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RESEARCH ARTICLE

τ-complemented and τ-supplemented modules

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Abstract. Proper classes of monomorphisms and short exact sequences were introduced by Buchsbaum to study relative homological algebra. It was observed in abelian group theory that complement submodules induce a proper class of monomorphisms and this observations were extended to modules by Stenström, Generalov, and others. In this note we consider complements and supplements with respect to (idempotent) radicals and study the related proper classes of short exact sequences.

1. Proper classes and τ-supplements

With the intention of formalising the theory of Ext functors depending on a specific choice of monomorphisms, Buchsbaum introduced in [4] certain conditions on a class of monomorphisms which are needed to study *relative homological algebra.* This lead to he notion of *proper classes* of monomorphisms and short exact sequences. Well-known examples of such classes are the *pure exact sequences* which can be defined by choosing a class $\mathcal{P}$ of (finitley presented) modules and considering those sequences on which $\mathrm{Hom}(P, -)$ is exact for each $P \in \mathcal{P}$. These techniques are outlined, for example, in Mishina and Skornjakov [10], Sklyarenko [13] and [16]. It was observed in abelian group theory that complement submodules induce a proper class of short exact sequences and this motivated the investigation of such questions for modules over arbitrary rings. First results in this direction were obtained, for example, by Stenström [14] and Generalov [5, 7]. For a more comprehensive presentation of the results and the sources we refer to E. Mermut's PhD thesis [9].

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Our interest in this approach to the structure theory of modules is based on an observation mentioned in Stenström [14], namely that over any ring, supplement submodules induce a proper class of short exact sequences. This may help to bring some order in the variations and generalisations of supplemented and lifting modules coming up recently by involving properties from torsion theory.

1.1. Proper classes. Let $\mathbb{E}$ be a class of short exact sequences in $\sigma[M]$. If

$$0 \longrightarrow K \xrightarrow{f} L \xrightarrow{g} N \longrightarrow 0$$

belongs to $\mathbb{E}$, then f is called an $\mathbb{E}$-*mono* and g is said to be an $\mathbb{E}$-*epi*. The class $\mathbb{E}$ is called *proper* if it satisfies the conditions

P.1 $\mathbb{E}$ is closed under isomorphisms;

P.2 $\mathbb{E}$ contains all splitting short exact sequences in $\sigma[M]$;

P.3 the class of $\mathbb{E}$-monos is closed under composition;
if f', f are monos and $f' \circ f$ is an $\mathbb{E}$-mono, then f is an $\mathbb{E}$-mono;

P.4 the class of $\mathbb{E}$-epis is closed under composition;
if g, g' are epis and $g \circ g'$ is an $\mathbb{E}$-epi, then g is an $\mathbb{E}$-epi.

The class of all splitting short exact sequences in $\sigma[M]$ is an example of a proper class.

1.2. Purities. Let $\mathcal{P}$ be a class of modules in $\sigma[M]$. Denote by $\mathbb{E}^{\mathcal{P}}$ the class of all short exact sequences in $\sigma[M]$ on which $\mathrm{Hom}(P, -)$ is exact for each $P \in \mathcal{P}$. It is straightforward to prove that $\mathbb{E}^{\mathcal{P}}$ is a proper class. This type of class is called *projectively generated* and its elements are called $\mathcal{P}$-*pure sequences.*

The "classical" purity is obtained by taking for $\mathcal{P}$ all finitely presented modules in $\sigma[M]$; for $M = R$ this is the *Cohn purity* (e.g., [16, § 33]).

1.3. Copurities. Let $\mathcal{Q}$ be a class of modules in $\sigma[M]$. Denote by $\mathbb{E}_{\mathcal{Q}}$ the class of all short exact sequences in $\sigma[M]$ on which $\mathrm{Hom}(-, Q)$ is exact for each $Q \in \mathcal{Q}$. Then $\mathbb{E}_{\mathcal{Q}}$ is a proper class. This type of class is called *injectively generated* and its elements are called $\mathcal{Q}$-*copure sequences* (see [16, § 38]).

1.4. Relative injectivity and projectivity. Let $\mathbb{E}$ be a proper class of short exact sequences in $\sigma[M]$.

A module $P \in \sigma[M]$ is called $\mathbb{E}$-*projective* if $\mathrm{Hom}(P, -)$ is exact on all short exact sequences in $\mathbb{E}$.

Dually, a module $Q \in \sigma[M]$ is called $\mathbb{E}$-*injective* if $\mathrm{Hom}(-, Q)$ is exact on all short exact sequences in $\mathbb{E}$.

It follows from standard arguments that the class of all $\mathbb{E}$-projective modules is closed under direct sums and direct summands, and the class of all $\mathbb{E}$-injective modules is closed under direct products and direct summands.

1.5. Class of complement submodules. *Let $\mathbb{E}_c$ be the class of short exact sequences $0 \to K \to L \to N \to 0$ in $\sigma[M]$ such that K is a complement (closed) submodule of L. Then:*

(1) *$\mathbb{E}_c$ is a proper class in $\sigma[M]$.*

(2) *Every (semi-) simple module (in $\sigma[M]$) is $\mathbb{E}_c$-projective.*

Proof. (1) The conditions P.1 and P.2 are easily verified.

P.3. Let $K \subseteq L$ and $L \subseteq N$ be closed submodules. Choosing $\widehat{K} \subseteq \widehat{L} \subseteq \widehat{N}$, we have

$$K = \widehat{K} \cap L = \widehat{K} \cap \widehat{L} \cap N = \widehat{K} \cap N,$$

proving that K is closed in N.

Consider $K \subseteq L \subseteq N$. If K is closed in N, then $K = \widehat{K} \cap N = \widehat{K} \cap L$, that is, K is closed in L.

P.4. The composition of two epimorphisms g and h can be presented by the commutative diagram with exact rows and columns,

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "="] & U \arrow[r] \arrow[d, "i"] & U/K \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & L \arrow[r, "g"] \arrow[d, "gh"'] & L/K \arrow[r] \arrow[d, "h"] & 0 \\
& & L/U \arrow[r, "="] & L/U, &
\end{tikzcd}$$

where $U/K = \operatorname{Ke} h$, for some $K \subseteq U \subseteq L$.

Assume g and h to have closed kernels. Suppose that $U = \operatorname{Ke} gh$ is not closed in L and denote by $\overline{U}$ the essential closure of U in L. Then K is closed in $\overline{U}$ and $\operatorname{Ke} h = U/K \trianglelefteq \overline{U}/K$, contradicting the assumption that h has a closed kernel.

On the other hand, assume $U = \operatorname{Ke} gh$ to be closed. Suppose that $\operatorname{Ke} h$ has a proper essential extension V in L/K. Then $U = (\operatorname{Ke} h)g^{-1} \trianglelefteq (V)g^{-1}$, contradiction our condition on $\operatorname{Ke} h$. This shows that $\operatorname{Ke} h$ is a closed submodule of L/K.

(2) Let $K \subseteq L$ be a closed submodule and $S \subseteq L/K$ any simple submodule. Then $S \simeq N/K$ for some submodule $K \subseteq N \subseteq L$. By

assumption, K is a maximal submodule and is not essential in N. Hence the map $N \to N/K \simeq S$ splits showing that $\mathrm{Hom}(S, L) \to \mathrm{Hom}(S, L/K)$ is surjective and that S is $\mathbb{E}_c$-projective.

Since direct sums of $\mathbb{E}_c$-projectives are again $\mathbb{E}_c$-projective, every semisimple module is $\mathbb{E}_c$-projective. □

1.6. τ-complement submodules. *Let τ be an idempotent preradical for $\sigma[M]$ with associated classes $\mathbb{T}_\tau$ and $\mathbb{F}_\tau$. Then for a submodule $K \subseteq L$ where $L \in \sigma[M]$, the following are equivalent:*

(a) *every $N \in \mathbb{T}_\tau$ is projective with respect to the projection $L \to L/K$;*

(b) *there exists a submodule $U \subseteq L$ such that*

$$K \cap U = 0 \text{ and } \tau(L/K) = (U + K)/K \simeq U;$$

(c) *there exists a submodule $U \subseteq L$ such that*

$$K \cap U = 0 \text{ and } \tau(L/K) \subseteq (U + K)/K \simeq U.$$

If this conditions are satified, then K is called a *τ-complement in L.*

Proof. (a)⇒(b) A pullback construction yields the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "i_K"] \arrow[d, "="] & \widetilde{K} \arrow[r] \arrow[d] & \tau(L/K) \arrow[r] \arrow[d, "i"] & 0 \\
0 \arrow[r] & K \arrow[r] & L \arrow[r, "g"] & L/K \arrow[r] & 0.
\end{tikzcd}$$

Since $\tau(L/K) \in \mathbb{T}_\tau$, there exists a morphism $h : \tau(L/K) \to L$ with $i = hg$. By the Homotopy Lemma (e.g., [16, 7.16]), this implies that the top row splits, that is,

$$\widetilde{K} = K \oplus U, \text{ for some } U \subseteq \widetilde{K} \text{ and } (K + U)/K = \tau(L/K).$$

(b)⇒(a) Let $N \in \mathbb{T}_\tau$ and $f \in \mathrm{Hom}(N, L/K)$. Then $\mathrm{Im}\, f \subseteq \tau(L/K)$ and it can be seen from the diagram in the proof of (a)⇒(b) that there is a morphism $h : N \to L$ with $f = hg$.

(b)⇔(c) This is easy to verify. □

1.7. Corollary. *Let τ be a preradical for $\sigma[M]$ and $K \subseteq L$ where $L \in \sigma[M]$.*

(1) *If K is a τ-complement in L and $L/K \in \mathbb{T}_\tau$, then K is a direct summand.*

(2) *If $L \in \mathbb{T}_\tau$, then every τ-complement submodule of L is a direct summand.*

For the preradical induced by the class of all (semi-)simple modules we find an interesting relationship with the complement submodules.

1.8. Neat submodules. A monomorphism $f : K \to L$ is called *neat* if any simple module S is projective relative to $L \to L/\operatorname{Im} f$, that is, the Hom sequence $\operatorname{Hom}(S, L) \to \operatorname{Hom}(S, L/K) \to 0$ is exact. The class of short exact sequences with neat monomorphisms is a projectively generated class in the sense of 1.2.

As shown in 1.5, all sequences in $\mathbb{E}_c$ are neat.

1.9. When neat submodules are closed in $\sigma[M]$. *For a module M the following are equivalent:*

(a) *every neat submodule of M is closed;*

(b) *a submodule of M is closed if and only if it is neat;*

(c) *for every $L \in \sigma[M]$, closed submodules of L are neat;*

(d) *for every essential submodule $U \subseteq M$, $\operatorname{Soc} M/U \neq 0$;*

(e) *every M-singular module is semi-artinian.*

Proof. (a)⇔(b) is clear since closed submodules are neat.

(c)⇒(a) is obvious.

(a)⇒(d) Let $U \trianglelefteq M$ be a proper submodule. Then U is not closed and hence not neat in M. Thus there exists a morphism $g : S \to M/U$ where S is simple, that can not be extended to a morphism $S \to M$. In particular, this implies that $\operatorname{Im} g \neq 0$, that is, $\operatorname{Soc} M/U \neq 0$.

(d)⇒(e) Let $U \subseteq V \subseteq M$. If $U \trianglelefteq M$ then $V \trianglelefteq M$ and hence (d) implies that every factor module of M/U has nonzero socle, that is, M/U is semi-artinian (see [6, 3.12]).

By [6, Proposition 4.3], the set $\{M/U \,|\, U \trianglelefteq M\}$ is a generating set of all M-singular M-generated modules and every M-singular module is a submodule of M-generated M-singular modules. Thus if all the M/U have nonzero socles then this is also true for all M-singular modules.

(e)⇒(c) Let $K \subseteq L$ be a neat submodule and assume that it has a proper essential extension $\overline{K} \subseteq L$. Then $\overline{K}/K$ is an M-singular module and hence, by assumption, contains a simple submodule $S = N/K$ where $K \trianglelefteq N \subseteq \overline{K}$. Now neatness of $K \subseteq L$ implies that the map $N \to N/K = S$ splits. This contradicts $K \trianglelefteq N$ proving that K is closed in L. □

For $M = R$ we obtain the following characterisation which was (partly) proved in [5, Theorem 5]:

1.10. When neat submodules are closed in R-Mod. *For a ring R the following are equivalent:*

(a) *every neat left ideal of R is closed;*

(b) *a left ideal of R is closed if and only if it is neat;*

(c) *for every left R-module, closed submodules are neat;*

(d) *for every essential left ideal $I \subseteq R$, $\operatorname{Soc} R/I \neq 0$;*

(e) *every singular module is semi-artinian.*

Rings with these properties are called *C-rings* (in [12]).

Dualising the notions considered above yields the following.

1.11. τ-supplement submodules. *Let τ be a radical for $\sigma[M]$ with associated classes $\mathbb{T}_\tau$ and $\mathbb{F}_\tau$. Then for a submodule $K \subseteq L$ where $L \in \sigma[M]$, the following are equivalent:*

(a) *every $N \in \mathbb{F}_\tau$ is injective with respect to the inclusion $K \to L$;*

(b) *there exists a submodule $U \subseteq L$ such that*

$$K + U = L \text{ and } U \cap K = \tau(K);$$

(c) *there exists a submodule $U \subseteq L$ such that*

$$K + U = L \text{ and } U \cap K \subseteq \tau(K).$$

If this conditions are satisfied, then K is called a *τ-supplement in L.*

Proof. (a)⇒(b) Consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "i"] \arrow[d, "p"] & L \arrow[r] \arrow[d] & L/K \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & K/\tau(K) \arrow[r] & L/\tau(K) \arrow[r] & L/K \arrow[r] & 0.
\end{tikzcd}$$

Since $K/\tau(K) \in \mathbb{F}_\tau$, there exists $h : L \to K/\tau(K)$ with $p = ih$. By the Homotopy Lemma (e.g., [16, 7.16]), this implies that the bottom row splits, that is,

$$L/\tau(K) = K/\tau(K) \oplus U/\tau(K), \text{ for some } \tau(K) \subseteq U \subseteq L.$$

This means $L = K + U$ and $U \cap K = \tau(K)$.

(b)⇒(a) By the given data, $L/\tau(K) = K/\tau(K) \oplus U/\tau(K)$. Let $N \in \mathbb{F}_\tau$ and $f \in \operatorname{Hom}(K, N)$. Then $\tau(K) \subseteq \operatorname{Ke} f$ and we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "i"] \arrow[d, "p"] \arrow[ddl, "f"'] & L \arrow[r] \arrow[d] & L/K \arrow[r] \arrow[d, "="] & 0 \\
& K/\tau(K) \arrow[r] \arrow[dl, "\bar{f}"] & L/\tau(K) \arrow[r] & L/K \arrow[r] & 0. \\
N & & & &
\end{tikzcd}$$

Since the middle row splits we obtain a morphism $h : L \to N$ with $f = ih$.

(b)⇔(c) One direction is trivial.

Assume $K + U = L$ and $K \cap U \subseteq \tau(K)$. Putting $U' = U + \tau(K)$ we have

$$K + U' = L \text{ and } K \cap U' = K \cap U + \tau(K) = \tau(K).$$

□

1.12. Corollary. *Let τ be a radical for $\sigma[M]$ and $K \subseteq L$ where $L \in \sigma[M]$.*

(1) *If K is a τ-supplement in L and $K \in \mathbb{F}_\tau$, then K is a direct summand.*

(2) *If $L \in \mathbb{F}_\tau$, then every τ-supplement submodule of L is a direct summand.*

(3) *If K is a τ-supplement in L and $X \subseteq K$, then K/X is a τ-supplement in L/X.*

Proof. (1) and (2) follow from the preceding observations. (3) Let $U \subseteq L$ be such that $K + U = L$ and $K \cap U \subseteq \tau(K)$. Then $K/X + (U + X)/X = L/X$ and

$$K \cap (U + X)/X = (K \cap U + X)/X \subseteq (\tau(K) + X)/X \subseteq \tau(K/X).$$

□

As a special case we consider the radical (for $\sigma[M]$) cogenerated by the simple modules.

1.13. Co-neat submodules. A monomorphism $f : K \to L$ is called *co-neat* if any module Q with $\operatorname{Rad} Q = 0$ is injective relative to it, that is, the Hom sequence $\operatorname{Hom}(L, Q) \to \operatorname{Hom}(K, Q) \to 0$ is exact. The class of short exact sequences with co-neat monomorphisms is an injectively generated class in the sense of 1.3.

1.14. Characterisation of co-neat submodules. *For a submodule $K \subseteq L$, the following are equivalent:*

(a) *$K \to L$ is a co-neat submodule;*

(b) *there exists a submodule $U \subseteq L$ such that*

$$K + U = L \text{ and } U \cap K = \operatorname{Rad} K;$$

(c) *there exists a submodule $U \subseteq L$ such that*

$$K + U = L \text{ and } U \cap K \subseteq \operatorname{Rad} K.$$

If these conditions are satisfied, then K is a Rad-supplement in L. If $\operatorname{Rad}K \ll K$, then K is co-neat (Rad-supplement) in L if and only if it is a supplement in L (see [9]).

1.15 Lemma. *A small submodule N of a module L is co-neat in L if and only if* $\operatorname{Rad} N = N$.

Proof. Let $N \ll L$. If N is co-neat in L, then there exists $K \subseteq L$ such that $N + K = L$ and $N \cap K = \operatorname{Rad} N$. Since $N \ll L$, $K = L$ and hence $N = N \cap L = \operatorname{Rad} N$. On the other hand assume $N \ll L$ and $\operatorname{Rad} N = N$. Then $N + L = L$ and $N \cap L = N = \operatorname{Rad} N$, thus N is a co-neat submodule of L. □

1.16. When are co-neat submodules coclosed. *Let M be a module. Then the following conditions are equivalent:*

(a) *every non-zero co-neat submodule of a module in $\sigma[M]$ is a coclosed submodule;*

(b) *every non-zero M-small module in $\sigma[M]$ is a Max module (resp. has a maximal submodule).*

Proof. (a)⇒(b) If $N \ll L$ and $\operatorname{Rad} N = N$, then, by the Lemma 1.15, N is a co-neat submodule of L. But by hypothesis co-neat submodules are coclosed submodules and hence not small - a contradiction.

(b)⇒(a) Let N be a co-neat submodule of a module $L \in \sigma[M]$. Then for any submodule $U \subseteq N$, N/U is co-neat in M/U. To see this let K be a submodule of L such that $N + K = L$ and $N \cap K = Rad(N)$. Then $N/U + (K+U)/U = L/U$ and

$$N/U \;\; (K+U)/U = ((N \;\; K)+U)/U = (\operatorname{Rad} N + U)/U \subseteq \operatorname{Rad} N/U.$$

Hence N/U is co-neat in L/U. Suppose $N/U \ll L/U$, then $\operatorname{Rad} N/U = N/U$ by the Lemma 1.15. But by hypothesis N/U is a Max module, and hence has a proper maximal submodule. Hence $N/U \not\ll L/U$ for all $U \subset N$ implies that N is coclosed in L. □

Let M be a cosemisimple module. Then $\operatorname{Rad} N = 0$ for any $N \in \sigma[M]$. Hence any non-zero submodule of a module in $\sigma[M]$ is coclosed. Moreover any non-zero module is a Max module. However a submodule N of a module L is co-neat if and only if it is a direct summand. Thus if M is not semisimple, there are coclosed submodules which are not co-neat. This shows that in general the dual statement of the statement for neat submodules (1.9 (b)$\Leftrightarrow$(e)) does not hold.

2. τ-supplemented modules

Throughout this section τ will denote a radical for $\sigma[M]$. Recall that a submodule $K \subseteq L$ is called a τ-supplement provided there exists some $U \subseteq L$ such that $U + K = L$ and $U \cap K \subseteq \tau(K)$ (1.11). To some extent the theory of supplemented, lifting and semiperfect modules can be transferred to the corresponding notions based on τ-supplements. This will be sketched in this section.

2.1. Definition. A module L is said to be *τ-supplemented* if every submodule $K \subseteq L$ has a τ-supplement in L, and it is called *amply τ-supplemented* if for any submodules $K, V \subseteq L$ such that $K + V = L$, there is a τ-supplement U for K with $U \subseteq V$.

2.2. τ-supplemented modules. *Let L be a τ-supplemented module in $\sigma[M]$.*

(1) *Every submodule $K \subseteq L$ with $K \cap \tau(L) = 0$ is a direct summand. In particular, if L is τ-torsion-free, then L is semisimple.*

(2) *Every factor module and every direct summand of L is τ-supplemented.*

(3) *$L/\tau(L)$ is a semisimple module.*

(4) *$L = U \oplus N$ where N is semisimple and $\tau(U) \trianglelefteq U$.*

Proof. (1) Recall that $\tau(K) \subseteq K \cap \tau(L)$ and then refer to 1.12.

(2) and (3) are also obvious consequences of 1.12.

(4) Let $N \subseteq L$ be a complement for $\tau(L)$, i.e. $N \cap \tau(L) = 0$ and $N \oplus \tau(L) \trianglelefteq L$. This implies $\tau(N) = 0$. By assumption, there exists $U \subseteq L$ such that $N + U = L$ and $N \cap U \subseteq \tau(U)$. By construction,

$$N \cap U = N \cap (N \cap U) \subseteq N \cap \tau(U) \subseteq N \cap \tau(L) = 0,$$

hence $L = N \oplus U$ and $\tau(L) = \tau(N) \oplus \tau(U) = \tau(U)$. Thus $N \oplus \tau(U) \trianglelefteq N \oplus U$ and this implies $\tau(U) \trianglelefteq U$. By (1), N is semisimple. $\square$

2.3. Sums of τ-supplemented modules. *Let $L \in \sigma[M]$.*

(1) *Let $L_1, U \subseteq L$ be submodules where L_1 is τ-supplemented. If L_1+U has a τ-supplement in L, then so does U.*

(2) *If L_1 and L_2 are τ-supplemented modules in $\sigma[M]$ and $L = L_1+L_2$, then L is τ-supplemented.*

(3) *Any finite sum of τ-supplemented modules is τ-supplemented.*

(4) *If L is τ-supplemented, then every finitely L-generated module is τ-supplemented.*

Proof. (1) By assumption, there exists $X \subseteq L$ such that $(L_1+U)+X = L$ and $(L_1+U) \cap X \subseteq \tau(X)$. Now $(U+X) \cap L_1$ has a τ-supplement in L_1, that is, some $Y \subseteq L_1$ with

$$(U+X) \cap L_1 + Y = L_1 \text{ and } (U+X) \cap Y \subseteq \tau(Y).$$

Since $U + X + Y = L$ we have that Y is a τ-supplement of $U + X$ in L. To prove that $X + Y$ is a τ-supplement of U in L is remains to show that $U \cap (X + Y) \subseteq \tau(X + Y)$. Now $Y + U \subseteq L_1 + U$, hence $X \cap (Y + U) \subseteq X \cap (L_1 + U) \subseteq \tau(X)$ and therefore

$$(X+Y) \cap U \subseteq X \cap (Y+U) + Y \cap (X+U) \subseteq \tau(X) + \tau(Y) \subseteq \tau(X+Y).$$

(2) Let $U \subseteq L$ be any submodule. Then $L_1 + L_2 + U = L$ trivially has a τ-supplement in L and hence, by (1), $L_2 + U$ has a τ-supplement in L. Again by (1), this implies that U has a τ-supplement in L. Thus L is a τ-supplemented module.

(3) and (4) are immediate consequences of (1) and 2.2(2). □

2.4. Amply τ-supplemented modules. *If $L \in \sigma[M]$ is an amply τ-supplemented module, then*

(1) *direct summands of L are amply τ-supplemented and*

(2) *factor modules of L are amply τ-supplemented.*

Proof. (1) Assume $L = K' \oplus K$ and let X, Y be submodules of K with $K = X + Y$. Since $L = K' + X + Y$, there exists $Y' \subseteq Y$ such that $Y'+K'+X = L$ and $Y' \cap (K'+X) \subseteq \tau(Y')$. Now $Y' \cap X \subseteq Y' \cap (X+K') \subseteq \tau(Y')$ and

$$\begin{aligned} K = K \cap L &= K \cap (Y' + (K' \oplus X)) \\ &= K \cap (K' \oplus X) + Y' = X + Y'. \end{aligned}$$

Hence Y' is a τ-supplement of X in K and $Y' \subseteq Y$.

(2) Assume $X \subseteq L$ and $L/X = K/X + K'/X$, where $X \subseteq K \subseteq L$, $X \subseteq K' \subseteq L$. Since $L = K+K'$, there exists $Y \subseteq K'$ such that $K+Y = L$ and $K \cap Y \subseteq \tau(Y)$. We will show that $Y + X/X$ is a τ-supplement of

K/X in L/X. It is clear that $(Y+X)/X + K/X = L/X$ and that $(Y+X)/X \subseteq K'/X$. Also we have

$$\begin{aligned} K/X \cap (Y+X)/X &= ((K \cap Y) + X)/X \\ &\subseteq (\tau(Y) + X)/X \subseteq \tau(Y + X/X). \end{aligned}$$

□

2.5. Corollary. *Let L be amply τ-supplemented in $\sigma[M]$.*

(1) *If K is a τ-supplement in L and $K \in \mathbb{F}_\tau$, then K is amply τ-supplemented.*

(2) *If $L \in \mathbb{F}_\tau$, then every τ-supplement submodule of L is is amply τ-supplemented.*

Proof. (1) and (2) follow directly from 1.12. □

An R-module L is called *π-projective* if for any submodules U, V of L such that $U + V = L$, there exists $f \in \mathrm{End}(L)$ with $\mathrm{Im}(f) \subseteq U$ and $\mathrm{Im}(1-f) \subseteq V$ (see [16, 41.14] for details).

2.6. Proposition. *Let $L \in \sigma[M]$.*

(1) *If every submodule of L is τ-supplemented, then L is an amply τ-supplemented module.*

(2) *If L is π-projective and τ-supplemented, then L is amply τ-supplemented.*

Proof. (1) Assume $U, V \subseteq L$ such that $L = U + V$. Since U is τ-supplemented, there exists $Y \subseteq U$ such that $U \cap V + Y = U$ and also $U \cap V \cap Y \subseteq \tau(Y)$. We know that $V \cap Y = U \cap V \cap Y \subseteq \tau(Y)$ and it is clear that $L = Y + V$.

(2) Assume $L = X + Y$, so there exists $e \in \mathrm{End}(L)$ such that $Le \subseteq X$, $L(1-e) \subseteq Y$. It is clear that $X(1-e) \subseteq X$. Since L is τ-supplemented, there exists $C \subseteq L$ such that $C + X = L$ and $C \cap X \subseteq \tau(C)$. Hence

$$\begin{aligned} L &= Le + L(1-e) \\ &= Le + (X + C)(1-e) \subseteq X + C(1-e). \end{aligned}$$

Therefore $L = X + C(1-e)$. It can be shown easily that $C(1-e) \subseteq Y$, and that $X \cap C(1-e) = (X \cap C)(1-e)$. Since $X \cap C \subseteq \tau(C)$, it follows that $X \cap C(1-e) \subseteq \tau(C(1-e))$. Hence $C(1-e)$ is a τ-supplement of X in L and $C(1-e) \subseteq Y$, i.e., L is amply τ-supplemented. □

2.7. Corollary. *For M the following are equivalent:*

(a) *every module in* $\sigma[M]$ *is* τ*-supplemented;*
(b) *every module in* $\sigma[M]$ *is amply* τ*-supplemented.*

2.8. τ-dense summands. *Let* $L \in \sigma[M]$. *For a submodule* $U \subseteq L$ *the following are equivalent:*

(a) *there is a decomposition* $L = X \oplus X'$ *with* $X \subseteq U$ *and* $X' \cap U \subseteq \tau(X')$*;*
(b) *there is an idempotent* $e \in \mathrm{End}(L)$ *with* $Le \subseteq U$ *and* $U(1-e) \subseteq \tau(U(1-e))$*;*
(c) *there is a direct summand* X *of* L *with* $X \subseteq U$ *and* $U/X \subseteq \tau(L/X)$*;*
(d) U *has a* τ*-supplement* V *in* L *such that* $U \cap V$ *is a direct summand of* U*;*
(e) *there is a decomposition* $U = X \oplus Y$*, such that* X *is a direct summand of* L *and* $Y \subseteq \tau(L)$*.*

In this case we say that U *contains a* τ*-dense direct summand.*

Proof. Compare [16, 41.11] and [15, Theorem 2.8]. □

2.9. τ-lifting modules. *A module* $L \in \sigma[M]$ *is called* τ*-lifting if every submodule of* L *contains a* τ*-dense direct summand.*

This concept has been introduced by P. F. Smith and I. Al-Khazzi in [1] for more general classes $\mathcal{X}$ of modules instead of a torsion theory τ.

2.10. Properties of τ-lifting modules. *Let* $L \in \sigma[M]$ *be a* τ*-lifting module. Then:*

(1) $\mathrm{Rad}\, L \subseteq \tau(L)$ *and if* $\mathrm{Rad}\, L \neq \tau(L)$*, then* L *has a nonzero direct summand that is* τ*-torsion.*
(2) *Any direct summand of* L *is* τ*-lifting.*

Proof. (1) By 2.2(3), $L/\tau(L)$ is semisimple and hence $\mathrm{Rad}\, L \subseteq \tau(L)$.

Suppose $\mathrm{Rad}\, L \neq \tau(L)$. Then there is a maximal submodule $K \subseteq L$ with $\tau(L) \not\subseteq K$. By assumption, K contains a τ-dense summand, that is a submodule $A \subseteq K$ with $L = A \oplus B$ and $K \cap B \subseteq \tau(B)$ and $K \cap B$ is a maximal submodule in B. Thus $\tau(B) = B$ or $K \cap B = \tau(B)$. In the latter case $\tau(L) = \tau(A) \oplus \tau(B) \subseteq K$, contradicting the choice of K. Thus $L = A \oplus B$ where $B = \tau(B)$.

(2) Assume $L = K \oplus K'$ and let $X \subseteq K$. Since L is τ-lifting, $L = N \oplus N'$ with $N \subseteq X$ and $X \cap N' \subseteq \tau(N')$. So $K = N \oplus (K \cap N')$ and

$$\begin{aligned} X \cap (K \cap N') &= X \cap N' \subseteq \tau(N') \\ &\subseteq \tau(L) = \tau(K) \oplus \tau(K') = \tau(N) \oplus \tau(K \cap N') \oplus \tau(K'). \end{aligned}$$

But $X \cap N' \subseteq K \cap N'$, hence $X \cap N' \subseteq \tau(K \cap N')$. So K is τ-lifting. □

2.11. τ-covers. An epimorphism in $\sigma[M]$ $f : P \to L$ is called a τ-cover provided $\mathrm{Ke}\, f \subseteq \tau(P)$. If P is projective in $\sigma[M]$, then f is called a *projective τ-cover* (in $\sigma[M]$).

The notion of (projective) Rad-covers was studied in [3, 17] under the name *generalised (projective) covers.* Nakahara [11] also studied ρ-covers under the name ρ-semicovers, for a (normal) preradical ρ.

The following lemma is needed to show general properties of τ-covers (compare [2, 8.17]).

2.12. Lemma. *If $f : L \to N$ is an epimorphism such that Ker$(f) \subseteq \tau(L)$, then $f(\tau(L)) = \tau(N)$.*

2.13. Properties of (projective) τ-covers.

(1) *If $f : P \to L$ is a projective τ-cover and $g : L \to N$ is a τ-cover, then $gf : P \to N$ is a projective τ-cover.*

(2) *If each $f_i : P_i \to L_i$, $i \in I$ is a (projective) τ-cover, then the map $\oplus_I f_i : \bigoplus_I P_i \to \bigoplus_I L_i$ is a (projective) τ-cover.*

Proof. (1) To show that $\mathrm{Ke}\, gf \subseteq \tau(P)$, let $x \in \mathrm{Ke}\, gf$, so $f(x) \in \mathrm{Ke}\, g \subseteq \tau(L) = f(\tau(P))$, hence $f(x) = f(t_1)$, where $t_1 \in \tau(P)$, which implies that $x - t_1 \in \mathrm{Ke}\, f \subseteq \tau(P)$, therefore $x \in \tau(P)$.

(2) This follows directly from the facts that τ commutes with direct sums and that a direct sum of projective modules is projective. □

From the definition, the following is clear: If $K \subseteq L$ is a τ-supplement for $U \subseteq L$, then $K \to K/K \cap U \cong L/U$ is a τ-cover.

Now we give a connection between τ-covers and τ-supplements.

2.14. Projective τ-cover and τ-supplements. *For $U \subseteq L$ with L in $\sigma[M]$ the following are equivalent:*

(a) *L/U has a projective τ-cover;*

(b) *U has a τ-supplement V which has a projective τ-cover;*

(c) *if $V \subseteq L$ and $L = U + V$, then U has a τ-supplement $V' \subseteq V$ such that V' has a projective τ-cover.*

Proof. (a)⇒(b) Assume that $f : P \to L/U$ is a projective τ-cover of L/U. Then there exists $g : P \to L$ such that $f = \pi g$ where $\pi : L \to L/U$ denotes the canonical projection. Denote by V the image of g, i.e. $V = \mathrm{Im}(g)$. Since $g^{-1}(U) = \mathrm{Ker}\, f$, we have $\mathrm{Ker}\, g \subseteq \mathrm{Ker}\, f$, i.e. P is a projective τ-cover of V. As $U \cap V = g(\mathrm{Ker}\, f) \subseteq g(\tau(P)) \subseteq \tau(g(P)) = \tau(V)$, we have V is a τ-supplement of U in L having the τ-projective cover P.

(b)⇒(a) Let V be a τ-supplement of U in L having a projective τ-cover $f: P \to V$. Consider $g: V \to V/U \cap V = L/U$, this is a τ-cover of L/U. By 2.13 it follows that $gf: P \to L/U$ is a projective τ-cover of L/U.

(a)⇒(c) Assume that $f: P \to L/U$ is a τ-cover and P is projective. Whenever $L = U + V$, then $V/(V \cap U) \simeq L/U$ and P is also a projective τ-cover of $V/(V \cap U)$. By (2), there exists $V' \subseteq V$ such that V' is a τ-supplement of $V \cap U$ in V, having a projective τ-cover. Since $V' \cap U = V' \cap (V \cap U)$ and

$$L = V + U = V' + (V \cap U) + U = V' + U,$$

V' is a τ-supplement of U in L.

(c)⇒(b) follows from $L = U + L$. □

2.15. τ-semiperfect. A module $L \in \sigma[M]$ is called τ-semiperfect (τ-perfect), if every factor module of L (any direct sum of copies of L) has a projective τ-cover.

From the above results we obtain the

2.16. Characterisation of τ-semiperfect modules. *For a module $L \in \sigma[M]$ the following are equivalent:*

(a) *L is τ-semiperfect;*

(b) *L is τ-supplemented by supplements which have projective τ-covers;*

(c) *L is amply τ-supplemented by supplements which have projective τ-covers.*

2.17. Corollary. *Let $L \in \sigma[M]$ be τ-semiperfect, then*

(1) *$L/\tau(L)$ is semisimple;*

(2) *if L is also τ-torsion-free, then L is semisimple*

If $M = R$ and L is projective in R-Mod then it is Rad-semiperfect (Rad-perfect) if and only if it is semiperfect (perfect). This was established in [3, 11, 17].

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