Towards the algebraization of Formal Concept Analysis over complete dioids

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Abstract

Complete dioids are already complete residuated lattices. Formal contexts with entries in them generate Concept Lattices with the help of the polar maps. Previous work has already established the spectral nature of some formal concepts for contexts over certain kinds of dioids. This paper tries to raise the awareness that linear algebra over exotic semirings should be one place to look to understand the properties of FCA over *L*-lattices.

Keywords: Semiring theory. Spectral theory of matrices. FCA: Semiring-valued extensions. Philosophical foundations.

1 INTRODUCTION

Several attempts have been made to generalise the basic framework of Formal Concept Analysis [9] or Galois Lattice theory [3] since it was conceived. Recall that this is a theory of concrete lattices arising from certain Galois connections between two sets induced by a binary incidence relation. If finds concrete applications in data mining and exploratory information retrieval, among others [18].

Perhaps the earliest and more developed generalisation is that of *L*-Fuzzy, or simply Fuzzy, Formal Concept Analysis (FFCA), where incidences are allowed to have values in a fuzzy algebra which is also a complete lattice [4, 5]. Such fuzzy algebras can alternatively be described as fuzzy semirings [11].

Recall that a *semiring* is an algebra $\mathscr{S} = \langle S, \oplus, \otimes, \varepsilon, e\rangle$ whose additive structure, $\langle S, \oplus, \varepsilon\rangle$, is a commutative monoid and whose multiplicative structure, $\langle S\backslash\{\varepsilon\}, \otimes, e\rangle$, is a monoid with multiplication distributing over addition from right and left and with additive neutral element absorbing for $\otimes$, i.e. $\forall a \in S,\ \varepsilon \otimes a = \varepsilon$ [11].

An independently motivated generalisation of FCA, $\mathscr{K}$-Formal Concept Analysis, uses an idempotent semifield $\mathscr{K}$—a kind of semiring with a multiplicative group structure—as the range of the relation [20]. Whereas fuzzy semirings are mostly used to capture a "degree of truth", semifields are used to capture the concept of "cost" or, dually, "utility".

It is intriguing that these algebras induce Galois Connections and Formal Concept Analysis inasmuch as idempotent semifields are as far as a naturally ordered semiring can be from prototypical fuzzy semirings like $\langle [0,1], \max, \min, 0, 1\rangle$—in a sense made evident in this paper. In fact, idempotent semifields do not fulfill some of the more restrictive or technical conditions for an algebra *L* to define an *L*-fuzzy set [10]: in particular, in an idempotent semifield the identity is never an infinity element.

However, it has already been determined that the condition for an algebra to induce a flavour of Formal Concept Analysis is that it be a complete residuated lattice [4]. Unsurprisingly, one of the notoriously overlooked abstractions of fuzzy semirings and idempotent semifields are dioids, or naturally-ordered semirings whose zero is the bottom in the order. The naturalness of the order comes from its compatibility with the multiplication.

$$\forall a, b, c \in S, a \leq b \iff c \otimes a \leq c \otimes b \ . \tag{1}$$

Dioids are already residuated so complete dioids are already complete residuated lattices (see Fig. 1), hence Formal Concept Analysis-inducing. Furthermore, semiring $\mathbb{B}$ is embedded in both fuzzy semirings and idempotent semifields. Note that [15] already asked for a revisiting of idempotent semifields and the investigation of their relationship to fuzzy algebras.

A more intriguing finding is that the formal concepts of such extensions seem to be related to the eigenvectors of certain operators related to the incidence of an *L*-valued formal context [19]. This subject has been picked up in more modern work [21], underscoring the importance of those concepts issuing from the broad field of linear algebra

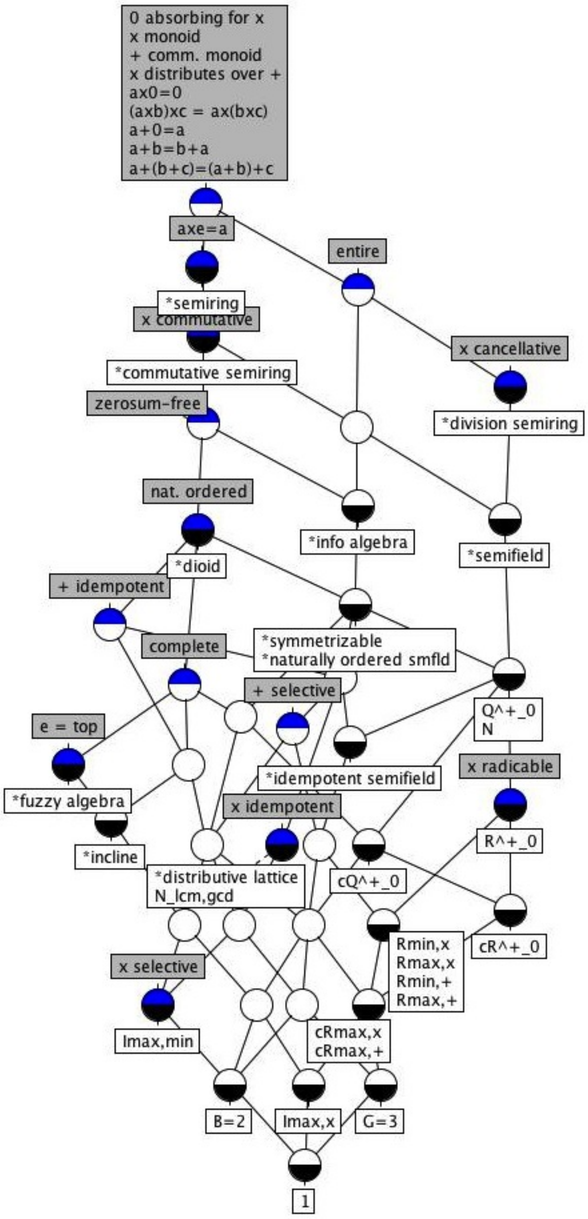

Figure 1: **Concept Lattice of dioids in the context of commutative semirings.** Note that rings—which are not naturally ordered—are missing in the picture, what makes the concept for attribute *complete* seem to be proper restriction of the concept for *dioid*. In a formal context including rings, this would not be so.

over "exotic" semirings for Formal Concept Analysis, and, in particular, the spectral theory of matrices.

2 SOME LINEAR ALGEBRA OVER DIOIDS

2.1 BASIC CONCEPTS

A semiring $\mathscr{S}$ is:

1. *zerosumfree* iff $a \oplus b = \varepsilon \Rightarrow a = \varepsilon$ and $b = \varepsilon$.
2. *entire* iff $a \otimes b = \varepsilon \Rightarrow a = \varepsilon$ or $b = \varepsilon$.
3. *positive or an information algebra* iff it is both zerosumfree and entire.
4. *partially-ordered* iff there is an order $\langle S, \preccurlyeq \rangle$ compatible with addition and multiplication, such that for all $a, b, c \in S$, if $a \preccurlyeq b$ then $a \oplus c \preccurlyeq b \oplus c$, $a \otimes c \preccurlyeq b \otimes c$ and $c \otimes a \preccurlyeq c \otimes b$.

Note that, in partially-ordered semirings, if $a_i \preccurlyeq b_i$ then $\sum a_i \preccurlyeq \sum b_i$. Furthermore, if $\mathscr{S}$ is a partially-ordered set, then it is *positive* if $\varepsilon = \bot$ is the infimum or *bottom* for this set $\bot \preccurlyeq a$ for all $a \in S$ [13]. If $\mathscr{S}$ is positive then also if $a_i \preccurlyeq b_i$ then $\prod a_i \preccurlyeq \prod b_i$ [13].

In a semiring, the *natural or canonical or difference pre-order* is for all $a, b \in S$, $a \preccurlyeq b \iff a \oplus c = b$ for some $c \in S$. A semiring $\mathscr{D} = \langle D, \oplus, \otimes, \bot, e \rangle$ is *a dioid—for* double monoid—*or naturally- or canonically- [16] or difference-ordered [11–13]* if this natural pre-order is actually a partial order.

Example 1. *The following are dioids:*

1. $\mathbb{B} \equiv \mathbb{2} \equiv \langle \{0,1\}, \vee, \wedge, 0, 1 \rangle$, *(Boolean algebra)*
2. $\mathbb{I}_{\max,\min} \equiv \langle [0,1], \max, \min, 0, 1 \rangle$, *(toll semiring) [2]*
3. $\mathbb{N}_{\min,+} \equiv \langle \mathbb{N} \cup \{0, \infty\}, \min, +, \infty, 0 \rangle$, *(*tropical semiring*)*
4. $\mathbb{R}_{\max,+} \equiv \langle \mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0 \rangle$, *(*max-plus, schedule or morphological algebra*) [1]*
5. $\mathbb{R}_{\min,+} \equiv \langle \mathbb{R} \cup \{\infty\}, \min, +, \infty, 0 \rangle$ *[1], (*optimization algebra*)*
6. $\mathbb{R}_{\max,\times} \equiv \langle \mathbb{R}_0^+ \cup \{\infty\}, \max, \times, 0, 1 \rangle$, *(*max-times semiring*) [16]*
7. $\mathbb{I}_{\max,\times} \equiv \langle [0,1], \max, \times, 0, 1 \rangle$,*(*fuzzy max-times algebra*) [16]*

Their relationships can be gleaned from Fig. 1 .

Note that the product of k dioids is also a dioid.

A big class of dioids is that of *(additively) idempotent semirings*. An idempotent semiring $\mathscr{D}$ is a semiring whose additive structure $\langle D, \oplus, \varepsilon \rangle$ is an *idempotent semigroup*, that is, $\forall a \in D, a \oplus a = a$. Idempotent semirings are all canonically-ordered and, if commutative, they are already $\vee$-semilattices (read *sup- or join-semilattice*), whose operation is compatible with the canonical order $a \oplus b = a \vee b$ and selects the *lowest upper bound, supremum or join* [16, Chap. 1, Theorems 1&2].

Another class of important dioids are those with a multiplicative group structure or *semifields*. These are all entire, and thence information algebras.

2.1.1 Completeness

In a semiring $\mathscr{S}$ for each $a \in S$, define

$$a^{+k} = a \oplus a^2 \oplus \dots a^k = \sum_{i=1}^{k} a^i \quad a^{*k} = e \oplus a^{+k} = \sum_{i=0}^{k} a^i \quad (2)$$

The *Kleene star* is the limit $a^* = \lim_{k \to \inf} a^{*k}$ when it exists. Likewise the *Kleene plus* is the limit $a^+ = \lim_{k \to \inf} a^{+k}$ when it exists. Since $a^{*k} = e \oplus a^{+k}$ and $a^{+k} = a \otimes a^{*(k-1)} = a^{*(k-1)} \otimes a$ the existence of one of the limits entails the existence of the other:

$$a^* = e \oplus a^+ \qquad a^+ = a \otimes a^* = a^* \otimes a \ . \qquad (3)$$

As the existence of such elements is crucial for our purposes after Theorem 3.1, completeness issues in semirings deserve a lengthier treatment: a *complete semiring* $\mathscr{S}$ [12] is a semiring where for every (possibly infinite) family of elements $\{a_i\}_{i \in I} \subseteq S$ we can define an element $\sum_{i \in I} a_i \in S$ such that

1. if $I = \emptyset$, then $\sum_{i \in I} a_i = \varepsilon$,
2. if $I = \{1 \dots n\}$, then $\sum_{i \in I} a_i = a_1 \oplus \dots \oplus a_n$,
3. if $b \in S$, then $b \otimes (\sum_{i \in I} a_i) = \sum_{i \in I} b \otimes a_i$ and $(\sum_{i \in I} a_i) \otimes b = \sum_{i \in I} a_i \otimes b$, and
4. if $\{I_j\}_{j \in J}$ is a partition of I, then $\sum_{i \in I} a_i = \sum_{j \in J} \left(\sum_{i \in I_j} a_i \right)$.

If I is countable in the definitions above, then $\mathscr{S}$ is *countably complete* and already zerosumfree [11, Prop. 22.28].

To investigate completion issues, call an element in a semiring $a \in S$ *infinite* iff $a \oplus b = a$ for all $b \in S$, and *strongly infinite* if also $a \otimes b = a = b \otimes a$ [11]. If $\overline{\mathscr{S}}$ is a complete semiring, then it has a (necessarily unique) infinite element [11, Prop. 22.27].

A dioid $\mathscr{D}$ is *complete*, if it is complete as a semiring, further complete as a naturally ordered set $\langle D, \preccurlyeq \rangle$ and left (L_a) and right (R_a) multiplications are lower semicontinuous, that is, join-preserving. In such cases, the *top* of the dioid is the unique infinite element $\top = \top \oplus a$, for all $a \in D$, whence $a \preccurlyeq \top$.

All the above can be seen instantiated in:

Example 2. 1. *In* $\mathbb{B}$ *we have* $0^* = \mathbb{1}^* = \mathbb{1}$*, hence* $\mathbb{1}$ *is (vacuously) strongly infinite.*

2. *In the schedule algebra* $\mathbb{R}_{\max,+}$ *we have* $a^k := k \cdot a$ *so* $e \oplus a^k := \max(0, k \cdot a)$*. That is, for* $a \leq 0, e \oplus a^k = e$ *and* $a^* := 0$*. On the other hand, for* $b > 0, e \oplus b^k = b^k$ *and such elements do not have a star, hence* $\mathbb{R}_{\max,+}$ *is incomplete. When we complete* $\overline{\mathbb{R}}_{\max,+}$ *with* $\top := \infty$*, for* $b > e$ *we have* $b^* = b^+ = \top$*.*

As a partially-ordered set, a $\vee$-semilattice is *complete* when the lowest upper bound operates on arbitrary subsets of S and likewise for complete $\wedge$-semilattices. Lattices are *complete* when both their $\vee$- and $\wedge$-semilattices are complete, hence they have both a top and a bottom. From a well-known order-theory theorem—a complete $\vee$-semilattice with bottom is also a complete lattice [8, Theorem 2.31, p. 47]—it is clear that complete idempotent semirings are already complete lattices.

A fortiori, selective semifields can all be completed, as, for instance, the (initially incomplete) maxplus and minplus semifields in Example 2

Example 3. 1. *The* completed Minplus semifield, $\overline{\mathbb{R}}_{\min,+} = \langle \mathbb{R} \cup \{-\infty, \infty\}, \min, \dot{+}, \infty, 0, -\infty \rangle$,

2. *the* completed Maxplus semifield, $\overline{\mathbb{R}}_{\max,+} = \langle \mathbb{R} \cup \{-\infty, \infty\}, \max, \underset{\cdot}{+}, -\infty, 0, \infty \rangle$.

These two completions are actually inverses $\overline{\mathbb{R}}_{\min,+} =$

$\overline{\mathbb{R}}_{\max,+}^{-1}$ *and order-dual [17]. Indeed they are better jointly called the* max-min-plus *semiring* $\overline{\mathbb{R}}_{\max,+}^{\min,\dot{+}}$. *We have* $-\infty + \infty = -\infty$ *and* $-\infty \dot{+} \infty = \infty$, *which solves several issues in dealing with the separately completed dioids*

The algebras above were first recorded as *blogs*, bounded, lattice-ordered groups [7, §4.1], although the name did not catch, and would be called *bounded ℓ-groups* nowadays. The lattice $\mathbb{B}$ can be embedded in any bounded ℓ-group, by restricting the carrier set to $\{\bot, \top\}$. The boolean operations would then be implemented as $\oplus$ and $\otimes$ restricted to such set. A richer structure is the $\dot{3}$-element bounded ℓ-group $\mathbb{3} = \langle \{\bot, e, \top\}, \oplus, \dot{\oplus}, \otimes, \dot{\otimes}, \bot, e, \top \rangle$.

Proposition 2.1. *[7, Th. 4.5] The* 3*-element blog can be embedded isomorphically in any blog.*

2.2 SEMIMODULES OVER DIOIDS

In this context, a *semimodule over a semiring*, is the analogue of a module over a ring [11, 12]: a *right $\mathcal{S}$-semimodule* is an additive commutative monoid $\mathcal{X} = \langle X, \oplus, \varepsilon_{\mathcal{X}} \rangle$ endowed with a *right action* $(x, \lambda) \mapsto x \odot \lambda$ such that $\forall \lambda, \mu \in S,\ \ x, x' \in X$. Following the convention of dropping the symbols for the scalar action and semiring multiplication we have:

$$
\begin{aligned}
x(\lambda\mu) &= (x\lambda)\mu & \qquad x\varepsilon &= \varepsilon_{\mathcal{X}} \\
(x \oplus x')\lambda &= x\lambda \oplus x'\lambda & \qquad xe &= x
\end{aligned}
$$

The definition of a *left $\mathcal{S}$-semimodule $\mathcal{Y}$* follows the same pattern with the help of a *left action*, $(x, \lambda) \mapsto \lambda \odot x$ and similar axioms. A *$(\mathcal{R}, \mathcal{S})$-semimodule* is a set M endowed with left $\mathcal{R}$-semimodule and a right $\mathcal{S}$-semimodule structures, and a *$(\mathcal{R}, \mathcal{S})$-bisemimodule* a $(\mathcal{R}, \mathcal{S})$-semimodule such that the left and right actions commute.

For $n, p \in \mathbb{N}$, the semimodule of finite matrices $\mathcal{M}_{n\times p}(\mathcal{S}) = \langle S^{n\times p}, \oplus, \mathcal{E} \rangle$ is a $(\mathcal{M}_n(\mathcal{S}), \mathcal{M}_p(\mathcal{S}))$-bisemimodule, with matrix multiplication-like left and right actions and entry-wise addition. Special cases of it are the bisemimodules of column vectors $\mathcal{M}_{p\times 1}(\mathcal{S})$ and row vectors $\mathcal{M}_{1\times n}(\mathcal{S})$. In the following we systematically equate left (resp. right) $\mathcal{S}$-semimodules and row (resp. column) semimodules over $\mathcal{S}$. This motivates calling them *(semi)vector spaces*.

In a semimodule $\mathcal{X}$ over a semifield $\mathcal{K}$ one can define an element-wise inversion operation $\cdot^{-1} : X \to X, x \mapsto x^{-1}$ such that $(x^{-1})_i = x_i{}^{-1}$. If the semifield is also a complete dioid, then the *inverse* semimodule is the order dual $\mathcal{X}^{-1} \cong \langle X, \leq^{\mathrm{d}} \rangle$. Hence, a *complete* semimodule [11] over an idempotent semifield is also a complete lattice, with join and meet operations fulfilling $v_1 \leq v_2 \iff v_1 \vee v_2 = v_2 \iff v_1 \wedge v_2 = v_2$. $v_1 \wedge v_2 = \left(v_1^{-1} \vee v_2^{-1}\right)^{-1}$ á la Boole.

Example 4. *Semimodules over* $\overline{\mathbb{R}}_{\max,+}$ *have inverses over* $\overline{\mathbb{R}}_{\min,+}$ *and vice versa. In particular* $\left(\overline{\mathbb{R}}_{\max,+}\right)^{-1} = \overline{\mathbb{R}}_{\min,+}$, *and dually. For* $\overline{\mathbb{R}}_{\max,+}$, *it is* $v_1 \wedge v_2 = v_1 \dot{\oplus} v_2 = (v_1^{-1} \oplus v_1^{-1})^{-1} = \min(v_1, v_2)$.

3 STUDY CASE: THE SPECTRAL THEORY OVER DIOIDS

Given a square matrix $A \in S^{n\times n}$ the *right (left) eigenproblem* is the task of finding the *right eigenvectors* $v \in S^{n\times 1}$ and *right eigenvalues* $\rho \in S$ (respectively *left eigenvectors* $u \in S^{1\times n}$ and *left eigenvalues* $\lambda \in S$) satisfying:

$$
u \otimes A = \lambda \otimes u \qquad\qquad A \otimes v = v \otimes \rho \tag{4}
$$

The left and right eigenspaces—$\mathcal{U}_\lambda(A)$ and $\mathcal{V}_\rho(A)$—and spectra—$\Lambda(A)$ and $\mathrm{P}(A)$—are the sets of solutions:

$$
\begin{aligned}
\mathcal{U}_\lambda(A) &= \{u \in S^{1\times n} \mid u \otimes A = \lambda \otimes u\} \\
\Lambda(A) &= \{\lambda \in S \mid \mathcal{U}_\lambda(A) \neq \{\varepsilon^n\}\}
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
\mathcal{V}_\rho(A) &= \{v \in S^{n\times 1} \mid A \otimes v = v \otimes \rho\} \\
\mathrm{P}(A) &= \{\rho \in S \mid \mathcal{V}_\rho(A) \neq \{\varepsilon^n\}\}
\end{aligned} \tag{6}
$$

Since $\Lambda(A) = \mathrm{P}(A^{\mathrm{T}})$ and $\mathcal{U}_\lambda(A) = \mathcal{V}_\lambda(A^{\mathrm{T}})$, from now on we will omit references to left eigenvalues, eigenvectors and spectra, unless we want to emphasize differences.

With so little structure it might seem hard to solve (4). Readily available techniques are combinatorial in nature and a very generic solution exists based on the following recurring concept. For a matrix over a semiring $A \in S^{n\times n}$ consider the sum $A^{*k} = I \oplus A \oplus A^2 \oplus \ldots \oplus A^k$, where addition and multiplication over matrices are intuitively obtained from those of the underlying semiring, A^k represents a product of k factors and I is the neutral element for matrix multiplication. The *Kleene star* of A is $A^* = \lim_{k\to\infty} A^{*k}$and the *Kleene plus* of A is $A^+ = A \otimes A^*$. Most of the results on eigenvalues and eigenvectors in this paper stem from the following fact:

Proposition 3.1 (Gondran and Minoux, Theorem 1 [14, 16])**.** *Let* $A \in \mathcal{S}^{n\times n}$ *be a matrix over a semiring. If* A^* *exists, the following two conditions are equivalent:*

1. $A^+_{\cdot i} \otimes \mu = A^*_{\cdot i} \otimes \mu$ *for some* $i \in \{1 \ldots n\}$, *and* $\mu \in S$.
2. $A^+_{\cdot i} \otimes \mu$ *(and* $A^*_{\cdot i} \otimes \mu$*) is an eigenvector of* A *for* e , $A^+_{\cdot i} \otimes \mu \in \mathcal{V}_e(A)$.

Regarding the structure of right eigenspaces, it is well-known that they are right subsemimodules of $\mathcal{S}^{n\times 1}$ [11, p.219],[6, §4.1.1]. The following corollary highlights the novelty afforded by completed dioids:

Corollary 3.2. *Let* $A \in \overline{\mathcal{K}}^{n\times n}$ *be a matrix with entries in a commutative complete idempotent semiring. For all eigenvalues* $\rho \in \mathrm{P}(A)$, $\mathcal{V}_\rho(A)$ *is a complete lattice.*

3.1 The improper spectrum

Without loss of generality, from now on call $\top$ any strongly infinite element of $\mathscr{S}$. Corollary 3.2 highlights the novelty afforded by complete dioids and preludes the following crucial difference between spectra in *completed* and incomplete semirings:

Proposition 3.3 (Improper spectrum)**.** *Let $A \in \mathscr{M}_n(\overline{\mathscr{S}})$ be a matrix with entries in an entire zerosumfree semiring with strongly infinite element $\top \in \overline{\mathscr{S}}$. If $\rho \neq \varepsilon$ and $v \in \mathscr{V}_\rho(A)$, $v \neq \varepsilon^n$ then,*

1. *$v \otimes \top \in \mathscr{V}_\rho(A) \cap \mathscr{V}_{\rho'}(A)$, with $\rho' \in \overline{S} \backslash \{\varepsilon\}$.*
2. $\mathrm{P}(A) \supseteq \overline{S} \backslash \{\varepsilon\}$.

The proposition applies, a fortiori, to dioids with a commutative multiplicative group structure. It essentially asserts that there is an underlying structure common to all possible eigenspaces in a matrix. But we know that these are all lattices, whence this underlying structure has to be a lattice.

This raises a terminological issue since we would like to distinguish between the *proper* eigenvalues, like those afforded by the spectral theory on matrices over incomplete dioids, and the induced or *improper* eigenvalues in Proposition 3.3. We propose to call an eigenvalue *proper* when it has at least one eigenvector with finite coordinates. The set of proper (left) eigenvalues is the *proper (left) spectrum*,

$$\Lambda^{\mathrm{P}}(A) = \{\lambda \in \Lambda(A) \mid \exists u \in \mathscr{U}_\lambda(A), \text{fin-supp}(u) \neq \varnothing\} \text{ and}$$
$$\mathrm{P}^{\mathrm{P}}(A) = \{\rho \in \mathrm{P}(A) \mid \exists v \in \mathscr{V}_\rho(A) \text{ fin-supp}(v) \neq \varnothing\} \; .$$

3.2 The null eigenspace

Some other results are combinatorial in nature: Call $e_i = I_{\cdot i}$ the i-th column of I, the unit in the semiring of matrices $\mathscr{M}_n(S)$—e_i is a vector whose coordinates are zero except for $e_i(i) = e$—and note that $A \otimes e_i = A_{\cdot i}$.

Lemma 3.4. *Let $A \in \mathscr{M}_n(\mathscr{S})$ over a semiring. Then:*

1. *If the i-th column of A is zero, then $e_i \in \mathscr{V}_\varepsilon(A)$, whence $\varepsilon \in \mathrm{P}^{\mathrm{P}}(A)$.*
2. *Further, if $\mathscr{S}$ is entire, then G_A has no cycles if and only if ε is the unique eigenvalue of A.*
3. *Further, if $\mathscr{S}$ is entire and zerosumfree and $\varepsilon \in \mathrm{P}(A)$, then A has at least one zero column.*

Call the columns of I selected by the set of zero columns of A, the *fundamental eigenvectors of A for ε*, $\mathrm{FEV}_\varepsilon(A) = \{I_{\cdot i} \mid A_{\cdot i} = \varepsilon^n\}$. The name is justified by,

Proposition 3.5. *Let $A \in \mathscr{M}_n(\mathscr{S})$ over an entire zerosumfree semiring. The null eigenspace is generated by the fundamental eigenvectors of A for ε.*

$$\mathscr{V}_\varepsilon(A) = \langle \mathrm{FEV}_\varepsilon(A) \rangle_{\mathscr{S}} \tag{7}$$

Recall that on completed dioids with multiplicative group structure this is actually a lattice. In fact it is the boolean lattice with as many atoms as independent eigenvectors.

3.2.1 The irreducible case

We can actually generate all eigenspaces in completed dioids with multiplicative group structure in this way: Recall that a matrix over a semiring is *reducible* if its rows and columns can be simultaneously permuted into a triangular block from. Otherwise it is *irreducible.*

Then an irreducible matrix A has a *maximal cycle mean* $\mu_\oplus(A)$ associated to the adjacency graph $\overline{G}_A$ of the matrix. In such case a finite skeleton lattice $\mathscr{L}_\rho(A)$ related to the improper eigenvalues, and another lattice related to the single proper eigenvalue $\mathscr{L}_{\rho(A)}(A)$:

Theorem 3.6. *Let $A \in \mathscr{M}_n(\overline{\mathscr{K}})$ be an irreducible matrix over a complete selective radicable semifield, $\overline{\mathscr{K}}$. Then,*

1. $\Lambda(A) = \overline{\mathscr{K}} \backslash \{\bot\} = \mathrm{P}(A)$.
2. $\Lambda^{\mathrm{P}}(A) = \{\mu_\oplus(A)\} = \mathrm{P}^{\mathrm{P}}(A)$.
3. *If $\rho \in \mathrm{P}(A) \backslash \mathrm{P}^{\mathrm{P}}(A)$, then $\mathscr{V}_\rho(A) = \{\bot^n, \top^n\} = \mathscr{L}_\rho(A)$.*
4. *If $\mu_\oplus(A) \prec \top$, then $\mathscr{V}_{\rho(A)}(A) = \langle \mathrm{FEV}_{\rho(A)}(A) \rangle_{\overline{\mathscr{K}}} \supset \mathscr{L}_{\rho(A)}(A) = \langle \mathrm{FEV}_{\rho(A)}(A) \rangle_{\mathbb{3}}$.*

The details for both the irreducible and the reducible case can be found in [21, 22].

3.2.2 Discussion

Although, promisingly, we have arrived to prove that there are (both continuous and discrete) lattices aplenty in the spectral theory of some commutative dioids with a multiplicative group structure, we have failed to prove our point in two aspects: that this applies to *commutative* dioids *in general* and that the lattices of FCA *actually arise in this form*. This is left for future work.

4 CONCLUSIONS

We have proven that there are lattices aplenty in the spectral theory of matrices over certain kinds of dioids. We believe that linear algebra over dioids provides an extended view into concept lattices beyond that provided by pure order theory, despite lacking the width and wealth of result that standard Linear Algebra has. Furthermore, it also widens the scope of Lattice Theory.

Although the comfort zone of handling a quite manageable set of results has to be abandoned in this pursuit, e.g. those extremely well captured in [9], we believe this avenue of research can help realize the true potential of Wille's initial grand view in placing a more constructive, data-aware version of Lattice Theory at the center of modern mathematical applications.

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