Some Recent Developments in Necessary Conditions of Optimality for Impulsive Control Problems

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There is already a long history in the optimization of trajectories for space systems by using the optimal impulsive control framework, [17, 24, 28]. In those early days, the minimization of fuel consumption was one of the main problems and this paradigm ensured the existence of the solution to the associated optimal control problem.

Since then, the diversity of optimal control problems grew substantially not only in what concerns the control objectives but also the classes of systems to be controlled. In particular, the knowledge improvement of the intervening phenomena enabled the incorporation of more sophisticated and complex models.

Thus, form the initially considered problems of midcourse guidance of space vehicles, [17, 20, 24, 28], the impulsive control formulation has been used for rendezvous problems of spacecraft, [9, 11, 12, 15, 18, 19, 35], automated docking, [15, 22], collision-avoidance trajectories, [8], precision guided munitions [5],control of formation flying spacecraft [7, 8, 29, 33], to name just a few.

Moreover, the recent extraordinary development of the various enabling technologies call for more powerful optimal control results required to address the new emerging control problems which involving more diversified objective functions, devices with more complex dynamics and satisfying more complex state and control constraints models, [13, 18, 22, 30, 35]. Among these, one should emphasize systems where activities require the cooperation of several vehicles and devices, such as formation flying, and cooperative docking, rendezvous and obstacle avoidance, [7, 9, 22], for which the hybrid control systems framework may open new perspectives.

In this article, we consider the following very general class of nonlinear impulsive dynamic control systems encompassing many of the above classes of applications.

$$dx(t) \in F(t, x(t))dt + \mathbf{G}(t, x(t))d\mu(t) \qquad \forall t \in [0, 1] \tag{1}$$

$$d\mu \in \mathcal{K} \tag{2}$$

where $F : [0, 1] \times I\!R^n \to \mathcal{P}(I\!R^n)$, and $G : [0, 1] \times I\!R^n \to \mathcal{P}(I\!R^{n\times k})$ are given set-valued maps, and $\mathcal{K} \subset C^*([0,1];K)$ is the set of control measures supported on $[0,1]$ with range in a given set $K \subset I\!R^k$. Note that the generality of the considered control measure allows this paradigm to encompass problems with conventional dynamics.

Several remarks are in order.

First, by considering $F(t,x) := \{f(t,x,u) : u \in \Omega\}$ and $\mathbf{G}(t,x) := \{G(t,x,u) : u \in \Omega\}$, where the measurable function u is the conventional control takes values in a given compact set $\Omega \subset I\!R^m$, and f and G are continuous in u, we address the class of impulsive optimal control problems where dynamics are specified by controlled differential equations. This allows the consideration of problems for which "conventional control action during an impulse" can be considered as an idealization. The main issue that arises here steams from the fact that, in this paradigm, the selection from the set-valued functions F and $\mathbf{G}$ on the support of the absolutely continuous component of the control measure have to correspond to the same conventional control. This issue is addressed in [27]. This optimal control problem has also been considered in [21], but the stated optimality conditions are of different character.

Second, another point of interest is the pertinence of this paradigm to hybrid control systems[1] that has already been perceived by a number of authors, [4, 6]. The current great interest in hybrid systems steams from the emergence of advanced controlled systems - for example, systems involving multiple autonomous devices and vehicles able to coordinate their activities in order to achieve the system's goals while satisfying given performance requirements - for which enabling technologies have been developing dramatically. To see the relation between hybrid automata, a popular model for hybrid systems, and the measure driven differential equation, just consider $x = col(y,z)$, a certain index set A, and $\mathcal{Z} = \{z_\alpha : \alpha \in A\}$, and note that the impulsive system

$$\begin{cases} \dot{y} = f(y,z,u), \quad u \in \Omega \\ dz = g(y,z)d\mu \end{cases}$$

models an hybrid system specified by a collection of conventional systems $\{\dot{y} = f_\alpha(y,u),\ u \in \Omega : \alpha \in A\}$, where $f_\alpha(y,u) := f(y,z_\alpha,u)$, and A is a given discrete set, being the evolution of the discrete variable α dictated by a transition automaton.

Third, the impulsive control paradigm can be regarded as an idealization of systems with fast and slow dynamics. This is pertinent to many of above mentioned advanced systems which are designed to perform multi-phase missions. The execution of a mission may imply the switching between different configurations of the system, being

[1]Systems whose state evolution is dictated by the interaction of time-driven dynamics and event-driven discrete dynamics.

each mission phase modeled by slow dynamics and the switching transition operation by fast dynamics. By considering the impulsive idealization, the overall system control optimization, encompassing both short and fast dynamics, can be formulated in the context of the paradigm considered in this article.

In spite of a, by now, large body of results on the optimal control of impulsive systems (see [10, 16, 21, 23, 26, 32, 34, 36] for a selected sample), in this article, we examine necessary conditions of optimality, recently derived in [2, 3, 25, 27], for which the concept of robust solution, [25, 31], plays a key role in their derivation and interpretation. The basic idea of this concept consists in filling in the "gaps" of the graph of trajectory of bounded variation by arcs that satisfy the singular (with respect to the Lebesgue measure) dynamics. This completion of the graph of a trajectory of bounded variation requires a time reparameterization so that the flow of the new time variable reflects, at each moment, the sum of the contributions of the original time and of the control measure variation. In particular, this yields the emergence of nonzero measure intervals whenever there is a discontinuity in the state trajectory, thus enabling the definition of an "equivalent" trajectory solution to an auxiliary conventional differential inclusion.

Let us assume that all the points in K have positive components. Define the reparameterization map $t \to \eta(t) := t + M(t)$, being $M(t) = \sum_{i=1}^{k} \int_{[0,t]} \mu_i(d\tau)$ for $t > 0$, and $M(0) = 0$, as well as the set-valued map $\bar{\eta}(t) := [\eta(t^-), \eta(t)]$.

Definition A *graph completion* of the vector-valued measure μ is any pair $(\theta, \gamma) : [0,1] \to I\!R^+ \times K$, where $\theta : [0,1] \to I\!R^+$ is the "inverse" of $\bar{\eta}$ in the sense that $\theta(s) = t$, $\forall s \in \bar{\eta}(t)$ and γ is defined $\forall s \in \bar{\eta}(t)$, $\forall t \in [0,1]$, by

$$\gamma(s) := \begin{cases} M(\theta(s)) & \text{if } \bar{\mu}(\{t\}) = 0 \\ M(t^-) + \int_{\eta(t^-)}^{s} \mathrm{v}(\sigma) d\sigma & \text{if } \bar{\mu}(\{t\}) > 0, \end{cases}$$

for some $\mathrm{v}(\cdot) \in V^t$, where $t = \theta(s)$ and V^t is the set of functions $\mathrm{v} : \bar{\eta}(t) \to K$ satisfying $\int_{\bar{\eta}(t)} \mathrm{v}(s) ds = \mu(\{t\})$, and $\dot{\theta}(s) + \sum_{i=1}^{k} \mathrm{v}_i(s) = 1$, $\forall s \in \bar{\eta}(t)$.

Definition. A trajectory x, with $x(0) = x_0$, is a robust solution to (1) if, $\forall\, t \in [0,1]$, $x(t) = x_{ac}(t) + x_s(t)$, where

$$\begin{cases} \dot{x}_{ac}(t) \in F(t, x(t)) + \mathbf{G}(t, x(t)) \cdot w_{ac}(t), \quad \mathcal{L}\text{a.e.} \\ x_s(t) = \int_{[0,t]} G_c(\tau) w_c(\tau) d\bar{\mu}_{sc}(\tau) + \int_{[0,t]} g_a(\tau) d\bar{\mu}_{sa}(\tau). \end{cases}$$

Here, $\bar{\mu}$ is the total variation measure associated with μ, and the μ_{sc}, μ_{sa} and μ_{ac} are, respectively, the singular continuous, the singular atomic, and the absolutely continuous components of μ. Moreover, w_{ac} is the time derivative of μ_{ac}, w_{sc} is the Radon-

Nicodym derivative of μ_{sc} with respect to its total variation, $G_c(\cdot)$ is a $\bar{\mu}_{sc}$ measurable selection of $\mathbf{G}(\cdot, x(\cdot))$ and $g_a(\cdot)$ is a $\bar{\mu}_{sa}$ measurable selection of the multifunction $\tilde{G}(t, x(t^-); \mu(\{t\}))$ whose values are the set of all vectors $\zeta = \dfrac{\xi_t(\eta(t)) - x(t^-)}{\bar{\mu}_{sa}(\{t\})} \in I\!R^n$ for which the process $(\xi(\cdot), \gamma(\cdot), \mathrm{v}(\cdot))$ satisfies:

$$\begin{cases} \dot{\xi}_t(s) = G(t, \xi_t(s))\mathrm{v}(s), \\ \dot{\gamma}(s) = \mathrm{v}(s), \end{cases} \quad \mathrm{v}(s) \in V^t, \text{ a.a. } s \in \bar{\eta}(t), \text{ with } \begin{cases} \xi_t(\eta(t^-)) = x(t^-), \\ \gamma(\eta(t^-)) = \mu([0,t)) \end{cases} \tag{3}$$

for some measurable selection $G \in \mathbf{G}$ continuous in t and Lipschitz in x, being the pair (θ, γ) a graph completion of μ with $\theta(s) = t$, $\forall\, s \in \bar{\eta}(t)$. Notice that we treat the trajectories of (1) as path-valued functions. In [25, 31], several properties, notably robustness, of this solution are proved and discussed. These are extremely useful in the derivation of the necessary conditions of optimality.

Now, we present necessary conditions discussed in [27] for the following fixed-time optimal control problem with state and control constraints.

$$(P) \qquad \text{Minimize} \quad h(x(0), x(1)) \tag{4}$$

$$\text{subject to} \quad (1),\ (2),$$

$$(x(0), x(1)) \in C, \text{ and } l(t, x(t)) \leq 0,\ \forall t \in [0,1], \tag{5}$$

where $h : I\!R^n \times I\!R^n \to I\!R$, and $l : [0,1] \times I\!R^n \to I\!R^q$ are given functions Lipschitz continuous in the state variable , $C \subset I\!R^n \times I\!R^n$ is a compact set.

As in [25], the optimality conditions presented for this problem are in the context of nonsmooth analysis. This enables not only the adoption of rather weak assumptions on the data of (P), notably, the mere Hausdorff Lipschitz continuity dependence of the dynamics on the state variable, (see [27] for details), but also the usage of powerful penalization and variational methods in the derivation of the optimality conditions, [14, 34]. However, this framework requires a number of concepts such as proximal normal vector, proximal and the limiting normal cones to C at c (respectively, $N_C^P(c)$ and $N_C^L(c)$), the Clarke and limiting generalized gradients of f at x (respectively, $\partial f(x)$ and $\partial^L f(x)$), whose definitions and properties we refer to [14].

Theorem. *Necessary Conditions of Optimality.*
Let (x^*, u^*, μ^*) be a solution to (P).
Then, there exists a multiplier $(\lambda, \zeta, q, \gamma, \nu, \bar{\zeta}^1)$, with $\lambda \geq 0$, $\zeta = \zeta^C + \zeta^h \in I\!R^n \times I\!R^n$, $\zeta^C = \mathrm{col}(\zeta^{C,0}, \zeta^{C,1}) \in N_C^L(x^*(0), x^*(1))$, $\zeta^h = \mathrm{col}(\zeta^{h,0}, \zeta^{h,1}) \in \partial^L h(x^*(0), x^*(1))$, a vector $\bar{\zeta}^1 \in I\!R^n$, $q \in BV([0,1]; I\!R^n)$, a set-valued measure ν and set-valued measur-

able function γ defined by

$$(d\nu(t),\gamma(t))=\begin{cases}\{(d\nu_c(t),\gamma_c(t))\} & \text{on } \mathrm{Supp}(\mu_c^*)\\ \{(d\tilde{\nu}_t(s),\tilde{\gamma}_t(s)):s\in\bar{\eta}(t)\} & \text{on } \mathrm{Supp}(\mu_a^*),\end{cases}$$

being $\nu_c\in C^+([0,1];I\!R^k)$ supported on the set S_0^{l,μ_c^*}, $\gamma_c(\cdot)$ a measurable selection of $\partial_x l(\cdot,x^*(\cdot))$ defined on S_0^{l,μ_c^*} and, for $\forall\, t\in S_0^{l,\mu_a^*}$, $\tilde{\nu}_t\in C^+(\bar{\eta}(t);I\!R^k)$ supported on the set $S_{0,t}^{l,\mu_a^*}\subset\bar{\eta}^*(t)$, and $\tilde{\gamma}_t(\cdot)$ a measurable selection of $\partial_x l(t,\xi_t^*(\cdot))$ defined on $S_{0,t}^{l,\mu_a^*}$, satisfying the following conditions:

$$\lambda+\|q\|_{BV}+\|\nu\|_{TV}\neq 0 \tag{6}$$

For all $t\in[0,1]$,

$$\begin{aligned}(-dq(t),dx(t))\in\partial[H_F(\Xi^*(t))+H_{\mathbf{G}}(\Xi^*(t))w_{ac}^*(t)]dt+\partial H_{\mathbf{G}}(\Xi^*(t))d\mu_{sc}^*(t)\\ +\overline{\partial H_{\mathbf{G}}}(\Xi^*(t);\mu_a^*(\{t\}))d\bar{\mu}_a^*(t),\end{aligned} \tag{7}$$

$$\zeta=(q(0),-q(1)-\bar{\zeta}^1) \tag{8}$$

$$0\geq\sigma_K(H_{\mathbf{G}}(\Xi^*(t))),\qquad \forall\, t\in[0,1] \tag{9}$$

$$0=\sigma_K(H_{\mathbf{G}}(\Xi^*(t)))=H_{\mathbf{G}}(\Xi^*(t))\mathrm{v}_{sc}^*(t),\ \ \mu_{sc}^*\text{-a.e.} \tag{10}$$

Moreover, $\forall\, t\in\mathrm{Supp}(\mu_a^*)$, the last term in the right hand side of (7) yields $(\xi_t,\chi_t,\mathrm{v}_t^*):\bar{\eta}(t)\to I\!R^n\times I\!R^n\times I\!R^k$ satisfying, a.e. on $\bar{\eta}(t)$,

$$(-\dot{\chi}_t(s),\dot{\xi}_t(s))\in\partial H_{\mathbf{G}}(t,\xi_t(s),\bar{\chi}_t(s;\tilde{\gamma}_t,\tilde{\nu}_t))\mathrm{v}_t^*(s) \tag{11}$$

$$0=\sigma_K(H_{\mathbf{G}}(t,\xi_t(s),\bar{\chi}_t(s;\tilde{\gamma}_t,\tilde{\nu}_t))=H_{\mathbf{G}}(t,\xi_t(s),\bar{\chi}_t(s;\tilde{\gamma}_t,\tilde{\nu}_t))\mathrm{v}_t^*(s), \tag{12}$$

with $\mathrm{v}_t^*\in V^t$ such that $\displaystyle\int_{\bar{\eta}(t)}\mathrm{v}_t^*(s)ds=\mu_a^*(\{t\})$, and

$$\begin{cases}(\xi_t,\chi_t)(\eta(t^-))=(x^*(t^-),\bar{q}(t;\gamma,\nu))\\ (x^*(t),\bar{q}(t;\gamma,\nu)+\gamma(t)\nu(\{t\}))=(\xi_t(\eta(t)),\bar{\chi}_t(\eta(t);\tilde{\gamma}_t,\tilde{\nu}_t)+\tilde{\gamma}_t(\eta(t))\tilde{\nu}_t(\eta(t))).\end{cases} \tag{13}$$

Here, the generalized derivative in (7) and (11) are taken with respect to the state and the adjoint variables, $\sigma_K(\zeta)=\sup_{c\in C}\{\langle\zeta,c\rangle\}$ is the support function function of the set C at ζ, Supp(ν) denotes the support of the measure ν, and $\bar{\nu}$ its total variation measure. Moreover,

- $d\mu^*(t)=d\mu_a^*(t)+d\mu_{sc}^*(t)+w_{ac}^*(t)dt$ is the canonical decomposition of the measure μ^*, being the continuous part denoted by μ_c^*, and v_{sc}^* the Radon-Nicodym derivative of μ_{sc}^* with respect to $\bar{\mu}_{sc}^*$,
- $\Xi^*(t)=(t,x^*(t),\bar{q}(t;\gamma,\nu))$, where, $\forall\, t\in[0,1]$, $\bar{q}(t;\gamma,\nu)=q(t)+\displaystyle\int_{[0,t)}\gamma(\tau)d\nu(\tau)$,

- $H_F(\Xi^*(t)) = \max_{\mathrm{v}\in F(t,x^*(t))} \{\langle \bar{q}(t;\gamma,\nu), \mathrm{v}\rangle\}$, and $H_{\mathbf{G}}(\Xi^*(t))$ is given by $\{h_{\mathbf{G}}(t)\}$ on $\mathrm{Supp}(\mu_c^*)$ and by $\{h_{\mathbf{G}}^t(s) : s\in\bar{\eta}(t)\}$ on $\mathrm{Supp}(\mu_a^*)$, where
$$\langle h_{\mathbf{G}}(t), w_{sc}^*(t)\rangle = \max_{w\in K, G\in\mathbf{G}} \{\langle \bar{q}(t;\gamma,\nu), G(t,x^*(t))w\rangle\},$$
$$\langle h_{\mathbf{G}}^t(s), \mathrm{v}_t^*(s)\rangle = \max_{\mathrm{v}\in V^t, G\in\mathbf{G}} \{\langle \bar{\chi}_t(s;\tilde{\gamma}_t,\tilde{\nu}_t), G(t,\xi_t^*(s))\mathrm{v}\rangle\},$$
- $\bar{\zeta}^1$ is given by $\gamma_c(1)\nu_c(\{1\})$ if $\mu_a^*(\{1\}) = 0$ and by $\int_{\bar{\eta}(1)} \tilde{\gamma}_1(s)d\tilde{\nu}_1(s)$ otherwise,
- $S_0^{l,\nu}=\{t\in[0,1]:l(t,x^*(t))=0\}\cap\mathrm{Supp}(\nu)$, and $\forall t\in S_0^{l,\mu_a^*}, S_{0,t}^{l,\mu_a^*}=\{s\in\bar{\eta}(t):l(t,\xi_t^*(s))=0\}$ and, $\forall s\in\bar{\eta}(t)$, $\bar{\chi}(s;\tilde{\gamma}_t,\tilde{\nu}_t) = \chi(s) + \int_{[\eta(t^-),s)} \tilde{\gamma}_t(\sigma)d\tilde{\nu}_t(\sigma)$.

Remark that the solution to (7) is precisely in the sense of the concept of robust solution to a measure driven differential inclusion presented in the previous section.

Although (10) and (12) seem to characterize only the support of the optimal control measure, these conditions together with remaining ones and the problem constraints allow for a full determination of the optimal control process. Moreover, these conditions also provide a detailed characterization of the paths joining the endpoints of any trajectory jump, thus offering a lot of information to support the analysis and the computation of solutions. These conditions extend the ones in [25] in that state constraints are, now, included, and nondegeneracy is ensured through regularity and controlability assumptions.

In this article, we will also discuss the nondegenerate first-order necessary conditions of optimality derived in [2]. The fact that a free time impulsive control problem with state constraints, besides the control constraints and the nonlinear equality and inequality endpoint state constraints, is considered in this reference brings in not only a lengthier statement of the conditions but also much more complex technical issues in their proof whose methods rely strongly on [1]. A key issue of this result is the nondegeneracy of the obtained conditions. However, the hypotheses assumed on the data of the control problem are smoother than those of (P), and the dynamics are given by a controlled differential equation, for which the vector fields multiplying the control measure depend on the time and state variables only.

A fixed-time optimal control problem without state or control constraints with dynamics similar to those in [2] satisfying even smoother assumptions[2] is considered in [3], where first-order and second-order necessary conditions of optimality are derived. On important point in this work is that second-order information is used in order to select a subset of multipliers from all those satisfying the first-order conditions so that nondegeneracy is ensured, thus, dispensing with any a priori normality assumptions.

[2]as well as the commutativity of the singular vector fields

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