposition 4.6 writes the call early exercise boundary $B^{ko(-1)}$ in term of the simpler one associated to an American-style down-and-out call option.

Proposition 4.6 *Under Assumptions 4.1 and 4.3, and if*

$$\frac{\partial B_t^{do(-1)}}{\partial t} \leq 0, \tag{4.42}$$

then the early exercise boundary $B^{ko(-1)}$ of an American-style double knock-out call option on the asset price S, with strike K, lower barrier level L, upper barrier level U, *and maturity at time T, is equal to*

$$B_t^{ko(-1)} = U \wedge B_t^{do(-1)}, \tag{4.43}$$

for all $t \in \mathcal{T}$, where $B_t^{do(-1)}$ denotes the exercise boundary of the American-style down-and-out call option defined in equation (4.5), and $L < K < U$.

Proof. The proof of Proposition 4.6 is similar to the one presented for Proposition 4.5, and is available upon request.

Equipped with Propositions 4.5 and 4.6, we can now price American-style double barrier knock-out options as long as we have a viable valuation method for American-style standard options as well as the knowledge of the optimal exercise boundaries for American-style up-and-out puts or down-and-out calls.

4.5 Numerical results

To test numerically Propositions 4.3, 4.4, 4.5 and 4.6, we need a pricing method that can cope with the valuation of the different types of contracts involved in such novel results. The SHP approach will provide a viable methodology under single-factor pricing models, namely under the GBM assumption, under the CEV process, and even under the jump to default extended CEV model of Carr and Linetsky (2006)—as recently shown by Ruas et al. (2012a). Nevertheless, and to allow the comparison with the previous literature on the pricing of American-style barrier options, our numerical analysis will be restricted to the GBM and CEV frameworks.

4.5.1 Further model assumptions and valuation procedures

Under the risk-neutral probability measure $\mathbb{Q}$, the CEV model assumes that the asset price $(S_t, t \geq t_0)$ is described by the following stochastic differential equation:

$$\frac{dS_t}{S_t} = (r - q)\, dt + \sigma(t, S)\, dW_t^{\mathbb{Q}}, \tag{4.44}$$

with

$$\sigma(t, S) := \delta S_t^{\frac{\beta}{2}-1}, \tag{4.45}$$

for $\delta, \beta \in \mathbb{R}$, and where $r \geq 0$ denotes the instantaneous riskless interest rate, which is assumed to be constant, $q \geq 0$ represents the dividend yield for the underlying asset price, and $W_t^{\mathbb{Q}} \in \mathbb{R}$ is a standard Brownian motion under $\mathbb{Q}$, initialized at zero and generating the augmented, right continuous and complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq t_0\}$. In the particular case $\beta = 2$, equations (4.44) and (4.45) yield the GBM process.

Under the above assumptions, standard American-style options are valued through the Chung and Shih (2009) procedure—enhanced by the Ruas et al. (2012a) extension to other values of the elasticity parameter (beta) besides $\frac{4}{3}$—and using closed-form solutions for the hedge ratios of the corresponding European-style options. American-style down-and-out puts and up-and-out calls are priced using the SHP extension (for American-style capped options) offered in Ruas et al. (2012a). The valuation of the rebates (4.11) and (4.28) is made through closed-form solutions for the particular case $\beta = 2$, and using the Derman et al. (1995) procedure for any other beta value. American-style down-and-in puts and up-and-in calls are valued using the algorithm offered by Chung et al. (2013), but with greeks (deltas and thetas) generalized for $\beta \neq \frac{4}{3}$.[4.5]

[4.5] The analytical solutions, for any value of beta, for the thetas of the binary options required by the Chung et al. (2013) scheme are available upon request.

To value American-style double knock-out options using Propositions 4.5 and 4.6, we propose a simple extension to the Chung et al. (2009) procedure for the pricing of American-style up-and-out puts and down-and-out calls. The procedure is implemented in two steps and illustrated below for American-style double knock-out put options (the call option case follows similarly). First, we must obtain the early exercise boundary of the American-style up-and-out put (or down-and-out call); then, the valuation of American-style double knock-out options follows easily from Propositions 4.5 and 4.6, as described in the following lines.

We divide the time to maturity of the option contract into n evenly-spaced time points such that $\delta t := (T - t_0)/n$. At each time $t_i := t_0 + i\delta t$ (for $i = n-1, \ldots, 1, 0$), the knock-out boundary conditions on the barriers as well as the value-matching and smooth-pasting conditions on the early exercise boundary imply the solution of following three recurrence conditions:

$$\begin{aligned} K - B_{n-i}^{ko(1)} &= ES_{t_{n-i}}\left(B_{n-i}^{ko(1)}, K, T; 1\right) \\ &+ \sum_{j=1}^{i} w_{n-j} \times ES_{t_{n-i}}\left(B_{n-i}^{ko(1)}, B_{n-j}^{ko(1)}, t_{n-j+1}; 1\right) \\ &+ \sum_{j=1}^{i} g_{n-j} \times ES_{t_{n-i}}\left(B_{n-i}^{ko(1)}, U, t_{n-j+1}; -1\right), \end{aligned} \tag{4.46}$$

$$\begin{aligned} 0 &= ES_{t_{n-i}}\left(U, K, T; 1\right) \\ &+ \sum_{j=1}^{i} w_{n-j} \times ES_{t_{n-i}}\left(U, B_{n-j}^{ko(1)}, t_{n-j+1}; 1\right) \\ &+ \sum_{j=1}^{i} g_{n-j} \times ES_{t_{n-i}}\left(U, U, t_{n-j+1}; -1\right), \end{aligned} \tag{4.47}$$

and

$$-\mathbb{1}_{\left\{B_{n-i}^{ko(1)}=B_{n-i}^{uo(1)}\right\}} = \left[\Delta_{ES_{t_{n-i}}\left(B_{n-i}^{ko(1)},K,T;1\right)} + \sum_{j=1}^{i} w_{n-j} \times \Delta_{ES_{t_{n-i}}\left(B_{n-i}^{ko(1)},B_{n-j}^{ko(1)},t_{n-j+1};1\right)} + \sum_{j=1}^{i} g_{n-j} \times \Delta_{ES_{t_{n-i}}\left(B_{n-i}^{ko(1)},U,t_{n-j+1};-1\right)}\right] \mathbb{1}_{\left\{B_{n-i}^{ko(1)}=B_{n-i}^{uo(1)}\right\}}, \tag{4.48}$$

where $ES_t\,(S_t, K, T; \phi)$ represents the time-t price of an European-style put (if $\phi = 1$) or call (if $\phi = -1$) on the asset price S, with strike K, and maturity at time T ($\geq t$).[4.6]

Finally, the time-t_0 SHP price of the American-style double knock-out put option, under the CEV model, is given by:

$$AKODB_{t_0}^{shp}\left(S_{t_0}, K, L, U, T, B^{ko(1)}; 1\right) = ES_{t_0}\left(S_{t_0}, K, T; 1\right) + \sum_{j=1}^{n} w_{n-j} \times ES_{t_0}\left(S_{t_0}, B_{n-j}^{ko(1)}, t_{n-j+1}; 1\right) + \sum_{j=1}^{n} g_{n-j} \times ES_{t_0}\left(S_{t_0}, U, t_{n-j+1}; -1\right). \tag{4.49}$$

In the remaining of the article, all computations involving the SHP method are based on 100 time steps (i.e. $n = 100$), and all numerical results are obtained through Matlab (R2009a) running on an Intel Core 2 Duo E8500 3.16GHz processor.

4.5.2 Results under the GBM assumption

Under the GBM assumption, we borrow the model' parameters from Gao et al. (2000) to construct Tables 4.1 and 4.2. Namely, we let $S_{t_0} \in \{42.5, 45, 47.5\}$ (time-t_0 asset price), $K = 45$ (strike price), $r = 4.88\%$ (interest rate), $\sigma = 20\%$ (constant volatility), and $(T - t_0) \in \{0.25, 0.5, 0.75, 1\}$ (time to expiration in years).

[4.6] Note that it is only necessary to compute the recurrence relation (4.48) while $B_{n-i}^{ko(1)} = B_{n-i}^{uo(1)}$; i.e. equation (4.48) is automatically satisfied if $B_{n-i}^{ko(1)} = L$.

Table 4.1 tests the in-out parity offered by Propositions 4.3 and 4.4, and is divided into three Panels (A, B and C) that correspond to the three cases shown in the proof of Proposition 4.3. Under Proposition 4.3, we set $q = 0\%$ (dividend yield) and $L = 35$ (lower barrier level) for Panel A; $q = 0\%$ and $L = 40$ for Panel B; $q = 7\%$ and $L = 37.5$ for Panel C. Under Proposition 4.4, we take $q = 7\%$ and $U = 60$ (upper barrier level) for Panel A; $q = 7\%$ and $U = 50$ for Panel B; $q = 3\%$ and $U = 52.5$ for Panel C. Columns 4 through 6 price a down-and-out, a standard, and a down-and-in American-style put, respectively, using the SHP approach. Column 7 shows the value of the "down rebate" (4.11), while column 8 checks Proposition 4.3 by presenting the difference between both sides of equation (4.13). Similarly, columns 10 through 12 price an up-and-out, a standard, and an up-and-in American-style call, respectively, using the SHP approach. Column 13 shows the value of the "up rebate" (4.28), and column 14 checks Proposition 4.4 by presenting the difference between both sides of equation (4.27). As expected, columns 8 and 14 show that both in-out parities are strictly satisfied by all contracts.

Table 4.2 implements Propositions 4.5 and 4.6, and prices American-style double knock-out options. Under Proposition 4.5, we set $U = 50$ for all contracts, while the lower barrier levels and the dividend yields are the same as those considered in Table 4.1 for Proposition 4.3. Under Proposition 4.6, we let $L = 40$ for all contracts, while the upper barrier levels and the dividend yields are the same as those considered in Table 4.1 for Proposition 4.4. Column 5 contains the prices of American-style up-and-out puts, whose early exercise boundary is then used in Proposition 4.5 to implement the SHP approach. Columns 6 and 7 value American-style double knock-out puts using the Ritchken (1995) method with 2 million time steps, and the SHP approach proposed, respectively.[4.7] Column 8 checks Proposition 4.5 by presenting the difference between the Ritchken (1995) and the SHP prices. Similarly, column 8 shows the values

[4.7] As pointed by Gao et al. (2000, Page 1805) the problem with the Ritchken (1995) method is that as the asset price approaches the barrier, the number of time steps required to value the knock-out option goes to infinity. This feature renders this numerical method difficult to apply. For the case under analysis (double barriers), the problem for American-style down-and-out calls, whose early exercise boundary is then used in Proposition 4.6. Columns 12 and 13 value American-style double knock-out calls using the Ritchken (1995) method (with 2 million time steps), and the SHP approach proposed, respectively. Finally, column 14 checks Proposition 4.6 by presenting the difference between the Ritchken (1995) and the SHP prices.

The results displayed in Tables 4.1 and 4.2 are remarkable. Even though the Ritchken (1995) and SHP schemes used to compute American-style option prices are subject to convergence and discretization errors, respectively, columns 8 and 14 of Tables 4.1 and 4.2 show that Propositions 4.3, 4.4, 4.5, and 4.6 hold for all contracts: The maximum difference obtained is equal to only 0.3 cents of a dollar. Concerning the CPU time (shown, in seconds, on the last line of Tables 4.1 and 4.2), note that Propositions 4.5 and 4.6 greatly enhance the efficiency associated to the valuation of American style double knock-out options: the SHP procedure proposed is about 508 times faster than the trinomial tree method of Ritchken (1995).

4.5.3 Results under the CEV assumption

Under the CEV assumption, we let $S_{t_0} = 100$, $K \in \{95, 100, 105\}$, $r = 6\%$, $\beta \in \{1, 3\}$, $\sigma(t_0, S_{t_0}) = 0.2$, and $(T - t_0) \in \{0.5, 1\}$ years.

Table 4.3 tests the in-out parity given by Propositions 4.3 and 4.4, and is divided into three Panels (A, B and C) that corresponds to the three cases contained in the proof of Proposition 4.3. Under Proposition 4.3, we let $q = 3\%$ and $L = 70$ for Panel A; $q = 3\%$ and $L = 87.5$ for Panel B; $q = 9\%$ and $L = 80$ for Panel C. Under Proposition 4.4, we set $q = 9\%$ and $U = 140$ for Panel A; $q = 9\%$ and $U = 115$ for Panel B; $q = 3\%$ and $U = 120$

is more serious since we have to deal with two barriers. To overcome the non-smooth convergence of the option price under this method, we use a very large number of time steps: 2 million.

for Panel C. Columns 5 through 7 price a down-and-out, a standard, and a down-and-in American-style put, respectively, using the SHP approach. Column 8 shows the value of the “down rebate” (4.11), while column 9 checks Proposition 4.3 by presenting the difference between both sides of equation (4.13). Similarly, columns 11 through 13 price an up-and-out, a standard, and an up-and-in American-style call, respectively, column 14 shows the value of the “up rebate” (4.28), and column 15 checks Proposition 4.4 by presenting the difference between both sides of equation (4.27). Again, columns 9 and 15 show that Propositions 4.3 and 4.4 are strictly satisfied by all contracts.

Table 4.4 implements Propositions 4.5 and 4.6, and prices American-style double knock-out options using our two-step procedure. Under Proposition 4.5, we let $U = 110$ for all contracts, while the lower barrier levels and the dividend yields are the same as those considered in Table 4.3 for Proposition 4.3. Under Proposition 4.6, we take $L = 90$ for all contracts, while the upper barrier levels and the dividend yields are the same as those considered in Table 4.3 for Proposition 4.4.

Column 6 of Table 4.4 contains the values of the American-style up-and-out puts whose early exercise boundary is then used in column 7 to price American-style double knock-out puts through equations (4.46) to (4.49). Similarly, columns 10 and 11 price down-and-out as well as double knock-out American-style calls, respectively, through the same SHP scheme. Note that, in Panel A of Table 4.4, the values of the double knock-out options are identical to the prices of the nested single barrier contracts because the early exercise boundary of the nested up-and out put (down-and-out call) is always above (below) the down (up) barrier level.

4.6 Conclusions

This paper provides three main contributions to the existent literature on option pricing. First, we derive new in-out parity relations for American-style puts with a down barrier and American-style calls with an up barrier. Second, and more importantly, we propose a novel representation for the early exercise boundary of American-style double knock-out options in terms of its barrier levels and as a function of the simpler optimal stopping boundary of a single barrier American-style option. Hence, we are able to reduce the valuation of American-style double barrier options to the same complexity level as the one faced to price the nested single barrier contracts. Finally, we extend the SHP approach, for the first time to the authors knowledge, to the valuation of American-style double barrier knock-out options.

All the previous results are based on only two mild assumptions: The absence of arbitrage, and the existence and uniqueness of an early exercise boundary for each American-style option contract that is also a continuous and monotone function of calendar time. Therefore, our results can be applied to the valuation of American-style single and double barrier options under several single factor diffusion and jump-diffusion option pricing models. We run an extensive numerical study—covering a large range of single and double barrier option contracts that can be used as a benchmark for future empirical analysis—and successfully validate all our results under both the GBM and CEV processes.

Appendix C

This appendix proves Proposition 4.1.

Using equation (4.3), equation (4.8) can be rewritten, for $\phi = 1$, as

$$\begin{aligned} & ADI_{t_0}\left(S_{t_0}, K, L, T, B^{di(1)}; 1\right) \qquad \text{(C-1)} \\ = \; & \mathbb{E}_{\mathbb{Q}}\left\{ \exp\left(-\int_{t_0}^{\tau_L} r_u du\right) \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{\tau_L}^{T\wedge\theta_{B^{s(1)}}} r_u du\right) \left(K - S_{T\wedge\theta_{B^{s(1)}}}\right)^+ \middle| \mathcal{F}_{\tau_L}\right] \right. \\ & \left. \mathbb{1}_{\{\tau_L \leq T\}} \middle| \mathcal{F}_{t_0} \right\}, \end{aligned}$$

where

$$\theta_{B^{s(1)}} := \inf\left\{t \geq \tau_L : S_t \leq B_t^{s(1)}\right\}. \qquad \text{(C-2)}$$

Clearly, equations (4.4) and (C-2) imply that

$$\theta_{B^{s(1)}} = \tau_{B^{s(1)}} \vee \tau_L, \qquad \text{(C-3)}$$

as long as $B_t^{s(1)}$ is a nondecreasing function of time: If $\tau_L < \tau_{B^{s(1)}}$, then $\theta_{B^{s(1)}} = \tau_{B^{s(1)}} = \tau_{B^{s(1)}} \vee \tau_L$; otherwise, $B^{s(1)}_{\tau_{B^{s(1)}}} > L$, and $\theta_{B^{s(1)}} = \tau_L = \tau_{B^{s(1)}} \vee \tau_L$ only if $L \leq B^{s(1)}_{\tau_L}$, i.e. if $B^{s(1)}_{\tau_L} \geq B^{s(1)}_{\tau_{B^{s(1)}}}$.

Combining equations (C-1) and (C-3), then

$$\begin{aligned} & ADI_{t_0}\left(S_{t_0}, K, L, T, B^{di(1)}; 1\right) \qquad \text{(C-4)} \\ = \; & \mathbb{E}_{\mathbb{Q}}\left\{ \exp\left(-\int_{t_0}^{\tau_L} r_u du\right) \mathbb{1}_{\{\tau_L \leq T\}} \right. \\ & \left. \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{\tau_L}^{T\wedge\tau_{B^{s(1)}}} r_u du\right) \left(K - S_{T\wedge\tau_{B^{s(1)}}}\right)^+ \mathbb{1}_{\left\{\tau_L \leq \tau_{B^{s(1)}}\right\}} \middle| \mathcal{F}_{\tau_L}\right] \middle| \mathcal{F}_{t_0} \right\} \\ & + \mathbb{E}_{\mathbb{Q}}\left\{ \exp\left(-\int_{t_0}^{\tau_L} r_u du\right) \mathbb{1}_{\{\tau_L \leq T\}} \right. \\ & \left. \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_{\tau_L}^{T\wedge\tau_L} r_u du\right) \left(K - S_{T\wedge\tau_L}\right)^+ \mathbb{1}_{\left\{\tau_L > \tau_{B^{s(1)}}\right\}} \middle| \mathcal{F}_{\tau_L}\right] \middle| \mathcal{F}_{t_0} \right\}. \end{aligned}$$

Using the tower law, and because $T \wedge \tau_L = \tau_L$ for the second term on the right-hand side of equation (C-4), then equation (4.10) follows immediately. ■

Appendix D

This appendix proves Proposition 4.2.

Next lines follow closely the proof of Gao et al. (2000, Theorem 6), but without imposing any specific asset price process. We consider three mutually exclusive cases.

First, if $L < B_t^{s(1)}$, for $\forall t \in \mathcal{T}$, then $B_t^{do(1)} = B_t^{s(1)} = L \vee B_t^{s(1)}$ because the down-and-out put will be exercised before it is knocked-out, and is, therefore, equivalent to a standard American-style put.

Second, if $L > B_t^{s(1)}$, for $\forall t \in \mathcal{T}$, then $B_t^{do(1)} = L = L \vee B_t^{s(1)}$ since the put option hits the barrier in-the-money (as $L < K$) and, therefore, it is preferable to exercise than to let the option be knocked out.

Third, if $L \in \left[B_{t_0}^{s(1)}, B_T^{s(1)}\right]$, and since $B_t^{s(1)}$ is a nondecreasing function of t, then there exists a unique $t^* := \left\{t \in \mathcal{T} : B_t^{s(1)} = L\right\}$. In the time-interval $[t^*, T]$, $B_t^{s(1)} > L$ and $B_t^{do(1)} = B_t^{s(1)} = L \vee B_t^{s(1)}$ because the down-and-out put will be exercised before it is knocked-out. In the time-interval $[t_0, t^*]$, $L > B_t^{s(1)}$ and $B_t^{do(1)} = L = L \vee B_t^{s(1)}$ because it is preferable to exercise the in-the-money down-and-out put than to let it be knocked out. ■

Table 4.1: In-out parity for American-style options under the GBM model ($K = 45$; $\sigma = 0.2$; $r = 4.88\%$)

		American-style put						American-style call					
S_{t_0}	$T - t_0$	L	ADO	AS	ADI	RD	P4.3	U	AUO	AS	AUI	RU	P4.4
Panel A													
42.5	0.25	35	3.001	3.001	0.449	0.449	0.000	60	0.705	0.705	0.006	0.006	0.000
42.5	0.5	35	3.413	3.413	1.449	1.449	0.000	60	1.249	1.249	0.151	0.151	0.000
42.5	0.75	35	3.716	3.716	2.225	2.225	0.000	60	1.659	1.659	0.470	0.470	0.000
42.5	1	35	3.957	3.957	2.798	2.798	0.000	60	1.992	1.992	0.848	0.848	0.000
45	0.25	35	1.571	1.571	0.099	0.099	0.000	60	1.675	1.676	0.044	0.044	0.000
45	0.5	35	2.104	2.104	0.617	0.617	0.000	60	2.292	2.292	0.456	0.456	0.000
45	0.75	35	2.472	2.472	1.190	1.190	0.000	60	2.733	2.733	1.040	1.040	0.000
45	1	35	2.757	2.757	1.683	1.683	0.000	60	3.081	3.082	1.603	1.603	0.000
47.5	0.25	35	0.715	0.715	0.018	0.018	0.000	60	3.197	3.197	0.227	0.227	0.000
47.5	0.5	35	1.220	1.220	0.242	0.242	0.000	60	3.734	3.734	1.135	1.134	0.000
47.5	0.75	35	1.586	1.586	0.606	0.606	0.000	60	4.136	4.136	2.025	2.025	0.000
47.5	1	35	1.875	1.875	0.979	0.978	0.000	60	4.460	4.460	2.750	2.750	0.000
Panel B													
42.5	0.25	40	2.996	3.001	2.596	2.591	0.000	50	0.703	0.705	0.437	0.435	0.000
42.5	0.5	40	3.339	3.413	3.246	3.171	0.000	50	1.215	1.249	1.072	1.038	0.000
42.5	0.75	40	3.535	3.716	3.621	3.440	0.000	50	1.552	1.659	1.541	1.434	0.000
42.5	1	40	3.664	3.957	3.894	3.601	0.000	50	1.791	1.992	1.907	1.706	0.000
45	0.25	40	1.570	1.571	1.088	1.088	0.000	50	1.665	1.676	1.308	1.298	0.000
45	0.5	40	2.080	2.104	1.859	1.835	0.000	50	2.191	2.292	2.118	2.016	0.000
45	0.75	40	2.391	2.472	2.322	2.240	0.000	50	2.493	2.733	2.629	2.389	0.000
45	1	40	2.603	2.757	2.654	2.499	0.000	50	2.693	3.082	3.011	2.622	0.000
47.5	0.25	40	0.715	0.715	0.375	0.375	0.000	50	3.144	3.197	2.920	2.867	0.000
47.5	0.5	40	1.213	1.220	0.981	0.974	0.000	50	3.480	3.734	3.624	3.371	0.000
47.5	0.75	40	1.551	1.586	1.421	1.386	0.000	50	3.661	4.136	4.075	3.599	0.000
47.5	1	40	1.796	1.875	1.755	1.676	0.000	50	3.775	4.460	4.420	3.735	0.000
Panel C													
42.5	0.25	37.5	3.378	3.389	1.790	1.779	0.000	52.5	0.822	0.824	0.257	0.255	0.000
42.5	0.5	37.5	4.028	4.102	3.230	3.155	0.000	52.5	1.507	1.530	1.014	0.991	0.000
42.5	0.75	37.5	4.477	4.670	4.115	3.922	0.000	52.5	2.021	2.107	1.708	1.623	0.000
42.5	1	37.5	4.804	5.152	4.764	4.416	0.000	52.5	2.419	2.604	2.295	2.110	0.000
45	0.25	37.5	1.885	1.888	0.613	0.610	0.000	52.5	1.876	1.884	0.920	0.912	0.000
45	0.5	37.5	2.673	2.701	1.777	1.748	0.000	52.5	2.632	2.700	2.099	2.031	0.000
45	0.75	37.5	3.232	3.325	2.667	2.573	0.000	52.5	3.141	3.333	2.931	2.739	0.000
45	1	37.5	3.654	3.848	3.358	3.164	0.000	52.5	3.510	3.868	3.579	3.221	0.000
47.5	0.25	37.5	0.917	0.918	0.171	0.171	0.000	52.5	3.447	3.478	2.385	2.355	0.000
47.5	0.5	37.5	1.663	1.673	0.894	0.884	0.000	52.5	4.078	4.248	3.716	3.547	0.000
47.5	0.75	37.5	2.238	2.282	1.640	1.595	0.000	52.5	4.481	4.866	4.543	4.158	0.000
47.5	1	37.5	2.696	2.802	2.285	2.180	0.000	52.5	4.762	5.395	5.173	4.541	0.000
CPU			5.81	11.11	12.31	0.00			5.36	11.30	12.40	0.00	

Table 4.1 tests, under the GBM model, the in-out parity given in Propositions 4.3 and 4.4 for American-style options. Columns 1, 2 and 3 show the asset price, the time to expiration, and lower barrier level, respectively. Under Proposition 4.3, the dividend yield is $q = 0\%$ for Panels A and B, and $q = 7\%$ for Panel C. Columns 4 through 7 implement equations (4.5), (4.3), (4.8) and (4.11), respectively, using the SHP approach. Column 8 checks Proposition 4.3 by presenting the difference between both sides of equation (4.13). Column 9 shows the upper barrier level used in Proposition 4.4. Under Proposition 4.4, the dividend yield is $q = 7\%$ for Panels A and B, and $q = 3\%$ for Panel C. Columns 10 through 13 implement equations (4.21), (4.3), (4.24) and (4.28), respectively, using the SHP approach. Column 14 checks Proposition 4.4 by presenting the difference between both sides of equation (4.27). Finally, the last line gives the CPU time in seconds taken to value the whole set of contracts considered.

Table 4.2: American-style double knock-out options under the GBM model ($K = 45$; $\sigma = 0.2$; $r = 4.88\%$)

		American-style put						American-style call					
S_{t_0}	$T - t_0$	L	U	AUO	R95	P4.5	diff	L	U	ADO	R95	P4.6	diff
						Panel A							
42.5	0.25	35	50	2.996	2.996	2.996	0.000	40	60	0.645	0.645	0.645	0.000
42.5	0.5	35	50	3.348	3.348	3.348	0.000	40	60	0.976	0.976	0.976	0.001
42.5	0.75	35	50	3.538	3.538	3.538	0.000	40	60	1.143	1.141	1.143	0.001
42.5	1	35	50	3.645	3.645	3.645	0.000	40	60	1.239	1.237	1.239	0.002
45	0.25	35	50	1.544	1.545	1.544	0.000	40	60	1.662	1.662	1.662	0.000
45	0.5	35	50	1.938	1.938	1.937	0.000	40	60	2.175	2.175	2.175	0.000
45	0.75	35	50	2.120	2.120	2.120	0.000	40	60	2.453	2.452	2.453	0.001
45	1	35	50	2.215	2.217	2.215	-0.001	40	60	2.620	2.619	2.620	0.001
47.5	0.25	35	50	0.616	0.616	0.616	0.000	40	60	3.194	3.194	3.194	0.000
47.5	0.5	35	50	0.862	0.863	0.862	0.000	40	60	3.687	3.687	3.687	0.000
47.5	0.75	35	50	0.969	0.969	0.969	0.000	40	60	3.991	3.991	3.991	0.000
47.5	1	35	50	1.022	1.022	1.022	0.000	40	60	4.186	4.186	4.186	0.001
						Panel B							
42.5	0.25	40	50	2.996	2.991	2.992	0.000	40	50	0.645	0.643	0.643	0.000
42.5	0.5	40	50	3.348	3.278	3.278	0.001	40	50	0.976	0.944	0.945	0.001
42.5	0.75	40	50	3.538	3.383	3.385	0.002	40	50	1.143	1.054	1.055	0.002
42.5	1	40	50	3.645	3.421	3.424	0.003	40	50	1.239	1.093	1.096	0.002
45	0.25	40	50	1.544	1.544	1.544	0.000	40	50	1.662	1.651	1.652	0.000
45	0.5	40	50	1.938	1.914	1.915	0.000	40	50	2.175	2.075	2.075	0.001
45	0.75	40	50	2.120	2.049	2.050	0.001	40	50	2.453	2.229	2.230	0.002
45	1	40	50	2.215	2.099	2.100	0.000	40	50	2.620	2.286	2.287	0.001
47.5	0.25	40	50	0.616	0.616	0.616	0.000	40	50	3.194	3.141	3.141	0.000
47.5	0.5	40	50	0.862	0.856	0.856	0.000	40	50	3.687	3.438	3.439	0.001
47.5	0.75	40	50	0.969	0.942	0.943	0.000	40	50	3.991	3.546	3.548	0.002
47.5	1	40	50	1.022	0.973	0.974	0.001	40	50	4.186	3.585	3.587	0.003
						Panel C							
42.5	0.25	37.5	50	3.384	3.373	3.373	0.000	40	52.5	0.755	0.754	0.754	0.000
42.5	0.5	37.5	50	4.034	3.959	3.959	0.000	40	52.5	1.206	1.184	1.184	0.001
42.5	0.75	37.5	50	4.473	4.282	4.283	0.001	40	52.5	1.470	1.395	1.397	0.002
42.5	1	37.5	50	4.790	4.456	4.458	0.002	40	52.5	1.649	1.502	1.504	0.002
45	0.25	37.5	50	1.859	1.856	1.856	0.000	40	52.5	1.869	1.862	1.862	0.000
45	0.5	37.5	50	2.505	2.477	2.477	0.000	40	52.5	2.570	2.502	2.503	0.000
45	0.75	37.5	50	2.891	2.800	2.801	0.000	40	52.5	3.011	2.824	2.825	0.001
45	1	37.5	50	3.152	2.976	2.975	-0.001	40	52.5	3.321	2.986	2.988	0.002
47.5	0.25	37.5	50	0.795	0.795	0.795	0.000	40	52.5	3.475	3.445	3.445	0.000
47.5	0.5	37.5	50	1.204	1.194	1.194	0.000	40	52.5	4.200	4.030	4.030	0.000
47.5	0.75	37.5	50	1.431	1.395	1.395	0.000	40	52.5	4.709	4.330	4.331	0.001
47.5	1	37.5	50	1.580	1.502	1.502	0.000	40	52.5	5.089	4.484	4.484	0.001
CPU				26.73	7567.17	14.89				26.28	7751.90	13.92	

Table 4.2 tests Propositions 4.5 and 4.6 under the GBM model. Columns 1 through 4 show the asset price, the time to expiration, the lower barrier level, and the upper barrier level, respectively. Under Proposition 4.5, the dividend yield is $q = 0\%$ for Panels A and B, and $q = 7\%$ for Panel C. Columns 5 through 7 implement equation (4.21), the Ritchken (1995) method with 2 million time steps, and equation (4.49), respectively. Column 8 checks Proposition 4.5 by presenting the difference between columns 6 and 7. Columns 9 and 10 show the lower and upper barrier levels used in Proposition 4.6. Under Proposition 4.6, the dividend yield is $q = 7\%$ for Panels A and B, and $q = 3\%$ for Panel C. Columns 11 through 13 implement equation (4.5), the Ritchken (1995) method with 2 million time steps, and equation (4.49), respectively. Column 14 checks Proposition 4.6 by presenting the difference between columns 12 and 13. Finally, the last line gives the CPU time in seconds taken to value the whole set of contracts considered.

Table 4.3: In-out parity for American-style options under the CEV model ($S_{t_0} = 100$; $r = 6\%$; $\sigma(t_0, S_{t_0}) = 0.2$)

			American-style put						American-style call					
K	β	$T - t_0$	L	ADO	ASP	ADI	RD	P4.3	U	AUO	AS	AUI	RU	P4.4
							Panel A							
95	3	0.5	70	2.810	2.810	0.134	0.134	0.000	140	7.426	7.426	0.774	0.774	0.000
100	3	0.5	70	4.968	4.968	0.161	0.161	0.000	140	4.908	4.908	0.688	0.688	0.000
105	3	0.5	70	7.861	7.861	0.187	0.187	0.000	140	3.124	3.124	0.602	0.602	0.000
95	3	1	70	4.376	4.376	1.165	1.165	0.000	140	8.766	8.766	3.199	3.199	0.000
100	3	1	70	6.641	6.641	1.398	1.398	0.000	140	6.489	6.489	2.844	2.844	0.000
105	3	1	70	9.445	9.445	1.631	1.631	0.000	140	4.753	4.753	2.488	2.488	0.000
95	1	0.5	70	2.927	2.927	0.443	0.443	0.000	140	7.569	7.569	0.284	0.284	0.000
100	1	0.5	70	4.956	4.956	0.531	0.531	0.000	140	4.920	4.920	0.253	0.253	0.000
105	1	0.5	70	7.714	7.714	0.620	0.620	0.000	140	3.008	3.008	0.221	0.221	0.000
95	1	1	70	4.530	4.530	2.124	2.124	0.000	140	8.988	8.988	1.942	1.942	0.000
100	1	1	70	6.606	6.606	2.549	2.549	0.000	140	6.524	6.524	1.726	1.726	0.000
105	1	1	70	9.216	9.216	2.974	2.974	0.000	140	4.603	4.603	1.510	1.510	0.000
							Panel B							
95	3	0.5	87.5	2.639	2.810	2.592	2.421	0.000	115	7.426	7.426	5.404	5.404	0.000
100	3	0.5	87.5	4.942	4.968	4.060	4.034	0.000	115	4.880	4.908	4.081	4.053	0.000
105	3	0.5	87.5	7.861	7.861	5.648	5.648	0.000	115	2.936	3.124	2.890	2.702	0.000
95	3	1	87.5	3.679	4.376	4.279	3.582	0.000	115	8.735	8.766	7.858	7.826	0.000
100	3	1	87.5	6.383	6.641	6.228	5.970	0.000	115	6.230	6.489	6.129	5.870	0.000
105	3	1	87.5	9.392	9.445	8.411	8.358	0.000	115	4.013	4.753	4.653	3.913	0.000
95	1	0.5	87.5	2.670	2.927	2.760	2.502	0.000	115	7.569	7.569	5.177	5.177	0.000
100	1	0.5	87.5	4.905	4.956	4.222	4.171	0.000	115	4.909	4.920	3.894	3.883	0.000
105	1	0.5	87.5	7.713	7.714	5.840	5.839	0.000	115	2.894	3.008	2.703	2.588	0.000
95	1	1	87.5	3.626	4.530	4.459	3.555	0.000	115	8.972	8.988	7.860	7.845	0.000
100	1	1	87.5	6.244	6.606	6.287	5.926	0.000	115	6.355	6.524	6.053	5.884	0.000
105	1	1	87.5	9.134	9.216	8.378	8.296	0.000	115	4.060	4.603	4.467	3.923	0.000
							Panel C							
95	3	0.5	80	3.663	3.698	1.996	1.961	0.000	120	8.955	9.053	5.410	5.313	0.000
100	3	0.5	80	6.159	6.184	2.639	2.614	0.000	120	6.146	6.277	4.382	4.250	0.000
105	3	0.5	80	9.315	9.333	3.286	3.268	0.000	120	3.985	4.185	3.387	3.188	0.000
95	3	1	80	5.895	6.194	5.086	4.787	0.000	120	11.208	11.736	9.804	9.277	0.000
100	3	1	80	8.665	8.870	6.588	6.383	0.000	120	8.407	9.138	8.153	7.422	0.000
105	3	1	80	11.854	11.995	8.119	7.979	0.000	120	5.975	7.016	6.607	5.566	0.000
95	1	0.5	80	3.771	3.830	2.494	2.435	0.000	120	9.097	9.176	4.879	4.800	0.000
100	1	0.5	80	6.146	6.184	3.285	3.247	0.000	120	6.178	6.277	3.940	3.840	0.000
105	1	0.5	80	9.185	9.211	4.085	4.059	0.000	120	3.911	4.051	3.019	2.880	0.000
95	1	1	80	5.941	6.379	5.590	5.152	0.000	120	11.458	11.910	9.530	9.077	0.000
100	1	1	80	8.586	8.874	7.157	6.869	0.000	120	8.539	9.138	7.861	7.262	0.000
105	1	1	80	11.639	11.830	8.778	8.587	0.000	120	6.005	6.827	6.268	5.446	0.000
CPU				673.49	1,287.52	1,412.61	29.79			762.24	1,349.38	1,457.27	31.90	

Table 4.3 tests, under the CEV model, the in-out parity given in Propositions 4.3 and 4.4 for American-style options. Columns 1 through 4 show the strike price, the beta value, the time to expiration, and lower barrier level, respectively. Under Proposition 4.3, the dividend yield is $q = 3\%$ for Panels A and B, and $q = 9\%$ for Panel C. Columns 5 through 8 implement equations (4.5), (4.3), (4.8) and (4.11), respectively, using the SHP approach. Column 9 checks Proposition 4.3 by presenting the difference between both sides of equation (4.13). Column 10 shows the upper barrier level used in Proposition 4.4. Under Proposition 4.4, the dividend yield is $q = 9\%$ for Panels A and B, and $q = 3\%$ for Panel C. Columns 11 through 14 implement equations (4.21), (4.3), (4.24) and (4.28), respectively, using the SHP approach. Column 15 checks Proposition 4.4 by presenting the difference between both sides of equation (4.27). Finally, the last line gives the CPU time in seconds taken to value the whole set of contracts considered.

Table 4.4: American-style double knock-out options under the CEV model ($S_{t_0} = 100$; $r = 6\%$; $\sigma(t_0, S_{t_0}) = 0.2$)

			American-style put				American-style call			
K	β	$T - t_0$	L	U	AUO	P4.5	L	U	ADO	P4.6
					Panel A					
95	3	0.5	70	110	2.582	2.582	90	140	6.675	6.675
100	3	0.5	70	110	4.402	4.402	90	140	4.563	4.563
105	3	0.5	70	110	6.673	6.673	90	140	2.973	2.973
95	3	1	70	110	3.463	3.463	90	140	6.922	6.922
100	3	1	70	110	5.016	5.016	90	140	5.316	5.316
105	3	1	70	110	6.849	6.849	90	140	4.030	4.030
95	1	0.5	70	110	2.724	2.724	90	140	6.677	6.677
100	1	0.5	70	110	4.467	4.467	90	140	4.504	4.504
105	1	0.5	70	110	6.680	6.680	90	140	2.830	2.830
95	1	1	70	110	3.674	3.674	90	140	6.893	6.893
100	1	1	70	110	5.135	5.135	90	140	5.196	5.196
105	1	1	70	110	6.882	6.882	90	140	3.811	3.811
					Panel B					
95	3	0.5	87.5	110	2.582	2.415	90	115	6.675	6.674
100	3	0.5	87.5	110	4.402	4.378	90	115	4.563	4.535
105	3	0.5	87.5	110	6.673	6.673	90	115	2.973	2.786
95	3	1	87.5	110	3.463	2.866	90	115	6.922	6.922
100	3	1	87.5	110	5.016	4.832	90	115	5.316	5.106
105	3	1	87.5	110	6.849	6.849	90	115	4.030	3.362
95	1	0.5	87.5	110	2.724	2.472	90	115	6.677	6.677
100	1	0.5	87.5	110	4.467	4.419	90	115	4.504	4.493
105	1	0.5	87.5	110	6.680	6.680	90	115	2.830	2.717
95	1	1	87.5	110	3.674	2.892	90	115	6.893	6.893
100	1	1	87.5	110	5.135	4.863	90	115	5.196	5.067
105	1	1	87.5	110	6.882	6.878	90	115	3.811	3.324
					Panel C					
95	3	0.5	80	110	3.420	3.385	90	120	8.204	8.107
100	3	0.5	80	110	5.536	5.511	90	120	5.867	5.737
105	3	0.5	80	110	8.036	8.018	90	120	3.994	3.796
95	3	1	80	110	5.003	4.727	90	120	9.377	8.912
100	3	1	80	110	6.879	6.692	90	120	7.591	6.916
105	3	1	80	110	8.914	8.791	90	120	6.013	5.042
95	1	0.5	80	110	3.579	3.521	90	120	8.184	8.106
100	1	0.5	80	110	5.617	5.579	90	120	5.790	5.692
105	1	0.5	80	110	8.064	8.039	90	120	3.829	3.691
95	1	1	80	110	5.246	4.840	90	120	9.309	8.908
100	1	1	80	110	7.026	6.763	90	120	7.434	6.883
105	1	1	80	110	8.981	8.816	90	120	5.746	4.983
CPU					2,798.87	1,607.51			2,898.60	1,727.81

Table 4.4 tests Propositions 4.5 and 4.6 under the CEV model. Columns 1 through 5 show the strike price, the beta value, the time to expiration, the lower barrier level, and the upper barrier level, respectively. Under Proposition 4.5, the dividend yield is $q = 3\%$ for Panels A and B, and $q = 9\%$ for Panel C. Columns 6 and 7 implement equations (4.21) and (4.49), respectively. Columns 8 and 9 show the lower and upper barrier levels used in Proposition 4.6. Under Proposition 4.6, the dividend yield is $q = 9\%$ for Panels A and B, and $q = 3\%$ for Panel C. Columns 10 and 11 implement equations (4.5) and (4.49), respectively. Finally, the last line gives the CPU time in seconds taken to value the whole set of contracts considered.

5. Conclusion

This thesis provides important results concerning the valuation of standard and barrier American-style options in three separate articles.

The most important theoretical contribution of the first paper is the generalization of the SHP procedure for valuing American-style standard and capped options under the JD-CEV model of Carr and Linetsky (2006). To accomplish this purpose, novel analytical representations were obtained for the hedge ratios of the corresponding European-style standard options, which can be used to jointly price equity and credit derivatives under this general and flexible modeling framework. The SHP approach is also implemented to price American-style standard and capped options under the unrestricted CEV model, thus accommodating both direct and indirect leverage effects typically observed by market practitioners. Furthermore, we extend the optimal stopping approach of Nunes (2009) for the pricing of American-style capped options, assuming that the recovery value associated to the put can be paid at the default time or at the maturity date of the option. Overall, the numerical experiments run have shown that the SHP pricing methodology is as accurate as but (generally) faster than the optimal stopping approach, thus offering a better speed-accuracy trade-off for pricing American-style standard and capped options under both the (single-factor) CEV and JDCEV models.

In the second paper, using the change of numéraire technique proposed by Geman et al. (1995) and used by Schroder (1999), we are able to derive the put-call symmetry for American-style single and double barrier options without imposing a previous assumption on the stochastic process followed by the underlying asset price. Our results are validated by applying the proposed symmetries to the CEV model.

Finally, the third paper provides three main contributions to the existent literature on option pricing. First, we derive new in-out parity relations for American-style puts with a down barrier and American-style calls with an up barrier. Second, and more importantly, we propose a novel representation for the early exercise boundary of American-style double knock-out options in terms of its barrier levels and as a function of the simpler optimal stopping boundary of a single barrier American-style option. Hence, we are able to reduce the valuation of American-style double barrier options to the same complexity level as the one faced to price the nested single barrier contracts. Finally, we extend the SHP approach, for the first time to the authors knowledge, to the valuation of American-style double barrier knock-out options. We run an extensive numerical study—covering a large range of single and double barrier option contracts that can be used as a benchmark for future empirical analysis—and successfully validate all our results under both the GBM and CEV processes.

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