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Understanding Linear Function in Secondary School Students: A Comparative Study between England and Shanghai

WANG, YUQIAN

Understanding Linear Function in Secondary School Students: A Comparative Study between

England and Shanghai

By

Yuqian Wang

Supervisors: Dr. Patrick Barmby

Dr. David Bolden

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Abstract

How to facilitate students' understanding of mathematics is a major concern for the mathematics education community as well as education authorities, especially in England, UK and Shanghai, China. However, research into such understanding in these two regions is still in its infancy. The aim of this thesis is to contribute to this research area by investigating how well students understand a particular mathematical concept, linear function, and describe how their understanding has been shaped. A model of understanding function is defined in terms of six levels: Variable Perspective, Dependent Relationship, Connecting Representations, Property Noticing, Object Analysis, and Inventising. These six levels are developed by examining the most prominent theories from existing Western and Eastern literature on understanding function. Using this model, three perspectives around understanding linear function are investigated: what the official documents expect; what students actually achieve; and teachers' views of how students' understanding of linear function develops. Mixed methods are adopted to portray a holistic view of understanding function in the two regions. The quantitative data analysis includes three curricula and seven selected textbooks to identify their characteristics and requirements. The main study also analyses student tests from 403 Year 10 Higher Level English students and 907 Grade 8 Shanghai students. Findings demonstrate that the Shanghai students have more abstract understanding than the English Higher Level students, and are more comfortable with algebraic expression, which is emphasised heavily in the Shanghai curriculum and textbook. The graphic representation dominates the Higher Level English students' solution approaches, which is again emphasised in their textbooks. This study recommends that the more emphasis should be on algebraic expression for understanding linear function in England and graphic representation in Shanghai.

Key words: comparative study, linear function, understanding

Declaration

I declare that this thesis, which I submit for the degree of Doctor of Philosophy, is my own

work and has not previously been submitted for a degree at this or any other university.

Statement of Copyright

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Afterwards

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Dedication

This thesis is dedicated to my daughter XU Jiayi (Vivian)

who is the fountain of my courage and genuine confidence to face challenges.

Contents

List of Tables

List of Figures

Part One: Research Context

This study is an analysis of students' understanding of linear function in England and Shanghai. It attempts to identify how to enhance students' learning with understanding, which remains one of the central aims in mathematics education around the world. From a comparative education perspective, the consistent superiority of Shanghai students' performance in mathematics (PISA specifically), as evidenced by their top ranking in international league tables, suggests that there must be something in the process of teaching and learning that Shanghai has developed which is absent in what England does now. However, what England could learn from Shanghai and what Shanghai could learn from England, in terms of the understanding of linear function, is an important contributory factor for this thesis. The league tables from large-scale cross-national projects for student assessment, which illustrate the gap in mathematics performance between England and Shanghai, also imply that what makes Shanghai students so successful should be related to how well they understand mathematical concepts and use them in real world situations. Therefore, this study aims to investigate the resemblances and discrepancies of mathematical understanding concerning both the pure knowledge and the real world application of that knowledge. Further inquiry into how these differences and similarities may have been shaped in the case of linear function will be probed by analyses of the curricula, the respective textbooks and interviews with selected teachers.

In seeking to explain the realities of understanding of mathematical concept in English and Shanghai junior secondary school students, this thesis is organised into three main parts. In the first part, I set out the research context, introduce relevant literature from similar comparative studies, and examine their methodologies to verify this study's methodology.

Chapter 1 Introduction

The purpose of the first chapter is to provide an overview of this study. It has six main sections and begins with what the thesis is about, followed by the rationale of this study. Then the research questions as well as the structure of the thesis will be addressed. Fifthly, the significance of the phenomenon being studied will be highlighted. The sixth section will outline the reasons for choosing case study approach.

1.1 The Focus of the Study

This study investigates the development of students' understanding of mathematical concept as well as identifying their strengths and weaknesses during the learning process, within England and Shanghai by considering a specific topic, namely linear function. The definition of understanding of linear function comes out by forming a general model of understanding function through summarising the most prominent theories from existing Western and Eastern (particularly in the Shanghai situation) literature. This model focuses primarily on the understanding of pure mathematics knowledge. The application in the real situation, however, will be explored as a consequence of the extent of understanding of pure knowledge. This use of knowledge shows the diverse approaches in the two regions towards understanding mathematical concept and reflects the ways in which students' understandings are shaped. Within each region, this study intends to measure students' developing understanding by examining four elements that constitute a coherent whole. These aspects are: the requirements of the compulsory curricula – England's national curricula and Shanghai's municipal curriculum; features of official textbooks; students' performance in two set of tests – one which tests pure knowledge understanding and a second which tests students' ability to apply that; and selected teachers' views.

Each aspect will be explored as follows:

1) The respective curriculum analysis. Both regions' curricula will be assessed in two ways: the background information concerned with distinctive features of curricula and the way towards understanding development, and the attainment targets for linear function.

2) The respective textbooks' analyses. Results will provide a detailed perspective of how these requirements of the curricula are interpreted, and how the topics are conceptualized and enacted towards real world situations.

3) Student tests. The present study will address the main barriers to each level of understanding - primarily focusing on what students could not do at the time. It will also examine which approach (algebraic or visual) students are more comfortable to use in order to solve problems in the real world, as which representation they prefer: the algebraic expression, the graphic representation, or the tabular representation.

4) Teacher interviews. Findings will provide teachers' views on what they perceive to be the barriers to understanding that students would encounter and how they plan lessons in order to overcome these barriers. Furthermore, teachers' beliefs of students' learning with understanding will be investigated to explain and justify their teaching approach.

This study therefore seeks to make evaluative comparisons by means of conducting a fair instrument and understandable comparisons advocated by Clarke (2003); addressing the interconnectedness of similarities and differences towards understanding between England and Shanghai; and attaching value to students' performances in terms of the model of understanding function to enforce the evaluative comparisons. It is hoped that a comparison of the results from the two regions will provide a holistic view of understanding linear function. The study will identify the inevitable gaps in the understanding of pure knowledge by applying the model that brings together the requirement of the national curriculum, the respective textbooks, and students' performances in understanding basic knowledge tests. Meanwhile, in the application section, the expectations of the textbooks and students'

preferred approaches to solutions in the tests will illustrate not only how students use knowledge, but also the reasons why they prefer a certain type of representation.

1.2 Rationale of this Study

In this section, the importance of this study is highlighted, based on four relevant research areas: (1) research on understanding; (2) research on algebraic learning in terms of the concept of function; (3) research on international comparative education, especially related to England and Shanghai; and (4) research on problem-solving.

1.2.1 Learning with understanding

How to evaluate understanding plays a central role in this study. Current thinking within the field of mathematical education suggests that learning with understanding is a well-accepted aim of mathematics education. Educators have sought to identify both the importance of understanding (Newton, 2000) and the definition of understanding (Hiebert & Carpenter, 1992; Sierpinska, 1990; Skemp, 1976). In order to facilitate the development of understanding, the process of understanding mathematics has been described by using different hierarchical models, for example the Pirie and Kieren (1994b) model and the APOS theory as proposed by Dubinsky and McDonald (2002). Although these models articulate the ways in which students understand mathematical knowledge in general, especially within Western culture, it is worthwhile to take a closer look at how understanding development takes place under different cultural dispositions in terms of a certain mathematical concept.

1.2.2 Learning of function in algebra

Algebra has been regarded as 'the most important gatekeeper in mathematics' (Cai, Ng, & Moyer, 2011, p. 26). From the 1980s, the letter-symbolic emphasis of algebra research has moved towards the study of function (Kieran, 2006). Function is a key algebraic topic in secondary schools (Brenner et al., 1997; Llinares, 2000) and a foundation of the whole curriculum around the world (Akkoç & Tall, 2005). In recent years, although researchers

have made efforts to study the learning of function through specially designed software (Schwartz & Yerushalmy, 1992) or the difficulties experienced by connecting multiple representations (Zaslavsky, Sela, & Leron, 2002), little is known of what shapes these difficulties and how the learning process is related to characteristics of official curriculum materials within different cultures.

1.2.3 International comparison

This study gathers data from two different countries. In the area of comparative education, Chinese students (mainly those in mainland China), and students from Hong Kong, Taiwan, Singapore, Japan and South Korea have consistently performed highly in the large-scale international comparative studies, e.g. those organised by the International Association for the Evaluation of Educational Achievement (IEA), and the Organization for Economic Cooperation and Development (OECD). The results of these cross-national studies have become indicators of success of the participating countries' school systems (Stanat & Ludtke, 2013). The success of Singapore has resulted in a decentralization of power to schools a little (Ng, 2013), while the strength of the South Korea school system is partly attributed to the extent of parents' involvement (Shin, 2013). However, the consistent higher performance of Chinese students led researchers to the features of Chinese learners (Biggs, 1996). Cultural factors have been underlined (Leung, 1998). In order to explore the Chinese learners' secrets, researchers have also focused on the common culture that these East Asian countries share, mainly being Confucian Heritage Cultures (CHC) (Lee, 1996). The findings from research on CHC students have expressed some possible contradictions, such as 'CHC students are perceived as using low-level, rote-based strategies', and on the one hand 'CHC students report a preference for high-level, meaning based learning strategies' (Biggs, 1996, p. 49). This contradiction needs to be examined in the light of handling a certain topic in mathematics, instead of tending to generalise for all mathematical problem-solving strategies.

Currently, the interest of international governments, academia, and the public in the reasons for East Asian students becoming the best in the world at mathematics is abundant and growing. This is especially so in the United Kingdom. The current UK government's 2010 White Paper, *The Importance of Teaching*, noted the importance of international comparisons: 'The truth is, at the moment we are standing still while others race past… The only way we can catch up, and have the world-class schools our children deserve, is by learning the lessons of other countries' success' (Department for Education, 24 November 2010, p. 3). Elizabeth Truss, Parliamentary Under Secretary of State for Education and Childcare at that time, claimed that mathematics was most important for England's future (Truss, 2013, September 18). Therefore, England seems prepared to learn lessons from a higher performing society in order to improve their own students' mathematics performance. The importance of learning from other societies is also embedded in the ancient Chinese saying such as, 'know yourself and your others'.

Chinese learners, particularly Shanghai students, have consistently outperformed other countries in the large-scale cross-country project - PISA (the Programme for International Student Assessment). Shanghai's top performance has been regarded as 'an important reference society' in terms of the schooling systems for the USA, England, and Australia (Sellar & Lingard, 2013, p. 479). The results of PISA 2012 (Mathematics, Reading and Science) revealed that Shanghai students 'have the equivalent of nearly 3 years of (extra) schooling' above the international average around which the performance of United Kingdom students were located (OECD, 2013, p. 17). This result has intrigued both education authorities and academics in England. In February 2014, Elizabeth Truss visited Shanghai to learn how mathematics was being taught there (Howse, 2014, Feb 18). Following her visit, an announcement that Shanghai maths teachers would be flown to England as part of an exchange project was released (Coughlan, 2014, March 12). This thesis therefore has

implications that reach beyond simply illuminating the mechanisms by which way mathematics is taught more effectively. Indeed, even though the features of better mechanisms might be identified, there are still a range of factors that impact on students' performances. Meanwhile, the triumph of the Shanghai situation or the lack of success of English students in PISA or the Third International Mathematics and Science Study (TIMSS) does not indicate how the understanding is developed which this study will do.

In the case of UK, although two projects presented contrasting results: PISA showed England's performance to be decreasing from 2000 to 2009, while TIMSS between 1995 and 2003 revealed 'steady improvements' (Brown, 2011, p. 153), English students' academic performances in both PISA and TIMSS have been seen as disappointing by the government and academia when compared with their counterparts in East Asian countries. Examining the reasons for this gap between England and East Asia countries, researches have approached the issue in different ways. Jerrim and Choi (2014) discerned the point in English students' schooling at which this gap appears, namely at the beginning or the end of middle school. In addition, most of the comparative studies related to England focus on features of official curriculum documents (Haggarty & Pepin, 2002; Park & Leung, 2006; Sun & Jia, 2003) and corresponding classroom interaction (Leung, 1992; L. Wilson, Andrew, & Below, 2006).

Researchers have carefully and systematically explained the outcomes from different perspectives. Learning outcomes are, however, influenced by complex factors (Broadfoot, Osborn, & Planel, 2000). Among these factors, the fact that understanding could facilitate learning has been well acknowledged (Newton, 2000) as discussed at the beginning of this chapter. Studies regarding how English students understand mathematical concept compared with their counterparts in East Asian countries are impoverished, although one recent example is the study by H. Li (2014) analysed English and Taiwanese students' performances (age 12 and 13) in case of fraction addition from procedural and conceptual knowledge

perspective. The present study attempts to explore some features of students' understanding development within a coherent picture of four perspectives which has been discussed in the first section, as originated in a general model of understanding function to explore the performance gap in more details.

1.2.4 Problem-solving

Problem-solving is considered to be the foundation of learning mathematics (P. Thompson, 1985), a basic trend for reform in curriculum and instruction (Hiebert et al., 1996), and also related to understanding (Kluwe, 1990). The meaning of problem-solving varies from 'working [with] rote exercises to doing mathematics as a professional' (Schoenfeld, 2009, p. 334). One of three aims in the national curricula in England from primary school to secondary school (KS1 to KS4) is to apply 'their mathematics to a variety of routine and non-routine problems with increasing sophistication' (Department for Education, 2013b, p. 3; 2013c, 2014). Although it has been noted that students' performance in school mathematics could not be used to predict their ability to solve non-routine problems (English, 1996), the underlying causes of these performance gaps should be investigated from how well students understand mathematical concepts and how to use these concepts.

Another report also released from PISA 2012, concerning students' skills in tackling real-life problems, highlighted that the mean score for Shanghai is 536 while 517 for England (OCED, 2014). This PISA 2012 mainly involves decision-making problems, system-analysis problems, resource-allocation problems, and so on, without requiring students' expert knowledge in mathematics. English students who do well in these tests demonstrate relative strength in the ability to apply their knowledge creatively in dealing with real-life context. It indicates that Shanghai students are relatively weaker in solving problems in a creative way, compared to the conventional use of mathematics as shown in PISA 2012. This test revealed that '86% of Shanghai students perform[ed] below the expected level in problem solving,

given their performance in mathematics, reading and science' (OCED, 2014, p. 70), whose items in PISA 2012 were related to specific subject knowledge in application. Of particular significance to tackling real-life problems was that Shanghai students showed their strength in knowledge-acquisition processes over knowledge-utilisation procedures, while English students show strengths in combining these two aspects. Students' strengths, to some extent, give an impression of how they approach, understand, and use the knowledge in each region.

When narrowing down to a certain topic within school mathematics, the present study not only focuses on the acquisition of knowledge and its application, but also the balance between the two in order to increase the effectiveness of problem-solving. The relationship between how well students understand basic knowledge and their application performance has previously remained under-researched.

1.3 Research Questions

This study investigates students' mathematical understanding within different cultural backgrounds. It is designed not only to explore the reality of what students understand, or what they do not, but also to look behind that 'what', to 'how' the understanding has been developed or shaped. One aim of this comparative study is to explore better way in the teaching and learning with understanding linear function for each region, as it is hoped that both should learn from each other.

To this end, Research Question 1 addresses the extent to which students in each region are expected to understand the topic and the different emphases placed on the topic by curricula and textbooks. Research Question 2 explores what students actually understand in the topic of linear function, as well as their approaches towards application in real world situations. Research Question 3 explores teacher views of teaching towards understanding mathematics. Through a comparison of the external differences between each group in terms of curricula, textbooks, and student tests, along with gaining the views of teachers, the

underlying reasons for the differences in understanding will be explored in Research Question 4. The four research questions are specifically put forward as follows:

Research Question 1: What are the requirements of the intended curriculum and officially used textbooks of the two regions in terms of linear function?

Research Question 2: What do English and Shanghai students actually achieve (the attained curriculum), with regards to linear function?

Research Question 3: What are teachers' views regarding the teaching and learning linear function?

Research Question 4: What shapes students' understanding of linear function?

1.4 Structure of the Thesis

In this section, I outline the structure of the thesis in three parts:

Part One consists of the broader research context, from Chapter 1 to Chapter 4. According to the title of this thesis, the first part is about comparative study while the second one is related to the understanding of function. Correspondingly, the two chapters of literature review will reflect on these two areas. Chapter 2 provides an overview of the education systems in the two regions, including the prevailing views on the nature of mathematics and current comparative mathematics education researches related either to England or to China. Chapter 3 contains a detailed literature review examining firstly the research on function and secondly the notion of 'understanding' (including models of understanding function) in order to form a general model of understanding function which can be applied as the theoretical framework at this study. Chapter 4 sets out the methodology used to approach the research questions, including considerations of ethics, validity, and limitations.

Part Two details the results from the four parts of the research: curriculum analysis in Chapter 5; textbook analysis in Chapter 6; student tests in Chapter 7; and findings from

teacher interviews in Chapter 8. Results from the curricula in the two regions in Chapter 5 and official textbooks in Chapter 6 will be used to answer the first research question. The findings from the student tests revealing the strengths and weaknesses within student understanding in each region, and also patterns within the application, are presented in Chapter 7 to answer Research Question 2. The selected teacher interviews are documented in Chapter 8 which explores the perceived barriers to understanding linear function and their views on teaching and learning in detail. These findings will answer Research Question 3.

Part Three includes a discussion of findings, Chapter 9, and the conclusion, Chapter 10. In the discussion, synthesising the important findings of each results chapter highlight several key issues explaining the performance gap between England and Shanghai from an understanding perspective, relating these issues to the existing literature, in answering Research Question 4 and the contribution for knowledge. The conclusion raises the issue of considering measurement processes used for comparative projects and the needs to focus on Shanghai students' understanding development using a more holistic picture of context in further studies.

1.5 Significance of the Study

As a result of the current interest in understanding mathematical concepts within the two different cultural contexts, one of the main values of the study is the step towards revealing the reasons for the gap in students' performance in mathematics, as generally shown in current large-scale cross-national projects. Although Alexander (2012, p. 12) stated that the quality of teaching matters most and proposed studies to explain how PISA high-flyers achieve their success using 'first-hand empirical data systematically and transparently presented', Elliott (2013, p. 454) additionally argued that research should examine what kinds of knowledge and skills were 'considered to be most worthy by students'. Therefore, this

study will also reveal how each region values the knowledge and skills involved in the concept of linear function.

Despite the fact that little mathematics comparative research has been undertaken between England and Shanghai, this study will increase the understanding of not only how the mathematical concept is handled in the two regions, but also the strengths and weaknesses of the approach that each region takes. Underlying the specific examples of mathematical concepts, unique social expectations, educational aims and social meanings of curricula are also central to the comprehension of these different requirements of understanding mathematics (Bishop, 1994). It suggests the importance of curricula in understanding development. Both commercial or compulsory textbooks represent the requirements of the respective curricula, and their role is deemed to link the curriculum with pedagogy (Pepin & Haggarty, 2001). Through an exploration of the emphasis of curricula and textbooks in terms of understanding, teachers, textbook writers, and national curriculum-makers will be able to reflect on current learning trajectories. The results of this study will enable them to evaluate what kind of change will be the most worthwhile to improve students' mathematical understanding development.

Another aim of this study is to explore students' mathematical understanding by using a general model of understanding function. This model is an innovative aspect of the research with the following benefits: (1) the facilitation of the diagnosis of students' understanding of linear function; (2) helping teachers invent suitable pedagogical activities and strategies to enhance students' understanding development; and (3) offering teaching guidance.

Finally, although the student performance gap has been explained by cultural factors, pedagogy and problem-solving strategies; few studies have examined how to present the nature of mathematical knowledge under these factors or strategies' influences. Shanghai or China advocates four basics of learning mathematics: basic knowledge, basic skill, basic idea

and method (simply called basic method), and basic experience as will be discussed at Chapter 5 and 8. How these basics shape the way mathematics is being taught will be revealed in case of linear function. Furthermore, how England develops students' understanding will be compared with the Shanghai approach. As noted earlier, this study is not only interested in students' understanding, but also in the way in which they are expecting to arrive at the higher levels of understanding. That is, the comparison of different approaches seeks to discover the most effective way to understanding function from a perspective of different representations.

1.6   Case Study

This study aims not only to explore students' understanding development through an international comparison perspective, but also to develop an understanding model which would be useful in other algebraic topics or other countries. Therefore, mixed methods will be adopted with an initial exploratory feature – the pilot study. In order to undertake the in-depth study of understanding development and what shapes this development in each area, a case study approach involving three schools in each education system will be chosen for the study.

1.6.1 The English case

In England, The Office for Standards in Education (Ofsted) inspects and regulates services that care for children and young people, and services providing education and skills for learners of all ages (Ofsted, 2015). Ofsted visits schools to help them improve, monitor the progress and share the best practice they find. The results for each school can be found on school websites as well as Ofsted website (http://reports.ofsted.gov.uk/). Ofsted grade descriptors have four grades: Grade 1 is Outstanding, Grade 2 is Good, Grade 3 is Requires improvement, and Grade 4 is Inadequate (Ofsted, 2014, p. 38).

Table 1 below shows the details of the sample schools from their latest school inspection report. Two sample schools (School SEN1 and SEN2) were related 'Good' (Grade 2, having very positive features of a school and serving its pupils well) during this research. Specifically, four elements: achievement of pupils, quality of teaching, behavior and safety of pupils, and leadership and management, were all considered as Grade 2 for these two schools. The third school, School SEN3, was rated as 'outstanding' (Grade 1, being highly effective and providing exceptionally well for all its pupils' needs) for overall effectiveness by Ofsted report with Grade 1 in both pupils' achievement and the extent to which they enjoy their learning, and the effectiveness of leadership and management in embedding ambition and driving improvement; and Grade 2 in terms of the quality of teaching.

Table 1

The Details of the English Sample Schools

School	School category	Age range of pupils	Appropriate authority	Grade for overall effectiveness
SEN1	Academy	11-18	Excel Academy Partnership	Good
SEN2	Voluntary aided		The governing body	Good
SEN3	Community		The governing body	Outstanding

The case in England, to some extent, represents how well students in 'good' and even 'outstanding' schools understand mathematical concept. With regards to student achievement in mathematics, the schools' GCSE rankings are considered and will be discussed in Chapter 4: Methodology. Meanwhile, the structure of GCSE mathematics being offered by the three main examination boards: AQA, EDEXCEL and OCR which will be discussed in Chapter 2. Three sample schools, however, have covered all three examination boards which have also reflected on their individual choice of the official used textbooks.

1.6.2 The Shanghai case

In Shanghai, the equivalent to Ofsted is Supervision Office. The supervision and inspection is envisaged as 'school developmental supervisory evaluation'. The reports they produce have three elements of school life: educational resources, school management, and student development. However, neither grades nor marks are shown in the report. The report only reflects on the progress or otherwise that the school has achieved since the last supervisory evaluation, what issues are remaining in the school, and their suggestions for further development. The key indicator for school quality is, however, considered to be students' academic outcomes and entrance levels to higher education, namely two standardized student assessments: Entrance examination for senior secondary school (*Zhongkao*) and Entrance examination for Higher Education (*Gaokao*) (Jung Peng, Thomas, Yang, & Li, 2006). These two assessments will be discussed in Chapter 2 section 1: Education in England and Shanghai.

Educational aims in China have heavily emphasised well-round individual development since 1990s, namely quality education rather than examination-oriented education. Moral education, physical, aesthetic education, education for ethnic minorities and so on, have been listed by the banner of quality education in the 1999 Action Plan (Department of Education, 1998). Although Dello-Iacovo (2009, p. 248) argues that examination-oriented nature of education in secondary school level 'remains unchanged', curriculum reforms in China have actually led to the emphasis of physical and aesthetic education. The three sample schools in Shanghai are all state secondary schools with common characteristics: arts and sports, but they have their own distinctive features. During the data collection period, sample schools' specialism was orchestra, harmonica, and drama respectively. Meanwhile, two of them have been both named as bilingual schools (Chinese - English) in Shanghai. In 2015, all of them were named as traditional sports school which 33

state schools (118 in total) have at district level. The case in Shanghai represents, to some extent, those schools which under the examination-oriented nature of education successfully promote quality education curriculum.

With regard to mathematics performance, the similar ranking corresponding to the English sample school will be addressed in Chapter 4: Methodology as well as compulsory textbooks used in Shanghai. The typical case study can 'make previously obscure theoretical relationship suddenly apparent' (Mitchell, 1984, p. 239). Therefore, from analytical interpretation perspective, this study is a typical (or paradigm) case rather than critical (or telling) case.

1.7 Summary

This chapter has presented the outline of this study, the rationale, the four research questions, the structure of the following chapters, and the importance and significance of the study. In the next chapter, the background of each education system and studies in comparative education will be explored.

Chapter 2 Research on Comparative Education

This comparative study focuses on two regions, England and Shanghai. It is necessary to review relevant comparative studies, so as to set out the rationale for my own methodology, which will be detailed in Chapter 4: Methodology. This current chapter has four main sections. The first section will introduce some of the important educational differences between England and Shanghai. This will involve the organisational factors of schooling and a discussion of their different cultural and systemic contexts. Based on these differences, a summary of current comparative mathematics education studies will be presented in the second section. This summary will set out the themes emerging from two large-scale cross-national projects, PISA and TIMSS, as well as other smaller scale studies concerning the respective national curricula and the used textbooks. Thirdly, this study will also focus on perceptions of the nature of mathematics as a way of explaining opposing stereotypical views of mathematical knowledge prevailing in the two regions. Finally, research about teachers' belief and practices related with two regions will be addressed.

Although Boote and Beile (2005) stated that a thorough examination of the field and the ability of synthesize the existing results were central for both literature review and methodology issue in doctoral research, Maxwell (2006, p. 29) argued that 'relevance, rather than thoroughness or comprehensiveness, is the essential characteristic to literature review in most scholarly works'. Two chapters' literature review of this thesis will be following this approach. Chapter 2 focuses on mathematics education area from a global perspective which will be divided into the four aforementioned sections. The first section aims to provide the overall view of education system. To do so, literature proposed by English academia was the main foci to highlight the different features. The following three sections will include literature from four perspectives related to each result part: student assessment, curriculum analysis and textbook analysis in the second section; and teacher's belief in the fourth section.

2.1 Education in England and Shanghai

The disappointing performance of England in mathematics and science among international studies since the 1960s led the Office for Standards in Education in England (the English national inspectorate) to review results from these studies. This review also included a review of other smaller scale studies on the processes and effectiveness of the English system itself. In the *Review of International Surveys of Educational Achievement involving England*, Reynolds, Farrell, and Britain (1996) explored the possible explanations for the students' performance gap in maths and science between England and the Pacific Rim countries. Attention fell on four potential explanatory factors: cultural factors, systemic factors, important school factors, and key classroom factors. This section focuses on the first two factors. The third factor, important school factors, partly referred to specialist teachers which are the same for the two regions at the secondary school stage. Some key classroom factors, such as the use of the same textbook, will be subsequently discussed in Chapter 6: Textbook Analysis.

Therefore, this section will briefly introduce the basic education systems in England and Shanghai from a schooling perspective in order to gain a broader picture. The second sub-section will address the culture factors, and this is followed by the systemic factors in the third sub-section.

2.1.1 Schooling

In both regions, the majority of state schools are non-selective in nature, having pupils of all abilities. The whole compulsory education period in England and Shanghai has been illustrated in Table 2 for the purposes of clarification.

Table 2

School Systems in England and Shanghai

		Primary				Junior secondary				
	Key Stage 1	Key Stage 2				Key Stage 3			Key Stage 4	
England	Year 1 (age 5)	Year 2	Year 3	Year 4	Year 5	Year 6 (age 11)	Year 7	Year 8	Year 9	Year 10	Year 11 (age 16)
		Primary				Junior secondary				
Shanghai		Grade 1 (age 6)	Grade 2	Grade 3	Grade 4	Grade 5 (age 11)	Grade 6	Grade 7	Grade 8	Grade 9 (age 15)

England. Most children in England attend a reception class at primary school from the age of 4 years old. Students receive six years of primary schooling (age 5 - 11) excluding reception, and five years of junior secondary schooling (age 11-16), where compulsory education leads up to a national examination for sixteen year olds, the General Certificate of Secondary Education (GCSE). Candidates will hopefully complete eight to ten individual subjects. Among these subjects, mathematics is one of the three compulsory subjects (English and science being the other two). If students continue post-secondary schooling (age 16 - 18), then they would be given the choice whether or not to continue learning mathematics. After two years, they will undertake examinations for the General Certificate of Education at Advanced and Advanced Supplementary (A/AS) level, which is regarded as the primary measurement for entry requirements for undergraduate study. The grade at GCSE mathematics that students gained, however, is also an indicator of eligibility in the undergraduate application.

Shanghai. Shanghai children start their schooling at the age of six, with five years at primary school (age 6 - 11) and four years at junior secondary school which covers the ages of 11 to 15 (Grade 6 - 9). After this compulsory education stage, students have to take the senior secondary school entrance examination (called *Zhongkao*) at age 15, one year earlier than that of English students in their GCSEs. There then follows three years of senior

secondary schooling (age 16 - 18) and, within this period, students take the Shanghai

Certificate of Senior Secondary examination (called *Huikao*). Most of them then participate

in the college or university entrance examinations (known as *Gaokao*). In senior secondary

school, mathematics which is the compulsory subject has two forms differing in their degree

of difficulty and these are referred to as the Arts and Science levels. The requirements of the

Science level will be higher than the Arts one.

In summary, the different settings in the respective region illustrate that mathematics

is an important subject for both but it is significant in Shanghai in pre-university schooling.

The end of compulsory education in England is also notably one year later than that in

Shanghai.

2.1.2 Cultural factors

This sub-section has three parts. The overview of cultural features will be briefly

introduced by how these values serve the education system in England as well as how values

of Confucian thought are embodied in the Chinese or Shanghai education system. Cultural

factors embody the characteristics of education systems (Alexander, 2012). Consequently,

studies on cultural differences provide an insight into how culture impacts on the learning

process. The role of the examinations in Shanghai will be discussed from a cultural

perspective. Following this view, the form of final assessments for compulsory education

stage in the two regions will be compared.

Cultural values. From a religious perspective, Sharpe (1997) suggests that England's

Protestant authority and democratic heritage has influenced the education system.

Historically, English students have been acknowledged as ones with individual needs and

abilities (Osborn, Broadfoot, McNess, & Ravn, 2003). As a result, the Education Act of 1994,

which has influenced the current structure of English schools, proposed that secondary school

students should be split into three broad groupings; but now, they are generally separated into

two groups, the Higher Level and Foundation Level. The evaluation system for student performance, GCSE, which will be introduced at following section, is the tiered examination which also implies or requires splitting pupils into two groups. Schools normally arrange students according to their ability levels. That is, students are typically organised into different hierarchical class groupings, called 'sets', such as four sets for Higher Level students in mathematics, within which students have the opportunity to change during school years. That is, students can move towards a higher set if they perform very well in their current set.

In Eastern countries, Confucianism or Confucian-heritage Cultures (CHC) have been regarded as traditional cultural values. This has deeply influenced the educational process for thousands of years. However, Wong (2004) argues that it is particularly the examination culture within East Asian societies that impacts on students' higher performances rather than that of Confucianism. From the 7th Century to the beginning of 20th Century, the Civil Service Examination, also called imperial examination, had been taken and its purpose was to select candidates for the state bureaucracy. Chinese education is therefore considered to be a 'scholar-nurturing education' with a rigorous examination system (An, 2004, p. 464).

In particular, Tan (2012a, p. 161) illustrated that Shanghai has had a 'dominant high-stakes exam-oriented culture'. Historically in China, exams have functioned like a conductor's baton which 'determines how the music score is to be played' (Tan, 2012b, p. 60). This means that each perspective of the education system would be influenced by the examinations, such as what kind of knowledge would be highlighted at the tests. Behind this exam-oriented system, there are three elements currently changing within the Shanghai education system: the selection function of suitable students which has not changed fundamentally but has been highly diversified; independence, as teachers and schools have

more autonomy working towards national, local, and school-based curricula; and justice, as migrant children have the right to compulsory education in Shanghai state schools.

The selection function is dependent on how students perform in the final examinations (called *Zhongkao)*. This will be discussed in more detail later. In terms of the second element, the newly-acquired independence of students and teachers is constrained by frequent standardized assessments at district level. Every term the district educational authority organises uniform tests for all of the state schools. During the term, some secondary schools whose academic performance is similar will join together to hold the same examinations at mid-term or even monthly tests. Therefore, there is actually not much room for the autonomy of teachers or schools in Shanghai. The nature of justice, the third element, is related to all students' equal rights to attend the *Zhongkao* and *Gaokao* in Shanghai, which is as yet unresolved as some of migrant students from other provinces in China still cannot attend the university entrance examinations in Shanghai (similar with A-level in England) (Deng & Zhao, 2014). All these three factors are tied to assessments. Therefore, the examinations still play an essential role in the whole education system. In line with the heavy emphasis on examinations in Shanghai, Burghes (1999) in fact recommended more regular mathematics assessments for English students to enhance the evaluation of students' learning outcomes.

The evaluation system for students' performance - GCSE and *Zhongkao*. Following the achievement tests, it is relevant to see how each education system monitors the final assessment when students finish their compulsory education stage.

In England, the evaluation system for students' academic performance is conducted by different examination boards, though all of them follow the requirements of the national curriculum. Therefore, the content of these examinations is basically the same. The exam boards are responsible for setting and awarding secondary education level qualifications including GCSEs. Three exam boards are widely used: AQA (Assessment and Qualifications

Alliance); OCR (Oxford, Cambridge and RSA Examinations); and Edexcel (Pearson Edexcel as of April 2013). All of the exam boards include Higher Tier and Foundation Tier assessments for students with different abilities. Each school or department is free to pick the appropriate exam board for their students. In the case of mathematics, students are required to attend three exams in mathematics: GCSE Mathematics (Linear) paper 1 (non-Calculator exam); GCSE Mathematics (Linear) paper 2 (Calculator allowed); and the exam for the GCSE Applications of Mathematics Unit. Student performance is indicated by grades ranging from A* to G and U, where Grade C and above is regarded as passes.

At the end of the compulsory education period, students in both regions have to undertake a final examination. The result of students' performance is not only used in their applications for further education, but also used to demonstrate the quality of schools. For example, all state schools in England are evaluated within a national league table according to GCSE performance at each subject. Although there is no official league table for state schools released in Shanghai, schools' performance in *Zhongkao*, such as how many students in the school can enroll in the key senior secondary schools, is also highly valued.

Compared with GCSEs, Shanghai's *Zhongkao* is a one-off examination and a student's performance is calculated by overall scores of six subjects: Chinese, Mathematics, English, Physics, Chemistry, and Physical Education (P.E). Calculators are not allowed in any subjects within *Zhongkao*. Full mark in the mathematics exams amounts to 150 points, nearly one quarter of the total available score: 630 points. Public examinations therefore play a leading role in the process of teaching and learning, and mathematics will be given much more attention than other subjects. M. Han and Yang (2001) put forward the criticism that students would, as a result of the exam system, be put in a passive position in order to gain proficiency skills in answering paper-and-pencil examinations. These skills may be one possible reason for Shanghai students obtaining a higher level of performance in PISA.

In conclusion, English students are divided into two ability levels in order to meet the different stages of assessment. In terms of two levels of assessments, basically there are two corresponding levels of textbooks in England. Some KS3 textbooks have several levels. It is notable that teachers in England are the lowest users of textbooks, apart from Iran and Islamic Rep. in TIMSS 2003 (Mullis, Martin, Gonzalez, & Chrostowski, 2004). In school, other resources, for example the Framework or the scheme of work, are important. Conversely, Shanghai's centralised education system leads to the uniform requirements for all Shanghai students within mixed ability classes. Shanghai students are expected to take only non-calculator assessments. Therefore, it is assumed that Shanghai teachers tend to heavily emphasise the importance of numeracy skills or procedural knowledge which will be discussed at Chapter 8: Teacher Interviews.

2.1.3 Systemic factors

Systemic factors mainly refer to mixed ability classes and high quantities of school time including cramming institutions (Reynolds et al., 1996). Mixed ability classes and frequent testing in core subjects in Pacific Rim countries were applied to enhance student attainment on achievement tests. So did Shanghai which has been discussed in the previous section, as was the evaluation system for students' performance.

Looking at other elements, an examination of the high quantities of school time will present the similarities and differences between organisational factors, especially time spent in mathematics classes. In English secondary schools, weekly mathematics is generally delivered over three lessons with one hour of homework, adding up to four hours overall as reported by the selected English teachers in Chapter 8: Teacher Interviews. Conversely, the Shanghai junior secondary schools normally arrange mathematics lessons and corresponding homework every day, amounting to roughly 1.5 hours of one school day in total. That comes to 7.5 hours per week, nearly double the time that the English students experience. The time

that students put into mathematics, differs between two regions, not merely for school maths. In addition, researchers have identified that private supplementary tutoring in cram schools is also highly demanded in China (Kwok, 2010), and more significant in East Asia than in any other area of the world (Bray & Kwok, 2003). These are additional hours. Therefore, that students spend more time in private tutoring can also provide a possible reason of the performance gap between England and Shanghai.

2.2 Comparative Mathematics Studies

This section contains three sub-sections pertaining to the comparative mathematics area. The first sub-section provides a general overview of students' performance within two large-scale cross-national studies, PISA and TIMSS. Both Shanghai and England have taken part in the former, while other East Asian countries and England have participated in the latter. Additionally, TIMSS offers an analytical framework for three different curricula: the intended curriculum, the implemented curriculum, and the attained curriculum, which is adopted in this study's research questions. In the second sub-section, the consideration of methodologies used for comparative studies will be addressed, because the results from these two projects have been criticised, especially with regards to the methodology they use and the tendency to generalise the findings. Finally, comparisons of official documentary materials related with England and China will be discussed to delineate trends for curriculum and textbook research area.

2.2.1 PISA and TIMSS

In the area of comparative research, large-scale cross-national projects have been employed over the last few decades. The results from these projects have uncovered students' performances and relevant factors which might influence their learning. These results provide a comparative overview of how well students do in mathematics on a country-by-country basis, as well as providing a reference point for policy makers and the educational

community to determine how to enhance students' learning outcomes. The purpose of PISA considers the question of 'what can you do', while TIMSS focuses on 'what do you know' (Hutchison & Schagen, 2007, p. 26). Although their scope and foci are different, the correlation of PISA and TIMSS has reached up to '0.91' in the Germany study (R. Adams, 2003, p. 380). League tables have been produced in order for academics and policy makers to see what is possible (Fan, 2011), but in fact, the results have been interpreted as the sole parameters of system quality (Broadfoot et al., 2000).

Over the years, the PISA test has taken on greater significance around the world as countries look to learn from others. The results of PISA 2003 suggest that there exists a positive relationship between a country's Gross Domestic Product (GDP) and students' mathematics performance (Anderson, Lin, Treagust, Ross, & Yore, 2007). On the basis of the league table, high-performance education systems or new reference societies are identified, such as Shanghai (Sellar & Lingard, 2013). These reference societies are able to offer other countries' policy-makers potential information which can be transferred to their local context (OECD, 2013).

PISA was launched by OECD in 1997 and started in 2000. The target sample is 'aged between 15 years 3 months and 16 years 2 months at the time of the assessment, and who have completed at least 6 years of formal schooling' (OECD, 2013, p. 26). The students' performances are evaluated from three perspectives: reading, mathematics, and science. There are three year cycles of PISA, each of which mainly focuses on one aspect, such as reading in 2000, mathematics in 2003, science in 2006, reading in 2009, and mathematics in 2012 (the fifth PISA cycle). PISA aims to look into the relationship between several factors related to learning and students' achievements; for example, student attitudes, home background variables, and school information (Anderson et al., 2007).

Either in mathematics performances or in the overall achievement for both PISA 2009 and 2012, Shanghai has remained in the top position, while Hong Kong (3rd position) and Chinese Taipei (4th position) also did very well in the area of mathematics at PISA 2012. England was around the international average in terms of mathematics performance: 28th in PISA 2009 (2 positions below the OECD average mean score), and 26th in PISA 2012 (exactly on the average).

The PISA 2012 assessments paid primary attention to mathematics proficiency which means the capacity of individuals to 'formulate, employ and interpret' mathematics in various contexts (OECD, 2013, p. 25). The PISA 2012 problem-solving skill assessment, however, pointed out that Shanghai students demonstrated weaknesses regarding 'curiosity, perseverance and creativity' (OCED, 2014, p. 91).

Although Tan (2012a) argued that questions in PISA were presented in a testing situation rather than a real-life situation, PISA did provide an outline of how well 15- and 16-year-old students are prepared for life's challenges in each participating country. TIMSS, however, may imply how to improve teaching and learning (M. Wu, 2009).

The IEA planned the first international surveys in education in 1958 and the Second International Mathematics Study (SIMS) in 1976. TIMSS in 1995 was the largest assessment at that time, with over 40 countries participating (Beaton & Robitaille, 1999). In the mathematics section, it focused on 9 year olds, 13 year olds, and the final year of secondary school. Six content areas were covered for 13 year olds (Grade 7 and 8): (1) fractions and number sense; (2) measurement; (3) proportionality; (4) data representation, analysis, and probability; (5) geometry; and (6) algebra. Among these, the percentage of algebra questions was 18% of all the content. Shanghai did not participate in TIMSS. There was a specific example of a given algebra question to solve a linear equation for *x* in TIMSS (Beaton et al., 1996, p. 77), as follows:

ITEM 15. If $3(x + 5) = 30$, then x=

A. 2   B. 5   C. 10   D. 95

Results revealed that in England the correct response was only 61% while the international average was 72%. This item assessed students' basic skills for solving a simple linear equation. These findings indicated that, in terms of this algebraic skill, English students were at a much lower level than the international average, though in terms of this item there was not reported the indications of statistical significance. The results on this item, however, suggest a weakness in English students' basic skills.

Four years later in 1999, the Third International Mathematics and Science Study-Repeat (TIMSS-R), namely the fourth comparison study sponsored by IEA, was conducted in order to track changes in achievement. Table 3 compares the average mathematics achievement of Grade 8 students in TIMSS-R (U.S. Department of Education, 2000, p. 86) as well as the mean mathematics achievement in TIMSS (Beaton et al., 1996, p. 22). The achievement of English students had improved, and was slightly above the international average.

Table 3

Mathematics Standardized Scores in Grade 8 Students in TIMSS and TIMSS-R

Country	Average in TIMSS	Average in TIMSS-R
England	496	506
International average	487 (38 nations)	513 (41 nations)

In summary, PISA and TIMSS including TIMSS-R have all demonstrated the lower mathematical performance of English students, although their performance has slowly improved. The algebra section has been in fact the weakness of English students compared with the average international achievement.

In addition to this, the TIMSS project divided the mathematics curriculum into three levels (see Figure 1 below): the intended curriculum, the implemented curriculum, and the

attained (or achieved) curriculum. These classifications have been widely acknowledged and applied by many comparative researches (Cai & Ni, 2011; Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002). The intended curriculum includes official curricula and syllabi, normally produced at a state level, such as statutory guidance in England, or at a provincial level in which the province is allowed its own educational system, such as the local curriculum in Shanghai. At the classroom level, teachers have to implement the curriculum according to the various levels of students' previous knowledge or backgrounds. Schools and teachers may 'go beyond' or 'ignore' some aspects of intended curriculum (Foxman, 1999, p. 5). Finally, students' learning outcomes would as a result be regarded as the achieved curriculum (Cai & Ni, 2011).

Results from TIMSS also reveal that the implemented curriculum is 'not identical' to the intended curriculum (Mullis et al., 2004, p. 164). Here, textbooks link the intended curriculum at the national level with the implemented curricula at the level of the classroom (Foxman, 1999; Y. Li, Chen, & An, 2009). Textbooks are also extensively used in the classroom around the world (Fan, Chen, Zhu, Qiu, & Hu, 2004). The comparison of the intended curricula, textbooks research, and the subsequent tendencies of how learners deal with problems will be discussed below. Before that the next sub-section focuses on the methodological approaches adopted by the large-scale cross-national projects.
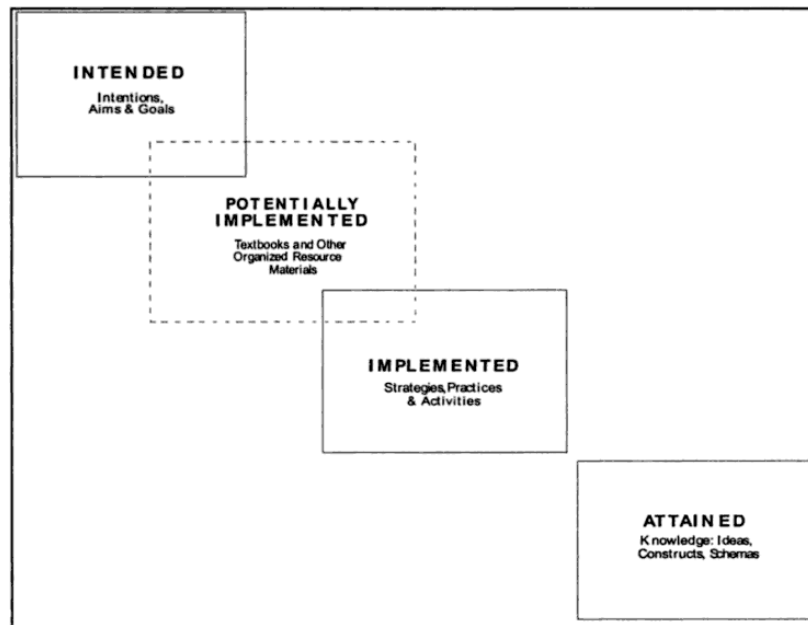
Figure 1. The Tripartite Model of curricula classification (Valverde et al., 2002, p. 13)

2.2.2 A rationale for methodology

To a great extent, the results from large-scale cross-national comparative studies in mathematics education have aroused widespread concerns regarding the methodology used within academia, namely quantitative-led methods. Therefore, there are two aspects in particular that have been the foci of these concerns: quantitative approach and generalisation. Actually, they are interwoven as the quantitative approach is associated with the problem of generalisation. Here, it is notable that Shanghai students' performance cannot present or be generalised towards the whole country, as they top the national group of pupils in China (Sellar & Lingard, 2013).

There are two main criticisms of quantitative research whose method dominated in PISA:

1. The reliance on instruments and procedures hinders the connection between research and everyday life.

2. The analysis of relationships between variables creates a static view of social life that is independent of people's lives (Bryman, 2004, pp. 159-160).

Basically, these criticisms cause the demands for qualitative researches. These large projects indeed value quantitative measures of performance over qualitative ones. The results have given rise to two types of methodological concerns: whether the sample could be representative of all students within each country, and whether a valid conclusion could then be generalised to describe the education reality in each country. In other words, the debates surrounding these projects lay stress on the generalised students' performances around the country, and the so-called several related major factors which influence students' achievements. One typical enquiry into these projects focuses on choosing the factors related to students' learning outcomes, what these factors are and why they are chosen.

In terms of the first concern, there was a debate within English academia regarding the disparate nature of English students' performances in the results of PISA 2000 compared with that of TIMSS 1995 and TIMSS 1999. First, Prais (2003) argued that the fewer representatives of schools in England would cause bias in reported average scores and representations of students; the strict 15-year-old criteria would exclude older students, so that it might impact on related factors, such as teachability. R. Adams (2003) then criticised Prais' incomplete understanding of methodology in large-scale assessments. He argued that even non-responses from schools could be caused by a variety of factors, but there was no significant relationship between schools' average score in GCSEs and whether or not they took part in the survey. Later, Prais (2004) still retained his previous judgement of poor reliability, due to the low response to these projects in England. These criticisms of the inadequate sample of respondents in fact reflect the issue of the representation of schools or individuals, because the purpose of a purely quantitative approach is to generalise the findings.

Particularly, several concerns for how to interpret the league tables have been made because PISA tested how to use knowledge instead of what certain mathematics knowledge

is. Moreover, the diverse foci of the curricula in each participating country influence students' performance in some ways. Some curricula place emphasis on the solving of problems; some prefer the recall of mathematical knowledge; and others might concentrate on justification and proof (Clarke, 2003). The results have also been criticised as the higher scores that students achieve in these questionnaires only reveal that they could correctly answer more questions. Clarke also proposed the curricular alignment to form the assessment which will be discussed at Chapter 10: Discussion. Secondly, there is the public's impression that whichever country achieves the higher ranking relates to the intelligence of their students. Press coverage within the countries that achieved low results showed their refusal to accept their less successful position, such as Canada (Stack, 2006). Similarly, the England's academic debate proved that the PISA outcome did not meet academic writers' preconceptions (R. Adams, 2003). It has been increasingly demanded that the quantitative results should be explained in comparative research (Creswell & Clark, 2007) and that other contextual differences should be acknowledged (Lin, Bumgarner, & Chatterji, 2014). This desire of explaining more has resulted in the quantitative data being reused.

Then, those data collected from large-scale projects has been used or combined with other data for other research purposes. In terms of discerning whether USA teaching changes between TIMSS 1995 and TIMSS 1999 had an impact, Jacobs et al. (2006) used videotapes from these two projects to look at eighth-grade (age 13/14) classrooms. In comparing how teachers handle the same topic between Hong Kong and Shanghai, Huang (2002) used the data from Hong Kong classrooms from the TIMSS-Repeat Video study, while Shanghai classrooms were videotaped following the procedure suggested by TIMSS-Repeat. In order to figure out when the students' performance gap between England and Eastern countries increases, Jerrim and Choi (2014) compared English students' and East Asian students' (Japan, Hong Kong, Singapore and Taiwan) achievements between 10-year-old and 16-year-

old students using the data from TIMSS (4th grade, age 9/10 in TIMSS 2003 and 8th grade, age 13/14 in TIMSS 2007) and PISA 2009 (aged 15/16). In summary, the studies that re-used data within a more contextualised approach are another branch of the comparative study movement, because these studies offered another view of interpreting students' performance. However, those findings are still not wholly satisfactory for academia, government, or public stakeholders who are eager to identify what can be done to improve students' performance in a practical way. This study will further examine the students' performance gap between England and Shanghai, but the gap shown in these large-scale projects is acknowledged as the background view to the present study.

In terms of the second concern, whether or not conclusions from quantitative research are valid for the reality of education has been a controversial issue. Firstly, it is questioned that the sum of each element, such as students' performance and teachers' questionnaires, could present the total of the educational reality in each country. Secondly, results from these questionnaires might lack consistency, namely in relation to how individual element operates with respect to others. Thirdly, the generalisation gained from quantitative results only explains the nature of the whole educational reality from a fragmental approach (Cohen, Manion, & Morrison, 2011) as opposed to considering education from a social science perspective.

In the next sub-section, narrowing towards curriculum comparisons, findings are able to provide more details about what has been identified so far in comparative studies.

2.2.3 Curriculum comparisons

Clarke (2003, p. 149) argued that international comparative studies should be 'evaluative', rather than a mere comparison of the similarities and differences. From this point of view, the following intended curriculum comparison and textbook research will document the literature findings related to England or China, or both.

The Intended Curriculum comparisons. Since TIMSS proposed the tripartite model of curricula (see Figure 1 above), researchers have sought to discover whether the coherence between the intended curriculum and the implemented curriculum matters with regards to successful learning outcomes. Oates (2011) proposed that it should be the foci of education reform in England, developing curriculum coherence and curriculum control as high-performance education systems did. Curriculum coherence is defined as the process in which 'standards move progressively towards the understanding of a deeper structure' (Schmidt, Wang, & McKnight, 2005, p. 529). The meanings of control and autonomy are opposite concepts in public education policy. Curriculum control policies might include 'textbook adoption, curriculum guidelines and testing' (Archbald & Porter, 1994, p. 22). Curriculum control is essential to curriculum coherence. Curriculum coherence includes the national curriculum content, textbooks, teaching content, pedagogy, assessment, or incentives which can reinforce one another. For example, Singapore's success is mainly secured by promoting curriculum coherence through the approval of textbooks and other teaching materials. Singapore's textbooks which follow its national framework facilitated students' understanding development compared with USA situation (Ginsburg, Leinwand, Anstrom, & Pollock, 2005). In Finland, textbooks and teaching materials are also tightly controlled. The curriculum coherence in terms of understanding linear function for both regions will be explored in Chapter 9: Summary and Discussion.

Cheng and Wong (1996) identified the features of conformity which shape the more uniform education systems such as Shanghai's: the single, consistent textbook and standardized assessments at the province or municipal levels. The Shanghai education system is highly centralised and controlled by the local municipal government rather than being regulated at the national level. In 1997, Shanghai was allowed to have its own curriculum. The local municipal (metropolis) curriculum and centralisation of textbook production are

both now subject to educational authority in the Shanghai government. Conversely, the English education system tends to be much less centralised 'in terms of educational experience provided' (Whitburn, 1995, p. 347).

Secondly, studies have been carried out to compare features of the intended curriculum in different countries. A national curriculum consists of concepts, principles, fundamental operations, and key knowledge. Elizabeth Truss, Parliamentary Under Secretary of State for Education and Childcare as mentioned at first chapter, advises that 'a rounded curriculum' is required for the curriculum reform in England (Truss, 2013, September 18). With respect to the content or structure of national curricula, Oates considered that the national curriculum in England has had 'significant structural problems' which should have been 'concept-led and knowledge-led, not context-led' (Oates, 2011, p. 132). Referring to how to introduce topics in the English curriculum, the intended curriculum has a distinctive attribute for arranging topics, which is separated into different years and becoming more complicated as students are allowed to progress and accumulate knowledge from year to year, which will be discussed at Chapter 5: Curriculum Analysis. A convoluted approach is adopted in USA as well. In comparison to the 'spiral curriculum' in USA schools' curriculum, the Chinese curriculum and instruction are much more 'sequential and non-repetitive' (Moy & Peverly, 2005, p. 253). Burghes (1999) suggested that topics in the mathematics curricula should have been treated more in-depth as well as with better organisation for the topics. The narrow yet deeper scope of the curriculum in China has also enabled students to gain basic knowledge and skills, but may not be appropriate for cooperative learning (Cai, Lin, & Fan, 2004).

It has been argued that the requirements of the mathematics curricula in East Asian countries are much more difficult than those in Western countries in terms of mastering the complexity of mathematics knowledge (Bao, 2002). For example, researchers examined the

difference between England and Japan regarding the solution of quadratic equations in junior secondary schools (Whitburn, 1995). The results showed that in England the approach to this topic is too limited, while in Japan it would be taught both algebraically and graphically. The approach towards mathematics in terms of linear function will be compared in Chapter 6: Textbook Analysis.

Thirdly, researchers also looked specifically at how students' understanding changed during the curriculum reform, especially in terms of algebra learning. For example, in the secondary school mathematics curricula in the USA, there was a fundamental change from the traditional emphasis on symbol-manipulation to a focus on problem-solving and application of mathematics knowledge. Comparing traditional curricula with the implemented reforms (to a standards-based mathematics curriculum), by using algebra and function as specific cases, Huntley, Rasmussen, Villarubi, Sangtong, and Fey (2000) found the algebra instruction that emphasised the use of graphing technology in order to solve authentic application problems (AAP), would be of benefit not only for students' problem-solving, but also for articulating abstract mathematical ideas. That is, the foci of curriculum changing towards solving real world situation facilitate students' understanding of mathematical topics, while how to balance these two perspectives should be carefully considered. For example, Cai and Wang (2006) have examined the difference between the traditional and reformed curricula in the USA in terms of their learning objectives, definition, and the development of equation-solving abilities in the case of linear function. From a mathematical perspective, they suggested that (1) the USA reformed curriculum should introduce more interpretations of the concept instead of just one explanation; (2) that the idea of 'variable as an unknown' in traditional curricula gave rise to confusion; and (3) that the application of linear function in real-world situations, such as weight and height, articulates the relationship between two variables. Therefore, the curriculum reform contributes to a deepening of students'

understanding of mathematics and successfully addresses the drawbacks from the traditional one. The question for policy makers is what will be the most effective changes to improve students' understanding for their own country, fitting in with different cultures, organisations and other realities.

In conclusion, curriculum coherence, emphasis on knowledge depth, and approaches to presenting a topic are the features of high performance education systems. The numerous ways in which to introduce the topic and being aware of the students' confusion raised by the arrangement of the curriculum is also becoming a trend of comparative education.

Textbook comparisons. The intended curriculum in all countries has been supported by ministry directives, instructional guides, school inspection, and recommended textbooks (Mullis et al., 2004). The importance of textbooks is embodied in several perspectives. First, textbooks not only act as a mediator between curricula policy and classroom instruction (Valverde et al., 2002), but they also link the curriculum and activities in classrooms (Johansson, 2003), though considerable gaps between curriculum standards and textbooks still exist (Fan & Zhu, 2007). For example, Johansson (2005) found that there was an objective gap of requirements between commercial textbooks and the national curriculum in the case of Sweden. Secondly, as a major conveyor of the curriculum (Fan, Zhu, & Miao, 2013), textbooks are the visible manifestations of the curriculum in most classrooms (Son & Senk, 2010). With regards to the implication of textbooks, they appear to influence teaching strategies (Fan, 2013). For example, classroom lessons have been compared in London, Beijing, and Hong Kong that have shown that teaching was highly influenced by the textbooks in these places (Leung, 1995). Thirdly, textbooks help teachers to learn not only the subject matter, but also the pedagogical knowledge (Collopy, 2003). Nicol and Crespo (2006) also found that prospective teachers felt that textbooks could offer more guidance on the basic teaching requirements, particularly because textbooks differed from the one with

which they were taught as students. The function of textbooks in England and Shanghai will be revealed by teacher interviews at Chapter 8.

Another trend of textbook comparative studies aims to discover what mathematics textbooks actually looks like, for example their layout. Compared with French and German textbooks, the layout of English textbooks has fewer questions and the structure is relatively brief (Pepin & Haggarty, 2001). That is, English textbooks are much concise on structure and the number of questions. Another kind of investigation is related with the content presented in textbooks. Eastern textbooks have focused on pure mathematics knowledge, while Western textbooks emphasise real-life situations. For example, Park and Leung (2006) compared the Grade 8 textbooks of Eastern countries (including China, Japan, and Korea) and Western countries (including England and the USA) and found the Eastern textbooks to be more beneficial for students when conveying an idea, but less successful in motivating students. The Western textbooks are effective in expressing the importance of mathematics in real-life, but unclear about the link between real-life situations and the mathematical concepts. Furthermore, focusing on characteristics of problems presented in textbooks, Zhu and Fan (2006, p. 614) argued that Chinese textbooks should present more authentic application problems (AAP) 'whose conditions and data are, indeed, from real-life situations or collected by students themselves from their daily lives'; whereas USA textbooks should consider more challenging problems for students with involving more steps in the solution, as China does. USA textbooks also include more visual information than Chinese ones. Furthermore, after comparing the content presentation of the addition and subtraction of integers between American and Chinese mathematics textbooks, the Chinese textbooks contain 'more problems with high level mathematics content' (Y. Li, 2000, p. 239).

The solution strategies in examples of Eastern textbooks such as China and Singapore are also less in number than in Western textbooks such as in the USA. Fan and Zhu (2007)

compared China, Singapore, and USA mathematics textbooks for problem-solving procedures in terms of two layers: general strategies referring to Polya's four-stage problem-solving model (understanding the problem; devising a plan; carrying out the plan; and looking back); and specific strategies. Chinese and Singapore textbook series merely presented the 'carrying out the plan'; while more than two-thirds of problem solving procedure presented in USA textbooks adopted at least two stages. This finding may partly explain why American students perform better than Chinese pupils in more open-ended problem-solving, as observed by Cai (1995).

Furthermore, due to the recent boom in textbook research, Fan (2013) proposed a framework for mathematics textbook research (see Figure 2 below). This framework involves three factors: the subject of the textbooks itself; textbooks as a dependent variable (how textbooks are affected by other factors); and textbooks as an independent variable (how they affect other factors). This framework has launched an appeal for the continuation of textbooks research. With regards to the dependent variable, the intended curriculum is supposed to be the main influence. In terms of the independent variable aspect of the continuation, studies have demonstrated 'how mathematics textbooks are used by teachers and students', 'how they impact the behaviour of teaching and learning of mathematics', and 'what the influences of textbooks on students' achievement in mathematics are (Fan, 2013, p. 772). Furthermore, Fan et al. (2013) pointed out that, although textbook studies have provided better understanding in terms of the role of textboooks, textbook analysis and comparison, and textbook use, research into the relationship between textbooks and students' learning outcome is still lacking. Especially in English situation, the relationship between the use of textbooks in England as discussed in TIMSS and student performance should be investigated further.
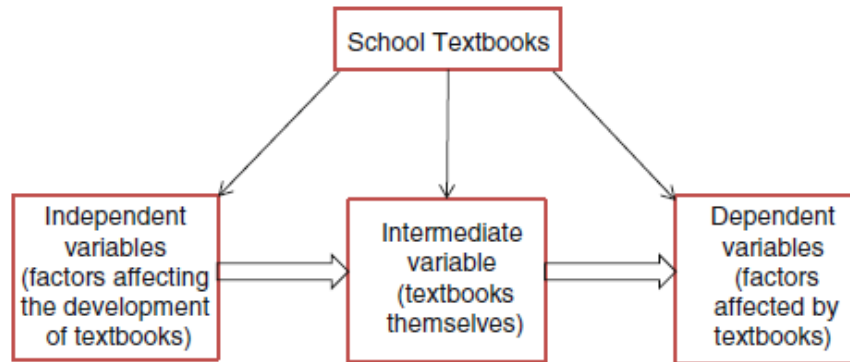
Figure 2. The framework of textbook research (Fan, 2013, p. 771)

2.3 Perceptions on the Nature of Mathematics

This study will not go into any further arguments about how the cultural and systemic factors (discussed in the first section) are able to point out the underlying reasons of the performance gap between the two regions. These two factors have been listed as an existing fact or background. To explain the gap, this study will be related to how the mathematical concept, in the case of linear function, is going to be taught and learnt. To do so, a discussion of the nature of mathematics needs to be addressed before the literature reviews on the concept of function.

2.3.1 Pure maths and applied maths

Mathematics has been considered to be the cornerstone of human knowledge. Schoenfeld (1992, p. 344) states that 'mathematics consists of systematic attempts, based on observations, study, and experimentation, to determine the nature of principles of regularities in systems defined axiomatically or theoretically (pure mathematics) or models of system abstracted from real world objects (applied mathematics)'. Here, two aspects of mathematics are outlined: the pure and the applied. Mathematical knowledge is considered to be either a kind of '*a priori* knowledge' or 'the paradigm of a certain knowledge' (Ernest, 1991, p. 4).

Furthermore, there are two main distinct views of mathematics: the absolutist view and the fallibilist one. The absolutist view regards mathematics as the type of knowledge

which is isolated and discrete from human knowledge. This isolation leads mathematics to become objective and value-free or, at least, not imbued with human values (Ernest, 1991). It is viewed as a pure knowledge. In an extreme case, it is considered that advanced mathematical thinking cannot be perceived by our common senses (Sfard, 1991). This view effectively contends that learning mathematics is to 'discover the already existing truths of formal logic' (Stigler & Baranes, 1988, p. 257). It leads to two kinds of worlds: a mathematics one and a human one. The danger of holding the absolute view of mathematics is to 'trivialize mathematics' as it was 'severely impoverished if it focused largely, if not exclusively, on issues of grammar' (Schoenfeld, 1992, p. 335). Schoenfeld also suggests that this view leads to the erroneous analogy that learning mathematics is to master a set of mathematical facts and procedural knowledge.

On the other hand, the fallibilist view considers mathematical knowledge to be 'corrigible and perpetually open to revision' (Ernest, 1991, p. 18), in line with applied mathematics. Lakatos (1922-1974) developed the quasi-empiricism school of the philosophy of mathematics which is not prescriptive but descriptive (Ernest, 1991). His logic of mathematical discovery is a cycle of conjecture, refutation, and new conjecture (Wilding-Martin, 2009). According to Lakatos' view, mathematical theories are fallible and may be 'the bottom-up retransmission of falsity' (Glas, 2001, p. 358). Wood (1995) also describes mathematics as the product of a taken-as-shared or human activity.

Nowadays, mathematics, in line with other forms of knowledge, is treated as 'domain-specific, context-bound and procedurally rooted' as well as influenced by culture (Stigler & Baranes, 1988, p. 258).

2.3.2 Culture and mathematics

Mathematics education in England takes an empiricist view which refers to and is rooted in useful experiences (Freudenthal, 1991). The implementation of this philosophy

influences and shapes the teaching and learning processes from a constructivist approach which will be discussed in greater detail in the next chapter: Research on Understanding Function. Constructivism lays the foundation for the freedom of activities designed in the classroom by teachers and the learner-centred teaching approach which will be reflected on Chapter 8: Teacher Interviews.

Socratic culture has significantly influenced western countries, including England, particularly from a religious perspective. Sharpe (1997) proposed that a Protestant outlook, which has enculturated democratic, individualistic, reductionist and utilitarian psychologies and philosophies has constrained education systems. Individualism was translated into child-centred educational philosophy. The overall aim was to 'develop in children an attitude to mathematics and an awareness of its power to communicate and explain which will result in mathematics being used wherever it can illuminate or make more precise an argument or enable the results of an investigation to be presented in a way which will assist clarity and understanding' (The committe of Inquiry into the teaching of Mathematics in primary and secondary schools in England and Wales, 1982, p. 96).

In terms of typical learning difficulties for English students, I concentrate primarily on the numeracy skills which in previous section TIMSS 1995 showed the weakness of English students. To investigate primary and secondary school mathematics in England and Wales, *Mathematics counts* known as Cockcroft report (The committe of Inquiry into the teaching of Mathematics in primary and secondary schools in England and Wales, 1982) in the 1980s has had a far-reaching significance for teaching mathematics. The report clearly demonstrated students' fundamental insufficiency of computational skills, especially in terms of 'fluency in mental arithmetic' (ibid.: 14), although primary schools did pay 'sufficient attention' to it (ibid.: 88). Here, computational skills comprised of three kinds of calculations: mentally, with pencil-and-paper and with a calculator (ibid.: 19) and also included 'the ability to carry out a

particular numerical operations' and 'the ability to know when' to use these operations (ibid:. 80)'.

When the People's Republic of China was established in 1949, initially the education system imitated the former Soviet Union. Notwithstanding its impact, this influence has gradually fallen away (Xu, 2013). Since the first set of unified syllabus and textbook, the style of the former Soviet Union, which paid more attention to the systematic nature and the rigors of knowledge, has influenced Chinese mathematics (Xu, 2013). Although textbooks in China have been revised several times and tend to focus more on problem-solving, this historical factor still remains in the mathematics curriculum, embodied by two basics (basic knowledge and basic skill), and extended towards four basics (the basic methods and basic experience), which are highly valued by teachers, as reported in Chapter 8. In turn, higher requirements for mathematics understanding in China and Shanghai also foster these characteristics. That is, achieving higher requirements causes the advanced expectations for students about their solid foundation knowledge, including proficiency in employing procedural knowledge or skills which will be addressed in Chapter 8: Teacher Interview. After several curriculum reforms, the last curriculum reform, the Curriculum Reform Guidelines for the Nine-Year Compulsory Education issued by the Ministry of Education 2001, focused on the construction of knowledge by students and enhancing real-life applications (Q. Li & Ni, 2011).

The dominant philosophy in China can be referred to as Confucianism. Leung (2001) identified six distinctive characteristics of mathematics education in East Asia: (1) the content, including two basics, is fundamental; (2) meaningful learning including memorization; (3) enjoying studying hard; (4) students' extrinsic motivations, such as familial and communal expectations; (5) whole class teaching rather than individualised learning; and (6) teachers' subject matter knowledge prior to pedagogy. The first two aspects will be illustrated in

Chapter 5: Curriculum Analysis as well as Chapter 8: Teacher Interviews. The middle two aspects are associated with social factors, namely the incentives from family or community expectations which are not the foci of this study. The final two aspects are related to the suggestion from Reynolds et al. (1996) for pedagogical solutions such as whole-class instruction should be more often taken up in England.

Whole-class interactive instruction was regarded as one of the key classroom factors for high achievement in Pacific Rim societies including Chinese by Reynolds et al. (1996). There are fundamentally different characteristics between Western and Eastern schools, such as a disparity between the two nations in terms of class size, and examination-oriented and expository teaching which became the initial negative impression of the Chinese approach from the Western educators' perspective (Biggs, 1996). These early comparisons showed how other societies were doing, or what they look like, but for certain mathematics knowledge how and what were unclear.

2.3.3 Mathematics and classroom

In the area of mathematics, Tall (2004) proposed three worlds of mathematics containing elementary mathematics and advanced mathematics: concept acquisition from the direct perception of physical or geometric systems and how these are interpreted; a perceptual world; and a formal axiomatic world, whose properties determine objects rather than vice versa. Focusing on the narrower perspective - the mathematical classroom, however, there are four linguistic domains: *Research Math*; *Inquiry Math*; *Journal Math*; and *School Math* (Richards, 1991). The inquiry approach to teaching and learning is grounded in 'constructivist epistemology' (Richards, 1991, p. 17). *School Math* in linguistic domains signifies that 'students had been treated as passive recipients of information' (Richards, 1991, p. 16). Meanwhile, Tan (2012a, p. 164) argued that the traditional approaches in Shanghai, such as memorisation, repeated practice, and didactic teaching, facilitates students ability to

'gain a deep understanding of the content knowledge, develop logical thinking, and possess strong application ability, as it might be used to explain the high performance in PISA'.

2.4 Teachers' beliefs and practices

This section introduces the research on teachers' beliefs and practices. First, two general mathematics teaching models will provide the background of the teaching approach, following research about English teaching approach as well as Shanghai's. Secondly, among teacher's knowledge, pedagogical content knowledge will be highlighted.

2.4.1   Mathematics teaching

Teacher education research has recently focused on the correlation between teachers' beliefs and their teaching practices (Fang, 1996). The relationship between the two aspects, however, is not directly one of cause-and-effect (A. Thompson, 1992, p. 140). There are different ways of teaching and learning mathematics as introduced first, and each area will have its unique view of the effective way, or the expected teaching approach in the second.

With regards to what teachers believed, there are two models of mathematics teaching in teacher education research which closely correspond with each other (A. Thompson, 1992). The first model of mathematics teaching identified by Kuhs and Ball (1986) which describes four distinctive categories of teacher beliefs: learner-focused; content-focused with an emphasis on conceptual understanding; content-focused with an emphasis on performance; and classroom-focused. In the second model of mathematics teaching, Ernest (1989) categorised three key elements on which the practice of teaching depended: (1) teachers' role in the intended outcome; (2) the use of curricular materials; and (3) the enacted model of learning mathematics (see Table 4).

Table 4

Ernest's Model

Key elements of mathematics	Patterns
Teacher's role in intended outcome	1. Instructor: skills mastery with correct performance 2. Explainer: conceptual understanding with unified knowledge 3. Facilitator: confident problem posing and solving
The use of curricular materials	1. The strict following of a text or scheme 2. Modification of the textbook approach, enriched with additional problems and activities 3. Teacher or school construction of the mathematics curriculum
Enacted model of learning mathematics	1. Mastery of skills model 2. Reception of knowledge model 3. Active construction of understanding model 4. Exploration and autonomous pursuit of own interests model

In line with the work of A. Thompson (1992), I linked the two models with

philosophical backgrounds as shown in Table 5.

Table 5

Models of Mathematics Teaching

View on teacher beliefs	Philosophy background	Teacher's role	Learning
Learner-focused	Constructivist	Facilitator	Active construction of understanding model
Content-focused with an emphasis on conceptual understanding	Platonist (Ernest, 1989)	Explainer	Reception of knowledge model
Content-focused with an emphasis on performance	Instrumentalist view of the nature of mathematics	Instructor	Mastery of skills model
Classroom-focused from teaching effectiveness studies	Not from any learning theory	Classroom activities	Active construction of understanding model

In the rise of constructivism in Western education community, it is well accepted that

knowledge is constructed by learners themselves rather than transmitted from teachers (Hoy,

Hughes, & Walkup, 2008). Along with these two main learning theories, the learning

environment has two extremes as well, student-centred view or teacher-centred view of

classroom instruction. Later, Swan (2005) summarised two extreme teaching approaches: transmission and challenging (see Figure 3). In the transmission approach, methods will be explained step by step and the teacher plays key role on the learning direction. This teaching approach fit with the belief of mathematics as pure maths. Swan (2005) also argued that the transmission approach can only be effective in short-term. The short-term learning outcome from students of the teacher-centred lesson would achieve significantly better than student-centred where the motivation was higher (Sturm & Bogner, 2008). Furthermore, Swan (2006) proposed to develop a collaborative orientation towards teaching instead of teacher-centred or transmission pedagogic practice for England through five teaching activities: classifying mathematical objects, interpreting multiple representations, evaluating mathematical statements, creating problems, and analysing reasoning and solution.

A 'Transmission' view

A given body of knowledge and standard procedures that has to be 'covered'.

Mathematics is ...

'Connected', 'challenging' view

An interconnected body of ideas and reasoning processes.

An individual activity based on watching, listening and imitating until fluency is attained.

Learning is ...

A collaborative activity in which learners are challenged and arrive at understanding through discussion.

Structuring a linear curriculum for learners.

Giving explanations before problems. Checking that these have been understood through practice exercises.

Correcting misunderstandings.

Teaching is ...

Exploring meaning and connections through non-linear dialogue between teacher and learners.

Presenting problems before offering explanations.

Making misunderstandings explicit and learning from them.

Figure 3. Two views of teaching (Swan, 2005, p. 5)

Between the extreme teacher-centred teaching approach and the learner-centred one, Land, Hannafin, and Oliver (2012, p. 8) suggested four core values and assumptions: '(a) centrality of the learner in defining meaning; (b) scaffold participation in authentic tasks and sociocultural practices; (c) importance of prior and everyday experiences in meaning construction; and (d) access to multiple perspective, resources, and representations'. Pampaka et al. (2012) divided the middle into three levels: Level 1, students-centered connectionist practice; Level 2, medium – teaching practices from both ends; Level 3, teacher-centred, transmissions, fast pace, exam orientated. Here, connectionism has two aspects: (a)

connecting teaching to students' mathematical understanding and productions; and (b) connecting teaching and learning across mathematics' topics, and between mathematics and other subject knowledge. Although the distinct teaching style cannot easily be identified by English large scale, qualitative studies (Askew, 2001), the connectionism is regarded as the desirable teaching approach in England.

In China, effective teaching and learning mathematics has been summarised by a theory named teaching with variation (Gu, Huang, & Marton, 2004) which explains the paradox of Chinese learners (discussed in the first chapter) from mathematics classroom perspective. The abstract mathematics concepts are built upon concrete and perceptual experience, while teaching with variation connects the experience and the concept.

Teaching with variation, this pedagogy has two forms: conceptual variation and procedural variation. The former one focuses on mastering the essential features of the mathematics concept by two forms: concept variation and non-concept variation. The concept variation varies the multiple perspectives of the concept (see Figure 4), standard figures and non-standard figures. Particularly, the non-standard figures provide different orientations to enhance the understanding of key characteristics of the concept. Another set of examples are related with the figures which do not belong to the concept, as called non-concept variation. Then, the essential of the concept could be understood by comparing with the non-concept examples (see Figure 5) or counterexamples. Teaching with non-concept variation can help student build upon the relationships between related concepts and clarify the confusion students might have. In case of linear function in the Shanghai textbook, a set of exercises requires the students to distinguish which algebraic expression belongs to linear function: $y = \frac{1}{x} + 1$, $y = -2x$, $y = x^2 + 2$, and $y = kx + b$. These samples are mainly of non-concept variation.

The conceptual variation, however, regards the concept as a static object during the teaching and learning process. Each concept might also have its own development process which the procedural variation emphasises on.
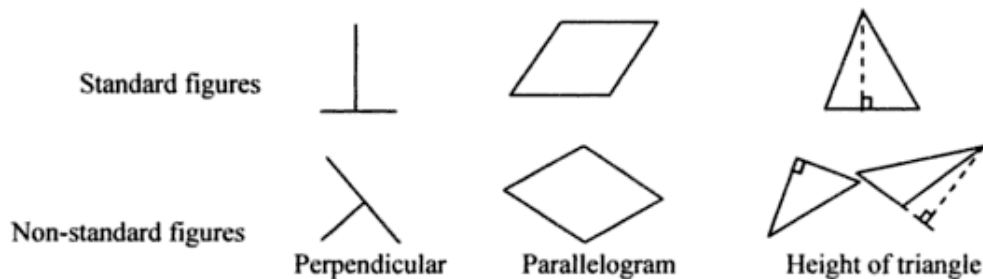
Figure 4. Examples of the concept variation(Gu et al., 2004, p. 317)

Figure 5. Non-concept example in terms of the concept of vertical opposite angles

The assumption that teaching with procedural variation will be effective is that experiencing the formation process of the concept helps students understand the concept step by step. For example, teaching the concept of equation can be formed by three scaffoldings: 'representing the unknown by concrete things'; 'symbolizing the unknowns'; and 'replacing unknown x with symbolic "□"' (Gu et al., 2004, p. 321). Meanwhile, Gu et al. proposed a framework to explain how procedural variation is applied for problem-solving (see Figure 6). For example, after learning alternate angles, corresponding angles and co-interior angles and angles in a triangle, students are expecting to solve problems from the first known-problem in Figure 7 to the rest of three unknown-problems through procedural variation for problem-solving teaching method.
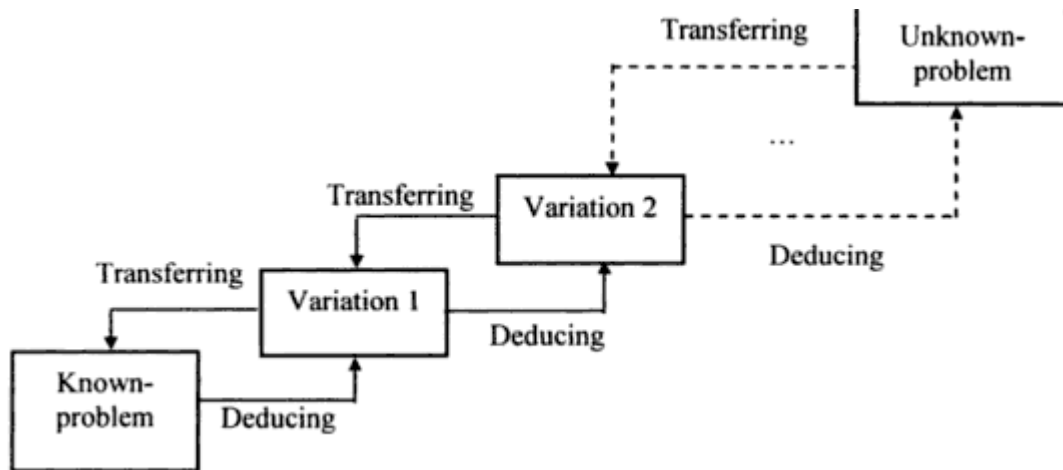
Figure 6. Procedural variation for problem-solving (Gu et. al., 2004, p.322)
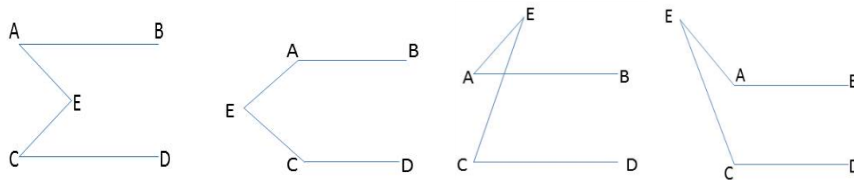
Figure 7. Examples of procedural variation for problem-solving

In sum, the teaching belief or effective teaching approach in England priorities students, and different activities; while Shanghai or China tends to focus on the mathematics itself.

2.4.2 Pedagogical content knowledge

Teachers' beliefs usually refer to their pedagogical beliefs about teaching, or the subject matter, or those beliefs that are of relevance to how students learn (Borg, 2001). Teachers' pedagogical beliefs dispose or guide teachers' behaviour in the classroom (Borg, 2001), and influence teachers' lesson plans as well as how they conduct the teaching process.

Many scholars have identified teachers' knowledge as a decisive way of enhancing student achievement (Hiebert, Gallimore, & Stigler, 2002; Hill, Rowan, & Ball, 2005; Monk, 1994). Shulman (1986) proposed three kinds of teacher knowledge: content knowledge as subject-matter knowledge; PCK as how teachers bring the content knowledge to their work with students; and curricular knowledge as how particular content relates to other forms of

knowledge in the curriculum. No strong relationship was found, however, between teachers' content knowledge and students' success (Mathematics Learning Study Committee, 2001), though teachers' content knowledge can provide 'an effective structure' for setting up activities to facilitate learning (Hodgen & Marshall, 2005, p. 169). Recently, Coe, Aloisi, Higgins, and Major (2014) state that PCK and the quality of instruction are proven to have a strong impact on students' performance. Due to Chinese students' consistently higher performance in several international assessments, the research on PCK related to Chinese teachers has become the foci of teacher knowledge research in comparative studies (An, Kulm, & Wu, 2004; Chee Mok, 2006; Y. Li & Huang, 2008; Y. Li & Shimizu, 2009). An et al. (2004) proposed a network (see Chapter 8 discussion) to explain secondary school teachers' PCK. In this network, teacher knowledge of students, in effectively analysing how their pupils think and learn, is related not only to how well students perform, but also how to teach (Mathematics Learning Study Committee, 2001). This model of teachers' pedagogical content knowledge (PCK) will be used in Chapter 8 to compare the differences and similarities of two groups of teachers.

2.5 Summary

In this chapter, the backgrounds of the education systems in both regions have been briefly introduced. Current comparative mathematics studies related to England or Shanghai, then accounted for students' learning outcomes in general, and some features of the mathematics curricula in particular. These provide a foundation for building up the view of comparative education taken in this current study, which will be discussed in Chapter 4: Methodology. The two views of the nature of mathematics have been reviewed. In the susequent chapter, the literature review on the concept of function and understanding will be presented.

Chapter 3 Research on Understanding Function

This chapter focuses on the literature related to three relevant research areas within this study: research on the concept of function, research on the definition of understanding and following the general view of understanding, and research on models of understanding. The first section of this chapter is about the concept of function. The second one will discuss the notion of understanding in general and the understanding of function in particular. The similarities and differences in the views on the development of understanding between the two cultural contexts – England and Shanghai, will also be explored to illustrate the potential influence of this difference on the process of teaching and learning mathematics in the two regions. In the third part of the chapter, a total of nine models of understanding will be explored: two predominant models of understanding in general; five prominent conceptual models of understanding function proposed by Western educators; and two models of understanding function for Shanghai situation. Based on these researches, a general model of understanding function will be proposed at the fourth section. This general model will be applied in this study as a theoretical framework to discern how curricula and the offical textbooks describe the requirements of understanding linear function, how well students understand linear function, and how the teachers predict what barriers students will encounter in their understanding development.

The main purpose of this chapter is to form the model of understanding function through the synthesis of arguments from Eastern and Western cultural approaches. It starts with studies on the concept of function from secondary school textbooks' definitions in England and Shanghai fitting with the target group in this thesis. Following the different forms of the definition, literature about the two different approaches to function and the issue of visualisation, therefore, will be selected to elaborate the nature of arguments for algebraic and graphic ways, so as to support further curriculum and textbook analysis. The second

section demonstrates the defining understanding from a wide range of literature, and Eastern and Western views of understanding will be used to provide initial embodiment from a cultural perspective. The models of understanding development shown in the third section represents from general ones in the algebraic area to specific ones for function. The specific models are either typical samples of different theoretical backgrounds in Western researches or precisely working for Shanghai situation.

3.1 Studies on the Concept of Function

When focusing on one of the seven key ideas of mathematics in secondary schools, namely functional relations between variables or function (A. Watson, Jones, & Pratt, 2013b), this section explores the nature of function. First, the definition of function will be outlined, mainly focusing on how the topic is introduced in junior secondary school's textbooks in England and Shanghai. Initially, the learning of function is approached in two ways: algebraically or graphicially (Leinhardt, Zaslavsky, & Stein, 1990). The disadvantages of each approach's ability to construct the concept will then be discussed. Particularly for the graphic method, the power of visualization for mathematics learning will be noted in the fourth sub-section. Meanwhile, visualization leads to another aspect, namely the benefits of technology in helping students' understanding of the concept of function at the fifth sub-section.

3.1.1 The definition of function

'The function is a mathematically powerful and pervasive idea'(Schwartz & Yerushalmy, 1992, p. 262). The fundamental element of this concept is a 'dependence' with univocal correspondence for ordered pairs (Piaget, 1977, p. 167). It has been considered as 'the univalence requirement: each element in the domain corresponds to exactly one element in the range', as one-to-one or many-to-one corresponding properties, with multiple representations (Dubinsky & Wilson, 2013, p. 85). There are three key aspects within this

defintion: (1) the 'dependence' and 'univalence requirement' are both embodied by the rule*,*

f ; (2) the rule *f* could be presented by multiple representations; and (3) the ordered pairs have

one-to-one corresponding property which means that the first 'one' refers to the independent

variable and the second 'one' is called as the dependent variable. This concept includes

mathematical relationships which 'can be presented in several different ways' (Stein, Baxter,

& Leinhardt, 1990, p. 651). Function is therefore a complex concept: (1) this concept has a

considerable number of sub-concepts associated with it, for example, the independent value

and one-to-one property; (2) it is connected to geometry and algebra from a representational

perspective; and (3) it is represented in a number of different settings (Dreyfus & Eisenberg,

1982).

The concept of function has evolved since Leibniz used this term in 1692. Two

definitions had far-reaching effects which were proposed by Dirichlet and Bourbaki

respectively. In 1837, Dirichlet developed an accurate definition of function by considering

'an arbitrary nature of function' (Kleiner, 2009, p. 20):

> *y* is a function of a variable *x*, defined on the interval $a<x<b$, if to every
>
> value of the variable *x* in this interval there corresponds a definite value of
>
> the variable *y*. Also, it is irrelevant in what way this correspondence is
>
> established.

The second sentence of Dirichlet's definition caused debate in the

mathematics community at the time and later in 1939, Bourbaki gave the formal

definition of function based on the meaning of *set* and *mapping*:

> Let *E* and *F* be two sets, which may or may not be distinct. A relation
>
> between a variable element *x* of *E* and a variable element *y* of *F* is called a
>
> functional relation in *y* if, for all $x \in E$ there exists a unique $y \in F$ which is
>
> in the given relation with *x*.

We give the name of function to the operation which in this way associates with every element $x \in E$ the element $y \in F$ which is in the given relation with x; y is said to be the value of the function at the element x, and the function is said to be determined by the given functional relation. Two equivalent functional relations determine the same function (cited in Kleiner (2009, p. 25)).

Turning to schools' mathematical definition of function, English students encounter the idea of function from primary school through to secondary school, while Shanghai students only begin to study the concept from junior secondary school and continue to senior secondary school. Therefore, the presentation of the concept of function was examined in commercial textbooks used in secondary schools in England, and complusory textbooks in Shanghai secondary schools.

The concept of function is developed using a visual approach in England. It is initially represented as a function-machine, such as in Figure 8, at primary school and then later as a flow diagram (see Figure 9) at KS3 and KS4. Words are often used to describe the rule in primary schools, while symbolic notation is applied in the secondary school.
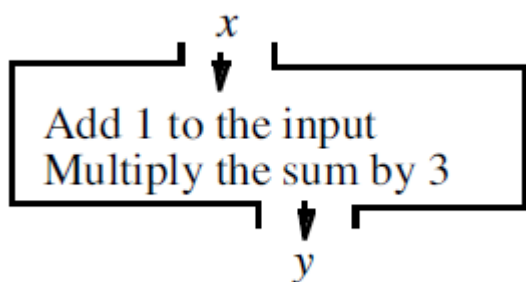
Figure 8. Function machine (McGowen, DeMarois, & Tall, 1999)
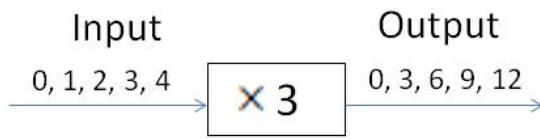
Figure 9. The definition of the concept of function in an English textbook (GCSE Maths 2 tier-foundation for AQA A)

Alternatively, secondary school textbooks in Shanghai (aged 12 to 18) offer two types of explanations for the concept of function: co-variation between two variables (rule-based) which tend to reflect Dirichlet's definition; and a correspondence between two sets (*mapping*) which fits more closely with Bourbaki's definition. Dirichlet's definition contained the concept of variable, while Bourbaki held the 'purely structural' view of function (Sfard, 1991, p. 15). The concept of function which appears first at Grade 8 (approx. aged 14) complusory textbook in junior secondary school is defined as a rule-based relationship:

> There are two variables, for example *x* and *y*; within the range of values
>
> allowed for *x*, variable *y* changes once *x* changes as they have a certain
>
> dependent relationship. Variable *y* is referred to as the function of
>
> variable *x*. *x* is referred to as the independent variable.

In Shanghai senior secondary school (approx. age 17), after the introduction of the concept of *set*, the concept of function is accommodated to a relation between two sets, *set* A and *set* B, with each member of *set* A exactly mapping onto one number of *set* B.

Sfard (1991) argued that the concept of function has two aspects, operational and structural, in line with the dual nature of mathematical concepts (process and structural). According to Sfard, the operational process is the first step towards a new notion of concepts (Kieran, 1997); therefore, Sfard's model was drawn from a process-oriented basis, through manipulating symbolic representations, and after this, the object-oriented stage by visual structural representations. From a psychological perspective, Sfard (1991, p. 18) also

proposed a three-stage model of concept development: 'interiorization, condensation, and reification'. In terms of function, through a process involving the function machine, variables and formulae have been acquired by students via interiorization. At the second stage, students focus on the relationship of input-output rather than actually undertaking the operations. This relationship also contains translations between different representations. These two stages lead to qualitative changes in the last stage which allow students to probe into some properties or to solve equations with parameters, referred to as reification. Similarly, Doorman, Drijvers, Gravemeijer, Boon, and Reed (2012, p. 1246) argued that there are 'three interrelated aspects of function': as an input-output assignment; as a dynamic process of co-variation; and as a mathematical object. In the first aspect, students carry out calculations and notice which one determines the patterns and the methods behind this determination. In the second aspect, the notions of independent variables and dependent variables are noticed in the form of a table and graph. In the third aspect, students have the full structural view of function at the global level to deal with complex problems related to other mathematics knowledge. In terms of junior secondary schools, the definition in England is mainly related to input-output assignment, while Shanghai emphasises the co-variation view. Both are operational-based. The Shanghai textbook offers rigourous mathematical definitions; however, England's textbooks use the visual approach.

Representations in function. The concept of function mainly contains three different and complementary representational systems: algebraic expression, tabular, and graphic representation. These are called the 'three standard types of representations' (Francesca, Dave, & Ornella, 2006, p. 255). These three systems jointly construct the definition of this concept, connecting three types of representations into a whole set. Students use one symbolic system to expand and understand others. There were two trends of research in these symbolic systems: one trend focuses on the alternative nature of an algebraic equation and a

graphic representation (Stein et al., 1990); and the other emphasised the role of tabular representation as an activity of generalization (Kieran, 2006).

3.1.2 Two approaches to the concept of function

In approaching the concept of function, there are two methods: one can start from the formal notational system of algebra in order to link with the graph; while the other approch stems from the system of Cartesian graphing to make sense of algebraic expression (Leinhardt et al., 1990). Researchers have tried to compare these two approaches in order to discover which would be more suitable in approaching the concept of function. The advocates of the algebraic method consider function to be the 'central content for school algebra' (Heid & Blume, 2008, p. 56). Other evidence, however, suggests that the second method is much easier for students to understand function (Francesca et al., 2006). The first method moves from an algebraic approch to a graphic one via ordered pairs, while the latter approach demonstrates a scientific presentation from observation using an array of data, through ordered pairs and graphs. Although it is better to approach this topic in a way that benefits students (Schwartz & Yerushalmy, 1992), the possible disadvantages of each approach should be identified and acknowledge in the teaching and learning processes.

Each approach to function has its own main representation. The algebraic process has the algebraic expression, while the graphical one has the graphical representation. Here, some drawbacks of each representation will be revealed, and it should be noted that the disadvantages of one might be the advantages of the other.

Guin and Trouche (1998) argued that students normally misinterpret the graphical representation of function. In the case of the gradient, one of the basic properties in linear function, there are two ways of exploring this property: visually and analytically (Zaslavsky et al., 2002). With the visual approach the gradient is computed as a quotient of segment-lengths. The drawback of the visual approach is that students sometimes encounter

difficulties in a non-homogeneous system (see Figure 10), because, in this instance, 'the

isomorphism between the graphic and the algebraic systems loses its utility, indeed its

meaning' (Zaslavsky et al., 2002, p. 136). That is, the gradient of these two lines has the same

value, but it does not look as the same steepness.
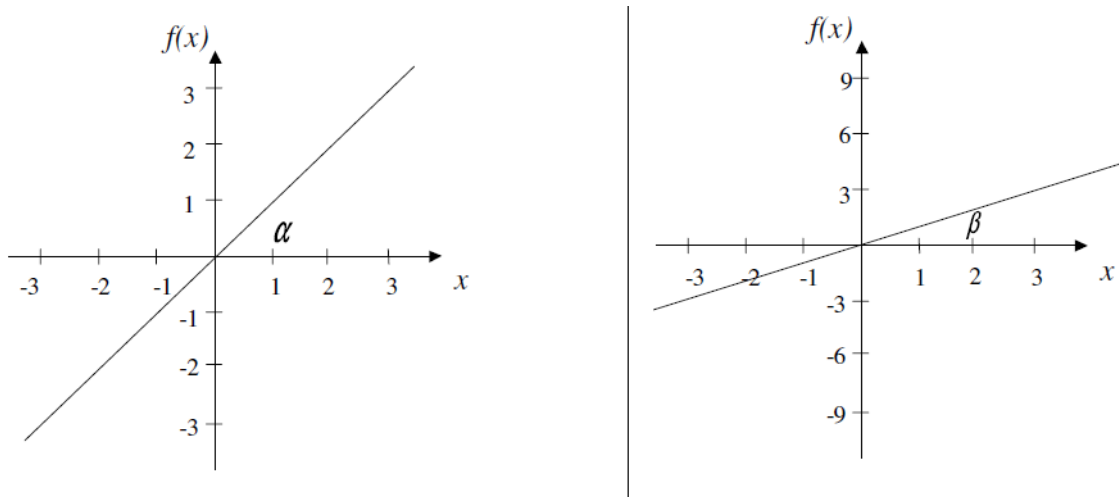
Figure 10. The gradient of the same function in two visual approaches (Zaslavsky et al., 2002,

p. 121)

A second disadvantage of the graphical approach is that students may find it difficult

to deal with abstract context of problem-solving. Leinhardt et al. (1990) argued that real-life

contexts did not always facilitate the learning process. Furthermore, A. Watson, Jones, and

Pratt (2013a, p. 179) argued that 'time' on the x-axis was drawn from 'chronological

progression rather than being seen as a variable'. Additionally, the utilisation of transport,

such as cars and trains, in examples of real-life situations is regarded much more successful,

as it is easy for students to imagine, and they could 'serve as a reification of linear function'

(Toom, 1999, p. 37). But students will encounter difficulty when dealing with reasoning tasks,

which are not physically accessible to them or when their intuitions and the definitions within

the question are conflicted (Edwards, Dubinsky, & McDonald, 2005).

In terms of the algebraic expression, children whose ages are between 11 and 14 years old can understand the algebraic concept at the formal operations stage (Dreyfus & Eisenberg, 1982). Besides the requirements of age, there are two perspectives which should be considered when using algebraic expression. First, compared with other represesentations, the algebraic expression is extremely difficult. Lue (2013, p. 450) concluded that the algebraic expression is the most challenging representation 'to be handled', even by Grade 10 students in Taiwan, after having examined the translations between representations within six types of elementary function, including linear function. Due to the difficult, there is a tendency for students to remember the procedure, computed as a coefficient in the analytic approach. As for the manipulation of algebraic expressions, some pupils could be in control of this manipulation and some appear to be controlled by it (Cottrill et al., 1996). That is, students could tackle the question in a mechanical or algorithmic way, namely devoid of meaning or understanding. One concern is that the overuse of these rules in learning function may contribute to 'structurally weak' understanding for students (Stein et al., 1990, p. 660). As a result, students tend to misunderstand that function must have an equation or rule (Rasmussen, 2001) in their future study at senior secondary school and university level, such as expecting a certain type of standard algebraic expression. On the other hand, the algebraic approach may also be dangerous for students' non-routine problem-solving. When relying on traditional rigid practices to develop students' procedural and conceptual knowledge in China, students experienced more difficulties with non-routine problems and tended to leave these blank (Ni, Li, Li, & Zhang, 2011).

The next subsection will consider how to construct the whole concept of function by integrating another approach.

3.1.3 How to construct the concept of function?

The concept of function is one of the 'big ideas' in mathematics (Confrey, 2002, p. 113). The construction of this concept is not simply a case of learning the algebraic and then graphical approaches, and as a result to have the sum of these two approaches, or vice versa.

First, the nature of the two approaches are not the same. The graphical world is not isomorphically the same as the algebraic expression (Leinhardt et al., 1990), as they are 'two different external systems of representation' of function (Goldin & Kaput, 1996, p. 405). The two approaches represent two differing perspectives. Students who use and understand one method do not automatically use and comprehend the other perspective (Leinhardt et al., 1990). The nature of symbolic representation and graphic representation will be further discussed in Chapter 9: Summary and Discussion.

Secondly, the concept of function has multiple representations that present the whole concept (Habre & Abboud, 2006). The main three ones (graphs, tables, and formulae) are regarded as separate static entities in students' minds (Schwarz & Dreyfus, 1995). Students regard different representations as individual tasks, instead of considering that these may represent the same idea (Gagatsis & Shiakalli, 2004). The perceived static nature of these approaches causes students' difficulties with connecting representations. Different psychological processes are employed when translating from formulae to graphs, and from graphs to formulae (Guin & Trouche, 1998), where the latter process would be more difficult for students than the former one. The lowest accuracy in connecting representations was between graphs and formulae (De Bock, Van Dooren, & Verschaffel, 2013). In particular, Gagatsis and Shiakalli (2004) argued that the low percentage of translating between different representations was related to the use of iconic representations.

In summary, constructing the concept involves two aspects of meaning: understanding the relationship between the different representations and the concept of function, and

connecting these representations. Once students overcome these difficulties in translating, another struggle is to be flexible in choosing the most appropriate representations in order to resolve tasks (Nistal, Van, Clarebout, Elen, & Verschaffel, 2009).

The algebraic approach is normally regarded as the triditional method, while the graphical approach is just as powerful. The next subsection will discuss the use of visualization in mathematics.

3.1.4 Visualization in mathematics

Mathematicians had a strong preference for symbolic representation in the nineteenth century, but this preference was challenged in the twentieth century. As a result, the power of the visual approach as a thinking process, and graphic representation as a product of that process, has been recognised (Stylianou & Silver, 2004). The term *visualization* was defined as a visual image in the person's mind when doing mathematics with or without the use of graphs (Presmeg & Balderas-Cañas, 2001). The graphical approach to function is one such visualisation, as Zarzycki (2004, p. 108) argued that 'visualization is the process of using geometrical illustrations of mathematical concepts'.

Within this meaning, visualization was regarded as an analytical process with the same function as the algebraic method. Visualization was, therefore, compared with the algebraic approach in order to illustrate its effectiveness in learning mathematics (Presmeg, 2006). The debate started with students' reluctance to think or use picutres which was initially reported around the late 1990s (Eisenberg, 1994). Arcavi (2003) proposed three causes for this reluctance: the cultural belief that visual proofs were not valued; the cognitive aspect, that reasoning with the visual approach cannot always be relied on; and sociological dificulties, such as whether it is encouraged by teaching process. Essentially, there were two levels of concern: the perception of the usefulness of visualization; and the ways in which visualization was explained as a method.

Research has confirmed that visualization can be effectively used. In solving non-routine mathematical problems, for example the importance of graphical representations has become well recognised (Pantziara, Gagatsis, & Elia, 2009). Also in the advanced mathematical area, both novices and experts were observed using a graphical representation in their solutions (Stylianou & Silver, 2004). Rösken and Rolka (2006) observed that Grade 12 (approx. 17 years old) German students showed their willingness to use visualization in solving problems, but were unable to solve it correctly. This led to another question regarding the accuracy of the answer when the visualization process is applied to solve mathematical problems.

When task complexity was higher, those primary school Australian students who preferred visual methods successfully solved problems without any gender differences (Lowrie & Kay, 2001). Furthermore, Presmeg and Balderas-Cañas (2001) investigated the question of whether, when, how, and why American doctoral Maths students used visualization. The results showed that some students used visual reasoning at the beginning of their problem-solving, even those students who chose an algebraic method for their solutions. Other students were observed using visual reasoning, only then to check their answer through the algebraic method.

In summary, the visual process is seen to be important. First, the use of visualization as a thinking method or part of a process to solve a problem is evident. Second, if the result of this visual problem-solving method is presented using the graphic representation, it may not be as reliable or credible as the answer provided by the algebraic method provided. The effectiveness of the result from visualization influenced the extensive use of it.

3.1.5   The benefits of software in learning function

In order to enhance the learning of the concept of function, the medium of computers has been used to facilitate the visual process. This method has sparked arguments within the

mathematical community regarding whether or not computer software aids learning and whether or not understanding is limited by such a visual approach.

Artigue (2002) noted that the use of software and computational tools was a constructivist approach. That is, this approach was not only a pedagogical instrument, but also changed the perspective from which students consider mathematics. In his research, students developed 'framing schemes' around the graph using the graphic calculator or within the computer environment. Students' conceptual image of function became window-dependent which would improve students' preference for the structrual view of the concept of function. The benefit of this window-dependency was that students can develop a global view of the graph without actually experiencing the process of construction.

The software can also help to construct the concept of function by linking different representations automatically. Schwarz and Dreyfus (1995) argued that technical software, such as Triple Representation Model (TRM), remedied the difficulty of constructing the concept of function (connecting representations) because the software could give students these notational systems of function immediately. Moreover, Hazzan and Goldenberg (1996) indicated that dynamic geometry environments (DGEs) can strongly impact on a robust understanding of function from two perspectives: (1) students could have two pictures of function together without reference to algebraic language or graphs; and (2) it could broaden students' ideas of function. In additional, computer software could certainly provide students with unprecedented visual capabilities (Habre, 2000).

Doorman et al. (2012) contended that the switch between an operational-based view and a more structural perspective was fundamental to the understanding of function. Computer tools were applied by their research to identify ways in which to encourage students to make progress in this transition. In this case, the advantages of software were that it simplified the calculation procedures. For example, software can automatically generate the

graph representation and linking with the algebraic expression (see Figure 11). Students were therefore able to highlight the 'reasoning abilities' and moved to the 'object' view more smoothly.
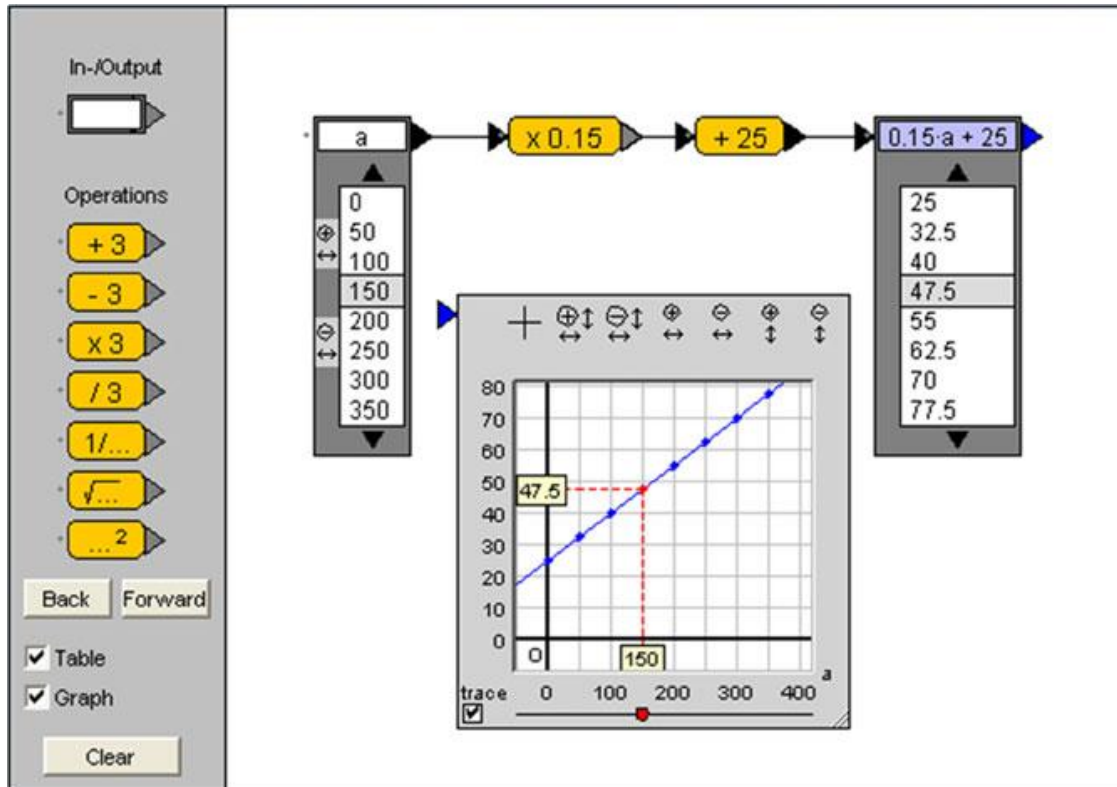
Figure 11. The computer tool use (Doorman et al., 2012, p. 1250)

Conversely, some researcheres investigated whether using software was more effective than traditional teaching methods, even in a symbolic computer-system context. From a graphic perspective, Asiala, Cottrill, Dubinsky, and Schwingendorf (1997) revealed that students had the better understanding within the graph of the concept of function compared with those who used traditional pencil-and-paper instruments. Meanwhile, O'Callaghan (1998) proposed four competencies within problem-solving ability related to function problems: modeling, interpreting, translating and reifying which will be further discussed at the final section of this chapter. Computer-Intensive Algebra (CIA) was used to test whether or not students improved in any of the competencies compared with the traditional way. The results showed that students' performances improved significantly in

three of the four competencies (only reifying did not improve significantly). Furthermore, students' attitudes towards mathematics was enhanced considerably in this case.

It has been noted that the most effective way to construct students' own mathematical understanding is for them to combine different resources including theoretical text, calculator, and calculation by hand, with the aid of a teacher's assistance (Guin & Trouche, 1998), as using diverse methods. The next section will introduce the research on understanding in order to clarify its features and to link with other studies of understanding function.

3.2 Studies of Understanding

This section begins by examining two views of knowledge which gave rise to two opposite learning theories: behaviourism and constructivism. The latter led to the necessity and definition of understanding in the learning process which will be depicted in the second sub-section by examining variations in the definition of understanding within Western and Eastern contexts. How Chinese educators perceive understanding in the learning process will then be delineated. Both sides of the literature agree that understanding is a process and involves a hierarchical division from obtaining the concept to being able to use the concept flexibly. The final sub-section will describe what is known about understanding function more specifically.

3.2.1 Behaviourism and constructivism

Learning theories aim to explain the ways in which students study new knowledge. It is important to note what knowledge is because different perceptions influence the approach to knowledge. In philosophy, the epistemology of knowledge has sought to grasp the 'inherent properties of knowledge' (Fagin, Halpern, Moses, & Vardi, 1995, p. 1). Differing views of 'inherent properties' have demonstrated how to access the structure of that knowledge. There are two primary opposing perspectives on knowledge: one is the absolutist view, while the other is the fallibilist view, as discussed in Chapter 2. Within the absolutist

outlook, knowledge represents an independent world (Von Glasersfeld, 1995) and is a finished product, expressed as a body of proposition (Ernest, 1991). Within the fallibility view, knowledge is an activity of knowing and changes along with the time and place (Ernest, 1991). These two contrasting views lead to two differing learning theories, namely behaviourism and constructivism. With regards to behaviourism, learners are passive as knowledge is delivered by others. Xu (2004) argued that some of the Shanghai students in primary school were passive learners and that their learning approach belonged to memorization. On the opposite extreme, constructivism involves learners actively constructing their knowledge. The question remains, then, as to whether behaviourism and constructivism are in fact incompatible in the process of learning mathematics from a practical perspective. The following will first explore the theories of behaviourism and constructivism a little deeper.

Behaviourism. Behaviourism reflects the view that knowledge has a structure and that students should incorporate this pre-existing structure into their own knowledge base (N. Adams, 2007). J. B. Watson, the father of behaviourism, defined the aim of learning as adjustment and the learning process as trial, error, and success (J. Watson, 1924). In essence, behaviourism is a strictly 'operational theory' (Suppes, 1975). Based on this perspective, knowledge or behaviour was accumulated and reinforced by drill and practice.

These results explain 'the learning of animals and simple human tasks', but not the more sophisticated process of learning knowledge (Watkins, 1996, p. 6). Critics also highlighted that behaviourists eliminated the distinction between training and teaching (Von Glasersfeld, 1995). The behaviourist view was a 'clinical' one (Harries & Spooner, 2000) as more mechanical than conceptual. However, this was not to deny the importance of the practice and class exercises for some of the skills that students would likely need to construct important conceptions and principles (Davis, Maher, & Noddings, 1990). In addition,

although teaching mathematics is largely related to real world situations, the 'esoteric domain of mathematics' must be taught (Dowling, 2008, p. 27). It is evident therefore that, to some extent, certain knowledge and skills should be taught along the lines of the behaviourism learning theory.

Constructivism. Constructivism holds another view that knowledge is contextualized rather than acquired (N. Adams, 2007). Knowledge is actively constructed by the individual (Bodner, 1986) and social interactions play an important role in knowledge construction (Hoy et al., 2008, p. 411). The learning process, in which every student possesses his or her own unique knowledge as a foundation, is recreated, reproduced, or restructured in the interaction of subject and object (Ernest, 1991).

There are two main schools of constructivism: 'psychological/individual constructivism' pioneered by Piaget; and 'social constructivism', pioneered by Vygotsky (Hoy et al., 2008). Piaget's psychological constructivism focuses on the internal factors in terms of the development of knowledge. Vygotsky's social constructivism positions the learning process as discovery together with the interaction between learners and the social community (Denis Phillips, 1995).

Psychological constructivism. Piaget (1972, p. 24) suggested that 'between stimulus and response there is the organism'. This view presented a model of equilibration and self-regulation to explain how the structure of organism comes to be. Knowledge construction is directed by internal processes (Hoy et al. 2008). This dynamic internal cognitive process has two aspects, a process of assimilation and accommodation, in order to adapt to the environment and organise internal structures. Assimilation results in a variation in quantity, while accommodation causes change in the qualitative characteristics of the knowledge. Accommodation could alter the old schema or set up a new one. When an individual cannot make sense of a situation with their current schema, real changes occur, involving the

assimilation and accommodation of the new information within the old schema, through the process of disequilibrium to equilibration. Thus, it is through this process that individuals develop their thinking and learning.

As for the teacher's role in the learning process, Piaget comments that 'what is desired is that the teacher cease being a lecturer, satisfied with transmitting ready-made solutions; his roles should rather be that of a mentor stimulating initiative and research' (Good, Kromhout, & Mellon, 1979, p. 430). Teachers listen to students' current thinking instead of 'correcting wrong answers' as within more behaviourist approaches (Hoy et al., 2008, p. 430).

Social constructivism. Vygotsky emphasised the importance of social interaction and cultural tools in the leaning process (Lerman, 1996). That is, students could construct their knowledge through acculturation and interaction with their teachers. In terms of teaching, Vygotsky (1978) proposed the concept of the zone of proximal development (ZPD) and scaffolding. ZPD is the gap between students' existing level and students' potential level of learning. Students need to cooperate with others for their cognitive development, as this is where the 'most efficient learning happens' (Wilding-Martin, 2009, p. 30). Teachers are expected to understand what students already know and to support them in achieving the learning objectives through scaffolding. Once students achieve the initial goal, a new zone develops accordingly (Kalina & Powell, 2009), and teachers scaffold again. The aim of the instructions from the teacher is to reorganise students' cognitive structures rather than transmitting information (Cobb, 1988). That is, teaching and learning is integrated instead of separated in behaviourism (Lerman, 1996). In terms of Vygotsky's constructivism, the teacher's role is also slightly different from the individual constructivist perspective. In the social constructivism view, teachers co-participant and co-construct different interpretations of knowledge (Hoy et al., 2008, p. 430), while in Piaget's constructivism, teachers normally listen to what students say. That is, for individual constructivism, this activity is built upon

individual cognition, while social constructivism relies on social processes (Denis Phillips, 1995).

Both branches of constructivism consider the learning process to be an activity. Teachers and students are active meaning-makers (Cobb, 1988). Constructivism forces us to think critically and imaginatively about the learning environment. Its function is to provide the setting, pose the challenges and offer support to students (Davis et al., 1990). On the other hand, Richards (1991, p. 38) contended that through teachers' tasks designs, students could 'ask questions, pose problems and set goals'. In addition, the learning environment should be created and maintained in a positive manner for students' ongoing active learning (Hoy et al., 2008). Mathematics education therefore focuses on how children engage in mathematical activity, and how teachers can promote this activity (Davis et al., 1990).

From a practical perspective, teaching basic skills is in fact based on 'memorization, drill and practice', and understanding comes after one has mastered the algorithms (Tobin, 1987, p. 297). This will be discussed later in Chapter 5. But it becomes clear that behaviourism and constructivism could therefore be reconcilable. From a mathematical perspective, rote learning and memorization is useful in a purely mechanical way. Yet, it is more important to invite students to consider the reasons why the particular conceptions or theories are deemed important (Von Glasersfeld, 1995). Western educators no longer think that memorization leads to understanding, while Chinese educators believe that memorization could be used not only to deepen but also to develop understanding (Cai, 2004). Two queries remain: the extent to which proficiency skills influence students' understanding of mathematical concepts, and the ways in which to balance the relationship between algorithmic learning and reasoning, and the elements of mastering skills or procedural knowledge with the purpose of understanding a mathematical concept, as will be addressed in Chapter 9: Summary and Discussion.

Furthermore, the constructivist view of examining a student's understanding of a mathematical concept was assumed to be as follows (Confrey, 2002, p. 115): 'a constructivist seeks to represent how a student approaches the mathematical content. S/He expects diversity – and idiosyncratic rationality. The interviewer's knowledge of the mathematical content, complete with multiple representations, competing interpretations, various applications […] create a model which may well transform the interviewer's own understanding of the mathematical content in fundamental ways'. At next section, the definitions of understanding will be explored.

3.2.2   The definitions of understanding in mathematics

Understanding is important in the learning process. First, the necessity of developing understanding exists not only to meet the demand of the individual but also of society. Newton (2000) summarised five aspects that understanding offers: it can satisfy the individual's need to seek for the 'why' as well as the 'what'; it can facilitate learning including the speed of learning new knowledge, retrieving the learned material, and flexibly applying knowledge in a new situation; it enables people to evaluate different situations; it can aid the construction of the information; and it can spur on creativity. In terms of the final point, creativity is a personal activity intended to produce something new (Bolden, Harries, & Newton, 2010). The function of understanding is to 'creatively use presented information to solve transfer problems' (Mayer, 1989, p. 43).

On the other side, understanding generates a sophisticated kind of memory (Hiebert & Carpenter, 1992). Behaviourism regards learners as passive receivers and views learners' memories to work like storage in a library. As a result, isolated and fragmentary knowledge has easily forgotten which lead to mechanical memorizing. Frederic Bartlett's research results (1932, as cited in Hiebert & Carpenter, 1992, p.74) demonstrated that memory was constructive. When information has been integrated into a whole network, misplacement is

less likely to happen. In this integrated network, information is linked with other information. When specific information is needed in a mental search, this integrated network becomes effective because there are already abundant routes of retrieval in existence.

There are two approaches to perceiving and interpreting the definition of understanding. One is focused on the naming of diverse types of understanding towards different mathematical knowledge. For example there exists mathematical knowledge that never being questioned or interrogated at primary or secondary school level, such as 1+1=2. Others pay attention to the explanatory function of understanding. The former aims to classify the different categories of mathematical knowledge that are expected to be understood by students. The latter describes what happens if understanding appears, for example having mental objects, as will be discussed below.

Focusing on types of understanding. Hiebert et al. (1996) suggested the understanding that can be identified at two periods of time as functional understanding and structural understanding. The functional view focused on the activity during class, while the structural view indicated what students acquired after class. Here, structural understanding revealed internal relationships between pieces of information, and therefore represented a much more holistic picture of the knowledge. Most researchers, however, divided understanding into types according to different forms of mathematical knowledge.

The categorization of understanding was initially proposed by Richard Skemp. Skemp (1976) first defined understanding in two ways: instrumental understanding and relational understanding. Instrumental understanding means that one can apply rules but be unaware of the reason why the rules work. For example, students can calculate the area of a rectangle by multiplying the length and breadth without questioning why these rules provide the correct answer. This kind of knowledge is delivered by teachers straightaway. Fundamentally, instrumental understanding means that pupils demonstrate their comprehension by showing

their ability to discern the function of rules, as where and when to use them. Instrumental understanding is related to procedural knowledge which can be gained through practice; first by building meaning for symbols and then practising rules (Hiebert & Carpenter, 1992). Although instrumental understanding indicates that learners acquire the meaning of the procedure through interpreting why this step is followed by the next one, the boundary between instrumental understanding and rote learning remains vague. When knowing both 'what' and 'why', relational understanding occurs. It is inappropriate to consider the understanding process to be merely relational understanding (Hiebert & Wearne, 1996). Usually, both instrumental and relational understandings are mixed during the understanding process.

Symbols involved in the process of understanding were also highly valued by Skemp. Later in his work, he proposed another type of understanding, symbolic understanding, which was defined as a mutual assimilation between a symbol system and a conceptual structure in mathematics (Skemp, 1982). He listed ten functions of symbols: (1) communication; (2) recording knowledge; (3) the formation of new concepts; (4) making multiple classifications straightforward; (5) explanations; (6) making possible reflective activity; (7) helping to show structure; (8) making routine manipulations automatic; (9) recovering information and understanding; and (10) creative mental activity (Skemp, 1971). It demonstrates the power of symbols but is unclear as to how symbolic understanding worked together with two other types of understanding.

Based on Skemp's work, other forms of understanding were identified, thereby forming certain ways of viewing understanding, which have become favoured by academics. For example, Newton (2000) identified three kinds of understanding: descriptive understanding, procedural understanding and causal understanding. Besides instrumental and relational understanding, Byers and Herscovics (1977) extended another two types of

understanding from what students are able to do to solve a problem: intuitive understanding as a direct insight to solve problems without analysis, and formal understanding as connecting mathematical symbolism and notation with relevant mathematical ideas and to combine these ideas into chains of reasoning. Furthermore, A. Watson (2003) proposed four forms of understanding based on Skemp's work as well, especially for teaching secondary mathematics: instrumental and procedural understanding; contextual understanding for contextual knowledge; relational understanding; and transformable, generalised and abstract understanding as a higher level of abstraction. The last type of understanding indicates the importance of abstract level of understanding.

Focusing on understanding itself. Dewey argued that understanding was a result of a thinking process (Sierpinska, 1990). This thinking process was mainly delineated by two interwoven aspects: the act and the linking. Newton (2000) described understanding as connecting new knowledge with the existing knowledge network in order to shape a new one. Using the idea of a schema, Skemp explained that to understand something means 'to assimilate it into an appropriate schema' (Skemp, 1971, p. 46). Here, the term, *schema*, refers to the mental structure used to organise the existing knowledge system in order to either solve new problems or acquire new knowledge. Skemp also used the concepts of 'assimilation' and 'accommodation', as proposed by Piaget, to account for the growth of schema: 'the individual organizes his past experience and assimilates new data to itself'; and 'the schema must accommodate to the new situation' (Skemp, 1971, p. 44). During the teaching process, the teacher is expected to be aware of when assimilation takes place and when accommodation occurs (Skemp, 1971). In this, Skemp chose the psychological perspective in order to interpret understanding following Piaget's theory. However, two views: Sierpinska who mainly focuses on the meaning of act, and Hiebert and Carpenter who stress the link, have profound implications for mathematics education area.

Sierpinska. Sierpinska (1990) described an act of understanding as grasping the meaning of a concept. An act of understanding is comprised of four components: the understanding subject; the object of understanding; the basis of understanding; and the operation of the mind (Sierpinska, 1994). The understanding subject refers to the person 'who understands'. The object of understanding signifies that which is expecting to be understood, for example concepts. The basis of understanding consists of four issues: representations, mental models, apperception, and through that. Apperception occurs in the highest level of abstract thinking and it aims to draw conclusions. Through that represents an individual's explanations of logical reasons. There are four types of mental operations: identification, discrimination, generalization, and synthesis. These components work in the following way: first, the understanding subject (the person) identifies the object of understanding (what the person wants to understand), then looks for the basis of understanding (four issues); and if the object of understanding is linked to some initial basis of understanding through the mental operations (four types), the object of understanding can be understood. In terms of making this link, Sierpinska (1994, p. 72) considered processes of understanding to be 'lattices of acts of understanding linked by reasonings'. The ability to prove reasoning as deductive or inductive reasoning could create these links, namely through logic. The acting of understanding is moving towards abstract thinking.

Hiebert and Carpenter. Hiebert and Carpenter (1992) offered the explanation of linking, related to external representations such as the object of understanding, and internal representations which were in learners' minds. Hiebert and Carpenter (1992, p. 67) elucidated that mental representation was part of 'a network of representations'. This framework of understanding based on a cognitive perspective is now predominantly recognised by the mathematics education community.

This framework for understanding involved three assumptions as follows:

(1) 'knowledge is represented internally, and that these internal representations are structured;

(2) there is a relationship between external and internal representations, but internal ones can be inferred by how external ones show, such as how to use written symbols and how to explain ideas;

(3) internal representations can be connected' (Hiebert & Carpenter, 1992, p. 66).

The internal network of connecting mental representations could be viewed as a 'spider's web' or 'vertical hierarchies' or a mix of spider's web and vertical hierarchies (Hiebert & Carpenter, 1992, p. 67). Understanding occurs when new information can make connections with existing networks, as in linking. This 'smooth cumulative' way of understanding (Hiebert & Carpenter, 1992) seemed simple in order to create the link, in line with the meaning of assimilation in Piaget's theory. In fact, the connections are more complex and could even be contrary to other existing connections. An existing network would therefore be adjusted in order to be reorganised. This kind of change is also similar to the meaning of accommodation in Piaget's theory. Through the connections, the network expands and its construction becomes more logically organised.

The progress of understanding does not move linearly when it occurs (Newton, 2000) and cannot be predictable (Hiebert & Carpenter, 1992). It may be spiral (Sierpinska, 1990), or even folding back (Pirie & Kieren, 1994b) when building internal representations. The levels of understanding are determined by the quantity of connections from one idea to another, and whether the connections are weak or strong (Hiebert & Carpenter, 1992). Thus the question turns to whether the understanding development of mathematics knowledge could be levelled towards abstract understanding. Hart (1981) investigated mathematics topics with over 10,000 secondary school children, looking at their understanding of number operations, place value and decimals, fractions, positive and negative numbers, ratio and proportion, algebra,

graphs, reflections and rotations, vectors and matrices. Every topic's results were provided with a 'matching of hierarchies' (Hart, 1981, p. 187). Likewise, Pirie and Kieren (1994b) stated that the process of understanding was levelled. Nickerson (1985) suggested that the nature of understanding was non-binary, and that understanding can vary in degree or completeness. Based on this type of opinion, mathematics education researchers have sought to build up the levels within model of understanding which will be introduced in the third section.

Good understanding. In mathematics education, 'to understand' often means to 'understand well' (Sierpinska, 1994, p. 117). Researchers proposed good understanding from three theoretical perspectives: schema; the connection; and representations. From a schema perspective, Skemp (1971, p. 40) suggested that 'the more other schemas we have available, the better our chance of coping with the unexpected'. It indicates the importance of structure, while within this structure, from the related networks view. Hiebert and Carpenter (1992) considered stronger and more numerous connections in existing networks as a way of determining whether students had a more thorough comprehension. More well-organised schemas or specifically more strengthened connections would denote better understanding. How to connect is related with representational perspective. Post, Wachsmuth, Lesh, and Behr (1985) suggested that the development of children's understanding is related to the flexibility of thought in representations and the transformation among these representations.

Another way of identifying good understanding is more practical. Sierpinska (1994, p. 124) regarded good understanding to be a significant act of understanding; that is, to overcome obstacles and finally reach an 'ideal' way of understanding the object. In line with this ideal version, Nickerson (1985) investigated the expert's view in certain knowledge areas and the nature of such expertise. Furthermore, he compared beginners with advanced students when they dealt with the same problems. He reached the conclusion that the ability to resolve

truly novel situations indicates a better understanding of the concepts involved. That is, 'being able to use a concept' exceeded simply 'having the concept' (Lesh, 1981). In this situation, students' ability to sort out certain novel situations is related to other mathematical knowledge. Students can flexibly choose relative knowledge with which to approach the solution in the complex problems. Next section will highlight differences on understanding between Eastern (China) and Western.

3.2.3   Understanding in the Chinese literature

Understanding is a process shown in Chinese literature as Chen and Weng (2003) explained that the understanding process should obtain the nature of the knowledge through updating, modifying, rearranging, and restricting the existing knowledge. Other researchers, however, followed Skemp's definition of understanding. H. Zhang (2006) put forward three types of understanding: operational understanding; relational understanding; and migratory understanding. The first two types share similarities with Skemp's instrumental understanding and relational understanding. The third type, migratory understanding, refers to the use of mathematical ideas and methods, and involving existing mathematical knowledge migrating to novel situations (the basic ideas and methods will be addressed in Chapter 5: Curriculum Analysis). The introduction of this third understanding reveals that the non-standard situation for problem-solving (non-routine problems) is one independent part of understanding in Chinese educators' views as well. Another opinion, however, is that migratory understanding was part of relational understanding as F. Ma (2001) proposed four levels of relational understanding:

(1) knowing - know the definitions, key attributes, typical examples, and differences of other definitions or key attributes;

(2) applying - apply some features in the similar situation; solve problems through following the examples; and know the justifiability of solutions;

(3) connecting – connect the new knowledge with existing mathematical concepts and expand the knowledge network;

(4) problem-solving – solve problems in the novel situation and use a new methods and ideas.

Furthermore, D. Zhang and Yu (2013) not only divided understanding into three types - instrumental understanding, relational understanding, and creative understanding, but also suggested that these three types were hierarchical. Additionally, each level of understanding was comprised of several types of understanding. In general, Chinese educators believe that understanding involves different forms and understanding itself can be levelled.

D. Zhang and Yu (2013) also indicated some differences between Western and Eastern studies regarding understanding from Eastern scholars' perspective, especially concerning views of procedural knowledge. In the case of the addition of fractions, if students could use visual methods to solve the problem, this kind of solution is regarded as understanding in the West. Zhang argued, however, that visual method which would take students too long compared with the pure algebraic approach was not enough for understanding, as an inefficient way. The underlying reason for this was that the Western educators regarded the visual method, specifically drawing, to be a significant part of understanding. Eastern educationalists, however, considered the drawing method to be a facilitator of understanding and that students' understanding should be shown without this facility by using the more abstract method. For example, in relation to the understanding of equivalent fractions, there were six tasks involved as follows (M. Simon & Tzur, 2004, pp. 97, 99, 100):

1. Draw a rectangle with $\frac{1}{2}$ shaded. Draw lines on the rectangle so that it is divided into sixths. Determine how many sixths are in $\frac{1}{2}$.

2. Draw a rectangle with $\frac{2}{3}$ shaded. Draw lines on the rectangle so that it is divided into twelfths. Determine $\frac{2}{3} = \frac{?}{12}$.

3. Draw diagrams to determine the following:

   a. $\frac{3}{4} = \frac{?}{8}$

   b. $\frac{4}{5} = \frac{?}{15}$

   c. $\frac{3}{4} = \frac{?}{20}$

4. Drawing diagrams to solve equivalent fractions problems is not much fun when the numbers get large. For the following do not draw a diagram. Rather describe what would happen at each step if you were to draw a diagram. Use that thinking to answer the following:

   a. $\frac{5}{9} = \frac{?}{90}$

   b. $\frac{7}{9} = \frac{?}{72}$

5. Without drawing a diagram, think in terms of cutting up a rectangle. Use a calculator to calculate the following. Write down each step that you do and the result you get. Justify each step in terms of how it is related to cutting up a rectangle.

   a. $\frac{16}{49} = \frac{?}{147}$

   b. $\frac{13}{36} = \frac{?}{324}$

6. Write a calculator protocol for calculating a problem of the form $\frac{a}{b} = \frac{?}{c}$.

Task 3 represents the Western preferable way towards understanding equivalent fractions, whereas Task 6 demonstrates the Eastern one. This raises two queries surrounding understanding: how to value the visual method in the Chinese definition of understanding which will be explored in Part 2: Results; and how Western educators view the role of rote

learning in understanding especially from the teacher perspective which will be discussed in Chapter 8: Teacher Interviews.

As well as the divergent Western and Eastern approaches to understanding, both sets of educators hold different views of teaching and learning mathematics. According to the dual nature of mathematical concepts (Sfard, 1991), Shiqi Li (1996) proposed two different views of learning mathematics between Western and the Eastern scholars. The Western way perceives drill as the school of behaviourism so that they do not adopt any more. Conversely, the Eastern way regarded rote learning to be a fundamental aspect of learning mathematics. Li explained this division from a psychological perspective. Later, he produced two other papers which profoundly influenced Chinese mathematics education, as warning of the disadvantages of overemphasising the implementation of drill: students would stupefy (Shiqi Li, 1999) even being bored by mathematics (Shiqi Li, 2000).

3.2.4   Understanding the concept of function

Hiebert and Carpenter (1992) considered mathematical understanding to be a network of internal representations. They argued that any idea, procedure, or fact can only be understood when represented in the mind as part of an internal network of representations. There are two further inquiries into a certain mathematical concept that require consideration in terms of internal network: the appearance, what it looks like; and the development, how to develop. In this subsection, relevant literature on understanding the concept of function will be addressed.

In relation to the first inquiry, Vinner (1983) put forward an explanation of the internal network, termed the concept image. A concept image refers to a person's mental picture of a particular concept, including any visual representation and a set of properties. The concept image of function by Grade 9 Israel students may involve linear and quadratic functions. This kind of concept image tended to have concrete types of function. Schwarz and

Hershkowitz (1999) summarised three aspects of the concept image of function by Grade 10 Israel students: prototypical, which means to use specific examples; part-whole reasoning, which means the ability to classify; and attribute understanding, which means to find attributes from representations. For first year college students and junior high school teachers, four categories of concept image were identified by Vinner and Dreyfus (1989): one-valuedness, which means one-to-one correspondence; discontinuity when the graph has a gap; split domain about piecewise function, such as $y = \begin{cases} 2x, & x \in (-\infty, 0] \\ 3x + 1, & x \in (0, +\infty) \end{cases}$; and exceptional point. Different grades of students demonstrated differing explanations of the definitions. The construction and development of these concept images are outlined by the second inquiry.

From a psychological perspective, the development of an internal network is related to how it occurs during the interaction between the information and the human brain. Cognitive psychology has established a series of theories with which to interpret learners' cognitive progress, such as the information processing model of learning. Mathematical concepts are logical, while cognitive progress cannot be conceived in a logical way (Tall & Vinner, 1981). This conflict between mathematical concepts and cognitive progress means that their progression is not simultaneous. This disparity between the development of mathematical concepts and cognitive progress explains why students experience potential cognitive conflict in their minds, especially when learning abstract mathematical concepts. The cognitive conflicts stem from the logical nature of mathematical concepts. The nature of these conflicts can be perceived through the different levels involved in the development of understanding. Models of understanding can clarify the problems with which learners are currently faced. The next sub-section therefore focuses on previous literature about models of understanding.

3.3 Models of understanding development

This section begins with the examination of two models of understanding mathematics which have mainly been applied in the algebraic area in general, and seven

models of understanding function in particular. The first two general models of understanding, Pirie and Kieren's model and the APOS theory, both proposed by Western educators, have been widely applied in China (Shuwen Li & Zhang, 2002; J. Wu, 2011; J. Zhang, 2011). Reviews of two general models' features also reveal that to understand mathematical concepts, especially in the algebraic area, there are two approaches: bottom-up and top-down. The model of understanding function will then be separated into two parts: those proposed by Western educators in the second sub-section, and those by Eastern educators, particularly in the case of Shanghai at the third sub-section.

3.3.1   Pirie and Kieren's Model and APOS

In describing two predominant models of the development of understanding, this sub-section has two objectives. The first introduces two predominant models of understanding models relevant to the area of algebra. Both models took on a constructivist perspective, by regarding understanding as a continual and never-ending process, and were based on observations of classroom teaching and interviews. Meel (2003) illuminated that both the APOS theory and Pirie and Kieren's model successfully met the eight standards for a theory proposed by Schoenfeld (2000): descriptive power, explanatory power, scope, predictive power, rigor and specificity, falsifiability, replicability, and triangulation respectively. The similarities and differences between the two models will then be compared.

Pirie and Kieren's Model. Pirie (1988) found that a student's understanding of the division of fractions could not be classified by understanding categories such as relational understanding or instrumental understanding. She therefore concluded that understanding was dependent on context and also involved many different levels. Later, Pirie and Kieren (1994a, p. 39) referred to mathematical understanding as a process 'grounded within a person, within a topic, within a particular environment'. At the same time, Pirie and Kieren (1994b) presented their growth of understanding model as a dynamic, levelled but non-linear and

recursive process. The model contained eight potential levels: *Primitive Knowing; Image Making; Image Having; Property Noticing; Formalising; Observing; Structuring*; and *Inventising* (see Figure 12).
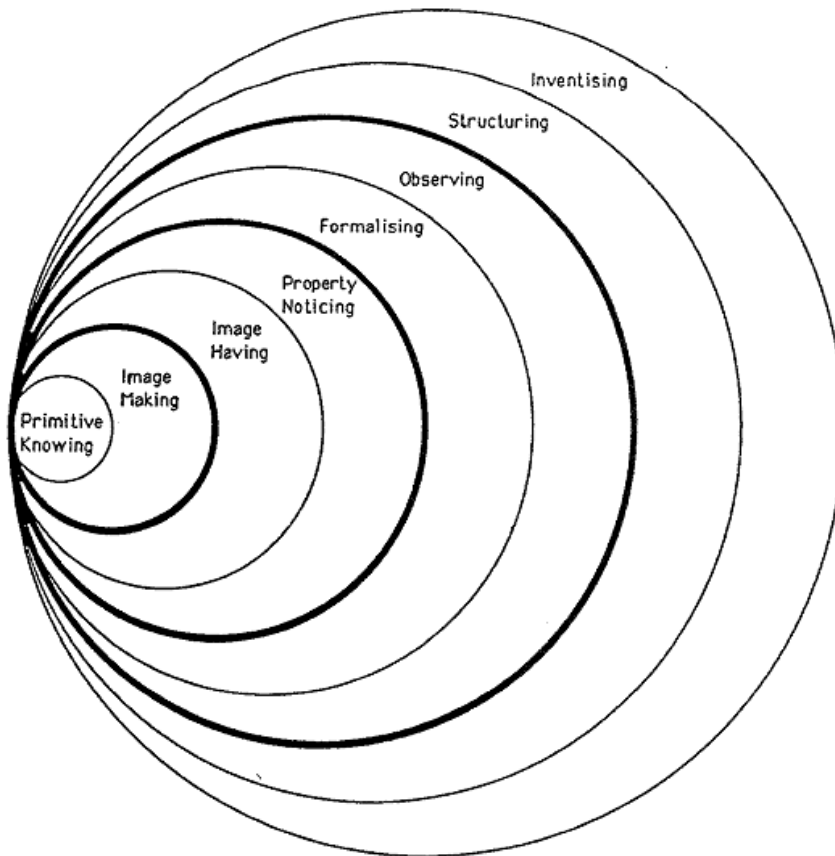
Figure 12. Pirie and Kieren model (Pirie & Kieren, 1994b, p. 167)

Level 1 is the starting point named *Primitive Knowing*, which involves pre-conception, prior knowledge, and any basic proficiency in or knowledge of the new mathematical information. Through pre-tests or oral tests, the teacher or researcher can identify whether students have some relative knowledge or skills related to this new concept. These tests can hardly judge, however, whether students have enough previous knowledge and skills. Level 2 and 3 both involve 'pictorial representation' about the new knowledge, but the word, *image,* does not connote a picture in the mind but any mental and physical activity. In *Image Making,* students use the new ways of representing ideas to reconsider their *Primitive Knowing*. In

Image Having, students have the initial new image already in their minds without having to retrieve the previous image. From Level 1 to Level 3, mathematical understanding is based on particular contexts and deals with specific problems (Pirie & Kieren, 1994a). The first three levels describe how students obtained the new knowledge, by linking with the old network and ready to assimilate a new one.

At Level 4 *Property Noticing*, students can identify the properties of this newly acquired knowledge in a more abstract way. This leads to the level of *Formalising* in which students formulate a formal mathematical definition using their own words and methods. Within these levels, students can therefore recognise general characteristics within informal patterns and apply proper methods in order to defend their views. Moving on to the level of *Observing*, the previous formal organisation of properties can be reorganised and recombined, while in the *Structuring* Level learners can justify and verify these restructured properties through mathematical proof. At this point, students have developed the new schema and know the structure of the new knowledge.

Finally, *Inventising*, the outermost layer, involves a fully structured understanding of a given mathematical area which can be used to generate new questions and process into the *Primitive Knowing* of a more abstract area of mathematical knowledge, forming a new cycle of understanding once more.

The Pirie and Kieren model has been widely used to analyse the process of students' understanding with primary school students (Thom & Pirie, 2006), low-ability students at secondary school (Pirie & Martin, 1997), undergraduates (Walter & Gibbons, 2010) and prospective teachers (Cavey & Berenson, 2005). It depicts the growth of understanding in detail from a cognitive perspective and is based on the subject's previous understanding or daily life experiences.

When students meet advanced mathematics in which concepts are more abstract, formal, and logical, they may find it hard to make connections with an existing network as there is little relationship between school maths and out-of-school maths, with school maths being far away from students' daily living experiences (school maths will be addressed at Chapter 8: Teacher Interview).

APOS Theory. Dubinsky and McDonald (2002) proposed a hierarchically ordered understanding theory, known as APOS, to probe undergraduate students' understanding. APOS was composed of four stages: *Action*; *Process*; *Object*; and *Schema*.

The first level forms an *Action* through repeated operations with the help of external stimuli. In fact, this step is normally algorithmic so that individuals can experience the relationship between operation and concept.

When learners are familiar with actions, they no longer require the external stimuli any more. Actions become interiorised to form an internal *Process*. Learners can then describe actions, reverse the steps, and infer other processes through coordination with other processes or through the reversal process. That is, in *Process*, students reflect on *Actions* by interiorising and encapsulating.

When learners regard the processes as a whole upon which to operate, the *Process* is encapsulated to become an *Object*. The mathematical concept comprises of properties. Understanding a mathematical concept stem from learning the properties of the concept, then identifying some 'salient elements' among these properties (Meel, 2003, p. 151). Once learners recognise the nature of the concept, properties, their understanding of concept can become an *Object*. Learners may return to the *Process* stage in order to perform new manipulations, namely to de-encapsulate.

Finally, learners have a mental structure of this concept, namely *Schema*. *Schema* is a generalisation of *Action*, *Process*, and *Object* combined in order to resolve a problem.

Schema is analogous to the notion of 'concept image' proposed by Tall and Vinner (1981) as discussed in the previous section: Understanding the concept of function. Furthermore, Clark et al. (1997) proposed a framework for schema development which includes three steps, *intra schema*, *inter schema*, and *trans schema*. For the single object, the learners' mind involves *intra schema.* For the different objects connected with each other, *inter schema* occurs. Once the *inter schema* comes into being as a continuous complete construct, *trans schema* appears.

Comparing the two models. The differences between the Pirie and Kieren model and APOS theory will initially be addressed and their similarities will then be presented.

Differences. Firstly, two models' approaches to understanding differ. APOS theory aims to establish that the advanced mathematical understanding processes are built on recognising the main mathematical characteristics of a concept. This kind of concept may be more abstract in mathematics while emphasising properties. Among these properties, the main features are distinguished. Instruction relies on the features of a certain concept rather than students' prior knowledge; Hiebert and Carpenter (1992) called this the top-down approach. Conversely, the Pirie and Kieren model tends to connect with prior knowledge, and Hiebert and Carpenter (1992) labelled this a bottom-up approach.

The second difference is related to the teacher's role. The role of the teacher differs in relation to fostering conceptual thinking. In the Pirie and Kieren model, teachers are expected to organise the classroom environment in order to encourage mathematical learning. The APOS theory is inclined to present dis-equilibrating environments so that teachers help the students to manage their frustration. For abstract concepts, learners' understanding is supposed to be developed from part to whole. Dubinsky (1991) used genetic decomposition to describe the possible construction of a concept. For example, the concept of function involves two approaches to construction as discussed earlier. When arriving at the full

structure, the process opens up a new horizon for learners. Teachers would therefore have to take a more leading role in the development of understanding.

Similarities. In addition to these differences, some similarities between the two models have been identified. First, the *folding back* in the Pirie and Kieren model is similar to the process of de-encapsulation in APOS theory (Meel, 2003). Secondly, the same phenomena are explained by both models, such as misconceptions and concept images. Pirie and Kieren's model described misconceptions as troublesome limited images in learners' minds, while the APOS theory considers it to be a form of cognitive dissonance between a narrow scope and the broader context. Finally, overcoming obstacles involves recognising non-conformity among existing connections and developing the more successful links in the Pirie and Kieren's model, while the APOS theory considers the process of overcoming obstacles to be a reflection on existing understanding and the integration of new elements.

The two models summarise the general understanding process in mathematics learning. The next section will focus on the understanding development within a certain concept, namely function, from the models proposed by Western educators.

3.3.2 Five models of understanding function in Western literature

The development of students' understanding of function is demonstrated by several steps in the understanding development model which reflected a significant shift in their cognitive progress. Five current models of understanding function provide the answers from different perspectives of how understanding is developed.

Model 1. This model revealed how 16-years old students understood the concept of function *f(x)* from a symbolic view. Sajka (2003) used the PROCEPT framework proposed by Gray and Tall (1994) to illuminate students' cognitive stages. The PROCEPT framework stemmed from revealing the cognitive progress of understanding in elementary mathematical concepts from a symbolic perspective. Gray and Tall (1994) demonstrated the ambiguity of

symbols. The same symbol could be interpreted in two ways, as a concept and a process. Concept referred to the production of process, namely the results of process. For example, this symbol itself, $f(x) = 2x + 1$, could be represented in two ways: one was the calculation of the value when given x, which meant the process; the other was the completed product for the general value of x, which meant the object, the concept of linear function. Gray and Tall (1994) combined the concept and process to define the notion of PROCEPT as follows:

An *elementary procept* is the amalgam of three components: a *process* that produces a mathematical *object*, and a *symbol* that represents either process or object.

A *procept* consists of a collection of elementary procepts which have the same object.

(Gray & Tall, 1994, p. 121)

Here, the 'elementary procepts' were similar to the 'process' stage in APOS theory, while PROCEPT would be the same as 'object' stage. Tall (1999, p. 5) acknowledged that 'the PROCEPT notion has strong links with APOS theory'.

Six steps were described in the PROCEPT of function in a study by Sajka (2003, p. 250):

Step 1: Absence of a notion of function; thinking in terms of numerical equations and unknowns;

Step 2: Function as the beginning of a new thought or new task;

Step 3: Function as a formula;

Step 4: Function as that which 'determines all the rest in the formula';

Step 5: Function as a computational process;

Step 6: Function as a kind of formula which leads to drawing a graph.

At Step 1, student's understanding remained at the process of construction. At Step 2, when the symbol f appears, function was understood as the rule to look for a value. Step 3 demonstrated that students could not distinguish between the concept of equation and

function. At the following step, students could tell the difference. Furthermore, f linked with 'value' x, namely *f(x)*, as it could determine the value of function in case of $f(x) = 2x + 1$. In the last step, students had the object aspect of algebraic expression, so that another operational aspect towards graphic representation emerged.

Model 2. The second model focused on different representations. Previously in this chapter, it had been discussed that one of the main learning difficulties regarding the concept of function is how to connect these representations. This model assumed that students had already grasped the meaning of individual representation, so that their understanding was at least at the '*Image Making*' level of Pirie and Kieren's model.

Hitt (1998) identified teachers' difficulties with representations and compared these with previous literature about students' struggles. He suggested five levels of understanding mathematical concepts to measure how to develop the understanding of representations:

Level 1: Imprecise idea about a concept;

Level 2: Identification of different representations of a concept;

Level 3: Translation with preservation of meaning from one system of representation to another;

Level 4: Coherent articulation between systems of representation;

Level 5: Coherent articulation of different systems of representation in the solution of a problem (Hitt, 1998, p. 125).

At Level 1, teachers showed their imprecise ideas of graphical representation when they were required to discern whether a certain curve corresponds to a function. This graphic representation was not as strong as the formal definition. At Level 2, the different ways of presenting the algebraic expression for the same function influenced the identification. The different writing approaches gave rise to misunderstanding; for example, whether $f(x) = 2$ was equal to $g(x) = \sqrt{4}$ for all $x \in R$. Level 3 required the ability to connect different

representations. Within Level 4, the different representations should be understood as they offered the same concept. But, teachers showed the barriers as they cannot successfully link the graphical representation with domain (the sub-concept of function). When placing graphical representations within a real context, it was difficult for teachers to articulate the relationship between different representations. Hitt's study demonstrated teachers' weaknesses when faced with a curve (as a type of graphical representation) in purpose of associated with the algebraic expression.

     Model 3. In this model, the emphasis was on the connection between graphical and symbolic representations based on the SOLO taxonomy (Biggs & Collis, 1991). SOLO taxonomy was one of the neo-Piagetian theories of cognitive development. Five levels (prestructural, unistructural, multi-structural, relational, extended abstract) and three models (previous, target, next) were categorised (see Table 6).

Table 6

SOLO Taxonomy (Biggs & Collis, 1991, p. 65)

Models	Levels	Structural level
Previous	1. Prestructural	The task is engaged, but the learner is distracted or misled by an irrelevant aspect belonging to a previous stage or mode.
	2. Unistructural	The learner focuses on the relevant domain, and picks up one aspect to work with.
Target	3. multi-structural	The learner picks up more and more relevant or correct features, but does not integrate them.
	4. Relational	The learner now integrates the parts with each other, so that the whole has a coherent structure and meaning.
Next	5. Extended abstract	The learner now generalizes the structure to take in new and more abstract features, representing a new and higher mode of operation.

     The first model (Previous) and first level (Prestructural) both signalled that 'the task is engaged' (Biggs & Collis, 1991, p. 65). For the concept of function, students can be engaged with the function machine such as a calculation game. The variable perspective at the prestructual stage provided students with an early understanding of function. Students

possessed the fundamental image of function, but this was not necessarily relevant to the kernel element of the concept of function.

The last model (Next) or the last level (Extended abstract) can also be regarded as the foundation for further learning. During the understanding development, not only was the content on function knowledge and skills carried over to the next level of study, but also the experience and feelings related to the concept of function. SOLO taxonomy, however, focused on the middle three levels: unistructural, multi-structural and relational. Zachariades, Christou, and Papageorgiou (2002) investigated the first year mathematics undergraduate students' cognitive development when they connected these representations in the concept of function. Three cognitive developmental levels of the concept of function were identified (see Table 7).

Table 7

Zachariades et al. (2002)'s Cognitive Levels about Function

Level	Cognitive levels about function	SOLO taxonomy
1	Identify some kinds of function representations	Unistructural
2	Discriminate and recognize symbolic and graphical functions in a consistent way	Multi-structural
3	Make precise connection between graphical and symbolic representations	Relational

At Level 1, the identification of some forms of function representation means that students can recognise these representations individually. At Level 2, students can connect corresponding representations, from graphical to algebraic expression and from algebraic expression to graphs. Level 3 demonstrated the degree of accuracy of translation. The precise connection may result from plenty of practice. Once students can integrate the relevant representations for function, the initial outline of particular types of function can appear in students' concept image. This can therefore be counted as a coherent structure and meaning.

Model 4. This fourth model delineated Grade 8, 9, 10 students' developing understanding of the algebraic expression. Compared with the other main representations

(tabular and graphic), the algebraic expression was a versatile way for students to understand function. The equation form was considered as a statement of equality first and a representation of relationship second. In this model, there were four big ideas as follows (Ronda, 2009):

Growth Point 1: Equations are procedures for generating values;

Growth Point 2: Equations are representations of relationships;

Growth Point 3: Equations describe properties of relationships;

Growth Point 4: Functions are objects that can be manipulated and transformed.

Each growth point included the strategies, knowledge, and procedures. At Growth Point 1, students focused on the results either for the value of x or that of y, without considering the relationship between x and y. Within Growth Point 2, students noticed the relationship between the variables. Moving to Growth Point 3, students recognised properties from the algebraic expression, for example the y-intercept and gradient in linear function. After identified these properties, students' understanding grew towards the object. By this point, they can regard the entire equation as the whole to manipulate or transform.

Model 5. Although the concept of function had three main representations: geometric (graph), numeric (tables), and symbolic (algebraic expression), DeMarois and Tall (1996) argued that the connections between representations should also be included within this list, such as the way of thinking about the connection and how to demonstrate the connection. From a much broader representations perspective, DeMarois and Tall (1996) proposed a two-dimensional structure, namely horizontal and vertical scaling to specify the understanding model of function (see Figure 13). Horizontal scaling was related to eight forms of representations. Besides three main representations, five other facets outlined various ways of presenting function such as notion *f(x)*, or expressing concepts such as written or verbal

descriptions - written, verbal (spoken), kinesthetic (enactive), colloquial (informal or idiomatic), and notational conventions (DeMarois & Tall, 1996, p. 2).
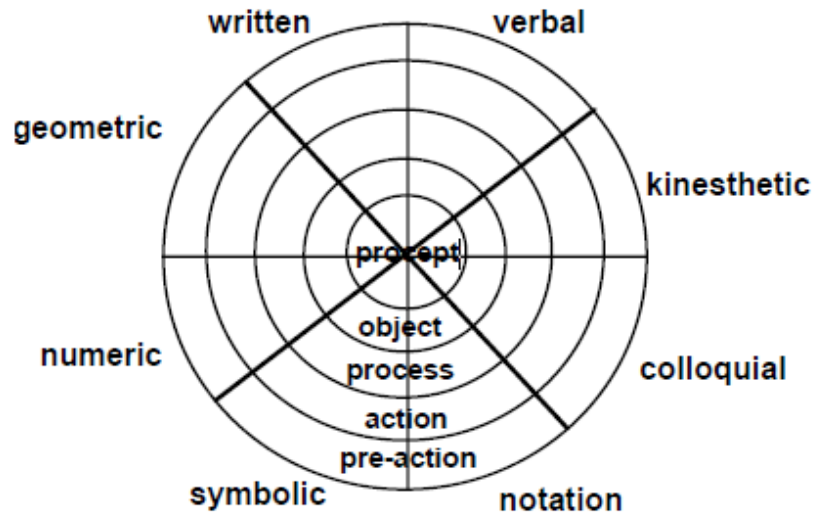
Figure 13. Facets and layers in Demarois and Tall's model

There were five layers in the vertical part of the model (pre-action, action, process, object and proceptual) with three of these aligning with the APOS theory, specifically *Action*, *Process*, and *Object*. The first layer, pre-action, indicated 'a ground floor' or a foundational basis for students' learning. Within the action layer, learners relied on the specific operation. In the process layer, learners began to understand the idea of 'input-output' without being aware of the concrete steps involved. For the object layer, learners can regard the process as an object, while in the last layer, proceptual, they demonstrated flexibility by shifting between different processes and object layers. This model was used to examine community college students' understanding of function.

In summary, each model was proposed from different considerations. The model proposed by Sajka (2003) was concerned the initial conceptualization of function. The other three models by Hitt (1998), DeMarois and Tall (1996), and Zachariades et al. (2002) mainly examined how to handle representations, while the model proposed by Ronda (2009)

especially paid attention to certain type of representation, the algebraic expression. These models were all proposed by Western educators. The next sub-section will introduce understanding models from the perspective of Shanghai.
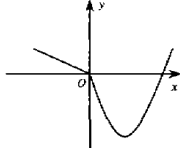
3.3.3   Two models of understanding function in Shanghai

As discussed at the beginning of this chapter, in Shanghai the concept of function underwent a change from the junior secondary school to the senior secondary school. Initially it was known as the variable view, but then altered to become known as the mapping view. The teaching aim of the variable view of function was to focus on the relationship between variables on a macro scale. The specific and concrete types of this function played a vital role in the understanding of function. At the mapping view stage, the major task of teaching function turned to focus on the meaning of domain, range, and correspondence as three main factors of function, instead of specified algebraic expression or mathematic terminology.

There were two models depicting secondary school students' cognitive processes proposed by Zeng (2002) and Jia (2004). Jia (2004)'s model was developed from Zeng (2002). Both considered students' understanding development of the concept of function during the whole secondary school stage. With more careful and clearer divisions, Jia's idea regarding the understanding process was more dynamic instead of an incremental shift. Jia considered students' understanding progression to be built upon cognition, feedback, and recognition. Table 8 shows the relationship between the two models with examples.

Table 8

Two Models of Understanding Function in Shanghai, China

	Zeng (2002)	Examples for Zeng (2002)	Jia (2004)	Examples for Jia (2004)
Junior secondary school (Year 11-15)	Level 1 Function as 'process of operation' (Year 14)	Connect function with equation. Cannot answer graph problem to judge if it is a function such as:	Stage 1 Understanding variable	Change view of $s = 100t$. If t changed, then s would also change.
			Stage 2 Stressing relations	$s = 100t$, students focus on the relationship between t and s, presenting as the algebraic expression
			Stage 3 Employing formula	Using functional view to re-examine the former concept, including equation, inequality and algebraic expression.
Senior secondary school (Year 16-18)	Level 2 Function as 'process of variation (Year 15)	Find out the maximum value and minimum value of quadratic function, $y = x^2 - 2x + 3$, $x \in [-1, 4]$.		
			Stage 4 Knowing correspondence Stage 5 Grasping formal description	Find endpoints in a certain piecewise function Explain the meaning of $y = f(x), x \in D$
	Level 3 Function as 'relation of correspondence' (Year 17)	Multiple choice Choose the right answer: A. function is just an expression; B. function reflects a changing process; C. function is a corresponding relationship.	Stage 6 Understanding it as an object	Draw a graph of $y = 2\sin(x + \dfrac{\pi}{6}) + 3$

At the beginning of learning the concept, namely Stage 1, Understanding variable (Jia, 2004), students were required to distinguish the differences between the constant and variable, which is in line with the first level of understanding in Sajka's model (Sajka, 2003). For example, in terms of the formula $s = vt$, if v was given by 2 miles per hour, then distance would be $s = 2t$. Before learning the concept of function, students would regard s and t as two constants, provided the value of t then calculated s, and vice versa such as in the input-output assignment in Doorman's model (Doorman et al., 2012). At this level of understanding, they should be able to recognise that the letter t was a set of numbers instead of a certain number as called variable. $2t$ became an object instead of merely a calculated number or a certain value, and the value of $2t$ was equal to the value of s. This description corresponded to the *elementary concept* of PROCEPT (Gray & Tall, 1994). With this new perspective, the relation between equation and function, which shared the same form, was reappraised. Algebraic expression or symbolic representation therefore became the primary impression of function for Shanghai students.

At Stage 2, Stressing relationship (Jia, 2004), this process demonstrated the nature of the process (Schwartz & Yerushalmy, 1992). If given the equation $x + y = 10$, students automatically re-arranged it as $y = -x + 10$, viewing it as one certain type of function, linear function. This process, however, only related to equation and not to graph. It implied the algebraic approach to function in Shanghai.

At Stage 3, Employing formula (Jia, 2004), students acquired the understanding that function can contain equation and inequality. They therefore gained the ability to link with other knowledge. For example, one unknown equation $x + 1 = 0$ can be explained as function $y = x + 1$ when the dependent variable equals 0; and one unknown inequality $x + 1 > 0$ can be regarded as above x-axis part at graph of function $y = x + 1$. Within Zeng's model, the function of this level was the 'process of operation'.

Both Stage 4, Knowing correspondence, and Stage 5, Grasping formal description (Jia, 2004) can be regarded as another form of Stage 2 Stressing relationship under the condition of *set*. Level 2 function as 'process of variation' at Zeng's model required not only connecting ability, but also the reasoning ability as shown in the example.

Compared with models proposed by Western scholars, the two Shanghai models show their strong links with the definition of concepts in textbooks and the tendency to use the algebraic approach towards the concept.

3.4 A General Model of Understanding Function

This section will present a general model of understanding function based on my own perspective alongside illustrative examples. The aspects of the five Western models and two Shanghai models are reflected in this general model. The general model will then link back to the Pirie and Kieren's model and the APOS theory.

3.4.1 The general model applied in this study

The general model divides the growth of students' understanding of the concept of function into six levels: Variable Perspective, Dependent Relationship, Connecting Representations, Property Noticing, Object Analysis, and Inventising. The understanding of function starts with two elements: 'variables; and the expression of the relationship between variables by the means of equations' (Kleiner, 2009, p. 15) as the first two levels in this general model.

Level 1 Variable Perspective. The first understanding step is to take views of the variables instead of unknown. There are two different approaches to explaining the variable perspective, visually or algebraically. Images, such as the input-output machine and graphs, present the importance of visualization in understanding function. On the other hand, algebraic expression demonstrates the rigor of the formal style.

Students start with an input-output assignment as an operational task. Slavit (1997) considered that this input-output view would be based on a point-to-point view, namely a coordinate which is a point in the graph. The function machine is the foundation of understanding the concept of function (McGowen et al., 1999). That is, input is a prototype of the independent variable, while output is a prototype of the dependent variable. The set of inputs is presented by a symbol x, while y signals the outputs. The notion of variables depends on 'the notion of domain' (Schoenfeld & Arcavi, 1988, p. 423) as the meaning of domain will be highlighted for further stage learning function (senior secondary school stage in Shanghai). Students now therefore make sense of the x represented in all of the inputs at this level of understanding function, regarding the input from 'constant quantities' into 'changing magnitudes' (Sfard & Linchevski, 1994, p. 200).

In Shanghai situation, the variable perspective is based on an equation. Function is developed from the concept of equation. In terms of the two variables in an equation such as $x + y = 2x - y + 4$, students should be able to differentiate between the views of function and equation (Oehrtman, Carlson, & Thompson, 2008). The understanding of the concept of the equation is expanded from 'thinking in terms of numerical equations and unknowns' as in Step 1 of the study of Sajka (2003), to regard the left and right sides of equation as two separate functions. At the beginning, students usually use two approaches, the numerical and the functional, to tackle the problems (Sfard & Linchevski, 1994).

Alternatively, graphs can be a form of input-output (see Figure 14) as follows (McGowen et al., 1999):

a. What are the output(s) if the input is 3?
   Answer:_____

b. What are the input(s) if the output is 0?
   Answer:_____

Figure 14. The example for Level 1

Meanwhile the relationship between the input and output can be visually obtained from graphs, while the process of discerning whether the relationship involves function occurs at the next level of understanding function, *Dependent Relationship*.

At Level 1 Variable Perspective, the concept of function is a pre-action aspect of DeMarois and Tall (1996)'s model, a new thought or new task at Step 2 in Sajka (2003)'s model, understanding variable at Stage 1 in Jia (2004)'s model.

Level 2 Dependent Relationship. At this level, the dependent relationship between two variables is the focus, while Level 1 is concerned much more with finding the unknown number in a given operation. The dependent relationship indicates a one-to-one or many-to-one property of the rule instead of one-to-many. Here, students' understanding is shaped by the action, in line with the second layer of DeMarois and Tall (1996). The rule is expressed by different representations. Hitt (1998) proposed that students acquired the different representations first and then developed an ability to identify them. Here, these two steps are interwoven together to develop students' understanding of the dependent relationship.

Three main representations (algebraic expression, tabular and graphic representations) show the dependent relationship from three perspectives. In the case of the algebraic expression, function can be seen initially as a formula; and this formula then determines all the rest; finally there occurs a computational process as in Steps 3, 4, and 5 in the model of Sajka (2003). Similarly, Ronda (2009) described how to develop the dependent relationship in terms of the algebraic expression: the first (a procedure to generate values) and second growth points (a product as a representation of relationships). Two Shanghai models also explained it as stressing relations (Stage 2 at Jia's model) and process of operation (Level 1 at Zeng's model).

There is an example provided by O'Callaghan of the development of the dependent relationship in terms of tabular representation. The following table (Table 9) gives the value

(*V*) in dollars of a car in the years (*t*) after it has been purchased (O'Callaghan, 1998). Students have to find out the dependent relationship between two variables first in order to solve the value of *V*.

Table 9

An example for Level 2

t	V
0	16800
2	13600
4	10400
6	7200
10	?

After this level of understanding, students should overcome the point-to-point understanding of function, and be ready to undertake the process perspective (Llinares, 2000) in the next level.

Level 3 Connecting Representations. Connecting representations requires students not only to translate one representation into another, but also to understand the relation between them as they present the same concept. For example, when given the algebraic expression $y = 2x + 1$, students could draw the graph, which is a straight line, through a table to produce two ordered pairs such as $(0,1)$ and $(-\frac{1}{2}, 0)$. That is, function becomes a kind of formula which leads to drawing a graph, as the sixth step in Sajka's model. Through this process, students can discriminate and recognise functions in form of symbolic and graphical as a consistent way, described as Level 2 in model of Zachariades et al. (2002).

Akkoç and Tall (2005) indicated that many students fail to connect different representations together. Particular difficulties with the translation from the graph representation to the algebraic rather than vice versa have been noted (Markovits, Eylon, & Bruckheimer, 1986). In this case, if the translation process is involved in recognising the property of function, it may be regarded as the next level of understanding.

Students can then make a precise connection between graphical and symbolic representations, as described in Level 3 in model of Zachariades et al. (2002). It leads to the preservation of meaning from one system of representation to another as in Stage 3 of Hitt's model. For further learning, students can defend their solutions via multiple representations which need more flexible views of function. This level of understanding can provide different perceptions of function's appearance.

Level 4 Property Noticing. This level is built on different representations that have been recognised and connected. Here, *Property Noticing* is the terminology that Pirie and Kieren used to describe the fourth layer in their understanding model (Pirie & Kieren, 1994b). Some local properties can be understood at this level and students can describe the y-intercept and the slope, in a similar way to growth point 3 as proposed by Ronda (2009): Equations describe properties of relationships. For example, with the standard form of linear function $y = ax + b$ $(a \neq 0)$, the intercepts are the points at which the graph meets the y-axis in the graphic representation. It can also be summarised by the standard form of point $(0, b)$. Students' understandings, therefore, are based on acquiring a coherent articulation between systems of representation, as described at Stage 4 in the model of Hitt (1998).

Level 5 Object Analysis. This term is similar to the object stage in the APOS theory. Within the APOS theory, after the process stage, students can develop the object stage. At Level 5, students can understand linear function as a whole thing; for instance, dealing with the transformation of the graph of $y = f(x)$ by a vector $\begin{pmatrix} 0 \\ a \end{pmatrix}$. For a concrete example, students can discern whether $y = 2x^2 + 3x + 1$ could move to $y = 2x^2 - 4x - 1$; and, if so, to which direction and how many units move in x-axis and y-axis at the Cartesian coordinate system.

In another example of this level, students would be able to articulate the relationship between $f(x + 3)$ and $f(x)$ through the perspective of transformation. Students should also

be able to articulate the similarities and differences based on a certain property. This is in line with the growth point 4 as proposed by Ronda (2009) as functions are objects can be manipulated and transformed. Meanwhile, this understanding level fits with the aspect of the concept of function as revealed by Doorman et al. (2012) which characterises understanding function as a mathematical object with differentiation or integration.

Distinguishing Level 4 from Level 5. As discussed in the first section, some properties have local features and some have global ones; the local features are in Level 4 and the global features are in Level 5 respectively. The former characterises individual pairs, such as intercepts, while the latter analyse 'the entire function', for example, monotonicity and period (Slavit, 1997, p. 264). If individual pair is concerned in the noticing of the properties, this understanding can be regarded as Level 4, while the whole 'picture' that emerges from the property can be considered Level 5.

Reason abilities (Doorman et al., 2012) are required at these two understanding levels. In the case of linear function $y = ax$ ($a \neq 0$), the more oblique the graph is, the bigger the value of a. The assumptions of the implications of this conclusion are based on the combination of the graphical representation and the algebraic expression (see Figure 15). The first step involves achieving the same value of x in each straight line, $y = a_1x$ and $y = a_2x$. Suppose, for example, that the lower point is B (x, y_1), namely(x, a_1x), while the higher point is C (x, y_2), namely (x, a_2x), with the same value of x. From the graph, the y-axis coordinate of point C, a_2x, is greater than that of B, a_1x. If $x > 0$, then $a_2 > a_1$. Similarly, if $x < 0$, the opposing conclusion can also be inferred.
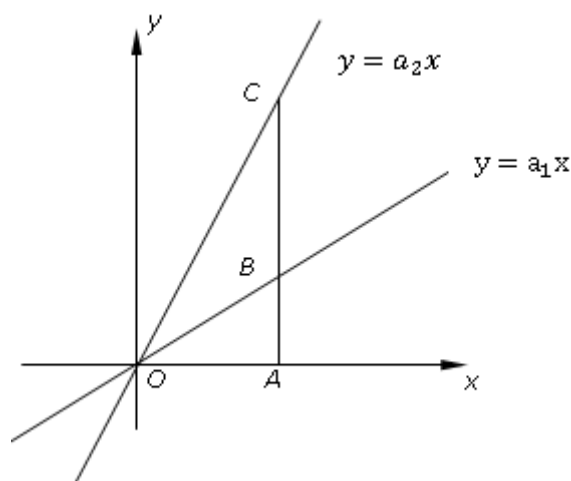
Figure 15. Example of gradient

From this example, the reasoning ability is still drawn from several related pairs. The understanding shown here is positioned in Level 4.

There is an example of Level 5. The general form of a quadratic graph is $y = ax^2 + bx + c, (a \neq 0)$. Investigate what happens to the graph if the values of a, b and c change (Rayner, 2006, p. 379). This question requires students to perceive the changing quadratic graphs as a whole.

Level 6 Inventising. At this final level of understanding, learners are more flexible when dealing with the concept of function. Students are capable of linking this knowledge to other mathematical knowledge and making new connections stronger. As a result, they can invent and explore new knowledge.

First, their internal network of function concepts connects to other mathematical knowledge such as inequalities and geometrical knowledge. Learners can apply the knowledge to novel situations or non-routine problems, displaying 'good understanding' as previously discussed. Here is an example chosen from the final examination of the first term in Grade 9 (age 15) in Shanghai, January 2013. As shown in Figure 16, in the rectangular coordinate system XOY, the graph of the quadratic function $y = -\frac{2}{3}x^2 + bx + 5$ has an

intersection point with the x-axis at point A (5,0) and the y-axis at point B, and there is a

point C in this graph in which the abscissa is 3. Students were asked to undertake the

following tasks':

   (1) Find out the algebraic expression of this quadratic function;

   (2) Find out the value of $\tan\angle BAC$;

   (3) If there is a point D in the graph of quadratic function and $\angle DAC = 45°$, please

find out the ordinates of point D.
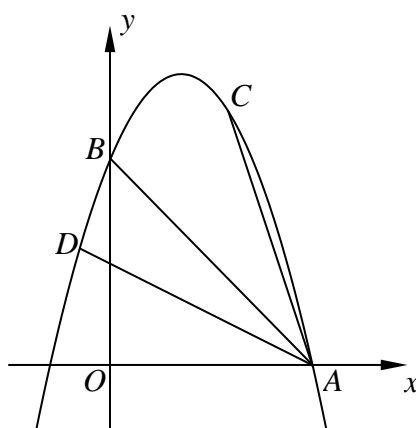
Figure 16. Example for Level 6

   In this question, trigonometric function and geometry knowledge, such as similar

triangles, are all linked to linear function. In order to solve this problem, students must have

all of the basic knowledge of linear function as well as quadratic function, and using

representations and properties in a much deeper way.

   Secondly, *Inventising* was the term used in Pirie and Kieren's model to distinguish the

outermost layer. The meaning is used here, i.e. it acts as a full structure of understanding and

a basis to develop new concepts. That is, students do not only develop new types of function,

but also defend their mathematical procedures. There is an illuminating example from Wells (1988, p. 414):

What can you predict about the graph of $y = (x - 1)(x - 2)(x - 3)$, before you draw it? To solve this problem, students should identify some properties, such as intersections, monotonicity and symmetry for this new function. There are three points of graph which will be located on the x-axis, $(1, 0), (2, 0), (3, 0)$ and one point on the y-axis, $(0, -6)$. As for monotonicity, when $x < 1$ or $x > 3$, y increases faster as x increases. As compared with the graph of $y = x^3$, when $x < 1$ or $x > 3$, the graph increases in magnitude. When the value of x is between 1 and 3, the graph would be smooth and point $(2, 0)$ is the point of symmetry. That is, the value of y when $x \in (-\infty, 2)$ is the opposite number of the value of y when $x \in (2, +\infty)$. There is the algebraic proof as follows: suppose there are two points, $(2 + t, y_1)$ and $(2 - t, y_2)$, to fill in the algebraic expression, i.e. $y_1 = (2 + t - 1)(2 + t - 2)(2 + t - 3) = (t + 1) \times t \times (t - 1)$ and $y_2 = (2 - t - 1)(2 - t - 2)(2 - t - 3) = (1 - t) \times (-t) \times (-1 - t)$. Simplify the second one $y_2 = (1 - t) \times (-t) \times (-1 - t) = [-(t - 1)] \times (-t) \times [-(1 + t)] = -(t - 1) \times t \times (t + 1) = -y_1$.

This process demonstrates not only that students can justify their judgment, but also can show the ability to develop the unknown concept, drawing the graph of $y = (x - 1)(x - 2)(x - 3)$ (see Figure 17), based on flexible applying the properties (local and global).
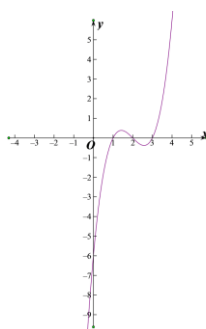
Figure 17. Example of exploring a new concept

Although students' understanding is not supposed to jump between the levels, the development of their understanding is not strictly sequential. Their learning does not 'proceed linearly' (M. Simon, 1995, p. 140). It is assumed in Pirie and Kieren's model that students would go back to former levels from time to time in order to expand their primitive knowledge or skills from the perspective of new levels. However, in this general model of understanding function, the preparative mathematical knowledge of the current level of understanding should be combined with the former levels of understanding, as a crucial aspect of the development of the current students' understanding level. For example, at Level 4, students are required to make sense of y-intercept. They would recall the characteristic of coordinate located in y-axis ($x = 0$), and combine it with algebraic expression with aids of graph to get this property.

This general model focuses on one specific concept rooted in Western and Shanghai models of understanding linear function. At the following sub-section this model will be linked with two general models.

3.4.2 A comparison with Pirie & Kieren's model and APOS

Pirie and Kieren model and the general model of understanding function. Table 10 shows the comparison between the Pirie and Kieren's model and the general model.

Table 10

Link with Pirie and Kieren's Model

The model of understanding function	Pirie and Kieren model
	Primitive Knowing
Variable Perspective	Image Making
Dependent Relationship	Image Having
Connecting Representations	
Property Noticing	Property Noticing
Object Analysis	Formalising
	Observing
Inventising	Structuring
	Inventising

In the Pirie and Kieren model, *Primitive Knowing* is regarded as a 'starting place' of understanding (Pirie & Kieren, 1994b, p. 170). Conversely, the model of understanding function considers more the preparative knowledge related to each level of understanding as discussed above.

Image Making is defined as making 'distinctions in previous knowing and using it in new ways' (Pirie & Kieren, 1994b, p. 170). As for function, the first step involves shifting the perspective, using the variable perspective to take an alternative look at quantity and equation as the level of Variable Perspective.

In *Image Having,* students can 'use a mental construct about a topic without having to do the particular activities' (Pirie & Kieren, 1994b, p. 170). This mental construction of function is where multiple representations connect. It is divided into two levels in the model of understanding function: Dependent Relationship and Connecting Representations. M. Wilson (1994, p. 347) proposed six crucial aspects of deep understanding for the concept of function, two of which are 'interpreting functions represented by graphs, situation descriptions, formulas, and tables', and 'translating among multiple representations of function'. The former aspect is about the rule (the dependent relationship), while the latter aspect describes the connections. Dependent Relationship and Connecting Representations should therefore be separate from *Image Having*.

Property Noticing does not have the exact same meaning in Pirie and Kieren's model and in the general model because of the global and local categorisation of the properties in function as discussed above.

Two levels in Pirie and Kieren's model, *Formalizing* and *Observing*, are combined into one level, Object Analysis in the general model. Reasoning ability is necessary for both of these levels in order to construct a coherent and systematic mental network. *Formalising* means that students could 'abstract the methods about how to find these properties' (Pirie &

Kieren, 1994b, p. 170). Here, students should have some skills or experience in the discovery of local properties. They can make sense of these different approaches to properties and then visualise 'class-like mental objects' (Meel, 2003, p. 145). *Observing* is described as 'looking for patterns' (Pirie & Kieren, 1994b, p. 171). Students can identify essential components and connect ideas (Meel, 2003). These two psychological processes are interwoven together to develop students' abilities of inference; for example, the transformation of the function.

     Structuring and Inventising are categorised under Inventising in the general model of understanding function. *Structuring* is involved in 'justification or verification' (Pirie & Kieren, 1994b, p. 171) and, in this process, students can make a connection 'across multiple domains' (Meel, 2003). Students are more flexible in dealing with non-routine problems which associate with other mathematics topics as Inventising Level.

     APOS theory and the general model of understanding function. Students have an image of Dependent Relationship at the input-output assignment, which also occurs in *Action* in the APOS theory (see Table 11). They follow the teacher's instructions in order to make sense of the dependent relationship in three main representations of function.

     In *Process*, students can reverse the process and combine other processes (Dubinsky & McDonald, 2002). Connecting Representations and Property Noticing link with *Process* in the APOS theory. The process of connecting the representations provides opportunities and is a foundation to identify properties. These properties have two types of meaning: algebraic and graphic. For example in linear function, y-intercept can be noticed as the point that the graph passed the y-axis. On the other hand, if given an algebraic expression, $y = ax + b, (a \neq 0)$, students can recognise that the intercept is $(0, b)$, rather than actually plotting a graph to discover the y-intercept. The way in which they note these properties is based on their automatic connection of the two main representations. In the case of the y-intercept of quadratic function $y = ax^2 + bx + c, (a \neq 0)$, it is $(0, c)$. Reciprocal function, however, has

no y-intercept point. In this case, property therefore stems from a graph and later is connected to the algebraic expression.

Considering function as an *Object* can be reflected in the fact that students can create graphic transformations in algebraic expressions and defend their statements. Object Analysis highlights the students' abilities of logic and reasoning. *Object* can be an outcome, whereas Object Analysis emphasises the process as well. Instinctively, action, process and object can 'be organized to form schemas' (Asiala et al., 1996, p. 8). *Schema* can be a medium through which to resolve non-routine problems such as Inventising level of understanding function.

Table 11

Links with APOS Theory

The model of understanding function	APOS theory
Variable Perspective Dependent Relationship	Action
Connecting Representations Property Noticing	Process
Object Analysis	Object
Inventising	Schema

In conclusion, from a theoretical perspective, the general model of understanding fits with or was reified two understanding models: Pirie and Kieren's model, and APOS theory.

3.4.3 Link with application

Whereas England's curricula require all the students to 'modify' or 'translate' the real life situation towards an algebraic explanation, the Shanghai curriculum simply emphasises applying pure knowledge into real life which will be introduced in Chapter 5: Curriculum Analysis. The main concern of application for both was how to connect the mathematics with real life. O'Callaghan (1998) proposed that this process consists of four competencies: modeling, interpreting, translating, and reifying in case of the concept of function (see Table 12).

Table 12

Four Competencies in the Function Model

Competencies	Description	Ability	Associated procedural skills
Modeling	Transition from a problem situation to a mathematical representation	Represent problem using functions	Three core representations: equations, tables, and graphs
Interpreting	The reverse procedure of modeling	Interpretation of different representations of the problem	Make different types of interpretations or different aspects of a graph
Translating	Move from one representation to another	Translating representations	Translating among three core representations
Reifying	Creation of a mental object	Possess certain properties or other higher level processes, such as composition	Conceptualization of functions

The task environment consists of the structure of facts in the real world, the related

mathematical concepts, and their interrelationships between the structure and the concepts

(H. Simon, 1978). The first two competencies, modeling and interpreting, as opposing

processes, are applied to probe into the interrelationships between the structure and concepts.

Modeling tries to extract mathematics information from the context using mathematical

representation system, while interpreting involves using these representations, for example to

explain the meaning of real-life graphs. The latter two competencies are related with

understanding function. In terms of the Translating competency, related with understanding

Level 3 Connecting Representations, these representations are required to be connected. As a

result, students have the mental object for the problem as shown by the fourth competency,

reifying, related with higher level of understanding, Object Analysis. In the case of function,

Sfard (1991) demonstrated reification as the proficiency in solving equations, articulating

with general properties, and recognising that computability does not play a vital role in

functions. In summary, the successful application also requires the higher levels of understanding function in the model of understanding function.

3.5 Summary

The literature discussed in this chapter has provided two in-depth perspectives of the research questions: the concept of function, including linear function; and the development of understanding. In addition, a general model of understanding function has been put forward as the theoretical framework for this study. This model aligns with seven models proposed by Western educators and slightly differs with two Shanghai models. The next chapter will now present the research design employed in this study.

Chapter 4 Methodology

This chapter sets out the chosen methodology of this research, consisting of six main sections. It starts with the theoretical background for the methodology, taking into account two extreme methodological perspectives. A discussion of their different ontological and epistemological assumptions is followed by a justification of realism as the philosophical foundation for this study, as well as views on comparative education. The second part describes the proposed analytical framework of mixed-methods as these methods are applied to the two phases of this study: the pilot (Phase 1), and the main study itself (Phase 2). Thirdly, the chosen methods for the four aspects of the comparative research will be discussed, as followed by a brief discussion of the data collection process. Issues of validity and ethics are then explored before ending with an exploration of the limitations of the chosen approaches.

4.1 Methodological Perspective

A methodology shows 'how research questions are articulated' (Clough & Nutbrown, 2012, p. 25). In other words, it clarifies the research decisions. Cohen et al. (2011, p. 3) highlight the principle of methodology as 'fitness for purpose' towards real world enquiry. This section illuminates the methodological consideration of realism and the corresponding standpoint on this comparative study.

4.1.1 Objectivism, subjectivism and critical realism

A decision concerning methodology entails a consideration of one's own view of social reality and this requires a discussion of two important concepts: ontology and epistemology. There are two opposite views concerning the nature of social science, both of which adopt very different ontological and epistemological assumptions: the traditional view (objectivist) and the interpretive view (subjectivist). The former regards social sciences and

natural sciences as having the same ability to 'discover the universal laws' (Robson, 2002, p. 5), while the latter focuses on the description and explanation of people's different behaviours rather than objects.

The objectivist approach holds a static view (also called standard view) of social science. Knowledge is assumed to exist outside of the knower as an independent existence, which is 'hard, objective and tangible' (Cohen et al., 2011, p. 6). It separates facts from values, and is known as positivism (Robson, 2002). Therefore, this ontology follows from an absolutist epistemology which has four characteristics: cause-and-effect thinking; select variables to interrelate; observations and measures; and theory verification (Creswell & Clark, 2007). Therefore, in a similar way to natural science or physical science, the objectivist view focuses on what we can see to uncover social laws (Johnson & Onwuegbuzie, 2004). However, few studies in social science would take this extreme view as it implies that data is time-, context- and value-free. Therefore, post-positivism as an extended approach emerged in the middle part of the 20th century. Post-positivism admits the limitations of research, especially from the researcher's perspective, such as the values they hold may result in research biases. But post-positivism still insists on scientific reasoning. From this objectivist approach, quantitative methods are applied extensively, mainly emphasising deduction and confirmation. Experiments and surveys are the main methods of quantitative research (Bryman, 2003). For example, an experimental group and a control group are used to examine the impact of independent variables on dependent variables, in the form of quasi-experiments in the educational research area. Surveys or tests by random sample are able to generalise the results to the entire population. The validity and reliability of the test or survey could be confirmed and improved through piloting. However, a problematic issue with these methods is the possibility of unrepresentative or skewed sampling (Cohen et al., 2011).

Subjectivism. At the opposite extreme, subjectivism argues that reality is value-bound and socially constructed, because the context legalizes the claimed truth. That is, knowledge can only be based on individual experience, as empirical knowledge. There is no external reality; however, reality is what the individual makes sense of. The epistemology of subjectivism has become known as constructivism or interpretivism. Advocates of subjectivism insist that value-free generalizations are impossible and inadvisable (Johnson & Onwuegbuzie, 2004). Meaning is shaped by social interaction with others and from their own personal histories (Creswell & Clark, 2007). The ontology, called nominalism, suggests 'there is no truth and facts' and are 'all human creations' (Easterby-Smith, Thorpe, & Jackson, 2012, p. 19). This dynamic view of ontology, which leads to a fallibilist epistemology, sheds light on individual experience to explore or interpret the reality, rather than seeking external reasons and universal laws to explain the world. Within the fallibilist epistemology, illustration is more important than proof (Cohen et al., 2011). This leads to grounded theory that aims to understand how individuals create, modify and interpret their world. From this subjectivist approach, qualitative methods such as interviews could explain the reality from multiple perspectives by understanding a small group of people in-depth. The reliability and validity of the research 'rarely seems appropriate or relevant' to qualitative research (Kirk, 1986, p. 14).

Each of the two extreme views holds an absolute approach to the real world, either purely emphasising on the proven unique truth or merely focusing on subjective personalised judgement. That is, objectivism tends to accept existing facts in real-world situations but excludes the function of individuality as it continues to build up the facts and meaning of this world. By contrast, subjectivism focuses on meaning or subject consciousness of the world but neglects the power of external structures in society. These two styles explore and understand social reality with two opposing lenses (Cohen et al., 2011).

Critical realism. Both objectivism and subjectivism have advantages and disadvantages. However, a combination of the two reflects the view of social reality that is taken in this study, namely a realist view. Between these two poles, critical realism adopts a middle ground (see Figure 18), as embracing both the existing knowledge and the importance of the context of justification. It represents the theoretical foundation of this research.

Figure 18. Critical realism as a middle ground between objectivist and subjectivist views

Specifically within the area of education research, critical realism has been proposed as the pragmatic approach. Pragmatists advocate that truth is 'what works' (Robson, 2002, p. 43) and this provides the best workable solution as using both qualitative and quantitative research (Johnson & Onwuegbuzie, 2004). It leads to and is consistent with a mixed-method approach for breadth and depth of understanding of social research. Mixed methods, which encourage multiple views of the world, effectively address the complexity of social research problems (Creswell & Clark, 2007).

My choice of research methodology, critical realism, is widely reflected by my view of mathematical concepts in schools. As I have previously discussed, although the concept of function has been socially developed within learning in schools, the absolute knowledge exists in the curricula, relevant textbooks and other teaching materials. As the learning object, knowledge is regarded as a static reality. That is, the presented information or resource that is required to be obtained by students. On the one hand, this study is going to discover the 'objective' differences in knowledge levels of the national curricula, the textbooks, and student performance. Within this, however, the different requirements of knowledge of the

curriculum in each country, to a large extent, show the divergent values of certain knowledge. Though these values have been influenced by a strong cultural and historical background, the context is also related to the process of teaching and learning, and views of the teachers. Therefore, a possible compromise, namely the pragmatic approach of realism, would be suitable in order to comprehend both what the differences are and the mechanisms behind performance, namely how these differences occur.

The chosen methodology also fits with my view of comparative education, laying stress on descriptions and explanations of the quantitative data as it will be discussed in the next section. This view is also built on the fundamental importance of comparison to understand other societies (Stenhouse, 1979).

4.1.2 Views of comparative education

Comparative education has its origins in the first half of the last century. This area reflected the need of global economic development in dealing with mutual issues that all societies and school systems could encounter (Robin, 2001). The development of comparative education research contains three dimensions, and the present study chose the global view. Following the realism methodology and standpoint of comparative education, the next section will analyse how to balance quantitative and qualitative approaches to suit the purpose of this comparative study.

Three dimensions. Arnove (2003) proposed three dimensions of comparative education: scientific, pragmatic, and global. Within the scientific dimension, researchers verified a positive relationship between 'education system and notional productivity' (Arnove, 2003, p. 4). As for the pragmatic dimension, it considered a better way to borrow some favourable elements from others and lend our own in the purpose of ameliorating or improving domestic policy and practice. Therefore, comparative education was able to have a corrective and supportive function (Robin, 2001). Japan and the United State were the first

successful examples of learning from others, although David Phillips (1989, p. 273), quoting William Cummings, commented that Japanese successes 'come at too high a price, a price Americans are unwilling to pay'.

The global labour market needs well-educated workers who have enough scientific knowledge, and the education system of each country tries to respond to this demand (S. Han & Jarvis, 2013). It has been increasingly the case that education is directed toward globalization (Tsui & Treagust, 2013). From the global dimension, comparative studies are based on a fairly large number of mainly quantitative tests (including pre- and post-tests) as well as qualitative and ethnographic investigations. Students' performance, their attitudes towards mathematics, opportunity to learn and so on are interwoven in PISA assessment. In addition, the IEA and the International Assessment of Educational Progress (IAEP) also look for the ideal pedagogies which include the most worthwhile teaching methods and best classroom practices.

Comparing pedagogy between different countries has gradually become the mainstream in the field of comparative education (Alexander, Broadfoot, & Phillips, 1999). At the same time, the interests of studies have moved toward not only the causal relationship between pedagogy and children's understanding, but also toward the process of how pedagogy promotes or fails to promote students' understanding (Robin, 2001). Therefore, the causal relationship might be assessed by the quantitative approach, while the qualitative approach enables an explanation of the process between the two variables. From this point, the mixed-method used by a global dimension of comparative research fits with the present methodological choice.

The balance of two approaches. Traditional views of comparative education, based on the macroscopic view (based on the state as unit) and the microscopic view (based on the school as unit) are inseparable and interacted (Dale, 2005). K. Watson (2001) criticised the

de-contextualized data and stated that statistics could not reflect the underlying educational philosophy of each country. Likewise, outcomes should not be isolated from the processes that have produced them (David Phillips, 1989).

The large volume of data available raises a further question: how can this quantitative data be explained, as Novoa and Yariv-Mashal (2003) argue that comparative education should be illustrative in purpose. Although international assessments, such as TIMSS, have included some observations in classrooms, Novoa and Yariv-Mashal (2003) suggest that observation and description have priority in this area of research rather than focusing on the results of students' performance. Consequently, explaining and describing have become extensively used for research in this area. Notwithstanding, Noah (1984, p. 562) asserts that the role of comparative education is to cultivate a 'rich understanding and knowledge of the other societies'. For example, researches about teacher beliefs (Cai & Wang, 2010), as well as teacher's content knowledge (L. Ma, 1999), have become another stream merely using qualitative methods such as interviews.

Ultimately, the function of qualitative data should be carefully balanced. Following this view of comparative education, this study will mainly use a quantitative approach to explore the reality of what students know, while the qualitative approach is mainly to provide some explanations for what the quantitative data is suggesting. The reason is that observation and description are necessary but insufficient to replace the strengths of quantitative results. The methods this study adopts will be explored further in the next section.

4.2 Overview

This section starts with how to combine the quantitative approach using document analysis and student tests, and the qualitative approach by interviews to answer the research questions. Before I justify my research design for each individual method, a literature review of how other small-scale comparative studies have been conducted and the main features of

their methodology will indicate the decisions for chosen methods. Finally, these methods will be presented chronologically.

4.2.1 The mixed-methods

Two approaches in this study. Both quantitative and qualitative approaches are involved in the study as follows:

Quantitative approach 1. Content and document analysis was used in order to delineate the official materials in both regions in respect to the basic knowledge contained and the expectations of how to apply this knowledge to real world situations. Here, the study considered national curricula at a regional level and textbooks used at the sample schools level as the official documents. The analysis of documents could be undertaken in both quantitative and qualitative ways (May, 2011). It is worth noting that Bryman (2004, p. 291) identifies content analysis as unable to answer any 'why' questions. This study, however, took the quantitative form. Instead, it investigates 'what', thereby fitting with this thesis' own tendencies to examine the 'what' over the 'why'.

The generated theoretical categories of the understanding model identified the requirements from the curricula, and the examples and exercises within textbooks at certain levels of understanding. In addition, the ways in which textbooks present the application questions and solutions was analysed from the perspective of the three main representations for linear function. Thus, each category was transformed into numbers in order for comparisons to be made between different regions.

Quantitative approach 2. Tests were undertaken in order to identify how well students understand linear function in each region using self-designed instruments. Student understanding was measured by the model of understanding function that was comprised from existing research and described in Chapter 3. Here, construct validity, which evaluates 'how well the measure conforms with theoretical expectations' (De Vaus, 2014, p. 54), needs

to be a focus. Therefore, a pilot study was conducted first. The self-designed tests at the pilot stage were based on the general model. Findings from the pilot tests would not only identify issues of validity, but also indicate necessary changes for the main research (Matthews & Ross, 2010). Later, the specific barriers to understanding basic knowledge and favoured or inferior solutions shown in application for each cohort would also be examined from the modified paper-pencil tests in the main study.

Qualitative approach. Teacher interviews were used in order to answer research questions about the contextual factors of teaching and learning processes. The results would reveal teachers' underlying beliefs towards the implemented curriculum, namely the lesson plan based on their perception of barriers to student's understanding. Harding (2013) described three frequently used interview types: biographical interviews; semi-structured interviews; and unstructured interviews. According to Bryman (2004, p. 439), if there is a 'fairly clear focus', then semi-structured interviews are best suited to the task. Thus, semi-structured interviews were chosen in this study. It was expected that the two groups of teachers would express their beliefs and understanding in their own way, using their own words as fully and as spontaneously as possible. Each interviewee was provided with an outline of the interview before conducting the data collection. The content of the interview focused on how teachers understand the official documents and perceive students' learning in order to gain a better understanding of the implemented curriculum.

Mixed-methods used in other small-scale comparative studies. There are three distinctive features involved in smaller scale comparative studies: (1) the tendency to employ several different questionnaires/tests for diversified mathematics rather than one assessment for the whole; (2) the importance of a pilot study; and (3) the trends towards systematic explanation as a continuum rather than as discrete categories.

Instead of one test to imply the whole of mathematical learning, many studies have made efforts to develop more targeted questionnaires that assess different types of mathematical performance, for example, comparing the reality of students' problem-solving and problem-posing between China and USA (Cai & Hwang, 2002); problem-solving for word problems between China and Singapore (Jiang & Chua, 2010); and problem-solving behaviour between Japanese and USA students (Becker, Sawada, & Shimizu, 1999).

Furthermore, more studies advocate a pilot study to form these assessments and qualitative methods in order to explain the results from tests. Harpen and Sriraman (2013) explored high school students' mathematical creativity and problem-solving and problem-posing in China and USA. Both the mathematical content test and the problem-posing test were conducted after several pilot phases. Meanwhile, follow-up interviews with students aimed to achieve explanations of performance within these tests.

Another series of studies was also related to Chinese and USA students in respect to their problem-solving performance. For example, Cai (1995) examined how students reacted to three types of problems: computation problem, simple problem, and complex problem. The assessment tasks were used after three pilot studies and were followed by interviews with their teachers. The purpose of the interviews was to find out a causal relationship between what students were taught and what they carried out on different types of problems. Therefore, the following interviews for either students or teachers played a supplementary role to the whole research. Meanwhile, the trend of current comparative education studies turned to investigate the holistic views of participants instead of one aspect so that more than one method of inquiry and more than one type of data were needed.

Furthermore, the multiple approaches used by those researchers also indicate that the findings from mixed methods could be verified with each other as measured by triangulation, which will be explained in a later section of this chapter: Validity of the study. Cai (2004)

investigated the relationship between students' problem-solving strategies and their use of representations. In order to figure out why students would have a particular tendency, interviews with teachers were used to discover their beliefs. Meanwhile, teachers' marking for each representation was collected to connect what teachers believe with what they actually did in a practical way. An et al. (2004) proposed a framework for secondary school teachers' pedagogical content knowledge which will be discussed at Chapter 8: Teacher Interviews. Mixed methods, such as questionnaires, interviews and observations, were applied for views of the Chinese and USA teachers in their study. The data from classroom observations and interviews was used to confirm the feedback from teachers' questionnaires.

Therefore, my chosen methods approach conforms to the main features of small-scale comparative studies. Two sets of student tests for understanding were devised and piloted: one to test pure knowledge and another to test application of that knowledge. To explore how students' understandings were shaped, curricula and textbooks provided explanations from a presenting mathematical knowledge perspective, while teachers offered their views to make students' understanding development clear. The procedure for the whole data collection will be discussed in the next section.

4.2.2 The procedure

Figure 19 demonstrates the framework of this research design as a pilot study at Phase 1 and main study at Phase 2.

```
                  ┌─────────────────────────────────────┐  ┌─────────────────────┐
                  │ Basic knowledge understanding test  │  │  Application test    │
                  └─────────────────────────────────────┘  └─────────────────────┘

         ┌──────────────────┐ ┌──────────────────┐ ┌──────────────────┐ ┌──────────────────┐
         │ England students │ │ Shanghai students│ │ England students │ │ Shanghai students│
         └──────────────────┘ └──────────────────┘ └──────────────────┘ └──────────────────┘
Phase 1
              ┌──────────────────────────────┐      ┌──────────────────────────┐
              │ Different understanding levels│      │   Different solutions     │
              └──────────────────────────────┘      └──────────────────────────┘
```

Phase 1

Phase 2

Document analysis

Student tests

Teacher interviews

Interpretation for each area

1. Curriculum
2. Textbooks analysis

1. Basic knowledge understanding test
2. Application test

1. Teaching of linear function
2. The perceived barriers

Comparison between two areas

Figure 19. A framework of this research

Phase 1: pilot study (June 2013). The functions of this pilot study include: (1) to gain feedback on the validity of the students' tests; (2) to check the time taken to complete the tests; (3) to identify which questions, if any, were too easy or difficult; and (4) to test the proposed coding system for data analysis (Cohen, Manion, & Morrison, 2013). Although Converse (1986) suggested two pre-tests were necessary for development, evaluation, and polishing tests, the limited time and cost only allowed one trial in this study. Findings of a pilot study would indicate possible category changes to the content of tests and the procedures of administration. The feedback from the teachers helped to check the testing time with regard to administration and the appropriate complexity of questions in the tests.

In the first two weeks of June 2013, pilot tests in Shanghai were completed. In the final two weeks of June 2013, pilot tests in England were implemented. Two groups of students were both given the tests near the end of the academic year.

Phase 2: main study (February 2014 - May 2014). After analysing the data from the pilot study, the schedule for data collection in Phase 2 was set up. In November 2014, I convened three brief meetings with the Heads of Maths at each sample school in England to explain the purposes of the Phase 2 data collection in order to gain their understanding and support. For the Shanghai context, Head Teachers at three sample schools cooperated fully in this phase of the study via telephone meetings.

Besides gaining support from mathematics departments, another purpose of the meetings was to discuss three aspects of conducting the research in school level: the arrangement of instruments, interviews with the Heads of Maths, and the details of the proposed classroom observations. During the meetings, I also gave the outline of interviews to each Head of Maths in England and emailed the translated Chinese versions to the selected Shanghai teachers. Before the end of January 2014, the Heads of Maths at the sample schools in both regions gave me a confirmation of an interview date and a time for the students' tests (at a time linear function/graph would be being taught). For the school visits, each sample school agreed to several classroom observations involving linear function/graph. In England, up to four classes were observed at each sample school for this topic. In addition, the English schools also offered other classroom observations involving other topics in different year groups. In Shanghai, one school offered all nine required and continuous Maths classes for this topic. Due to very similar teaching schedules, I managed to observe three classes at another school, but did not have time to observe more at the third school.

School visits were arranged in order to carry out participant observation. Participant observation aims to achieve 'intimate knowledge' of people (Matthews & Ross, 2010, p.

257). In this study, what happened in the classroom played a comprehensive role in the teaching and learning process at each region. Classroom observations provided further contextual information for the study. Meanwhile, teacher interviews following the classroom observation started with a concrete question about the lesson plan. Visiting was also beneficial in building positive relationships between the teachers and I.

During classroom observations, I acted as participant when students had their own tasks or group activities. During times of teacher input, I remained a complete observer. Expanded notes of the field experiences were made after each classroom observation. The function of these notes was to help me to inspect the different ways in which students would encounter difficulties during their classroom experience, and to be familiar with the teaching process especially in England. Three examples, one from an English classroom and another two from Shanghai classrooms, were used in the following Result part to make teacher's views clear with more details or to reveal relevant facts.

4.3 Methods

A method refers to an individual technique employed for data collection, while methodology refers to a set of methods used to inquire into the research questions (Hitchcock & Hughes, 1995) which are often underpinned by certain ontological and epistemological assumptions. Since quantitative and qualitative methods are 'compatible' (Tashakkori & Teddlie, 1998, p. 12), the second phase of design is to 'obtain different but complementary data on the same topic' (Morse, 1991, p. 122). Findings from qualitative and quantitative data could be combined to 'produce a general picture' of the reality (Bryman, 2003, p. 137). Through this triangulation process this research aims to illuminate the interrelationships between the development of students' understanding and the wider education system. It attempts to do so via interrogation of four sources of information. First, documentary research of the official national curricula would examine the findings from the intended

curriculum. Clarke (2003, p. 156) argued that the similarities and differences in mathematics curricula would be a 'signature of international comparative' research. Secondly, document analysis of the textbooks would provide how textbook writers explain the requirements of curricula/syllabi and their hypotheses about students' learning and thinking, referred to as the 'hypothetical learning trajectory' (M. Simon, 1995, p. 133). Thirdly, students' tests would describe the outcomes of the attained curricula. Finally, teacher interviews would provide insights into the implemented curricula. Table 13 shows the samples in terms of the four perspectives used in the whole study.

Table 13

The Outlining of the Samples

Perspective		England	Shanghai
Curriculum analysis		Mathematics programmes of study: Key Stages 3 and Key Stage 4 National Curriculum in England	The Shanghai City Primary and Secondary Mathematics Curriculum Standard
Textbook analysis		Collins New GCSE Maths for Edexcel Modular: Foundation 1 and Higher 1 Collins GCSE Maths 2 tier-foundation and tier-higher for AQA A; Foundation and Higher GCSE Mathematics: Revision and Practice;	Shanghai nine-year compulsory education textbook: Mathematics Grade 8 (Volume 2).
Student test	Pilot study	96	292
	Main study	561	907
Teacher interviews		3 Heads of Maths	3 Heads of Maths and 1 research teacher

The detailed method will be addressed throughout this section. Each sub-section is broadly divided into three parts: the brief research design; the selection of the sample; and the data collection process and analysis.

4.3.1 Curriculum analysis

The research design. Recent studies have examined existing literature or official documents through content analysis. It has two trends, comparing the reformed curriculum

with previous curricula in the same country; or with other countries' curricula as the features of national curricula between different countries are compared (Bao, 2002; Cai et al., 2011). This approach to curriculum analysis which this research adopts only traced current official documents, with introduced little for the historical background.

Selection of the sample. Within this study, content analysis is conducted on national curricula documents from England and the local one for Shanghai. It aims to delineate the requirements of intended curricula in the case of linear function in both regions. Specifically the latest released curricula in England will be analysed. From 1th September 2013, the national curriculum programmes of mathematics at Key Stages 3 and 4 have no longer applied. The new national curriculum will be applied from September 2014. During the academic year of 2013, 'schools are free to develop their own curriculums for mathematics that best meet the needs of their pupils, in preparation for the introduction of the new national curriculum' (Department for Education, 2013a). However, the draft programme of study for KS2 to KS4 was provided for the 2013 academic year.

In September 2013 the new mathematics curriculum in England, *Mathematics programmes of study: Key Stages 3 National Curriculum in England* (Department for Education, 2013c) was released. Later, in July 2014, the *Mathematics programmes of study: Key Stages 4 National Curriculum in England* (Department for Education, 2014) were launched. As to content of the new curricula, however, there were minor improvements made to earlier draft programmes (Steers, 2014). Therefore, this new KS3 national curriculum instead of the draft was chosen. In terms of linear function, the requirements in the draft and the formal curriculum were similar as well. Thus, in line with KS3, this new formal KS4 curriculum was chosen as well.

The Shanghai City Primary and Secondary Mathematics Curriculum Standard (Shanghai City Education Committee, 2004) was chosen for this study. In 1997, Shanghai

was allowed to have its own curriculum instead of following the national curriculum in China (See Chapter 2 for a fuller discussion of this). Four years later, Shanghai implemented the second edition of the mathematics curriculum. In 2004, Shanghai modified it again and called it the new mathematics curriculum which is still in use.

Data collection. The statutory guidance including the statutory programmes of study and attainment targets for mathematics at Key Stages 1 to 4, can be accessed from UK government website (see https://www.gov.uk/government/publications/national-curriculum-in-england-mathematics-programmes-of-study). As a former secondary school teacher in Shanghai, the Chinese curriculum was provided by Maths departments in schools. The analysis for the data will be addressed in Chapter 5.

4.3.2 Textbook analysis

The research design. Due to the commercial textbook market in England, the textbook research is expected to examine not only the books' characteristics but also how to use them (Fan, 2013). The textbooks analysis focuses on the features, while its function for lesson plans will be illuminated in Chapter 8: Teacher Interviews.

Selection of the sample. All the selected textbooks were officially used by the sample schools in England and Shanghai. Due to limited time and resources, a convenience sample of chosen schools was selected for this research. The limitation of the convenience sample would be that it would 'not represent any group apart from itself' (Cohen et al., 2013, p. 164). As discussed in later sections regarding the limitations of this study, I will claim that the findings do not intend to be generalised into a theory.

In England, there exists a variety of mathematics textbooks used for classroom teaching and learning. The English textbook series were developed for two different ability levels: Foundation level and Higher level, in line with the 'additional mathematical content' which is expected to 'be taught to more highly attaining pupils' (Department for Education,

2013c, p. 3; 2014, p. 3). Both followed the same national curriculum in England. Each school, however, has a considerable degree of autonomy to choose the appropriate series of textbooks for their students.

Three secondary schools located in the North East of England volunteered to take part in this study. According to the national league table for mathematics based on the qualification results at the end of secondary schooling in 2012, students' performance in these three schools were within the top 30% of all secondary state schools in England which will be explained thoroughly at next part. Each sample school, however, had a different series of school mathematics textbooks. Therefore, all of the textbooks used in these schools were included in this study. Thus, although the selected textbooks may provide a typical pattern for presenting knowledge in England, I acknowledge that they are a convenience sample and not representative.

In contrast, the choice of textbook is not flexible in Shanghai. The uniform textbooks were developed based on *The Shanghai City Primary and Secondary Mathematics Curriculum Standard* (Shanghai City Education Committee, 2004) instead of the national curriculum in China, but it also remains a centralised education system in Shanghai. Therefore, mandatory textbooks are widely used by all Shanghai students at state schools as well as at public schools during the compulsory education stage (from age 7 to 15). Each term in the school year has one separate mathematics textbook. Linear function is introduced in the second term of Grade 8 (approx. age 14) and, therefore, the one appropriate Shanghai textbook was selected in the present study.

Therefore, the following seven textbooks containing linear function were examined in this study:

England:

1. Collins New GCSE Maths for Edexcel Modular (Foundation 1), published by Collins, 2010;

2. Collins New GCSE Maths for Edexcel Modular (Higher 1), published by Collins, 2010;

3. Collins GCSE Maths 2 tier-foundation for AQA A, published by Collins, 2006;

4. Collins GCSE Maths 2 tier-higher for AQA A, published by Collins, 2006;

5. Foundation GCSE Mathematics: Revision and Practice, published by Oxford University Press, 2006;

6. Higher GCSE Mathematics: Revision and Practice, published by Oxford University Press, 2006;

   Shanghai:

7. Shanghai nine-year compulsory education textbook: Mathematics Grade 8 (Volume 2), 2007.

Data collection. English textbooks were collected at the meeting with the sample schools in November 2013, and I have already possessed the compulsory Shanghai textbooks from the junior secondary school stage. Again, details concerning the analysis of the content of linear function/graph will be set out in Chapter 6.

4.3.3 Student tests

The research design. Students' performance will be analysed using paper-and-pencil tests, including the results from both the pilot study and the main study. This study focuses on how well students understand a certain topic and what supports or constrains the development of this understanding. As both the Year 10 English students and Grade 8 Shanghai students are still teenagers, it might be hard for them to evaluate their own understanding, or to reflect with their barriers to that understanding, or to articulate the development of their

comprehension of a certain concept. Therefore, student interviews were not included in this study.

Selection of the sample. In this research, North-East England and Shanghai were selected as two regions to be investigated for students understanding (see Chapter 1 for justification). Linear function/graph is covered in Grade 8 (approx. age 14) in Shanghai, and in Year 8, 9, and 10 in England. The detailed arrangements of topic within the curricula will be illuminated in Chapter 5: Curriculum Analysis. Year 10 students in England (approx. age 15) were selected for this research as they are supposed to have covered the knowledge for linear function required by the KS4 national curriculum. I acknowledge, however, that there was an avoidable one-year difference between the two samples which will be discussed at later section: Limitations.

English participants came from three state schools as discussed in the methods section for the textbooks analysis. In the three sample schools, all Higher Level and the majority of Foundation Level students in Year 10 took part in the study. The students in the lowest set of Foundation Level students, however, were not selected for this study on the recommendation of the Heads of Maths.

After the English sample was chosen, the Shanghai sample was selected correspondingly, i.e. with schools at a similar percentage of school performance in mathematics. The Shanghai sample was drawn from the Pudong District which is the biggest district with about one-fifth of total Shanghai students. Shanghai has no uniform league table for the senior secondary School Entrance Examinations (*Zhongkao*) whose function is similar to GCSEs in England as discussed in Chapter 2. However, students' academic performance in mock examinations could be regarded as providing a similar ranking of schools, especially in the second mock exam whose timing is roughly two months before *Zhongkao*. Therefore, according to the district league table for mathematics based on the second mock examination

in 2012, three schools that ranked at around the top 30% among all state schools in the Pudong District were selected for this research. All Grade 8 registered students in the three sample schools participated in this research. Due to mixed-ability classes in the Shanghai education system, the lower ability students were also included in the study.

Data collection. In the main study, students' tests were conducted once students had finished learning the topic. The tests were administered by the students' regular classroom mathematics teachers in both regions.

In Shanghai, two schools participated in the pilot study. The feedback from teachers suggested that the two tests should be combined into one test so that it would be more convenient for students to answer. I adopted this idea and the time given to students was increased to one hour to complete this combined test. Due to the uniform teaching schedule in Shanghai, two Shanghai schools finished teaching this topic on Friday 21st February 2014 and another on the following Monday, 24th February 2014. After that, all three schools immediately arranged an agreed time for the whole Grade 8 students to complete the test.

When back in England, I presented the Heads of Maths with the combined test and they were all happy with the layout. Two of the three sample schools also took part in the pilot study so were familiar with the administration procedure. I had a brief meeting with the Head of Maths at the third school. Due to the different teaching schedule in England, one school finished teaching this topic at the beginning of April 2014, one at the end of April 2014, and one in the middle of May 2014 respectively. Once they finished the topic, each respective Year 10 mathematics teacher then immediately arranged the tests.

4.3.4 Teacher interviews

Selection of the sample. In terms of teacher selection, the Head of Maths in each sample school was asked to participate in an interview. These teachers are well recognised

within their respective schools. Moreover, their beliefs often represent accepted values from the sample schools with regards the process of the teaching and learning of mathematics.

In general, the Head of Maths in English schools is in charge of the Scheme of Work which every mathematics teacher within the whole department will follow. The Scheme of Work prescribes the arrangement of topics in each Key Stage, for example how long a certain topic should be taught and the different requirements for corresponding levels of students, reported by the selected English teachers at their interviews. The Head of Maths in Shanghai normally pays more attention to difficulties in the teaching and learning in each grade, which is discussed at every departmental meeting usually scheduled every two weeks. Therefore, the views either of English or Shanghai Heads of Maths will be more informed than other mathematics teachers.

In addition, Shanghai has a unique researcher-teacher system that is also in operation in other parts of China. One researcher-teacher in the Pudong District was also interviewed. The researcher-teachers in Shanghai (normally they were in-service maths teachers before this role) are mainly responsible for the teaching and learning of mathematics at the district level. The research topic of linear function was arranged in Grade 8, and therefore, the Grade 8 researcher-teacher participated in the interview. His views about teaching and learning represent another level of expert perspective. However, in Chapter 8, Teacher Interviews, I did not distinguish the views from the expert and Heads of Maths, because the purpose of the interview was not to discern the differences between them. Both the Shanghai sample and English sample of teachers are specialists who only teach mathematics subject.

Data collection. The Head of Maths in each sample school was interviewed separately. Interview data was collected during the classroom observation period. Before the interview began, all the interviewees signed consent forms (see Appendix A) according to the requirements of the Ethics Committee at Durham University (discussed in more detail at

section: Ethics). Meanwhile, every interviewee was informed that I would send them the transcription of each interview for the purpose of verification. In Shanghai, linear function would be taught from 11th February 2014, so I returned to Shanghai on 7th Feb 2014. The teaching schedule of mathematics is approximately nine classes over two weeks. During these two weeks, four interviews in Shanghai were conducted in schools and all were audio-recorded. About two days after the interview, each teacher in England and Shanghai received the transcription but there were no significant modifications proposed by any interviewees. Before the student tests data collection in Shanghai, the interview transcriptions and confirmations had been received.

After the Shanghai trip, interviews at the three sample schools in England were carried out from the end of March to the beginning of April. The data collection procedure in English sample schools was the same as that in Shanghai.

4.3.5 The equivalence of transcriptions

This research has involved two languages, Chinese and English. The issue of meaning equivalence in different languages and appropriateness in different cultures might lead to bias (May, 2011). Therefore, in terms of the two sets of instruments and teachers' interview outlines, it is essential to ensure transcription equivalence.

Students' tests were first formed in English. Both the pilot study and the main study tests were discussed with Heads of Maths in the sample schools to ensure English students were comfortable with the expression of these questions. The collaboration between two regions' Head of Maths to develop tests could conform to the values of diverse cultures in this study.

Later, I translated the English version into Chinese and then checked with the researcher-teacher and Heads of Maths in the Shanghai sample schools in order to guarantee that the mathematical terms would fit with the expressions with which students were familiar.

With regard to the teachers' interview questions, they were also initially composed in English. Later, a process of Chinese-back translation was carried out. That is, after translating the interview outline into Chinese, I sent the Chinese version to an English teacher working in a Shanghai secondary school. She translated this Chinese version back to English. Then we compared the two English versions, before and after translation, in order to ensure more accurate expression.

A similar translation process was carried out with the Shanghai teachers' interview. The translation of Shanghai teachers' interview transcription principally used key words in Chinese in order to maintain equivalence, as Lawrence (1988, p. 102) commented that the interviewer should understand that 'his job is to end up with an enlarged understanding, not to teach other people English'.

4.4 Ethics

As mentioned above, this comparative study has required access to schools in each area. In England, I have visited schools based in County Durham, North East of England, since November 2012, which has provided the foundation for this comparative study.

At the beginning of the research, I sent an email to the Head Teacher or the Head of Maths in order to explain the purposes of this research in both regions. Having confirmed their interest in this research, I had a brief meeting with the Heads of Maths at each possible school. Data collection with student tests, teacher interviews, and classroom observations involved possible ethical issues.

First, data would be collected using two student tests, which would last approximately sixty minutes in total, and be taken twice due to the pilot and main study stages. Both tests would include questions that students were expected to experience in their usual day-to-day learning in their maths classes. As a result of discussions with the Heads of Maths, it was deemed unnecessary to acquire informed consent from students' parents or guardians, as the

tests would be completely anonymous. Neither student nor school identities would be revealed. Furthermore, all data would be held under the rules of the Data Protection Act (1998) and the British Educational Research Association's Revised Ethical Guidelines for Educational Research (2004).

Secondly, the teacher interview would examine two aspects: the barriers faced by students with regards the research topic; and general information about how teachers plan lessons. Neither aspect would be related to highly personal or sensitive information. Informed consent was requested from six Heads of Maths in all sample schools and one researcher-teacher in Shanghai before the interviews began.

Furthermore, all selected sample schools agreed to the classroom observations during maths lessons on linear function. Two English schools provided classroom observations for linear function, mainly in Year 10, and other topics, as well as different year groups, specifically Year 7, Year 8, and Year 11. I have obtained an Enhanced Criminal Record Certificate in the UK since 2013, and have had teaching certification for secondary schools in China. These allowed me to work with students in the classroom if necessary.

The Ethics Committee in the School of Education at Durham University approved this study before my data collection.

4.5 Validity of the study

Validity is defined as 'the extent to which measures and research findings provide accurate representation of the things they are supposed to be describing' (Easterby-Smith et al., 2012, p. 347). All researches are striving for the maximum degree of validity. There are many types of validity. Among them, internal validity, content validity, and cultural validity were involved within this study.

Internal validity considers the accuracy of explanation sustained by the data. This research, from the realist methodological perspective, uses triangulation to map out

secondary school students' understanding to enhance the internal validity of the research. Methodological triangulation means two or more methods used in the data collection of one study (Cohen et al., 2011). It could help to overcome 'method-boundedness' (Cohen et al., 2011, p. 142), where method-boundedness means if there was only one particular method used, it might not be certain that the method chosen was as a result of a researcher's habit, or whether the researcher considered this method as superior to others methods. If the outcome of different methods corresponds to each other, the findings could be more confidently assumed to be valid by researchers.

Content validity demonstrates that the instrument should cover the domain of research. In this research, each requirement of curricula had been located at a certain understanding level in the model of understanding function. Each understanding level that curricula and textbooks required had been covered by the student tests. Each item tests different aspects of basic knowledge. Furthermore, Heads of Maths at sample schools examined the tests to approve the content in line with the research topic, linear function. Meanwhile, interview questions were aimed at students' learning of linear function which related to teaching and teachers' views about their pupils' understanding barriers. Further, the reliability and validity of the tests I designed have been checked to verify what teachers view.

Cultural validity is concerned with cross-cultural comparative research in order to ensure that research is fair and sensitive within different cultural contexts. There are two issues within this study: the theoretical framework to assess understanding and the researcher, me. This thesis investigates students' understanding based on the model of understanding function which was developed from current models proposed by Western and Chinese educators. In a previous chapter, the model has been shown to fit well with five Western models and cover two Eastern models in junior secondary school stage. It is unclear whether

this model could be applied for assessing both regions' students. Therefore, a pilot study was necessary to verify its applicability.

Additionally, my personal perspective has been shaped by Eastern culture rather than that of Western cultures. It might, therefore, influence the objectivity of the findings, especially from England. To minimize the threat of the researcher impinging on the main research, I always keep in mind that the analysis of the data is my priority without making any prior assumptions. This will be discussed in the limitation section below.

4.6 Limitations

Although the methodology chosen has fitted with the purposes of this study, there have been four limitations: (1) no generalization is possible either from the sample or from linear function to any other mathematical topic; (2) the possible threat from the one-year gap between English and Shanghai students; (3) the possible bias from me, the researcher; and (4) the possible limited scope due to the time for this study.

First, although this research has been conducted using a considerable sample of students' tests and sample schools with similar backgrounds, this study is not representative of students' performance within the top 30% of schools in both countries. Due to the small-scale nature of the study, this research does not generalise the findings for all schools in the top 30% and their students. However, the intention of this research is to show possible examples of students' understanding of linear function and provide a better understanding of how to handle this topic in each country.

In addition, this research focuses on specific content- linear function. The findings regarding the advantages or disadvantages of the ways in which each region possibly handles the teaching of the topic cannot be applied to other mathematical content. Furthermore, working within one content area cannot infer students' understanding for other concepts or the whole mathematics subject.

Secondly, there is a one-year age gap between the England and Shanghai student samples which may slightly distort the comparability of the data from the students. Due to the spiral structure of the English curriculum, students would not learn all the basic knowledge that the curriculum requires until Year 10 (age 15). The data needs to reflect students' understanding after they have mastered all the basic knowledge so that the comparison can be considered much fairer than focusing on the same age, i.e. age 14. Therefore, slightly different age groups of students were sampled to solve this potential issue but the one-year age difference must be recognised.

Thirdly, one primary problem in comparative research is the ability of researchers to well understand other cultural or social beliefs (May, 2011). My previous teaching experience in Shanghai may influence the objectivity of the model of understanding or the interpretation of this research. I have, however, been working in Shanghai for over a decade, so I have had a better understanding of Shanghai students and teachers than of their English counterparts. This might cause a diversity of depth and breadth of the conclusions when conducting the analysis of documents and interviews in both countries. In order to increase my experience in the English education system, I insisted on undertaking classroom observations even after the data collection finished. Meanwhile, other actions, such as talking about my findings with school teachers here, presenting my work at several international conferences, department PG seminar and college seminar, and lecturing undergraduates, also resulted in a better understanding of the English situation and getting a balance between my previous experience in Shanghai and current experience in England.

Finally, the current curricula used in each country and textbooks used in the sample schools fit with four criteria for the validity and reliability of the data: 'authenticity (unquestionable sources), credibility (free from error and evasion), representativeness (typicality) and meaning (clear and comprehensible)' (Scott, 1990, p. 6). Since each country

changes their curriculum at certain points in time, these altering factors will influence classroom practice, the content of textbooks, student learning outcomes, and so on. The English schools may change their textbooks in the future when new curricula are introduced from 2014 academic year. The changing of these official documents would be a limitation of the current research's implications. But in keeping with the realism view, it is acknowledged that knowledge is a social and historical product which can be specific to a particular time, culture or situation (Robson, 2002). This study depicted the particular year 2013 when English schools had to make their own preparations for new curricula.

4.7 Summary

This chapter has explained the methodology chosen for this research, including the justification of the views of a comparative study. It articulated both quantitative and qualitative approaches and the design of the two phases of data collection. Furthermore, details of the four individual methods – curriculum analysis, textbook analysis, student tests, and teacher interviews – were described. Finally, the issue of ethics, validity, and limitations from a methodological perspective were presented. Having explained the methodology, data collection, and data analysis, the next chapter will present and interpret the data.

Part Two: Results

Having described the research context, related literature, theoretical framework of understanding development for the concept of function, and research design in Part One of this thesis, Part Two will set out the findings from the four sources discussed above. Chapter 5 will present an analysis of the respective curricula. In doing so, the topic of linear function will be scrutinised in terms of the intended curricula of both regions. In addition, how each region views understanding in mathematics will also be compared. Chapter 6 will focus on the analysis of the respective textbooks in order to get further insight into the governing levels of understanding linear function in each region, distinct approach towards each level, and how this understanding is embodied in the application part. These two chapters provided the learning trajectories towards understanding linear function from official documents. Chapter 7 discusses the analysis of the student tests. Here the attained curriculum will be compared under the general model of understanding function first. It is hoped that the results will also illustrate the strengths and weaknesses each group of students have in terms of understanding pure knowledge and application. Finishing off the discussion of results, Chapter 8 presents the analysis of the teacher interviews. This focuses on how the selected teachers perceive the teaching and learning process of mathematics in general as well as linear function in particulars.

Chapter 5 Curriculum Analysis

This chapter aims to answer the first part of research question of the thesis in terms of the curriculum's perspective: 'What are the requirements of the intended curriculum and officially used textbooks of the two regions in terms of linear function?' Here, the intended curriculum consists of three official documents: two national curriculum documents from England setting out the requirements of Key Stage 3 (Department for Education, 2013c) and Key Stage 4 (Department for Education, 2014); and one curriculum document from Shanghai setting out its local curriculum (Shanghai City Education Committee, 2004). The curriculum document from Shanghai applies to the pre-university stage (from primary school to senior secondary school) but the analysis will mainly focus on the junior secondary school stages as mentioned in Chapter 4. The analysis assesses understanding in two ways: the general meaning of understanding mathematics, and a more focused and specific meaning of understanding linear function. With regards to the first, the analysis will be drawn from the background information from the general aims in the curriculum to show the bigger picture of how understanding is stated. A narrower perspective will be discussed with regards to the particulars of understanding linear function which provides a more detailed, specific examination of this concept.

The first section of the chapter will, therefore, provide the detailed analytical framework. The second one will present a comparison of results from the general aims. The third one will set out the outcomes of the more specific findings concerning understanding of linear function. The discussion in the fourth section will address the importance of understanding function to successfully solve the real world situation, ending with a summary for further exploration.

5.1 An Analytical Framework

A curriculum acts as a guide to the prescribed content of what pupils learn in school (Kirst & Walker, 1971). Analysis of different curricula plays a vital role in realising the differences of students' learning (Nie, Cai, & Moyer, 2009). Therefore, what the statutory curricula proposed in both regions will be compared in the case of general understanding of mathematics first. The results will examine similarities and differences of how each educational system conceives understanding mathematics. To do so, this first part of the analysis will explore how each curriculum defines understanding and then draw out some of the most distinguishing differences between the two regions. It is hoped that this general analysis will provide some insight into the views of structure of mathematical topics in each region, including the certain types of function that each intended curriculum covers.

Following on from the large-scale cross-national projects discussed earlier, Oates (2011) suggested that curricula should focus on the essential areas of knowledge in key subjects in order to improve schooling systems. This raises the question of what essential knowledge is, a query answered by Alexander (2012) who recommended an investigation into how high performing education systems do as discussed in Chapter 2. Such general analysis would look into the definition of essential knowledge, and will form the basis of the following section. The importance of the concept of function has been discussed in Chapter 1 and now, a comparison of the varying types of function between England and Shanghai's curricula will not only illustrate the background but also justify the chosen research topic, linear function.

This chapter's focus then turns to a detailed examination of the attainment targets in particular for understanding linear function. The content of linear function is split into two parts to investigate the requirements: pure mathematical knowledge, and the application of

that knowledge in real-life situations. The whole analytical framework is therefore put forward as follows:

A. From the background information:

1. Two distinctive features: the arrangement of subject knowledge including function; and the focus of applying the knowledge;

2. The expected way in which to develop mathematical understanding in the two regions.

B. From the particulars of understanding linear function:

1. Understanding levels in the requirements of basic knowledge by the model of understanding function;

2. The emphasis of applying knowledge to real world situations.

5.2 General Aims of the Curricula

The overview of aims in both regions' curricula demonstrates that both are interested in developing students' positive attitudes towards mathematics. The development of their attitudes and interests in the process of teaching every subject is valued in the Shanghai curriculum; while 'an appreciation of the beauty and power of mathematics, and a sense of enjoyment and curiosity about the subject' is stated in England's curricula (Department for Education, 2013c; 2014, p. 3).

As discussed in Chapter 2 regarding the issue of curriculum control, the Shanghai curriculum also elaborates some more specific issues, such as student assessments, teaching materials, and the schedule of classes, which verifies the centralised characteristic; while England's curricula do not include these, as Oates (2011) argues that some degree of curriculum control is necessary in England discussed in Chapter 2.

5.2.1 Two distinctive features

The arrangement of subject content. This sub-section starts with how the basic content in general is arranged in the respective curricula, and how it is then exemplified by the concept of function. One way of doing this, developed by TIMSS, was to use an analysis called *Topic Trace Mapping* (Foxman, 1999, p. 13), developed by Schmidt (1992). It aims to examine the depth and breadth of topic in the curriculum, including for how long (how many academic years) a certain topic is covered.

Repeated or non-repeated approach. The two English curricula (KS3 and KS4) arrange the subject content in a spiral pattern, while Shanghai shows a non-repeated approach as discussed in Chapter 2. Both the KS3 and KS4 curricula applying at the junior secondary school stage in England categorise mathematics into the same six parts of the subject content: (1) Number; (2) Algebra; (3) Ratio, proportion and rates of change; (4) Geometry and measures; (5) Probability; and (6) Statistics. Topics in KS3 are further explored and extended in KS4, although KS4 also introduces new topics. That is, there are large amounts of overlap of mathematics topics between the two Key Stages.

Similarly, the Shanghai curriculum is divided into two stages: Grade 6 to 7; and Grade 8 to 9. The four sections of the subject content, however, are slightly different in each stage (see Table 14). As mentioned in Chapter 2, the subject content is arranged in a more sequential and non-repetitive pattern in China's curriculum, which Shanghai follows. It means that the topics in each strand would be separated by two stages of learning. For example, in the case of equations in the strand of *Equation and algebraic*, Grade 6 to 7 contains three types of equation: linear equation, simultaneous linear equations such as,

$\begin{cases} x + y = 9 \\ 2x - y = 6 \end{cases}$ and ternary linear equations such as $\begin{cases} a + b = 5 \\ b + c = 6 \\ a + c = 7 \end{cases}$. Meanwhile, Grade 8 to 9 will

introduce another type of equation, the quadratic equation, excluding the previous types

shown in Grade 6 and 7. There is little overlap of topics between these two stages.

Table 14

Subject Content in Shanghai Curriculum

Grade	Strands				
6 to 7	Number and Operation	Equation and Algebra	Graph and Geometry		Data process and Probability; and Statistics
8 to 9				Function and Analysis	

 Function. In England, two types function are introduced at KS3: linear and quadratic.

As well as deepening these two types, KS4 also extends to simple cubic functions such as

$y = x^3 + x^2$, and 'the reciprocal function $y = \frac{1}{x}$ with $x \neq 0$' for all the students (Department

for Education, 2014, p. 8), making a total of four types of functions. Additionally, higher

attaining students are expected to learn 'the exponential function $y = k^x$ for positive values

of k, and the trigonometric functions (with arguments in degrees) $y = sinx$, $y = cosx$ and

$y = tanx$ for angles of any size' (Department for Education, 2014, p. 8).

 In Shanghai, the concept of function is introduced in the strand of Function and

Analysis (Grade 8 to 9) with three specific types: linear function $y = kx + b, (k \neq 0)$

including proportional function $y = kx, (k \neq 0)$; reciprocal function $y = \frac{k}{x}, (k \neq 0)$; and

quadratic function $y = ax^2 + bx + c, (a \neq 0)$ for all students.

 Besides the fact that the first type of function students would have to learn is linear

function, another reason that linear function was chosen for this study is because other types

of function are not suitable for a fair comparison due to the different depth of the

requirements in the curricula. The overlap of the types of function between the two regions

occurs for linear, quadratic, and reciprocal function for all the students in two regions. The

reciprocal function is taught as a standard form ($y = \frac{k}{x}$) in Shanghai but as a specific form

($y = \frac{1}{x}$) in England, and is therefore not considered an appropriate topic to draw an analogy

of learning approach. In terms of the quadratic function, the Shanghai curriculum introduces

more properties, such as symmetry, and heavily emphasises links with other knowledge of

both algebra and geometry. The KS4 statutory guidance requires students to 'identify and

interpret roots, intercepts and turning points of quadratic functions' (Department for

Education, 2014, p. 8) as merely introducing properties. Therefore, quadratic function is also

not suitable to measure or evaluate the similarity or dissimilarity between two regions.

The aims for applying the knowledge. In general, all curricula pay a great deal of

attention to applying knowledge. England prefers students to solve non-routine problems

where application in the real world is included, while Shanghai emphasises the significance

of real-world situations.

In England, an overall aim of the national curricula is to 'solve problems by applying

their mathematics to a variety of routine and non-routine problems' (Department for

Education, 2013c, p. 3; 2014, p. 3). Generally, in order to solve routine problems, a standard

step by step solutions is used (Harskamp & Suhre, 2007), while there is no straightforward

solution for non-routine problems (Elia, van den Heuvel-Panhuizen, & Kolovou, 2009). Non-

routine problems were initially regarded as novel or non-standard problems in the learning

process. This division of problems separates out what students are taught to do, and what

students could deal with after their learning.

In contrast, applying knowledge in Shanghai is set within meaningful contexts. Two

of the four criteria in choosing the content of teaching materials are to value these real world

situations (Shanghai City Education Committee, 2004, p. 93):

- The content should be closely associated with real life. The introduction of mathematics knowledge and its concept development should emphatically derive from real life, students' previous knowledge, or other subjects.

- Application should be enhanced as well as problem-solving, projects, and practical activities.

To sum up, England emphasises the importance of solving non-routine problems to show what students can do to apply the knowledge they have learned. Many studies have examined children's strategies in solving non-routine problems (Pantziara et al., 2009; Selter, 2009). For example, English (1996) found that the process of solving non-routine problems could help construct mathematical understanding. In her research, even if students lacked formal domain knowledge; they still could combine domain-general strategies and their existing informal models to generate a solution, as a sign of applying what they know. On the other hand, Shanghai merely focuses on the real world application problems as a way to show what students can do to use that knowledge.

The next sub-section will discuss the differences of developing mathematical knowledge stated in the curricula.

5.2.2 The definition of understanding

Aims for fundamental mathematics. Both the curricula prioritise the learning of the fundamentals of mathematics as their first aim. England focuses on conceptual understanding and knowledge growth through graphic representations. In Shanghai, fundamental mathematical concepts consist of three basics: (1) basic knowledge; (2) basic skills; and (3) basic ideas and methods (as simply called basic methods). The development of understanding or conceptual understanding builds on basic methods through algebraic expressions in Shanghai.

In England, the first overall aim of KS1 to KS4 is to 'become fluent in the fundamentals of mathematics, including through varied and frequent practice with increasingly complex problems over time, so that pupils develop conceptual understanding and the ability to recall and apply knowledge rapidly and accurately' (Department for Education, 2013c, p. 3). It essentially indicates the importance of conceptual understanding in mathematics. Understanding refers to the connection of mathematical knowledge to form a bigger and better connected network of concepts as discussed in Chapter 3. In addition, conceptual understanding refers to 'an integrated and functional grasp' of isolated mathematical ideas and methods (National Research Council, 2001, p. 118). It can be examined by how students use representations, the ability to provide different solutions, and being aware of the strengths and weaknesses of each representation.

Within the English curricula furthermore, the development of understanding mathematics is built on the use of graphic representations. For example, in terms of linear function, both KS3 and KS4 require students to 'find approximate solutions of simultaneous linear equations' by using the graphical representation of linear function (Department for Education, 2013c, p. 7; 2014, p. 8). That is, the graphical representations link the knowledge of linear equation to other knowledge, such as solving simultaneous linear equations. For example, pupils discover the solution of the simultaneous linear equations $\begin{cases} x + y = 10 \\ y = x \end{cases}$ from graphs. English students, therefore, have been provided graphic way to solving simultaneous linear equations. Basically, the England's curricula show that using visual representations (through the use of graphs) aids the understanding of mathematics knowledge.

English students' understanding links between functions and equations is enhanced through the graphical representation of different types of function and using these to solve equations.

Conversely, in the Shanghai case, the different types of equations are the basics of learning function developed through algebraic expressions. The first aim of mathematical study is to have the 'three basics' (Shanghai City Education Committee, 2004, p. 32). That is, learning mathematics includes three aspects: the concept; the skills involved in grasping that concept; and the idea and method linked with the concepts. Here, basic skills consist of calculation, plotting, reasoning; communication including speaking, listening and writing; and data handling including using calculators for the junior secondary school stages (Shanghai City Education Committee, 2004, p. 35). The development of mathematical knowledge means not only the connection between concepts such as linear equation ($ax + b = 0, a \neq 0$) and the linear function ($y = ax + b, a \neq 0$) through symbolic ways of representation: algebraic expressions, but also the same methods or strategies that were used in understanding mathematics during previous school years.

The meaning of the basic idea and methods (also called basic methods) will be addressed first and followed by an exemplified case to clarify it. Four main types of basic methods are stipulated by the Shanghai curriculum in the junior secondary school stage: the *Xiaoyuan* method, known as *Elimination method* in English textbooks, which could be used to solve simultaneous linear equations with two variables; the *Peifang* method, known as *Completing the square* in English textbooks; the *Huanyuan* method, which is to substitute and exchange the same value; and the *Daidingxishu* method, which is to find out the coefficients of the equations, for example the pure algebraic approach to gradient in linear function as shown in the next section.

Memory and understanding in the Shanghai curriculum. In Western countries, memory is less emphasised when it comes to developing understanding; however, in China, it can be intertwined with understanding (see Figure 20) (Marton, Dall'Alba, & Tse, 1996).
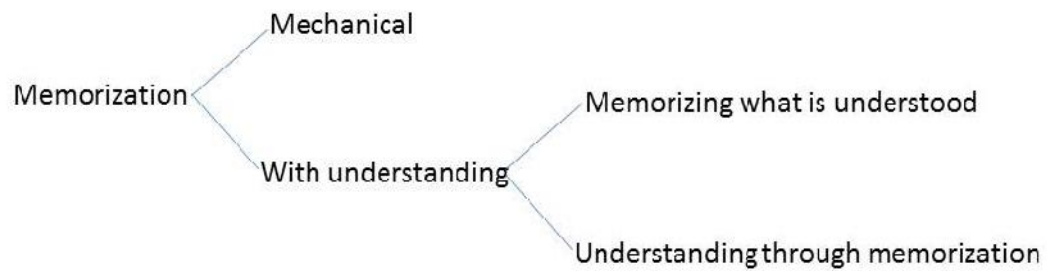
Figure 20. The relationship between memorization and understanding (Marton et al., 1996, pp. 76-77)

The Shanghai curriculum illuminates three cognitive stages: memorization; understanding for explanation; and understanding for inquiry. These stages are hierarchical, sequential, and relative. The memorization stage requires students to discern or remember some mathematical fact as well as to apply procedures simply in routine contexts or to imitate examples (Shanghai City Education Committee, 2004, p. 30). Memory is regarded as the prerequisite to understanding. The curriculum acknowledges the importance of memory in the development of understanding and regards it as a necessary path towards that understanding. Meanwhile, the memorization stage was analogous with two categories of performance expectations in the TIMSS Mathematics Curriculum Framework: Knowing: Recalling Mathematical Objects and Properties; and Using Routines: Performing Routine Procedures (Foxman, 1999).

With regards to the concept of function, the first learning objective is to 'acknowledge' variables, both independent and dependent variables within real life; to 'know' the concept of function and domain, and the value of function and range; and to 'know' the constant function (Shanghai City Education Committee, 2004, p. 66). Two verbs used here, '*acknowledge*' and '*know*', indicate that the cognitive requirement is at the memorization level (Shanghai City Education Committee, 2004, p. 30). The understanding of the concept, therefore, starts with the memorization.

5.3 Results from Linear Function in Attainment Targets

5.3.1 Analysis of understanding levels

      Statutory guidance in England. Both the KS3 and KS4 national curricula have the attainment targets for each topic. Linear function is part of algebra. Although there are the same requirements for both Higher Level and Foundation Level students in KS3, the KS4 curriculum stipulates that 'additional mathematical content [is] to be taught to more highly attaining pupils' (Department for Education, 2014, p. 3). Table 15 summarises not only these attainment targets, but also how this concept is expected to develop from KS3 to KS4. According to the general model of understanding, the greatest level of understanding of linear function is reached in KS3 (see the second column in Table 15). KS4 deepens the knowledge of linear function in KS3 and completes the understanding of Level 6 Invertising.

      With regards to algebra, in the later 1990s The Royal Society / Joint Mathematical Council (1997) recommended that algebraic symbols should be a powerful way as solutions. At the revised national curriculum, the concept of linear graph is initially developed from sequences and patterns, while the algebraic method is highlighted by KS3 stage, shown in Level 2 in Table 15.

Table 15

The General Model Applied to the Statutory Guidance of KS3 and KS4

The model of understanding function	KS3 Algebra Pupils should be taught to	KS4 Algebra, in addition to consolidating subject content from KS 3. Pupils should be taught to
Level 2 Dependent Relationship	reduce a given linear equation in two variables to the standard form $y = mx + c$ (Department for Education, 2013c, p. 7)	
Level 3 Connecting Representations	recognise, sketch and produce graphs of linear and quadratic functions of one variable with appropriate scaling, using equations in x and y and the Cartesian plane (Department for Education, 2013c, p. 6)	recognise, sketch and interpret graphs of linear functions, quadratic functions, simple cubic functions, the reciprocal function $y = \frac{1}{x}$ with $x \neq 0$, {the exponential function $y = k^x$ for positive values of k, and the trigonometric functions (with arguments in degrees) $y = \sin x$, $y = \cos x$ and $y = \tan x$ for angles of any size}
Level 4 Property Noticing	calculate and interpret gradients and intercepts of graphs of such linear equations numerically, graphically and algebraically (Department for Education, 2013c, p. 7)	find the equation of the line through two given points or through one point with a given gradient (Department for Education, 2014, p. 8)
Level 5 Object Analysis		use the form $y = mx + c$ to identify parallel {and perpendicular} lines (Department for Education, 2014, p. 8)
Level 6 Inventising	use linear and quadratic graphs to estimate values of y for given values of x and vice versa, and to find out approximate solutions of simultaneous linear equations (Department for Education, 2013c, p. 7)	solve two simultaneous equations in two variables (linear/linear {or linear/quadratic}) algebraically; find approximate solutions using a graph (Department for Education, 2014, p. 8)

At Level 2, numeracy skill will be required to shape the standard form. Level 3

indicates the ability to connect representations, mainly from the algebraic expression to

graphical representation. English students are expected to understand linear function by

actually drawing the linear function in the Cartesian plane during KS3, while KS4 students

have built up the relationship between the straight line and the algebraic expression. When

connecting these two representations, however, the tabular representation acts as a bridge. Whether the role of tabular representation in this connecting process is necessary is unclear both in KS3 and KS4. Here is an example (see Figure 21) from a previous GCSE test (No. 13, GCSE, Mathematics Syllabus A, Paper 1, Foundation Tier, 12 January 2010). The tabular representation and an instruction of how to draw the graph is a given in final assessment. The question is whether or not the table is a key point to go through this understanding level.

Example 1 Complete this table for $y = x + 1$

X	-2	0	3	6
Y			4	

On the grid, draw the graph of $y = x + 1$ for x from -2 to 6.
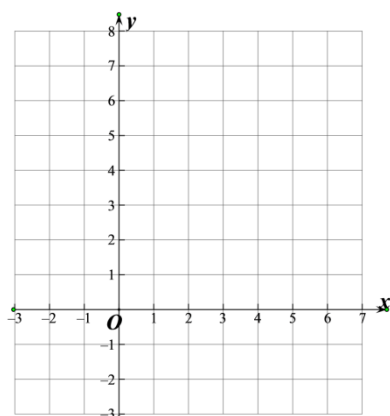
Figure 21. The graph of the GCSE example in Level 3

The way in which students are expected to construct the algebraic expression from two points of the graph is unclear because the property, gradient, might be involved in this connection. The requirements of KS4 also require students to form the algebraic expression through two points, moving from the graphical representation to the algebraic expression. There are two ways of establishing the formula: graphical or algebraic. One approach is related to the meaning of gradient learned in KS3 (see Figure 22) by constructing a right triangle with points ABC in the Cartesian plane. Through the graphic representation, y-intercept should be noted to form the algebraic expression.
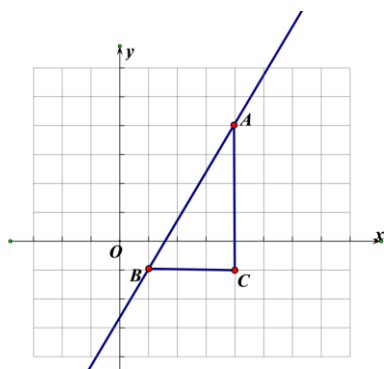
Figure 22. Graphical approach to gradient

The other approach is by the purely algebraic calculation as follows:

Put these two pairs of points into the algebraic expression: $y = ax + b, (a \neq 0)$.

$$\begin{cases} y_1 = ax_1 + b & (1) \\ y_2 = ax_2 + b & (2) \end{cases}$$

$$(1) - (2): y_1 - y_2 = ax_1 - ax_2$$

$$y_1 - y_2 = a(x_1 - x_2)$$

$$a = \frac{y_1 - y_2}{x_1 - x_2}$$

When getting the value of a, the value of b can be solved. It is the possibility for students to work out the value of *a* and *b* by merely manipulating the symbols without being aware of the meaning of *a* and *b*.

It is not specified how English students are expected to use either the graphical or algebraic approach to solve these kinds of problems. However, the meaning of the gradient (m) is introduced in KS3 with its meanings interpreted under the graphical representation. It is assumed that this property is calculated by the graphical approach as $\frac{\Delta y}{\Delta x}$ by constructing a right-angled triangle ABC (Figure 22) in the Cartesian plane. Level 4 will be discussed more in the next chapter: Textbook Analysis.

Level 5 understanding in the general model of understanding function appears in KS4. Here, two global properties are introduced, parallel for all English students while the perpendicular property is added for Higher Level students.

Level 6 understanding shows the connection with other mathematical knowledge, especially with simultaneous linear equations. Students are expected to make sense of the relationships between linear function, linear equation, and simultaneous linear equations through the graphical approach, by drawing the graphical representation of two linear functions.

Requirements in Shanghai. Appendix I shows the requirements of the Shanghai curriculum in Chinese and English translation. The content for linear function includes the concept of linear function, the graph of linear function and properties, application of linear function, and the expression of function. The last one, the expression of function, is highly related to the concept of function in general, so this study does not analyse it as well as the corresponding requirements and advice (point 4 in appendix I).

In general, the learning requirements (see Table 16) are more concise than that of England. It is unclear what kind of property is introduced in linear function at Level 4. In terms of Level 6, the basic ideas and methods have been discussed in the definition of understanding subsection of this chapter, in the case of solving quadratic equations by the square root method. The basic method here is about the combination of symbolic-graphic. There was an example of this basic method from one of the classroom observations in Shanghai.

Example 2: Linear function $y = kx + 3$, the value of y increases if x increases. Its graph, x-axis, and y-axis consist of a triangle. If the area of this triangle is $\frac{9}{2}$, find out the corresponding value of k.

Solution: $y = kx + 3$ passes by (0, 3), the value of y increases if x increases. So the graph should roughly look like Figure 23.

The area of triangle OAB $= \frac{1}{2} \times OA \times OB$

$$\frac{9}{2} = \frac{1}{2} \times OA \times 3$$

$$OA = 3$$

Because A (-3, 0) is located at the line
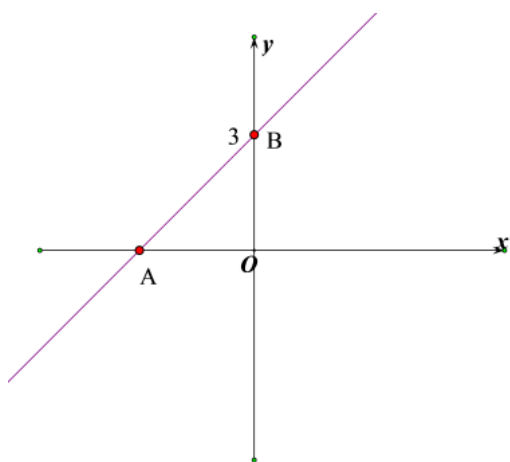
$$y = kx + 3,$$

then $0 = k \times (-3) + 3$

So, $k = 1$.

Figure 23. Graph of $y = kx + 3$

This example shows the tactical requirement to draw the graph according to the first sentence 'linear function $y = kx + 3$, the value of y increases if x increases', as revealing the property, monotonicity, for linear function. Meanwhile, this problem is related with other mathematical knowledge such as the area of a triangle. The basic method is based on a full structural understanding of current knowledge as located at Level 6.

Cognitive development levels are also indicated at each requirement. Two verbs that are used to describe the requirements of linear function are '*know*' and '*master*'. The former verb shows the memory stage for the requirement of this knowledge as previously mentioned in the cognitive levels. In addition, verbs such as 'understanding' or 'explain' are to describe the second stage of cognitive level: Understanding for Explanation; while verbs such as 'master' or 'proof' are to describe the third stage of cognitive level: Understanding for Inquiry (Shanghai City Education Committee, 2004, p. 30). In case of Level 4 Property Noticing, the Shanghai curriculum demonstrates it through the two verbs: 'know' and 'master', which also indicate the understanding development begins with memorization (the

first cognitive stage) – knowing the rule, and then moving towards 'Understanding for

Inquiry' (the last cognitive stage).

Table 16

Requirements in the Shanghai Curriculum alongside the General Model

The model of understanding function	Learning objectives
Level 2 Dependent Relationship	To understand the concept of linear function
Level 3 Connecting Representations	To establish the relationship among linear function, linear equation for two unknowns and straight line
	To plot the graphical representation of linear function
Level 4 Property Noticing	To know and master the properties of linear function using the graph
Level 5 Object Analysis	To grasp the relationship between the motion of straight line and b in algebraic expression $y = kx + b$
Level 6 Inventising	To further experience the basic method: the combination of symbolic-graphic

5.3.2 The content of application

In terms of application, the common point between the two regions is that they both

attach great importance to mathematical modeling. Modeling refers to the process that

abstracts the real world problem to the mathematics (Blum & Niss, 1991). It acts as the first

step towards application. Students are able to construct a mathematical model that fits the

essence of the elements and the relations in the context (De Corte, Verschaffel, & Greer,

2000). To create relationships is a hallmark feature of problem solving (P. Thompson, 1985).

Modeling is intrinsically an open-ended task which could reflect students' inadequate

knowledge (Sleeman & Smith, 1981). However, it is unclear in the Shanghai curriculum

where and how to apply knowledge into real life (see Table 17). In England's situation, the

mathematical modeling within both KS3 and KS4 takes place within graphical contexts in

order to generate the algebraic expression in terms of distance, speed and acceleration

problems. Furthermore, three particular types of graph are required for the higher level

students: distance-time graphs, velocity-time graphs and graphs in financial contexts.

Table 17

Requirements in terms of Application in Two Regions' Curricula

KS3 Algebra Pupils should be taught to	KS4 Algebra, in addition to consolidating subject content from KS3, pupils should be taught to	Shanghai learning objective
model situations or procedures by translating them into algebraic expressions or formulae and by using graphs (Department for Education, 2013c, p. 6) find approximate solutions to contextual problems from given graphs of a variety of functions, including piece-wise linear, exponential and reciprocal graphs (Department for Education, 2013c, p. 7)	translate simple situations or procedures into algebraic expressions or formulae (Department for Education, 2014, p. 8) plot and interpret graphs (including reciprocal graphs {and exponential graphs}) and graphs of non-standard functions in real contexts, to find approximate solutions to problems such as simple kinematic problems involving distance, speed and acceleration (Department for Education, 2014, p. 8) For Higher Level students, {calculate or estimate gradients of graphs and areas under graphs (including quadratic and other non-linear graphs), and interpret results in cases such as distance-time graphs, velocity-time graphs and graphs in financial contexts} (Department for Education, 2014, p. 8)	To apply linear function into real world situation and formalise initial function model

5.4 Summary

This chapter has presented some of the features of the two regions' curricula with regards to their overall aims, and more specifically, the aims regarding linear function. Generally, between the features of the curricula and students' performance, there does not exist any cause-effect relationship (Alexander, 2012). That is, higher students' performance does not indicate that every feature of the respective curriculum is better than others. The

curriculum analysis however provides the foundation of how to make sense of what each region is expecting of their students' learning outcomes and the approaches taken towards the mathematical concept.

First, the analysis of the curricula justifies three aspects: topic chosen, test design, and sample selected. The topic chosen, linear function, is the paradigm case within the concept of function due to the depth of requirement being similar for pure mathematical knowledge – i.e. involves the same range of levels of understanding in both regions, from understanding Level 2 Dependent Relationship to the highest Level 6 Inventising. The test design should be two sets of tests as the mathematics knowledge itself and what they can do (application in real world situation). The sample students chose as English students should be in KS4 stage while Shanghai students should be in Grade 8 or Grade 9.

Secondly, the two regions' curricula represent different approaches to learning fundamental mathematics. England prefers the graphic representation while Shanghai emphasises the use of the basic method in Shanghai. Furthermore, it is worthwhile to investigate how this concept is presented at each understanding level in the selected textbooks. In addition, the following specific questions will be clarified in the next chapters:

In terms of basic knowledge understanding,

1. How is the concept of linear function initially presented or defined; in the algebraic expression or as a graph? (Textbooks analysis)

2. When English students connect the representations, is the tabular representation important for their understanding? (Student tests)

3. What kind of properties do Shanghai students learn? (Textbooks analysis)

In terms of application,

1. How do Shanghai students explore the modeling? (Textbooks analysis)

These questions will be explored in the subsequent chapters.

Chapter 6 Textbook Analysis

The previous chapter has examined the intended curriculum, including the KS3 and KS4 national curricula in England, and the local curriculum in Shanghai. This chapter turns to the respective textbooks that the Maths departments in the sample schools currently provide for their teachers and students in the classroom. It presents an analysis of important features of linear function in the selected seven mathematics textbooks used by sample schools. It hopes to answer the second part of the Research Question 1: 'what are the requirements of …the officially used textbooks of the two regions in terms of linear function?' Following on from the curriculum analysis framework, the general features examined in this chapter will be related to the background information of linear function, and the specific features will narrow towards understanding linear function. All of these selected textbooks reveal that there are also two contexts in which linear function are displayed: basic knowledge; and applying this knowledge to real-life situations. In terms of basic knowledge, the general model of understanding function preciously discussed is applied in order to measure how each understanding level is presented. The application part examines the preferred approach that each region takes towards using mathematical concepts.

The first section of this chapter, therefore, will address the detailed analytical framework and will be followed by an explanation of how the data was analysed. The results will be split into two sections: background information and then particulars of linear function. This will be followed by a discussion concerning the different approaches drawn from the findings from the data analysis, and their implication for the test design in this comparative study. The last section will summarise the main findings, the potential gap between the textbooks and the respective curricula, and point to three further questions to be clarified in the following chapter: Student Tests.

6.1 Method

6.1.1 An analytical framework

Results from the curriculum analysis in the previous chapter argued that linear function was the only possible type of function which could be used to make comparisons. It is still worthwhile to investigate, however, whether linear function has a similar importance among all mathematics topics in the textbooks. To do this, the percentage of pages covering the topic in the textbooks from different countries was counted. This has been done previously by Y. Li et al. (2009) whose research analysed the similarities and differences between conceptualising and organising the division of fractions among Chinese, Japanese, and USA textbooks. The percentage of page usage was calculated using the number of pages devoted to linear function and dividing that by the total number of pages in the selected textbooks used. In addition, the study further explored how this concept was presented or defined at the beginning, relating with previous knowledge. This background stage was, therefore, analysed from the following two perspectives:

A. From the background information:

1) The percentage of pages allocated;

2) Previous knowledge related with the concept of function.

The particulars of the linear function analytical framework is drawn primarily from the work of TIMSS (Bianchi & Wolfe, 2002). TIMSS used 'blocks' instead of sections in characterising textbooks. There were a total of ten blocks, for example, narrative blocks, graphic blocks, exercise and question sets, activities, worked examples, and an 'other' block (Valverde et al., 2002). On the other hand, Love and Pimm (1996, p. 386) suggested that the most frequent used organisation in textbooks was the 'exposition – examples – exercises' model. The different choice of 'exposition' however represented what learning theory textbooks' authors took. Therefore, this analysis was carried out just for the examples and

exercises which were clearly marked as 'Example' or 'Exercise' in the selected textbooks. Especially, the worked examples indicated the detailed solution strategy to a problem where they 'presuppose that students will follow the flow of that pursuit' (Valverde et al., 2002, p. 142).

The particulars stage contained two parts related to linear function: pure knowledge and application. The former was examined using the model of understanding function, while the latter focused on the ways in which real world problems were presented and expected to be solved. With regards to pure knowledge, the Shanghai curriculum has not indicated which property of function should be introduced so that it can be answered by this chapter. Similarly, the Shanghai curriculum only stated the application in general, how students were expected to model the real life problem in Shanghai so that the application context was examined in the textbooks. Therefore,

B. From the particulars of understanding linear function:

1) Pure knowledge would be investigated from

   a) Understanding levels of Examples and Exercises on basic knowledge were examined according to the model of understanding function;

   b) Approaches towards each understanding level (how to present the knowledge).

2) Application would be analysed based on:

   a) How examples and exercises are presented (for example, only pure word problems or with a visual approach);

   b) How students were expected to solve them with regards to the three main representations of linear function (algebraic expression/equation, graphical representation, tabular representation).

6.1.2 Data analysis

In terms of background information, the format of the textbooks' content was examined. This would look at the general arrangement of the topic in terms of page use, as well as identifying whether there was a separate chapter on linear function and, if not, how many sections were allocated. Secondly, the background knowledge context for linear function and how this concept was introduced at the beginning of the section/chapter was also explored.

The particulars of linear function focused on the examples and exercises. In accordance with the model of understanding function, each example and exercise was merely allocated to a certain level of understanding. All the selected textbooks in England started with drawing the graphs from a concrete algebraic expression which is at Level 3. The first section of the Shanghai textbook concerned the definition of linear function to distinguish them from other types of functions, which the English textbooks did not cover, so that examples or exercises from this first section in the Shanghai textbook were not included in the study.

The data coding of levels of understanding for examples and exercises was a two-steps process. The first step involved identifying the number of examples and exercises at each level and the second involved calculating the percentage of these examples and exercises among the total of examples and exercises. Here, each example and exercise was designated at a certain level from the understanding model (see appendix J). If an example or exercise included a set of questions, the highest level of understanding conveyed within the example or exercise was assigned. At the same time, each understanding level had a certain number of examples and exercises in each textbook. Numbers at each understanding level in three Higher English textbooks were put together and calculated as a whole. Then, the percentage of these examples and exercises at each level in the textbooks could be calculated

and compared. The same procedure was also undertaken for the three Foundation textbooks. That is, the seven selected textbooks are divided into three types: Higher Level, Foundation Level, and Shanghai.

With regard to application, only two Higher Level textbooks (New GCSE Maths Edexcel Modular and Collins GCSE Maths 2 Tier-Higher for AQA A) included independent sections on the '*use of graph*' for linear graph. Meanwhile, these two textbooks arranged the same order in three sections: (1) linear graphs; (2) uses of graphs; and (3) parallel and perpendicular lines. Application and pure knowledge were interwoven together within the sections or chapter in other selected textbooks. The Shanghai textbook had an independent section to highlight the application part. As a result, these two English textbooks were chosen to compare with the Shanghai textbook for application.

The section entitled '*use of graphs*' has two types of knowledge application: discovering approximate solutions of simultaneous linear equations; and applying these in real life situations. It was argued earlier (Chapter 5: Curriculum Analysis) that solutions of simultaneous linear equations were regarded as the highest level in the model of understanding function. In a practical way, this knowledge is regarded as application of linear function in England as another method to solve simultaneous linear equations. But in the Shanghai textbook, solving simultaneous linear equations was presented as part of solving equation in Grade 6-7, which was arranged only before linear function. This difference was rooted in the different approaches towards the concept: algebraic method was highlighted in Shanghai. Therefore, this topic would not be suitable for forming standardised tests in this study. Solving simultaneous linear equations therefore would not be compared, neither as part of application, nor as pure knowledge in this chapter. The application in this study only included the real world context.

As for representing problems, two types were identified: using only pure word problems; or with visual approaches included tabular and graphical representations. The total number of examples and exercises of each type were examined. The tabular representation could not be found, however, in presenting any example or exercise within the selected textbooks. The visual approach is therefore only indicated by the graphical representation. As for the kind of solutions the textbooks were expecting, these answers were initially coded according to the three types of representations that were possibly involved. In fact, tabular representations could not be found here either. Meanwhile in the Shanghai textbook, some problems required both types of presentations within the full answer, namely algebraic expressions and graphs, while some could be solved flexibly by any method the students preferred. The approaches were therefore re-coded as using graphic representation, algebraic expression, both types, or as flexible choice.

6.2 Background information

6.2.1 The percentage of pages allocated

Table 18 compares the amount of content in the English and Shanghai textbooks. This table does not imply the equivalent proportion of time, only showing how textbooks writers arranged the proportion of this topic. This content includes the knowledge substance part and sections and chapters related to the topic of linear function.

Four of the English textbooks arranged linear function in the chapters entitled *Algebra: Graph* or *Graphs*. The ways in which the English textbooks presented different types of functions were based on the different shapes of graphs, in other words, from their graphical representations.

It was found that the average percentage of page usage in the Foundation Level, Higher Level textbooks from England, and the Shanghai textbook was similar, as the difference between Higher Level and Shanghai was roughly around 0.4% and the difference

between Shanghai and Foundation Level was around 0.3%. In particular, the presence of

linear function in one of the selected English textbooks, Collins GCSE Maths 2 tier-higher

for AQAA, was the highest among these seven textbooks.

Table 18

Linear Function Content Placement in Textbooks

Textbooks		Content organization	Pages (%)
England Foundation level in textbooks	New GCSE Maths Edexcel Modular (Foundation)	A section of Chapter Algebra: Graph	18 (2.13%)[1]
	Collins GCSE Maths 2 tier-foundation for AQA A	A section of Chapter Graphs	8 (1.43%)[2]
	Foundation GCSE Mathematics: Revision and Practice	A section of Chapter Algebra 1	10 (1.95%)[3]
		Average	1.84%
England Higher level textbooks	New GCSE Maths Edexcel Modular (Higher)	Three sections of Chapter Algebra: Real-life graphs	18 (1.94%)
	Collins GCSE Maths 2 tier-higher for AQA A	A Chapter: Linear graphs and equations (4 sections)	21 (3.5%)
	Higher GCSE Mathematics: Revision and Practice	Two sections of Chapter: Algebra 2	10 (2.16%)
		Average	2.53%
Shanghai textbook		A Chapter: Linear function	20 (2.16%)[4]

6.2.2 Previous knowledge

With regards to the expected background knowledge for function, results showed that

English students have more experience in relating the concepts to real world situations.

Meanwhile, their counterparts in Shanghai have more experience in algebraic approaches.

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[1] The percentage of page usage is calculated in terms of all two textbooks for Foundation/Higher. There are two volumes of textbooks in Foundation level and higher level of this series respectively. For example in Foundation 1, the percentage of page use age is 4.24% while in the two textbooks that is 2.13%.
[2] There is only one volume of textbook in Foundation level/ higher level in this series.
[3] There is only one volume of textbook in Foundation level/ higher level in this series.
[4] This percentage calculated is based on all the page usage in Shanghai secondary school textbooks. There are eight volumes of textbooks in Shanghai secondary school.

English students are firstly introduced to straight-line distance-time graphs, straight-line velocity-time graphs, or conversion graphs, and then turn to other types of graphs. Linear function was therefore presented under this graphical approach. The term 'linear function' itself, however, was not shown in any selected English textbooks. On the other hand, in Shanghai, before the linear function chapter, students would have tackled the general concept of function and were familiar with two types of function: proportional function $y = kx$ $(k \neq 0)$; and reciprocal function $y = \frac{k}{x}$ $(k \neq 0)$, while proportional function belongs to linear function. When learning these two types of function, Shanghai students started with mathematically rigorous definitions involving algebraic expressions. Their graphic representations, properties including intercepts and monotonicity, and application to real-world problems were introduced later. This same arrangement was then applied for linear function.

The two regions initially used quite opposing ways to present the concept of linear function, with England adopting a graphical method and Shanghai using an algebraic process. The initial presentation of linear function in two series of English textbooks (New GCSE Maths Edexcel Modular and Collins GCSE Maths) was related to the real life graphs. Here, the linear graph built upon the straight-line distance-time graphs which present 'how far someone or something has travelled over a given time period'. For example, in travel graphs, the formula of average speed implies the meaning of gradient. Another series, GCSE Mathematics Revision and Practice, started with straight-line graphs, horizontal and vertical lines, and then related lines with x and y, the coordinates. These two sub-sections paved the way toward drawing graphs. Therefore, the linear graph started with drawing a concrete graph such as $y = 4x + 5$ for values of x from 0 to 5. The Shanghai textbook gave an algebraic definition of linear function as $y = kx + b$ $(k \neq 0)$ and offered some examples to discern if a relationship belongs to linear function, while the learning process was based on

the special type of linear function, proportional function $y = kx$ $(k \neq 0)$ arranged at the previous term. Shanghai students had mastered basic knowledge and skills such as how to connect representations before starting the linear function chapter.

6.3 Particulars of understanding linear function

Looking further at the structure of the concept in all the different sets of textbooks, both the English and Shanghai textbooks looked at students' ability to deal with real world situations, although there were some differences. All of the textbooks, however, paid more attention to the basic knowledge.

Application was intertwined with the basic knowledge in the English textbooks, both at the Foundation and Higher levels. Exercises after each part presented one or two real-life problems corresponding with the basic knowledge which had been introduced. The approximate ratios were calculated of pages about pure mathematics knowledge to pages about application for the two chosen compared High level textbooks: in the New GCSE Maths Edexcel Modular it was 2.2:1; and in the Collins GCSE Maths 2 Tier-Higher for AQA A was it was 7.5:1. Application was highly valued by the former textbook with nearly one third of the coverage of pages. Pure mathematics knowledge, however, played a more significant role in the other textbook.

In Shanghai, application was the last section of the chapter on linear function. The ratio of pages about basic knowledge to application was 3.5:1. The basic knowledge section was demonstrated in a purely mathematical way, unrelated to any real-life context. There was a clear boundary between emphasis on basic mathematical knowledge and application in the Shanghai textbook. Here, again, a higher proportion of pure mathematics knowledge in the ratio suggests that it is more highly valued than the application.

6.3.1 Analysis of understanding levels

This subsection will explore the overall distributions of examples and exercises across the understanding levels first. The differences of approaches towards each understanding level will be examined later.

Overall distributions of understanding levels. *Examples.* A conceptual framework about understanding levels was previously established as a model of understanding function. Here, Table 19 shows the percentage of examples used in the textbooks at each level of understanding function. Both the Shanghai textbook and selected Higher level textbooks covered Levels 3 to 6 of the understanding model. For the more abstract understanding levels, namely Level 5 Object Analysis and Level 6 Inventising, the Shanghai textbook provides double the percentage of examples compared to the English Higher level textbooks, with particular emphasis on Level 5. The examples from selected Foundation level textbooks, however, evidently placed emphasis on Level 4, without presenting any examples from Level 5 or Level 6. All the English textbooks emphasised Level 4 Property Noticing. Lower down, at Level 3 Connecting Representations, the Shanghai textbook contained more examples than the English Higher level textbooks, but fewer than the Foundation level ones.

Particularly in the Shanghai textbook, the analysis suggests a big jump from Level 3 to Level 5. The percentage at Level 4 was lower than that of any of the English textbooks due to only one property, y-intercept, being introduced here. The meaning of gradient was simply explained as how steep the straight line was, while there was only one example provided by the textbook that indicated how to calculate the gradient by a purely algebraic approach, namely solving simultaneous linear equations, using the basic method *Daidingxishu*, one of the basic methods mentioned in Curriculum Analysis chapter. Without a detailed introduction to the concept of gradient, however, the Shanghai textbook quickly moved on to how to apply

this concept in order to identify parallel lines, and focused on the higher level of understanding, Level 5 Object Analysis, which was the most prominent understanding level.

By contrast, the English textbooks heavily emphasised Level 4, particularly the meaning of the gradient by drawing the graph. There are two methods to draw a line with this property: the gradient-intercept method, and drawing a line with a certain gradient. Methods used to draw a line in each region offered different ways of understanding this property.

Table 19

The Percentage of Examples at Each Level

	Level 3: Connecting Representations	Level 4: Property Noticing	Level 5: Object Analysis	Level 6: Inventising
English Higher level textbooks (total 24 examples)	13.6%	45.4%	36.4%	4.5%
English Foundation level textbooks (total 10 examples)	22.2%	77.8%	0%	0%
Shanghai textbook (11 examples)	18.2%	9.1%	63.6%	9.1%

Exercises. Table 20 reveals that exercises in all the selected textbooks included the highest level of understanding. All the textbooks contained higher percentages of exercises at Level 6 Inventising, compared with the fact that there were no examples at this level for the Foundation level textbooks. The distribution of examples and exercises was quite different for England and Shanghai based on the model of understanding function. Selected Foundation and Higher English textbooks paid more attention to Level 3 and Level 4 understanding, whereas the Shanghai textbook emphasised Level 5 and Level 6 understanding in terms of teaching and learning linear function.

The Foundation level textbooks heavily emphasised Level 3, with almost three times the percentage of exercises than that of examples. The Higher level textbooks emphasised

exercises at Level 4 in line with the situation of the examples, but contained more than

double the percentage of exercises compared to examples at Level 3 Connecting

Representations. In comparison to examples at Level 5 Object Analysis, there was half the

percentage of exercises. The Shanghai textbook reduced the exercises at Level 3 but added

more at Level 4.

Table 20

The Percentage of Exercises at Each Level

	Level 3: Connecting Representations	Level 4: Property Noticing	Level 5: Object Analysis	Level 6: Inventising
English Higher Level textbooks (total 124 exercises)	32.8%	41.2%	16.0%	10.1%
English Foundation Level textbooks (total 92 exercises)	72.5%	19.8%	5.5%	2.2%
Shanghai textbook (16 exercises)	6.3%	18.8%	50.0%	25.0%

Different approaches towards each level. *Level 3*. Both regions' textbooks started

with a concrete example, using an algebraic expression, such as $y = \frac{2}{3}x - 2$ in the Shanghai

textbook, and $y = 4x - 5$ for values of x from 0 to 5 in English textbooks. Here, the

difference was the domain (value of x) that English textbooks specified while Shanghai

textbook did not. The graph in the Shanghai textbook therefore was a straight line, while part

of a line between two points (segment) was shown in England. The initial expectation of

drawing a linear graph/function therefore differed.

Level 4. GCSE Mathematics Revision and Practice (Foundation) gave some exercises

to translate graphic representation to algebraic expression using two ways: indicating the

gradient in the graph (see Figure 24) or in words, for example to write the equation of the line

with gradient 4 and y-intercept -3.

Gradient = 2
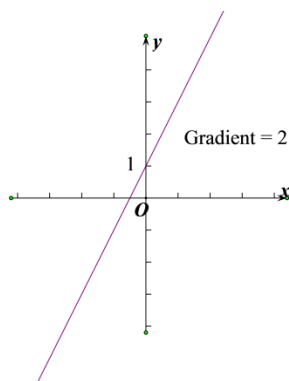
Figure 24. An exercise for Level 4

Particularly in the Shanghai textbook, the analysis from Table 19 suggests a big jump

from Level 3 to Level 5. The percentage of examples at Level 4 in Shanghai was lower than

that of any of the English textbooks due to only one local property, y - intercept, being

introduced here. The meaning of gradient was simply explained as how steep the straight line

was, while there was only one example provided by the textbook that indicated how to

calculate the gradient using a purely algebraic approach of solving simultaneous equations.

For example, the straight line $y = kx + b$ passes through points A (-20, 5), B (10, 20), find

out (1) the value of k and b; (2) the points that this straight line cut the axes (of a Cartesian

coordinate system). The solution of the first question was related with pure algebraic method

to work out the gradient, because the straight line $y = kx + b$ passes through points A (-20,

5), B (10, 20), so

$$\begin{cases} -20k + b = 5, \\ 10k + b = 20. \end{cases}$$

Solving the simultaneous equations, $\begin{cases} k = \frac{1}{2}, \\ b = 15. \end{cases}$

The solution of the second question is related with the concept of y-intercept, as this

straight line cuts the y-axis at (0, 15).

Without detailed introduction to the graphical approach towards gradient, Shanghai

textbooks quickly moved on to how to apply this concept in order to identify parallel lines,

and focused on the higher level of understanding, Level 5 Object Analysis, which was the most prominent understanding level in Shanghai.

By contrast, the gradient in the English textbooks was calculated by constructing a right triangle from a graph, e.g. $Gradient = \frac{Differences\ in\ y}{Differences\ in\ x} = \frac{AC}{BC}$ as discussed in previous chapter and indicated that a line which slopes upwards to the right has a positive gradient while upwards to the left has a negative gradient.

Essentially, the two methods used by England and Shanghai are identical. If given point A (x_1, y_1), and point B (x_2, y_2) in Figure 22, the algebraic way to solve the gradient of straight line AB, $Gradient = \frac{y_1 - y_2}{x_1 - x_2}$, as discussed in Curriculum Analysis chapter. The graphic meaning of the gradient of a straight line AB in Figure 22 is $\frac{AC}{BC}$, as the coordinate of point C is (x_1, y_2). The length of AC is $y_1 - y_2$ while the length of BC is $x_1 - x_2$. Therefore,

$$Gradient = \frac{Differences\ in\ y}{Differences\ in\ x} = \frac{AC}{BC} = \frac{y_1 - y_2}{x_1 - x_2}.$$

Both regions' textbooks illustrate the meaning of gradient as the steepness of the line. From this perspective, the algebraic approach cannot explain how this method links with the steepness. This property therefore is presented as rule-based procedure knowledge for instrumental understanding in the Shanghai textbook. The English textbooks are one step closer to conceptual knowledge for relational understanding, because the deepness rooted in the graph is determined by the degree of angle ABC in Figure 22. As the degree increases, the steeper line is.

Level 5. The Foundation Level textbooks did not introduce the property of parallel lines in worked examples, which was in the statutory guidance in the England's curricula. The exercises questioned the relationship between two parallel lines whose gradient was the same. In the Foundation GCSE Mathematics Revision and Practice, a note of exercise

pointed out 'the parallel lines have the same gradient'. The selected Higher Level textbooks followed the statutory guidance to introduce parallel and perpendicular lines.

The common global property introduced by England and Shanghai textbooks was the parallel. The approach towards this property however was opposing as well. In the Shanghai textbook, Example 4 required students to draw two straight lines, $y = -\frac{1}{2}x + 2$ and $y = -\frac{1}{2}x$, and then describe the geometrical relationship between the lines. The algebraic expression was offered, and the solution was obtained by observing the two lines on the Cartesian coordinate system. Conversely, the English textbooks gave the parallel lines first, and then required students to work out their algebraic expressions in order to find the same coefficient of x. In the selected Higher Level English textbooks, the two lines were presented by graphical representations, and the example required students to: (1) find the equation of each line; (2) describe the geometrical relationship between the lines; and (3) describe the numerical relationships between their gradients.

In summary, moving from the algebraic to the graphical was emphasised in the Shanghai textbook, while the opposite approach was taken in the selected English textbooks.

Level 6. The Shanghai textbook showed the link between linear function and inequalities from both the algebraic and graphical approach. In case of one example, namely for the given linear function $y = \frac{2}{3}x + 1$, (1) when $y = 5$, find out the value of x; (2) when $y > 5$, find out the value of x; (3) in the Cartesian Plane, there are some points located in the straight line $y = \frac{2}{3}x + 1$, as well as under the x-axis, find out the range of abscissa for these points. The textbook gives two approaches of solution by using the algebraic method first: (1) $\frac{2}{3}x + 1 = 5$, then $x = 6$; (2) $\frac{2}{3}x + 1 > 5$, then $x > 6$; (3) $\frac{2}{3}x + 1 < 0$, then $x < -\frac{3}{2}$.

Following this solution, the textbook presented the graphical representation of $y = \frac{2}{3}x + 1$, explaining the solution from the graph.

Conversely, the Higher Level textbooks in England merely linked linear function to geometrical knowledge, the midpoint in both examples and exercises. There was an example in the New GCSE Maths Edexcel Modular, Higher 1, Example 12 (p.376) which shows a graph of AB in the Cartesian plane. The point A is (2, -1) and the point B is (4, 5) and students were expected to (a) find the equation of the line parallel to AB and passing through (2, 8); and (b) find the equation of the line perpendicular to the midpoint of AB. The solution of the second question first pointed out that the midpoint of AB was (3, 2), and then linked with the meaning of gradient of the perpendicular line. In Foundation Level textbooks, one exercise linked linear graph to the area of the triangle which was formed by the three lines, such as $y = 4$, $y = x$, and $x = 1$.

From these two examples and one exercise, the complexity of questions differed at the same understanding level.

6.3.2 The content of application

How application questions were presented. Table 21 summarises the ways in which problems were presented, including both examples and exercises related to application, using pure word problems or incorporating a graph. In general, the selected Higher level textbooks in England used the graphical representation to find a formulae in application; for example, a conversion graph between temperature in ℃ and ℉ in the first quadrant. Given its graphical representation, it required students to find out the rule that was able to convert ℃ to ℉.

Conversely, the Shanghai textbook put the emphasis on pure word problems to model the situation by generating formulae. Some application problems involved two kinds of situations and required students to choose which one was more suitable for the context. Therefore, Shanghai students' reasoning abilities might be highlighted through defending their solutions.

Table 21

How Examples/Exercises was Presented and Their Purposes

Textbook	Examples/Exercises Word problem without graph	Having graph	Purposes
New GCSE Maths Edexcel Modular (higher)	0	9	• Finding formulae or rules • Making sense of gradient and intercept in the graph
Collins GCSE Maths 2 tier-higher for AQA A	0	6	• Finding formulae • Making sense of gradient and intercept in the graph
Shanghai textbook	6	1	• Comparing with two conditions to enhance reasoning ability • Modeling the situation

How problems are expected to be solved. Table 22 demonstrates that selected Higher level textbooks in England required a single type of solution, namely algebraic expression and equation. The Shanghai textbook additionally required students to use two kinds of representation (algebraic expression and graph) to answer the problem stated as 'both two types' in Table 22, while there are two problems without this requirement but can be solved by two approaches, stated as 'flexible'.

Table 22

Expected Solution

Textbooks	Graph	Algebra	Both two types	Flexible
New GCSE Maths Edexcel Modular (higher)	1	8	0	0
Collins GCSE Maths 2 tier-higher for AQA A	0	6	0	0
Shanghai Textbook	0	3	2	2

In the Higher level English textbooks, problems tended to be presented in a graphical way and were solved through algebraic expression. In contrast, the Shanghai textbook mainly

used pure word problems and required both kinds of representations, algebraic expression and a graph within the solution. Shanghai students were therefore given more opportunities for using different representations in terms of application. This raises the questions: when there is a choice, which kind of solution do the Shanghai students prefer? This will be examined in the next chapter: Student Tests.

6.4 Discussion

This section discusses two issues: different approaches each region taken towards presenting linear function; and the implications for designing student tests.

6.4.1 Different approaches towards presenting linear function

Building on the findings from the curriculum analysis presented earlier, the selected textbooks in the two regions also show two distinct approaches to learning linear function: graphic approaches in England, and algebraic approaches in Shanghai. It is believed in the West that real-world situations and visual representations 'help students learn the abstract ideas of mathematics with understanding' (Fennema & Franke, 1992, p. 154). This distinct approach, however, is embodied by three perspectives: in what way to introduce linear function, how to schedule the whole topic content, and by what means to demonstrate the relevant pure mathematical knowledge.

First, the topic is introduced with a slightly different meaning in the two regions. Linear function as proposed by England's curriculum is explained as linear graph in the English textbooks. Technically, there are two slightly different concepts. Essentially, linear function describes a proportional relationship between two variables. The dependence rule for this relationship is presented by an algebraic expression, $y = kx + b, (k \neq 0)$, and a straight line in the Cartesian plane. Not all straight lines belong to the concept of fuction, however; for example, $x = 1$ which does not conform to the one-to-one property required by the concept of function. That is, one value of x ($x = 1$) corresponds to infinite or any values of y

and does not fit with the univalence requirement. Pictorially, any vertical line in the Cartesian coordinate system with $x = a$ (a could be placed by any real number) does not represent the concept of a function. In this study, two extreme examples of graphs (vertical and horizontal lines) are excluded, as it would not affect the research questions.

Secondly, the English textbooks start from real-life graphs, moving to the system of Cartesian graphing, and then to form the formal notational symbol system of algebra. The concept of linear function is therefore initially based on the real world. In Shanghai, the approach to the issue of function progresses from an algebraic definition to a graphical representation and then moves to resolve the real-world problems. The method taken in Shanghai indicates that the abstract algebraic approach is the priority. These differences can be explained as differences in the cultural characters of the two education systems. For example, the English curricula emphasises the enrichment of contexts as the use of mathematics or the applicability (Brown, 2011), while the Chinese curricula pay more attention to mathematical knowledge structures and systems (Bao, 2002).

Thirdly, the method of connecting representations and the meaning of property are both influenced by two approaches. The English textbooks offered a graphical definition for the meaning of gradient, especially from graphical representation to algebraic expression. In contrast, the Shanghai textbook merely uses an algebraic way to demonstrate either how to translate graphical representation to an algebraic expression, or discerning the property of gradient, even though the graphical meaning of gradient would be introduced in Grade 11 (approx. age 17).

6.4.2 Implications for designing student tests

Before investigating student performance in linear function or linear graph in the two regions, it was important to give some thoughts to form the assessment taken in this study. This included the way in which problems were to be presented.

Textbooks largely embody the 'student performance expectations presented in content standards' (Valverde et al., 2002, p. 10). The selected English textbooks including Higher and Foundation Levels focus on the graphical approach to the construction of the meaning of linear function; and, based on graphs, the rules are generated in order to form an algebraic representation in the real-life application. Therefore, the English textbooks represent real-world problems using graphical representations, while the Shanghai textbook uses pure word problems. The application problems in the English textbooks require students to explain the graph so that the formula can be generated. The assumed answer in England is merely in the form of algebraic expression (equation). Conversely, the Shanghai textbook expects mixed methods, algebraic and graphic answers, to solve problems, as Confrey (2002) advocated diversity in mathematical solutions instead of uniformity ones. This textbook analysis therefore suggests that the use of pure word problems might negatively influence the reliability of the tests for England's part.

6.5 Summary

This chapter has discussed features of the selected textbooks in England and Shanghai in terms of understanding linear function. First, the findings have been summarised under three broad heading:

 Point 1: Similar importance of the topic. Finding from the percentage of content covered suggests the importance of linear function in junior secondary school stage is similar as the percentage is around 2%. Therefore, this finding confirms that the topic chosen is suitable for the comparative study.

Point 2: Different foci of understanding level. Findings also suggested that, although the curricula in two regions have the same depth of understanding requirement, the model of understanding function demonstrates that the more abstract understanding for linear function is indeed highlighted in the Shanghai textbooks more than the English textbooks.

The examples from the textbooks suggest it is reasonable to speculate that the Shanghai students might be encouraged to move towards more abstract levels of understanding linear function or be given much more opportunities to work on the questions located at higher levels of understanding. This potential deeper expected understanding of mathematics in Shanghai students might lead to better performance. On the other hand, the English students' understanding development is constrained by the requirement of curriculum or textbooks in general.

Point 3: Application. In terms of the application part, finding reveals the English Higher Level textbooks tended to present knowledge with graphs and expected the algebraic expression as the solution or answer. Conversely the Shanghai textbook heavily emphasised word problems and encouraged the two ways as solution: algebraic expression and graphic representation.

Secondly, linking findings of the textbook analysis with the findings of the previous chapter (Curriculum Analysis), the coherence between the curriculum and the textbook in Shanghai is less diverse than in England. That is, the selected English textbooks do not follow the statutory guidance so closely. Linear function was presented as an official term in both the KS3 and KS4 curricula, but interpreted to be linear graph by the form of graph rooted in real-world situations. Furthermore, the Higher Level textbooks contain the Inventising Level of understanding, linking to geometry knowledge, midpoint of a segment, and by examples as well as exercises, which expand the requirements of the curricula.

Thirdly, the selected Foundation Level textbooks do not reach all the statutory requirements for all students in the KS4 curriculum, as the parallel lines are not manifested in worked examples, though introduced in exercises. Conversely, Shanghai demonstrates uniformity between these two official sources.

The textbook analysis showed, however, that each region has their own patterns for expected teaching methods, which means that students' understanding may 'remain deficient' (Pepin & Haggarty, 2007, p. 13). The pros and cons of each approach were weighted by the impact on students' understanding development in linear function, from its positive and negative perspectives. Consequently, there are three questions that require clarification in the next chapter:

1. What kind of role will tabular representations play in the learning of linear function/graph?

2. Within the meaning of gradient, in which way will students perform better, algebraic or graphic?

3. In terms of application, which kind of solution will Shanghai students choose: algebraic or graphic?

In addition, the next chapter, which sets out the work around the student tests, will explore how students show their understanding of linear function in the two sets of understanding tests, pure knowledge and application in real life.

Chapter 7 Student Tests

The previous document analysis examined the differences in the requirements for linear function within each education system in general, and how the textbooks present this knowledge in detail. However, these findings only revealed what students might be taught and in which possible way. Therefore a further examination of what students actually achieve will be presented in this chapter. Two types of tests are involved in the examination of students' understanding of linear function: basic knowledge and application. In order to formalise the appropriate tests, a pilot study was conducted as detailed in Chapter 4: Methodology. The present chapter therefore consists of six sections. First, the following four aspects of method will be addressed: sample chosen; the criteria of how questions in each test were selected; data coding; and instrument reliability. Then the results are reported in three sections: the results from basic knowledge tests; the results from application tests; and the relationship of performances between the two tests respectively. The discussion section will explore two perspectives: the barriers at each understanding level and the English students' weakness of numeracy skills. The final section will summarise the findings that emerge from the student results.

7.1 Method

7.1.1 Samples

Participants in the Pilot Study. In each region, two of the three sample schools took part in the pilot study. In total, the pilot study involved 96 English students in Year 10 and 292 students from Shanghai in Grade 8 (see Table 23 for further details concerning these two samples). Boys and girls were approximately equal in distribution throughout both groups.

Table 23

A Profile of Sample in Pilot Study

Area	Boys	Girls	Unstated gender	Total	Mean age
England	34 (35.42%)	54 (56.25%)	8 (8.3%)	96	15.25
Shanghai	166 (56.8%)	126 (43.2%)	0	292	14.32

As mentioned previously, English students are divided into Higher and Foundation Levels. Within each level there are several hierarchical sets according to the students' abilities. The two English schools both offered the top set of each level in Year 10 respectively. With regards to these 96 England students, 45 students were from the top set of the Higher Level, while 51 students came from the top set of the Foundation Level.

Shanghai classes contain mixed-ability students. One sample school provided a list of all registered students in Grade 8. Another school offered half of the Grade 8 students who were selected from three classes taught by three different mathematics teachers.

Participants in the Main Study. The participants for the main research phase were 561 Year 10 English students, including 158 students from the Foundation Level and 403 students from the Higher Level; and 907 Grade 8 Shanghai students (See Table 24). The profile of participants, such as the mean age and sex ratio, was very similar to that of the pilot study. All the sample schools were able to offer all students in the sample year or grade, apart from the bottom set of English students. During the fieldwork in England, there was one sample school, where the teaching of linear graph was arranged before the half-term break (which was one week) so that not all the teachers successfully managed to conduct the tests before the break. Therefore, I only chose the students whose teachers had been able to arrange the tests before the break. This minimized the possibility of English students forgetting mathematics knowledge over the break, as reported in selected English teacher interviews at Chapter 8.

Table 24

A Profile of Subjects in Main Study

	Higher Level students	Foundation Level students	Shanghai students
Numbers	403	158	907
The number of boys and girls	189 Boys 202 Girls	66 Boys 66 Girls	440 Boys 467 Girls
Ratio of boys to girls	1:1.07 (12 unstated)	1:1 (26 unstated)	1:1.06
Mean age	15.26	15.14	14.04
Std. Deviation of age	0.31	0.32	0.52

7.1.2 Instruments

During the pilot stage, the content (questions) of the two types of tests was the same in each region, although with different language versions. However, the content was altered a little bit for the main study, as there were three types of basic knowledge tests: for English Higher Level, English Foundation Level, and Shanghai students respectively; and two types of application tests: for English and Shanghai students. The application tests used in the main study for English students did not cater for the two levels of students because similar types of questions appear in the selected Foundation and Higher Level textbooks which would be addressed in the application test subsection. All of the sample students were provided with two pencil-and-paper tests where calculators were not permitted.

Piloting the basic knowledge test. Initially in the pilot study, the tests consisted of 9 mathematical questions that covered all the understanding levels for linear function. As mentioned in Chapter 6, linear function was defined in terms of two different types of representation: algebraic expression in Shanghai; and graphic representation in England, as the corresponding relationship had been presented already. That is, the first two levels of understanding levels were omitted. The questions therefore started from Level 3 Connecting Representations, and ended at Level 6 Inventising, which was in line with the range of requirements of the intended curriculum and official used textbooks in both regions, as

measures of students' achievement should be aligned with the respective curricula (Clarke, 2003). All of the questions were selected either from standardized tests for GCSE, or from the final examinations of Grade 8 in the Pudong District, Shanghai. Appendix B shows the English version of the 9 questions using during the pilot study.

The basic knowledge test during the main study. The main phase of the research aimed to gather the information from all sets from the Higher and most of sets from the Foundation Level students. Based on the results from the pilot study, therefore, tests in the main study were separated for the three groups, due to additional content of linear function knowledge for higher ability students within KS4 curricula. In each group, the tests featured the same number of questions (five). Each test used three different questions from the pilot study. Table 25 summarises the distribution of questions at each understanding level.

Table 25

Numbers of Questions at Each Understanding Level

	Level 3	Level 4	Level 5	Level 6
Higher Level test	0	2 (Item 1, 2)	2 (Item 3, 5)	1(Item 4)
Foundation Level test	1(Item 1)	3 (Item 2, 3, 4)	1 (Item 5)	0
Shanghai test	0	0	3 (Item 1, 2, 3)	2 (Item 4, 5)

In terms of the English students' tests, the two additional questions in the main study for both Higher Level and Foundation Level came from the corresponding level textbooks used by the sample schools. The reason for choosing these two new examples was that students might be more comfortable with types of questions from the textbooks which resembled their daily class activities. The findings from textbook analysis showed that worked examples from Level 5 Object Analysis and Level 6 Inventising were introduced by the textbooks for Higher Level students only. Thus, the Higher Level test covered Levels 4 to 6, while the Foundation Level assessment included a question on Level 3 Connecting Representations, for the purpose of examining the role of tabular representation. Another purpose of this design was to identify which factors influenced the process of Connecting

Representation for English Foundation students; their basic numeracy skills or basic knowledge of plotting in the Cartesian plane. Appendix C shows the test used for Higher Level students and Appendix D presents the Foundation Level assessment.

The two additional questions for Shanghai students were selected from previous assessments for the final examination of eighth-grade pupils used by all state schools in the Pudong District. Question selection was based on two criteria: (1) requiring higher understanding, Levels 5 and 6; and (2) having different mathematics knowledge linked with linear function in terms of Level 6. These two criteria demonstrated that questions represented the two levels: 5 and 6. The reliability and validity of the questions from these previous formal examinations had been previously checked during the usual standardisation process. These examinations, designed by experts, were well acknowledged by the education authority and schools. Therefore these types of questions were thought suitable for the requirements of the curriculum. However, worked examples in the uniform Shanghai textbook were not used, because students would have been overly familiar with them as teachers' lesson plan has heavily relies on the examples in textbook reported by the selected Shanghai teachers at Chapter 8. Appendix E shows the test for Shanghai students in the Chinese version with English translation.

Piloting the application test. Both phases of the study consisted of four questions that were designed to assess how students applied their knowledge to real life situations and the solution processes used.

Questions originated from the following sources: (1) the 'match' question was from one of the selected textbooks - Higher GCSE Mathematics: Revision and Practice; (2) The 'delivery' question was from GCSE, Mathematics A, paper 1, No calculator, Higher Tier, 11 June 2012; (3) the 'time-distance' question was also from one of the selected textbooks (Higher), Collins GCSE Maths 2 tier-higher for AQA A; and (4) the 'long word' question

was taken from instrument used by Doorman et al. (2012) whose model was used to develop the model of understanding function described in an earlier chapter. Appendix F shows the application test during the pilot study and which was used for both regions.

The 'match' question was developed from the KS1 and KS2 Algebra Statutory requirements: 'generate and describe linear number sequences' (Department for Education, 2013b, p. 138). This is sometimes referred to as 'guess my rule' in primary school maths circles. Therefore, this question was thought suitable for the top set of the Foundation Level students as well. Although the second question - 'delivery' question, was drawn from the Higher Tier GCSE assessment, the heads of Maths thought it would be acceptable for the Foundation Level students as well. In terms of the third question, the selected Foundation Level textbooks provided distance-time graphs problems before introducing the concept of linear graphs. This kind of question also appeared after linear graphs shown in the 'Foundation examination questions' section. Question 4, the 'long word' question, is an example of a piece-wise linear function. According to the requirement of the KS3 curriculum, all students are required to 'find approximate solutions to contextual problems from given graphs of a variety of functions, including piece-wise linear, exponential and reciprocal graph' (Department for Education, 2013c, p. 7). All four questions were consistent with the requirements of the Shanghai curriculum – to apply linear function into real world situation. Although similar types of questions are not found in the Shanghai textbook, the Head of Maths in Shanghai considered these questions as appropriate for the Shanghai students.

The main study for application test. The tests in the main study did not distinguish between the two levels of English students. These tests were developed from the pilot study test as well (see Table 26).

Table 26

The Changes of Application Test in the Main Study

	Match question	Delivery question	Time-distance question (replace a new one)	Long word question
England	Remain in the main study		Five sub-questions	Changes to 'Hire charge' question
Shanghai			Four sub-questions	Remain in the main study but is slightly changed

In the main study, the 'time-distance' question was replaced by another question of a similar type due to the former's low success rate in the pilot study for the English students. In this new 'time-distance' question, there were also slight differences in the sub-questions between the two regions. There were five sub-questions for English students, while the last sub-question was shortened for the Shanghai students as four sub-questions in total. That was because the final sub-question, which involved the graphical meaning of gradient, exceeded the requirements of the Shanghai textbook.

The 'long word' question in the pilot study was not suitable for English students who were more familiar with a graphical approach, as confirming the speculation in the Textbook Analysis chapter, and verified by the pilot study. The alternative question in the main research was therefore tailored to explore particular issues regarding each country's situation. The Shanghai test retained the 'long word' question used in the pilot study, which was consistent with the main type of examples and exercises in the Shanghai mandatory textbook. In the second sub-question, both algebraic and graphical approaches to solutions were required which was slightly different from the pilot study in which students had the flexibility of choice for the solution. This question, which involved the concept of piecewise function, not only assessed students' performances in the two approaches, but also compared their achievements between two approaches.

In terms of the English test, a 'hire charge' problem drawn from an exercise in one of the selected textbooks, New GCSE Maths Edexcel Modular (Higher), assessed two properties, the y-intercept and the gradient, and then generate the corresponding algebraic expression. This was in line with the KS4 requirement for all students, i.e. to 'translate simple situations or procedures into algebraic expressions' (Department for Education, 2014, p. 8), but the labelling of axis in the graph was different. In solving this problem, students were expected to understand the meaning of labels in the x-axis and y-axis which differed from the students' usual method of simply calculating the gradient as shown in the basic knowledge test. Appendix G reveals the application test for English students during the main study and Appendix H was the test used for the Shanghai students in Chinese version with English translation.

The Process of Translation. Both tests in the pilot study were initially taken from the English version and then translated into Chinese. In the process of translating to Chinese for the application test, two of the background contexts were changed for Chinese students: the subjects' names, for example 'Bill' and 'Ed' in English were replaced with 'Xiaohua' and 'Xiaobai' in Chinese; and the currency, for example the pound as the monetary unit in England was changed to RMB (Yuan) for the Shanghai tests. These changes would not affect the mathematical difficulty and contextual meaning of both tests. Questions in the main study were built upon those in the pilot study which had been tested for the issue of translation equivalent. The additional questions in the basic knowledge Shanghai test were taken from previous Shanghai examinations which had already been in Chinese version. In terms of the application test, the new 'time-distance' question was translated into Chinese from the English main study assessment.

7.1.3 Data coding

All the students who took the tests were included in the analysis. Data coding involved three stages: (1) marking their performance in each question; (2) identifying their level of understanding in the basic knowledge test; and (3) ascertaining the representation tendency in the application test. The coding criteria were the same for the three groups of students.

Marking students' performance. In terms of the basic knowledge test, every question has a unique right answer. Therefore, each student's response was coded as correct or incorrect. If a student omitted an item, the student response on this item was coded as incorrect. Students' scores were marked as 1 for corrected answers and 0 for uncorrected answers or leaving blank.

With regards to the application test, each student response was marked on a scale of 0 to 2. Students were given a score of 2 if the solution showed a correct and complete understanding of the problem. In order to achieve a score of 1, a student's main processing of the item was essentially right except for a minor error. If an answer showed no understanding or had been left blank, the response was scored as 0.

The reason that the application tests were marked as three levels instead of two: right or wrong, was that reasoning processes were involved in two questions: the last question, and the second question. In the case of the last question, at the initial data coding, around 20 per cent of the students got the question correct until the final step, because they chose the bigger number as the answer which did not fit with the best benefit of the customers' financial (the cheaper price). This kind of situation was marked as 1.

Identifying the Level of understanding. Due to the hierarchical levels of understanding linear function, students were clustered into one of five levels: four levels from Level 3 to Level 6; and another one for those 'not reaching' Level 3. There were three steps

involved in this identification process. First was to mark the total score in each understanding level. Each understanding level was represented in several questions. For example, if Level 3 was contained in two questions, the marks gained by a student in these two questions would be added together. This process was continued for each level where every student was given a mark. The second step was to identify the cut-off score which delineated each level. In attempting this, the method that Nicolaou and Pitta-Pantazi (2014) used in their study for fraction understanding levels was applied. Their study deemed that half the maximum score was a useful cut-off score for the respective ability. Finally, the identification of students' understanding levels began with the highest level due to the hierarchy in the model of the understanding function. That is, if students could achieve a higher level of understanding, it can be assumed that they have already progressed through the lower levels.

For example, if a student's score at Level 6 exceeded half of the overall achievable mark in that level, his or her understanding would be considered to be at Level 6. The highest achievable mark in Level 6 was 4, but, if the student gained a mark of 3, then this student's understanding was still regarded as Level 6. If a student achieved 2 or less, the understanding level would not be Level 6. The score of the lower level, namely Level 5, would then be checked using the same method. Table 27 below shows a hypothetical example of a student's mark at each level of understanding. According to the level coding method, the level of understanding of this theoretical student would be coded as Level 3.

Table 27

A Hypothetical Example of Understanding Level Coding

	Overall achievable mark	A student's actual mark
Level 3 Connecting Representations	7	4
Level 4 Property Noticing	3	1
Level 5 Object Analysis	4	2
Level 6 Inventising	4	2

In the main study, this issue was raised in Higher Level understanding test as well. There were in total 11 students whose understanding achieved Level 6, but did not get expected marks in Level 4. Six of them got 0 point and 5 of them got score 1 at Level 4. But according to the coding procedure which was described for the pilot study, these 11 students' understandings were coded as Level 6. Table 28 states the corresponding items and total mark for each testing understanding level in England. Appendix K gives an example of coded data for one English Higher Level student's understanding test. According to the data coding procedure, the sample sheet was marked as Level 5.

Table 28

The Distribution for Understanding Test in England

Understanding Level	Item(s)	Full mark
Level 4	1 and 2	3
Level 5	3 and 5	2
Level 6	4	1

Table 29 shows the details of the Shanghai understanding test. According to the same procedure, a student sample in Appendix L was marked as Level 5.

Table 29

The Distribution for Understanding Test in Shanghai

Understanding Level	Items	Full mark
Level 5	1, 2, and 3	3
Level 6	4 and 5	6

This marking approach solved a dilemma which came out of the results from the pilot study in England. During the pilot study, five students were found to have the right answer in Level 5 but failed to get the correct answer in Level 4. Level 4 was about two properties: gradient and intercept. Most of them left it blank or did not calculate correctly. In this case, their understanding jumped to Level 5. The reason for marking it as Level 5 was that the basic knowledge test was designed to test how well students understood linear function or linear graph. Notwithstanding that calculation skills play a key role in effective performance

in secondary mathematics (Brown, 2011). The questions in Level 4 focused on following the right procedure to arrive at the correct answers, namely reproducing the rules or formulae (how to use it was irrelevant at Level 4). The suspicion was that in answering this question students might simply memorise a sufficient procedure and reproduce it but with no real understanding of the meaning and why it works. The analogy to that suspicion was the Searle's (1987, p. 213) story about the 'Chinese room'. Searle supposed that someone who did not know Chinese at all, but with instructions in the particular 'room', could translate the English into Chinese correctly. Searle (1987, p. 214) argued that there was 'a distinction between manipulating the syntactical elements of languages and actually understanding the language at a semantic level'. Questions at Level 5 discerned how students use the meaning of gradient or intercept. The data coding could minimize the influence of numerical skill and discern if their understanding can achieve structural view. This situation however did not occur in the Shanghai sample.

With regards to students' tendency to use a certain type of representation in the application test, data coding involved the three main representations: algebraic expression, tabular representation, and graphical representation. This type of coding, however, did not calculate whether students achieved the right answer. Every type of representation shown in each student's answer was counted.

7.1.4 Reliability of the tests

Reliability during the pilot study. In examining the reliability of the basic knowledge tests of the pilot studies in both regions, the Cronbach α values for the understanding test was found to be 0.85 in the English sample and 0.89 in the Shanghai sample. The Cronbach α values for the application test were 0.81 in the English sample and 0.85 in the Shanghai sample. The Cronbach alpha values were above 0.7 in each case and

therefore the values could be regarded as acceptable, indeed, values greater than 0.8 could be considered desirable (Pallant, 2010).

Reliability of the tests for Higher Level English and Shanghai students. All of the data analysis in the main study was based on the entirety of the English and Shanghai samples. All of the tests were developed from the pilot study which had been validated on the basis of a small group of students. Table 30 summarises the Cronbach alpha coefficients of the two tests for the English Higher Level students and the Shanghai students. The numbers in parentheses represent the standard deviations for each test.

Table 30

General Quantitative Results in Main Study

	Understanding questionnaire	Application questionnaire
Higher Level students (403)	0.840 (2.068)	0.765 (5.166)
Shanghai students (907)	0.813 (2.092)	0.795 (4.796)

Reliability for the Foundation Level English students. The Cronbach alpha coefficient for the Foundation Level understanding test was low at 0.53 and that of the application test was 0.51. The lower Cronbach alpha values, however, may be due to the insufficient number of students, 158 Foundation Level students. As in the case of Foundation Level, the mean inter-item correlation for the understanding test was 0.14 and for the application test was 0.08. Both are lower than the optimal level, which fell between the range of 0.2 to 0.4 as recommended by Briggs and Cheek (1986), as lower than 0.1 is unlikely representable. It demonstrated that the two tests did not have an acceptable degree of reliability for the Foundation Level students. Most of the Foundation students only answered the first question in the understanding test, which asked them to draw the graph with having being given a table. With regards to the application test, they mainly answered the first question as well. As a result of its lower reliability the Foundation data was not included in the latter result sections of the study.

It was evident that the Foundation students could not progress beyond the beginning of both tests. In the first question in the basic knowledge test, 69% of the Foundation students could correctly translate the algebraic expression $y = x + 1$ into the tabular representation. The main error made by the rest of the students was that they incorrectly calculated the value of *y* when value of *x* equals -2; on average, they gave the answer as -3, whereas the correct answer was in fact -1. Their numeracy weakness, especially when dealing with negative numbers, was observed. For those correctly converting the algebraic expression into the table, 78% of them could successfully draw the graph. That is, most of the Foundation students could reach the understanding Level 3 Connecting Representations only if they possessed strong numeracy skills. At Level 4, which involved assessing the meaning of the gradient, the responses were very low and the rate of success was below 4%. Similarly in the application test, most of the students left the questions blank apart from the first question which asked them to translate the real life example to a table. The Foundation Level data was therefore not analysed any further in the main study.

7.2 Results from Basic Knowledge Test

This section includes two parts of results: pilot and main study. Details of the pilot study are provided. There are two purposes: to indicate the certain understanding levels that the main study would assess for each area; and the adjustment of the same level's question in main test.

7.2.1 Results in the pilot study

The general performance of the English and Shanghai students will first be described and compared, focusing on areas such as the distribution and descriptive statistics. According to the two types of data coding for the basic knowledge tests, marking their performance for each question will demonstrate the comparison in detail; and identifying the understanding

level will lead to a picture of their understanding for each level. Finally, the barriers to each level of understanding will be addressed.

General quantitative results. In general, the students in Shanghai far outperformed the English students in the pilot study. Figures 25 and 26 below show the distribution of scores for English and Shanghai students respectively. On each graph, the horizontal scale shows the students' scores in the test; and the vertical scale shows the percentage of students answering correctly. Comparing Figure 25 with Figure 26 reveals that a large percentage of Shanghai students achieved full marks (17) on basic knowledge understanding.
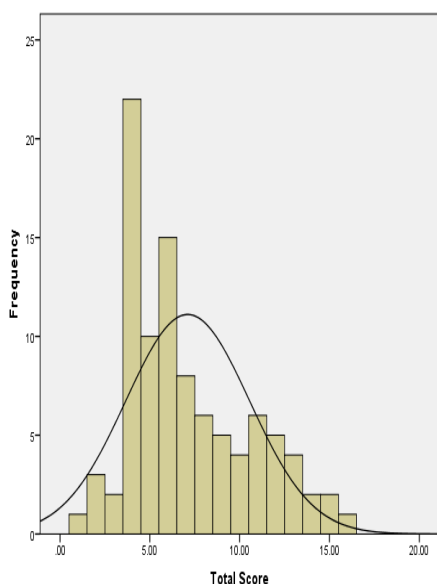
Figure 25. Total score of basic knowledge for English students
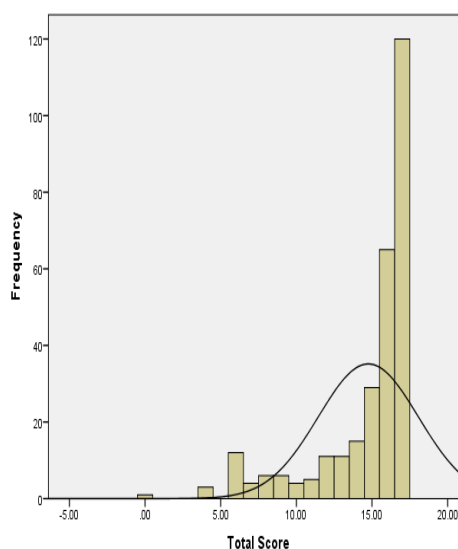
Figure 26. Total score of basic knowledge for Shanghai students

The mean score of the Shanghai students (M=14.76, SD=3.31) was much better than their counterparts in England (M=7.10, SD=3.45). Before checking whether the difference in means were statistically significant, statistical analysis for normality was assessed through examination of the values of the Kolmogorov-Smirnov (K-S) statistic. For the K-S test a significant result ($p < .001$) indicated non-normality. Mann-Whitney U test instead of t-test

for independent sample was therefore conducted. Results revealed that there was a significant difference between England and Shanghai (z (388) = -12.867, p=0.000, two-tailed). The effect size was calculated, as $\gamma = 0.65$ which led to a large effect.

Detailed results from each question. Table 31 summarises the percentage of students who answered correctly in each question geared towards a certain level of understanding function. Four English students and two Shanghai students failed to provide the right answer in the Level 3 questions. As these students comprised only a small proportion of the whole group, their low level of understanding positioned them as outliers.

Table 31
A Comparison of Students' Understanding Function in Two Regions

The general model	The basic knowledge assessed in each question	Correct percentage of students	
		England	Shanghai
Level 3 Connecting Representations	No. 1a From algebraic expression to a table	91.7%	96.9%
	No. 1b From tabular to graphic representation	47.9%	79.5%
	No.2 To generate algebraic expression using two pairs[5] (presented by word question)	51%	88.7%
	Average	63.5%	88.4%
Level 4 Property Noticing	No. 3 Intercept in algebraic expression	20.8%	83.9%
	No.4a Gradient in a graph (positive)	28.1%	77.7%
	No. 4b Gradient in a graph (negative)	15.6%	76.7%
	Average	21.5%	79.4%
Level 5 Object Analysis	No. 5 Parallel and intercept presented in algebraic form	5.2%	84.2%
	No. 6 Parallel and intercept in graphic approach	31.3%	91.1%
	No. 7 Transformation of the graph	34.4%	91.4%
	No. 8 Parallel and intercept presented in algebraic form, but intercept has been pointed out	28.1%	94.5%
	Average	24.75%	90.3%
Level 6 Inventising	No. 9 Related with geometry knowledge	1%	58.2%

[5] To translate two pairs which were supposed to be presented in Cartesian plane to algebraic expression, there are two methods; either using the graphic meaning of gradient that England taken, or solving the simultaneous equations the way Shanghai students did. In this question, two pairs were not presented in the graph. Neither solution got involved in the meaning of gradient to form the algebraic expression in English students, so that this question was still located in understanding Level 3.

In each item, the Shanghai students outperformed the English students. It indicates that a majority of the English students were struggling to understand properties such as gradient and intercept, while few if any Shanghai students had difficulty with questions at Level 5 or below. At Level 5 in particular, the English students performed well when the question presented them with the graphical approach, as opposed to word problems. In order to approach the meaning of gradient at Level 4, all of the English students unsuccessfully used the geometrical method for calculating the gradient; while the Shanghai students used the algebraic approach, successfully.

Identification of understanding level. The higher mean score of the Shanghai students indicates their higher level of understanding. Figure 27 shows the percentage of each level of understanding in both areas, applying the data coding set out above. More than two-third of the English students achieved Level 3 Connecting Representations (64.6%), while the majority of the Shanghai students reached Level 6 Inventising (72.7%). Most English students were grouped around the lower understanding levels whereas Shanghai students were centralised around the higher levels.
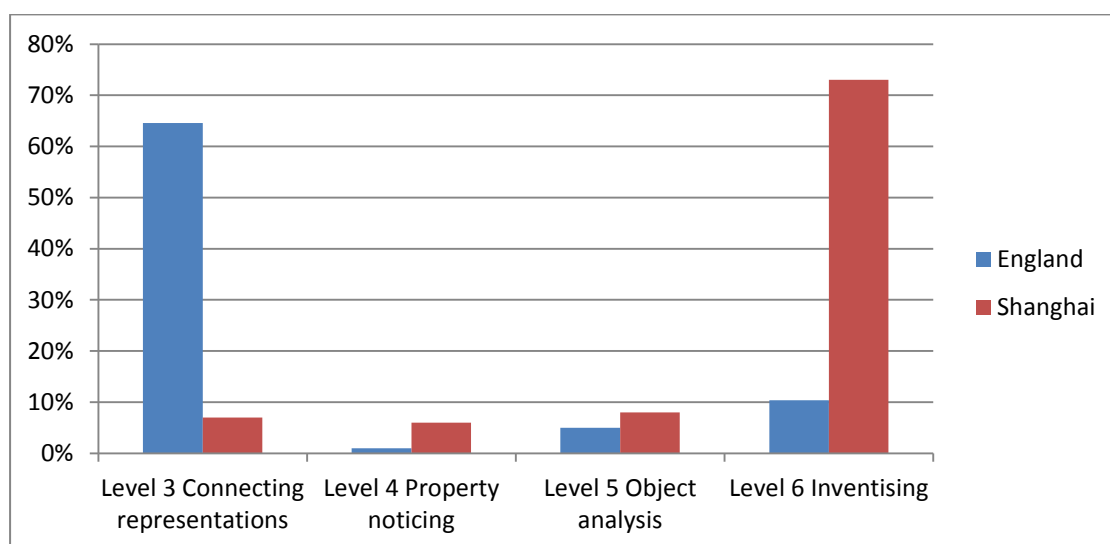

Figure 27. Percentages achieving each understanding level in England and Shanghai

Barriers to each level of understanding. The two cohorts of students have shown their distinct distribution of understanding levels. Their performance at each level of understanding was compared in order to find out what hampered the development of understanding linear function.

Level 3 Connecting Representations contained the first two questions. The first question investigated the conversion of the algebraic expression to a graphical representation by providing students with a table as follows:

Question 1: Complete this table for $y = x + 1$

x	-2		3	6
y		1		

On the grid, draw the graph of $y = x + 1$ for x from -2 to 6.

In the second step of the question, connecting the tabular representation with the graphical representation, 20.8% of the English students left the graph blank; and they were all Foundation Level students. All of the Higher Level students from England presented it accurately. In the main study, the Higher Level students therefore would not be given this type of question, while the Foundation Level tests had it as the first question. 79.5% of the Shanghai students gave the correct straight line.

The second question: 'A straight line passes through the point (0, 2) and (-2, 0). Find the equation of this line', was a word problem without aids of graphic representation. Half of the English students (51%) could not provide the answer. And none of answer had been found by using a graphic representation. It assumes that English students might be not comfortable with the pure word expression. That is, the graph might be necessary for them to present the question. This speculation came from the findings that the knowledge of linear graph was presented alongside the graph at the previous chapter: Textbook Analysis. This question was therefore decided that this should be deleted in the main study. All the Shanghai

students used algebraic method to solve it. Without indicating any property, this question was deemed to be Level 3.

Level 4 included two properties, intercept and gradient. Within intercept, only 20.8% of English students could simplify the linear function $y = 2(x − 1) + 5$ into $y = 2x + 3$, and then identify the value of y-intercept as 3. Their primary error was classifying 5 as the y-intercept. It was hard to conclude whether the reason for this stemmed from the students' lack of numeracy skills which would have enabled them to rearrange the algebraic expression to the standard form, or that they did not understand this property. The concept of y-intercept would however be reflected in the application test as discussed below.

In terms of the concept of gradient, the English students performed better with a positive value gradient than a negative value. They normally drew a triangle in the graph then calculated the lengths of the two sides in order to get the value of gradient. This approach conformed to the presentation of gradient shown in the English textbooks. This solving process, however, differed from their counterparts' method in Shanghai. The Shanghai sample showed that the students achieved a consistently accurate rate for both the positive and negative values of gradient. All of the Shanghai students solved the problem by constructing simultaneous equations, although some of them made a few computing mistakes. It is evident that the teaching process of this concept in Shanghai focused on the use of the algebraic approach, also in line with the findings from the textbook analysis. The coherence between how textbooks presented the knowledge and students' solution for both regions was revealed.

7.2.2 Results in the main study

Findings from the textbook analysis and the results from the pilot study demonstrated that the English students preferred the graphical presentation when expressing questions, while the Shanghai students were more experienced with the pure word questions. Therefore,

in the main study, the questions in the English tests all included a graph, while word problems dominated the Shanghai test. The results from the English Higher Level students and the Shanghai students will be addressed separately. Findings from overlap questions would be shown in the Shanghai students' performance.

Understanding of the English Higher Level students. Table 32 shows the percentage distributions of understanding of the English Higher Level students. Generally, the English Higher Level students illustrated their weakness in understanding gradient and their strength in dealing with problems that required the highest understanding connecting to other mathematical knowledge.

Table 32

The English Students' Performance Compared with the Pilot Study

The general model	The basic knowledge assessed in each question	The correct percentage of students
Level 4	No.1 From graphic representation to algebraic expression (New question)	44.4%
	No. 2a Gradient (positive)	36.7% (28.1% in pilot)
	No. 2b Gradient(negative)	16.4% (15.6% in pilot)
	Average	32.4%
Level 5	No.3 Parallel and intercept in word problem	32% (5.2% in pilot)
	No 5. Transformation	40.4% (34.4% in pilot)
	Average	36.2%
Level 6	No.4 connect to other mathematics knowledge, midpoint (New question)	29%

England and Shanghai have different approaches to translating from graphical representations to algebraic expressions. Nearly half of the English Higher Level students successfully used the graphical meaning of gradient and y-intercept to generate the algebraic expression.

Looking further at how the students calculated the gradient, the English students were more successful in finding a positive gradient in line within the findings of the pilot study. The correct percentage for finding the positive gradient (36.7%) was more than doubles that of discerning the negative one (16.4%). These students knew how to calculate the gradient,

but were less successful in understanding why there were two conditions: the positive and the negative. Furthermore, the English students either understood this property (gradient) fully or not at all. In the case of the positive gradient question, 28.6% of students left it blank. One student commented that he or she 'cannot remember how to do it' (see Figure 28). The rest of the pupils (37.1%) showed no understanding of this property, with Figure 29 and Figure 30 revealing the typical answers. Only 3% of students used the opposite way round ($\frac{\Delta x}{\Delta y}$ instead of $\frac{\Delta y}{\Delta x}$) to calculate the gradient.
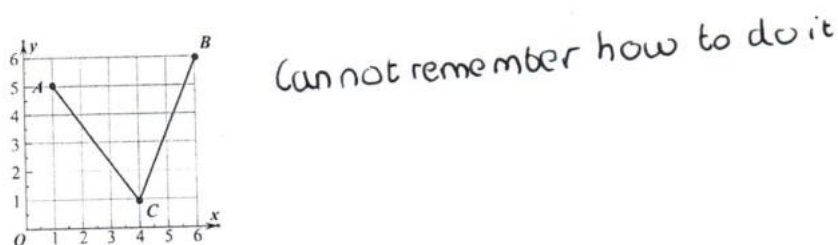

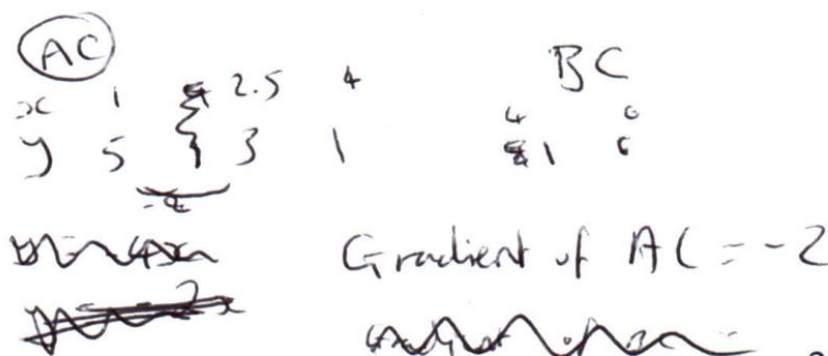*Figure 28.* An explanation of why the pupil could not answer this question

Figure 29. No understanding of gradient example 1

Figure 30. No understanding of gradient example 2

At understanding Level 5, the requirement of the English curriculum for the Higher Level students included the meanings of parallel and perpendicular. Perpendicular would be assessed at the next understanding level. Two questions were related to Level 5 in this test: one for the meaning of parallel, and another for the transformation of the whole graph. Within the first question, 32% of the students successfully combined the meaning of parallel and y-intercept to form a new algebraic expression. In terms of the second question, the English students showed their ability to deal with non-routine problems. In terms of the transformation of the graph, 40.4% of the students could regard the graph as a whole object rather than focusing on the individual pairs.

The question at Level 6 required students to make sense of the meaning of midpoint within a segment and perpendicular in order to form the algebraic expression of a new straight line. Although the question came from one exercise in the selected English textbook, it cannot be sure that all of the students had previously encountered similar questions. The function of textbooks will be discussed further in the next chapter: Teacher Interviews. Nearly 30% of students could achieve the highest level of understanding, Level 6 Inventising.

In conclusion, more than half of the Higher Level students successfully dealt with complex problems and achieved the more abstract levels: Object Analysis and Inventising, while the remaining pupils could not progress beyond Level 4. Questions in Level 4 were straightforward in that they required procedural knowledge and numeracy skills to solve it. In addition, understanding the specific knowledge of linear function was the important step in successfully solving complex problems in the higher level of understanding (see Table 33).

Table 33

English Higher Level Students' Overall Understanding

	Not reach Level 4	Level 4	Level 5	Level 6
Percentage of English students	45.9%	3.5%	21.6%	29.0%

Shanghai students' understanding. Half of the Shanghai students could answer every item correctly. Table 34 reveals their performance across each question and presents the comparison between the two regions' answers in the tests. Although the majority of students could achieve at least an understanding of Level 5 Object Analysis, beyond 40% of the students could not reach the highest level required by the curriculum. All of the questions at Level 5 were required by the curriculum, and the similar types were also presented in the compulsory textbook. Almost all of the Shanghai students showed their solid basic understanding in the case of linear function.

Table 34

The Shanghai Students' Performance Compared with That of the English Students

The model of understanding function	The basic knowledge required	The percentage of correct answers	Percentage of correct answers from English students for comparison
Level 5	No. 1 Parallel and intercept in word problem	93.2% (84.2% in pilot)	32%
	No. 2 Transformation	95.8% (91.4% in pilot)	40.4%
	No. 3 Monotonicity (new question)	88.8%	N/A
	Average	92.6%	N/A
Level 6	No. 4 Related with geometry knowledge	60.2% (58.2% in pilot)	N/A
	No. 5 Related with algebraic knowledge (new question)	45.4%	N/A
	Average	52.8%	N/A

Two questions were related to Level 6: one linking to the area of a triangle using geometry knowledge and another linking to the reciprocal function as algebra knowledge. The students' performance indicated that they faced few barriers in dealing with linear function; however, they did express some difficulties in linking the linear function with or understanding other mathematical knowledge at Level 6.

Especially in terms of the second question, which linked the linear function with the reciprocal function, the students were not given an existing graph. Most of the students (73.5%) were able to form the correct simultaneous equations in order to calculate points of intersection of the reciprocal function and the linear function, and got two correct points of coordinates. However, the requirement of the point was in the third quadrant, which means they must choose the suitable one. The students were solely unsuccessful in the last step – to pick out the right one in these two points. It is reasonable to assume that none of the students attempted to draw the graph in solving this question; because if they did, they would have been careful to discern that only one point in the third quadrant was required, instead of two points in their answer which was drawn from the calculation. The main barriers to their understanding were therefore seldom related to the concept of linear function or finding out the intersection for the two types of functions. Instead, their primary obstacle was their failure to read the requirements of the question carefully enough, and merely relying on their habits by using the algebraic method. They may treat the graph and algebraic method as two separated ways, so that the answers from the algebraic way did not link with geometrical knowledge, the concept of quadrant which was relevant to the Cartesian plane. It also indicated that these Shanghai students might not use visual representations or actually draw the graph to help them connect the other knowledge with linear function.

In summary, the Shanghai students demonstrated a higher general level of understanding in comparison to English students (see Table 35) in line with the conclusions of the pilot study.

Table 35

Overall Results of Shanghai Students' Understanding

	Did not reach Level 5	Level 5	Level 6
Percentage of Shanghai students	3.7%	38.4%	57.9%

7.3 Results from Application Test

Findings from the application test will reveal the strengths and weaknesses of students when they were faced with real-world situations in mathematical questions.

7.3.1 Results in the pilot study

This section consists of three sub-sections. First, general quantitative results will provide the overviews of how well students did. Secondly, the performance across the four questions used in the pilot study will be scrutinized and compared with each other. Finally, according to the data coding of solution tendency, the results will reveal the preferred representation students in each region used. An analysis of the data from the main study at next sub-section will examine if the alternative method would be a challenge for students in each region.

General quantitative results. The Shanghai students consistently outperformed their counterparts in England in this test in the pilot study. The superiority of Shanghai students' performance, however, was not as pronounced in the application test compared with the basic knowledge test. In other words, the difference between distributions was smaller in this test (see Figure 31 and Figure 32).
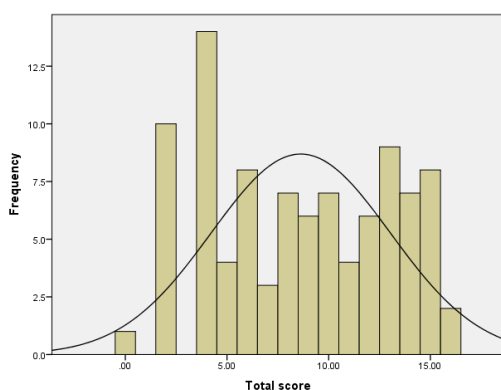

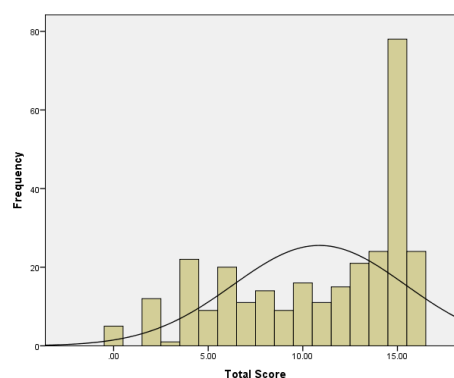*Figure 31*. Application performances of the England students in the pilot

Figure 32. Application performances of the Shanghai students in the pilot

There was also a significant difference between England (M= 8.63, SD= 4.41) and Shanghai (M= 11.01, SD= 4.44; z (388) = -4.789, p=.000, two-tailed). The effect size is 0.25 which indicates the small size. Thus, the Shanghai students were more successful in the basic knowledge test and less successful on the application assessment compared with the Higher Level English students.

Results in detail. Table 36 summaries how well the two groups of students performed across each question. It was notable that in questions 2, 3b, and 4b, there was more than the 10% difference between the two cohorts in the percentage of correct answers.

In No. 2a, students can use the graph below to find the total cost of having a parcel delivered by Bill: Bill works for a company that delivers parcels. For each parcel Bill delivers there is a fixed charge plus £1.00 per mile he has to drive thereafter (see Figure 33), how much is the fixed charge?

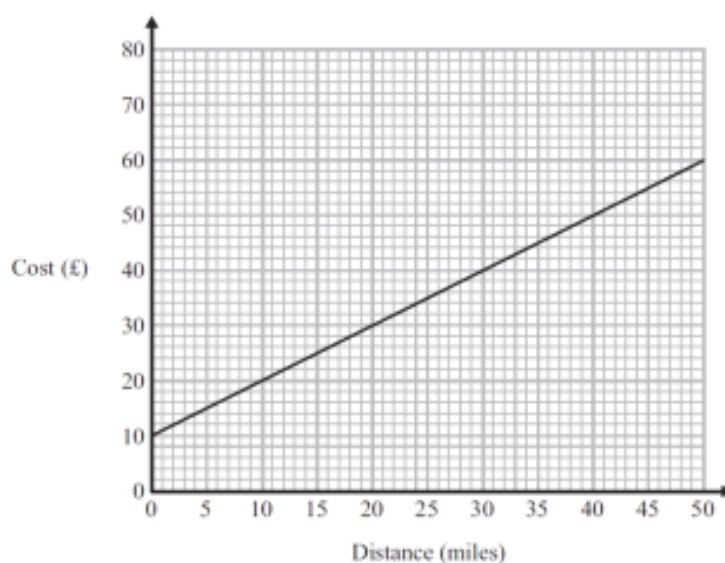*Figure 33.* The graph of application test No.2

Most of the English students showed their understanding of the concept of y-intercept. Linking back to the lower correct rate of the related question in the basic knowledge test in the pilot study, No. 3 (find the y-intercept of the straight line $y = 2(x - 1) + 5$) required students to use the algebraic expression and find out the y-intercept; as the English students

performed worse. It was unclear in the basic knowledge test why students showed these difficulties. Now, No.2a in the application test demonstrated that the English students' mathematical knowledge about *y*-intercept was not necessarily insufficient. They had difficulties in reducing the algebraic expression into a standard form as their numeracy skills constrained their presentation of understanding.

Table 36

English Students' Performances in the Application Test

	The purpose of each sub-question	Percentage of correct answers	
		England	Shanghai
No.1 Match question	No. 1a From real life situation to a table	96.9%	95.2%
	No. 1b From table to algebraic expression	64.6%	72.9%
	Average	80.8%	84.1%
No.2 Deliver question	No. 2a To discern intercept (the property) in graph	70.8%	88.4%
	No. 2b To solve simultaneous equations	42.7%	56.9%
	Average	56.8%	72.7%
No.3 Time-distance question	No. 3a To plot graphs	46.9%	54.8%
	No. 3b To derive an equation and solve the equation	21.9%	54.5%
	Average	34.4%	54.7%
No. 4 Long Word question	No. 4a To translate simple situations or procedures into algebraic expressions or formulae	52.1%	69.9%
	No. 4b To model situations or procedures by translating them into algebraic expressions or formulae, and by using graphs	8.3%	16.4%
	Average	30.2%	43.2%

Apart from No. 2a, the other questions, No. 2b, No. 3b and No. 4b, all involved generating an algebraic expression. The comparatively lower percentage of correct answers implied that the English students met some challenges, and the main study would investigate it in detail.

Solution tendency. When looking further at the ways in which English students demonstrated their solutions in terms of representations, they tended to use visual

representations, namely graphic and tabular representations, rather than algebraic expressions, a method on which the Shanghai students mainly focused.

The Shanghai students modeled the situation by forming the algebraic expression for each linear function and then solving the two linear equations (see Figure 34). In the case of No. 2b, the number of Shanghai students using abstract algebraic expressions was nearly 2.5 times that of the graphical representation. None of Shanghai students resolved problems by using tabular representations. In comparison, quite a few English students chose a tabular representation (see Figure 35). Figure 36 shows the percentage distributions of the English and Shanghai students who used the three types of representation in No. 2b.

Figure 34. An example of Shanghai students' solution tendency for No.2b

Figure 35. An example of England students' solution tendency for No. 2b
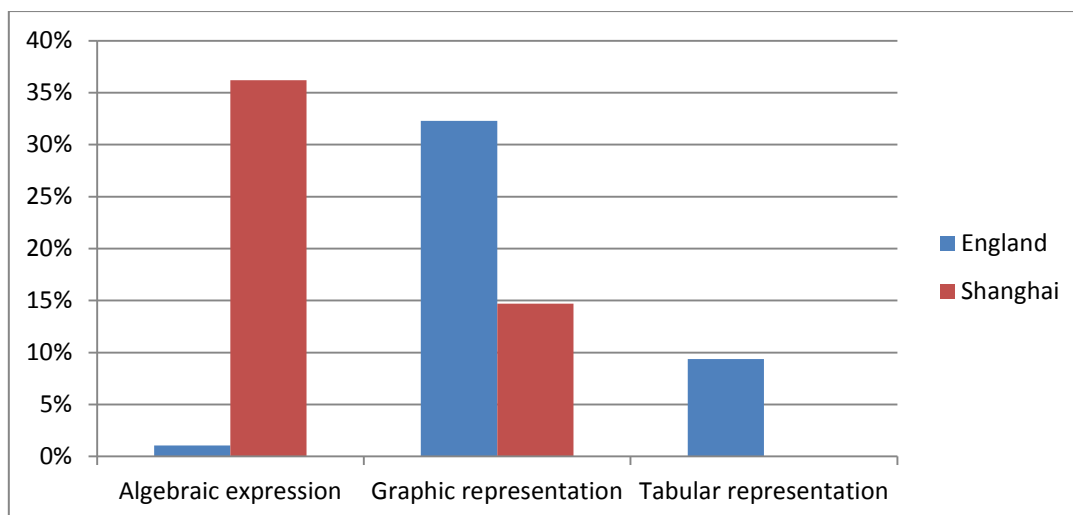
Figure 36. Percentages of England and Shanghai students who used different representations in No. 2b

In No. 4b, English students showed a strong preference for the use of tabular representations with pure word problems, whereas, once again, the algebraic approach dominated in the Shanghai students' solution-making processes (see Figure 37).
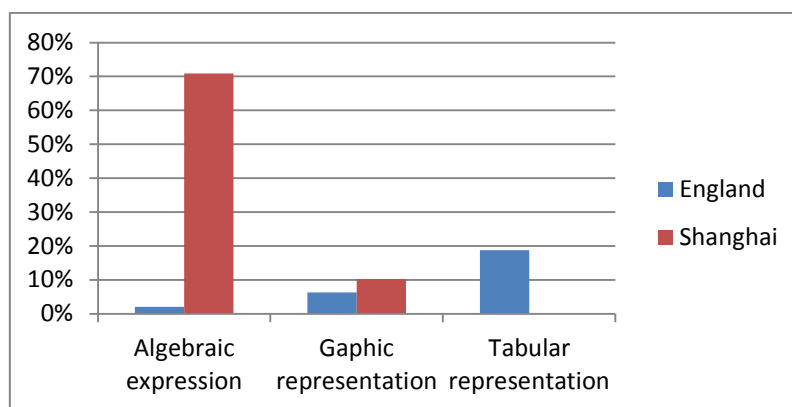
Figure 37. Percentages of English and Shanghai students who used different representations in No. 4b

In summary, the tabular representation was the popular strategy for English students (see Figure 38), while the algebraic expression was the primary choice of Shanghai students (see Figure 39), shown in No.4b. The Shanghai students' superior performances in application are likely a result of their use of the algebraic approach. The algebraic approach

was therefore more effective than the more visual method, proven by the higher percentage of correct answers in the Shanghai sample.

Figure 38. English students' main solution for No.4b

Figure 39. Shanghai students' main solution for No.4b

7.3.2 Results in the main study

In this section, the performance of each region's students will be presented first. There were three questions that both groups participated in so the second sub-section will analyse the results in detail. Results from the pilot study revealed that graphical or tabular representations were preferred by the English students. Therefore, the third sub-section will explore how the English students worked on generating an algebraic expression. Conversely, the Shanghai students opted for the algebraic methods shown in the pilot study. Findings from the Textbook Analysis (Chapter 6) demonstrated the requirement for two methods, algebraic expression and graphic representation. Consequently, the fourth sub-section will investigate how the Shanghai students dealt with the two methods when they were both required.

General performance. In the main study the Shanghai students performed better in most of the questions than their English counterparts (see Table 37).

Table 37

The English and Shanghai Students' Performances in the Application Test at Main Study

	Purpose of the item	Percentage of English students with correct answer	The Shanghai students' correct percentage
No. 1 Match question	No. 1a Translation from real life situation to a table	99.3% (96.9% in pilot)	93.6% (95.2% in pilot)
	No. 1b Translation from tabular representation to algebraic expression	48.4% (64.6% in pilot)	76.6% (75.9% in pilot)
No. 2 Deliver question	No. 2a To discern intercept (the property) in graph	81.6% (70.8% in pilot)	96.4% (88.4% in pilot)
	No. 2b To solve simultaneous equations	41.2% (42.7% in pilot)	71.6% (52.9% in pilot)
No. 3 Time-distance question (revised)	No. 3a – No. 3d Basic graph for linear function	14.4% (mean percentage)	22.6% (mean percentage)
	No. 3e The graphic meaning of gradient	39.5%	N/A
No. 4 Hire charge question (only for English students)	No. 4a Gradient in graph	10.9%	N/A
	No.4b Intercept in graph	70.9%	N/A
	No. 4c To form the algebraic expression	8.2%	N/A
No.4 Long word question (only for Shanghai students)	No.4a To translate simple situations or procedures into algebraic expressions or formulae	N/A	70% (69.9% in pilot)
	No.4b To model situations or procedures by translating them into algebraic expressions or formulae and by using graphs	N/A	10.6% in algebraic expression 13.9% in graphs 6.6% in both ways

Differences in performance across the three questions. Three questions were presented to students from both regions: a 'match' question; a 'delivery' question; and a 'time-distance' question. Table 38 summarises the general quantitative results: mean scores, results of nonparametric independent t-tests, and effect sizes. The Shanghai students clearly outperformed the English students in the mean scores for all these questions. For each question there was a significant difference in the means between English and Shanghai

students as shown by Mann-Whitney U test. The magnitude of the differences in the means was medium for the 'delivery' question and small for the two others.

Table 38

General Quantitative Results of the Three Questions

	Match question	Delivery question	Time-distance question
England (n=403)	2.97 (1.02)	2.50 (1.40)	4.44 (2.16)
Shanghai (n=907)	3.43 (1.14)	3.37 (1.06)	5.10 (2.18)
Full Score	4	4	8
Mann-Whitney U Test	z (1310) =-9.14, p=.000, two-tailed	z (1310) = -11.53, p=.000, two-tailed	z (1310) =-5.13, p=.000, two-tailed
Effect	$\gamma = 0.25$	$\gamma = 0.32$	$\gamma = 0.14$
Effect Size	Small	Medium	Small

The 'match' question. This question assesses how well students represent situations within a table and then how they translate from the tabular representations to algebraic expressions. In the first step, depicting situations within a table, the English students were more successful than the Shanghai students as almost everyone got the right answer, while the Shanghai students performed far better in the second part of the question. Half of the English students and nearly a quarter of the Shanghai students struggled to form the algebraic expression. The translation to an algebraic expression from a table can therefore be deemed as a universal problem for secondary school students.

The most frequent wrong answer found in the second sub-question was $y = kx + 3$ (various different numbers of k in the students' answers) for both groups. The reason why students gave the answer as plus 3 in the algebraic expression was assumed from these sequential visual pictures, i.e. that three matches were added sequentially (see Figure 40). The students were therefore directly linked the increasing values of y (3) with the algebraic expression without identifying the corresponding x value. It indicates that these students cannot actually understand the dependent relationship in form of algebraic expression.
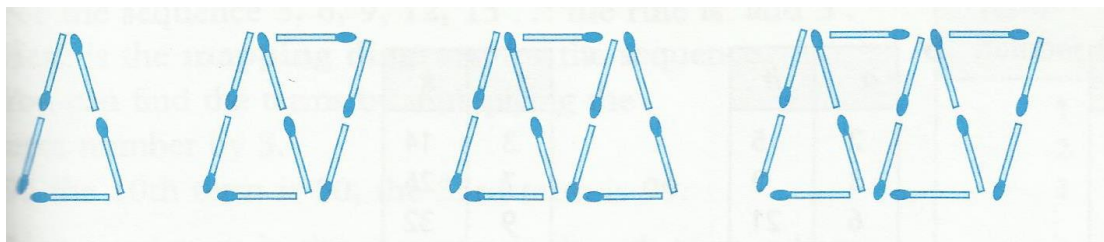
Figure 40. The match question

The 'delivery' question. This question was meant to evaluate the ways in which students understand the meaning of the y-intercept in graphical representations, as well as identifying students' preferred solutions. A majority of the students for both regions (81.6% English Higher Level and 96.4% Shanghai students) therefore understood this property.

Table 39 summarises their solution tendencies towards certain representation. In line with the findings of the pilot study, the Shanghai students demonstrated their strong preference for algebraic expressions and the English students used the visual approach such as graphical and tabular representations to solve the problem. Most of the English students using tabular representation chose several integer numbers to represent the distance in miles, such as 5, 10, 15, 20, 25, etc., to discover the cheaper delivery cost. It is unclear as to how the English students would perform if the right answer was a decimal number. Particularly in terms of graphical representation, three of the English students could not discern the right intersection point due to their imprecise drawing. Their answer was 18 or 19, rather than the correct one: 20. This implied that the conclusion drawn from the graphical representation was not as precise as the algebraic method. There were 7.1% Shanghai students who used the graphical representation. These pupils, however, all re-examined the answer (20) given by the graph, by calculating through an algebraic method again to ensure that the delivery fee was the same in the 'delivery' question.

Table 39

A Comparison of Students' Solution Tendencies in the Delivery Question

	No answer	Algebraic expression	Graphic representation	Tabular representation
% of English students	41.9%	0.7%	23.8%	33.5%
% of Shanghai students	22.6%	70.3%	7.1%	0

The 'time-distance' question. This question included four sub-questions for both groups, which assesses a student's ability to identify information through the use of the whole graph. Table 40 summarises the percentages of correct answers in each sub-question for both regions. There was a similar proportion of correct answers in both the first and forth sub-questions between England and Shanghai.

The second sub-question examined English and Shanghai students' understanding of gradient by providing distance and time in a graph in order to calculate the speed. There were three steps involved: to ascertain the distance; to discover the time; and, from these, to find the speed. According to the results from the fourth question, which involved discovering the distance travelled in a certain time, it was clear that that the same percentages of students in the fourth question did not have problems with identifying distance. The difficulty stems from the meaning of speed or deciphering the time in the x-axis. Additionally, as calculators were not allowed, 5% of English students could list the proper formula, but did not achieve the right answer, while only 0.4% of their counterparts in Shanghai gave the wrong answer. It also shows the higher proficiency of numeracy skill in Shanghai students.

With regards to the third sub-question, the main problem that the students faced was their inability to understand the meaning of average speed. The confusion for students for both groups was to understand the average speed as being for the whole journey. That is, they

calculated the average of each segment of the speed as $\frac{v_1+v_2+v_3}{3}$ instead of the whole distance

divided by the corresponding whole time.

Table 40

A Comparison for the Time-distance Question

Absolute correct answer	The point of intersection two straight line cross	The meaning of speed	The meaning of average speed	The distance at a certain time
% of English students	92.8	28	34.2	63.5
% of Shanghai students	94.5	37.5	50.2	69.7
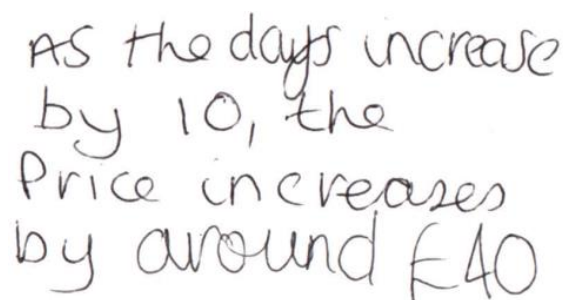
How English students explore the algebraic expression. Results from the pilot

study illustrated that very few English students preferred to use the algebraic expression as

their chosen solution, which is in line with that the understanding test found that the English

students felt comfortable with the visual approach to presenting the question. It was also

important in the main study to investigate whether the English students struggled with the

algebraic approach, and if they did, what their main barrier to their understanding was. Table

41 summarises English students' performances in the 'fire charge' problem. They showed

their limited understanding in creating an algebraic expression and were unsuccessful in

mastering the meaning of the gradient.

Table 41

How the English Students Explore the Algebraic Expression

	No answer	No understanding	Partial understanding	Full understanding
The meaning of gradient	53.1%	16.4%	19.6%	16.4%
The meaning of y-intercept	28.8%	1%	0	70.2%
To form an algebraic expression	63.8%	27.8%	0.2%	8.2%

Only 8.2% of the students could successfully generate the algebraic expression and 63.8% of the students left it blank. A student's inability to form the algebraic expression (27.8% of students) was shown in three ways: the use of words (17.9% of 27.8%); informal algebraic expressions (8.4% of 27.8%); and tabular representation (1.5% of 27.8%). In terms of using words, most students demonstrated their difficulty in comprehending the meaning of the *letter*. They described the algebraic expression as 'basic charge + daily charge', '£100 for 30 days', or 'for every 10 days up to £20 is added'. Figures 41 and 42 both show this kind of description. These students were primarily unable to come up with a solution or use formal mathematical language to express the solution. In terms of informal algebraic expressions, students therefore had to use words because they could not understand the symbolic letters (see Figure 43). A few students attempted to describe the rule by depicting a tabular representation, but were unaware of the limitations of a table, as it can only present particular variables.

AS the days increase
by 10, the
Price increases
by around £40

Figure 41. An answer for the algebraic expression 1

Figure 42. An answer for the algebraic expression 2

Figure 43. An answer for the algebraic expression 3

With regards to partial understanding, students knew the components and format of an algebraic expression, but produced the wrong gradient in their calculation; for example, '$y = 20x + 20$' or '$y = \frac{5}{8}x + 20$'. This signified, however, that the students had at least progressed in their understanding towards the algebraic expression. The meaning of gradient was proved to be the main barrier for these students.

This led to a further inquiry into the way in which English students make sense of the meaning of gradient, despite the fact that they did not perform well in this concept in the basic knowledge tests. There were two sub-questions related to the concept of gradient: the first sub-question in the 'fire charge' question; and the fifth sub-question in the 'distance-time' question. The 'fire charge' question required students not only to calculate the gradient

according to the graphical representation, but also to make sense of the different meaning of labelling between the x-axis and y-axis.

The final question of the 'time-distance', however, aimed to investigate how well students understood the meaning of gradient from a visual perspective instead of using calculations. In this question, they were required to discover the steepest straight line.

In the 'fire charge' question, over half of the English students (53.1%) left it blank. This percentage was much higher than that of the basic knowledge test in main study (around one-third in terms of positive gradient). The different response rate suggested that the English students had no preference for problem-solving situations over pure mathematical problems. Nearly one-fifth (19.6%) of the students who showed their partial understanding did not notice that the labelling of the x-axis and y-axis had different meanings. Actually, one grid for the x-axis was presented as 5 (days) while one grid of the y-axis was (£) 20. The wrong answer was therefore $\frac{5}{8}$ which should have been calculated as:

$$gradient = \frac{the\ differences\ on\ y\ axis}{the\ differences\ on\ x\ axis} = \frac{5\ grids\ on\ y\ axis}{8\ grids\ on\ x\ axis} = \frac{5 \times 20}{8 \times 5} = \frac{100}{40} = 2.5.$$

For the fifth sub-question of the 'time-distance' question, students were expected to identify that the steepest line represents the greatest speed. The number of blank responses resembled the proportion shown in the 'fire charge' problem. The remaining answers were all correct, proving that students understood the graphical meaning of gradient; namely the steeper the line, the greater the gradient's value.

The findings from the application tests have verified that the meaning of gradient was the main barrier for the Higher Level English students and influenced students' performances in generating algebraic expressions.

How Shanghai students explored the graphical representation. Although the Shanghai students showed their strong preference for the algebraic approach in the pilot study, it would be inaccurate to assume that they were proficient in using the graphical approach as

well. In the 'long word' question, students were required to find the solution by using both methods – algebraic and graphical. Table 42 summarises the percentage distribution of each approach.

Table 42

Shanghai Students' Representational Tendency

	No answer	No understanding	Partly understanding	Full understanding
% students using Algebraic approach	28.4	10.6	50.4	10.6
% students using Graphical approach	34	20.9	31.2	13.9

All of the Shanghai students initially solved the problem through the algebraic expression; they then translated it into a graphical representation. In other words, the graphical representation was built upon the algebraic one.

No understanding. For those who exhibited no understanding of the graphical representation, the main error was that they could not correctly translate the algebraic expression into the graphical representation, even though they could identify the correct algebraic expression (see Figure 44). In addition, they did not notice that the value in the payment had to be a positive number. The graph also illustrated that the more time a customer called, the less he or she paid which did not fit with common sense. This error indicated that students' understanding of pure mathematical knowledge was no guarantee that they could automatically and correctly apply the pure knowledge into real world situations. Alternatively, they might superficially apply the knowledge without considering too much of what the algebraic expression or graphic representation meant in the real world.
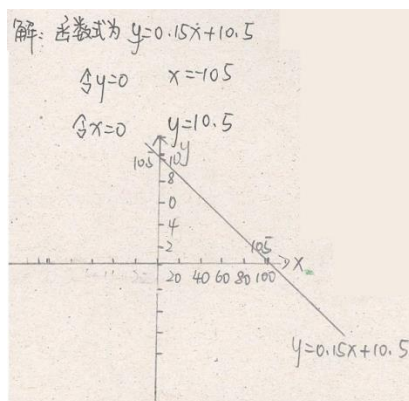
Figure 44. An example of *no understanding of graphic representation*

Partial understanding. Partial understanding represented those who successfully

completed the conversion process, but met challenges in the real world situation meaning.

These students faced four types of obstacles. First, students could not foresee that the values

of both the minutes and the money in the solution were not negative numbers, in a similar

situation to that shown in Figure 44. Almost half of the students with a partial understanding

(45.6%), however, provided negative answers (see Figure 45). That is, they completed the

correct translation without considering the meaning of *y* and *x*, despite writing in the first line

of their answer that 'the value of *y* is the fee and the value of *x* is the minutes'.
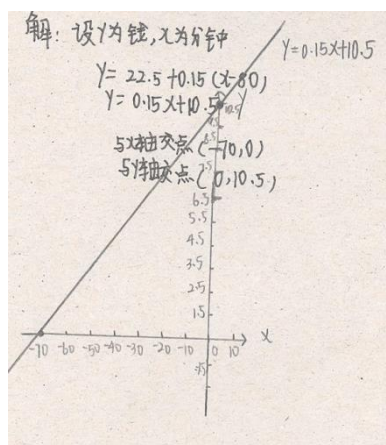
Figure 45. Partial understanding: example 1

Secondly, four students provided the correct algebraic expression, proving their

ability to translate the algebraic into the graphical expression. These students also displayed a

deficient understanding of the graphical representation. However, they were incapable of

comprehending the domain of the *x* value in the real world situations (see Figure 46). These

students also noted that the domain of *x* had two situations, $x > 80$ and $x \leq 80$, which

showed their awareness of the domain in the purely algebraic method instead of the graphical

form.
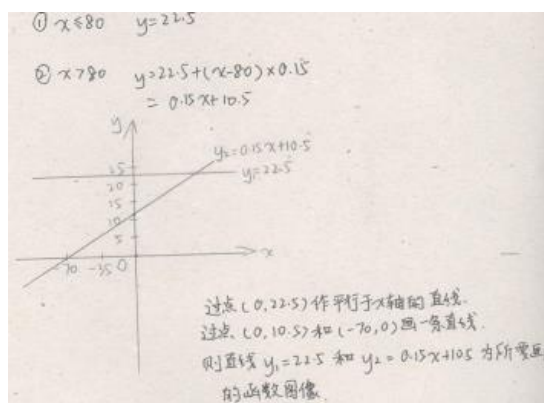
Figure 46. Partial understanding: example 2

Thirdly, nearly one-third of partial understanding students (29.7%) did not understand

that there would be two different offers for the phone charge (see Figure 47). The graph

showed that, if there was no call at all, the fee would be 10.5 RMB instead of 22.5 RMB. It

was unclear as to why students failed to recognise the discrepancy between the answer and

the statement in the question.
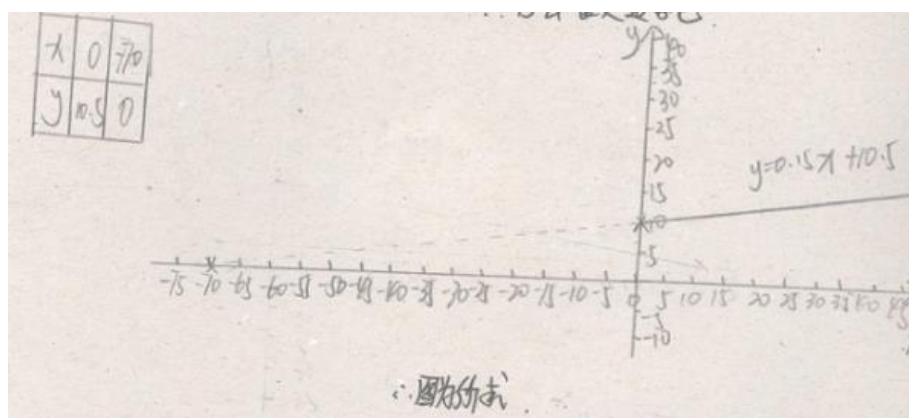
Figure 47. Partial understanding: example 3

Fourthly, a number of students (13.1%) noticed that the meaning of the domain was x>80 (see Figure 48). The graph lacked the domain for x between 0 and 80. The students' answers implied that the concept of the piecewise function comprised of two linear graphs, which caused difficulty for the Shanghai students.
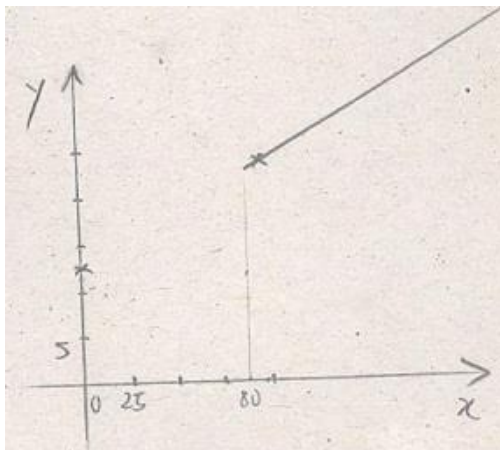
Figure 48. Partial understanding: example 4

In general therefore, the Shanghai students met challenges with the graphical representation. They also showed that they could solve the correct algebraic approach, without fully understanding the meaning. They generally ignored the meaning of domain and range in real world situation.

7.4 The relationship between the two types of tests

In this section, the relationship between the basic knowledge test and the application test will be examined. The first sub-section will initially explain the reasons why the relationship is necessary to explore, and later in Chapter 9, the relationship will be addressed again concerning the distinct teaching approach in each region. Based on this justification, the results of the pilot study will demonstrate whether there is positive or significantly positive relationship between what students know and what they can do with this knowledge, even though the positive relationship between PISA and TIMSS has been found as mentioned in Chapter 2, and at Curriculum Analysis chapter, the application model has been discussed

as the higher level of understanding required as one competency. Furthermore, data from the main study will illustrate which type of knowledge is more worthwhile in terms of understanding performance: the higher abstract levels of understanding, or the higher performance of application.

7.4.1 Why explore the relationship

During the discussion in Chapter 5: Curriculum Analysis, four competencies in application of function proposed by O'Callaghan (1998) illustrated that two competencies, translating and reifying, were related with how well students understand basic knowledge. To solve application questions requires basic knowledge understanding. The question is whether the higher level of understanding plays a crucial role in the higher performance of application. The application test from the Shanghai sample in the main study showed that Shanghai students whose understanding level were significant higher than the English sample might superficially apply knowledge by using algebraic method. It implies that too much emphasis on pure knowledge seemed to constrain students' application ability. On the other hand, the English students' understanding of pure knowledge hampered their performance in the application test. Therefore, the balance point between pure knowledge and application should be investigated.

The findings from the Textbook Analysis (Chapter 6) revealed that the arrangement of the content of linear function in the English textbooks was rooted in real-life situations, however Shanghai focused on rigorous knowledge first. That is, understanding mathematical concept started with making sense of a concept in the real world in England, while understanding stemmed from rigorous mathematical definition in Shanghai. If students finally achieved the structural stage through either way, namely at understanding Level 5, Object Analysis, and Level 6, Inventising, it is worthwhile to compare their performance in

an alternative way in order to discern which approach: starting with real world or rigorous definition, to introducing the topic is more effective for students' understanding.

7.4.2 Strong positive relationship

The relationship was investigated in two ways: between the total scores of basic knowledge and that of application, and between the understanding levels shown in basic knowledge test and total score of application test.

The relationship between the two tests' totals scores. In the pilot study, the relationship of students' performances on the basic knowledge test and application test was investigated using the Spearman correlation coefficient. With regards to the Shanghai data, there was a strong, positive correlation between these two variables, $\rho =.572$, n=907, p<.001, with a higher performance of understanding basic knowledge associated with higher ability levels in application. The English data provided confirmation of this statement, $\rho =.557$, n=403, p<.001. Data from England and Shanghai suggested a relatively strong performance relationship between understanding basic knowledge and application. In addition, the amount of shared variance for the coefficient of determination was 41.5% ($r^2 = .415$) in the Shanghai data and 32.9% ($r^2 = .329$) in the English data. That is, mastering pure knowledge seems to be more important for application performance in Shanghai than that in England.

The relationship between understanding levels and the total score of the application test. Besides the relationship between the performance of understanding and application, there was also a strong connection between understanding levels and application by Spearman correlation coefficient, with $\rho=.501$, n=907, p<.001 for Shanghai data. The English data showed the same conclusion, $\rho=.542$, n=403, p<.001, with higher levels of understanding associated with higher performance of application. R^2 showed how much of the difference in the dependent variables (application performance) was explained by the

understanding levels. For the English results, the value was 0.266; and for the Shanghai case, it was 0.289.

It was evident that the bivariate correlation between the performance of understanding test and the levels of understanding was straight and significant, ρ =.886, n=907, p<.001 for Shanghai data and ρ =.868, n=403, p<.001 for English data, as the tests were designed according to the general model of the understanding function.

7.4.3 Mastering basic knowledge or use of knowledge

In the main study, an additional investigation was undertaken to identify and examine the independent variable of students' performance in two types of tests. This further inquiry considered whether the higher understanding levels, as the independent variable, can determine the higher application performance, or vice versa.

The study explored the success of the students who reached Levels 5 and 6 in the application test, by calculating their average score in the application test. For example, the English students who achieved the understanding Level 5 were selected and their corresponding total scores in the application test were used to provide the average score of this group, which was 13.2. Using the same method, the Shanghai students whose understanding level was marked as Level 5 were picked out first and their average mark in the corresponding application test was then calculated (12.6). Due to different full marks in the two regions (24 for England and 22 for Shanghai), the percentage of their mean scores was compared, as 55% (13.2 ÷ 24) in England and 57.3% (12.6 ÷ 22) in Shanghai. The same method was applied for the Level 6 scenario as well. Figure 49 therefore shows the percentages of the Level 5 and Level 6 students' mean scores in the application test. Both the English and the Shanghai groups had very similar mean score percentages in the application tests at these two levels. It also indicated that, if students understanding was at Level 6, their

application performance was much better (nearly 15%) than those with an understanding at Level 5 (see Figure 49).
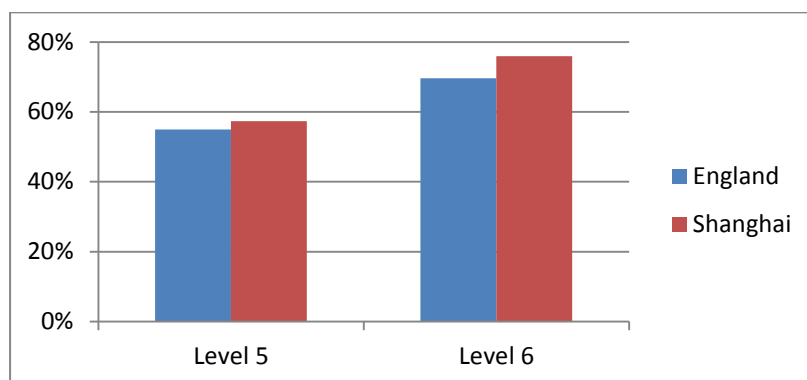
Figure 49. Percentage mean scores in application test with understanding Levels 5 and 6

On the other hand, higher performance in the application test, as the independent variable, was compared with the higher understanding levels. A total score above 80 per cent of the respective full marks was deemed to be a high performance. The per cent of the students who can achieve 90 percent at application test was only 5.7 in England, while 11.4 percent beyond 80 per cent. Due to lower proportion of 90 percent, this study chose the 80 percent as the benchmarking for higher performance at application test.

Those students whose score exceeded 80 per cent in the respective application test were examined. For example, 80 per cent of the full mark of English students in the application test was 19.2. The students who achieved a mark of 20 or higher were selected to discern their understanding level. Their corresponding understanding level was then identified (see Table 43 for the English situation). Due to the diverse number of students at each level, the identified number was divided by the total number of students in the corresponding understanding level. For example, there were 37 English students at Level 6 reaching the higher performance in application; and the whole number of English students in Level 6 was 117. The percentage, therefore, was 31.4 for understanding Level 6. The same method was also applied for the Shanghai situation.

Table 43

The English Situation for Higher Performance in the Application Test

	Not beyond Level 4	Level 4	Level 5	Level 6
Numbers of students who have the higher performance in application tests	2	0	7	37
The numbers of total students at each Level	185	14	87	117
The percentage	1.1	0	8.0	31.4

In the Shanghai situation, in terms of those students with higher performance in the application test, their understanding level was located either in Level 6 or in Level 5. Therefore, only Level 5 and Level 6 are highlighted. Figure 50 shows the distribution of students' understanding levels. Shanghai got higher percentage of students within the abstract levels compared with their counterparts in England.
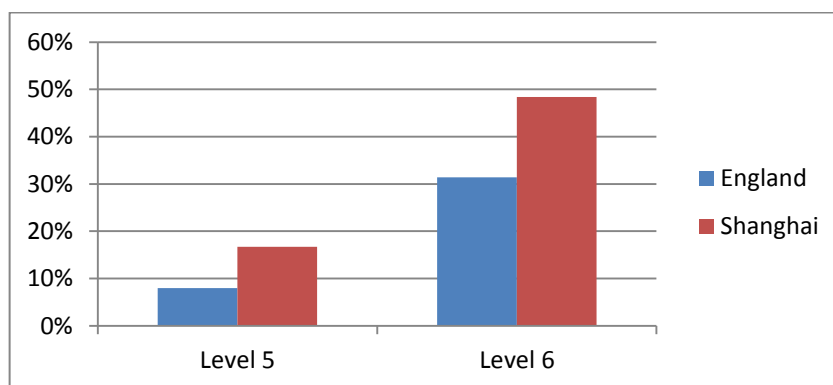
Figure 50. The understanding levels' distribution with the higher performance in application

In summary, the findings from the previous chapter, Curriculum Analysis, showed that the English curricula places an emphasis on more meaningful contexts related to real life in order to facilitate students' understanding of mathematics, while the Shanghai curriculum focuses on a more abstract understanding of pure mathematical knowledge. Figure 49 shows the similarly successful performances in the application test for students who had achieved abstract understanding in both regions. Conversely, Figure 50 reveals that students who attain a higher performance in the application test in the two regions have a more diverse

percentage within the abstract understanding levels, especially at Level 6. There is a disparity

between two regions. It indicates that the application approach does not facilitate the

understanding development towards the highest understanding level in a satisfactory way.

7.5 Discussion

Both the pilot and main studies of students' understanding basic knowledge tests

illustrate that, in general, the English students show a varying distribution of understanding

levels and Shanghai's performance demonstrates a more unified picture. This results comply

with that of PISA 2012 that Shanghai has the 'largest proportion of top performers' (OECD,

2013, p. 18), and the lowest percentage, '3.8%', for all the countries (OECD, 2013, p. 19).

In this section, first, each of the level's barriers that students encounter will be

summarised. Two examples taken from English classroom observations will be described to

show how students get confused with the property of gradient. The numeracy skills will then

be addressed for English students.

7.5.1 Understanding barriers at each level

Level 3. The Foundation Level English students have unsuccessfully translated

representations from the algebraic expression to a graphic representation despite being given

a table as a transmitting stage. Their numeracy skill is the first obstacle which appears before

learning the linear graph. This weakness embodies the difficulty of the ability to translate

from the algebraic expression to a tabular representation during the learning process.

Level 4. The graphic meaning of gradient is difficult for English students to

understand, even those in the Higher Level. This barrier is partly derived from the definition

in the textbooks: $Gradient = \frac{Differences\ in\ y}{Differences\ in\ x}$. During a classroom observation of this topic in

one sample school, two questions were proposed by students: one asked for the reason why

minus was used instead of plus to discern the differences of x or y; and the other was the

reason why it was a rule to calculate the gradient. The first question that he proposed shows

that he faced a big challenge in terms of understanding what the segment in the axis meant and how to measure this segment under the Cartesian plane. The second question relates to the mathematical explanation of steepness of the graph: the steepness is presented by the degree of the angle (θ) with one side as a line parallel with the x-axis and another side as the straight line. This angle can be calculated by $tan\,\theta$, one of the three trigonometric functions ($sin\,\theta$, $cos\,\theta$, and $tan\,\theta$) which are introduced in the KS4 English curricula for higher attaining students. Furthermore, the meaning of gradient is related to $tan\,\theta$. At students' current learning stage, these two concepts have not been related with each other. Only by asking the question of 'how', rather than 'why', are students unable to understand. This concept is based on an instrumental view of understanding for now, but in their future learning on trigonometric functions, the rule could be clarified by asking 'why'.

At another classroom observation in an English school, three different types of problems were used to discern the value of the gradient and to enhance students' understanding: (1) if given the equation; (2) if given the graph; and (3) when given no graph or equation, just two points such as *(a, b)* and *(c, d)* presented in the word problems. In terms of type (3), without the aid of a graph, most of the students went astray due to the meaning of 'differences' as shown in the rule. For example, two points without the graph, (-2, 2) and (1, 4), were provided by the teacher to calculate the gradient. The most common incorrect solution for the pupils in the class was $\frac{4-2}{-2-1}$ instead of the correct solution of $\frac{2-4}{-2-1}$ or $\frac{4-2}{1-(-2)}$. The underlying reason may be that students avoid the appearance of negative numbers in the denominator. An alternative reason might be that this concept came from the graph; but when the question is separated from the graph, there appears a difficulty for students to seek the meaning of difference in a pure word description. When facing a non-graph situation, students showed their confusion regarding the significance of the relationship between

'differences in *x*' and 'differences in *y*'. The English students could not note that there was a strict rule as shown in the algebraic formula $Gradient = \frac{y_1 - y_2}{x_1 - x_2}$ or $Gradient = \frac{y_2 - y_1}{x_2 - x_1}$.

In fact, over two-fifths of the English Higher Level students were unable to acquire Level 4 Property Noticing, as they failed to discern the value of gradient under a graphic representation; while half of them achieved Level 5 Object Analysis. The percentage of the students at Level 4 was quite small (6.7%) which indicates that this understanding level can be marked as a watershed in English Higher Level students' understanding.

The approach to linear function is dramatically different in England and Shanghai, especially when dealing with the concept of gradient. The symbolic method, as used in Shanghai, could be judged as more successful for students' learning outcomes rather than the graphic way used in England. Healy and Hoyles (1999, p. 83) have pointed out, however, that only when using symbolic aspects would students lose the opportunity to 'exploit the visual, to explain or justify their symbolic constructions, or to develop the capacity to move flexibly between representations'. Meanwhile, Huang (2002, p. 241) questioned that teachers in Shanghai who used 'abstract representation and logical reasoning' should use 'more visual and concrete representation to blend the process of learning'.

The English students approached the slope in the coordinate system, which involves both interpreting the graph and shifting from graph to equation. The core of understanding mathematics involves a 'flexible and competent translation back and forth between visual and analytic representations of the same situation' (Arcavi, 2003, p. 235). On the contrary, Shanghai students adopted an algebraic calculation that omitted the 'Cartesian connection' (Leinhardt et al., 1990, p. 36). That is, Schoenfeld's work, as noted by Leinhardt et al. (1990), claimed that the algebraic way to slope was fragile in its meaning. The effectiveness of the formula method, the generation of simultaneous linear equations, was observed in the

Shanghai students' tests. The far-reaching influence of the formula approach in the Shanghai pupils' responses was valuable to the investigation in the future study.

Level 5. Understanding function shifts towards the structural view since this level. Although the concept of linear function is based on input-output views of function in England, most of the Higher Level English students who attempted to answer this level's questions normally could provide the right answer which shows their full understanding of the linear function, as a change in the quality of their understanding development.

Level 6. The main barriers are rarely associated with understanding this topic. The barriers derive from other knowledge or the ability to identify the required information within the complex problem context for the Shanghai students.

7.5.2 Numeracy skills

The English students showed their weakness such as dealing with the negative number, reducing a linear function to a standard form, and an inadequate knowledge of gradient. In contrast, the essence of Chinese mathematics education emphasises that a solid foundation of basic knowledge with proficient numeracy skills is necessary before and while learning a new topic (Xu, 2010). The findings from Shanghai students confirmed their consolidated basic knowledge and skills.

Theoretical perspective. Taking the numeracy skills from a theoretical perspective, Breidenbach, Dubinsky, Hawks, and Nichols (1992, p. 279) indicated that students normally failed to construct processes in their minds for the concept of function and suggested 'de-encapsulating the objects and representing these processes'. In terms of process, Schwartz and Yerushalmy (1992, p. 263) argued that the symbolic representation could effectively lead students to make sense of the 'process' nature of function, while the graphical representation would result in the 'entity' nature of the function, i.e. the shape. During the process, in order to master symbolic representation, the symbol manipulator was required. Sfard and

Linchevski (1994) explained that function tied 'the arithmetical processes (primary processes) and formal algebraic manipulations (secondary processes)' together, and that both related to relational understanding. The numeracy skills of students, as the primary processes, were the primary aspect in the English student tests that constrained their understanding development which will be discussed in Chapter 9 in detail.

Foundation Level students. Figure 49 and 50 suggested that if a student's understanding of linear function reached Level 5 Object Analysis, both their visual and algebraic approach to a mathematical problem of either basic knowledge or application, could be identified as representing their successful understanding. This resonated with earlier findings of Stylianou and Silver (2004, p. 381), who noted that novices failed to use visual representations because they could not 'treat mathematical concepts as objects' from a reflective abstraction theory perspective. At this study, the Foundation Level English students had not yet developed a rich structure of mathematical concepts; as a result, they could not progress in either their understanding of basic knowledge or application via a visual approach.

7.6 Summary

This chapter has provided a thorough exploration of and the delineation between English and Shanghai students' understanding of the concept of linear function. Although the age group of the Shanghai sample was one year younger than the English sample, they surpassed the English students in both the pilot and main studies. One possible reason for this discrepancy is that the Shanghai students are used to more abstract methods, primarily the algebraic approach. In addition, the correlations between the higher level of basic knowledge understanding and higher performances in the application test were relatively higher. These correlations also suggested that an ideal understanding of basic knowledge should achieve the Object Analysis level.

The next chapter will investigate the selected teachers' perceptions of the process of teaching and learning linear function, focusing especially on: what teachers view the role numeracy skills play in students' understanding; and what kind of barriers the teachers identify in the students' learning of linear function.

Chapter 8 Teacher Interviews

This chapter will present the views of the selected teachers in order to perceive the reality of the classroom, focusing on the possible disparity between the intended curriculum and the attained curriculum, as well as exploring the ways in which students' understanding was shaped. First, the details of the teachers' semi-structured interviews will be addressed, including the outline of the interview content, limitations of the chosen method, and the analysis of the qualitative data. Secondly, two parts of the findings will be explored: the first issue focuses on how the selected teachers intended to carry out their lesson plans; the second issue will look at how they facilitate their students' understanding development based on their experience of the potential barriers to students' understanding of linear function. In terms of the former element, the interview starts with a set of questions related to how they plan a lesson. In looking at this issue, underlying their individual interpretations, teachers' cultural beliefs concerning the process of teaching and learning are described and compared between England and Shanghai. With regard to the second issue, it will also be necessary to develop a clear perception of English and Shanghai teachers' cultural perspectives concerning their definition of understanding. Thirdly, three points: the teachers' beliefs about mathematics, the analysis of their pedagogical content knowledge, and the function of textbooks in each region, will be further addressed in the discussion section. Ultimately, it is hoped that teachers' responses in the interviews will clarify how the description of their beliefs aligns with the findings from the study's previous document analysis and the results from the students' tests which will be discussed in the next chapter: Summary and Discussion.

8.1 Semi-Structured Interview

In the Methodology chapter, the reason for using semi-structured interviews and the ways in which these interviews were conducted in England and Shanghai were both

described. This section focuses on outlining the semi-structured interview process, some of the limitations of this method, and the process of data analysis.

8.1.1 The outline of the semi-structured interview

The questions used in the semi-structured interviews were designed to explore two important issues: how teachers approach the teaching of linear function; and the perceived barriers that impede students' understanding of linear function. When considering the specific beliefs that teachers hold, the possible underlying cultural differences on viewing teaching and learning between England and Shanghai will be explored. Within Part 1, as the role of the textbooks differs between English and Shanghai contexts, the teachers were asked to express explicitly their views on how mathematics should be taught. It was followed up with the question on how mathematics should be learned. Part 2 focuses on the perceived barriers to students' understanding linear function, and the teachers' views of understanding will be examined. All of the questions below were included in the interview:

Part 1: The teaching of linear function

- How do you plan lessons?

- Which curriculum materials do you use most regularly? Do you follow the textbook(s) closely?

- In your opinion, how can mathematics be best taught?

- And how can mathematics be best learned?

  Part 2: The perceived barriers to students' understanding of linear function

- What are the main barriers when students learn linear function?

- What are your views on how to aid students' understanding of linear function?

- Many people believe that, in order to learn mathematics, understanding is essential. What do you think?

- How do you define 'understanding'?

8.1.2 Limitations of the semi-structured interview

Qualitative research is considered to have some weaknesses, such as problems of representativeness and generalizability of findings, and problems of objectivity and detachment (Sarantakos, 1993). There were two main concerns regarding the use of semi-structured interviews in the present study. First, the interview is limited by its scope and sample. It was confined to a small scale, with a total of seven teachers. Although their views could represent the main values or culture of mathematics teaching and learning in the sample schools, the findings were not representative of all teachers' beliefs, neither within the sample schools nor in both regions.

Secondly, Cohen et al. (2013) suggested that the practical way to achieve a valid interview is to minimize the possible impact of researcher and respondent bias in order to address the issue of objectivity. Although this point has been generally discussed in Chapter 4, I feel it is also necessary to address it especially from the perspective of the teacher interviews. The main concern for the Shanghai interviews was that interviewees would regard me as an expert or a peer, and assume that I hold certain opinions about these questions to expect specific answers. My working experience in a secondary school in Shanghai might influence Shanghai interviewees' responses. I also had a personal connection to all the sample schools in Shanghai. In order to minimize this potential bias, the interview started with specific questions related to a previous classroom observation, such as how to choose examples during a lesson.

In England, the teachers were aware of the goals of this study and my teaching background in Shanghai secondary school. The primary concern in this situation was that they might hold a reserved attitude towards the interview. In order to overcome this obstacle, I needed to build rapport, to ensure and reassure the teachers of the confidentiality of their responses, and to establish trust. Before the interviews, I observed at least one class in each

sample school, mainly with the Head of Maths in each school. Even after the interviews and tests which had been conducted, I continued classroom observation in order to gain a fuller understanding of the teaching and learning processes in the English schools. Although the interview data would be compared with quantitative data from the tests, which could balance the bias, during the interview I always asked for clarifications, including examples of statements made by the interviewee.

Finally, due to the limited interview sample and focusing merely on certain topic, linear graph, the results of what kind of software English teachers would use or how often they would use might contradict with other researches. For example, the Bretscher's survey (Bretscher, 2011) illustrated that (1) interactive whiteboards would be the most accessible hardware in England; (2) English teachers do not routinely use graphing software; and (3) MyMaths is frequently used source. However at this study, the teachers did not mention the interactive whiteboards and in terms of this topic, the software was frequently used to explain the meaning of gradient.

8.1.3 Data Analysis

Cohen et al. (2013) suggest four stages in the analysis of interview data: (1) generating natural units of meaning; (2) classifying, categorizing, and ordering these units of meaning; (3) structuring narratives to describe the interview contents; and (4) interpreting the interview data.

I adopted this four - phase process to code and analyse the transcribed data. The first phase was an open-coding approach in order to develop categories for each part and to find all of the themes emerging from the data using a paper-and-pencil method. The second phase was to re-examine all of the data using a start list of codes by the Nvivo 10 software, which could easily retrieve the context (see Appendix M for two examples of a section for English and Shanghai interview transcript respectively). The purpose of using Nvivo instead of a

manual check again was to classify and re-examine if these codes covered everything. During this phase, the reappearance of specific themes, the similar and different points mentioned by the teachers, were noted. In the third phase, four themes were identified: (1) views of lesson plans; (2) views of teaching and learning; (3) views of the definition of mathematical understanding; and (4) barriers to students' understanding of linear function. Finally, I compared the similarities and differences between England and Shanghai teachers' views in each theme, and then read back through the transcripts to ensure that the findings reflected what the interviewees said. Meanwhile, the results of the question on predicted students' barriers would be linked back to the model of understanding function in the next chapter.

8.2 Findings from the Teacher Interviews

Four themes were identified from the teacher interviews as discussed above. Apart from the third theme, namely the views on the definition of mathematical understanding, the remaining three themes will be examined below in turn by firstly exploring the English and Shanghai results separately, and then comparing the two. In terms of Theme 3, teachers' comments will be compared from the start, rather than firstly being explored separately. Teachers' words will be used to support the existence of these themes, and I will use the following protocol in referring to teachers: the three English teachers will be referred to as ENT1, ENT2, and ENT3; and the Shanghai teachers, SHT1, etc.

8.2.1 Theme 1: Views of lesson plans

It is important to obtain teacher views because they 'always bring their own frames of understanding and their knowledge of the local context to bear on how they use curricular materials' (Stein, Remillard, & Smith, 2007, p. 324). All of the interviews were followed by a classroom observation. The process started with the teachers' familiar daily scene - the planning of the lesson. Teachers' responses to this part were used to identify patterns in

lesson planning. From their responses, teachers' primary consideration in lesson planning could be revealed.

English teachers: focusing on students' ability. *Point 1: Starting with the Scheme of Work.* The Scheme of Work written by each department in English schools indicates what should be taught and when it should be taught in the current academic year. The ENT3 gave a clear explanation of the function of the Scheme of Work:

> So, a Scheme of Work means that all the staff - all teachers - follow the same order of programme… it allows students to move between classes.

For each level of students, rough learning objectives were listed in the Scheme of Work, as the ENT2 stated:

> Within the same topic, [for example] equations, we have three columns which say that most kids, children, should learn this. So, for example, children in lower sections should definitely learn this, pupils in higher sections should learn this, so we have three different sections within each topic. So if you have a lower class, you know the type of things you have to do. You might then want to help them to be able to do the next section, so it's not laid down. When you have a class, it does not say Set 5 does this, Set 4 does this. It just says that the main part of Year 8 will do this, and if they're very bright they will do this or if they're very slow they'll do this.

The Scheme of Work also provided teachers with 'other documents, their links to resources, and links to textbook pages they can use to help them plan it' (ENT1). These links were only suggestions for teachers to use in the lesson plan.

Point 2: Students' ability. The expectation of achievement within a certain level was already factored into a teacher's lesson plan, but in quite a basic way. The teacher had to consider what challenges students would meet and how to allocate lesson time. Developing students' ability was the teacher's priority.

> And as a teacher, you would tailor it to your particular class. For example, I would look at the Collins that has 'C' grade working because that is where my students are targeting. And I

would look at the 'C' grade topics and I might go through it and write on it that they couldn't

do this very well, or I need to do another lesson on this, or whatever it might be. (ENT2)

Point 3: Looking for teaching materials. Results from the interviews showed that

apart from the Scheme of Work, there was no mandatory resource that teachers must strictly

follow in the English schools. Many open resources were available for teachers, enabling

them to choose any that teachers thought would help their lesson planning. The selected

English teachers tended to use online searches including: (1) online websites such as the well-

known TES site: http://www.tes.co.uk/, or Boardworks: http://www.boardworks.co.uk/; and

(2) worksheets designed by teachers based on various materials. Or they will make their own

worksheet (ENT3). When facing topics related to graphs, a commercial graph package such

as *Autograph* was a popular choice. In summary, teachers had the flexibility to structure their

own lessons, as long as it met the requirements for their students located within various sets

or levels.

Notably, textbooks played a very limited role in the planning of lessons, e.g.:

It is not linked at all with the textbook (ENT2)

Textbooks we don't use as resource here (ENT1);

The ENT3 used textbooks for a 'particular lesson' such as 'drawing graphs'. The

reason that teachers dismissed the textbooks was that they were not satisfied with the

structure of the textbooks provided for the topics: 'because textbook lessons tend to be quite

dry, quite less discussion-based' (ENT1). In terms of linear function, the textbooks'

arrangement showed that 'the strength is there are plenty of different types of questions. But

there are not necessarily enough of the same types for students to practice. So there is not

enough to consolidate, it moves on quite quickly' (ENT1).

Shanghai teachers: The requirements of the curriculum. There were three

documents officially provided to in-service teachers in Shanghai: the local curriculum; the

compulsory textbook; and official teaching guidance. The teaching guidance included

detailed information about learning objectives, the emphasis, and the difficulties students might encounter for each lesson. The Shanghai teachers would carefully study the textbook before planning the lesson, as the SHT3 argued:

> First of all, we will identify what kind of position this class has in the whole topic, students' previous knowledge, and after this what new knowledge students will learn.

Then, regarding a certain class, the SHT1 said:

> Teachers basically focus on textbooks to understand the mathematical content.

The textbook not only provided the content; the examples also embodied the requirements of this content in detail. The SHT2 gave an explanation of how to work out the requirements from examples in the textbook:

> When reading the textbooks, we need to understand the writer's tendency. For example, in terms of rational number addition, there are two examples in textbooks,
>
> （1）16+（-25）+24+（-32）, （2）$0.125 + 2\frac{1}{4} + \left(-2\frac{1}{8}\right) + (-0.25)$
>
> The trick of the first one is to calculate the positive number together while the second one is to add 0.125 to $2\frac{1}{8}$, and add $2\frac{1}{4}$ to -0.25, so that the fraction part could be counteracted. Every example has its purpose which needs to be found out in planning a lesson. Textbooks would not remind us. But teachers must know about the purpose of each example in the textbook. (SHT2)

Although teaching guidance contained the specific difficulties and emphasis for each class, the SH2 commented that new teachers tended to read the teaching guidance carefully; while experienced teachers had profound knowledge of the guideline content, and they would focus on other materials such as previous assessments, as the SHT1 said:

> We focus on the objectives, difficulties and key points (in the teaching guidance). According to these requirements, teachers will find out some examples which could be in a textbook or from previous assessments in order to test if students could achieve these requirements.

A comparison of the two regions' way to lesson plan. This sub-section described the ways in which mathematics teachers planned their daily lessons in England and Shanghai. It was found that the planning of lessons in both regions had a particular context: the English teachers were allowed to be flexible to schedule the lessons within the topic, while the Shanghai textbook or teaching guidance provided the basic structural context that the teachers were followed with. When narrowing to a certain topic, the lesson plans in each region had a very similar format: firstly looking for the objectives of a lesson; then deciding on the content of a lesson in order to achieve these objectives; and finally identifying the suitable teaching materials for a lesson (see Table 44).

Table 44

A Comparison of Two Regions' Way to Lesson Plan

Structure of lesson planning	England	Shanghai
Objectives for a lesson	Different objectives for different level students in the Scheme of Work	The same objective for all students in textbook or teaching guidance
The content of a lesson	Decided by teachers according to their students' ability	Described by textbook or teaching guidance
Teaching materials for a lesson	Open resource, such as website and worksheet	Mainly textbook

Within this similar structure, the following differences were noted:

- In relation to the function of textbooks in lesson planning, the English teachers seemed to hold negative attitudes towards the textbooks, even though the school and department had considerable autonomy to choose a suitable textbook. Meanwhile, the Shanghai teachers relied heavily on the compulsory textbook and teaching guidance. The findings from England clashed with one assumption of the usefulness of textbook analyses research that 'teachers of mathematics in all countries rely very heavily on textbooks in their day-to-day teaching' (Robitaille & Travers, 1992, p. 706).

- From the requirements or learning objectives perspective, England had a multi-level structure for the different abilities of students, whereas Shanghai had a uniform one-size-fits-all approach.

- With regards to the presentation of a topic, the English teachers would consider students' ability first, what they can do or cannot do, and often intensively seek the help of open resources. The Shanghai teachers would initially study the requirements that the textbook presented or questions in previous assessments which embodied the requirements of the intended and attained curricula. Thus, it was evident that students' ability had the priority in England, while the requirements of the curriculum including textbook was emphasised in Shanghai.

- In the case of linear function, the English teachers frequently used commercial graph packages, as Patterson and Norwood (2004) recommended that the display of visual representations (tabular and graphical) in technological tools may assist in learning the various concepts.

The striking difference of the function of textbooks between England and Shanghai was indeed affected by each country's education policy as discussed in Chapter 2. Furthermore, it is necessary to discover the underlying cultural beliefs that the teachers possess. The following section explores teachers' cultural perspectives on how mathematics should be best taught and learned.

8.2.2 Theme 2: Views of teaching and learning

The objective of this section is not merely to document teachers' perspectives on the process of teaching and learning in the two regions, but also to indicate whether these models fitted with teaching and learning patterns in England and Shanghai. This section's findings will explain the results from the previous section, views of lesson plans.

Based on the specific questions about lesson plans, I explored teachers' general approaches to teaching and learning. Teaching and learning cannot be separated into two distinct parts. In terms of a particular question on how students could best learn mathematics, the answers may reflect how teachers view effective teaching methods from other perspectives. However, teachers' responses might be limited by the previous question of how they planned the lesson for learning new knowledge and what kind of materials they used frequently. Therefore, their replies mainly focused on how to teach a new mathematics topic rather than from a broader background, for example how to do the revision.

Two teachers from both regions agreed that the premise of successful teaching was that students had basic foundation knowledge for a new topic. The ENT1 stated that the ideal foundation for teaching and learning a new topic was that students already had a 'consolidated basis':

> Basically when you teach a new topic, it is important you go, you make sure students have the basics; if I am teaching straight line [linear graph], making sure that they can choose numbers to make a table, and make the negative numbers, that they could plot the coordinates, make all the basics before you go on to the next step.

This view indicated that English teachers highly valued relevant previous knowledge and skills before learning new mathematical knowledge. That does not imply that the English teacher would pursue students' accurate numeracy skills, namely the proficiency of skills. On the other hand, Shanghai teachers emphasised the need for students to obtain the two basics: basic knowledge and basic method in the teaching process.

English approach: learner-focused with multiple activities. *Point 1: Best taught.* With the flexibility to structure their own classes, the English teachers placed emphasis on designing multiple activities in the classroom for students, as ENT3 said:

But I honestly believe that a mixture of visual aids, fun activities and written work - because I think a combination of everything works best…

The ENT2 preferred to use visual verification for students' discovery, especially in the linear function topic:

If we do gradient, for example, we will tell pupils how to set it out but not what they'll find, so we ask them to plot $y = 2x, y = 3x, y = 4x$… what do you notice, are the lines getting steeper? They discover then the number in front of the x, the meaning of gradient. You do not tell them that is the gradient; you let them discover it's the gradient as that number gets bigger. They often write down what they notice - that as the number gets bigger, the line gets steeper... Then you would do a set of ones in which the intercept stays the same, $y = 2x + 3$, $y = 3x + 3, y = 4x + 3$, so the gradient changes but the intercept stays the same.

The teachers would keep 'questioning students' (ENT1) during these activities in order to probe any confusion:

…making sure also when you are teaching them, you are not just telling them how to do something, you are getting them to tell you with great questions. (ENT1)

It was found that all teachers emphasised students' mathematical exploration with various activities rather than a clear structure of lessons or effective instruction which was one assumption of a classroom-focused view in Table 5 at Chapter 2. From the interview responses and classroom observations, it was evident that the teachers in England paid great attention to the visual approach, especially graphical representations, when introducing the concept of linear function. They were all aware that, as teachers, they should avoid adopting the role of instructor or explainer. The goals of these activities were viewed as ultimately to be able to uncover students' inadequacies in their understanding through students' feedback in the classroom, rather than teacher's dominating by monitoring and correction.

Point 2: Best learned. In terms of teaching beliefs, the English teachers held a relatively positive attitude towards the function of practice; for example, the ENT3 argued that:

> I think… by them doing it - I think with maths, the more practice you have, the better you get. I honestly believe that. While it is important to hook them into a topic, you also need to have the practice as well.

This perspective is closely aligned with that of Cai and Wang (2010, p. 275): 'USA and Chinese teachers see practicing as a key to consolidating knowledge and facilitate understanding'.

Meanwhile, it is reasonable to assume that a 'lack of time' (ENT1) had affected the efficacy of teaching and learning in England:

> We only have three hours Maths in the school per week and one hour homework. So one week, just 4 hours for Maths. It might be fair enough they are forgetting. It is hard, they do basically need time. (ENT1)

Apart from the importance of practice, the English teacher mentioned 'having a consolidated teacher' (ENT2) for students' best learning:

> Give them a clear explanation of what's going up, not just saying do this and do it now, actually explaining to them. I am going to put this here, why I am going to do that, because I explain. It is really important, I think. Students like that. And also to give students the opportunity to ask questions and not feeling stupid to ask questions. Making mistakes is important; they could deal with that and learn from it. Have a consolidated teacher in classroom. (ENT2)

From this statement, the teacher acted as facilitator, not only providing students with answers to their questions but also explaining mathematical processes.

Shanghai approach: basic method – focused. Teacher-dominated classrooms and rote learning were two common western misconceptions of teaching and learning under Confucian-Heritage Culture (CHC) for Chinese students (Biggs, 1998). Huang and Leung

(2005) argued that teacher-centric and student-centric approaches were not dichotomous but integrative to the reflection of the characteristic of CHC mathematical classrooms. However, Kuhs and Ball (1986) suggested that Shanghai classrooms tended to mainly concentrate on content.

All of the selected Shanghai teachers described 'teaching with purpose' as the best method for teaching. The 'purpose' initially stemmed from the content or requirements of assessments which were based on that of the curriculum:

> There are two approaches. One is the traditional teaching style. Based on knowing what students have already known, teachers express new knowledge using the language and examples which students could accept, skilfully and patiently. This method could pass on lots of knowledge. The key is if teachers' expression is clear and students could digest… Teachers should think about certain strategies and methods in advance, aiming at this topic, based on students' ability. In fact, under the classroom teaching system (mix-ability students), teachers barely control the progress to meet every student's situation. Therefore, teachers should choose the proper methods including mixing these two methods. This way of tailoring the teaching approach hardly comes true. But for some students who have more difficulties, teachers could mobilize the initiative of good learners (their classmates) to help them to meet the requirement. (SHT1)

> Teaching with purpose means teachers have a certain purpose to teach what kind of concept or idea, for example, understanding concept or mastering properties. Teachers pay attention not only to mathematical knowledge, but also ideas, how to introduce a basic method as combination of symbolic-graphic, or classified discussions through examples. (SHT3)

The SHT3 explained that the best way to learn mathematics from a student's perspective was by 'learning consciously' which means that students must be aware of what kinds of requirements they should meet:

> Learning consciously means that students initially learn the lesson by themselves from the textbook before being taught the content by the teacher. They could therefore make sense of

the new knowledge when teaching occurs in the classroom. Of course, not all of the students could achieve the requirement, but high ability students could do that. (SHT3)

Teachers divided mathematics into two categories: basic knowledge; and basic methods (basic ideas and basic methods). The former, basic knowledge, could apply to the teacher-centred and students-centred teaching approaches; while the latter, basic methods, could only be taught by teachers:

Teacher-centred does not mean cramming education. For example, when teaching how to calculate $|-x| = 3$, teacher clarifies to regard $-x$ as 'the whole'. If the teacher did not indicate that guidance, students might have the right answer by only depending on feeling and experience. However, students are not clear about basic ideas and methods. When meeting $|x - 3| = 3$, it requires students to consider $x - 3$ as 'the whole' as well. The idea and methods are from the teacher's guidance. (SHT2)

Basic ideas and methods need teachers to introduce them. For example, the basic method of combination of symbolic-graphic is not clearly expressed in the textbook. Students could not know it from a higher level when looking at the examples. Teacher-centred is more important. Maybe one or two classes could be open for students' exploring, but it still needs to be controlled. It also depends on the average student's ability level in the class; sometimes it would not have a good effect. For teachers, we could enhance the teaching effectiveness, choosing some typical examples, focusing on teaching design, but it needs students to be actively learning. Teachers and students working together is a key to improve the learning outcome. (SHT3)

From the teachers' view, these methods could not be learned or discovered by students themselves. The teacher had to control what the students were supposed to understand or learn by his or her careful lesson planning towards the basic methods. Meanwhile, it was verified that the requirement of the curriculum focused on basic method as discussed in the Curriculum Analysis chapter. The selected Shanghai teachers therefore held a negative attitude for the effectiveness of students-centred approach to teaching and learning

mathematics:

> Student-oriented definitely has much more benefits but is not effective. Students' inquiry might give rise to taking detours. At the end, knowledge might not be investigated thoroughly. In fact, it is impossible for students to go through so much mathematics knowledge one by one. (SHT1)

On the other hand, because basic methods were unreachable, highly qualified experienced teachers who mastered these basic methods were crucial to meet with the precondition of best teaching.

There are two ways to support in-service teacher professional development in Shanghai: the mentor system working within the school for new teachers and the research-teacher system in the subject area at district level for all the in-service teachers. The mentor system for new teachers in Shanghai ensured that every new teacher could get support from the senior teachers. One of these supports was about mathematical professional knowledge, such as which basic methods should be delivered to students in a certain topic and how to make sense of the worked examples in the textbook. Meanwhile, another researcher-teacher system for professional development provided all the in-service teachers with a communication platform and opportunities to gain advice from experts or colleagues in other schools. That is, highly qualified teachers were regarded as playing a significant role in the best teaching. It was assumed that students' learning outcomes heavily relied on the performance of their teacher, or how profoundly the teachers could help these outcomes. As a result, this opinion could reflect how teachers perceived the best way to learn mathematics, 'depending on teaching' basic methods (SHT3):

> During the learning process, a student is required to master certain mathematical methods. For example, with regards to the concept of absolute value, the basic method of classified discussion needs to be taught. If students did not master this method and focused on practice, it would not be enough. Learning mathematics depends on students themselves, especially for

junior secondary school students, which will be unpractical. The mathematical methods are taught by teachers. Mathematics content or knowledge could be learned by students superficially. (SHT2)

Students' interest was also regarded as a foundation of the best learning. Learning processes can be seen as to 'explore the unknown', so that students were required to develop mathematical 'ability' for future study.

During the teaching and learning process, teachers should cultivate students' ability, as knowledge is a kind of medium. Knowledge is explicit which could be testable while ability is implicit which could not be reflected at the current stage. (SHT1)

Based on the Shanghai teachers' responses to the best methods for teaching and learning mathematics, the following main findings were identified:

- After the teachers carefully studied the curriculum materials, including textbooks, teaching guidance, and previous assessments, they separated the content into the underlying basic methods to handle a certain topic.

- The Shanghai teachers directly led the teaching of basic methods in the classroom because they believed that basic methods could only be taught by teachers.

A comparison of the two countries' views of teaching and learning. Simply labelling classrooms as teacher-centred (teacher-dominated) or students-centred (learner-centred) cannot explain the whole picture in terms of patterns of interaction, especially in East Asian classrooms (Chee Mok, 2006). Further, Mok indicated that the content-oriented and teacher-dominated image in Shanghai classrooms was not negative for learning, because the teachers valued the students' responses, and it actually was an 'alternative form of student-centredness' (Chee Mok, 2006, p. 141). Table 45 shows different views on teaching and learning mathematics within a cultural context for the two areas.

Table 45

Views of Teaching and Learning in the Two Regions

	England	Shanghai
Teachers' beliefs	Learner-centred with multiple activities	Content-centred emphasis on performance as well as conceptual understanding
Teachers' role	Facilitator and explainer	Instruction and explainer
The use of curricular materials	Provide Learning Objectives	Mainly follow curricular materials enriched with additional problems
Learning mathematics	Active construction of understanding model	Active construction of understanding model for mathematics knowledge, but reception of mathematics method

Essentially, the English teachers believed that active exploration was the best way to learn, while the Shanghai teachers preferred direct instruction and skills practice, because their views of teaching and learning for mathematics were different as it will be further discussed in the next chapter. These two opposing approaches would lead to different pedagogical intents (Stein et al., 2007) as addressed in the point: *pedagogical content knowledge* in the discussion section.

8.2.3 Theme 3: Views on the definition of mathematical understanding

Hiebert and Carpenter (1992, p. 67) defined understanding as a way that 'information is represented and structured'. Teaching or learning with understanding has been widely acknowledged by the mathematical education community. Llewellyn (2013) argued, however, that the definition of understanding within the education system slightly differed from that of the research area. That is, the research area considered understanding to be important; however, although in a practical way understanding might be deemed desirable, mastering of skills would be more highly valued to enhance students' achievement in assessments. The following sub-section shows the definition of understanding in practical teaching from the teachers' perspectives.

The results from Table 46 revealed that the England and Shanghai teachers' conceptions of mathematical understanding were closely associated with their respective views on of the content of mathematics.

Table 46

Views on Mathematics Understanding

Understanding mathematics	Views of English teachers	Views of Shanghai teachers
A process with understanding how and why	Agreed	Agreed
Linking with other knowledge	When introducing new topic	New topic beginning and finishing
Procedural knowledge	Not part of understanding	Part of understanding
How to probe students' understanding	Speaking	Written work
The nature of mathematic knowledge	Not mentioned	Four basics; Three languages translating

What did understanding mathematics mean to the teachers? Answers to this question varied, depending on whether opinions were taken from a mathematical or educational perspective. Bishop (1992) examined the relationship between mathematics and education within mathematics education research, an exploration that requires more consideration than has been previously given. Each contending attitude towards the identity of mathematics tends to consider teaching and learning from different directions. In general, the Shanghai teachers primarily focused on the nature of mathematics. All of the English and Shanghai teachers agreed that learning with understanding is essential, but held different views on mathematical understanding. These views embodied their attitudes towards procedural knowledge and the ways in which they examined students' understanding.

Point 1: As a process. The two groups proposed that understanding would be a process instead of a one-off event. The ENT3 stated:

I think time tells as well. So you know, one lesson does not always say everybody understood everything. But over a series of a fortnight, you would expect them to see that understanding as well.

As for a specific topic or problem, the development of understanding combines the reason, 'why', and the process, 'how'.

Point 2: Linking with other knowledge. The capability of linking to other knowledge can be identified as understanding, which matched with the highest level of the general model of understanding. The two regions, however, had different ways of articulating the act of linking knowledge. England tended to connect knowledge when introducing a new topic, while Shanghai focused on 'the beginning' and 'the after'. Especially after finishing a new topic, Shanghai teachers would deepen students' understanding by giving them complex problems in order to strengthen their knowledge links.

Point 3: Instrumental understanding and relational understanding. Conceptual knowledge is defined as 'a connected web of knowledge', while procedural knowledge means formal language including symbolic representation and algorithms (Hiebert & Lefevre, 1986, p. 3). The English teachers focused on understanding conceptual knowledge rather than procedural knowledge. That is, instrumental understanding was not part of understanding as ENT1 did not believe that mastering the procedural knowledge could be part of understanding because students were told to do it. She provided an example:

But I am convinced that there are still some techniques that just have to be learned. For example, division of decimal, 6.38 divided by 2, got 3.1, then 18, but this is not really 18. They are not really doing it from understanding.

Learning procedural knowledge was considered by the English teachers as either knowing or not knowing. Alternatively, the Shanghai teachers were deeply influenced by the four basics (basic knowledge, basic skill, basic method, and basic experience of fundamental action which will be discussed at Point 5), so that in Shanghai the subject of mathematics

consisted of these basics. Therefore the basic skills or procedure knowledge is much more highly valued in Shanghai than England.

Point 4: Speaking or writing. In order to probe into students' understanding, the English teachers preferred to question them, especially through students' oral expression:

> They could explain it back to me, how and why and they can spot the steps. If they can, their understanding is good. If they cannot, they do not understand it. (ENT1)

> By questioning them, for example "tell me where you got the answer from", "talk to me about your solution, where you get it", "why is this?" - Lots of questions. (ENT2)

> I think through questioning them mainly, so asking them and having that discussion going between them. So there were two girls at the back who, when one of them got the answer and the other one said "I do not understand how you got that", I said "right, you explain it to her". They will then go away and explain it to each other and that demonstrates their understanding. (ENT3)

On the contrary, the Shanghai teachers preferred to check students' written work because of the more abstract level of requirement for mathematical concepts:

> Mathematical concepts normally are quite abstract... Doing (solving questions) could make sense of it. Students will have experience. When they correct their mistakes, they will be close to understanding well. (SHT3)

> When students are doing, we could give some comments. (SHT4)

The Shanghai teachers also emphasised application:

> If students could abstract the real world problems into mathematics problem, they could be regarded as understanding. (SHT2)

Chen and Weng (2003) suggested three levels to probe into students' understanding: (1) the sign of *understanding* was that students could explain mathematics knowledge through speech; (2) the sign of *exact understanding* was that students could solve problems, namely through judgement, calculation, deduction, and proof; (3) the sign of *deep understanding* was that students could apply knowledge to real world situations. According

to this classification, the English teachers tended to use speech to confirm students' basic comprehension, while the Shanghai teachers placed more emphasis on *exact understanding* and *deep understanding* of mathematical knowledge. In Shanghai, paper-and-pencil was a primary approach to examining students' understanding, so that 'reading ability' was proposed by the SHT4 as a crucial element of understanding, especially reading long word questions.

Understanding was also highly valued by the education authority in Shanghai. The Shanghai Teaching and Research Office proposed to 'focus on dialogue in classrooms, and a promotion of students' mathematics understanding' as the emphasis of year 2014, while in 2012 the theme was an 'emphasis on mathematics reading, and to enhance understanding ability'. Therefore, the speaking or communicating maths has become much weighed.

Point 5: The nature of mathematics knowledge. Three of the four selected Shanghai teachers defined their understanding of the nature of mathematics when asked 'what mathematics understanding is'. The SHT2 believed that 'it needs to understand two basics' (basic knowledge and basic skills), while the SHT1 mentioned 'four basics'. Wang, Wang, and Wang (2008) explained the relationship between the 'four basics' (see Figure 51), that the above two basics and the basic experience of fundamental action would be refined as the basic mathematics method.
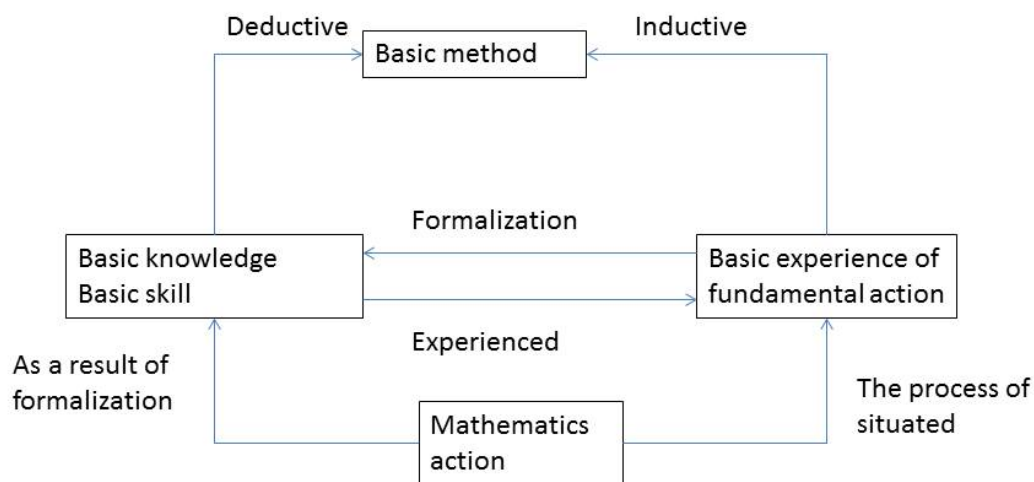
Figure 51. The relationship between the four basics

Additionally, the SHT2 expressed her unique understanding of mathematics knowledge, which consists of translating three languages. The understanding or learning of mathematics involved the translation of these three languages. She also thought that this understanding could address students' main barriers in the comprehension of linear function:

Actually, learning mathematics is to translate three kinds of language: word, symbolic and graphic. For example, 'absolute values' is for word; '| |' is for symbolic language; and graphic (geometry) language is to show the distance between this point and origin (zero). These three languages should be combined together (to understand). Once seeing the word, students could reflect in symbolic and the distance in the number axis between two points; or seeing the symbolic representation, students could think about the word and the number axis. If students could figure it out clearly, mathematics would be easier for them. Another example, if seeing $a > 0$, students could understand it as a positive number in terms of word, and in graphic representation it is located at the right side of zero. If three languages could be used well then it would not be a big problem for students' learning. I think mathematics understanding is to translate three languages.

8.2.4 Theme 4: Barriers to students' understanding of linear function

The results for this theme are presented in three parts: the predicted English students' barriers to understanding, the predicted Shanghai students' barriers to understanding, and connecting these barriers to the model of understanding function. The teachers' views on what caused these barriers to understanding were interwoven with their descriptions of what these barriers were. Furthermore, their predictions would determine the direction of the teaching they intended to promote, in view of facilitating students to overcome these understanding barriers.

English students' barriers to understanding. The English teachers summarised students' barriers from three perspectives: (1) previous related knowledge and skills; (2) connecting representations; and (3) the meaning of gradient. 'Forgetfulness' as shown in the English student test where one student commented 'cannot remember it' (see Figure 28 at Chapter 7), and 'fear of algebra' (ENT1) were also described as barriers to understanding by the selected teachers, but would not be discussed or compared with each other as not fitting with the purpose of this study.

Point 1: Previous related knowledge and skills. In identifying the students' barriers to understanding, the first thought of all three English teachers was about the students' previous related knowledge and skills, as ENT3 pointed out:

> I think linear graph is a very accessible topic. I think students of all abilities can access it at some level... The biggest barrier is their prior knowledge and what skills they have acquired previously to; then build on to get to the new topic.

Previous knowledge, especially considering the coordinate system, confused the English students from three aspects:

(1) The meaning of substitution into algebra:

> Where they've got the equation, actually the substitution into the algebra, their misconceptions, $2x$ means 2 times x, you know, that kind of thing. (ENT3)

(2) Labelling coordinates in the square sheets frequently used in English classrooms:

… Confusion is the number being in the middle for the bar chart (see Figure 52) as opposed to being on the line for the coordinate system. Another is that they need the same distance between 1 and 2, or 2 and 3 – they don't realise it matters (Figure 53). (ENT2)
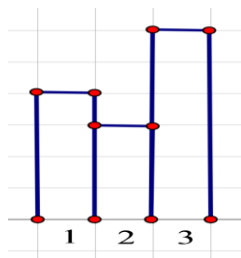
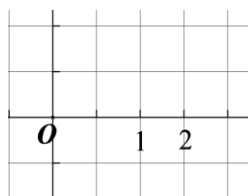

Figure 52. Labelling in bar chart

Figure 53. Labelling in coordinate system

(3) The meaning of coordinate:

Understanding the connection between the x in the table and those coordinates. They don't always make that connection. (ENT1)

They have trouble with plotting as well. At primary school they are taught to do it "along the corridor and up the stairs". For example, (3, 4), they just find 3 at anywhere in the diagram, maybe in the y-axis. They get it the wrong way round regularly…The other confusion is… we write it as $y = 2x$. If you told them that, they come to the coordination for that, typically they will say (6, 3) instead of (3, 6). Because the x is the right and the y is to the left. (ENT2)

So things like if we are going to do $y = 2x + 1$ with a negative x value, they will go the wrong way up the number line, that kind of thing. Actually, those kinds of things hinder their understanding of linear graph. Despite the fact that they might understand what a linear graph is - they might understand the gradient and the y-intercept - but actually being able to draw the graph in the first place can sometimes be a bit tricky. (ENT3)

All teachers expressed their concerns for students' numeracy skills, specifically dealing with negative numbers:

> When plotting linear graphs, how to calculate negative numbers. (ENT1)
>
> They're very bad with negative numbers. (ENT2)
>
> For foundation groups, generally speaking, their numeracy work is weaker. (ENT3)

This weakness has an effect on the ability of correct calculations for procedure knowledge. Apart from prescribing more practice in order to deal with the lack of numeracy skills, the ENT2 provided a tip that enabled the students to double check their answers:

> Another thing I did recently which I'm very proud of is when you do $y = 2x + 1$ in our lower group, you have a table that you're trying to do in four quadrants. You've got negative numbers at the start of the table, but they're very bad with negative numbers, so they need to practice it. If you tell them not to fill it from left to right, but from right to left, then after they've done two or three, they can spot what the pattern is and then they just continue the pattern downwards. For example, $y = 3x + 1$, when you fill in the tables from right to left, then pupil know it just keep down the 3, as follows:

x	-1	0	1	2	3
y				7	10

During their interviews on handling these barriers, all of the English teachers mentioned the importance of practice. Two of the three teachers emphasised students' misunderstanding or insufficiency of related previous knowledge as a major barrier to progression and comprehension.

Point 2: Connecting Representations. Two teachers noted that students could only connect representations, from the algebraic expression to the tabular representation, and then from the tabular representation to the graphical representation, but failed to link the algebraic expression to the graphical representation directly, namely without being given a table in the question. For example, the ENT3 explained that 'linking a graphic topic and an algebraic topic together they find quite difficult'. More specifically, the ENT1 stated that:

> So if I just give them the linear equation in Year 10, they do not know how to plot without a table. But as soon as you give them tables, they could figure it out and link them, because they recognise the table. I don't think they necessarily understand what they are doing.

None of them mentioned whether students would have problems connecting the converse approach, namely translating a graphic representation to an algebraic expression (equation). The results from the last chapter showed that the majority of the Foundation Level English students had encountered difficulties with this process.

Point 3: The meaning of gradient. The ENT1 argued that students could not actually understand the meaning of gradient from procedure perspective:

> They do it, but they don't understand what they are doing. Another one is gradient calculated, if you got the straight line and calculate gradient, you take two points, some of them will take any points regardless ... Not the intersection point, they just go for anywhere, [and] they don't understand why they cannot calculate it. They don't actually understand the concept of gradient.

Shanghai students' barriers to understanding. Two main barriers were highlighted by Shanghai teachers: long word questions and stereotype of solution in application; and the basic method: the combination of symbolic-graphic in understanding knowledge.

Point 1: Application. Two of the four teachers not only presented clear descriptions of two main barriers in application, but also were aware the underlying reasons: these obstacles were derived either from examples in the textbook or the arrangement of topics.

First, the SHT3 commented that '(application) questions are too long so that students have barriers to read through it'. As discussed in Theme 1, the Shanghai teachers heavily relied on textbooks for their daily lesson planning. Findings from the textbook analysis chapter verified that examples and exercises in the textbook were mainly presented as long word problems. Secondly, the SHT2 considered broader barriers that students would face

after learning all types of function. The students might be clear at individual topic of application, but meet difficulty to choose the proper type of function to solve the question.

> In application, students have had a prototype for solutions. During this time, they would consider all application related with linear function. Because all exercises in the textbook choose the solution related with algebraic expression of linear function. Students lost opportunities to choose which kind of function to solve the problem. (SHT2)

Point 2: Basic method of the combination of symbolic-graphic. All the selected Shanghai teachers thought that complex problems which were interwoven with other mathematics knowledge were main barriers for the Shanghai students' learning of linear function. This linking was embodied in the basic method: the combination of symbolic-graphic as involving visualization.

Visualization, as discussed in an earlier chapter, had two specific meanings in case of linear function: students could derive information from a graphical representation and then translate it to an algebraic expression; or students could translate information from an algebraic expression or words to a graphical representation.

The students' weaknesses in the utilization of the graphical representations were initially observed from their solution tendency towards the algebraic solution in the application tests, as set out in Chapter 7: Student Tests. Especially in connecting linear function with linear inequality of one unknown $(ax + b > c, a \neq 0)$ shown in the compulsory textbook, the SHT4 argued:

> In classroom teaching, both two methods (algebraic and graphical approach) are taught. But when facing the actual problems, they will tend towards the algebraic approach... Students are quite familiar with the algebraic one so it is unnecessary to use the graphic one.

There was an example of this basic method from the observation of SHT4's classroom. This question could be solved by two approaches as graphic approach would be more effective, but almost every student chose the algebraic one.

Linear function $y = kx + b$ passes through (-1, -12) and (33, 2). Find out the quadrants that the graph passes through.

Solution 1: Algebraic approach

Because $y = kx + b$ passes by (-1, -12) and (33, 2),

So $\begin{cases} -12 = -k + b \\ 2 = 33k + b \end{cases}$

then $k = \dfrac{7}{17}, b = -\dfrac{197}{17}$

Because $k > 0, and\ b < 0$

Then this graph passes through the first, third and fourth quadrant.

Solution 2: Graphical approach

Draw the straight line through two given points in Cartesian plane (Figure 54). From the graphical representation perspective, this graph passes through the first, third and fourth quadrant.
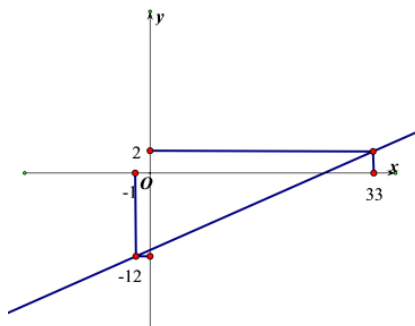

Figure 54. Graphic representation in the example

Although the algebraic approach solution was not easy to calculate correctly, only two students in that class (35 students in the whole class) initially used the graphical approach. This example also indicated that the students had the higher proficiency of numeracy skills.

On the other hand, the teachers highlighted this basic method mainly because it played the vital role on assessments, as the SHT3 stated that 'in the assessment or final examination or high school entrance examination, the requirements for this method are higher and lots of questions will be related to it'.

The expression of these types of questions usually provided solely algebraic expression of linear function without presenting the graph so that students were not able to notice the necessary of drawing a graph to effectively solve it. In order to overcome this barrier, SHT4 suggested:

> … by trial-and-error. From the start, students are not familiar with the idea of drawing the graph, and then slowly by slowly, they could know that, if they draw the graph, they could resolve the problems. Geometry knowledge emphasises on the graph, while the algebraic one focuses on calculation. For the concept of function, we need to combine both. Students do not get used to doing it at the beginning. Both graphical representations of proportional function and inverse proportional function are simple; if students do not draw the graph, they could resolve most of questions. But for linear function, they need to develop a habit [drawing the graph].

A comparison of barriers on understanding pure knowledge. Table 47 reveals that the English teachers' main concern focused on Level 3 and 4, with the basic knowledge, gradient and basic skill of connecting representations without a table. Conversely, the Shanghai teachers emphasised that basic method.

Table 47

Understanding Barriers in the Model of Understanding Function

Model of understanding function	View of English students' understanding barriers	View of Shanghai students' understanding barriers
Level 3 Connecting Representations	Linking algebraic expression with graphic representation directly	
Level 4 Property Noticing Level 5 Object Analysis	The meaning of gradient	
Level 6 Inventising		Mathematical method of combination of symbolic-graphic

8.3 Discussion

In this chapter, teachers' views on the teaching and learning mathematics, including linear function, portray the different considerations of the two regions. The first sub-section will address mathematics in classroom, as the teachers hold certain beliefs as to what the nature of mathematics in the classroom should be. Secondly, teachers' pedagogical content knowledge (PCK) in two areas will be depicted according to the model discussed in literature review. Finally, the function of textbooks will be compared.

8.3.1 Mathematics in the classroom

This study suggests that English teachers prefer to use Inquiry Math; while findings from the Shanghai teachers revealed that they mainly applied School Math in the classroom.

The selected English teachers demonstrated their tendency to use *Inquiry Math* in their classroom, which fitted with 'three aspect of inquiry by design' (Richards, 1991, p. 38): 'creat[ing] an inquiry environment' by using multiple activities; students inquiring actively, as the ENT1 said 'getting them to tell you [something]'; and 'students [acting] as designer', with the ENT3 mentioning that students were encouraged to make their own questions based on what they learned. Speaking is highlighted by the English teachers, as it promotes mathematical thinking and learning which is a Western assumption (Jin, 2004). The inquiry approach to *Inquiry Math* contains the use of blended instructions as the English teachers advocated. All of the selected Shanghai teachers talked of the best teaching method as 'teaching with purpose'. This purpose normally indicates a basic method which requires teachers to deliver information to students. This 'delivery' belief demonstrates that the teaching approach is didactic. This didactic traditional method shows the invaluable role teachers play in governing students' understanding development, because students cannot construct these basic methods by themselves. According to the didactic teaching approach to basic method, within the classroom dialogue, it is reasonable to speculate that teachers adopt

the behaviourism learning theory, playing the role of information transfer. As with the example of the rational number addition that the SHT2 talked about, this example can be solved by other methods, but the teacher emphasised a certain procedure and evaluated it as the only proper solution.

8.3.2 Pedagogical content knowledge (PCK)

While comparing PCK between Chinese and USA teachers in middle school, Chinese teachers emphasised developing knowledge, including procedural and conceptual, while USA teachers focused on various activities (An et al., 2004). Findings from the selected Shanghai teacher interviews within this study, as teachers tend to be content-focused, are aligned with Chinese teachers. Meanwhile, the English teachers are inclined to use multiple activities, like the USA teachers. I use four perspectives from *Knowing Students' Thinking* (see Figure 55) in their network to demonstrate the similarities and differences between the selected England and Shanghai teachers.
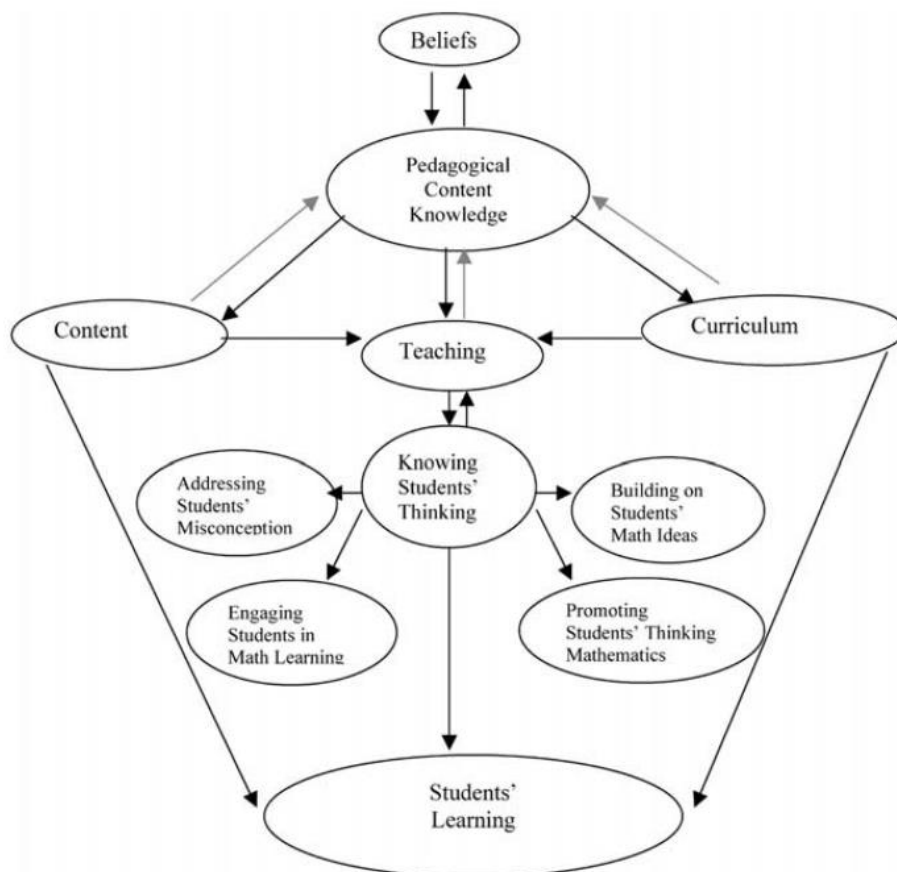
Figure 55. An et al. (2004, p. 147) network of teachers' pedagogical content knowledge

First, in terms of *Building on students' Math Ideas*, understanding mathematical

concepts was underlined by the English teachers, as ENT1 noted that 'they don't actually

understand the concept of gradient' (at understanding Level 4). The Shanghai teachers

indicated the basic method: the combination of symbolic-graphic (at understanding Level 6),

which proved to be an effective way of solving complex problems.

Secondly, with regards to *Promoting Students' Thinking Mathematics*, questioning

students, encouraging students to ask questions and multiple activities are all approaches

adopted by the English teachers; the Shanghai teachers lay stress on examining their students'

written work and directly delivering the basic methods. The basic method is tacit, so that

teachers' content knowledge is highly valued in order to develop students comprehension in

Shanghai, evidenced by L. Ma (1999) appeal for USA primary school teachers to obtain a

profound understanding of fundamental mathematics. It can be inferred from this study that in the Shanghai situation, the importance of teachers' content knowledge, especially in terms of basic methods, cannot be underestimated.

Thirdly, regarding *Engaging Students in Math Learning*, however, a solid previous knowledge is essential to the development of new concepts concerning the teachers in both regions.

Fourthly, looking upon *Addressing students' misconceptions*, findings concerning how the teachers perceive students' barriers to understanding reveal teachers' strategies when addressing them. These barriers influence teachers' decisions, and cause them to design the implemented curriculum, as ENT2 mentioned regarding the lesson plan: 'what students could do and could not do'. All of the selected English teachers also offered a particular perspective on the causes of these barriers which have been noted. For example, when drawing axes in the graph, ENT2 identified students' confusion between the continuous numbers and discrete set of numbers in labels, which were in line with the finding from A. Watson et al. (2013a). The evaluation of students' barriers, however, have already been contextualised (Clarke, 2003).

In summary, it is impossible to make an outright, clear cut judgment as to which PCK knowledge the English or Shanghai teachers have is the best. The differences shown in teacher interviews illustrate that the teaching approach in each region is also constrained by the textbooks, as Elliott (2014) contends that pedagogic practices have their underlying factors (such as the culture of schooling, and students' academic motivation and engagement) and cannot be imported to other cultures effectively.

8.3.3 The function of textbooks

Research has shown that textbooks largely influence how teachers portray a mathematical topic and implement their understanding of students' learning trajectories in a

classroom, as there is a 'statistically significant' relationship between textooks and classroom instruction (Valverde et al., 2002, p. 10). Many researchers have revealed that textbooks are closer to the classroom than the national curricula (Howson, 2013), as embodied in teaching strategies or designing activities (Fan, 2013; Fan et al., 2013; Johansson, 2003; Leung, 1995; Son & Senk, 2010).

Textbook use in the two regions has very different roles in the teaching and learning process. In terms of textbook use in English classrooms, initial reports from TIMSS revealed that, through the key stages, the use of textbooks increased considerably from 66% of Year 5 pupils to 84% in Year 9 students, when teachers use textbook schemes over half of their teaching time (Foxman, 1999). However, in the case of linear function, ENT1 was dissatisfied with the lack of the number of the same type of practice questions.

The data from English teachers' suggests that the role of textbooks within lesson plans is limited, as some of the selected textbooks are even designed for students to use for exam preparation instead of initial classroom learning. Instead, teachers work from the Scheme of Work. Within English schools, the Scheme of Work is written by the Head of Maths and taken from the respective exam broad. Every exam broad is required to follow the National Curriculum. Those textbooks designed espcially for the GCSE examination, all explain these requirements from the authors perspective, as Howson (2013, p. 652) noted that England textbooks are written 'not by experienced teachers, but by experienced examiners'. The English teachers might not follow the textbooks in their entirety, but their approach to topics, such as students' performance expectations, instructional features, and their perspectives, are not different to the textbooks. In addition, teachers' lesson plan is reflected by their understanding of learning trajectories. This understanding is also based on teachers' experiences as a student (Fennema & Franke, 1992), specifically how they were taught and how their own teachers presented the knowledge. The selected English teachers' attitudes

towards instrumental understanding might be explained by their own lack of attention towards this procedural knowledge during their secondary school education.

The Shanghai teachers regarded the compulsory textbook as the foundation of the lesson plan. Y. Li and Huang (2008) suggest that further investigations into 'what exactly Chinese teachers do and learn with the textbooks' are needed. Although Chinese teachers study textbooks very carefully and classrooms are textbook-based (L. Ma, 1999), the SHT2's idea, as discussed earlier, of mathematics involving the translation of three languages was not listed in the textbook. The function of the textbooks in Shanghai is likely to provide the content; teaching is then built on, but not limited to this content.

In conclusion, these three issues: the kinds of mathematics in classrooms, the PCK, and the function of textbooks in lesson plans within the two regions differed markedly, due to the different views of mathematics itself, the students' reality, and the education policy.

8.4 Summary

By comparing the views on lesson plans, teaching and learning processes, the nature of mathematical understanding, and students' barriers to understanding linear function (four themes), this chapter has identified some similarities and differences between England and Shanghai.

In terms of education policy, high control and coherence among teaching materials in Shanghai, for example the same requirements for all the students, and uniform standard tests, leads to a content-centred emphasis on teaching (Theme 2) shown in their lesson plan (Theme 1). The mathematics content is explained as four basics (Theme 3) so that the teachers focus on students' abstract levels of understanding as well as the basic skills. Students' written work is stressed rather than their verbal expression. Among these basics, the basic method in linear function was proved to be the main barrier of the Shanghai students within the pure knowledge section, while in real life problems, reading long word problems is difficult for

students (Theme 4). The teachers also pointed out that the arrangement of application misleads students into undertaking all of the application questions through linear function simultaneously, which hindered their selection of solutions when having to decide which kind of function was more suitable to solve the real life situation.

The English education system is more flexible than that of Shanghai. Different levels of requirements lead to a student-centred lesson plan (Theme 2) with multiple activities (Theme 1). Procedural knowledge or instrumental understanding (Theme 3) is less valued by the English teachers than in Shanghai. As a result, one of the English students' barriers is drawn from a lack of instrumental understanding (Theme 4).

Behind these four themes, the views of mathematics that the two regions' teachers have hold differed: *School Math* in Shanghai and *Inquiry Math* in England. This underlying belief of mathematics led to the views of understanding mathematics (Theme 3), while these views caused diverse approaches to lesson planning (Theme 1). Teachers' pedagogical knowledge is also shaped by both the policy and views of mathematics. From the discussion section, the reason why the teaching approach (Theme 2) is inappropriate to import from each other has been addressed from a theoretical perspective. Students' barriers (Theme 4) were shaped by the preferred ways of handling the topic at each region.

As well as documenting cultural differences, comparative studies should accommodate the interrelationship of student achievement, curriculum content, and teachers' approaches (Clarke, 2003). A discussion related to all of the findings will now be presented in the next chapter.

Part Three: Discussion and Conclusion

The preceding chapters have offered a thorough exploration of the expectations of curricula, the ways in which respective textbooks convey these statutory requirements, and the Heads of Maths' views of the teaching and learning process, focusing on understanding mathematics in general and understanding linear function in particular. Meanwhile, what students actually achieve in terms of pure knowledge and application has been illustrated. These findings which comprise Part Two of this thesis not only individually answer the first three research questions, but also offer a closer look at the bigger picture, in terms of factors that support or constrain students' understanding development, providing an answer to the fourth research question. Part Three of the thesis (Chapters 9 and 10) will present a discussion of the key findings and assess the answers to the research questions in light of the extant research. The key findings will also serve as a mechanism for drawing out implications for future practice, for other researchers as well as suggesting recommendations for further study. The limitations of this study will also be addressed in the last chapter.

Chapter 9 Summary and Discussion

This chapter has three main sections. The key findings from the four result chapters will be briefly listed in the first section of this chapter. Answers to the four research questions will then be combined with the literature to explore the underlying issues. Finally, the main contribution will be highlighted.

9.1 Key Findings with respect to the four Result chapters

The key findings will be summarised according to the analytical framework set out in each results chapter. The first two sub-sections relate to Chapter 5 and 6, and will address results from the curricula and textbooks in terms of the background information as well as the particulars of understanding linear function. The key findings from student performance will be categorised by the pure knowledge test and application test in the third sub-section. Finally, three key findings from the teacher interviews will be highlighted.

9.1.1 Key findings of the Curriculum Analysis

The findings from Chapter 5 justified three decisions made in this study: the sample chosen (as in key findings 1.1 below), the topic chosen (key findings 1.5), and two sets of test design (key findings 1.2). The different meaning of highest understanding level was identified (key findings 1.3) as implying the expectations of understanding at curriculum level. Particularly, the Shanghai curriculum pointed to the importance of memorization (key findings 1.4) in the process of understanding development.

In terms of background information of linear function in the curriculum.

Key findings 1.1: the sample chosen. The English students will be taught all the required knowledge at KS4 (Year 10 and Year 11) while linear function is arranged at Grade 8 in Shanghai. In order to test the understanding of this topic, there was an unavoidable one-year age gap between two groups.

Key findings 1.2: the two sets of tests design. The content of linear function has been split into two parts by both regions' curricula: pure mathematics knowledge and application. Therefore, the student tests corresponded to these two parts.

In terms of understanding linear function.

Key findings 1.3: the approach towards the highest understanding level. Graphic representations play a central role in relating other mathematics knowledge in England; while Shanghai pays more attention to the algebraic expression within the basic method: the combination of symbolic-graphic.

Key findings 1.4: memorization. It is deemed to be part of the understanding process in Shanghai as the first level of cognitive development.

Key findings 1.5: the topic chosen. Linear function is the only suitable concept in function for the two regions, because of the same depth of requirement for understanding pure mathematical knowledge according to the general model of understanding function.

9.1.2 Key findings of the Textbook Analysis

The textbooks explain in detail the aims of learning within the respective curricula. Along with the justification of the topic chosen, content coverage in the textbooks indicated the similar importance for the topic in both regions (key findings 2.1). The differences of how to structure this topic combining real life situation with pure knowledge (key findings 2.2) offered the distinct views of handling linear function in each region. Looking further, the different emphasis of a certain understanding level (key findings 2.3) and preferred representation (key findings 2.4) illustrated what textbooks bring the focus and how each region handled this topic.

In terms of background information of linear function in the selected textbooks.

Key findings 2.1: content coverage. The selected Higher Level textbooks from England paid more attention to linear graph than the Shanghai compulsory textbook, while

the Foundation Level introduced less content concerning linear function. However, overall, the percentage of content coverage for linear function was roughly similar, at around 2%.

Key findings 2.2: the arrangement of linear function. The selected English textbooks started with real life situations, moving towards Cartesian graphing, ending with the symbolic system of algebra, while the rigorous definition was introduced in the Shanghai textbook first, mainly keeping with the algebraic approach towards the basic method, and ending with application to real life situations.

In terms of understanding linear function.

Key findings 2.3: different foci of understanding levels. The worked examples in the selected English textbooks including Foundation and Higher Levels primarily focused on Level 4 Property Noticing while the Shanghai textbook focused on Level 5 Object Analysis.

Key findings 2.4: different approaches. In the pure knowledge part, graphs were noticeable in the English textbooks. Conversely, the algebraic expressions dominated the Shanghai textbook. In the application part, the English textbooks required students to generate the algebraic expression from real world situations which were represented with the aid of the graphic representation while the generation of the rule in Shanghai was rooted in long word problems with the expectation that students can use two representation systems: algebraic and graphic.

9.1.3 Key findings from the Student Tests (main study)

Generally, the Shanghai students did significantly better than the English Higher Level students in both sets of tests. Results also portrayed the distribution of understanding levels for each group (key findings 3.1). Furthermore, probing into the understanding levels, what students succeeded at and what they failed at were diagnosed (key findings 3.2 for the English Higher Level students and key findings 3.3 for the Shanghai students). From the application test, the preferred representations (key findings 3.4) noticeably differed between

the two groups of students, as another way to reflect how their understanding was shaped. The relationship between achieving higher levels of understanding and higher performance in application (key findings 3.5) is statistically significant.

In terms of understanding pure knowledge tests.

Key findings 3.1: overall views of understanding function. The distribution of understanding levels that the English Higher Level students showed was more diverse than their counterparts in Shanghai. The Shanghai students mainly were located at Levels 5 and 6; while the English Higher Level students were dispersed from below Level 4, Level 4, and Level 5 to Level 6.

Key findings 3.2: the main strength and weakness of the English Higher Level students. The main strength of their understanding was shown in the highest level of understanding. Nearly 30 per cent of the students could successfully solve the questions related to geometry knowledge which were beyond the requirements of the respective curriculum. Their main weakness appeared at Level 4, the meaning of gradient. The positive value of gradient was fully understood by one third of the students while the percentage for the negative value of gradient was only one-sixth of the students.

Key findings 3.3: the main strength and weakness of the Shanghai students. The Shanghai students heavily relied on algebraic method to solve the complex question. This strong preference caused their insufficient understanding of graphic representation embodied in the basic method: combination of symbolic-graphic required by the Shanghai curriculum.

In terms of application tests.

Key findings 3.4: preference for representations. The solutions offered by the English Higher Level students demonstrated their tendency of graphic representations, while the strong preference of algebraic expression was observed from the Shanghai students. In

other words, the English students were not good at generating algebraic expressions while the Shanghai students showed their difficulties in using graphic representations in their solutions.

Key findings 3.5: understanding and application. Understanding Level 5 has been identified as a benchmark for higher performance of application in both regions. On the other hand, the higher performance of application showed less successful influences of highest level of understanding.

9.1.4 Key findings from the Teacher Interviews

Interviews with the selected teachers examined issues around students' understanding: what they prioritised when doing the lesson plan (key findings 4.1), what they perceived to be the barriers to students' understanding of linear function (key findings 4.3), and how they facilitated students' understanding (key findings 4.2).

Key findings 4.1: lesson plan. The selected English teachers tended to focus on students' ability in order to design a suitable lesson, along with flexible seeking of suitable materials; while the Shanghai teachers would carefully consider the requirements of the textbook first.

Key findings 4.2: understanding development. Both groups of teachers agreed that understanding was a process and students were supposed to understand both how mathematics knowledge works and why. The teachers in England particularly drew attention to students' spoken language and using multiple activities to engage their learning with understanding; while written work was highly valued by the Shanghai teachers because of the more abstract understanding involved in the tasks. In terms of procedural knowledge, the English teachers did not regard it as a part of understanding; while the Shanghai teachers perceived it as basic skills, as one of four basics which understanding was divided into.

Key findings 4.3: perceived barriers to students' understanding. The English teachers predicted that their students would meet the challenges of linking algebraic

expressions with graphic representations without being given tabular representation (understanding Level 3), and the meaning of gradient (understanding Level 4), but also identified the lack of related previous knowledge (such as the meaning of substitution, coordinates) and basic numeracy skills such as dealing with negative numbers. The Shanghai students were expected to overcome the basic method: the combination of symbolic-graphic, located at understanding Level 6. Long word problems in application were difficult to handle for Shanghai students.

9.2 Discussion of the research findings in response to the Research Questions

The answers for each research question from each chapter have been partly stated in the first section, and will not be repeated here. But findings emerging from the overlapping issues in each results chapter will be synthesised in this section and related to the literature. There are four key discussion points in response to the four research questions: the coherence of requirements between the curricula and the selected textbooks in terms of understanding linear function; the coherence between achieved curriculum and intended curriculum; different teaching approaches towards understanding; and the holistic map of student understanding development in the two regions, such as the notion of mathematical proficiency in Shanghai and different emphases on representations. Therefore, each sub-section consists of three parts: a restatement of research question, additional answers to the question, and discussion of that research question in terms of the related literature.

9.2.1 Research question 1

What are the requirements of the intended curriculum and officially used textbooks in the two regions in terms of linear function?

In response to Research Question 1. The key findings from Chapter 5: Curriculum Analysis and Chapter 6: Textbook Analysis in the previous section, have briefly answered Research Question 1.

Particularly, due to the different arrangement of simultaneous linear equations in the two regions, this knowledge was excluded by this study as discussed in Chapter 6: Textbook Analysis. Therefore, the requirements of the intended curriculum (KS4) for all English students reached understanding Level 5 for pure knowledge understanding. With regards to worked examples, there was incoherence between the national curriculum and commercial textbooks in England. Additionally, linear function was introduced as linear graph in the English textbooks, and they were different concepts. Conversely, the coherence of understanding levels between official documents was present in the Shanghai situation.

Discussion: two approaches towards mathematics. As discussed in Chapter 2: Research on Comparative Education, there are two types' views of mathematics, pure and applied mathematics. The English textbooks avoid defining the formal concept of linear function as well as the concept of function. Furthermore, linear graph is built upon particular contexts such as time-related graphs. The textbooks therefore cultivate the pragmatical approach towards mathematics. Conversely, mathematical concepts are rigorous in the Shanghai situation. Linear function has the rigorous definition in the form of $y = ax + b$ $(a \neq 0)$, distinguished from the constant function $(y = c)$. Shanghai's intended curriculum emphasises three basics in line with this formalist view of mathematics. Furthermore, teacher interviews reveal that basic methods are highly valued in the process of teaching and learning. Meanwhile, these basic methods do not rely on knowledge which students can bring from outside of school, and can only be taught as a separate subject. The findings from the Shanghai student tests confirm that students mainly use the formal symbolic method. Therefore the learning trajectory in Shanghai curriculum and textbook conforms to the description of the pure view towards teaching mathematics.

9.2.2 Research question 2

What do English and Shanghai students actually achieve (the attained curriculum), with regards to linear function?

In response to Research Question 2. The findings from the Student Tests have answered this question. In this section, the students' learning outcomes will be compared with what the curricula expect according to the general model of understanding function, to ascertain how close students achieve these required criteria.

In terms of the consistency between the intended and the attained curriculum in England, the understanding level required by the statutory guidance sits at Level 5, and 50.6% of the English Higher Level students (21.6% at Level 5 and 29.0% at Level 6) have reached the understanding requirements of the intended curriculum. In the Shanghai situation, this percentage is 57.9%. Although Shanghai students have done better than their counterparties in England in general, the percentage of students who have fulfilled the respective curriculum requirements does not demonstrate the disparity between the English Higher Level and the Shanghai situations. It is reasonable to assume that these students' understanding is highly related to the depth of knowledge students are expected to master in each region. This relation is correlation rather than causality. It does not imply that to increase the complexity of curriculum is a possible effective way to enhance English Higher Level students' achievement. This finding only gives another perspective to interpret the students' performance gap.

Discussion: the more effective representation to solve problems. This study's findings of students' performance in application tests are in line with the results of PISA. In additional, the English and Shanghai students adopted opposing approaches and showed differing preferences for representation system. The visual approach dominated the Year 10 (15 years old) English students' test responses; but a study of outstanding senior secondary school students (16-17 years old) showed that they were always non-visualisers, preferring to

use methods other than visualisation in order to solve problems (Presmeg, 1986). To examine this issue further, the inquiry is whether the algebraic method or algebraic expression has helped the Shanghai students to achieve better results. That is, which representation could be the most effective in terms of students' learning outcomes: the graphical one or the algebraic expression?

Advocates for the symbolic representation. With respect to learning outcomes, it appears that the symbolic or algebraic approach is better than the visual ones, graphic and tabular. Indeed, students who used symbolic representations showed higher mean scores on tests than those who used other representations, such as visual representations (Cai & Lester Jr, 2005). Nistal, Van Dooren, and Verschaffel (2012) illustrated that students who worked with formulae produced a higher accuracy rate than those who used tables. The tabular representation was unable to list all variables, and findings from the application test (No.4b) in the pilot study demonstrated the superiority of algebraic expression.

Gagatsis and Shiakalli (2004) suggested that the use of symbolism is fundamental to enhancing understanding. Similarly, Aspinwall, Shaw, and Presmeg (1997) examined the function of graphic representation in college-level calculus in order to understand the derivative function via analytic modes of thinking, with results showing that graphic instruction handicapped students' understanding. In terms of the understanding tests, the Shanghai students revealed their statistically significant better performance in this study. Symbolic representation facilitates their understanding development.

On the other hand, British students struggle with algebraic expressions (Herscovics & Linchevski, 1994), which resonates with this study's findings from the application tests in main study (with only 8.2% displaying an ability to form an algebraic expression). Most of the Higher Level English students failed to use the formal symbolic method to construct an algebraic expression. Apart from their insufficient understanding of gradient, there were two

other concepts associated with difficulties: the meaning of variable, and the '=' sign. The notion of a variable is normally 'associated with algebraic symbols' (Leinhardt et al., 1990, p. 22). Nearly 30% of the English Higher Level students in the application test in the main study ('fire charge' question) showed that they avoided the use of symbols when generating the rule. It implies that the perspective of variable should be more focused in England.

Although students' understanding of the '=' sign in the algebraic expression was unclear in this research, 15-year-old British students were reported to regard this sign as working out a result rather than demonstrating a relation (Sfard & Linchevski, 1994), and the main barrier in moving from arithmetic to algebra is the '=' sign (Mathematics Learning Study Committee, 2001, p. 379). Additionally, Herscovics and Linchevski (1994, p. 75) found a considerable cognitive gap between arithmetic and algebra for gifted British students, as they could not 'operate with or on the unknown'. Operating 'spontaneously with or on the unknown' was viewed as the demarcation between arithmetic and algebraic understanding (Herscovics & Linchevski, 1994, p. 63). The results of the study suggest that further research should investigate whether these English students' understanding still remains at the arithmetic level.

After examining students' preferences for visual solutions in mathematics in three countries, South Africa, Sweden, and the United States, Presmeg and Bergsten (1995, p. 59) argued that visualization was not given enough attention in the classroom or subject system, and visualization might be downplayed or devalued in certain classrooms or systems', but that, if given the opportunity, students would prefer visual methods. Results from the Shanghai students did not show this preference. However, the partiality for algebraic expression might have weakened their graphical understanding or visualization which was also related to their insufficient grasp of the meaning of gradient. The Shanghai students' responses implied that this tendency also weaken their understanding development in graphs.

Advocates of the graphic representation. Although Cox and Brna (1994) argued that graphical representations limit the presented information and are less expressive of abstraction, findings from the application test illustrated that only three English students got the inaccurate answer from graphic representation. It verified the power of visualization in case of linear function.

On the other hand, the symbolic approach towards solving problems has been criticised. The dangers with the algebraic approach, as Sfard and Linchevski (1994) warned, were that students can easily focus simply on the automatic symbolic manipulations, resulting in their inability to explain or justify their problem-solving methods and reasoning. Further, Abdullah (2010) argued that students who operated superficially with symbols showed difficulties when using the Cartesian graph. The Shanghai students did show their difficulties when linking to basic knowledge in the Cartesian plane in the highest understanding level question.

Focussing on the property of gradient, the visual approach to the meaning of this property could seemingly lead to some degree of confusion, because the scale of the axes could be changing in different contexts, while the algebraic approach would not meet this problem (Zaslavsky et al., 2002). Ayalon, Watson, and Lerman (2014) argued that English students only at Year 13 showed strong understandings of gradients. Results from the understanding tests in terms of gradient showed that graphic meaning of gradients was unsuccessful for English students to understand and their teachers also expressed their worries.

Connecting algebraic expression with graphical representation. Translation from algebraic expression to graphical representation is much easier than the opposite way round (Markovits et al., 1986). It was recommended that the crucial steps of the transition from visual representations to internalized abstract representations, such as algebraic expression,

should be a focus (National Mathematics Advisory Panel, 2008). In this case, the algebraic method was much more effective, as the Shanghai students outperformed the English students in the conversion of a tabular representation to an algebraic expression in terms of the 'match' question in the application test. This finding was analogous to the study by Ayalon et al. (2014) about generating a formula from the similar series of figures that compared Israeli and English school students, their results showing that only 50% of the English students successfully resolved the question. The researchers assumed that the formal algebraic approach to function used by the Israeli students had more advantages than the adaptive reasoning used by the English students. In this case, it is reasonable to speculate that Shanghai students used two pairs of points in the table to form simultaneous equations to discover the appropriate formula. The English students had to undertake the deduction from the pairs of points in the table to form the algebraic expression, which was much harder. In this respect, the algebraic approach played a crucial role in the Shanghai students' superior performance in connecting representations.

Therefore, ideally, textbooks should use a variety of representations to promote understanding (Gagatsis & Shiakalli, 2004), and introducing different approaches leads students to pay more attention to how to get the solution instead of merely solving the problem without any comprehension (Xu, 2004). The textbook in Shanghai not only presents more opportunities for students to use the two types of representations (algebraic and graphic), but also encourages the use of two kinds of solutions in their answers. If more than one representation is used in problem-solving, students' performances would be higher than those students merely using one (Ainsworth, 1999).

9.2.3 Research question 3

What are teachers' views regarding the teaching and learning of linear function?

In response to Research Question 3. The findings from the Teacher Interviews have answered this question.

Discussion: two approaches towards understanding development. As discussed in the Literature Review, there are two approaches to teaching mathematics knowledge: bottom-up, such as the Pirie and Kieren model (Pirie & Kieren, 1994b), or top-down, such as APOS theory (Dubinsky & McDonald, 2002) by didactic approach. These two approaches were in line with the findings from this chapter: the English teachers perceived mathematics classroom as inquiry maths based on activities that students undertake, i.e. bottom-up; while school maths was applied in the Shanghai teachers, i.e. top-down.

The bottom-up approach (mathematising) – otherwise known as the 'realistic approach', emphasises the actual 'doing' of mathematics which involves 'solving real life problems' so that various contextual problems not only play an essential role in learning with understanding, but are also integrated into the curriculum from the start (Gravemeijer, 1997, p. 330). This method, the bottom-up approach, is well rooted in the English situation. On the other hand, the top-down approach (transfer), also known as the 'information processing approach', separates the process of learning mathematics into 'learning formal mathematical knowledge and learning to apply it' (Gravemeijer, 1997, p. 330), as adopted in Shanghai. In terms of the bottom-up approach, the doubts of the selected Shanghai teachers stem from two perspectives: this method is criticised as 'not effective' for teaching by SH1 and SH3, and as a means of 'taking detours' for learners by SH1.

Although the Shanghai students' performances are generally better than the English students in this study, it cannot be concluded that the bottom-up method fails in its pragmatic approach to understanding mathematics in secondary school level. There are two reasons for this, as shown in the student tests. First, when comparing what percentage of the students achieved of the requirements the intended curriculum, there was not a great difference, with

50.6% of the English Higher Level students and 57.9% of the Shanghai students reaching the necessary point in the respective course. Secondly, the relationship between higher performance in understanding (total score and understanding levels) and higher performances of application is statistically positive. However, findings from the Student Tests showed that in terms of learning outcomes, the top-down approach was more effective in terms of the highest level understanding than the bottom-up one.

9.2.4 Research question 4

What shapes students' understanding of linear function?

In response to Research Question 4: differences between predicted barriers by teacher and actual barriers by students. When referring to effective support for students, the views of teachers about students' barriers are important. The selected teachers for both groups have demonstrated their accurate prediction of students' barriers to understanding which were verified by the student tests in the main study (see Table 48 and Table 49). Approximately two-thirds of English Higher Level students successfully overcame their problem-solving obstacles. In Shanghai, however, the majority of students could not fully overcome their barriers to application, and beyond one-quarter of students struggled in the highest level of understanding – the basic method, even though Shanghai has a higher expectation of students' deep understanding of mathematics should be considered.

Table 48
The English Teachers' Prediction and Students' Real Barriers

Predicted by English teachers		Affected at understanding test	Affected at application test
Point 1 Previous knowledge and skills	The meaning of substitution		27.8% of the Higher Level students
	The weak numeracy skills (negative number)	31% of the Foundation Level students	
Point 2 the meaning of gradient		73.5% of the Higher Level students	

Table 49

The Shanghai Teachers' Prediction and Student' Real Barriers

Predicted by Shanghai teachers	Affected at understanding test	Affected at application test
Basic method of combination of symbolic-graphic	26.5% of students had no habit to draw a graph when solving the complex questions	
Application for long word problem		Around 10% of students could fully deal with the long word problem

In summary, the coherence between students' barriers as shown in the tests and teachers' predictions was dissimilar in the two regions. This indicates to what extent students might receive support to overcome these barriers, although acknowledging the complex and higher levels understanding of barriers for Shanghai students.

Discussion: holistic views of understanding. The underlying reasons why the Shanghai students had the more abstract understanding using the preferred algebraic representations will be explored from three perspectives. First, the comparisons of the main aspect of definition and the preferred representation will illustrate the less effective way of presenting the knowledge in England. In Point 2, a comparison of the view of understanding will be addressed from the notion of a mathematical proficiency perspective. Following the divergent views of understanding, the importance of skill-algorithm in understanding development will be discussed.

Point 1: main aspect of definition and the preferred representation. In the England's situation, the KS4 English curriculum requires that students 'where appropriate, interpret simple expressions as functions with inputs and outputs' (Department for Education, 2014, p. 7), revealing the input-output view in the concept of function. Correspondingly, the input-output assignment aspect of function is introduced by the junior secondary textbook as discussed in the Literature Review. The results from the Textbook Analysis revealed that the

graphic representation is widely applied at each understanding level as well as the application part in England. Schwartz and Yerushalmy (1992, p. 263) pointed out that 'the graphical representation of the function is relatively more effective in making salient the nature of the function as an *entity*'. The graph itself offers a tendancy to a global analysis view of the mathematical object. In other words, the process aspect might be missing in England.

In the case of the concept of gradient, how to present this knowledge and how to teach it in England confirmed this issue. First, there is seemingly a conflict between the nature of function and the graphical representation for English students as well as for teachers to cope with. That is, students might find it easier to grasp the meaning of gradient: the steepness of the line, when regarding the graph as the object. However, how to work out the value of the gradient is through constructing three coordinates within a right triangle. Therefore, understanding the gradient requires two perspectives together: intuitive understanding of steepness and calculation from points. These two seperate issues might cause the students to experience difficulties in mastering this property, in terms of how to go through the process. Secondly, the English teacher interviews illustrated that commercial software was widely used in their classrooms to help students understand the concept of gradient. Researchers (Artigue, 2002; Schwarz & Dreyfus, 1995) have confirmed the positive influence of a computer environment in the enhancement of balancing these two perspectives as discussed in Chapter 3. The less successful English student performance in this study did not imply the disagreement of the benefit of computer environment. However, the reason why the meaning of the gradient became one of the main barriers for the English students can be speculated as the gap between the non-calculator test environment in student tests, which means that students have to go through the process stage, and the commercial software environment teachers used in the teaching and learning process, which would automatically go through the process stage.

In addition, the variable aspect of function (Doorman et al., 2012) is missing in England with respect to function, although both the KS3 and KS4 national curriculum documents have pointed out the concept of variable in the section titled 'Reason mathematically of Working mathematically': 'identify variables and express relations between variables algebraically and graphically' in KS3 (Department for Education, 2013c, p. 4) and 'extend their ability to identify variables and express relations between variables algebraically and graphically' in KS4 (Department for Education, 2014, p. 5). The concept of variable was introduced in another chapter in the selected textbooks titled *Algebra*: 'this is what the letters used to represent numbers are called' and 'the variables are treated just like a set of (x, y) coordinates'. But the link between the concept of variable and function or linear function was weak. Schwartz and Yerushalmy (1992, p. 263) argued that 'the concept of function and the concept of variable are intimately linked and we believe that each can serve to shed light on the other'. This missing of the variable view in function led to the English Higher Level students' difficulty of generating the rule, as shown in the application tests, where only 8.2% of pupils presented a full understanding of the form of an algebraic expression. This resonates with the fact that British students have been struggling with algebraic expressions (Herscovics & Linchevski, 1994) as discussed above.

In the Shanghai situation, the dynamic process of co-variation aspect of function was introduced in Chapter 3, where the algebraic expression played a crucial role in understanding function. Schwartz and Yerushalmy (1992, p. 263) put forward that 'the symbolic representation of the function is relatively more effective in making salient the nature of the function as a *process*'. This coherence helped students to build the operational view of function which is the first step towards understanding (Sfard, 1991).

Point 2: different perspectives on understanding. The Chapter 3 revealed the different perspectives on understanding between the West and China. The Shanghai

curriculum, as discussed in a previous chapter, is built on the development of proficiency, like most top-performing countries.

Mathematical proficiency has five strands: (1) conceptual understanding; (2) procedural fluency; (3) strategic competence; (4) adaptive reasoning; and (5) productive disposition (Kilpatrick, 2001, p. 107; National Research Council, 2001, p. 5). These five strands match with the four basics emphasised in the Shanghai classroom (basic knowledge, basic skills, basic methods, and basic experience). Conceptual understanding relates to understand conceptual knowledge. Procedural fluency can correspond to mastering basic skills or procedural knowledge. Strategic competence and adaptive reasoning are relevant to basic methods as students have the capability to evaluate the question first, then to identify the appropriate strategy, and finally to defend their solution. The last strand, productive disposition, which describes students' attitudes and beliefs towards and regarding mathematics, is similar to basic experience to shape their values towards mathematics. The common factor between mathematics proficiency and the four basics is that they can be developed together. In England's curriculum, only conceptual understanding is stated.

Point 3: understanding and skill-algorithm. Findings from the teacher interviews also revealed that the English teachers did not consider numeracy skills to be part of understanding mathematics. Instead, they separate the procedural knowledge and conceptual knowledge. The latter is regarded as developing understanding and is highly valued by the English teachers. Conversely, the Shanghai teachers believe that both forms of knowledge contribute to understanding development. Procedural knowledge contains numeracy skills. It is unnecessary for teachers and students, however, to understand why a calculator or computer algebra system can solve mathematical problems (Howe, 1999). But using procedural knowledge is also to indicate the ability to choose the right procedure to solve the problems.

Moreover, Fan and Bokhove (2014) proposed an algorithms learning model which consists of three hierarchical cognitive development: (1) knowledge and skills; (2) understanding and comprehension; and (3) evaluation and construction. The first level is a straightforward employing of the rules; for example, how to calculate the gradient of a linear graph, or how to use the given gradient to draw a straight line. It leads to a view of rote learning. The English students' weakness in numeracy skills is largely related to this level according to their performance in the tests. The second level involves comprehending the procedure knowledge. In the case of gradient, the steepness rooted in the graph is determined by the degree of angle between the graph and horizon line. As the angle increases, the steeper the line is. The angles of any size can be positively related with trigonometry tanα which equals the graphic meaning of gradient. The third level, evaluation and construction, entails between choosing several algorithms in a complex situation to solve the problem. The latter two levels involve more reasoning abilities which are built on the first level, knowledge and skills. The lack of the first cognitive level of algorithms and teachers' views of pure knowledge will weaken the procedural fluency as well as the adaptive reasoning which are two aspects of mathematics proficiency. Results from the student tests showed that the majority of the Foundation Level and a considerable number of the Higher Level English students cannot go through the first level, so that it might be hard to improve their other two higher cognitive levels of algorithms learning.

9.3 Contribution

This section will highlight two main contributions of this study to the knowledge in comparative mathematics area and understanding development area. First is, to what extent, the cultural features in China support abstract mathematics understanding type. Secondly, the model of understanding is an innovative aspect of the understanding research area. The strengths of the model have been validated in the result part of the thesis and the limitations

will be discussed in next chapter. The further use of this model will be illustrated by its potential value and aware of its weaknesses.

9.3.1 Understanding mathematics and cultures

The first main contribution of this study is to link the main characteristics of Chinese culture with that of mathematics education, specifically shown in terms of curriculum, textbooks, and teacher's belief. Furthermore, these impacts from the cultural perspective directly or indirectly have shaped students' understanding development in Shanghai. Findings from this study have identified the main reason that Shanghai students performed better is their more abstract understanding of mathematics than their counterparts in England. The algebraic approach instead of the graphic way of approaching mathematics, in terms of linear function, can be viewed as the consequence of the requirements of abstract understanding and the foci of the basics.

Leung (2006) stated that according to TIMSS and TIMSS-R reports, the higher achieving East Asian students had surprisingly held lower positive attitudes towards maths than the Western students. It appears that this achievement is not strongly influenced by features of education system (centralised system or instruction in and out school). The cultural perspective in comparative studies with regards to mathematics teaching and learning, however, has been widely used to interpret the results (Leung, 1995, 2005, 2006; Leung & Park, 2002). Leung (1995) summarised four perspectives of CHC culture to explain the striking differences on mathematics education between China and West countries: (1) the integration and harmony social orientation of the Chinese, (2) the importance and necessity of memorization and practice, (3) the high expectations on student achievement and the parents' attribution, and (4) diligent and persevering attitude towards study.

The third perspective - higher expectation and parents' involvement, brings about the considerable possibility that students are encouraged by parents in the family, teachers in the

classroom, and ethos of the school to obtain higher performance from education traditional aspect. Competitive examinations in China would correspondingly have more complex questions. A key to the success of these important assessments is to acquire abstract understanding of maths. The degree of complexity has been embodied in curriculum, the compulsory textbooks, and teaching strategies towards higher understanding levels shown in this study. The result part has indicated that Shanghai had put more efforts in facilitating students' abstract understanding development than England did.

In terms of the forth perspective, the whole community appreciates the diligence and perseverance which lay the foundation for students to be proficient in three basics: basic knowledge, basic skills, and basic method from mathematics tradition. Students' efforts especially for mastering basic skills in China are more appreciated than their ability (Shiqi Li, 2006). Similarly, the findings from teacher interview show different teacher attitudes towards procedural knowledge.

The function of memorization from second cultural perspective has been written and acknowledged by the Shanghai local curriculum as the preliminary step towards understanding. Leung and Park (2002) replicated the study of profound understanding conducted by L. Ma (1999) to investigate Hong Kong and Korea primary teachers' competence in mathematics. Hong Kong and Korea students' performance both have consistently been in the top performance group in PISA and TIMSS while still being lower than Shanghai students. The teaching strategies used by their teachers tended towards procedurally understanding instead of conceptual understanding one which Shanghai teachers follow, according to Ma's study (Leung & Park, 2002). The findings also explained that the reason that East Asian students and teachers achieved mathematical competence would be 'through repeated practice of well-designed exercises that the learner progressively gains

conceptual understanding' (Leung & Park, 2002, pp. 127-128). In other words, the importance of memorization actually expedites the conceptual understanding.

Concerning Chinese social orientation, Chu and Choi (2011, p. 267) suggest that Chinese culture tends towards horizontal collectivism where social relationship development 'focus[es] on close bonding with great influence on attitudes, norms, and behaviours'. This kind of bonding has certainly nurtured the effectiveness of the whole-classroom instruction and large size of class. Meanwhile, the nature of harmony or bonding is to balance opposing views of mathematics, for example 'the application of maths and the formal nature of maths' (Zheng, 2006, p. 385). From the negative influence of culture, due to the high pressure of examination, teachers in China lacked adequate awareness of the application part and allowed little space to develop students' creativities (Zheng, 2006). In this study, compared with the outperformance of understanding tests in Shanghai (large effect size), results from the application did not have that stronger advantage (middle or small effect size). Conversely, English mathematics teaching has given much more attention to real-world examples (Kaiser, Hino, & Knipping, 2006).

The Chinese culture factors have gradually fostered the abstract mathematics understanding which conversely reinforces those cultural features. There is an on-going debate in academia about what can be learned in comparative studies due to completely different cultural background - the Western and the Eastern (Fan, 2011; Ginsburg et al., 2005; Jeynes, 2008). In case of America and East Asia, parent-teacher relationships and advocacy of effort instead of ability are considered, as America can learn and transfer from the Eastern culture (Jeynes, 2008). This suggestion is drawn from the culture perspective or education background. Similarly, what England can do is to rethink the role of procedural knowledge in understanding development and the importance of effort, even though English students are levelled by ability throughout their junior secondary school. The thesis, however, shows the

pragmatic approach towards the mathematics background: the balance point at understanding level, Object Analysis. It is important for Shanghai to stress more in the application part and release a little from the Inventising level of understanding; while England should emphasis more the in Object Analysis level of understanding.

In future research, the two areas under different cultures should collaborate with each other to improve students understanding development. To do so, this study is a start or a pilot to set out this collaboration. It does not mean comparative assessments or other kinds of collaborations were unimportant. A more fundamental aspect of comparative study is to focus on mathematics itself, the concept and how to present the concept. The potential differences in these two perspectives of mathematics present us with the chance to enrich and complete ourselves.

9.3.2 The model of understanding mathematics

The second main contribution is the model of understanding mathematics which has particular relevance in the areas of understanding development not only from student perspective, but also for teaching materials. As a result, student performance can be explained by mathematical aspect. After comparing the East Asian classroom with the West, the East Asian mathematics content was regarded much 'more complex and advanced' (Leung, 1995, p. 210). The model, however, has indicated a method to measure the complexity and advance within three perspectives: curriculum, textbooks and student performance. From the competition perspective, it is important to explore the degree of this complexity to which the model offers useful perspectives for understanding how the understanding develop and towards which level. Furthermore, the model of understanding developed shows the correlation among these three perspectives.

Secondly, the model gives the level of understanding function as well as the certain method at each level. It demonstrates that a certain method - algebraic one, seems to aid a

higher level of understanding development. In England, the CSMS study (Concept in Secondary Mathematics and Science) in the 1970s has had a far-reaching influence on investigating mathematics understanding (Hart, 1981). This study was designed to level the understanding instead of probing certain methods. The model, however, combines these two perspectives.

Thirdly, the model can be used to monitor the change of curriculum in terms of understanding development. Hodgen, Brown, Kuchemann, and Coe (2010) conducted the ICCAMS (Increasing Competence and Confidence in Algebra and Multiplicative Structures) as replicating CSMS in terms of three topics: Algebra, Decimals and Ratio 30 years later. Although algebra results illustrated that fewer Year 9 English students reached the higher Levels as possible negative trend. ICCAMS did not explore more at this point and the results cannot be explained well.

The role of curriculum should be listed as an indicator of investigating understanding development. In addition, this study shows that different arrangement in curriculum for learning mathematical concept. The algebraic concept in CSMS study was considered as variable and generalised number. These two weak understandings in algebraic concept shown in ICCAMS can also be found in this thesis which clearly indicates that English students, even in Year 10 Higher Level, have been facing difficulties in understanding variable due to lack of enough awareness of the variable perspective in the concept of function in the curriculum and selected textbooks.

The strength of the model will be the importance of analysing curriculum and textbook requirements; while this strength also causes the possible weakness. The standpoint of the model is based on the concept of function because the term 'linear function' is listed in both curricula. The textbooks, however, differ in their approach to linear function, with Shanghai predominantly exploring the context of function, and England introducing graphs

within the context of modeling real world situations. In the CSMS study, understanding linear graph had three levels: Level1 'which involve plotting points, others on interpreting block graphs, recognition that a straight line represents a constant rate and simple interpretation of the scattergram'; Level 2 which 'includes simple interpolation from a graph, recognition of the connection between rate of growth and gradient, use of scales shown on a graph, interpretation of simple travel graphs and awareness of the effect of changing the scale of a graph'; Level 3 which 'consists of items that require an understanding of the relation between a graph and its algebraic expression' (Hart, 1981, p. 134). Normally, the concept in mathematics and science are often thought to be the same across all contexts. The start understanding level obviously differs between linear function and linear graph. Further comparative study should compare the similarities and differences of the concept, not only in the curriculum, but also in the textbooks and how teachers deliver it.

9.4 Summary

Based on the summary of the key findings from the data, the student performance gap between England and Shanghai from the understanding mathematics viewpoint was explained from three perspectives: representations (discussion of research question 2), the role of skill-algorithm, and mathematics proficiency (both in discussion of research question 4). The different backgrounds of understanding mathematics were distinguished from official documents (discussion of research question 1) and the teaching approach (discussion of research question 3). In the next chapter, it is hoped that findings from the current study can be used to enhance students' understanding in each region and also indicate further research threads to be developed.

Chapter 10 Conclusion

In the first section of this chapter, the additional limitations after having conducted the current study will be addressed. The second section will discuss the implications of the study for three groups of potential stakeholders: teachers, researchers and education authorities. General recommendations for future studies in the mathematics education area related to England and Shanghai will also be put forward.

10.1 Limitations of the Current Research

The students' achievement can be influenced by many factors, for example the different motivation towards learning, parental engagement, attribution, life outside of school, and peer influence (Elliott, Hufton, Ilyushin, & Willis, 2005), but this study only focused on student learning with understanding of the mathematical concept. After having conducted all the data collection and data analysis, there were three further restrictions which will be identified as follows:

First, the model of understanding function was successfully applied for the Higher Level English and the Shanghai students. However, due to the low reliability of the tests for English Foundation Level students, their understanding development did not fit well with this model. This failure mainly resulted from the complexity of questions which proved to be too difficult for them since the starting level of understanding linear function was Level 3 Connecting Representations. This might have resulted from the fact that in the pilot study, only the highest set of Foundation Level students were involved. Therefore, the findings from their performance failed to apply to other sets in the Foundation Level. It is recommended that further study should focus on Foundation Level students' understanding.

Secondly, calculators were not allowed for all the participants in order to balance the two regions' situations, partly because no sophisticated calculation was involved in any self-

design test. Another reason was that the uniform assessments in Shanghai for all grades of junior secondary school students did not allow the use of calculator, even though the English GCSE mathematics tests included both calculator and non-calculator types. This will have likely affected the English students' performances more than that of the Shanghai students. The English students, especially at the Foundation Level, showed their weakness in numeracy skills, and the selected teachers mentioned this as their main barrier for learning mathematics. It is unclear if students would performance better with the aid of calculators.

Thirdly, although some examples from the classroom observations in both England and Shanghai were used in this study, the whole classroom analysis was not involved. There may have been a gap between the teachers' planned activities and what they actually implemented in the classroom, and the effectiveness of these activities or instructions. But this study predominately focused on students' understanding rather than what teacher actually did. Further research can explore which instruction is better for certain level of students.

10.2 Implications

10.2.1 Practical implication

Better handling with linear function. How to overcome Level 3 for the Foundation Level English students, Level 4 for the Higher Level and the Shanghai students' weaknesses of graphic representation will be addressed to provide suggested ways that teachers can do better.

At Level 3, the English Foundation students met challenges in calculations related to negative numbers. The solution as identified in an English teacher's interview was to fill the table from right to left, as Fan and Bokhove (2014, p. 489) recommended 'direct teaching', which involves telling, demonstration, drill-and-practice, and remediation. Another way was to do more practice. The selected English teachers indicated that several similar types of practice could reinforce these skills.

At Level 4, Shanghai students are not expected to understand the graphic meaning of gradient, while English students are not required to comprehend the algebraic meaning. The approach has restricted Shanghai students' understanding of the graphic issues, such as the basic method: the combination of symbolic-graphic. On the other hand, as the previous discussion on the drawbacks of graphic representations noted, the formation of algebraic expressions may be rooted at this level for English students. It is therefore recommended that teachers provide the alternative approach for students to explain the meaning of gradient. By using the different approaches in order to make sense of the gradient, the concept would be reinforced. Furthermore, Shanghai students may counteract their graphical defects and lay a foundation for the basic method. The algebraic approach can not only be complementary, but also exemplified by the graph and vice versa. It seems that striking a balance between algebraic and graphic is the possible answer that will enable the facilitation of understanding development. Such awareness of the drawbacks of the textbooks, however, could hardly derive from the teachers themselves, because their views were constrained by the curriculum and textbooks as discussed in the Summary and Discussion chapter.

Through the comparative study, how to properly handle the topic leads to an inquiry of borrowing good practice from others.

Borrowing good practice. If good practices are adopted for other regions' students, a detailed examination of these practices will be necessary to figure out the obstacles and to determine the extent to which students could make sense of these (Sparapani, Perez, Gould, Hillman, & Clark, 2014). Shanghai handles the topic in the traditional symbolic way, while England does it both pictorially and with time-based graphs. It is reasonable to speculate that simply borrowing either practices or textbooks cannot effectively work for both English and Shanghai students due to their different approaches to basic knowledge. Furthermore, this speculation is analogous to that of Leung (2005) as the practice include cultural values which

are difficult to transfer. Instead of borrowing practice from other countries, it is recommended to extend the understanding of the alternative way to presenting the knowledge. Further research can investigate how other mathematical concepts were introduced first, and then the supplementary way can be found as well as the corresponding practice.

10.2.2 Methodological implications

The selection of sample. Results from the different arrangements of the subject content indicate that scheduling of the sample of students leads to the consideration that what students are expecting to learn is based on priority rather than age- or grade-based when investigating the understanding of mathematics knowledge. This justifies my chosen sample, which placed emphasis on when the linear function has been fully learned in two regions. In a comparative study, therefore, this knowledge- based factor in related to the sample chosen should be considered.

How to form a fair assessment. Clarke (2003, p. 153) considered three issues related to international comparative tests: 'curricular alignment', 'equity', and 'data aggregation'. First, the degree of alignment between the test and the mathematics curriculum influences student achievement. The test in this study was drawn from not only matching the statutory requirements in the England and Shanghai curriculum, but also how the official textbooks presented the research topic. It increased the validity of tests as a measure of understanding basic knowledge and application. Secondly, students' particular characteristics of mathematical understanding were considered as an equity issue, for example how to present the questions. It leads to the data aggregation issue. For example, the question at Level 5 produced different requirements for the English and Shanghai students between the two phases of the tests. The process of how to choose the questions has also indicated a way of measuring students' understanding more fairly. As Cai (1995, p. 106) recommended, the use of 'a wide array of mathematical tasks' in comparative studies should be carried out, with

questions being selected considering the students' familiarity with certain expressions, ensuring that the problems are in line with questions they had faced in the class and therefore enabling them to achieve the requirements of their syllabus, textbooks, and assessments. Before designing the tests, a thorough examination of related mathematical topics is recommended.

10.2.3 Implications for education authorities

England: teachers' professional development support. From the previously mentioned English teacher interviews, the Head of Maths (ENT1) would convene a meeting for all the maths teachers once a half term. Conversely, China has a more formal and sophisticated system for in-service teacher professional development (Huang & Li, 2008). Chinese teachers have gotten the chance to develop substantial subject knowledge and are offered ample opportunities to observe others classrooms. Such a research-teacher system is available at the town, city, and provincial levels, as it plays a crucial role within in-service teachers' professional development (J. Li, 2008).

English teachers do not have much settled time for their reflection, although they refer to a specified website for advice or support. The issue remains as to what extent the in-service teacher professional development is beneficial for teachers' lesson plans compared with the website in England on which teachers can share their experience, find resources, and acquire guidance. The implication for educational authorities is to establish more effective face-to-face network support for teachers' professional development.

Shanghai: the curriculum approach. The English curriculum approach is a spiralling kind, while Shanghai follows a linear non-repeated approach. Leung (1987) argued that a linear approach leads to a limited scope of mathematics. This was in line with the drawback of this approach reported by the Shanghai teacher that students might not be able to discern which type of function should be used in the real world situation. Students would

automatically draw on the linear function for all questions. Leung suggested that the ideal solution to this restriction would be the acquisition of a broader knowledge at a lower level alongside a deeper knowledge of one or two topics. There remains the question of how much deeper and what kinds of topics should be focussed upon. Findings from this study suggest that the ideal level is located at Level 5 Object Analysis in linear function.

10.3 Recommendations for Further Study

This study can be extended and enriched, for example, in order to examine the effects of the general model for further study and could reproduce it in research with other countries. Apart from this, another two potential studies will be addressed in this section: the balance approach towards linear function, and the Chinese learning theory.

Practically speaking, the purpose of identifying students' barriers to understanding is for them to learn this topic more effectively. Some tentative suggestions for overcoming these barriers have been provided for Levels 3 and 4. In terms of the higher levels, Levels 5 and 6, which are based on the solid foundations of Levels 3 and 4, the Shanghai students' understanding barriers are possibly caused by a deficiency in the comprehension of graphic meaning, while English students lack an understanding of the algebraic meaning. It is therefore recommended for researchers to design an experimental study; for example, by comparing English students who are provided with an algebraic meaning with students with the traditional English approach, or Shanghai students who would be offered the graphic meaning compared with those only experiencing the algebraic approach, in order to discern how well the balanced way would facilitate their understanding development of linear function.

Western educators, who use Western methods to interpret the success of Chinese learners, have produced opposing conclusions, called the paradox of the Chinese learner by Biggs and Watkins (1996). Furthermore, Huang and Wong (2007) have made an appeal to

establish a learning theory based on cultural and societal factors for the Chinese community instead of duplicating Western theories.

Previous researches have investigated different learning environments and teaching approaches, as Biggs (1996) clarified misconceptions held by the West for proficient learning under CHC classroom, and Huang (2002) identified characteristics of Chinese mathematics teaching by comparing Hong Kong and Shanghai classrooms. This study indicated the importance of the four basics for students' understanding development. A feasible way to establish learning theory in Shanghai or China is to combine these previous studies with understanding development by highlighting the role of the four basics, after examining several comparable topics, by taking the western perspective as a mirror to reflect on what Shanghai or China is doing.

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Appendices

Appendix A Informed Consent

School of Education

Leazes Road

Durham, DH1 1TA

Researcher: Yuqian (Linda) Wang

Email: yuqian.wang@dur.ac.uk

PhD Research Project Informed Consent Form

Thesis Title: Understanding Linear function in Secondary School Students: A Comparative Study between England and Shanghai

Research description:

This research focuses on students' understanding in mathematics. Its main objective is to identify (1) When learning linear function/graph, what barriers are experienced by students? and (2) To what extent can students be supported through these barriers?

I have been informed provided with the project information and give my full consent to participate in this research project. I have had the opportunity to discuss my concerns with the researcher, and have understood the aims and objectives of the research. All information that I provide will be confidential, and the completed PhD thesis will be available through the School of Education, Durham University.

I understand that I am free to withdraw from the study at any time, without having to give a reason, and that I may withdraw any information provided.

I agree to information that I provide to being digitally recorded.

SIGNED: .... **DATE:** ....

Appendix B Pilot test for basic knowledge

Name:

D.O.B. (DD/MM/YY):

Circle a or b: a. Male   b. Female

1. Complete this table for $y = x + 1$

X	-2		3	6
Y		1		

On the grid, draw the graph of $y = x + 1$ for x from -2 to 6.
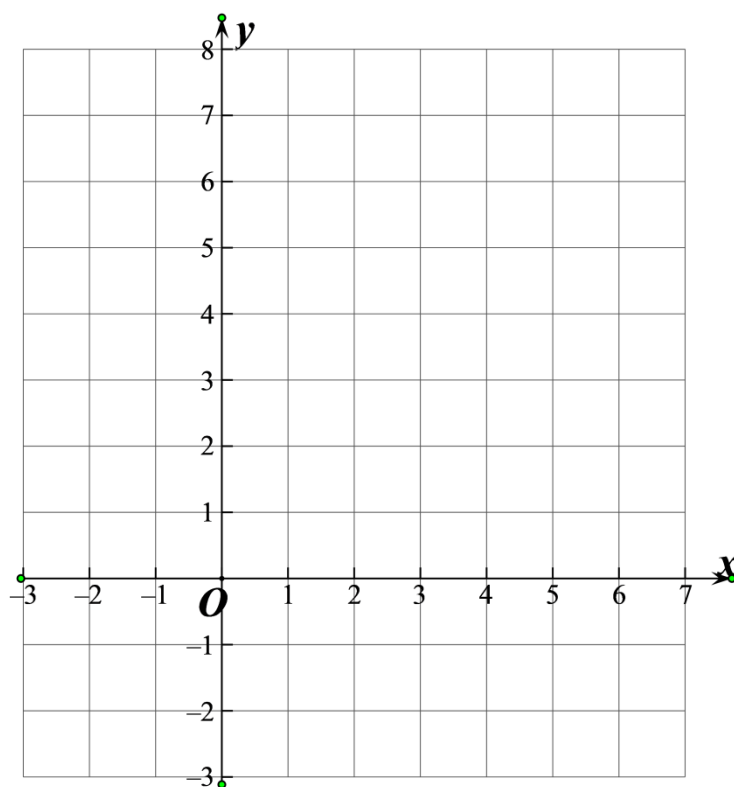
2. A straight line passes through the point (0, 2) and (-2, 0). Find the equation of this line.

3. Find the intercept of the straight line $y = 2(x - 1) + 5$.

4. Find the gradients of BC and AC.
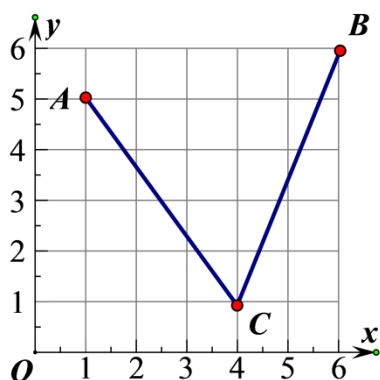
5. The diagram shows lines A and B. The equation of the line A is $y = 3x + 5$. The straight line B is parallel to A. Find the value of p.
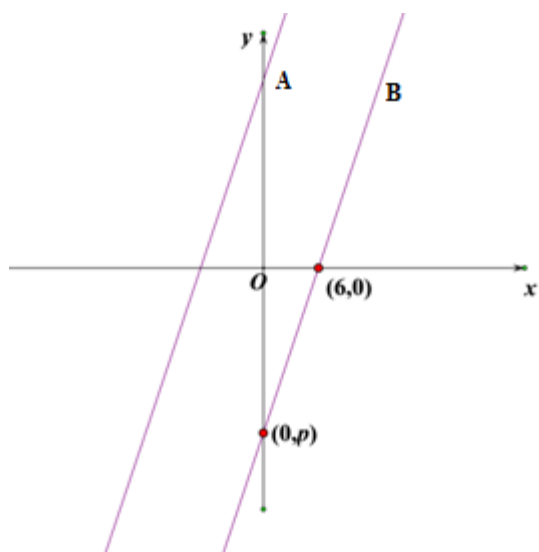
6. A straight line passes through the point (0, 3) and is parallel to $y = -2x + 1$. Find the equation of this straight line.

7. A straight line $y = 2x - 1$ (as seen below) will be translated upward 4 units. Find the equation of the new line.
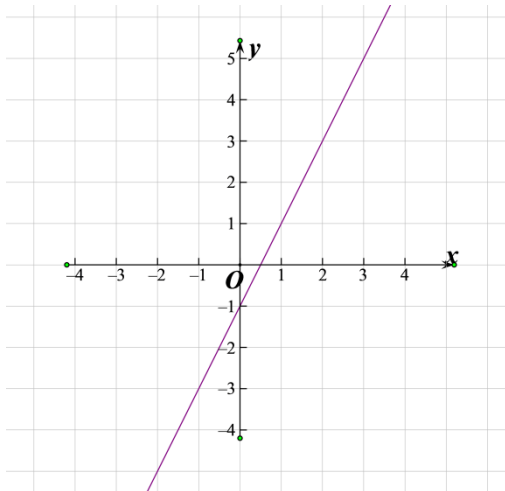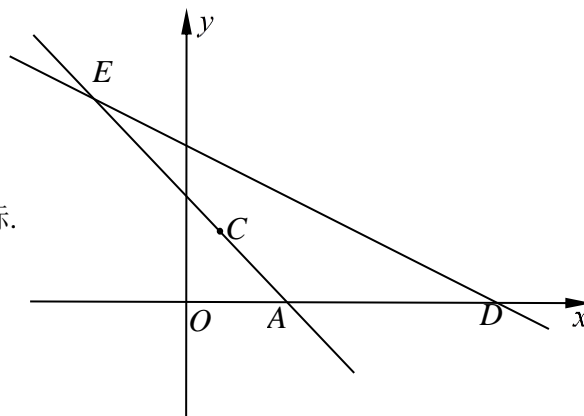
8. A straight line $y = kx + b$ is parallel to another straight line $y = 3x + 4$. The intercept of this straight line is 3. Find the equation of this straight line.

9. A straight line $y = -x + b$, passes through the point C (2, 4) and meets the x-axis at point A. Another straight line DE meets the x-axis at point D (18, 0). The straight lines DE and AC have the point of intersection E. Point E is located at the second quadrant.

    1) Find b.

    2) Find the coordinate of point A.

    3) Find the length of segment DA.

    4) If the area of triangle DAE is 72, find the coordinate of point E.
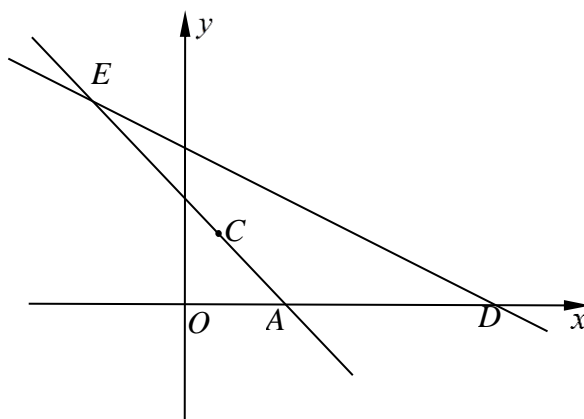
Appendix C Main test for Higher Level students – basic knowledge

Birthday (DD/MM/YY):   **Circle a or b: a Male b Female**

1. Find the equation of the line shown in diagram. Show how you found your answer.
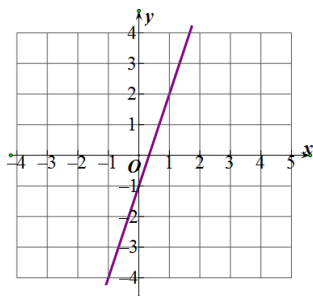
2. Find the gradients of BC and AC.
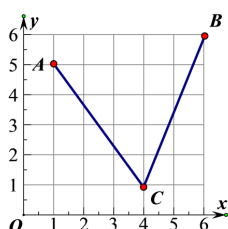
3. A straight line passes through the point (0, 3) and is parallel to $y = -2x + 1$. Find the equation of this straight line.

4. A is the point (1, 5). B is the point (3, 3). Find the equation of the line perpendicular to AB and passing through the midpoint of AB.
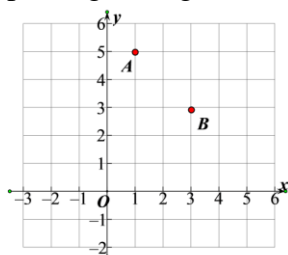
5. A straight line $y = 2x - 1$ (as seen below) will be translated upward 4 units. Find the equation of the new line.
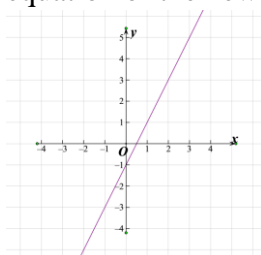
Appendix D Main test for Foundation Level students – basic knowledge

Birthday (DD/MM/YY):   **Circle a or b: a Male   B Female**

1. Complete this table for $y = x + 1$

X	-2	0	3	6
Y				

On the grid, draw the graph of $y = x + 1$.
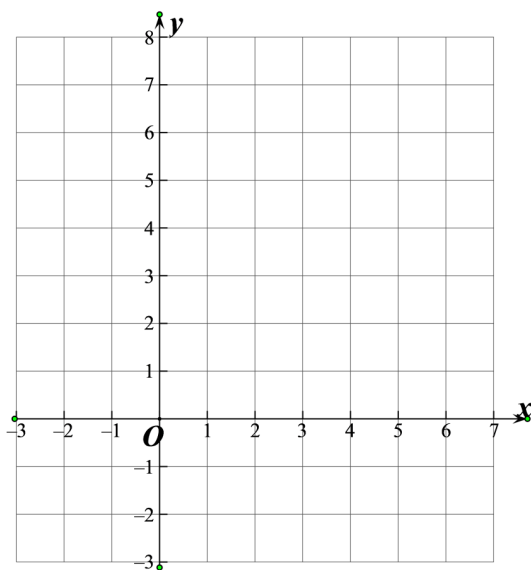
2. Other than the method of finding points, could you find another way of plotting $y = 2x - 1$? Explain your method.
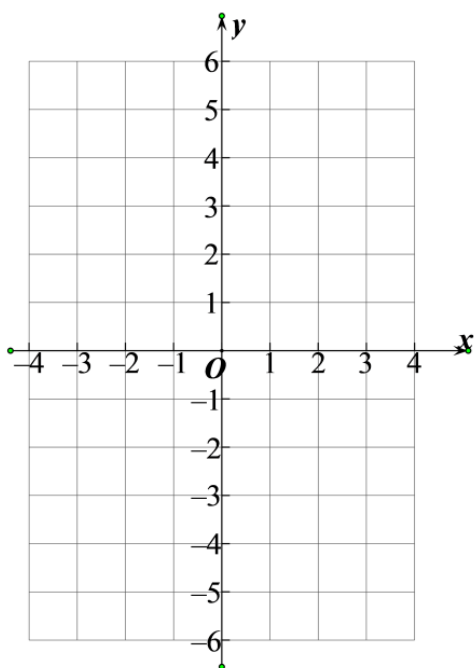
3. Find the equation of the line shown in diagram. Show how you found your answer.
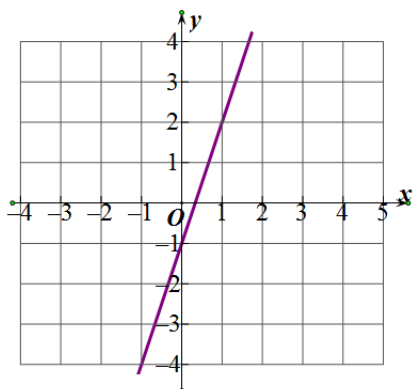
4. Find the gradients of BC and AC.
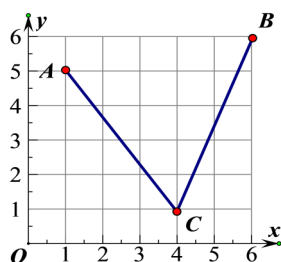
5. A straight line $y = 2x - 1$ (as seen below) will be translated upward 4 units. Find the equation of the new line.

Appendix E Main test for Shanghai students (the Chinese version with English translation) – basic knowledge

姓名(name):   班级(class)：   学号(Enrolled No.)：

性别(Gender)：   出生年月日(Birth: YY/MM/DD):

1. 一条直线经过点 (0,3) 并且平行于直线 $y = -2x + 1$. 求这条直线的表达式.

    (A straight line passes through the point (0, 3) and is parallel to $y = -2x + 1$. Find the equation of this straight line.)

2. 直线 $y = 2x - 1$ 向上平移 4 个单位. 求平移后的直线表达式.

    (A straight line $y = 2x - 1$ will be translated upward 4 units. Find the equation of the new line.)

3. 一次函数 $y = (k-1)x + k$ 中，y 随着 x 的增大而减小，求 k 的取值范围.

    (Linear function $y = (k-1)x + k$, when the value of x increase, the value of y increases as well. Find out the range of k.)

4. 如图，在平面直角坐标系中，直线 AC: $y=-x+b$ 经过点 C（2,4），与 x 轴相交于点 A，直线 DE 与 x 轴交于点 D（18,0），直线 DE 与直线 AC 都经过点 E，且点 E 在第二象限.

（1）求 b；
（2）求点 A 坐标；
（3）求线段 DA 长度；
（4）若△DAE 的面积为 72，求点 E 坐标.

(A straight line, $y = -x + b$, passes through the point C (2, 4) and meets the x-axis at point A. Another straight line DE meets the x-axis at point D (18, 0). The straight lines DE and AC have the point of intersection E. Point E is located at the second quadrant.

1) Find b.

2) Find the coordinate of point A.

3) Find the length of segment DA.

4) If the area of triangle DAE is 72, find the coordinate of point E.)

5. 已知一次函数 $y=x+2$ 与反比例函数 $y=\dfrac{k}{x}$，其中一次函数 $y=x+2$ 的图象经过点 P$(k,5)$.

(1) 试确定反比例函数的表达式；
(2) 若点 Q 是上述一次函数与反比例函数图象在第三象限的交点，求点 Q 的坐标.

(The linear function $y = x + 2$, and the reciprocal function $y = \dfrac{k}{x}$, the graph of linear function $y = x + 2$ passes by the point P$(k, 5)$.

(1) Find out the algebraic expression for this reciprocal function;

(2) If the point Q is the intersection of the linear function and reciprocal function at the third quadrant, find out the coordinate of point Q.)

Appendix F A pilot test for application

Name:

1. Here are some matches formed into a sequence of triangles.
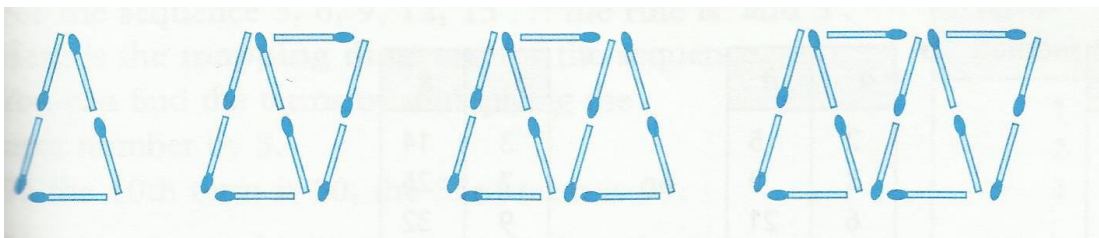
a. Complete this table.

Triangles (t)				
Matches (m)				

b. Find a formula connecting m and t

Formula: m=

2. Bill works for a company that delivers parcels.

For each parcel Bill delivers there is a fixed charge plus £1.00 per mile he has to drive thereafter.

You can use the graph below to find the total cost of having a parcel delivered by Bill.

a. How much is the fixed charge?

Ed works for a rival delivery company.

There is no fixed charge but for each parcel Ed delivers it costs £1.50 for per mile.

b. Compare the cost of having a parcel delivered by Bill with the cost of having a parcel delivered by Ed. At what point would you rather have a parcel delivered by Bill? At what point are the costs of having a parcel delivered by Bill and Ed identical?

3. Steve travelled from home to school by walking to the bus stop and then catching the bus to school.

a) Use the information below to construct a travel graph showing Steve's journey.

Steve left home at 8.00 am.

He walked at 6 km/h for 10 minutes. He then waited for 5 minutes before catching the bus.

The bus took him a further 8 km to school at a steady speed of 32 km/h.
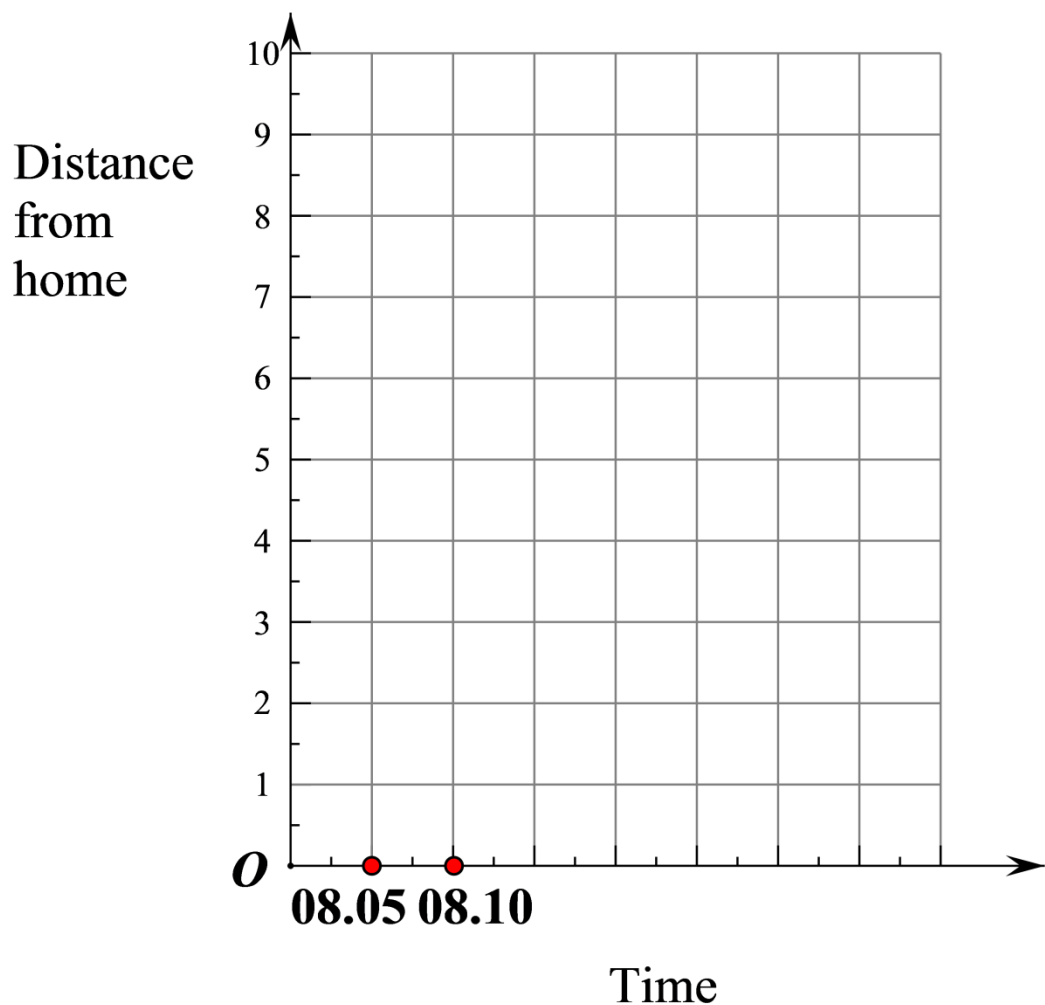
b) How long would it take Steve to cycle from home to school at an average speed of 15 km/h? Give your answer in minutes.

4. A mobile phone company offers a newest mobile phone either of the following plans:

- The Seldom Plan: this involves a monthly subscription charge of £7.50, plus 25p per minute on phone calls. The first 30 call minutes are free.

- The Often Plan: this involves a monthly subscription charge of £22.50, plus 15p per minute on phone calls. The first 80 call minutes are free.

  a. Nadia usually uses her phone for about 100 minutes per month. Which plan should Nadia choose the Seldom Plan or the Often Plan? Explain your answer.

  b. Find a way to show how the costs of the Often Plan vary when you call less or more minutes each month.

Appendix G Main test for English students – application

6. Here are some matches formed into a sequence of triangles.
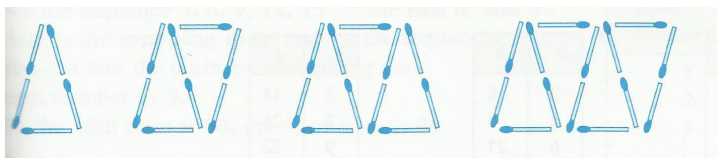
   a. Complete this table.

Triangles (t)				
Matches (m)				

   b. Find a formula connecting m and t

   Formula: m=

7. This graph shows the hire charge for heaters over a number of days.

   a) Calculate the gradient of the line.

   b) What is the basic charge before the daily hire charge is added on?

   c) Write down the rule used to work out the total hire charge.
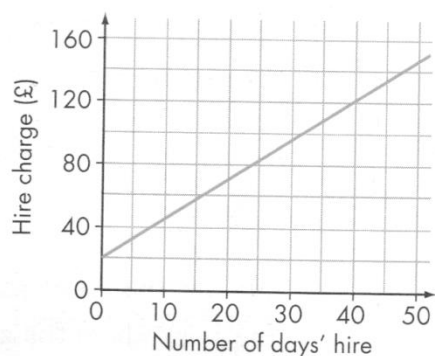
8. Bill works for a company that delivers parcels. For each parcel Bill delivers there is a fixed charge plus £1.00 per mile he has to drive thereafter.

You can use the graph below to find the total cost of having a parcel delivered by Bill.

a. How much is the fixed charge?

Ed works for a rival delivery company. There is no fixed charge but for each parcel Ed delivers it costs £1.50 for per mile.

b. Compare the cost of having a parcel delivered by Bill with the cost of having a parcel delivered by Ed. At what point would you rather have a parcel delivered by Bill? At what point are the costs of having a parcel delivered by Bill and Ed identical?

9. The graph shows the journeys of a bus and a car along the same road. The bus goes from Leeds to Darlington and back to Leeds. The car goes from Darlington to Leeds and back to Darlington.
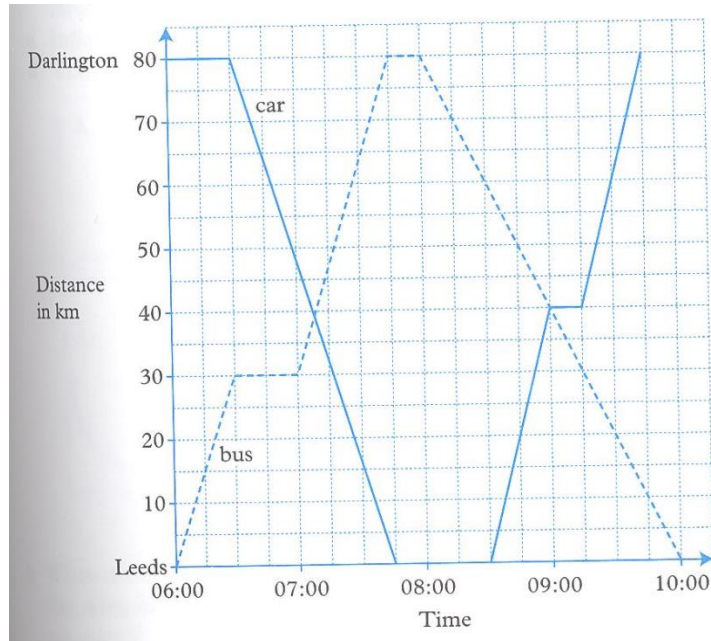
i. When did the bus and the car meet for the second time?

ii. At what speed did the car travel from Darlington to Leeds?

iii. What was the average speed of the bus over its entire journey?

iv. Approximately how far apart were the bus and the car at 09.45?

v. What was the greatest speed attained by the car during its entire journey?

Appendix H Main test for Shanghai students (the Chinese version with translation of English) - application

姓名：   学号：   班级：

6. 用若干根火柴棒拼成如下图所示的三角形，
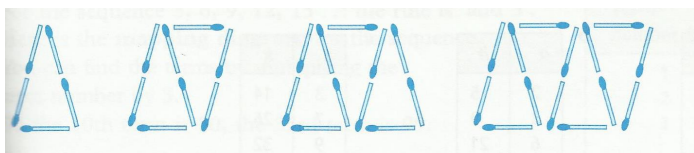
根据图示填表，并写出 Y 关于 X 的函数表达式：

三角形数量 x (个)	1	2	3	4	...
火柴棒数量 y (根)					...

函数表达式为

Here are some matches formed into a sequence of triangles.

a. Complete this table.

Triangles (t)				
Matches (m)				

b. Find a formula connecting m and t

Formula: m=

7. 小白用货车运送快递，每英里收取 1 英镑钱的运费。下图显示了小白运送的总费用和英里数的函数图像.

Xiao Bai works for a company that delivers parcels. For each parcel Xiao Bai delivers there is a fixed charge plus £1.00 per mile he has to drive thereafter. You can use the graph below to find the total cost of having a parcel delivered by Xiao Bai.

运费（英镑）

路程（英里）
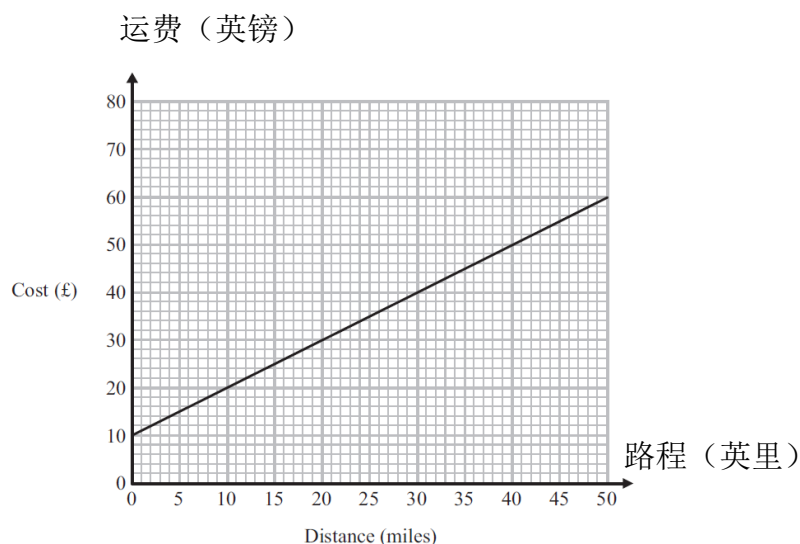
a. 小白运送快递的起步价是多少英镑？

How much is the fixed charge?

小花运送快递没有起步价，但是每英里收取 1.5 英镑钱的运费.

b. 比较小白和小花的快递运费用后回答：什么情况下选用小白运送快递、什么情况下选用小花运送快递运费合算？

Xiao Hua works for a rival delivery company. There is no fixed charge but for each parcel Xiao Hua delivers it costs £1.50 for per mile.

Compare the cost of having a parcel delivered by Xiao Bai with the cost of having a parcel delivered by Xiao Hua. At what point would you rather have a parcel delivered by Xiao Bai? At what point are the costs of having a parcel delivered by Xiao Bai and Xiao Hua identical?

8. 下图显示了巴士和轿车行驶同一旅程的情况。巴士从利兹（Leeds）到达灵顿（Darlington）， 然后返回利兹（Leeds）. 轿车从到达灵顿（Darlington）到利兹（Leeds）， 然后返回达灵顿（Darlington）.

The graph shows the journeys of a bus and a car along the same road. The bus goes from Leeds to Darlington and back to Leeds. The car goes from Darlington to Leeds and back to Darlington.
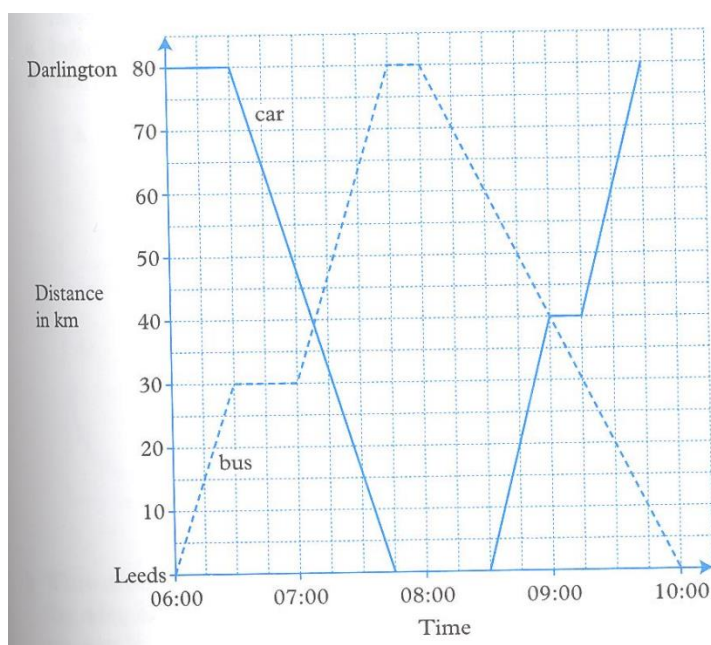
   i.   什么时候巴士和轿车第二次相遇？

When did the bus and the car meet for the second time?

   ii.   求轿车从到达灵顿到利兹的行驶速度.

At what speed did the car travel from Darlington to Leeds?

   iii.   求巴士在整个旅程中的平均速度.

What was the average speed of the bus over its entire journey?

   iv.   在 9：45 分时，求轿车和巴士的距离.

Approximately how far apart were the bus and the car at 09.45?

9. 某通讯公司对手机收费推出两种不同的套餐：

A: 每月月租为 7.5 元, 接听免费，打出电话则每分钟 0.25 元. 前 30 分钟打出电话免费.

B: 每月月租为 22.5 元, 接听免费，打出电话每分钟 0.15 元. 前 80 分钟打出电话免费.

A mobile phone company offers a newest mobile phone either of the following plans:

A: this involves a monthly subscription charge of RMB7.50, plus 25p per minute on

phone calls. The first 30 call minutes are free.

B: this involves a monthly subscription charge of RMB22.50, plus 15p per minute on

phone calls. The first 80 call minutes are free.

1) 小明每个月大约要打出电话 150 分钟. 哪个套餐更适合他? 为什么？

Xiao Ming usually uses her phone for about 100 minutes per month. Which

plan should Xiao Ming choose, the Plan A or the Plan B? Explain your

answer.

2) 对于 B 套餐 求函数解析式并画出图像.

Using algebraic expression and drawing the graph to show how the costs of the

Plan B vary when you call less or more minutes each month.

Appendix I The Requirement of Shanghai Curriculum in terms of Linear Function

2	学习主题	以丰富实例为背景,学习一次函数以及利用图像直观研究函数的方法。重视一次函数、二元一次方程、直线之间的沟通,体会知识的联系和整体性。在认识具体函数的基础上,理解函数的意义,总结表示函数的常用方法,了解抽象的函数符号的意义
	一次函数 (13 课时)	在应用一次函数解决实际问题的过程中,认识建立函数模型的方法和意义 在一次函数的图像研究和实际应用中,重视利用 DIMA 平台
学习内容		学 习 要 求 及 活 动 建 议
一次函数的概念		1. 以实例为背景引入一次函数,理解一次函数的概念,建立一次函数、二元一次方程、直线之间的联系;掌握直线平移与一次函数解析式 $y = kx + b$ 中的 b 之间的关系。从中感知辩证的观点,进一步体会数形结合思想
一次函数的图像和性质		2. 会画一次函数的图像,并借助图像直观,认识和掌握一次函数的性质
一次函数的应用		3. 选取实例讨论一次函数的实际应用,初步认识函数模型
函数的表示方法		4 通过实例分析以及正比例函数、反比例函数、一次函数等案例,理解函数的意义,知道函数的表示方法有解析法、列表法、图像法,知道符号"$y = f(x)$"的意义(说明)
说明		有关求函数值的问题,不涉及繁复的计算;关于函数的值域,只要求了解其含义,不涉及求值域的问题

Figure 1. The requirements in terms of linear function in the Shanghai curriculum (p.66)

学习内容 (content)	学习要求及活动建议 (The requirements and advice)
一次函数的概念 (The concept of linear function)	1. 以实际为背景引入一次函数,理解一次函数的概念, (linear function is introduced from real life background, understand the concept of linear function)
一次函数的图像和性质 (The graph of linear function and the properties)	建立一次函数,二元一次方程,直线之间的关系, (To establish the relationship among linear function, linear equation for two unknowns and straight line) 掌握直线平移与一次函数解析式$y = kx + b$中 b 之间的关系。 (To grasp the relationship between the motion of straight line and *b* in algebraic expression $y = kx + b$)
一次函数的应用 (Application of linear function)	
函数的表示方法 (The expression of function)	从中感知辩证的观点,进一步体会数形结合的思想 (to perceive dialectical ideas, To further experience the basic method: the combination of symbolic-graphic) 2. 会画一次函数的图像, (To plot the graphical representation of linear function) 并借助图像的直观,认识和掌握一次函数的性质 (and from the intuition of graph, to know and master the properties of linear function using the graph) 3. 选取实例讨论一次函数的实际应用,初步认识函数模型

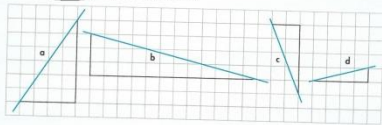|	(from the chosen examples to apply linear function into real world situation and formalise initial function model) 4. 通过实例分析以及正比例函数，反比例函数，一次函数等案例，理解函数的意义， (through analysis of examples and cases of proportional function, inverse proportional function, linear function and so on, to understand the meaning of function) 知道函数的表示方法有解析法，列表法，图像法，知道符号 'y = f(x)' 的意义 (to know there are analytical, tabular, graphic way to present function, to know the meaning of symbol 'y = f(x)')

Figure 2. The English translation of the requirements

Appendix J An example of how to allocate the understanding level in textbook

Remember: Take care when finding the differences between the coordinates of the two points. Choose one point as the first and the other as the second, and subtract in the *same order* each time to find the difference. When a line slopes *down from right to left* (/) the gradient is always positive, but when a line slopes *down from left to right* (\) the gradient is always negative, so you must make sure there is a minus sign in front of the fraction.

EXAMPLE 6   L4

Find the gradient of each of these lines.

In each case, a sensible choice of triangle has already been made.

a y-difference = 6, x-difference = 4   Gradient = $6 \div 4 = \frac{3}{2} = 1.5$

b y-difference = 3, x-difference = 12   Line slopes down from left to right,

so gradient = $-(3 \div 12) = -\frac{1}{4} = -0.25$

c y-difference = 5, x-difference = 2   Line slopes down from left to right,

so gradient = $-(5 \div 2) = -\frac{5}{2} = -2.5$

d y-difference = 1, x-difference = 4   Gradient = $1 \div 4 = \frac{1}{4} = 0.25$
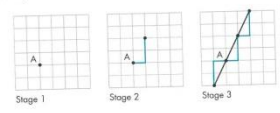
Drawing a line with a certain gradient

To draw a line with a certain gradient, you need to 'reverse' the process described above. Use the given gradient to draw the right-angled triangle first. For example, take a gradient of 2.

Start at a convenient point (A in the diagrams opposite). A gradient of 2 means for an x-step of 1 the y-step must be 2 (because 2 is the fraction $\frac{2}{1}$). So, move one square across and two squares up, and mark a dot.

Repeat this as many times as you like and draw the line. You can also move one square back and two squares down, which gives the same gradient, as the third diagram shows.
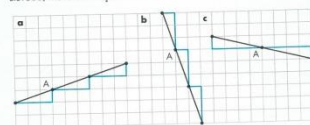
Stage 1   Stage 2   Stage 3

Remember: For a positive gradient you move across (left to right) and then *up*. For a negative gradient you move across (left to right) and then *down*.

EXAMPLE 7   L4

Draw lines with these gradients **a** $\frac{1}{3}$ **b** -3 **c** $-\frac{1}{4}$

a This is a fractional gradient which has a y-step of 1 and an x-step of 3. Move three squares across and one square up every time.

b This is a negative gradient, so for every one square across, move three squares down.

c This is also a negative gradient and it is a fraction. So for every four squares across, move one square down.

$y = mx + c$

All linear graphs can be expressed in the form $y = mx + c$ where m is the gradient and c is the value of the y-intercept. This means that if you know the gradient, m, of a line and its intercept, c, on the y-axis, you can write down the equation of the line immediately.
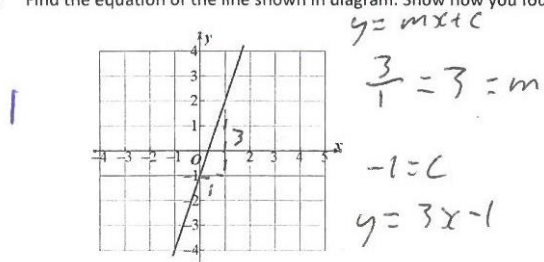
For example, if $m = 3$ and $c = -5$, the equation of the line is $y = 3x - 5$. This gives a method of finding the equation of any line drawn on a pair of coordinate axes.
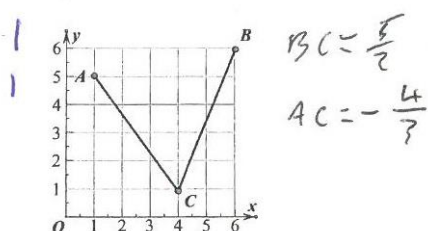
Appendix K An example for one English Higher Level student's answer in the understanding test

1. Find the equation of the line shown in diagram. Show how you found your answer.

$y = mx + c$

$\frac{3}{1} = 3 = m$

$-1 = C$

$y = 3x - 1$

2. Find the gradients of BC and AC.

$BC = \frac{5}{2}$

$AC = -\frac{4}{3}$

3. A straight line passes through the point (0, 3) and is parallel to $y = -2x + 1$. Find the equation of this straight line.
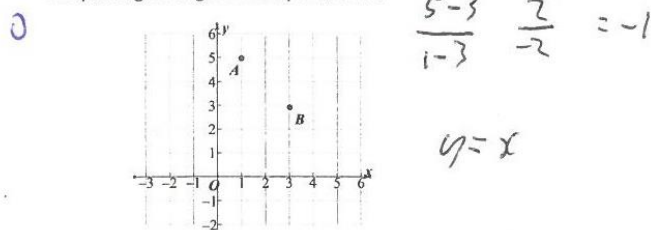
-2

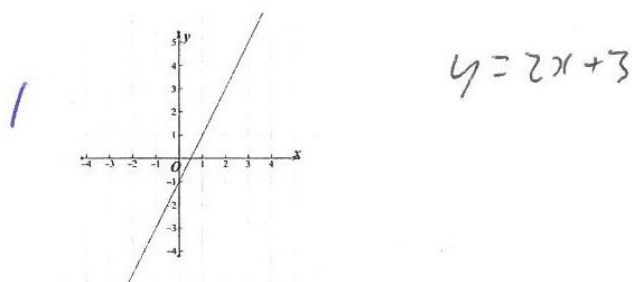$3 = -2 \times 0$ $3 = 0 + 3$

$y = -2x + 3$

4. A is the point (1, 5). B is the point (3, 3). Find the equation of the line perpendicular to AB and passing through the midpoint of AB.

$\frac{5-3}{1-3}$ $\frac{2}{-2} = -1$

$y = x$

5. A straight line $y = 2x - 1$ (as seen below) will be translated upward 4 units. Find the equation of the new line.

$y = 2x + 3$

Appendix L An example for one Shanghai student's answer in the understanding test

1. 一条直线经过点 (0,3) 并且平行于直线 $y = -2x + 1$. 求这条直线的表达式.

设 $y = -2x + b$
把 $x = 0, y = 3$ 代入, $\therefore y = -2x + 3$
$\therefore 3 = b$.

2. 直线 $y = 2x - 1$ 向上平移 4 个单位. 求平移后的直线表达式.

$y = 2x + 3$

3. 一次函数 $y = (k-1)x + k$ 中, y 随着 x 的增大而减小, 求 k 的取值范围.

$k - 1 < 0$
$k < 1$

4. 如图, 在平面直角坐标系中, 直线 AC: $y = -x + b$ 经过点 C (2,4), 与 x 轴相交于点 A, 直线 DE 与 x 轴交于点 D (18,0), 直线 DE 与直线 AC 都经过点 E, 且点 E 在第二象限.

(1) 求 b;
(2) 求点 A 坐标;
(3) 求线段 DA 长度;
(4) 若△DAE 的面积为 72, 求点 E 坐标.

(1) 把 $x = 2, y = 4$ 代入 $y = -x + b$.
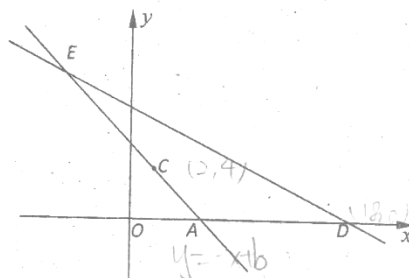$4 = -2 + b$
$b = 6$

(2) 把 $y = 0$ 代入 $y = -x + b$
$\therefore 0 = -x + b$
$x = 6$. $\therefore A(6, 0)$

(3) $18 - 6 = 12$. $DA = 12$

(4) $S_{\triangle DAE} = 6 \times h \times \frac{1}{2} = 72$ 把 $y = 24$ 代入 $y = -x + b$.
$h = 24$. $24 = -x + b$
                                $x = -18$ $\therefore E(-18, 24)$

5. 已知一次函数 $y = x + 2$ 与反比例函数 $y = \frac{k}{x}$, 其中一次函数 $y = x + 2$ 的图象经过点 P(k, 5).

(1) 试确定反比例函数的表达式;
(2) 若点 Q 是上述一次函数与反比例函数图象在第三象限的交点, 求点 Q 的坐标.

(1) 把 $x = k, y = 5$ 代入 $y = x + 2$. $\therefore y = \frac{15}{x}$ 把 $x = -5$ 代入 $y = x + 2$.
$5 = k + 2$ $y = -5 + 2$
$k = 3$ (2) $\frac{15}{x} = x + 2$ $y = -3$
$\therefore P(3, 5)$ $15 = x^2 + 2x$ $\therefore Q(-5, -3)$
$\therefore$ 把 $x = 3, y = 5$ 代入 $y = \frac{k}{x}$. $x^2 + 2x - 15 = 0$
$5 = \frac{k}{3}$ $(x+5)(x-3) = 0$
$k = 15$. $x_1 = -5$ $x_2 = 3$.

Appendix M Raw data extracts from Nvivo Sections for teacher interview

L: What are the strength and weakness of the textbook in terms of linear function?
Q: The chapter, linear function is built on previous knowledge and basic method related with the concept of function, proportional function (y=kx, $k \neq 0$) and inverse proportional function ($y = \dfrac{k}{x}, k \neq 0$). Linear function lays the foundation for quadratic function. Therefore, this chapter serves as contextual information. Generally speaking, the content of linear function in textbook fits with students' cognitive development and mathematical knowledge regularity so that the arrangement of linear function is reasonable. However, due to complication of textbooks and the limitation of students' cognitive ability, the idea of combination of symbolic and graphic is the weakness in textbook, I think. Meanwhile, the requirement of application is lower.

L: The requirement of application is lower?
Q: In terms of applying into the real-life situation, students always show their difficulties. Ideal classroom should make effort on the process of situation creation, introducing concept, and then based on understanding, go back to real-life situation. Compared with other countries, we just emphasis on basic knowledge, and two basics (basic knowledge and basic skills). Now, four basics was proposed, namely added 'basic methods and ideas' and 'basic experience of activities'.

L: Since we focused on two basics, what approach do teachers take to mathematics teaching and learning?
Q: First of all, it is the learning process. During this learning process, students' development is priority; teacher is a dominant player and assistor in this process. Students will meet some challenges (some difficulties that students could tell but some they could not perceive). For those challenges or difficulties which could not be perceivable for students, teacher's guidance and assistance would be necessary to improve learning effectiveness. In the whole learning and teaching process, teachers spell out not only how to do it, but also why and other solutions. That means, beside mastering the basic knowledge, students' ability about analysis and resolving needs to improve through the teaching and learning process. Under this requirement, whether mathematical thinking and methods permeates in the teaching
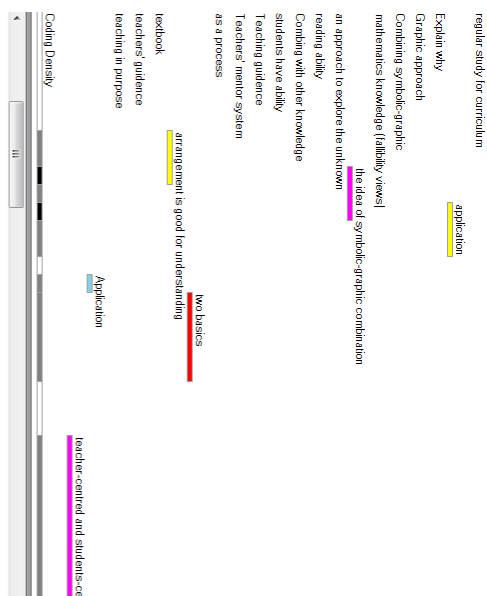
Figure 1. An English sample

K: We do both, we do AQA and Edexcel. We do Edexcel for the majority of our students but for our top groups we do AQA A.

L: Why?

K: Simply because we feel that it gives them a better base knowledge for going onto A-level and university. The styles of the questions are much more open and proof-based. So we feel that programme of content gives them a better knowledge for going on to A-level.

L: So the students could choose?

K: No, I would choose. If a student was in Set 3, they would do Edexcel, but if they move up to Set 2, they will do AQA. It is the same content. Maths is maths and they all do maths. And we teach them exactly the same as we would always do. The only difference is the assessment model. For us, we feel the accessibility of Edexcel for the majority of our students is better, but for the very top students, we have got some very very bright students, they need a bit more proof and algebraic skills that come with the assessment of AQA. So we do a kind of combination.

L: That is good. Another part of (interview) is about understanding. We talk about the specific understanding first, because you just taught the linear graph and what kind of difficulties you think, or barriers for students to learn linear graph?

K: With linear graph, if were thinking of drawing linear graphs. I think the biggest barrier is overcoming the fear of algebra, so where they've got the equation, actually the substitution into the algebra, their misconceptions, 2x means 2 times x, you know, that kind of thing. That is the biggest barrier, because linking a graphic topic and an algebraic topic together they find quite difficult. Using the skills of substitution, particularly for foundation groups, it is not so much an issue for higher groups, who will be able to access that without problem, but for foundation groups, generally speaking, their numeracy work is weaker. So things like if we are going to do y=2x+1 with a negative x value, they will go the wrong way up the number line, that kind of thing. Actually, those kinds of things hinder their understanding of linear graph. Despite the fact that they might understand what a linear graph is - they might understand the gradient and the y-intercept - but actually being able to draw the graph in the first place can sometimes be a bit tricky.
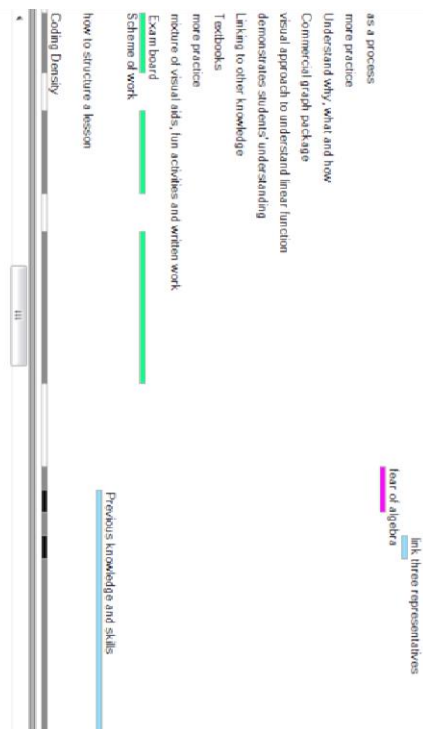
Figure 2. A Shanghai sample