Nonparametric Predictive Inference for System Failure Time

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Dedicated to

my parents

for all the love, prayers, infinite support, encouragement and belief in me

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Abstract

This thesis presents the use of signatures within nonparametric predictive inference (NPI) for the failure time of a coherent system with a single type of components, given failure times of tested components that are exchangeable with those in the system. NPI is based on few modelling assumptions and here leads to lower and upper survival functions. We also illustrate comparison of reliability of two systems, by directly considering the random failure times of the systems. This includes explicit consideration of the difference between failure times of two systems. In this method we assume that the signature is precisely known. In addition, we show how bounds for these lower and upper survival functions can be derived based on limited information about the system structure, which can reduce computational effort substantially for specific inferential questions. It is illustrated how one can base reliability inferences on a partially known signature, assuming that bounds for the probabilities in the signature are available. As a further step in the development of NPI, we present the use of survival signatures within NPI for the failure time of a coherent system which consists of different types of components. It is assumed that, for each type of component, additional components which are exchangeable with those in the system have been tested and their failure times are available. Throughout this thesis we assume that the system is coherent, we start with a system consisting of a single type of components, then we extend for a system consisting of different types of components.

Declaration

The work in this thesis is based on research carried out at the Department of Mathematical Sciences, Durham University, UK. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all the author's original work unless referenced to the contrary in the text.

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Chapter 1

Introduction

1.1 Overview

One of the basic problems in reliability theory is prediction of the failure time of a system consisting of multiple components, each of which has a random failure time. Assessing the reliability of a coherent system requires knowledge about the structure function of the system as well as the probability distribution of component failure times. System reliability can be studied at the structural level by building a relationship between the system reliability and the reliability of its components. Throughout this thesis, we restrict attention to coherent systems, a system is coherent if each of its components is relevant and its structure function is monotonously increasing [12].

The theory of system signatures [61] provides a powerful framework for reliability assessment for systems consisting of exchangeable components. For a system with m components, the signature is a vector containing the probabilities for the events that the system fails at the moment of the j-th ordered component failure time, for all $j = 1, \ldots, m$. In this thesis, The use of signatures for system reliability is explored in the generalized theory of uncertainty quantification where lower and upper probabilities (also called 'imprecise probability' [65] or 'interval probability' [67]) are used instead of precise probabilities. This thesis presents the use of signatures within Nonparametric Predictive Inference (NPI), a statistical framework which uses few modelling assumptions enabled by the use of lower and upper probabilities to quan-

tify uncertainty. Mainly, we introduce NPI with the use of signatures to derive lower and upper survival functions for the failure time of systems with exchangeable components, given failure times of tested components that are exchangeable with those in the system. In addition, comparison of reliability of two systems is presented by directly considering the random failure times of the systems. However, deriving the system signature is computationally complex. We present how limited information about the signature can be used to derive bounds on such lower and upper survival functions and related inferences.

The system survival signature [26] is a generalisation of the system signature to systems with multiple component types. We also present the use of survival signatures within NPI for the failure time of a coherent system, which can consist of different types of components. It is assumed that, for each type of component, additional components which are exchangeable with those in the system have been tested and their failure times are available.

In this thesis the NPI method for the system survival function using the system signature is presented in Chapters 2 and 3, and the NPI method for the system survival function using the survival signature is presented in Chapter 4.

In Section 1.2 we briefly review the concept of system signature. Section 1.3 presents the main idea of NPI. Section 1.4 briefly presents NPI for order statistics of m future real-valued observations given n observations, which will be used in Chapters 2 and 3. Section 1.5 presents a brief overview of NPI for Bernoulli data, which will be used in Chapter 4. Finally, a detailed outline of this thesis is given in Section 1.6, with details of related publications.

1.2 System signatures

In recent decades, system signatures have proven to be a powerful tool for qualifying structures of coherent systems consisting of exchangeable components, which can be used to quantify aspects of reliability of systems such as their failure time distribution [61]. Consider a system consisting of components which have exchangeable random failure times [40]. It is convenient to call these ‘exchangeable components’,

informally they can be said to be all 'of the same type'. As an example, consider batteries of the same brand; their failure times will not be identical, but not knowing the individual batteries' failure times, the exchangeability assumption implies that the information about the failure time of one specific battery is the same as the information about the failure time of any other specific battery. It should be emphasized that such failure times are not statistically independent, as for example learning that one battery's failure time is small will provide important information about the random failure time of another battery. A standard situation where such an exchangeability assumption is reasonable, and indeed implicit to many standard statistical methods, is when the components (batteries) for which failure times are observed had been chosen by simple random sampling from a batch of components, with interest in predicting the failure times of one or more of the other components from the same batch.

Birnbaum [14] presented the foundations for the study of system reliability via the structure function. For a system with m components, let state vector $x = (x_1, x_2, \ldots, x_m) \in \{0,1\}^m$, where for each i, $x_i = 1$ if the ith component functions and $x_i = 0$ if not. The structure function $\phi : \{0,1\}^m \to \{0,1\}$ is a mapping such that $\phi(x) = 1$ if the system functions and $\phi(x) = 0$ if the system does not function for state vector x. The structure function for series and parallel systems is trivial. The series structure functions only if every component is functioning, while the parallel structure functions as long as at least one component is functioning. The structure function for an m-component series system is given by

$$\phi(x) = \prod_{i=1}^{m} x_i \tag{1.1}$$

while for a parallel system it is given by

$$\phi(x) = 1 - \prod_{i=1}^{m} (1 - x_i) \tag{1.2}$$

A system is coherent if each of its components is relevant and its structure function is monotonously increasing. Throughout this thesis we assume that the system is coherent, which means that $\phi(x)$ is not decreasing in any of the components of x, so system functioning cannot be improved by worse performance of one or more

of its components. We assume that $\phi(0) = 0$ and $\phi(1) = 1$, so the system fails if all its components fail and it functions if all its components function [12].

Let the random failure time of a system consisting of m components be T_S and let $T_{j:m}$ be the j-th order statistic of the m random component failure times, for $j = 1, \ldots, m$, with $T_{1:m} \leq T_{2:m} \leq \ldots \leq T_{m:m}$. The system's signature [61] is defined to be the m-vector q with j-th component

$$q_j = P(T_S = T_{j:m}) \tag{1.3}$$

so q_j is the probability that the system failure occurs at the moment of the j-th component failure. It is natural to assume that $\sum_{j=1}^{m} q_j = 1$; this assumption implies that the system functions if all components function, has failed if all components have failed, and that system failure can only occur at times of component failures. The essential feature of the calculation of a signature is counting of the orderings of the m potential component failure times that correspond with system failure upon the jth failure time among the m components.

The signature provides a qualitative description of the system structure that can be used in reliability quantification [61]. For example, the survival function of the system failure time can be derived by

$$P(T_S > t) = \sum_{j=1}^{m} q_j P(T_{j:m} > t) \tag{1.4}$$

and the expected value of T_S can be derived by

$$E(T_S) = \sum_{j=1}^{m} q_j E(T_{j:m}) \tag{1.5}$$

The system signature was introduced by Samaniego in 1985 [60] and has become a useful tool to compute the system reliability [16], and to compare different systems when all components are exchangeable. A comprehensive discussion and an excellent review of the results based on system signatures obtained since 1985 and their applications in engineering reliability can be found in a book by Samaniego [61].

In recent years, many authors have discussed theory and applications of system signatures, for example, signatures were used in [52] and [57] to study system comparison based on stochastic, hazard rate and likelihood ratio orderings. Boland [15]

presented the signature in terms of the number of path sets in the system as well as the number of ordered cut sets. Andronova et al. [5] applied signatures to a queueing system with an unreliable server. However, signatures have thus far been mostly considered from a probability perspective, not related to statistical inference. The main exception is the recent PhD thesis by Aslett [9], who presents a Bayesian approach for inference on system reliability for several scenarios, with the system structure taken into account through the signature. A particularly interesting feature of his work is the possibility to learn about the system structure in case only system level failure data are available, so without data on individual components. Such inverse inference is useful for black-box systems and requires powerful simulation-based computational methods, for which the Bayesian approach is very suitable. In this thesis, nonparametric predictive inference is used, which provides a frequentist statistics alternative to the Bayesian approach, suitable for inference about system reliability based on failure data for individual components. Throughout the thesis, the system structure is assumed to be known, the presented approach is not suitable for the kind of inverse inferences mentioned above. Knowing the system structure, however, does not necessarily imply that the system signature is readily available, we therefore will also consider inference using only partially known signatures.

Computation of the system signature is a combinatorial exercise. However, this does not mean that it is quite easy, it just means there is a well organized body of knowledge and tools that can be applied to such problems. It is obvious that there are 2 coherent systems of order 2 (series and parallel systems). Shaked and Suarez-Llorens [62] proved that there are 5 different coherent systems consisting of 3 components and 20 different coherent systems consisting of 4 components. They computed the signatures of these systems and used them to study some ordering properties. Navarro and Rubio [56] provide an algorithm to compute the number of coherent systems with a given number of components. They show that there are 180 different coherent systems with 5 components and also computed the signatures of these systems and their expected lifetimes. According to Samaniego [61] the number of coherent systems with m components grows exponentially, there are more than a billion coherent systems with 30 components.

One may be interested in comparing two systems which do not have the same number of components. Samaniego [61] provides a formula to convert the smaller system into an equivalent larger system with exactly the same failure time distribution as the smaller system. Let $q = (q_1, \ldots, q_m)$ be the signature of a coherent system with m components which have independent and identically distributed (*i.i.d.*) failure times. Then the system with $m+1$ components with *i.i.d.* failure times, with the same distribution as for the smaller system, and with signature

$$\begin{aligned} q^* &= \left(\frac{m}{m+1}q_1, \frac{1}{m+1}q_1 + \frac{m-1}{m+1}q_2, \frac{2}{m+1}q_2 + \frac{m-2}{m+1}q_3, \ldots, \right. \\ &\quad \left. \frac{m-1}{m+1}q_{m-1} + \frac{1}{m+1}q_m, \frac{m}{m+1}q_m\right) \end{aligned} \tag{1.6}$$

has identical random system failure time as the system with m components and signature q. Eryilmaz [44] presented an algorithm for computing the signature of consecutive k-out-of-m:F systems, which fail if and only if k of components fail. Such systems have received much attention in the reliability literature in recent years, particularly also with focus on their signatures [45, 55].

Da et al. [48], show how signatures for subsystems can be combined to derive a system's signature in case of two subsystems in series or parallel configuration, which we will use in Chapter 3. Suppose that the system consists of two subsystems A and B with m_a and m_b components each. Let $q^a = (q_1^a, ..., q_{m_a}^a)$ and $q^b = (q_1^b, ..., q_{m_b}^b)$ be the signature vectors of subsystems A and B, respectively. The aim is to derive the signature vector q of the overall system based on signatures q^a and q^b. First consider a system which consists of subsystems A and B in parallel configuration. Then the overall system has signature vector q with jth component q_j given as follows [48]. Since the system is the parallel of two subsystems, it is clear that the system will not fail at the first component failure, which leads to $q_1 = 0$. For further component failures, the signature can be derived for the following cases.

For $2 \leq j \leq m_a$:

$$\begin{aligned} q_j = \binom{m_a+m_b}{m_a}^{-1} \left[\sum_{i=1}^{j-1} \binom{j-1}{i} \left[\left(q_{j-i}^a \sum_{k=1}^{i} q^b\right) \binom{m_a+m_b-j}{m_b-i} \right.\right. \\ \left.\left. + \left(q_{j-i}^b \sum_{k=1}^{i} q^a\right) \binom{m_a+m_b-j}{m_b-i}\right]\right] \end{aligned} \tag{1.7}$$

For $m_a < j \leq m_b$:

$$q_j = \binom{m_a+m_b}{m_a}^{-1} \left[\sum_{i=j-m_a}^{j-1} \left(q_{j-i}^a \sum_{k=1}^{i} q^b \right) \binom{j-1}{i} \binom{m_a+m_b-j}{m_b-i} + \sum_{i=1}^{m_a} \left(q_{j-i}^b \sum_{k=1}^{i} q^a \right) \binom{j-1}{i} \binom{m_a+m_b-j}{m_b-i} \right] \quad (1.8)$$

For $m_b < j \leq m_a + m_b$:

$$q_j = \binom{m_a+m_b}{m_a}^{-1} \left[\sum_{i=j-m_a}^{m_b} \left(q_{j-i}^a \sum_{k=1}^{i} q^b \right) \binom{j-1}{i} \binom{m_a+m_b-j}{m_b-i} + \sum_{i=j-m_b}^{m_a} \left(q_{j-i}^b \sum_{k=1}^{i} q^a \right) \binom{j-1}{i} \binom{m_a+m_b-j}{m_b-i} \right] \quad (1.9)$$

For a system consisting of subsystems A and B in series configuration, the system's signature vector q has jth component q_j which can be derived as follows. Since the system is the series of two subsystems, it is clear that $q_{m_a+m_b} = 0$. The other components of q can be derived for the following cases.

For $1 \leq j \leq m_a$:

$$q_j = \binom{m_a+m_b}{m_a}^{-1} \left[\sum_{i=0}^{j-1} \binom{j-1}{i} \left[\left(q_{j-i}^a \sum_{k=i+1}^{m_b} q^b \right) \binom{m_a+m_b-j}{m_b-i} + \left(q_{j-i}^b \sum_{k=i+1}^{m_a} q^a \right) \binom{m_a+m_b-j}{m_b-i} \right] \right] \quad (1.10)$$

For $m_a < j \leq m_b$:

$$q_j = \binom{m_a+m_b}{m_a}^{-1} \left[\sum_{i=j-m_a}^{j-1} \left(q_{j-i}^a \sum_{k=i+1}^{m_b} q^b \right) \binom{j-1}{i} \binom{m_a+m_b-j}{m_b-i} + \sum_{i=0}^{m_a-1} \left(q_{j-i}^b \sum_{k=i+1}^{m_a} q^a \right) \binom{j-1}{i} \binom{m_a+m_b-j}{m_b-i} \right] \quad (1.11)$$

For $m_b < j \leq m_a + m_b$:

$$q_j = \binom{m_a+m_b}{m_a}^{-1} \left[\sum_{i=j-m_a}^{m_b-1} \left(q_{j-i}^a \sum_{k=i+1}^{m_b} q^b \right) \binom{j-1}{i} \binom{m_a+m_b-j}{m_b-i} + \sum_{i=j-m_b}^{m_a-1} \left(q_{j-i}^b \sum_{k=i+1}^{m_a} q^a \right) \binom{j-1}{i} \binom{m_a+m_b-j}{m_b-i} \right] \quad (1.12)$$

1.3 Nonparametric predictive inference (NPI)

Nonparametric predictive inference (NPI) is a statistical approach to learning from data in the absence of prior knowledge, which requires only few modelling assumptions [21]. NPI gives a direct conditional probability for one or more future observable random quantities, conditional on observed values of related random quantities [7, 20, 21]. NPI uses lower and upper probabilities, also known as imprecise probabilities, to quantify uncertainty [8, 30, 65, 67] and has strong consistency properties from frequentist statistics perspective [7, 21]. NPI provides a solution to some explicit goals formulated for objective (Bayesian) inference, which cannot be obtained when using precise probabilities [20], and it never leads to results that are in conflict with inferences based on empirical probabilities. Imprecise probabilities provide many exciting opportunities for reliability quantification [31, 63, 64]. The NPI method has already been used for system reliability [1, 23, 37, 53], but only for systems with quite restricted structures. NPI has been developed for a variety of problems in operational research and statistics, including predictive analysis for queueing problems [25], replacement problems [38], and decision making under uncertain utilities [51].

NPI is based on Hill's assumption $A_{(n)}$ [49] which gives direct probabilities [41] for one or more real-valued future random quantities, based on observations of n related random quantities. These probabilities are such that all orderings of the future random quantities among the observed random quantities are equally likely; for more details we refer to Coolen [21]. NPI is a framework of statistical theory and methods that use $A_{(n)}$-based lower and upper probabilities [30, 31]. In classical probability theory, a single probability $P(E) \in [0, 1]$ is used to quantify uncertainty about an event E. Lower and upper probabilities generalize the standard theory of ('single-valued' or 'precise') probability and provide a powerful method for uncertainty quantification [64]. A lower (upper) probability $\underline{P}(E)$ ($\overline{P}(E)$) can be interpreted as supremum buying (infimum selling) price for a gamble on the event E, or as the maximum lower (minimum upper) bound for the probability of E. An informal way to interpret lower and upper probabilities is as follows; a lower prob-

ability for an event E reflects the evidence in available information in favour of the event E, the corresponding upper probability for this event reflects the evidence in available information against this event. These are logically linked by the conjugacy property $\underline{P}(E) = 1 - \overline{P}(E^c)$, where E^c is the complementary event to E [30].

To introduce the assumption $A_{(n)}$, we first need to introduce some notation. Suppose that $T_1, \ldots, T_n, T_{n+1}$ are positive, continuous and exchangeable random quantities. Let the ordered observations of $T_1, \ldots, T_n$ be denoted by $t_1 < t_2 < \ldots < t_n$. For ease of notation, define $t_0 = 0$ and $t_{n+1} = \infty$. These n observations partition the non-negative real-line into $n+1$ intervals $I_i = (t_{i-1}, t_i)$ for $i = 1, \ldots, n+1$. The assumption $A_{(n)}$ is that the future observation T_{n+1}, based on n observations, will fall in the open interval I_i with probability $1/(n+1)$, for each $i = 1, \ldots, n+1$,

$$P(T_{n+1} \in (t_{i-1}, t_i)) = \frac{1}{n+1} \tag{1.13}$$

These $A_{(n)}$-based probabilities are specified for T_{n+1}, but also hold for any future observation T_{n+i}, $i \geq 1$, as long as one considers these future observations to be exchangeable [40]. However, such future observations are not independent [49], learning the value of one of them will change the probabilities for other future observations. Hill [50] discusses $A_{(n)}$ in detail. $A_{(n)}$ does not assume anything else, and can be considered to be a post-data assumption related to exchangeability. Inferences based on $A_{(n)}$ are predictive and nonparametric, and can be considered suitable if there is hardly any knowledge about the random quantity of interest, other than the data, which consists of n observations, or if one does not want to use such further information. $A_{(n)}$ is not sufficient to derive precise probabilities for many events of interest, but it provides optimal bounds for probabilities for all events of interest involving T_{n+1}.

It should be noted that, to avoid notational complexity, we assume throughout this thesis that there are no tied observations. Any tied observations can be dealt with by breaking ties by adding small values to one or more of the tied observations. The method can be generalized to allow ties by breaking the ties in all possible ways and obtaining the overall NPI lower and upper probabilities as the minimum and maximum, respectively, of the lower and upper probabilities corresponding to each way of breaking the ties [50].

1.4 NPI for order statistics

For the scenario in Section 1.3, we are now interested in $m \geq 1$ future observations, T_{n+j} for $j = 1, \ldots, m$. We link the data and future observations via Hill's assumptions $A_{(n)}$, $A_{(n+1)}$,..., $A_{(n+m-1)}$, see [7,20,27] for more details. Arts et al. [6] considered NPI for m future observations, and showed that, with $S_j = \#\{T_l \in I_i,\ l = 1, \ldots, m\}$, these assumptions lead to

$$P(\bigcap_{j=1}^{n+1}\{S_j = s_j\}) = \binom{n+m}{n}^{-1} \tag{1.14}$$

for all $(s_1, \ldots, s_{n+1})$ with s_j non-negative integers and $\sum_{j=1}^{n+1} s_j = m$. For any event involving the m future observations, Equation 1.14 implies that the number of orderings of the m future observations among the n data observations, for which this event holds, can be simply counted as all such orderings have equal probability.

Generally in NPI a lower probability for the event of interest is derived by counting all orderings for which this event has to hold, while the corresponding upper probability is derived by counting all orderings for which this event can hold [7,20]. The order statistics of the m future observations $T_1, \ldots, T_m$ are the ordered component failure times, denoted by $T_{1:m} \leq T_{2:m} \leq \ldots \leq T_{m:m}$. The following probabilities for $T_{j:m}$, for $j = 1, \ldots, m$, are derived by counting the relevant orderings [27], and hold for $i = 1, \ldots, n+1$,

$$P(T_{j:m} \in I_i) = \binom{i+j-2}{i-1}\binom{n-i+1+m-j}{n-i+1}\binom{n+m}{n}^{-1} \tag{1.15}$$

NPI provides a precise probability for the event $T_{j:m} \in I_i$, as each of the $\binom{n+m}{n}$ equally likely orderings of n test observations and m future observations has the j-th ordered future observation in precisely one interval I_i. Therefore, we must have $i-1$ test observations and $j-1$ future observations in any order before time $T_{j:m} \in I_i$, which can occur in $\binom{i+j-2}{i-1}$ different orderings, and $n-(i-1)$ test observations and $m-j$ future observations in any order after time $T_{j:m} \in I_i$, which can occur in $\binom{n-i+1+m-j}{n-i+1}$ different orderings.

As an example, suppose that one is interested in the minimum $T_{1:m}$ of m future observations. Formula 1.15 gives $P(T_{1:m} \in I_i) = \binom{n-i+m}{n-i+1}\binom{n+m}{n}^{-1}$, so e.g.

$P(T_{1:m} \in I_1) = \frac{m}{n+m}$. Clearly, the event $T_{1:m} \in I_1$ occurs if the smallest of all $n+m$ observations considered, so the n data observations and m future observations, is among the m future observations, which indeed occurs with probability $\frac{m}{n+m}$ due to the assumed exchangeability. Another special case of interest is $P(T_{1:m} \in I_{n+1}) = \binom{n+m}{n}^{-1}$, following from the fact that there is only one ordering for which all n data observations occur before all m future observations.

The probabilities 1.15 straightforwardly lead to the following NPI lower and upper survival functions for $T_{j:m}$, these are the sharpest bounds for the probability of the event $T_{j:m} > t$ that can be justified without further assumptions. The NPI lower survival function for $T_{j:m}$ is

$$\underline{S}_{T_{j:m}}(t) = \underline{P}(T_{j:m} > t) = \sum_{l=i+1}^{n+1} P(T_{j:m} \in I_l) \qquad \text{for } t \in (t_{i-1}, t_i] \tag{1.16}$$

and the corresponding NPI upper survival function is

$$\overline{S}_{T_{j:m}}(t) = \overline{P}(T_{j:m} > t) = \sum_{l=i}^{n+1} P(T_{j:m} \in I_l) \qquad \text{for } t \in [t_{i-1}, t_i) \tag{1.17}$$

At an observed data value t_i these NPI lower and upper survival functions are equal, that is $\underline{S}_{T_{j:m}}(t_i) = \overline{S}_{T_{j:m}}(t_i)$, while $\underline{S}_{T_{j:m}}(0) = \overline{S}_{T_{j:m}}(0) = 1$. Beyond the largest data observation, the NPI lower survival function is equal to zero but the NPI upper survival function remains positive,

$$\underline{S}_{T_{j:m}}(t) = 0 \quad \text{and} \quad \overline{S}_{T_{j:m}}(t) = P(T_{j:m} \in I_{n+1}) = \prod_{l=j}^{m} \frac{l}{n+l} > 0 \quad \text{for } t > t_n$$

This reflects that there is no evidence in favour of observations greater than t_n actually being able to occur, this is reflected by the lower survival function being equal to zero; but the evidence against this is limited as there are only n observations thus far, this is reflected by the upper survival function being a positive decreasing function of n. In this thesis NPI for order statistics is used in Chapters 2 and 3. In the next section we introduce NPI for Bernoulli random quantities, which is used in Chapter 4.

1.5 NPI for Bernoulli quantities

NPI for Bernoulli random quantities, as introduced by Coolen [19], is summarized in this section. Suppose that there is a sequence of $n+m$ exchangeable Bernoulli trials, each with 'success' and 'failure' as possible outcomes, and data consisting of s successes in n trials. Let Y_1^n denote the random number of successes in trials 1 to n, then a sufficient representation of the data for the inferences considered is $Y_1^n = s$, due to the assumed exchangeability of all trials. Let Y_{n+1}^{n+m} denote the random number of successes in trials $n+1$ to $n+m$. Let $R_t = \{r_1, \ldots, r_t\}$, with $1 \leq t \leq m+1$ and $0 \leq r_1 < r_2 < \ldots < r_t \leq m$, and, for ease of notation, define $\binom{s+r_0}{s} = 0$. Then the NPI upper probability for the event $Y_{n+1}^{n+m} \in R_t$, given data $Y_1^n = s$, for $s \in \{0, \ldots, n\}$, is

$$\overline{P}(Y_{n+1}^{n+m} \in R_t | Y_1^n = s) = \binom{n+m}{n}^{-1} \sum_{j=1}^{t} \left[\binom{s+r_j}{s} - \binom{s+r_{j-1}}{s} \right] \binom{n-s+m-r_j}{n-s} \quad (1.18)$$

The corresponding NPI lower probability can be derived via the conjugacy property

$$\underline{P}(Y_{n+1}^{n+m} \in R_t | Y_1^n = s) = 1 - \overline{P}(Y_{n+1}^{n+m} \in R_t^c | Y_1^n = s) \quad (1.19)$$

where $R_t^c = \{0, 1, \ldots, m\} \backslash R_t$.

These NPI lower and upper probabilities are the maximum lower bound and minimum upper bound, respectively, for the probability for the given event based on the data, the assumption $A_{(n)}$ and the model presented by Coolen [19].

1.6 Outline of thesis

This thesis is organized such that each chapter addresses one main inference problem, and is related to papers that have been published in academic journals. In Chapter 2 we introduce the use of signatures in the study of system reliability with lower and upper probabilities. We present the comparison of the reliability of two systems by directly considering the random failure times of the systems, including explicit consideration of the difference between failure times of two systems. This chapter

is closely related to the paper "Nonparametric predictive inference for failure times of systems with exchangeable components" which appeared in Journal of Risk and Reliability in 2012 [24].

In Chapter 3 we present how bounds for lower and upper survival functions can be derived based on limited information about the system signature and related inferences. We present the comparison of the reliability of two systems considering the random failure times of the systems using partially known signatures. This chapter closely resembles the paper "Nonparametric predictive inference for system failure time based on bounds for the signature", which appeared in Journal of Risk and Reliability in 2013 [4].

In Chapter 4 we present the use of survival signatures for NPI for the failure time of a coherent system, which can consist of different types of components. In addition, we present how survival signatures of subsystems can be combined to derive a system's survival signature, and we present how limited information about the survival signature can be used to derive bounds on such lower and upper survival functions and related inferences. This chapter forms part of the paper "Nonparametric predictive inference for system reliability using the survival signature", which appeared in Journal of Risk and Reliability in 2014 [28]. We summarize our main results with some concluding remarks in Chapter 5. All computations in this thesis were performed using R.

Parts of this thesis have been presented at several conferences and short papers have appeared in related conference proceedings. Chapter 2 has been presented at The 19$^{\text{th}}$ Advances in Risk and Reliability Technology Symposium (Stratford-upon-Avon, UK, April 2011) [2]. A part of Chapter 3 was presented at The Statistical Models and Methods for Reliability and Survival Analysis and Their Validation (Bordeaux, France, July 2012) [3]. In addition, results related to Chapter 4 were presented (by Prof. Frank Coolen) at The 20$^{\text{th}}$ Advances in Risk and Reliability Technology Symposium (Nothingham, UK, May 2013) [29].

Chapter 2

Failure time of a system consisting of exchangeable components

2.1 Introduction

This chapter presents the use of signatures for nonparametric predictive inference (NPI) [7, 20] about the failure time of a system consisting of exchangeable components, given failure times of tested components. A useful feature of describing system structures through signatures is the possibility to compare the reliability of different systems based on stochastic ordering of their signatures, as long as the components in these systems are all exchangeable [61]. This chapter also presents an alternative way to compare the reliability of different systems by directly considering the random system failure times.

We assume in this chapter that the signature is precisely known, in Chapter 3 we will consider the case of a partially known signature. Section 2.2 presents the use of system signatures to derive NPI lower and upper survival functions for a system. In Section 2.3 comparison of reliability of two systems is presented by directly considering the random failure times of the systems. This includes explicit consideration of the difference between failure times of two systems. Section 2.4 contains some concluding remarks.

2.2 Predicting system failure time

This section presents the NPI lower and upper survival functions for systems with exchangeable components, derived by generalizing Equation 1.4 to lower and upper probabilities. In order to combine NPI with system signatures, it is important to explain a key ingredient of theory of lower and upper probabilities, namely a set $\mathcal{P}$ of precise probability distributions, each denoted by $P \in \mathcal{P}$, which corresponds to the assessed values and which is such that the lower probability of an event E is $\underline{P}(E) = \inf_{P \in \mathcal{P}} P(E)$ and the corresponding upper probability is $\overline{P}(E) = \sup_{P \in \mathcal{P}} P(E)$. In his theory of interval probability, Weichselberger [67] calls such a set a 'structure', see [7] for more details and strong consistency properties of inferences based on such a construction of lower and upper probabilities.

Generally, in NPI the assumption $A_{(n)}$ provides precise probabilities for some events involving one or more future observations, and the corresponding structure consists of all precise probabilities which assign those values to all those events. So, the structure $\mathcal{P}_j$ for $T_{j:m}$, for $j = 1, \ldots, m$, consists of all precise probability distributions which assign $P(T_{j:m} \in I_i)$, as given in Equation 1.15 in Section 1.4, to interval I_i, for each $i = 1, \ldots, n+1$. As interest is in the system failure time T_S, let $\mathcal{P}_S$ be the structure corresponding to NPI for T_S. $\mathcal{P}_S$ is derived directly from the $\mathcal{P}_j$, $j = 1, \ldots, m$, by the logical relationship that exists based on Equation 1.4 for the precise probability distributions in the respective structures. This means that for each probability distribution $P_S \in \mathcal{P}_S$, there is a combination of probability distributions in the structures $\mathcal{P}_j$ that, by Equation 1.4, leads to P_S. Also the reverse relation holds, namely that any combination of probability distributions in the structures $\mathcal{P}_j$ lead, by application of Equation 1.4, to a probability distribution P_S which belongs to $\mathcal{P}_S$. The NPI lower and upper survival functions for T_S are derived by minimisation and maximisation, respectively, of the probabilities for events $T_S > t$ over the structure $\mathcal{P}_S$. While in general this would be non-trivial optimisation problems, NPI provides a simple solution as explained below.

As mentioned in Section 1.4, suppose that in a test of n components, exchangeable with those in the system considered, the observed failure times were $t_1 < t_2 < \ldots < t_n$. Consider reliability of a system with m components, so interest is in the m

failure times of those components, say $T_1, \ldots, T_m$. The test data and $T_1, \ldots, T_m$ are linked via the assumptions $A_{(n)}$, $A_{(n+1)}$,..., $A_{(n+m-1)}$. The order statistics of the m future observations $T_1, \ldots, T_m$ are the ordered component failure times.

The NPI lower and upper survival functions for the failure time T_S of a coherent system consisting of m exchangeable components, with the system structure represented by signature q, can be derived by the following generalizations of Equation 1.4. The NPI lower survival function is

$$\begin{aligned} \underline{S}_{T_S}(t) &= \underline{P}(T_S > t) = \inf_{P_S \in \mathcal{P}_S} P_S(T_S > t) = \inf_{P_S \in \mathcal{P}_S} \sum_{j=1}^{m} q_j P_S(T_{j:m} > t) \\ &= \sum_{j=1}^{m} q_j \inf_{P_j \in \mathcal{P}_j} P_j(T_{j:m} > t) = \sum_{j=1}^{m} q_j \underline{P}(T_{j:m} > t) \end{aligned} \tag{2.1}$$

The corresponding upper survival function is

$$\begin{aligned} \overline{S}_{T_S}(t) &= \overline{P}(T_S > t) = \sup_{P_S \in \mathcal{P}_S} P_S(T_S > t) = \sup_{P_S \in \mathcal{P}_S} \sum_{j=1}^{m} q_j P_S(T_{j:m} > t) \\ &= \sum_{j=1}^{m} q_j \sup_{P_j \in \mathcal{P}_j} P_j(T_{j:m} > t) = \sum_{j=1}^{m} q_j \overline{P}(T_{j:m} > t) \end{aligned} \tag{2.2}$$

The crucial step in the derivations of (2.1) and (2.2) is the fourth equality. In general theory of lower and upper probabilities [7, 20], we only have

$$\inf_{P_S \in \mathcal{P}_S} \sum_{j=1}^{m} q_j P_S(T_{j:m} > t) \geq \sum_{j=1}^{m} q_j \inf_{P_j \in \mathcal{P}_j} P_j(T_{j:m} > t) \tag{2.3}$$

and

$$\sup_{P_S \in \mathcal{P}_S} \sum_{j=1}^{m} q_j P_S(T_{j:m} > t) \leq \sum_{j=1}^{m} q_j \sup_{P_j \in \mathcal{P}_j} P_j(T_{j:m} > t) \tag{2.4}$$

so justification of the fourth equalities in (2.1) and (2.2) is required. The argument is given for the case of the NPI lower survival function, justification of the NPI upper survival function follows the same steps. For the equality to hold in (2.3), the probability distributions in $\mathcal{P}_j$ which minimise $P_j(T_{j:m} > t)$ for all t must be attained simultaneously for all $j = 1, \ldots, m$. That this holds follows from the derivation of (1.15), as given in [27] and discussed in Section 1.4, which is based on the $\binom{n+m}{n}$ equally likely orderings of the n data observations and m future observations. Each NPI lower survival function for a $T_{j:m}$, for all $j = 1, \ldots, m$, can be

derived by considering, for each of the equally likely orderings, the situation with all future observations assigned to interval $I_i = (t_{i-1}, t_i)$, by the specific ordering, to actually be located immediately to the right of t_{i-1} (so to the left of $t_{i-1} + \epsilon$ for any $\epsilon > 0$) with all their probability mass for this interval. This construction clearly corresponds to the NPI lower survival function for $T_{j:m}$, and can be used in each interval to get all these NPI lower survival functions, so for all $j = 1, \ldots, m$, simultaneously.

Example 2.1

Figure 2.1 presents the signatures of six coherent systems with $m = 4$ exchangeable components. Suppose that $n = 4$ components exchangeable with those in such a system were tested, leading to ordered failure times $t_1 < t_2 < t_3 < t_4$, which create the partition $I_1, \ldots, I_5$ of the positive real-line. Table 2.1 presents the probabilities, as given by Equation 1.15 and denoted by ${}_jP_i = P(T_{j:4} \in I_i)$ for $j = 1, \ldots, 4$ and $i = 1, \ldots, 5$, together with the NPI lower and upper survival functions for $T_{j:4}$ as given by Equations 1.16 and 1.17, respectively.

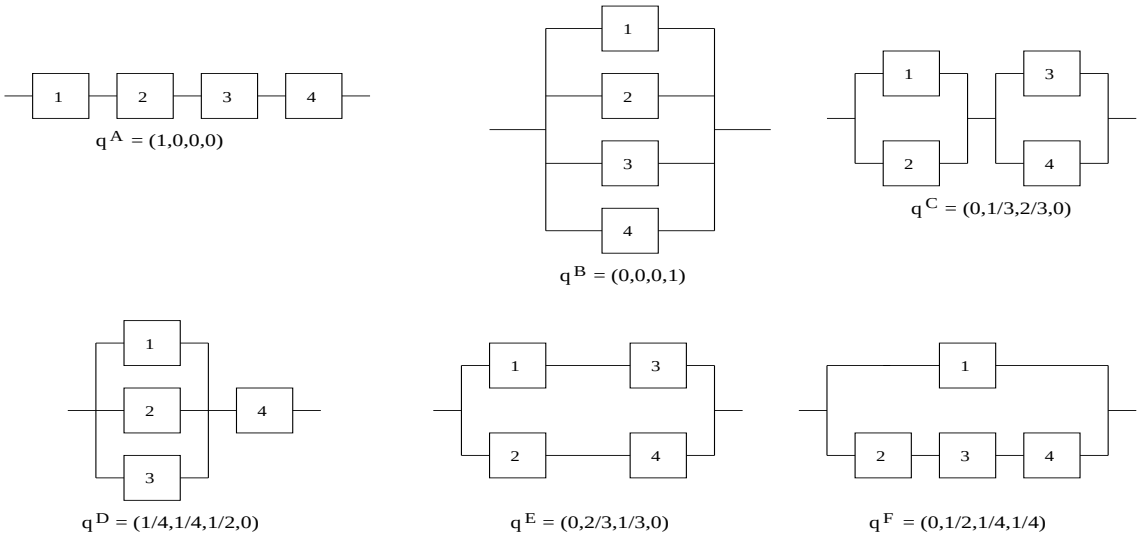

Figure 2.1: Coherent systems with 4 exchangeable components

Table 2.2 presents the NPI lower and upper survival functions, $\underline{S}_{T_S}(t)$ and $\overline{S}_{T_S}(t)$, for the system failure time T_S, from Equations 2.1 and 2.2, for the systems presented in Figure 2.1. We will also consider these systems in Chapter 4 to compare the results

	$j=1$			$j=2$			$j=3$			$j=4$		
i	${}_1P_i$	$\underline{S}_{T_{1:4}}$	$\overline{S}_{T_{1:4}}$	${}_2P_i$	$\underline{S}_{T_{2:4}}$	$\overline{S}_{T_{2:4}}$	${}_3P_i$	$\underline{S}_{T_{3:4}}$	$\overline{S}_{T_{3:4}}$	${}_4P_i$	$\underline{S}_{T_{4:4}}$	$\overline{S}_{T_{4:4}}$
1	0.500	0.500	1	0.214	0.786	1	0.071	0.929	1	0.014	0.986	1
2	0.286	0.214	0.500	0.286	0.500	0.786	0.171	0.757	0.929	0.057	0.929	0.986
3	0.143	0.071	0.214	0.257	0.243	0.500	0.257	0.500	0.757	0.143	0.786	0.929
4	0.057	0.014	0.071	0.171	0.071	0.243	0.286	0.214	0.500	0.286	0.500	0.786
5	0.014	0	0.014	0.071	0	0.071	0.214	0	0.214	0.500	0	0.500

Table 2.1: ${}_jP_i$, $\underline{S}_{T_{j:4}}(t)$ and $\overline{S}_{T_{j:4}}(t)$ for $t \in I_i$, for $n = 4$ and $m = 4$

in Table 2.2 with the use of survival signature, as presented in that chapter. Table 2.2 illustrates that the upper survival function for the system failure time is always equal to one in the first interval and the corresponding lower survival function is less than one. Of course, these lower and upper survival functions decrease at each observed failure time of a component in the test. The lower survival function is zero after the largest observation while the upper survival functions always remains positive. Tables 2.1 and 2.2 show that the upper survival function in interval I_i is equal to the lower survival function in interval I_{i-1}. This is a property that generally holds for the lower and upper survival functions in this chapter, and which follows directly from Equations 1.16 and 1.17.

Figures 2.2 and 2.3 present the NPI lower and upper survival functions for the six systems in Figure 2.1 based on $n = 30$ observations of component failure times, simulated from the Weibull distribution with shape parameter 2 and scale parameter 1. The 30 ordered simulated component failure times are given in Table 2.3.

The signatures of systems C and F are not stochastically ordered (see Section 2.3), which leads to their NPI lower and upper survival functions crossing as is illustrated in Figure 2.2, and the same applies for systems D and E, shown in Figure 2.3. These lower and upper survival functions clearly indicate the differences in the system reliability for these six systems. However, one may wish to quantify the differences in reliability more precisely, a new approach that can be used for this will be presented in Section 2.3.

q	$(1,0,0,0)$		$(0,0,0,1)$		$(0,\frac{1}{3},\frac{2}{3},0)$	
i	$\underline{S}_{T_S}$	$\overline{S}_{T_S}$	$\underline{S}_{T_S}$	$\overline{S}_{T_S}$	$\underline{S}_{T_S}$	$\overline{S}_{T_S}$
1	0.50	1	0.99	1	0.88	1
2	0.21	0.50	0.93	0.99	0.67	0.88
3	0.07	0.21	0.79	0.93	0.41	0.67
4	0.01	0.07	0.50	0.79	0.17	0.41
5	0	0.01	0	0.50	0	0.17
q	$(\frac{1}{4},\frac{1}{4},\frac{1}{2},0)$		$(0,\frac{2}{3},\frac{1}{3},0)$		$(0,\frac{1}{2},\frac{1}{4},\frac{1}{4})$	
i	$\underline{S}_{T_S}$	$\overline{S}_{T_S}$	$\underline{S}_{T_S}$	$\overline{S}_{T_S}$	$\underline{S}_{T_S}$	$\overline{S}_{T_S}$
1	0.79	1	0.83	1	0.87	1
2	0.56	0.79	0.59	0.83	0.67	0.87
3	0.33	0.56	0.33	0.59	0.44	0.67
4	0.13	0.33	0.12	0.33	0.21	0.44
5	0	0.13	0	0.12	0	0.21

Table 2.2: $\underline{S}_{T_S}(t)$ and $\overline{S}_{T_S}(t)$ for $t \in I_i$

0.086	0.167	0.277	0.319	0.394	0.400	0.402	0.481	0.494	0.599
0.601	0.642	0.642	0.712	0.720	0.732	0.790	0.832	0.863	1.023
1.088	1.097	1.172	1.185	1.334	1.336	1.620	1.851	2.060	2.329

Table 2.3: 30 simulated component failure times for Ex. 2.1

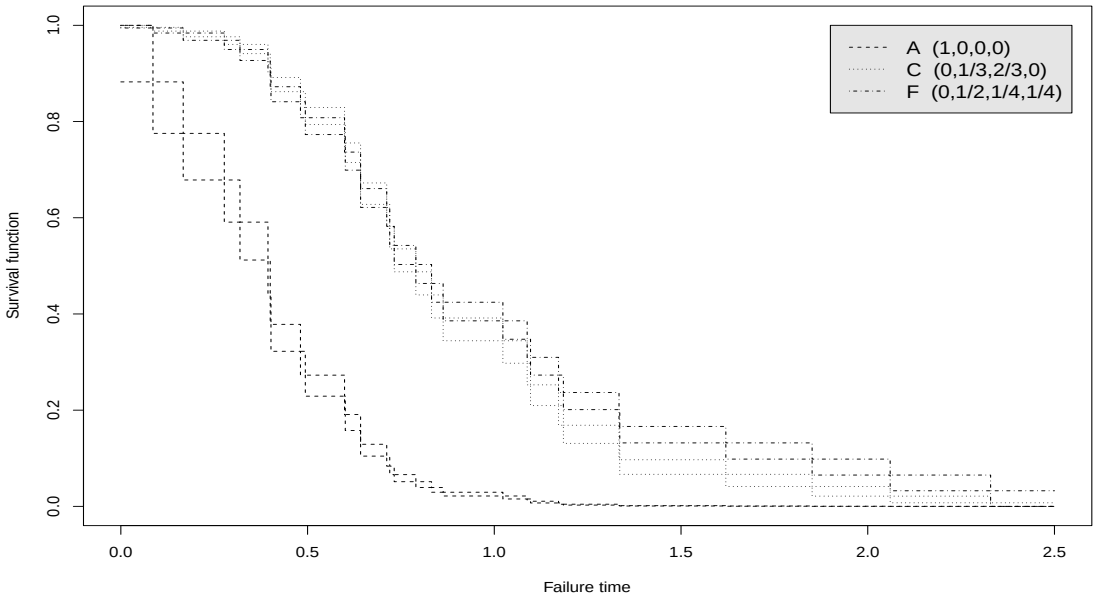

Figure 2.2: NPI lower and upper survival functions for system A, C and F (Ex. 2.1)

Example 2.2

To further illustrate the NPI lower and upper survival functions for systems presented in this chapter, consider linear and circular consecutive k-out-of-m:F systems, which fail if and only if k or more linearly or circularly ordered components fail [44, 45, 55]. Table 2.4 gives $n = 30$ component failure times simulated from a Weibull distribution with shape parameter 3 and scale parameter 1. Figure 2.4 presents the NPI lower and upper survival functions, based on these data, for both a linear and circular consecutive 2-out-of-4:F system, for which the signatures are also given in the figure. The circular system fails for all neighbouring pairs of failing components for which the linear system fails, but in addition it also fails if only the first and last ordered components fail. This results in the circular system being less reliable than the linear system, as shown in Figure 2.4. Figure 2.5 presents similar NPI lower and upper survival functions for the linear and circular consecutive 3-out-of-6:F systems based on the same component failure data. These systems are clearly more reliable early on than the 2-out-of-4 systems. For all these four

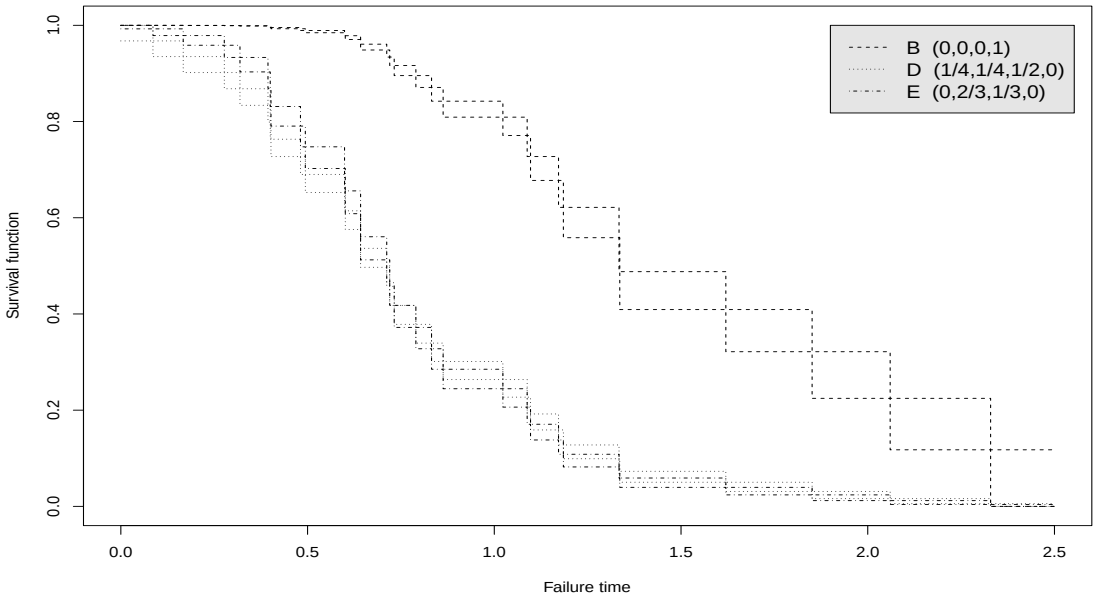

Figure 2.3: NPI lower and upper survival functions for system B, D and E (Ex. 2.1)

0.223	0.265	0.372	0.419	0.564	0.630	0.675	0.685	0.709	0.727
0.747	0.798	0.807	0.824	0.850	0.887	0.914	0.921	0.981	0.987
0.994	1.008	1.073	1.115	1.167	1.182	1.275	1.397	1.400	1.425

Table 2.4: 30 simulated component failure times for (Ex. 2.2)

systems considered, the lower survival function is zero beyond the largest observed component failure time, $t = 1.425$, reflecting that the data provide no evidence in favour of survival beyond this time, yet the corresponding upper survival functions are positive reflecting the fact that such survival cannot be deemed to be impossible on the basis of the 30 observations only. Figure 2.6 and 2.7 present the NPI lower and upper survival functions for linear consecutive 2-out-of-4 and 3-out-of-6 systems, and for circular consecutive 2-out-of-4 and 3-out-of-6, respectively.

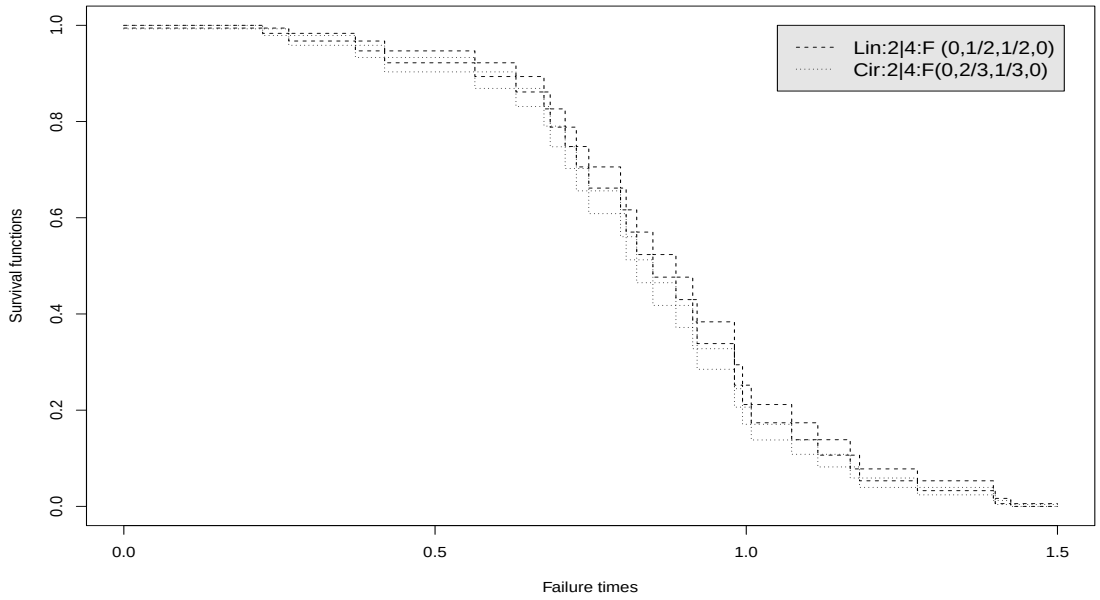

Figure 2.4: NPI lower and upper survival functions for the linear and circular consecutive 2-out-of-4:F systems (Ex. 2.2)

2.3 Comparing failure times of two systems

System signatures provide a straightforward way to compare the reliability of two systems with m exchangeable components (so both systems having components of the same single type) if the signatures are stochastically ordered [61]. Let the signature of system A be q^a and of system B be q^b, and let the failure times of these systems be T^a and T^b, respectively. If $\sum_{j=r}^{m} q_j^a \geq \sum_{j=r}^{m} q_j^b$ for all $r = 1, \ldots, m$ then $P(T^a > t) \geq P(T^b > t)$ for all $t > 0$. Such a comparison is even possible if the two systems do not have the same number of components, as one can always increase the length of a system signature in a way that does not affect the corresponding system's failure time distribution [61], hence one can always make the two systems' signatures of the same length. For example, the signatures $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$ and $(0, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, 0)$ do not have the same number of components. Using Equation 1.6 in Section 1.2, we find that the 5-component system with signature $(\frac{2}{10}, \frac{2}{10}, \frac{3}{10}, \frac{3}{10}, 0)$ is stochastically equivalent to the original system with signature $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$, and

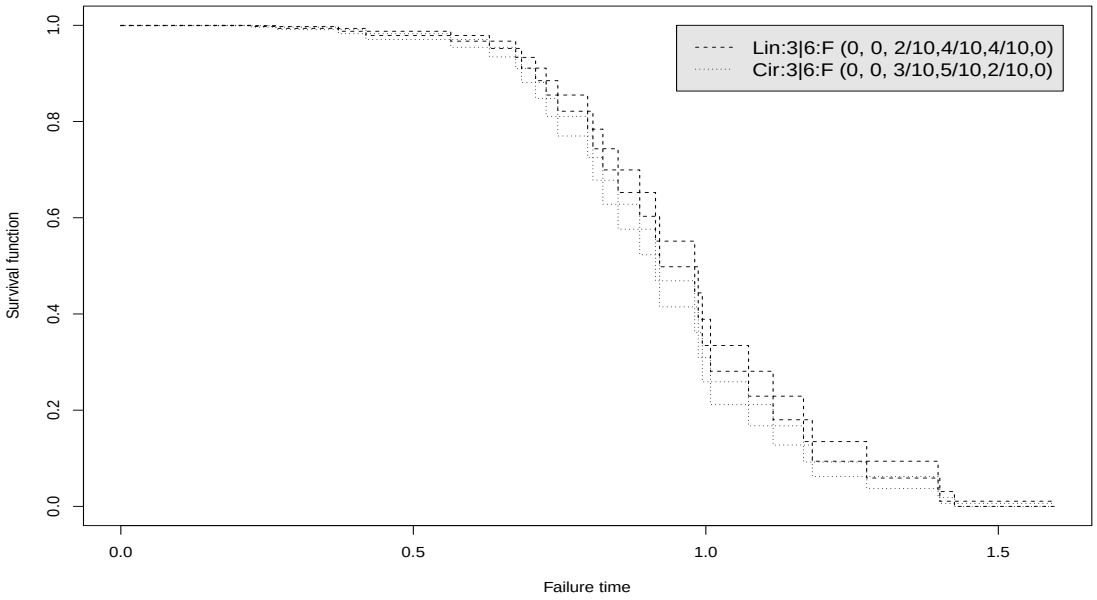

Figure 2.5: NPI lower and upper survival functions for the linear and circular consecutive 3-out-of-6:F systems (Ex. 2.2)

stochastically ordered with the signature $(0, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, 0)$. However, many systems' structures do not have corresponding signatures which are stochastically ordered. For example, the signatures $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$ and $(0, \frac{2}{3}, \frac{1}{3}, 0)$ in the example in Section 2.2 are not stochastically ordered.

This section presents a different way to compare the random failure times T^a and T^b of two systems A and B within the NPI framework, namely by considering the event that system B does not fail before system A, so $T^a \leq T^b$. This has the further advantage of being applicable to any two independent systems, so also to systems that each only have a single type of components but with the components of system A of a different type than those of system B. Subsection 2.3.1 presents NPI lower and upper probabilities for the event $T^a \leq T^b$ for two systems that share the same type of components, followed in Subsection 2.3.2 by such results for two systems with different types of components. Subsection 2.3.3 generalizes this by considering the event $T^a \leq T^b + \delta$ for any real-valued constant δ, and how the NPI

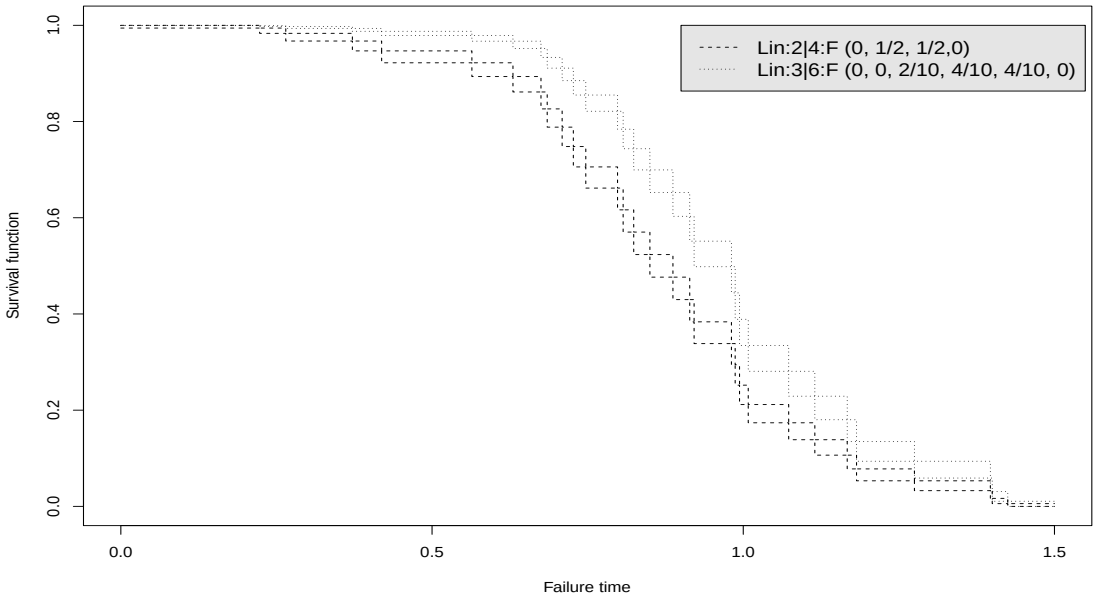

Figure 2.6: NPI lower and upper survival functions for the linear consecutive 2-out-of-4:F and 3-out-of-6:F systems (Ex. 2.2)

lower and upper probabilities for this event behave as a function of δ. This enables a more detailed insight into the actual difference between the random lifetimes of the systems A and B.

2.3.1 Two systems with components of a single type

Consider two systems A and B with m components each and all their components assumed to be exchangeable, so both systems share components of a single type. Using the results presented in Section 2.2, it is easily seen that a similar result holds for the NPI lower and upper probabilities as for precise probabilities mentioned above, namely if $\sum_{j=r}^{m} q_j^a \geq \sum_{j=r}^{m} q_j^b$ for all $r = 1, \ldots, m$ then $\underline{P}(T^a > t) \geq \underline{P}(T^b > t)$ and $\overline{P}(T^a > t) \geq \overline{P}(T^b > t)$ for all $t > 0$. If the signatures q^a and q^b are not stochastically ordered, a different way to compare the systems' failure times is needed, and indeed it is natural to consider the event $T^a \leq T^b$. This does not require both systems to have the same number of components, so let system A consist of m_a

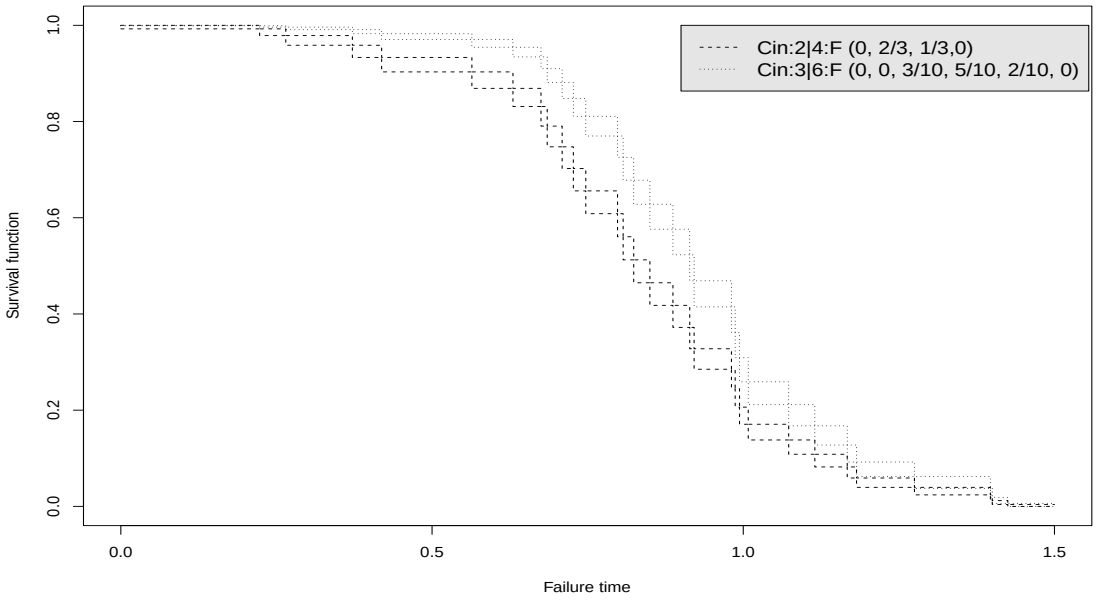

Figure 2.7: NPI lower and upper survival functions for the circular consecutive 2-out-of-4:F and 3-out-of-6:F systems (Ex. 2.2)

components and system B of m_b components, where the failure times of all $m_a + m_b$ components are assumed to be exchangeable. Let the ordered random failure times of the components in system A be $T^a_{1:m_a} \leq T^a_{2:m_a} \leq \ldots \leq T^a_{m_a:m_a}$ and let the ordered random failure times of the components in system B be $T^b_{1:m_b} \leq T^b_{2:m_b} \leq \ldots \leq T^b_{m_b:m_b}$. Using the signature q^a and q^b of these systems, the following equality holds [61]

$$P(T^a \leq T^b) = \sum_{i=1}^{m_a} \sum_{j=1}^{m_b} q^a_i q^b_j P(T^a_{i:m_a} \leq T^b_{j:m_b}) \tag{2.5}$$

This equality can be used directly in NPI without depending on observed failure times, due to the assumed exchangeability of the failure times of all $m_a + m_b$ components. The probabilities in the sum on the right-hand side of (2.5) are precise-valued in NPI, so no use of lower and upper probabilities is required. These probabilities are

$$P(T^a_{i:m_a} \leq T^b_{j:m_b}) = \binom{m_a + m_b}{m_a}^{-1} \left[\sum_{l=0}^{j-1} \binom{i-1+l}{i-1} \binom{m_a - i + m_b - l}{m_a - i} \right] \tag{2.6}$$

This follows by a straightforward counting argument, using the fact that exchangeability of the $m_a + m_b$ component lifetimes includes that their orderings are all equally likely. This implies that the $\binom{m_a+m_b}{m_a}$ different orderings of the lifetimes of the m_a components in system A and the m_b components in system B, neglecting the specific role played by each component in the system (note that this is taken into account by the signatures), are all equally likely. For the event $T^a_{i:m_a} \leq T^b_{j:m_b}$ to occur, the number of components in system B failing before $T^a_{i:m_a}$, so before the failure time of the i-th failing component in system A, can be at most $j-1$. For a value of $l \in \{0, 1, \ldots, j-1\}$, the corresponding term in the sum in Equation (2.6) counts all equally likely orderings of the component failure times with precisely l such failure times for components in system B occurring before $T^a_{i:m_a}$.

Table 2.5 presents the probabilities as given in Equations 2.5 and 2.6 for the events $T^a \leq T^b$ for all combinations of two systems out of the six presented in Figure 2.1. Consider, for example, the systems D and E in Figure 2.1, which have signatures that are not stochastically ordered. Let their failure times be denoted by T^D and T^E, respectively, then $P(T^D \leq T^E) = 0.519$ as shown in Table 2.5, which can be interpreted as indicating that these two systems are about equally reliable, with system E slightly more reliable than system D. While for systems A and B in Figure 2.1, which have signatures that are stochastically ordered, we have $P(T^A \leq T^B) = 0.986$, which clear shows that system B is far more reliable than system A.

2.3.2 Two systems with different types of components

Let system A consist of m_a exchangeable components, and system B of m_b exchangeable components, with the components of the different systems being of different types and their random failure times assumed to be fully independent, which means that any information about components of the type used in system A does not contain any information about components of the type used in system B, and vice versa. The ordered random failure times of the components in system A and of those in system B are denoted as in Subsection 2.3.1. Suppose that n_a components exchangeable with those in system A have been tested and had ordered failure times

(a,b)	$P(T^a \leq T^b)$	(a,b)	$P(T^a \leq T^b)$
(A,B)	0.986	(D,A)	0.214
(A,C)	0.881	(D,B)	0.871
(A,D)	0.786	(D,C)	0.595
(A,E)	0.833	(D,E)	0.519
(A,F)	0.871	(D,F)	0.607
(B,A)	0.014	(E,A)	0.167
(B,C)	0.167	(E,B)	0.881
(B,D)	0.129	(E,C)	0.586
(B,E)	0.119	(E,D)	0.481
(B,F)	0.214	(E,F)	0.595
(C,A)	0.119	(F,A)	0.129
(C,B)	0.833	(F,B)	0.786
(C,D)	0.405	(F,C)	0.481
(C,E)	0.414	(F,D)	0.393
(C,F)	0.519	(F,E)	0.405

Table 2.5: Pairwise comparisons of six systems from Figure 2.1

$t_1^a < t_2^a < \ldots < t_{n_a}^a$, and similarly that ordered observed failure times of n_b tested components, exchangeable with those in system B, are $t_1^b < t_2^b < \ldots < t_{n_b}^b$. Using the signatures q^a and q^b of these systems, a result similar to Equality 2.5 holds for the NPI lower probability for the event $T^a \leq T^b$, namely

$$\underline{P}(T^a \leq T^b) = \sum_{i=1}^{m_a} \sum_{j=1}^{m_b} q_i^a q_j^b \underline{P}(T_{i:m_a}^a \leq T_{j:m_b}^b) \tag{2.7}$$

where, as presented in [27]

$$\underline{P}(T_{i:m_a}^a \leq T_{j:m_b}^b) = \sum_{l=1}^{n_a} P_l^{a,i} [\underline{P}(T_{j:m_b}^b \geq t_l^a)] \tag{2.8}$$

with $P_l^{a,i} = P(T_{i:m_a}^a \in (t_{l-1}^a, t_l^a))$. The summation in (2.8) does not include a term for $l = n+1$ because $\underline{P}(T_{j:m_b}^b \geq \infty) = 0$. Let $v_l \in \{1, \ldots, n_b + 1\}$ be such that $t_{v_l-1}^b < t_l^a < t_{v_l}^b$, then

$$\underline{P}(T_{j:m_b}^b \geq t_l^a) = \sum_{v=v_l+1}^{n_b+1} P(T_{j:m_b}^b \in (t_{v-1}^b, t_v^b)) \tag{2.9}$$

The justification of Equation 2.7 is similar to that of Equation 2.1 in Section 2.2, effectively the NPI lower probabilities for the events $T^a_{i:m_a} \leq T^b_{j:m_b}$, for $i = 1, \ldots, m_a$ and $j = 1, \ldots, m_b$, can all be attained simultaneously for the same underlying configuration of observed and future failure times for components of type A (all future observations 'at' the right end-point of each interval) and the same underlying configuration of observed and future failure times for components of type B (all future observations 'at' the left end-point of each interval) [27]. The corresponding NPI upper probability for the event $T^a \leq T^b$ is derived and justified similarly, and is

$$\overline{P}(T^a \leq T^b) = \sum_{i=1}^{m_a} \sum_{j=1}^{m_b} q^a_i q^b_j \overline{P}(T^a_{i:m_a} \leq T^b_{j:m_b}) \tag{2.10}$$

where

$$\overline{P}(T^a_{i:m_a} \leq T^b_{j:m_b}) = \sum_{l=1}^{n_a+1} P^{a,i}_l [\overline{P}(T^b_{j:m_b} \geq t^a_{l-1})] \tag{2.11}$$

and

$$\overline{P}(T^b_{j:m_b} \geq t^a_l) = \sum_{v=v_l}^{n_b+1} P(T^b_{j:m_b} \in (t^b_{v-1}, t^b_v)) \tag{2.12}$$

Example 2.3

The pairwise comparison results presented in this section are illustrated using the six systems from Example 2.1, each with four exchangeable components but with the different systems considered having different types of components with independence of failure times assumed. Table 2.6 presents the NPI lower and upper probabilities for the events $T^a \leq T^b$, as presented in Equations 2.7 and 2.10, for the failure times T^a and T^b for all combinations of two systems out of the six presented in Figure 2.1. For all these 30 events, it is assumed that $n_a = 3$ components exchangeable with those in the system with failure time T^a and $n_b = 2$ components exchangeable with those in the system with failure time T^b have been tested and that the ordering of the test data is $t^a_1 < t^b_1 < t^a_2 < t^b_2 < t^a_3$. Of course, the NPI lower and upper probabilities in Table 2.6 show that system A is the least reliable and system B the most reliable of these systems. Notice that the comparisons of systems A, B, C, F with either system D or E (whose signatures are not stochastically ordered) give very similar results, yet they all indicate that system E is slightly more reliable than

(a,b)	$\underline{P}(T^a \leq T^b)$	$\overline{P}(T^a \leq T^b)$	(a,b)	$\underline{P}(T^a \leq T^b)$	$\overline{P}(T^a \leq T^b)$
(A,B)	0.724	0.983	(D,A)	0.110	0.657
(A,C)	0.514	0.950	(D,B)	0.444	0.923
(A,D)	0.438	0.937	(D,C)	0.294	0.810
(A,E)	0.457	0.941	(D,E)	0.257	0.781
(A,F)	0.524	0.951	(D,F)	0.304	0.816
(B,A)	0.017	0.276	(E,A)	0.097	0.650
(B,C)	0.059	0.543	(E,B)	0.423	0.924
(B,D)	0.049	0.476	(E,C)	0.272	0.810
(B,E)	0.050	0.486	(E,D)	0.229	0.770
(B,F)	0.063	0.562	(E,F)	0.283	0.815
(C,A)	0.076	0.577	(F,A)	0.077	0.556
(C,B)	0.350	0.903	(F,B)	0.343	0.890
(C,D)	0.185	0.717	(F,C)	0.219	0.743
(C,E)	0.190	0.728	(F,D)	0.184	0.696
(C,F)	0.230	0.771	(F,E)	0.190	0.706

Table 2.6: Pairwise comparisons of six systems from Figure 2.1 (Ex. 2.3)

system D, the same conclusion as drawn in Subsection 2.3.1. This is an attractive way to compare the random failure times of two systems, which takes both the system structures and the information from the test data directly into account and considers a natural event of interest. The NPI lower probability reflects the evidence in favour of the event $T^a \leq T^b$ while the corresponding upper probability reflects the evidence in favour of the complementary event $T^a > T^b$. The difference between corresponding upper and lower probabilities, also called the 'imprecision', is due to the limited information available and the relatively weak modelling assumptions. In Table 2.6 the imprecision of most events is large, which is due to there being only 5 observations in total. If more test data are available, the imprecision typically becomes smaller, it would decrease to 0 if the numbers of test data in both groups go to infinity.

Table 2.7 presents the NPI lower and upper probabilities for the pairwise com-

Data ordering	$\underline{P}(T^D \leq T^E)$	$\overline{P}(T^D \leq T^E)$
$t_1^d < t_2^d < t_3^d < t_1^e < t_2^e$	0.548	1
$t_1^d < t_2^d < t_1^e < t_3^d < t_2^e$	0.442	0.940
$t_1^d < t_2^d < t_1^e < t_2^e < t_3^d$	0.371	0.869
$t_1^d < t_1^e < t_2^d < t_3^d < t_2^e$	0.328	0.852
$t_1^d < t_1^e < t_2^d < t_2^e < t_3^d$	0.257	0.781
$t_1^e < t_1^d < t_2^d < t_3^d < t_2^e$	0.219	0.757
$t_1^e < t_1^d < t_2^d < t_2^e < t_3^d$	0.149	0.686
$t_1^d < t_1^e < t_2^e < t_2^d < t_3^d$	0.181	0.675
$t_1^e < t_1^d < t_2^e < t_2^d < t_3^d$	0.072	0.580
$t_1^e < t_2^e < t_1^d < t_2^d < t_3^d$	0	0.466

Table 2.7: Pairwise comparisons of systems D and E with $n_D = 3$ and $n_E = 2$

parison of systems D and E, considering the event $T^D \leq T^E$ with $n_D = 3$ observed failure times for components exchangeable with those in system D and $n_E = 2$ observed failure times for components exchangeable with those in system E, and all possible orderings of these observed failure times. These lower and upper probabilities vary of course for the different data orderings, and also the imprecision varies. If the three tested components of type D all failed before the two components of type E, the data do not contain any evidence against the possibility that components of type D will always fail before components of type E, which is reflected in $\overline{P}(T^D \leq T^E) = 1$ in this case. Similarly, the other extreme data ordering does not provide any evidence in favour of the possibility that components of type D will ever fail before components of type E, as reflected by $\underline{P}(T^D \leq T^E) = 0$ for the final ordering in Table 2.7.

2.3.3 Difference between failure times of two systems

The method presented in Subsection 2.3.2 compares the random failure times of two systems by considering the event that one fails before the other, but it does not provide insight into the actual difference between these failure times. Therefore, the

approach of Subsection 2.3.2, using the same setting of two systems with different types of components, is now generalized by considering the event $T^a \leq T^b + \delta$, for real-valued δ. Of course, the setting of Subsection 2.3.1 can be similarly generalized. The following generalization of Equation 2.5,

$$P(T^a \leq T^b + \delta) = \sum_{i=1}^{m_a} \sum_{j=1}^{m_b} q_i^a q_j^b P(T_{i:m_a}^a \leq T_{j:m_b}^b + \delta)$$

is proven in the same way as Equation 2.5, and is intuitively logical because adding the constant value δ to the random lifetime of a system can be thought of as adding it to the lifetimes of all its components, doing so will not change the signature of the system. This immediately carries through to the NPI lower probability for this event, which is

$$\underline{P}(T^a \leq T^b + \delta) = \sum_{i=1}^{m_a} \sum_{j=1}^{m_b} q_i^a q_j^b \underline{P}(T_{i:m_a}^a \leq T_{j:m_b}^b + \delta) \tag{2.13}$$

with the NPI lower probabilities in the sum on the right-hand side equal to

$$\underline{P}(T_{i:m_a}^a \leq T_{j:m_b}^b + \delta) = \sum_{l=1}^{n_a} P_l^{a,i}[\underline{P}(T_{j:m_b}^b + \delta \geq t_l^a)] \tag{2.14}$$

Let $v_{l,\delta} \in \{1, \ldots, n_b + 1\}$ be such that $t_{v_{l,\delta}-1}^b < t_l^a - \delta < t_{v_{l,\delta}}^b$, then

$$\underline{P}(T_{j:m_b}^b + \delta \geq t_l^a) = \sum_{v=v_{l,\delta}+1}^{n_b+1} P(T_{j:m_b}^b \in (t_{v-1}^b, t_v^b)) \tag{2.15}$$

The corresponding NPI upper probability for the event $T^a \leq T^b + \delta$ is

$$\overline{P}(T^a \leq T^b + \delta) = \sum_{i=1}^{m_a} \sum_{j=1}^{m_b} q_i^a q_j^b \overline{P}(T_{i:m_a}^a \leq T_{j:m_b}^b + \delta) \tag{2.16}$$

where

$$\overline{P}(T_{i:m_a}^a \leq T_{j:m_b}^b + \delta) = \sum_{l=1}^{n_a+1} P_l^{a,i}[\overline{P}(T_{j:m_b}^b + \delta \geq t_{l-1}^a)] \tag{2.17}$$

and

$$\overline{P}(T_{j:m_b}^b + \delta \geq t_{l-1}^a) = \sum_{v=v_{l,\delta}}^{n_b+1} P(T_{j:m_b}^b \in (t_{v-1}^b, t_v^b)) \tag{2.18}$$

Compared to the NPI lower and upper probabilities presented in Subsection 2.3.2, which correspond to those for $\delta = 0$ here, calculation of these NPI lower and upper probabilities just follows from shifting the m_b test observations for components

exchangeable to those in system B by adding δ, or alternatively by subtracting δ from each observation t_l^a. For changing values of δ, these NPI lower and upper probabilities only change if δ is large enough to change the ordering of the $t_1^b, \ldots, t_{n_b}^b$ relative to the values $t_1^a - \delta, \ldots, t_{n_a}^a - \delta$, such a change of the ordering can happen for at most $n_a \times n_b$ different values of δ. Therefore, $\underline{P}(T^a \leq T^b + \delta)$ and $\overline{P}(T^a \leq T^b + \delta)$ can have at most $n_a \times n_b + 1$ different values (including the case $\delta = 0$), and as function of δ these lower and upper probabilities are step functions which change value at the same $n_a \times n_b$ points, making their computation straightforward unless $n_a \times n_b$ is very large.

Example 2.4

Systems D and E of Figure 2.1 have been of interest as their signatures are not stochastically ordered. Assume now that they have different types of components, and that $n_d = n_e = 30$ components exchangeable with those of each type in the respective system have been tested, leading to the failure times in Table 2.8. These ordered failure times for system D were simulated from a Weibull distribution with shape parameter 3 and scale parameter 1, and for system E from a Weibull distribution with shape parameter 2 and scale parameter 1.

System D			System E		
0.223	0.747	0.994	0.154	0.585	1.076
0.265	0.798	1.008	0.155	0.598	1.169
0.372	0.807	1.073	0.347	0.642	1.239
0.419	0.824	1.115	0.402	0.692	1.248
0.564	0.850	1.167	0.483	0.738	1.327
0.630	0.887	1.182	0.512	0.822	1.421
0.675	0.914	1.275	0.513	0.843	1.569
0.685	0.921	1.397	0.548	0.848	1.643
0.709	0.981	1.400	0.563	0.863	1.735
0.727	0.987	1.425	0.574	0.938	2.565

Table 2.8: Simulated ordered component failure times for Example 2.4

Figure 2.8 presents the NPI lower and upper probabilities for the event $T^d \leq T^e + \delta$ as function of δ. In the top-left figure, Figure 2.8.1, these functions are given for the data in Table 2.8. For these data, these functions remain constant for values of δ less than -2.342 or greater than 1.271, as in these cases the two data sets are completely non-overlapping, which shows in the fact that the NPI lower probability for this event is equal to zero for $\delta < -2.342$ and the NPI upper probability for this event is equal to one for $\delta > 1.271$. Actually, the changes in these NPI lower and upper probabilities at δ equal to -2.342 or 1.271 are very small and not well visible in Figure 2.8.1. The same is true at other values of δ close to these minimal and maximal ones at which the NPI lower and upper probabilities change. At $\delta = -2.342$, the NPI lower probability $T^d \leq T^e + \delta$ increases from 0 to 0.00013 and the NPI upper probability increases from 0.03630 to 0.03656, while at $\delta = 1.271$ the lower probability increases from 0.98702 to 0.98717 and the upper probability increases from 0.99996 to 1.

The three further figures included in Figure 2.8 show the effect of substantial changes to the actual observations, that is changes that actually change the order of the observations, and hence they show how the NPI lower and upper probabilities for the event $T^d \leq T^e + \delta$ adapt to changes in the component test data. First, the largest observed failure time for system D, 1.425, is replaced by 3.425, which makes it the largest observed value in both sets of data. The resulting NPI lower and upper probabilities for the event $T^d \leq T^e + \delta$ as functions of δ are presented in Figure 2.8.2, but the effect on the figures is not well visible when compared to the original situation in Figure 2.8.1. Figures 2.8.3 and 2.8.4 show the NPI lower and upper probabilities with the largest 4 and 10, respectively, values for System D, as given in Table 2.8, changed by adding 2 to the original data values, which implies that these all become larger than the largest observation for System E. Now the effect is clear in both figures, and of course substantially stronger in case 10 observations have been changed. Figure 2.9 presents the same functions of Figures 2.8.1 and 2.8.4, so for the original data and with 10 values changed, on a larger scale to see the differences more clearly. While the differences for the larger values of δ are obvious, this figure shows that there have also been some small changes for δ

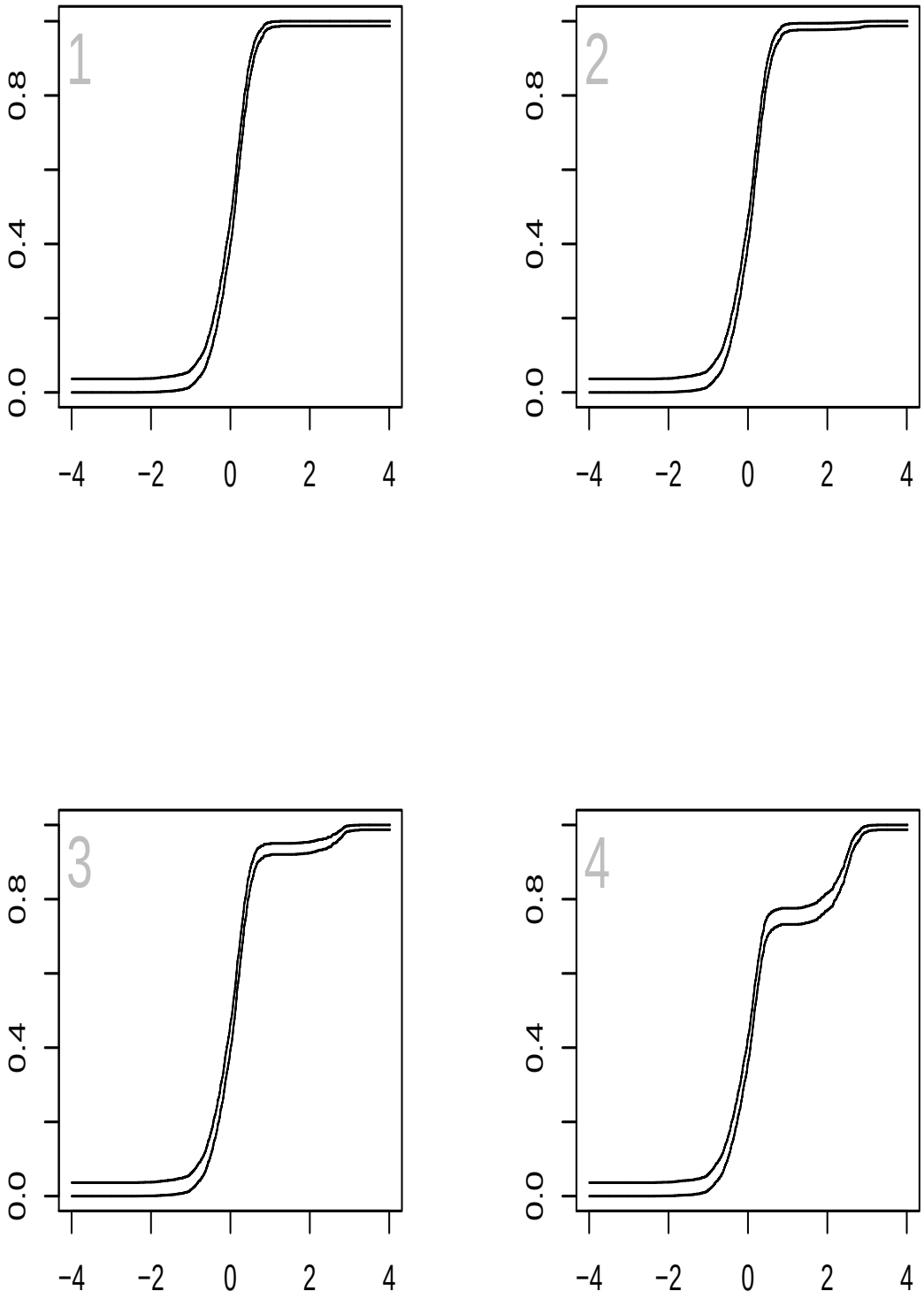

Figure 2.8: The difference of failure times of two systems (Ex 2.4)

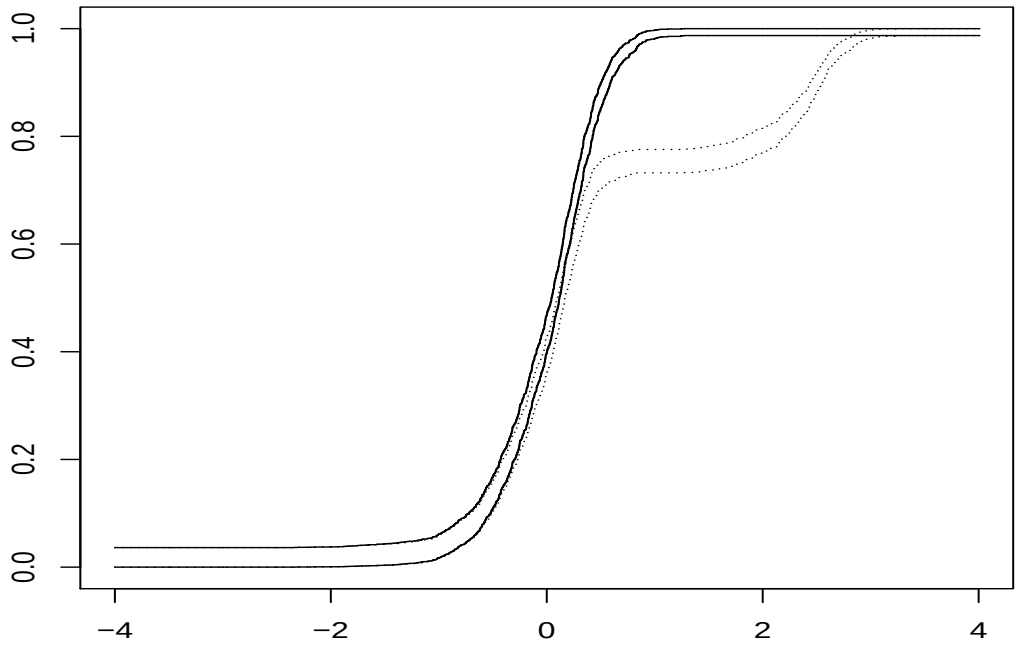

Figure 2.9: Difference of failure times of two systems (Figures 2.8.1 and 2.8.4)

close to 0 and even for negative values of δ.

2.4 Concluding remarks

In this chapter we have introduced the use of signatures in the study of system failure times with lower and upper probabilities. There are many related research challenges, for example a slightly more challenging topic is simultaneous comparison of more than two systems' failure times. The NPI lower and upper probabilities for pairwise comparisons, as presented in Section 2.3, cannot be combined directly into such quantifications for multiple comparisons. For example, it may be of interest to consider the event that a particular one of the systems considered is the most reliable in the sense of its random failure time being the largest of all systems' failure times, so it is of interest to generalize the method presented in Section 2.3 to derive NPI lower and upper probabilities for such events. This can be done in NPI along

the lines of such multiple comparisons as presented in [32].

There are major research challenges to the general theory of signatures, solutions to which may be of particular interest when working with lower and upper probabilities. However, its generalization to systems with multiple types of components is very complicated, if not impossible. In Chapter 4, we use the concept of 'survival signatures' presented by Coolen and Coolen-Maturi [26], as a more powerful alternative to derive NPI lower and upper survival functions for a system consisting of different types of components.

Chapter 3

System failure time based on bounds for the signature

3.1 Introduction

In Chapter 2 we have presented the use of signatures for nonparametric predictive inference (NPI) for system reliability [24]. In NPI for system reliability, lower and upper survival functions are derived for the system's failure time, these reflect the limited knowledge about reliability of the components, using only the information from component tests. However, deriving the system signature is computationally complex. This chapter presents how limited information about the signature can be used to derive bounds on such lower and upper survival functions and related inferences.

Derivation of the signature is not straightforward, even for relatively basic systems. However, for specific inferences it may not be necessary to compute the exact signature. If computation of signatures is stopped before the exact signature is derived, one typically has bounds for the elements of the signature vector, so the probabilities q_j. We explore the use of such bounds in NPI, leading to lower and upper bounds for the NPI lower and upper survival functions. For specific inferences, these bounds may already be conclusive, meaning that no further computation is needed. The basic results for the use of such bounds in NPI are presented in Section 3.2. Section 3.3 presents the explanation of the possible use of information on

signatures for subsystems. In Section 3.4 comparison of reliability of two systems is presented by directly considering the random failure times of the systems using partially known signatures. Section 3.5 contains some Concluding remarks.

3.2 Partially known signatures

Computation of the system signature is a complex problem due to the fact that $m!$ orderings in which the m components can fail must be considered [15, 61]. Explicit expressions for the signature of some specific system structures are available [44], but general algorithms to compute signatures have not received much attention in the literature, with the noticeable exception of a logical approach presented by Boland [15] which uses the concept of minimal ordered cut sets, reducing the total number of orderings that need to be counted by grouping together orderings which share the same minimal ordered cut set. However, as any computational method has to deal with the very large number of orderings, it is interesting to consider if one really needs to know the exact signature for a specific inference on the system's reliability. It is likely that any method for computing the signature, if ended before the exact signature has been derived, will provide bounds for the probabilities q_j of the signature. In this section the use of bounds on q_j is explored in NPI. The method presented can be applied throughout the process of computation of the signature and can indicate when further computation is not required.

Assume that bounds $\underline{q}_j$ and $\overline{q}_j$, for $j = 1, \ldots, m$, for the elements of signature $q = (q_1, \ldots, q_m)$ have been derived, with $0 \leq \underline{q}_j \leq q_j \leq \overline{q}_j \leq 1$. Assume that $\sum_{j=1}^{m} \underline{q}_j \leq 1$ and $\sum_{j=1}^{m} \overline{q}_j \geq 1$, so at least one signature (with elements summing to one) exists between these bounds. We also assume that, for all $j = 1, \ldots, m$

$$\underline{q}_j \geq 1 - \sum_{\substack{l=1 \\ l \neq j}}^{m} \overline{q}_l \quad \text{and} \quad \overline{q}_j \leq 1 - \sum_{\substack{l=1 \\ l \neq j}}^{m} \underline{q}_l \tag{3.1}$$

If these inequalities are not satisfied then $\underline{q}_j$ can be increased or $\overline{q}_j$ decreased, to the value which gives equality in the corresponding inequality without any change to the set of signatures q whose elements are all within these bounds.

Suppose that we want to derive the NPI lower and upper survival functions

(2.1) and (2.2) based on the observed failure times of n tested components, which are exchangeable with those in the system. If the exact system signature is not known, but bounds $\underline{q}_j$ and $\overline{q}_j$ are available for each probability q_j, then these can be used to derive lower and upper bounds for these NPI lower and upper survival functions, which are the tightest possible bounds corresponding to these bounds for the elements of the signature. Because $\underline{S}_{T_{j:m}}(t)$ and $\overline{S}_{T_{j:m}}(t)$ are increasing functions of j, for all $t > 0$, it is clear that we can derive two signatures with all their elements within the bounds and such that one of them provides the maximum lower bound for both $\underline{S}_{T_S}(t)$ and $\overline{S}_{T_S}(t)$, and the other provides the minimum upper bound for both $\underline{S}_{T_S}(t)$ and $\overline{S}_{T_S}(t)$, for all $t > 0$. This corresponds to the link between the stochastic ordering of random failure times of systems and the stochastic ordering of their signatures [61]. We call the signature within these bounds that provides the maximum lower bound for the NPI lower and upper survival functions, the 'pessimistic signature' and we denote it by q^p. Similarly, we call the signature within these bounds that provides the minimum upper bound for the NPI lower and upper survival functions, the 'optimistic signature', denoted by q^o. These terms follow the logical interpretation of 'pessimistic' and 'optimistic' in terms of survival of the system and the lack of knowledge of the actual NPI lower and upper survival functions as the exact signature is not known.

To apply this in practice, define $r_j = \underline{q}_j$ for $j = 1, \ldots, m$, then calculate $\sum_{j=1}^{m} r_j$:

- If $\sum_{j=1}^{m} r_j > 1$ then $\underline{q} = (q_1, \ldots, q_m)$ is not a lower bound for the signature.
- If $\sum_{j=1}^{m} r_j = 1$ then we already know the whole signature.
- If $\sum_{j=1}^{m} r_j < 1$ then, for the pessimistic signature, we put the probability mass that is flexible according to the given bounds $\underline{q}_j$ and $\overline{q}_j$ as far to the left as possible, so to elements with lower values of j, hence making earlier system failure more likely. The optimistic signature puts this probability mass as far to the right as possible, so to elements with higher values of j, hence making later system failure more likely.

Algorithms to derive q^p and q^o are easy to implement. To determine the pessimistic

signature vector q^p of the system based on $[\underline{q}_j, \overline{q}_j]$ for $j = 1, \ldots, m$, we use the following algorithm.

1. Set $r_j = \underline{q}_j$ for $j = 1, \ldots, m$

2. Determine how much probability mass has not yet been assigned, we define this by

$$f = 1 - \sum_{j=1}^{m} r_j \tag{3.2}$$

   Now we assign this probability mass f in the most pessimistic way, staying within the bounds $[\underline{q}_j, \overline{q}_j]$. We initiate the following step by setting $j = 1$.

3. If $\overline{q}_j - \underline{q}_j < f$, then set $r_j = \overline{q}_j$ and go to step 4.
   If $\overline{q}_j - \underline{q}_j \geq f$, then set $r_j = \underline{q}_j + f$ and stop the algorithm;

4. Set $f = 1 - \sum_{j=1}^{m} r_j$ and assign the extra probability mass to the next component, by setting j to $j + 1$ and repeating step 3.

5. This process is repeated until the total probability mass f has been assigned. It is easy to see that the process will terminate.

For $j_p \in [1, \ldots, m]$, the components q_j^p of the pessimistic signature vector q^p based on $[\underline{q}_j, \overline{q}_j]$ are given by

$$q_j^p = \begin{cases} \overline{q}_j & (j = 1, \ldots, j_p - 1) \\ 1 - \sum_{j=1}^{j_p-1} \overline{q}_j - \sum_{j=j_p+1}^{m} \underline{q}_j & (j = j_p) \\ \underline{q}_j & (j = j_p + 1, \ldots, m) \end{cases}$$

For the optimistic signature vector q^o of the system based on $[\underline{q}_j, \overline{q}_j]$ for $j = 1, \ldots, m$, we can put the probability mass as far to the right as possible. To do this we use the following algorithm

1. Set $r_j = \underline{q}_j$ for $j = 1, \ldots, m$

2. Determine how much probability mass has not yet been assigned, we define this by f;

$$f = 1 - \sum_{j=1}^{m} r_j \tag{3.3}$$

Now we assign this probability mass f in the most optimistic way, staying within the bounds $[\underline{q}_j, \overline{q}_j]$. We initiate the following step by setting $j = m$.

3. If $\overline{q}_j - \underline{q}_j < f$, then set $r_j = \overline{q}_j$ and go to step 4.
If $\overline{q}_j - \underline{q}_j \geq f$, then set $r_j = \underline{q}_j + f$ and stop the algorithm;

4. Set $f = 1 - \sum_{j=1}^{m} r_j$ and assign the extra probability mass to the next component by setting j to j_1, and repeating step 3.

5. This process is repeated until eventually the total probability mass f has been assigned. It is easy to see that the process will terminate.

For $j_o \in [1, \ldots, m]$, the components q_j^o of the optimistic signature vector q^o based on $[\underline{q}_j, \overline{q}_j]$ are given by

$$q_j^o = \begin{cases} \overline{q}_j & (j = 1, \ldots, j_o - 1) \\ 1 - \sum_{j=1}^{j_o-1} \underline{q}_j - \sum_{j=j_o+1}^{m} \overline{q}_j & (j = j_o) \\ \underline{q}_j & (j = m, \ldots, j_o + 1) \end{cases}$$

These algorithms to derive q^p and q^o are easy to implement. The assumptions (3.1) ensure that the j_p, j_o are unique and $q_{j_p}^p \in [\underline{q}_{j_p}, \overline{q}_{j_p}]$ and $q_{j_o}^o \in [\underline{q}_{j_o}, \overline{q}_{j_o}]$.

The lower and upper bounds for the NPI lower and upper survival functions for T_S follow immediately, in line with the results from Equations 2.1 and 2.2 in Section 2.2 and using the pessimistic and optimistic signatures q^p and q^0. The lower and upper bounds for the NPI lower survival function for T_s are

$$\underline{S}_{T_S}^p(t) = \sum_{j=1}^{m} q_j^p \underline{P}(T_{j:m} > t) \tag{3.4}$$

$$\underline{S}_{T_S}^o(t) = \sum_{j=1}^{m} q_j^o \underline{P}(T_{j:m} > t) \tag{3.5}$$

and the lower and upper bounds for the NPI upper survival function for T_S are

$$\overline{S}^{p}_{T_S}(t) = \sum_{j=1}^{m} q_j^p \overline{P}(T_{j:m} > t) \tag{3.6}$$

$$\overline{S}^{o}_{T_S}(t) = \sum_{j=1}^{m} q_j^o \overline{P}(T_{j:m} > t) \tag{3.7}$$

In this notation, p and o again indicate pessimistic and optimistic bounds, respectively. These are the sharpest possible bounds for the NPI lower and upper survival functions for T_S corresponding to the bounds $\underline{q}_j$ and $\overline{q}_j$ for q_j, for $j = 1, \ldots, m$. Due to the construction of these bounds, it is clear that they can actually be attained.

If the real signature q is only known up to such bounds for its individual elements, it follows that the NPI lower and upper survival functions for T_S are between their respective bounds as given by Equations 3.4 to 3.7, and nothing more can be deduced without additional assumptions or indeed without further computation of the signature. Further computation which falls short of deriving the exact signature will lead to new bounds for the NPI lower and upper survival functions which are within the corresponding earlier bounds. This may be useful for deciding if further computation is required for a specific inferential problem. For example, if one is interested in the system's reliability at time t^* and requires a minimum probability of p^* for the system to function at time t^*, then $\underline{S}^{p}_{T_S}(t^*) \geq p^*$ would imply that the reliability requirement is certainly met without need for further computation of the signature. Similarly, if $\overline{S}^{o}_{T_S}(t^*) \leq p^*$ then the reliability requirement is certainly not met. In the other situations one cannot draw a firm conclusion about whether or not the reliability requirement is met and one may want to continue computation of the system signature.

Even with the exact signature it is possible that no firm conclusion can be drawn, namely if $\underline{S}_{T_S}(t^*) < p^* < \overline{S}_{T_S}(t^*)$. In such a case one would either require more test data or use additional information, insights or assumptions in order to reach a conclusion. We consider it an advantage of the use of lower and upper probabilities that such situations can occur, as they reflect the limits to the amount of information in test results. The use of these lower and upper bounds at different levels of computation of the system signature, so with increasingly accurate bounds, will

be illustrated in Example 3.1. In all examples throughout this chapter, we will concentrate on the optimal lower bound for the NPI lower survival function and the optimal upper bound for the NPI upper survival function, which are likely to be of most relevance for inferences.

As a special case of the above presented method, suppose that we have computed exactly the values of $q_1, \ldots, q_l$ for a value $l < m$, but we have not performed any computations for further elements of q. Then define the following two signatures, with $q_{(l+1,m)} = 1 - \sum_{j=1}^{l} q_j = \sum_{j=l+1}^{m} q_j$, the optimistic signature is

$$q_l^o = (q_1, \ldots, q_l, 0, \ldots, 0, q_{(l+1,m)}) \tag{3.8}$$

and the pessimistic signature is

$$q_l^p = (q_1, \ldots, q_l, q_{(l+1;m)}, 0, \ldots, 0) \tag{3.9}$$

It is clear that, compared to any signature with known $q_1, \ldots, q_l$ but further elements unknown, these two signatures above are indeed 'optimistic' and 'pessimistic', respectively, with regard to system survival (this follows from the stochastic orderings). Using these optimistic and pessimistic signatures, we can derive bounds for the system's NPI lower and upper survival functions as presented in Equations 3.4 to 3.7.

Example 3.1

For the system in Figure 3.1, computing the signature involves determining for all of the 7! = 5040 orderings of the failure times of the 7 components, at which of these ordered times the system fails. Of course, all 6! = 720 orderings with failure of Component 1 occurring first lead to immediate failure, from which we can conclude the lower bound $\underline{q}_1 = 0.143$. It is easy to see that no other component's failure will lead to immediate system failure if it is the first to fail, hence also the upper bound $\overline{q}_1 = 0.143$. In addition, it is easy to see that the system cannot function with at most two functioning components, this leads to the upper bounds $\overline{q}_6 = \overline{q}_7 = 0$. This information, using conditions (3.1) but without further computation, can be reflected by $\underline{q} = (0.143, 0, 0, 0, 0, 0, 0)$ and $\overline{q} = (0.143, 0.857, 0.857, 0.857, 0.857, 0, 0)$. The cor-

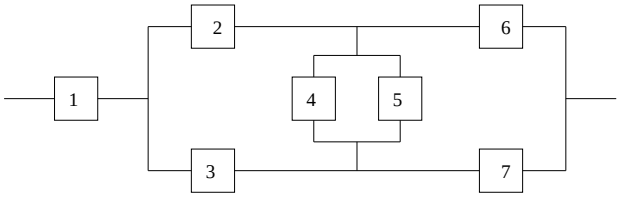

Figure 3.1: A system with 7 components (Exs. 3.1, 3.2 and 3.4)

responding pessimistic and optimistic signatures are $q^p = (0.143, 0.857, 0, 0, 0, 0, 0)$ and $q^o = (0.143, 0, 0, 0, 0.857, 0, 0)$. Computation of signatures by counting orderings typically leads to information in the form of lower bounds $\underline{q}_j$ for individual elements of the signature. To illustrate the method presented in Section 3.2 further, Table 3.1 provides, in addition to the first case just mentioned, three more combinations of lower and upper bounds for this system's signature as occurred at different stages of its computation, with increasing amount of information in Cases 1 to 4. For each case the pessimistic and optimistic signatures q^p and q^o, respectively, are also presented in this table.

Test component failure times were simulated for this example, with $n = 100$ observations taken from the Weibull distribution with shape parameter 3 and scale parameter 1. The corresponding lower bounds for the NPI lower survival function, $\underline{S}^p_{T_S}(t)$ as given in Equation 3.4, and the upper bounds for the NPI upper survival function, $\overline{S}^o_{T_S}(t)$ as given in Equation 3.7, are presented in the plots in Figure 3.2, where each plot also presents the NPI lower and upper survival functions based on the exact signature, which is $q = (1/5040) \times (720, 1200, 1392, 1440, 288, 0, 0) = (0.143, 0.238, 0.276, 0.286, 0.057, 0, 0)$. These plots illustrate the use of the bounds as presented in this chapter, and also show that the lower bound of the NPI lower survival function moves up if more details about the signature become known, in which case the upper bound for the NPI upper survival function moves down.

As possible use of these bounds in order to determine when no further computation for the signature is needed, suppose a reliability requirement that the system's failure time should exceed 0.5 with probability at least 0.8. With the bounds for the signature in Case 1, the upper bound for the NPI upper survival function at 0.5 is greater than 0.8 and the corresponding lower bound for the NPI lower survival

Case 1	$\underline{q}$	(0.143, 0, 0, 0, 0, 0, 0)
	$\overline{q}$	(0.143, 0.857, 0.857, 0.857, 0.857, 0, 0)
	q^p	(0.143, 0.857, 0, 0, 0, 0, 0)
	q^o	(0.143, 0, 0, 0, 0.857, 0, 0)
Case 2	$\underline{q}$	(0.143, 0.143, 0, 0, 0, 0, 0)
	$\overline{q}$	(0.143, 0.857, 0.714, 0.714, 0.714, 0, 0)
	q^p	(0.143, 0.857, 0, 0, 0, 0, 0)
	q^o	(0.143, 0.143, 0, 0, 0.714, 0, 0)
Case 3	$\underline{q}$	(0.143, 0.143, 0.076, 0, 0, 0, 0)
	$\overline{q}$	(0.143, 0.781, 0.714, 0.638, 0.638, 0, 0)
	q^p	(0.143, 0.781, 0.076, 0, 0, 0, 0)
	q^o	(0.143, 0.143, 0.076, 0, 0.638, 0, 0)
Case 4	$\underline{q}$	(0.143, 0.143, 0.152, 0.157, 0, 0, 0)
	$\overline{q}$	(0.143, 0.548, 0.557, 0.562, 0.405, 0, 0)
	q^p	(0.143, 0.548, 0.152, 0.157, 0, 0, 0)
	q^o	(0.143, 0.143, 0.152, 0.157, 0.405, 0, 0)

Table 3.1: Bounds, pessimistic and optimistic signatures (Ex. 3.1)

function is less than 0.8, but for the bounds in Case 2, based on some additional computations, the upper bound for the NPI upper survival function at 0.5 is less than 0.8, so it is clear that the reliability requirement cannot be met and hence that no further computation of the signature is needed. Similarly, if one only requires that the system's failure time should exceed 0.5 with probability at least 0.3 then one needs no more computation once the bounds in Case 4 have been derived, as the corresponding lower bound for the NPI lower survival function at 0.5 exceeds 0.3, hence this reliability requirement is certainly met.

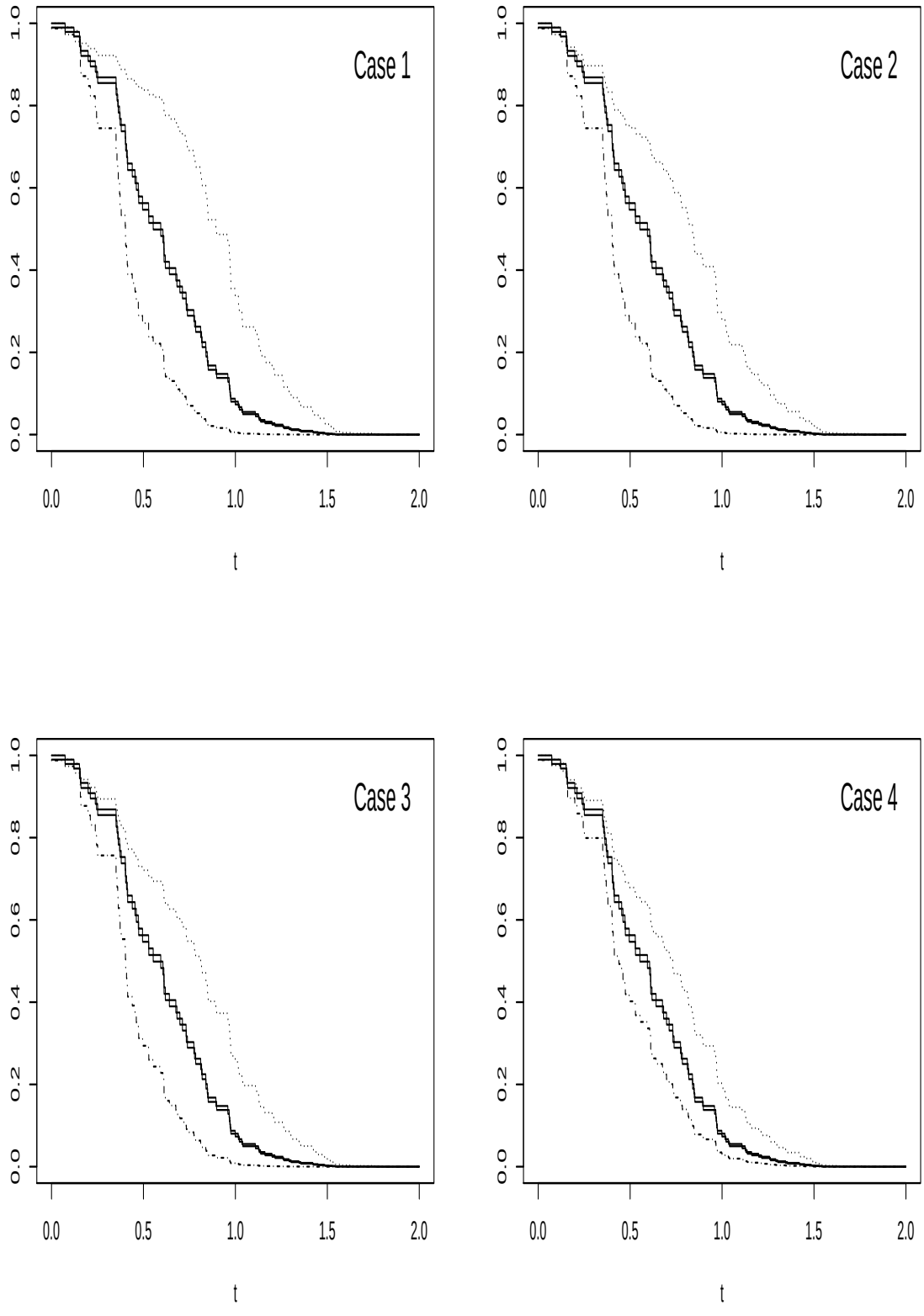

Figure 3.2: NPI lower and upper survival functions (Ex. 3.1)

Example 3.2

In this example, three data sets, each consisting of n=100 observations of component failure times, are simulated from the Weibull distribution with shape parameter 3 and scale parameter 1. The data sets are presented by boxplots in Figure 3.3. We focus again on the system consisting of 7 components presented in Figure 3.1, and assume that we have the signatures q^p = (0.143, 0.548, 0.152, 0.157, 0, 0, 0) and q^o = (0.143, 0.143, 0.152, 0.157, 0.405, 0, 0). Figure 3.4 presents the lower bound for the NPI lower survival function and the upper bound for the NPI upper survival function using these three different data sets in order to see how much these different data sets affect these bounds for the NPI lower and upper survival functions. The first 3 plots show that, of course, there are some differences due to the different data sets. In the top left of Figure 3.4, the functions are given for data set 1 in Figure 3.3. Figure 3.4.1 for data set 1 shows relatively early drop in the lower bound for the NPI lower survival function and the upper bound for the NPI upper survival function due to the fact that data set 1 has relatively more small data values than data sets 2 and 3. Also, it is clear in Figure 3.4.2 that the lower bound for the NPI lower survival function and the upper bound for the NPI upper survival function drop a bit at a very low value of t due to the first value in data set 2, which is much smaller than all values in data sets 1 and 3. Figure 3.4.4 presents the same functions of Figures 3.4.1, 3.4.2 and 3.4.3 in order to see the differences more clearly.

3.3 Computation using bounds for the signatures of two subsystems

For the NPI approach, bounds for the signatures of two subsystems in parallel or series configuration can be used to derive bounds for the full system's signature, using the same algorithms as presented by Gaofeng et al. [48] and discussed in Section 1.2. The reason for this is the assumption that the system is coherent, which implies that a decrease (increase) in reliability of a component can never lead to increased (decreased) reliability of the system, therefore a decrease (increase) in reliability of a subsystem can never lead to increased (decreased) reliability of the

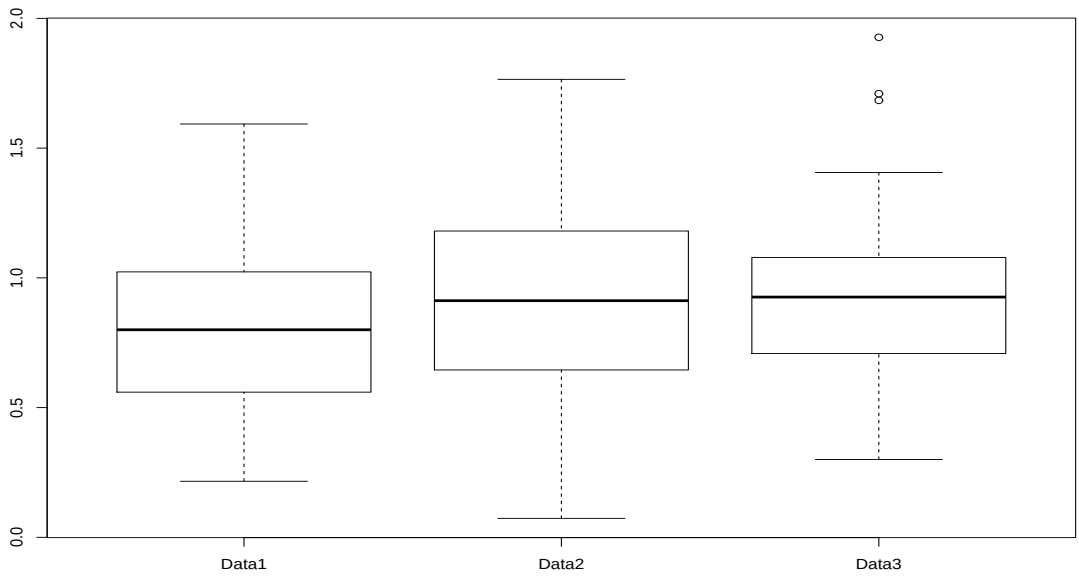

Figure 3.3: different data sets (Ex. 3.2)

system. The pessimistic signatures for the two subsystems can be combined to give the pessimistic signature for the full system, and combining the optimistic signatures for the two subsystems leads to the optimistic signature for the full system. Example 3.3 illustrates this approach.

Example 3.3

Figure 3.5 shows a coherent system consisting of 17 exchangeable components, which consists of two subsystems in parallel configuration. Subsystem A is the same system, consisting of 7 components (number 1-7), as considered in Example 3.1. Subsystem B consists of 10 components (number 8-17). While the exact signature for this full system can be obtained by using the given signature for Subsystem A together with repeated use of the algorithm presented by Gaofeng et al. [48] for Subsystem B and for the combination of the two subsystems, we assume, in order to illustrate the use of the bounds on signatures presented in this chapter, that the signatures of subsystems A and B have only been derived partially, with the bounds

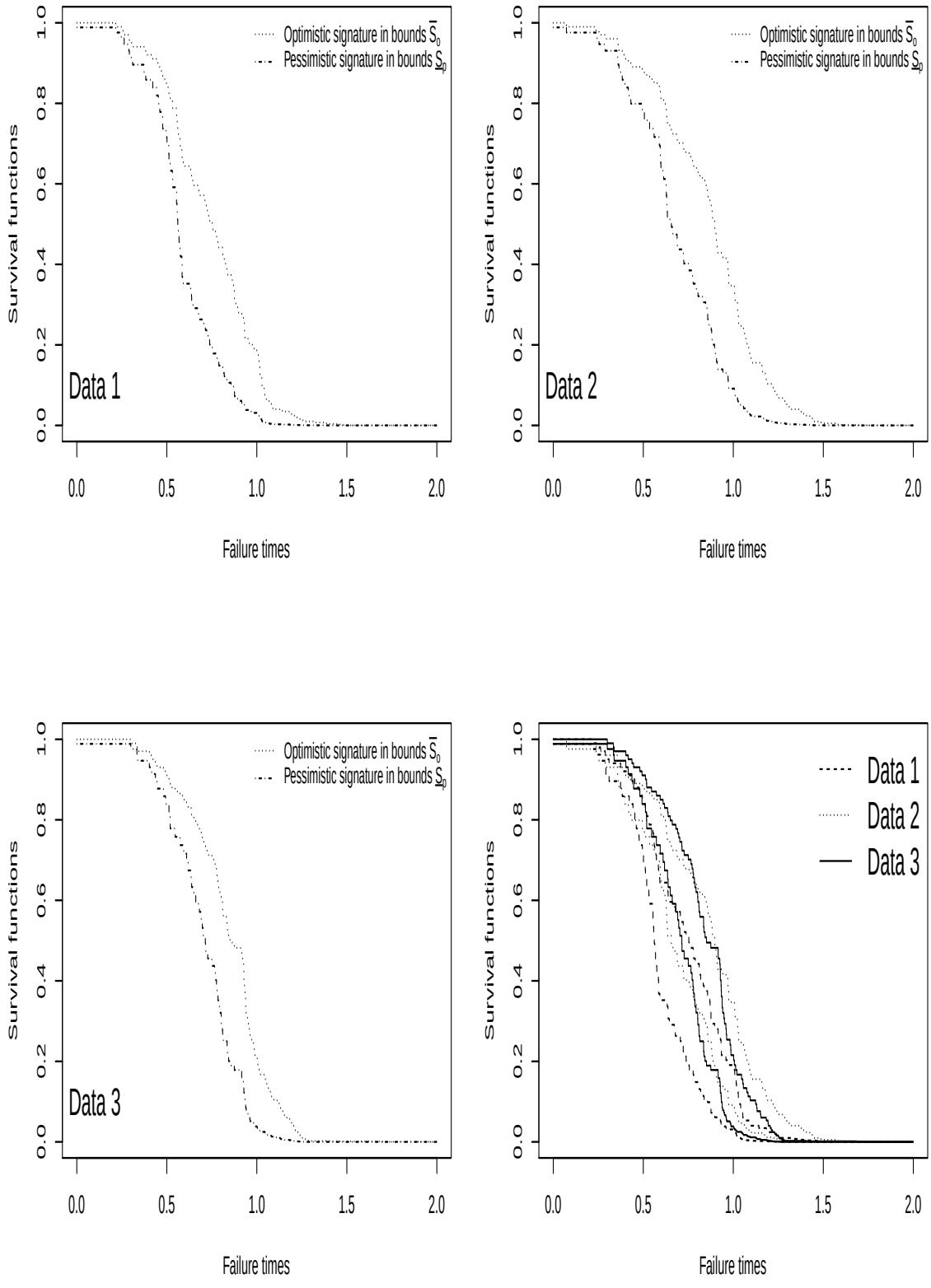

Figure 3.4: NPI lower and upper survival functions (Ex. 3.2)

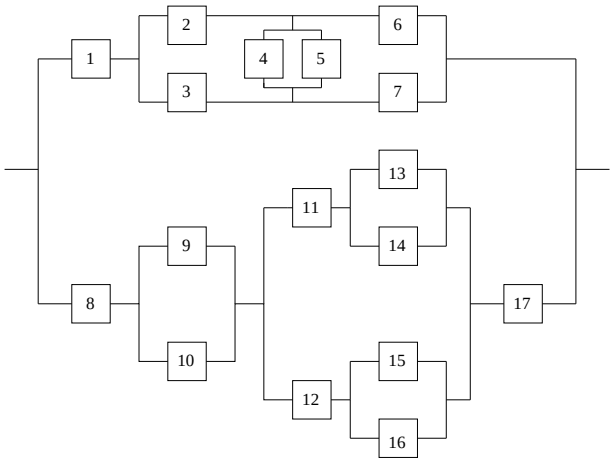

Figure 3.5: Two subsystems in parallel (Ex. 3.3)

and corresponding pessimistic and optimistic signatures as presented in Table 3.2.

The pessimistic signature for the full 17-component system is derived by application of the algorithm of Gaofeng et al. [48], as presented in Section 1.2, with the use of the pessimistic signatures $q^{a,p}$ and $q^{b,p}$, which leads to

$$\begin{aligned} q^p &= (0, 0.015, 0.050, 0.099, 0.161, 0.158, 0.136, 0.109, \\ &\quad 0.084, 0.064, 0.048, 0.035, 0.023, 0.013, 0.005, 0, 0) \end{aligned}$$

Applying the same algorithm with the optimistic signatures $q^{a,o}$ and $q^{b,o}$ leads to

$$\begin{aligned} q^o &= (0, 0.015, 0.031, 0.040, 0.046, 0.051, 0.061, 0.078, \\ &\quad 0.106, 0.128, 0.164, 0.128, 0.084, 0.047, 0.021, 0, 0) \end{aligned}$$

In Figure 3.6, the left plot presents the lower bound for the NPI lower survival function and the upper bound for the NPI upper survival function, both for the failure time of the full system and based on $n = 10$ failure times of tested components which are exchangeable with those in the system (simulated from the Weibull distribution with shape parameter 2 and scale parameter 1).

The right plot in Figure 3.6 is included for comparison with the following situation: Suppose that one would apply the NPI method presented in this chapter directly to each subsystem individually, using the bounds given in Table 3.2, but neglecting the fact that all components in both subsystems are exchangeable. Making

A	$\underline{q}^a$	(0.143, 0.143, 0.152, 0.157, 0.100, 0, 0)
	$\overline{q}^a$	(0.143, 0.448, 0.457, 0.462, 0.405, 0, 0)
	$q^{a,p}$	(0.143, 0.448, 0.152, 0.157, 0.100, 0, 0)
	$q^{a,o}$	(0.143, 0.143, 0.152, 0.157, 0.405, 0, 0)
B	$\underline{q}^b$	(0.200, 0.222, 0.072, 0.100, 0.046, 0.013, 0, 0, 0, 0)
	$\overline{q}^b$	(0.200, 0.222, 0.419, 0.447, 0.393, 0.360, 0, 0, 0, 0)
	$q^{b,p}$	(0.200, 0.222, 0.419, 0.100, 0.046, 0.013, 0, 0, 0, 0)
	$q^{b,o}$	(0.200, 0.222, 0.072, 0.100, 0.046, 0.360, 0, 0, 0, 0)

Table 3.2: Bounds, pessimistic and optimistic signatures for subsystems A and B (Ex. 3.3)

this mistake, one could continue by calculating bounds for the full system's survival function following the standard way for simple parallel systems (effectively using '$1 - (1 - S_a)(1 - S_b)$', with self-explanatory notation). The resulting lower and upper survival functions are greater than (or equal to) the correctly derived bounds for the NPI lower and upper survival function, because for the correct method the dependence of the components in both systems is taken into account. An intuitive explanation is as follows: The parallel system will only fail if both subsystems fail, and if one subsystem is known to fail this contains some information that suggests that the components are not very reliable, which as a consequence increases the (lower and upper) probability that the second subsystem also fails (when compared to the situation with the wrongly assumed independence between the two subsystems). This example shows the importance of taking the dependence of the exchangeable components, due to the limited information about their reliability from the test results, carefully into account, as is done by the NPI approach with the use of (bounds of) signatures.

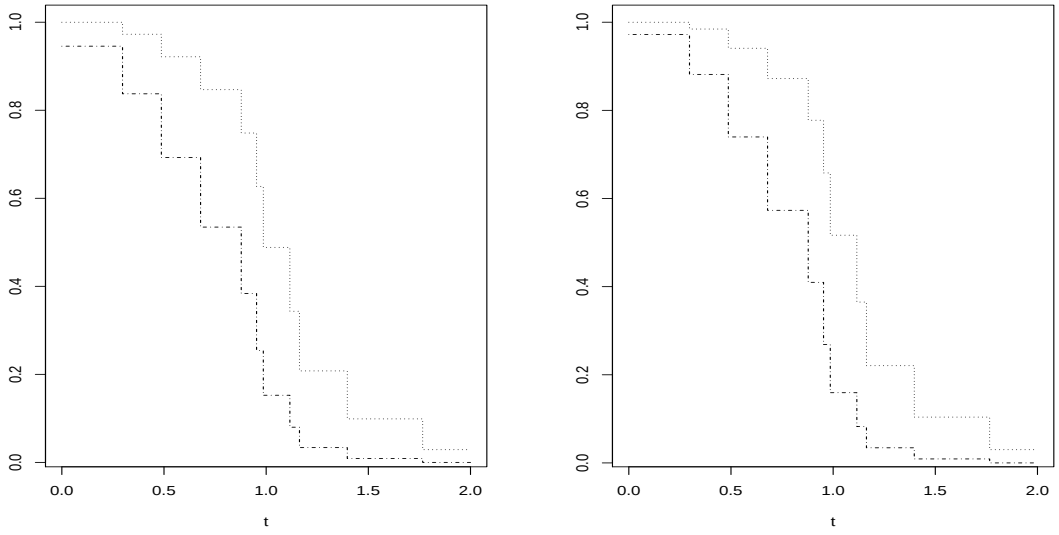

Figure 3.6: Bounds on NPI lower and upper survival functions (Left); similar but resulting from wrongly assumed independence of subsystems (Right) (Ex. 3.3)

3.4 Comparing failure times of two systems

In addition to the survival time of a system consisting of exchangeable components, other inferences can be considered. In Section 2.3.3 we considered the comparison of the failure times of two coherent systems, each consisting of exchangeable components. It is assumed that the failure times of the components in the different systems are fully independent, so any information about components' failure times of one system does not affect (lower and upper) probabilities involving only failure times of components of the other system. Due to the monotonicity of this comparison with regard to the systems' signatures, such a comparison with exactly known signatures, following the results from Equations 2.13 and 2.16) in Section 2.3.3, can be generalized to partially known signatures. If the exact signatures are not available but instead bounds $\underline{q}^a$ and $\overline{q}^a$ for q^a and $\underline{q}^b$ and $\overline{q}^b$ for q^b have been derived, which are assumed to satisfy conditions (3.1), then the optimal lower bound for the NPI lower probability for the event $T^a \leq T^b + \delta$ is derived using the optimistic signature

$q^{a,o}$ for System A and the pessimistic signature $q^{b,p}$ for System B, leading to

$$\underline{P}^{l}(T^a \leq T^b + \delta) = \sum_{i=1}^{m_a} \sum_{j=1}^{m_b} q_i^{a,o} q_j^{b,p} \underline{P}(T^a_{i:m_a} \leq T^b_{j:m_b} + \delta) \tag{3.10}$$

The optimal upper bound for the NPI upper probability for $T^a \leq T^b + \delta$ is derived using the pessimistic signature $q^{a,p}$ for System A and the optimistic signature $q^{b,o}$ for System B, leading to

$$\overline{P}^{u}(T^a \leq T^b + \delta) = \sum_{i=1}^{m_a} \sum_{j=1}^{m_b} q_i^{a,p} q_j^{b,o} \underline{P}(T^a_{i:m_a} \leq T^b_{j:m_b} + \delta) \tag{3.11}$$

These bounds follow from the monotonicity of these NPI lower and upper probabilities with regard to the signatures. The lower bound for the NPI lower probability for this event corresponds to maximum optimism about the lifetime of System A and maximum pessimism about the lifetime of System B, which is fully in line with intuition, and of course the other way around for the upper bound for the NPI upper probability. The upper bound for the NPI lower probability and the lower bound for the NPI upper probability are of course derived by taking the alternative optimistic or pessimistic signatures, but these are less likely to be of interest.

Example 3.4

Consider the systems of Figures 3.7 and 3.1, called System A and System B, respectively. Assume that each system consists of exchangeable components but these are different for the two systems, and we assume independence of the failure times of components in the different systems. Assume that bounds $\underline{q}^a$ and $\overline{q}^a$ are available for the signature of System A and bounds $\underline{q}^b$ and $\overline{q}^b$ for the signature of System B, as given in Table 3.3, which also presents the pessimistic and optimistic signatures corresponding to these bounds. Assume further that $n_a = n_b = 30$ components exchangeable with those of each type in the respective systems have been tested, leading to the failure times in Table 3.4. The optimal lower bound for the NPI lower probability and the optimal upper bound for the NPI upper probability for the event $T^a_S \leq T^b_S + \delta$ are presented in Figure 3.8 as functions of δ. This figure also gives the NPI lower and upper probabilities for this event corresponding to the exact signatures, which for System B was given in Example 3.1 and for System A

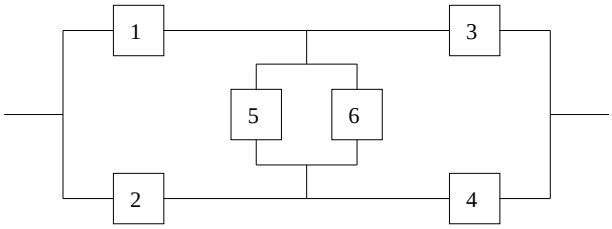

Figure 3.7: System A

System A	$\underline{q}^a$	(0, 0.133, 0.267, 0.044, 0, 0)
	$\overline{q}^a$	(0, 0.133, 0.267, 0.600, 0.556, 0)
	$q^{a,p}$	(0, 0.133, 0.267, 0.600, 0, 0)
	$q^{a,o}$	(0, 0.133, 0.267, 0.044, 0.556, 0)
System B	$\underline{q}^b$	(0.143, 0.143, 0.152, 0.157, 0.100, 0, 0)
	$\overline{q}^b$	(0.143, 0.448, 0.457, 0.452, 0.405, 0, 0)
	$q^{b,p}$	(0.143, 0.448, 0.152, 0.157, 0.100, 0, 0)
	$q^{b,o}$	(0.143, 0.143, 0.152, 0.157, 0.405, 0, 0)

Table 3.3: Bounds, pessimistic and optimistic signatures (Ex. 3.4)

is equal to $q^a = (1/720) \times (0, 96, 192, 336, 96, 0) = (0, 0.133, 0.267, 0.467, 0.133, 0)$. Figure 3.8 gives a good impression of the actual difference between the failure times of these two systems, where it should be remarked that the bounds based on the partial information are still relatively wide compared to the NPI lower and upper probabilities based on the exact signatures, as the vertical distances between the functions at specific values of δ must be considered.

System A			System B		
0.223	0.747	0.994	0.154	0.585	1.076
0.265	0.798	1.008	0.155	0.598	1.169
0.372	0.807	1.073	0.347	0.642	1.239
0.419	0.824	1.115	0.402	0.692	1.248
0.564	0.850	1.167	0.483	0.738	1.327
0.630	0.887	1.182	0.512	0.822	1.421
0.675	0.914	1.275	0.513	0.843	1.569
0.685	0.921	1.397	0.548	0.848	1.643
0.709	0.981	1.400	0.563	0.863	1.735
0.727	0.987	1.425	0.574	0.938	2.565

Table 3.4: Component failure times (Ex. 3.4)

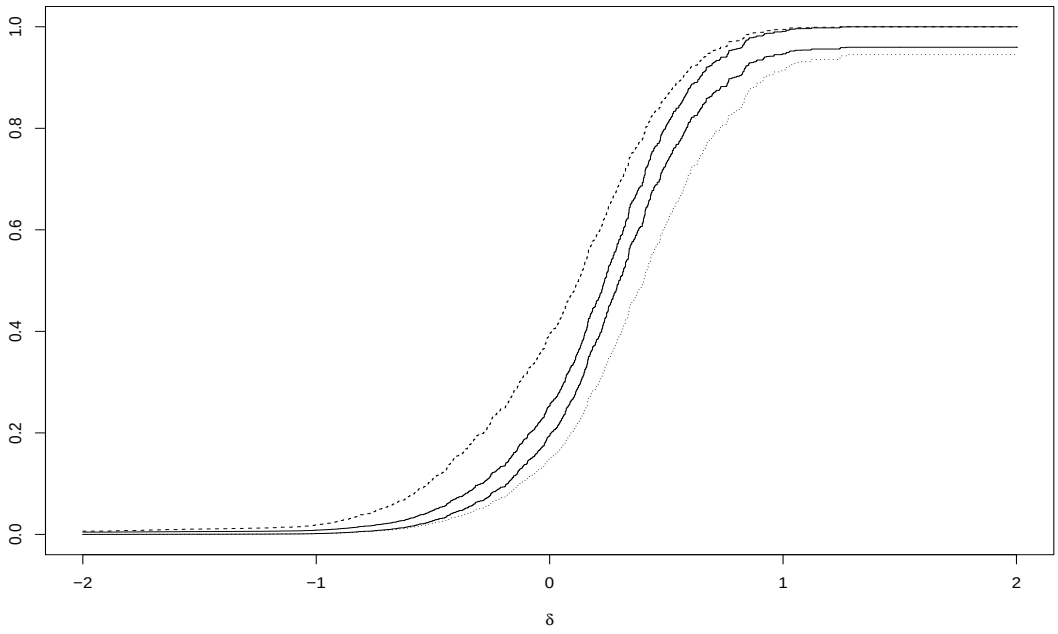

Figure 3.8: (Bounds on) NPI lower and upper probabilities for $T_S^A < T_S^B + \delta$ (Ex. 3.4)

3.5 Concluding remarks

While the concept of system signature and its use for reliability quantification has received increasing attention in the literature in recent years, the computation of the signature has received relatively little attention and is complex for most systems. In this chapter, it is illustrated how one can base reliability inferences on a partially known signature, assuming that bounds for the probabilities in the signature are available. Such bounds may typically result from computations that are based on counting all the orderings, where any further computations lead to sharpening of the bounds. The method introduced by Da et al. [48] to derive a system signature from the signatures of subsystems, if these are in either series or parallel configuration, which was presented in Section 1.2, can also be used with partially known signatures, as illustrated in this chapter. An interesting topic for further research is whether such results can also be derived for subsystems that are in different configurations.

The bounds on signatures considered in this chapter could be interpreted as imprecise probabilities [30, 31]. For the inferences considered in this chapter, the bounds corresponded to logical and well-identifiable signatures within the bounds, called the optimistic and pessimistic signatures. Of course, one may be interested in other inferential problems for which this nice monotonicity with regard to the signature does not hold, for example if one would be interested in a system failing in its second year of operation then the bounds would be less easy to derive. One could still apply the ideas presented in this chapter, but deriving the bounds for the inferences that correspond to the bounds for the signature would be formulated as constrained optimisation problems that may require numerical solution methods.

As indicated in the examples, the lower bound for the NPI lower survival function and the upper bound for the NPI upper survival function are most likely to be of main interest. However, the two other bounds presented can also be useful, particularly as the lower and upper bounds for the NPI lower survival function provide a clear indication of the accuracy with which, at any specific stage of computation, the real NPI lower survival function can be approximated (and similar of course for the NPI upper survival function). This may also be useful to provide an indication of the value of additional calculations to derive the signature.

An interesting further question is whether it is possible to learn about the system signature from failure observations. Aslett [9] has made interesting contributions to Bayesian learning of the system signature when only data for the whole system are available. This is important for 'black-box' systems, where it is not possible to construct the signature on the basis of available information. In such cases, system failure data can enable learning about some aspects of the system signature and hence of the actual structure of the system.

Chapter 4

Failure time of a system with multiple types of components

4.1 Introduction

In Chapters 2 and 3 we have presented the use of system signatures to derive non-parametric predictive inference (NPI) based lower and upper survival functions for the failure time of a system, with attention restricted to systems consisting of a single type of components. In this chapter, these results are generalized by presenting NPI for the failure time of a coherent system which can consist of different types of components. It is assumed that, for each type of component, additional components which are exchangeable with those in the system have been tested and their failure times are available. As in Chapters 2 and 3, we present NPI-based lower and upper survival functions. In those chapters signatures were used for quantification of reliability of coherent systems consisting of components with exchangeable failure times, which can be regarded informally as components of 'a single type'. However, the restriction to systems with a single type of components prevents its application to most practical systems.

Coolen and Coolen-Maturi [26] recently introduced an alternative concept, called the survival signature, which is closely related to the signature and has similar characteristics. However, the survival signature can be used for systems consisting of multiple types of components. In Section 4.2 we briefly review the main idea of

the concept of survival signature. Section 4.3 presents the use of survival signatures to derive NPI lower and upper survival functions for a system with a single type of component, in order to relate this method to the earlier chapters. Section 4.4 presents the use of survival signatures to derive NPI lower and upper survival functions for a system with multiple types of components. In Section 4.5 we present formulas for computing the survival signature of systems by using the survival signatures of subsystems, which appear either in series or in parallel configuration in the system. Section 4.6 presents how limited information about the survival signature can be used to derive bounds on such lower and upper survival functions. Section 4.7 contains some concluding remarks.

4.2 The survival signature

The signature was introduced to assist reliability analyses for systems consisting of one type of components, as discussed in Section 1.2. and used in Chapters 2 and 3. The signature is used to model the structure of a system, separating this from random failure times of the components. Coolen and Coolen-Maturi [26] introduced the following alternative to the signature which can achieve a similar task, and which is related to the signature. However, their concept, called the 'survival signature' is easily generalized to systems with multiple types of components.

As mentioned in Chapter 1, for a system with m components, let state vector $\underline{x} = (x_1, x_2, \ldots, x_m) \in \{0,1\}^m$, with $x_i = 1$ if the ith component functions and $x_i = 0$ if not. The labelling of the components is arbitrary but must be fixed to define $\underline{x}$. The structure function $\phi : \{0,1\}^m \rightarrow \{0,1\}$, defined for all possible $\underline{x}$, takes the value 1 if the system functions and 0 if the system does not function for state vector $\underline{x}$. Throughout this thesis it is assumed that the system is coherent [12]. We further assume that $\phi(\underline{0}) = 0$ and $\phi(\underline{1}) = 1$, so the system fails if all its components fail and it functions if all its components function. For a system consisting only of components with exchangeable, the survival signature, denoted by $\Phi(l)$, for $l = 1, \ldots, m$, is defined as the probability that the system functions given that *precisely* l of its components function [26]. For coherent systems, $\Phi(l)$ is an increasing function of

l, and we assume that $\Phi(0) = 0$ and $\Phi(m) = 1$. There are $\binom{m}{l}$ state vectors $\underline{x}$ with precisely l components $x_i = 1$, so with $\sum_{i=1}^{m} x_i = l$; we denote the set of these state vectors by S_l. These state vectors are equally likely to occur due to the exchangeability assumption for the components' failure times, hence

$$\Phi(l) = \binom{m}{l}^{-1} \sum_{\underline{x} \in S_l} \phi(\underline{x}) \tag{4.1}$$

Coolen and Coolen-Maturi [26] called $\Phi(l)$ the survival signature because, by its definition, it is closely related to survival of the system. Let $C(t) \in \{0, 1, \ldots, m\}$ denote the number of components in the system that function at time $t > 0$. The probability for the event that the system functions at time $t > 0$ can be derived by

$$P(T_S > t) = \sum_{l=0}^{m} \Phi(l) P(C(t) = l) \tag{4.2}$$

It is clear from Equation 4.2 that the system structure is taken into account through the survival signature $\Phi(l)$, which models how the system's functioning depends on the functioning of its components, while the term $P(C(t) = l)$ takes the random failure times of the components into account. Separating these two essential parts in order to determine the survival function for the system failure time is similar to the use of system signatures as discussed in Section 1.2, and used in the previous chapters. The survival signature and the system signature are closely related, it is easily seen that the following equality holds [26]

$$\Phi(l) = \sum_{j=m-l+1}^{m} q_j \tag{4.3}$$

Equation 4.3 is logical as the right-hand side gives the probability that the system failure occurs at the moment of the $(m-l+1)$-th ordered component failure time or later. This is exactly the moment at which the number of functioning components in the system decreases from l to $l-1$, hence the system would have functioned with l components functioning.

Generalizing the signature to multiple types of components is very complicated while keeping it separate from the component lifetime distributions, as this would always require computation of probabilities for orderings of order statistics from

different probability distributions, corresponding to the different types of components, as discussed by Coolen and Coolen-Maturi [26]. However, they show that the survival signature can easily be generalized for systems with multiple types of components. Consider a system with $K \geq 2$ types of components, with m_k components of type $k \in \{1, 2, \ldots, K\}$ and $\sum_{k=1}^{K} m_k = m$. Assume that the random failure times of components of the same type are exchangeable, while full independence is assumed for the random failure times of components of different types. Due to the arbitrary ordering of the components in the state vector, components of the same type can be grouped together, leading to a state vector that can be written as $\underline{x} = (\ underline{x}^1, \underline{x}^2, \ldots, \underline{x}^K)$ with $\underline{x}^k = (x_1^k, x_2^k, \ldots, x_{m_k}^k)$ the sub-vector representing the states of the components of type k. Let the ordered random failure times of the m_k components of type k be denoted by $T_{j_k:m_k}^k$. The survival signature for such a system is denoted by $\Phi(l_1, l_2, \ldots, l_K)$, for $l_k = 0, 1, \ldots, m_k$, and is defined to be the probability that the system functions given that *precisely* l_k of its components of type k function, for each $k \in \{1, 2, \ldots, K\}$ [26].

There are $\binom{m_k}{l_k}$ state vectors $\underline{x}^k$ with precisely l_k of its m_k components of type k functioning, so with $\sum_{i=1}^{m_k} x_i^k = l_k$; we denote the set of these state vectors for components of type k by S_l^k. Let $S_{l_1,\ldots,l_K}$ denote the set of all state vectors for the whole system for which $\sum_{i=1}^{m_k} x_i^k = l_k$, for $k = 1, 2, \ldots, K$. Due to the assumption that the failure times of the m_k components of type k are *iid*, so also exchangeable, all the state vectors $\underline{x}^k \in S_l^k$ are equally likely to occur, hence

$$\Phi(l_1, \ldots, l_K) = \left[\prod_{k=1}^{K} \binom{m_k}{l_k}^{-1}\right] \times \sum_{\underline{x} \in S_{l_1,\ldots,l_K}} \phi(\underline{x}) \tag{4.4}$$

Let $C_k(t) \in \{0, 1, \ldots, m_k\}$ denote the number of components of type k in the system that function at time $t > 0$. The probability that the system functions at time $t > 0$ is

$$P(T_S > t) = \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \left[\Phi(l_1, \ldots, l_K) \prod_{k=1}^{K} P(C_k(t) = l_k)\right] \tag{4.5}$$

Calculation of (4.5) is quite straightforward if $\Phi(l_1, \ldots, l_K)$ is known. The survival signature $\Phi(l_1, \ldots, l_K)$ must be derived for all $\prod_{k=1}^{K}(m_k + 1)$ different $(l_1, \ldots, l_K)$, but this information must be extracted from the system anyhow and is only required

to be calculated once for any system, similar to the (survival) signature for systems with a single type of component. The advantage of Equation 4.5 is that the information about the system structure is again separated from the information about the components' failure times, and the inclusion of the failure time distributions is straightforward due to the assumed independence of failure times of components of different types.

4.3 Single type of component

This section presents the NPI lower and upper survival functions for the failure time T_S of a system consisting of a single type of component, derived by generalizing Equation 4.2 to lower and upper probabilities. We also relate this method to the NPI method presented in Section 2.2 for the system survival function using the signature, for systems with a single type of components.

The NPI lower and upper survival functions for the failure time T_S of a coherent system consisting of m exchangeable components, with the system structure represented by survival signature $\Phi(l)$, can be derived by the following generalizations of Equation 4.2. We now present NPI lower and upper survival functions for the failure time T_S of a system consisting of a single type of component, using the system signature combined with NPI for Bernoulli data [19]. This enables the NPI method to be applied to, in principle, all systems, so this methodology widely generalizes the earlier results on NPI for system reliability as presented in chapters 2 and 3. NPI is used for learning about the components in the system, from data consisting of failure times for components that are exchangeable with those in the system. We assume therefore that such data are available, for example resulting from testing or previous use of components. Let n denote the number of components for which test failure data are available, and let $s(t)$ denote the number of components which still function at time t.

The NPI lower survival function is derived as follows. Remember that $C(t)$ denotes the number of components in the system which function at time t, where it is assumed that failure ends the functioning of a component and it is not repaired or

replaced. Under the assumptions for the NPI approach [19], we derive the following lower bound for the survival function

$$P(T_S > t) \geq \sum_{l=0}^{m} \Phi(l) \overline{D}(C(t) = l)$$

where

$$\begin{aligned} \overline{D}(C(t) = l) &= \overline{P}(C(t) \leq l) - \overline{P}(C(t) \leq l-1) \\ &= \binom{n+m}{n}^{-1} \binom{s(t)-1+l}{s(t)-1} \binom{n-s(t)+m-l}{n-s(t)} \end{aligned}$$

In this expression, $\overline{P}$ denotes the NPI upper probability for Bernoulli data [19]. The function $\overline{D}$ ensures that maximum possible probability, corresponding to NPI for Bernoulli data [19], is assigned to the event $C(t) = 0$, so $\overline{D}(C(t) = 0) = \overline{P}(C(t) = 0)$. Then, $\overline{D}(C(t) = 1)$ is defined by putting the maximum possible remaining probability mass, from the total probability mass available for the event $C(t) \leq 1$, to the event $C(t) = 1$. This is achieved by $\overline{D}(C(t) = 1) = \overline{P}(C(t) \leq 1) - \overline{P}(C(t) = 0)$. This argument is continued, by assigning for increasing l the maximum possible remaining probability mass $\overline{D}(C(t) = l)$. As the survival signature is increasing in l for coherent systems, as assumed in this chapter, and the resulting $\overline{D}$ is a precise probability distribution, the right-hand side of the inequality above is indeed a lower bound and it is the maximum possible lower bound. As such, it is the NPI lower probability for the event $T_S > t$, giving the NPI lower survival function for the system failure time (for $t > 0$)

$$\underline{S}_{T_S}(t) = \underline{P}(T_S > t) = \sum_{l=0}^{m} \Phi(l) \overline{D}(C(t) = l) \tag{4.6}$$

The corresponding NPI upper survival function for T_S is similarly derived, using the upper bound

$$P(T_S > t) \leq \sum_{l=0}^{m} \Phi(l) \underline{D}(C(t) = l)$$

where

$$\begin{aligned} \underline{D}(C(t) = l) &= \underline{P}(C(t) \leq l) - \underline{P}(C(t) \leq l-1) \\ &= \binom{n+m}{n}^{-1} \binom{s(t)+l}{s(t)} \binom{n-s(t)+m-l-1}{n-s(t)} \end{aligned}$$

In this expression, $\underline{P}$ denotes the NPI lower probability for Bernoulli data [19]. This construction ensures that minimum possible weight is given to small values of $C(t)$, resulting in the NPI upper survival function for the system failure time (for $t > 0$)

$$\overline{S}_{T_S}(t) = \overline{P}(T_S > t) = \sum_{l=0}^{m} \Phi(l)\underline{D}(C(t) = l) \tag{4.7}$$

For systems with a single type of components, NPI theory for the system survival time using the signature was presented in Chapter 2. This used NPI for future order statistics of real-valued observations [27]. It is not trivial that this leads to the same inferences as the method using the survival signature and NPI for Bernoulli quantities [19] as presented in this section. However, the resulting inferences for such systems, from these two different NPI approaches, are identical. The proof that these two approaches lead to the same NPI lower survival function is as follows,

$$\begin{aligned}
\underline{P}(T_S > t) &= \sum_{l=0}^{m} \Phi(l)\overline{D}(C(t) = l) \\
&= \sum_{l=0}^{m}\Big(\sum_{j=m-l+1}^{m} q_j\Big)\overline{D}(C(t) = l) \\
&= \sum_{j=1}^{m+1} q_j\Big[\sum_{l=m-j+1}^{m} \overline{D}(C(t) = l)\Big]
\end{aligned}$$

where

$$\begin{aligned}
\sum_{l=m-j+1}^{m} \overline{D}(C(t) = l) &= \sum_{l=m-j+1}^{m} \big[\overline{P}(C(t) \leq l) - \overline{P}(C(t) \leq l-1)\big] \\
&= \overline{P}(C(t) \leq m) - \overline{P}(C(t) \leq m-j) \\
&= 1 - \overline{P}(C(t) \leq m-j) \\
&= \underline{P}(C(t) > m-j) \\
&= \underline{P}(C(t) \geq m-j+1) \\
&= \underline{P}(T_{j:m} > t)
\end{aligned}$$

so

$$\underline{P}(T_S > t) = \sum_{j=1}^{m+1} q_j \underline{P}(T_{j:m} > t)$$

which is the NPI lower survival function as specified in Section 2.2. The proof for the corresponding NPI upper survival function follows the same steps.

Example 4.1

The six systems with $m = 4$ exchangeable components in Figure 4.1 were used to illustrate the use of the signature to derive NPI lower and upper survival functions in Section 2.2. We now use these systems to illustrate the survival signature method presented in this section. Figure 4.1 also gives the survival signature of each of these systems. Suppose that $n = 4$ components exchangeable with those in such a system were tested, leading to ordered failure times $t_1 < t_2 < t_3 < t_4$, which create the partition $I_1, \ldots, I_5$ of the positive real-line.

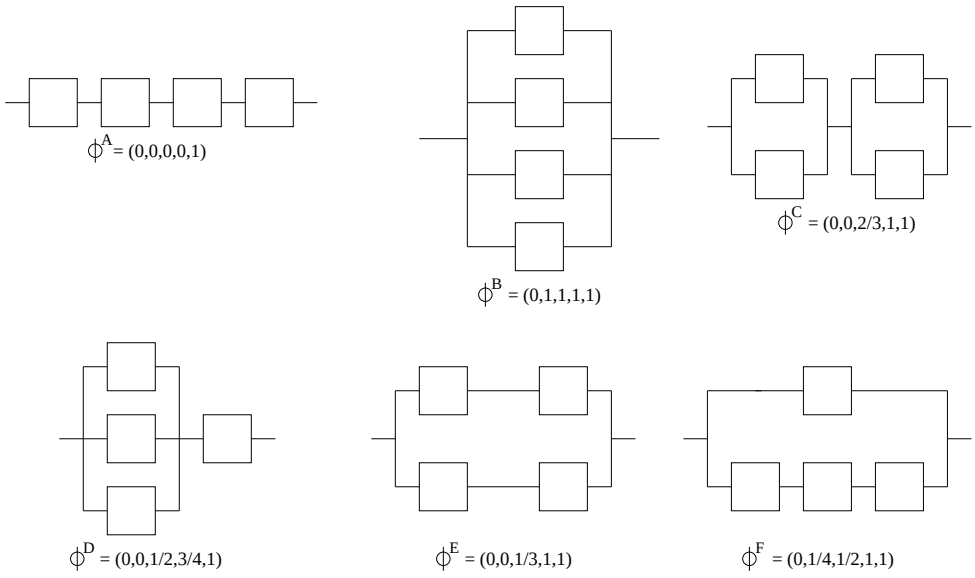

Figure 4.1: Coherent systems with 4 exchangeable components

Table 4.1 presents the NPI lower and upper survival functions $\underline{S}_{T_S}(t)$ and $\overline{S}_{T_S}(t)$ for the system failure time T_S, including the survival signature and the system signatures, as discussed in Section 2.2 for each of the systems presented in Figure 4.1. The main purpose of presenting this example is to show that we can achieve the same results either using signature or survival signature. These results in Table 4.1 illustrate that NPI lower and upper survival functions using the survival signature are identical to those using the signature, as presented in Section 2.2.

q	$(1,0,0,0)$		$(0,0,0,1)$		$(0,\frac{1}{3},\frac{2}{3},0)$	
Φ	$(0,0,0,0,1)$		$(0,1,1,1,1)$		$(0,0,\frac{2}{3},1,1)$	
$t \in$	$\underline{S}_{T_S}(t)$	$\overline{S}_{T_S}(t)$	$\underline{S}_{T_S}(t)$	$\overline{S}_{T_S}(t)$	$\underline{S}_{T_S}(t)$	$\overline{S}_{T_S}(t)$
$(0,t_1)$	0.50	1	0.99	1	0.88	1
(t_1,t_2)	0.21	0.50	0.93	0.99	0.67	0.88
(t_2,t_3)	0.07	0.21	0.79	0.93	0.41	0.67
(t_3,t_4)	0.01	0.07	0.50	0.79	0.17	0.41
(t_4,∞)	0	0.01	0	0.50	0	0.17
q	$(\frac{1}{4},\frac{1}{4},\frac{1}{2},0)$		$(0,\frac{2}{3},\frac{1}{3},0)$		$(0,\frac{1}{2},\frac{1}{4},\frac{1}{4})$	
Φ	$(0,0,\frac{1}{2},\frac{3}{4},1)$		$(0,0,\frac{1}{3},1,1)$		$(0,\frac{1}{4},\frac{1}{2},1,1)$	
$t \in$	$\underline{S}_{T_S}(t)$	$\overline{S}_{T_S}(t)$	$\underline{S}_{T_S}(t)$	$\overline{S}_{T_S}(t)$	$\underline{S}_{T_S}(t)$	$\overline{S}_{T_S}(t)$
$(0,t_1)$	0.79	1	0.83	1	0.87	1
(t_1,t_2)	0.56	0.79	0.59	0.83	0.67	0.87
(t_2,t_3)	0.33	0.56	0.33	0.59	0.44	0.67
(t_3,t_4)	0.13	0.33	0.12	0.33	0.21	0.44
(t_4,∞)	0	0.13	0	0.12	0	0.21

Table 4.1: $\underline{S}_{T_S}(t)$ and $\overline{S}_{T_S}(t)$ for systems in figures 2.1 and 4.1

In the following section, the NPI method for the system survival function using the survival signature will be extended to the general case with $K \geq 1$ types of components, the main ideas are the same as for this case with $K = 1$.

4.4 Multiple types of components

This section presents the NPI lower and upper survival functions for the failure time T_S of a system consisting of multiple types of components, using the system signature combined with NPI for Bernoulli data [19]. This enables the NPI method to be applied to, in principle, all systems, so this methodology widely generalizes the earlier results on NPI for system reliability. The failure times of components of different types are assumed to be independent. NPI is used for learning about

the components of a specific type in the system, from data consisting of failure times for components that are exchangeable with these. We assume therefore that such data are available, for example resulting from testing or previous use of such components. Assume that these are $K \geq 1$ different types of components in the system. For $k \in \{1, \ldots, K\}$, let n_k denote the number of components of type k for which test failure data are available, and let $s_k(t)$ denote the number of these components which still function at time t.

The NPI lower survival function is derived as follows. Remember that $C_k(t)$ denotes the number of components of type k in the system which function at time t, where it is assumed that failure ends the functioning of a component and it is not repaired or replaced. Under the assumptions for the NPI approach [19], we derive the following lower bound for the survival function

$$P(T_S > t) \geq \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \cdots, l_K) \prod_{k=1}^{K} \overline{D}(C_k(t) = l_k) \tag{4.8}$$

where

$$\begin{aligned} \overline{D}(C_k(t) = l_k) &= \overline{P}(C_k(t) \leq l_k) - \overline{P}(C_k(t) \leq l_k - 1) \\ &= \binom{n_k + m_k}{n_k}^{-1} \binom{s_k(t) - 1 + l_k}{s_k(t) - 1} \binom{n_k - s_k(t) + m_k - l_k}{n_k - s_k(t)} \end{aligned}$$

In this expression, $\overline{P}$ denotes the NPI upper probability for Bernoulli data [19]. For each component type k, the function $\overline{D}$ ensures that maximum possible probability, corresponding to NPI for Bernoulli data [19], is assigned to the event $C_k(t) = 0$, so $\overline{D}(C_k(t) = 0) = \overline{P}(C_k(t) = 0)$. Then, $\overline{D}(C_k(t) = 1)$ is defined by putting the maximum possible remaining probability mass, from the total probability mass available for the event $C_k(t) \leq 1$, to the event $C_k(t) = 1$. This is achieved by $\overline{D}(C_k(t) = 1) = \overline{P}(C_k(t) \leq 1) - \overline{P}(C_k(t) = 0)$. This argument is continued, by assigning for increasing l_k the maximum possible remaining probability mass $\overline{D}(C_k(t) = l_k)$. As the survival signature is increasing in l_k for coherent systems, as assumed in this chapter, and the resulting $\overline{D}$ is a precise probability distribution, the right-hand side of inequality 4.8 is indeed a lower bound and it is the maximum possible lower bound. As such, it is the NPI lower probability for the event $T_S > t$, giving the NPI lower survival function for the system failure time (for $t > 0$)

$$\underline{S}_{T_S}(t) = \underline{P}(T_S > t) = \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \cdots, l_K) \prod_{k=1}^{K} \overline{D}(C_k(t) = l_k) \qquad (4.9)$$

The corresponding NPI upper survival function for T_S is similarly derived, using the upper bound

$$P(T_S > t) \leq \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \cdots, l_K) \prod_{k=1}^{K} \underline{D}(C_k(t) = l_k)$$

where

$$\begin{aligned} \underline{D}(C_k(t) = l_k) &= \underline{P}(C_k(t) \leq l_k) - \underline{P}(C_k(t) \leq l_k - 1) \\ &= \binom{n_k + m_k}{n_k}^{-1} \binom{s_k(t) + l_k}{s_k(t)} \binom{n_k - s_k(t) + m_k - l_k - 1}{n_k - s_k(t)} \end{aligned}$$

In this expression, $\underline{P}$ denotes the NPI lower probability for Bernoulli data [19]. This construction ensures that minimum possible weight is given to small values of $C_k(t)$, resulting in the NPI upper survival function for the system failure time (for $t > 0$)

$$\overline{S}_{T_S}(t) = \overline{P}(T_S > t) = \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \cdots, l_K) \prod_{k=1}^{K} \underline{D}(C_k(t) = l_k) \qquad (4.10)$$

Next, we illustrate these NPI lower and upper survival functions in three examples. In Example 4.2, we focus on a system with $K = 2$ types of components, with $m_1 = m_2 = 3$ components of each type and using two different orderings of the observed failure times. In Example 4.3, we consider a system with $K = 2$ types of components, with $m_1 = m_2 = 5$ components of each type. In Example 4.4, we consider a system with 3 types of components.

Example 4.2

We consider the use of the survival signature for the system with $K = 2$ types of components, Type 1 and 2 as presented in Figure 4.2. With $m_1 = m_2 = 3$ components of each Type, the survival signature $\Phi(l_1, l_2)$ must be specified for all $l_1, l_2 \in \{0, 1, 2, 3\}$; this is given in Table 4.2, it is easily verified by checking all possible combinations of the specific components of each type which function or not.

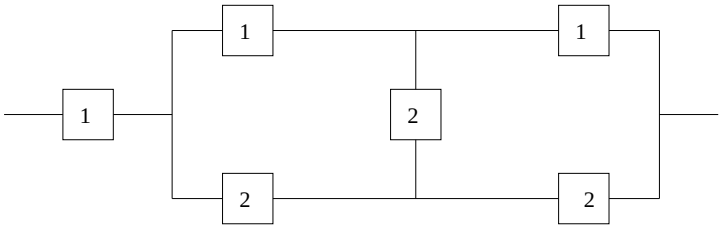

Figure 4.2: System with 2 types of components

To illustrate its derivation, let us consider $\Phi(1,2)$ and $\Phi(2,2)$ in detail. The state vector is $\underline{x} = (x_1^1, x_2^1, x_3^1, x_1^2, x_2^2, x_3^2)$, where we order the three components of Type 1 from left to right in Figure 4.2, and similar for the three components of Type 2. To calculate $\Phi(1,2)$, we consider all such vectors $\underline{x}$ with $x_1^1 + x_2^1 + x_3^1 = 1$ and $x_1^2 + x_2^2 + x_3^2 = 2$, so precisely 1 component of Type 1 and 2 components of Type 2 function. There are 9 such vectors, for only one of these, namely $(1,0,0,1,0,1)$, the system functions so, $\Phi(1,2) = 1/9$. To calculate $\Phi(2,2)$ we need to check all 9 vectors $\underline{x}$ with $x_1^1+x_2^1+x_3^1 = 2$ and $x_1^2+x_2^2+x_3^2 = 2$. For 4 of these vectors the system functions, namely $(1,1,0,1,0,1)$, $(1,1,0,0,1,1)$, $(1,0,1,1,1,0)$ and $(1,0,1,1,0,1)$, so $\Phi(2,2) = 4/9$.

Suppose that $n_1 = 2$ components exchangeable with those of Type 1 and $n_2 = 2$ components exchangeable with those of Type 2 were tested. Suppose that ordered failure times as shown in Table 4.3 and Table 4.4 are observed. These tables present NPI lower and upper survival functions $\underline{S}_{T_S}(t)$ and $\overline{S}_{T_S}(t)$ for the system failure time T_S, for the system presented in Figure 4.2 and with the given orderings of the data. In the first interval in Table 4.3 we have not yet seen a single failure in the test data, so the NPI upper probability that the system will function is equal to one. In the second interval, one failure of Type 2 has occurred before, but we do not have any evidence from the data against the possibility that a component of Type 1 will certainly function at times in this interval, so the upper probability remains one. In the fourth interval, both Type 2 failures have occurred but only one of Type 1 component has failed before. In this interval, to consider the lower survival function the system is effectively reduced to a series system consisting of three components of Type 1, with one 'success' and one 'failure' as data, denoted by $(2,1)$. As such a

l_1 \ l_2	0	1	2	3
0	0	0	0	0
1	0	0	1/9	3/9
2	0	0	4/9	6/9
3	1	1	1	1

Table 4.2: Survival signature $\Phi(l_1, l_2)$ of the system in Figure 4.2

series system only functions if all three components function, the NPI lower survival function within this fourth interval is equal to $\underline{S}_{T_S}(t) = \frac{1}{3} \times \frac{2}{4} \times \frac{3}{5} = 0.100$, which follows by sequential reasoning, using that, based on n observations consisting of s successes and $n - s$ failures, denoted as data (n, s), the NPI lower probability for the next observation to be a success is equal to $s/(n + 1)$ [19]. The NPI lower probability for the first component to function, given test data $(2, 1)$, is equal to $1/3$. Then the second component is considered, conditional on the first component functioning, which combines with the test data to two out of three components observed (or assumed) to be functioning, so combined data $(3, 2)$, hence this second component will also function with NPI lower probability $2/4$. Similarly, the NPI lower probability for the third component to function, conditional on functioning of the first two components in the system, so with combined data $(4, 3)$, is equal to $3/5$. In the last interval, we are beyond the failure times of all the tested components, so we no longer have evidence in favour of the system to function, which is reflected by $\underline{S}_{T_S}(t) = 0$, but the system might of course still function, as represented by $\overline{S}_{T_S}(t) = 0.148$.

Table 4.4 also presents the NPI lower and upper survival functions $\underline{S}_{T_S}(t)$ and $\overline{S}_{T_S}(t)$ for the system failure time T_S, for the same system presented in Figure 4.2 but with the different data ordering $t_1^1 < t_1^2 < t_2^1 < t_2^2$. We have $\overline{S}_{T_S}(t) = 0.667$ in the second interval, where one failure of type 1 has occurred in the test data. In the fourth interval, both tested components of type 1 have failed, leading to $\underline{S}_{T_S}(t) = 0$. Both of these values are directly related to the required functioning of the left-most component in Figure 4.2.

$t_1^2 < t_1^1 < t_2^2 < t_2^1$		
$t \in$	$\underline{S}_{T_S}(t)$	$\overline{S}_{T_S}(t)$
$(0, t_1^2)$	0.553	1
(t_1^2, t_1^1)	0.458	1
(t_1^1, t_2^2)	0.148	0.553
(t_2^2, t_2^1)	0.100	0.458
(t_2^1, ∞)	0	0.148

Table 4.3: $\underline{S}_{T_S}(t)$ and $\overline{S}_{T_S}(t)$ for the system in Figure 4.2

$t_1^1 < t_1^2 < t_2^1 < t_2^2$		
$t \in$	$\underline{S}_{T_S}(t)$	$\overline{S}_{T_S}(t)$
$(0, t_1^1)$	0.553	1
(t_1^1, t_1^2)	0.230	0.667
(t_1^2, t_2^1)	0.148	0.553
(t_2^1, t_2^2)	0	0.230
(t_2^2, ∞)	0	0.148

Table 4.4: $\underline{S}_{T_S}(t)$ and $\overline{S}_{T_S}(t)$ for the system in Figure 4.2

Example 4.3

We consider the use of the survival signature for the system with $K = 2$ types of components in Figure 4.3. With $m_1 = m_2 = 5$ components of each type, the survival signature $\Phi(l_1, l_2)$ must be specified for all $l_1, l_2 \in \{0, 1, 2, 3, 4, 5\}$; this is given in Table 4.5. For example, to calculate $\Phi(3, 2)$ we need to check all 100 vectors $\underline{x}$ with $x_1^1 + x_2^1 + x_3^1 + x_4^1 + x_5^1 = 3$ and $x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 2$. For four of these vectors the system functions, namely $(1, 1, 1, 0, 0, 0, 0, 1, 0, 1)$, $(1, 1, 1, 0, 0, 0, 0, 0, 1, 1)$, $(1, 1, 0, 1, 0, 0, 1, 0, 0, 1)$ and $(1, 1, 0, 0, 1, 0, 1, 0, 0, 1)$, so $\Phi(3, 2) = 4/100$.

Suppose that $n_1 = 2$ components exchangeable with those of type 1 and $n_2 = 2$ components exchangeable with those of type 2 were tested, leading to failure times with ordering $t_1^2 < t_1^1 < t_2^2 < t_2^1$. Table 4.6 presents the NPI lower and upper survival

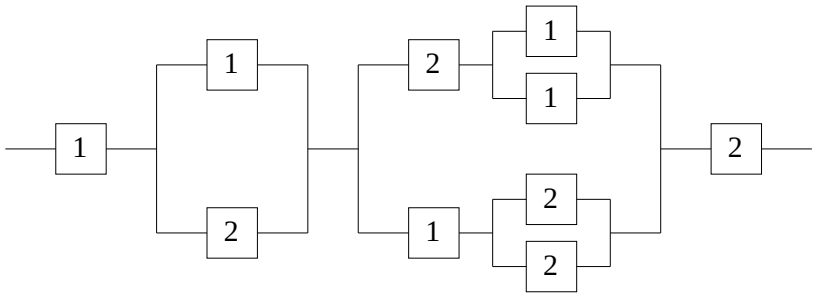

Figure 4.3: System with 2 types of components

l_1 \ l_2	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	0	0	0	4/100	14/100	30/100
3	0	0	4/100	18/100	36/100	60/100
4	0	0	14/100	18/100	56/100	80/100
5	0	0	30/100	60/100	80/100	1

Table 4.5: Survival signature $\Phi(l_1, l_2)$ of the system in Figure 4.3

$t_1^2 < t_1^1 < t_2^2 < t_2^1$		
$t \in$	$\underline{S}_{T_S}(t)$	$\overline{S}_{T_S}(t)$
$(0, t_1^2)$	0.355	1
(t_1^2, t_1^1)	0.134	0.633
(t_1^1, t_2^2)	0.046	0.355
(t_2^2, t_2^1)	0	0.134
(t_2^1, ∞)	0	0.046

Table 4.6: $\underline{S}_{T_S}(t)$ and $\overline{S}_{T_S}(t)$ for the system in Figure 4.3

functions $\underline{S}_{T_S}(t)$ and $\overline{S}_{T_S}(t)$ for the system failure time T_S for this system and based on this ordering of the test observations. In the second interval one failure of Type 2 has already occurred, which is reflected by $\overline{S}_{T_S}(t) = 0.633$. In the fourth interval, both failures of Type 2 have occurred, which is reflected by $\underline{S}_{T_S}(t) = 0$, due to the right-most component in Figure 4.3.

Example 4.4

We consider the use of the survival signature for the system with $K = 3$ types of components presented in Figure 4.4. With $m_1 = m_2 = m_3 = 2$ components of each type, the survival signature $\Phi(l_1, l_2, l_3)$ must be specified for all $l_1, l_2, l_3 \in \{0, 1, 2\}$; this is given in Table 4.7. Clearly, if no component of Type 1 functions, the system will not function no matter how many components of the other types function, due to the left-most component in Figure 4.4. To calculate $\Phi(1, 1, 2)$, for example, we need to check all 4 vectors $\underline{x}$ with $x_1^1 + x_2^1 = 1$, $x_1^2 + x_2^2 = 1$ and $x_1^3 + x_2^3 = 2$. For only one of these vectors the system functions, namely $(1, 0, 1, 0, 1, 1)$, so $\Phi(1, 1, 2) = 1/4$.

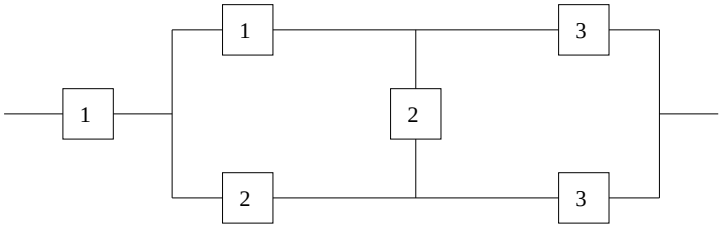

Figure 4.4: System with 3 types of components

Computation of the survival signature is complicated for systems of realistic size. In Section 4.5 we present results that can simplify computation in specific situations. It may not be needed to compute a system's survival signature exactly for a specific inference, as bounds resulting from partial computations may be sufficient, similar to the use of bounds for signatures as presented in Chapter 3. As the survival signature of a coherent system is non-decreasing in all its components, the use of such bounds is pretty straightforward; we present this in Section 4.6.

l_1	l_2	l_3	$\Phi(l_1,l_2,l_3)$	l_1	l_2	l_3	$\Phi(l_1,l_2,l_3)$	l_1	l_2	l_3	$\Phi(l_1,l_2,l_3)$
0	0	0	0	1	0	0	0	2	0	0	0
0	0	1	0	1	0	1	0	2	0	1	1/2
0	0	2	0	1	0	2	0	2	0	2	1
0	1	0	0	1	1	0	0	2	1	0	0
0	1	1	0	1	1	1	0	2	1	1	1
0	1	2	0	1	1	2	1/4	2	1	2	1
0	2	0	0	1	2	0	0	2	2	0	0
0	2	1	0	1	2	1	2/4	2	2	1	1
0	2	2	0	1	2	2	1/2	2	2	2	1

Table 4.7: Survival signature of system in Figure 4.4

4.5 Combining survival signatures of subsystems

Computation of the survival signature is complicated for systems of realistic size. As discussed in Section 1.2 and used in Section 3.3, Da et al. [48] showed how the system signature can be derived from the signatures of two subsystems, if the system consists of these two subsystems in either series or parallel configuration. Repeated application of their method enables quite straightforward computation of the signature of a system consisting of any number of subsystems, if the overall system's structure can be created through a sequence of series or parallel configurations.

In this section we present a similar method for the survival signature of a system consisting of two subsystems in either series or parallel configuration. By repeated use this enables the survival signatures for quite a substantial range of systems to be computed relatively easily. Suppose that a system consists of $R = 2$ subsystems for which the survival signatures are known. Let the system consist of $K \geq 1$ types of components, with m_k components of type k, for $k = 1, \ldots, K$, of which $m_k^r \geq 0$ are in subsystem r, for $r = 1, 2$. Let subsystem r consist in total of m^r components, so $m^r = \sum_{k=1}^{K} m_k^r$. We denote the survival signature for subsystem r by $\Phi^r(l_1^1, \ldots, l_K^r)$, for $l_k^r = 0, 1, \ldots, m_k^r$. For ease of notation, we define $\Phi^r(l_1^1, \ldots, l_K^r) = 0$ if $l_k^r > m_k^r$ for one or more $k \in \{1, \ldots, K\}$. Before presenting the general results for any number

R of subsystems and K of component types in Section 4.5.4, the case of a system consisting of $R = 2$ subsystems, with each subsystem consisting of the same single type of components is considered in detail in Section 4.5.1, followed in Section 4.5.2 by the case of a system with $R = 2$ subsystems and $K = 2$ types of components. Section 4.5.3, we consider the case of a system with $R = 2$ subsystems and more than two types of components.

4.5.1 2 subsystems with the same single type of components

Consider a system with m components of a single type, consisting of two subsystems in series configuration, where subsystem r $(r = 1, 2)$ consists of m^r components of the same type, so $m = m^1 + m^2$. The system survival signature, denoted by Φ_s to emphasize the series structure, can be derived by

$$\Phi_s(l) = \binom{m}{l}^{-1} \left[\sum_{l^1=0}^{l} \binom{m^1}{l^1} \binom{m^2}{l-l^1} \Phi^1(l^1)\Phi^2(l-l^1) \right] \tag{4.11}$$

The combinatorial terms in Equation 4.11 follow from the hypergeometric distribution giving the probability for the event that l^1 of the l functioning components are in subsystem 1 and the other $l - l^1$ functioning components are in subsystem 2. For a system consisting of two subsystems in parallel configuration the survival signature, denoted by Φ_p, can similarly be derived by

$$\Phi_p(l) = \binom{m}{l}^{-1} \left[\sum_{l^1=0}^{l} \binom{m^1}{l^1} \binom{m^2}{l-l^1} \left(1 - (1 - \Phi^1(l^1))(1 - \Phi^2(l-l^1))\right) \right] \tag{4.12}$$

4.5.2 2 subsystems with 2 component types

Consider again a system with $R = 2$ subsystems but now with $K = 2$ types of components. If the two subsystems are in series configuration, then the survival signature of the system can be derived, for $0 \leq l_k \leq m_k$, k=1,2, by

$$\begin{aligned} \Phi_s(l_1, l_2) \;=\; & \sum_{l_1^1=0}^{l_1} \sum_{l_2^1=0}^{l_2} \Bigg[\Phi^1(l_1^1, l_2^1)\Phi^2(l_1 - l_1^1, l_2 - l_2^1) \times \\ & \prod_{k=1}^{2} \binom{m_k^1}{l_k^1} \binom{m_k^2}{l_k - l_k^1} \binom{m_k}{l_k}^{-1} \Bigg] \end{aligned} \tag{4.13}$$

Similarly, if the two subsystems are in parallel configuration, then the survival signature of the system can be derived, for $0 \leq l_k \leq m_k$, k=1,2, by

$$\Phi_p(l_1, l_2) = \sum_{l_1^1=0}^{l_1} \sum_{l_2^1=0}^{l_2} \Big[\{1 - (1 - \Phi^1(l_1^1, l_2^1))(1 - \Phi^2(l_1 - l_1^1, l_2 - l_2^1))\} \times \prod_{k=1}^{2} \binom{m_k^1}{l_k^1} \binom{m_k^2}{l_k - l_k^1} \binom{m_k}{l_k}^{-1} \Big] \quad (4.14)$$

These results follow from straightforward combinatorial arguments similar to the justification of Equation 4.11, together with assumed independenet of components of different types.

Example 4.5

We calculate the survival signature for the two systems in Figure 4.5, which both have $K = 2$ types of components. The survival signature of each subsystem is equal and presented in Table 4.8. Equation 4.13 leads to the survival signature for the left system in Figure 4.5, with the subsystems in series structure, as presented in Table 4.9. Equation 4.14 leads to the survival signature for the right system in Figure 4.5, with the subsystems in parallel structure, as presented in Table 4.10.

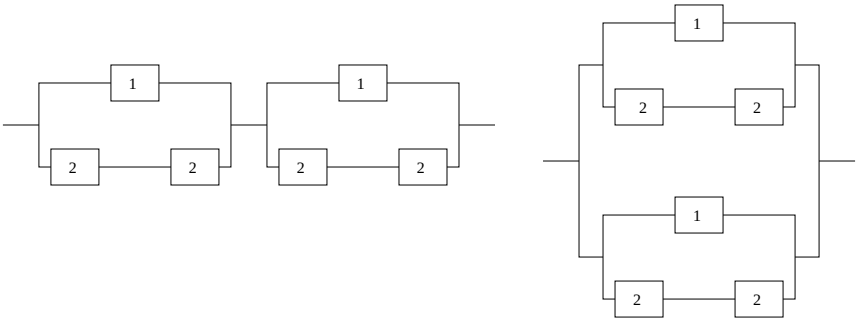

Figure 4.5: Two systems with 2 types of components

To illustrate its derivation, let us consider $\Phi_s(1, 2)$ and $\Phi_p(0, 2)$ in detail. To calculate $\Phi_s(1, 2)$ in Table 4.13, precisely 1 component of type 1 and 2 components of type 2 function. There are 12 such vectors, for only two of these, namely $(1, 0, 1, 1, 0, 0)$ and $(0, 1, 0, 01, 1)$, the system functions so, $\Phi_s(1, 2) = 2/12$. To cal-

l_1	l_2	$\Phi(l_1, l_2)$
0	0	0
0	1	0
0	2	1
1	0	1
1	1	1
1	2	1

Table 4.8: Survival signature of each subsystem in Figure 4.5

l_1 \ l_2	0	1	2	3	4
0	0	0	0	0	1
1	0	0	1/6	1/2	1
2	1	1	1	1	1

Table 4.9: Survival signature $\Phi_s(l_1, l_2)$ of left system in Figure 4.5

l_1 \ l_2	0	1	2	3	4
0	0	0	1/3	1	1
1	1	1	1	1	1
2	1	1	1	1	1

Table 4.10: Survival signature $\Phi_p(l_1, l_2)$ of right system in Figure 4.5

culate $\Phi_p(0,2)$ in Table 4.14, we need to check 6 vectors, for two of these vectors the system functions, namely $(0,0,1,1,0,0)$ and $(0,0,0,01,1)$, so $\Phi_p(0,2) = 2/6$.

4.5.3 2 subsystems with $K > 2$ component types

We now gerenalize the situation of the previous section to consider more than two types of components, while still restricting attention to two subsystems. So consider a system with $K > 2$ types of components and two subsystems. If the two subsystems are in series configuration, then the survival signature of the system can be derived, for $l_k \in \{0, 1, \ldots, m_k\}$, $k = 1, \ldots, K$, by

$$\Phi_s(l_1, \ldots, l_K) = \sum_{l_1^1=0}^{l_1} \cdots \sum_{l_K^1=0}^{l_K} \Bigg[\Phi^1(l_1^1, \ldots, l_K^1) \Phi^2(l_1 - l_1^1, \ldots, l_K - l_K^1) \times \prod_{k=1}^{K} \binom{m_k^1}{l_k^1} \binom{m_k^2}{l_k - l_k^1} \binom{m_k}{l_k}^{-1} \Bigg] \tag{4.15}$$

Similarly, if the two subsystems are in parallel configuration, then the survival signature of the system can be derived, for $l_k \in \{0, 1, \ldots, m_k\}$, $k = 1, \ldots, K$, by

$$\Phi_p(l_1, \ldots, l_K) = \sum_{l_1^1=0}^{l_1} \cdots \sum_{l_K^1=0}^{l_K} \Bigg[\{1 - (1 - \Phi^1(l_1^1, \ldots, l_K^1))(1 - \Phi^2(l_1 - l_1^1, \ldots, l_K - l_K^1))\} \times \prod_{k=1}^{K} \binom{m_k^1}{l_k^1} \binom{m_k^2}{l_k - l_k^1} \binom{m_k}{l_k}^{-1} \Bigg] \tag{4.16}$$

These results follow from similar combinatorial arguments as Equation 4.11, together with assumed independence of components of different types.

4.5.4 $R > 2$ subsystems with $K \geq 2$ component types

For a system consisting of $R > 2$ subsystems with $K \geq 2$ component types, using Equations (4.15) and (4.16), one can start by combining the survival signature of pairs of subsystems. This combination can be applied repeatedly to derive the system's survival signature for quite complicated systems, as long as they can be built up by a sequence of pairwise combinations of subsystems, either in series or

parallel configuration, similarly as discussed in Section 1.2 and used in sections 3.3 and 4.5 [48]. This is illustrated in the following two examples.

Example 4.6

We consider computation of the survival signature for the system with $K = 3$ types of components as presented in Figure 4.6, with $m_1 = 1$ and $m_2 = m_3 = 4$ components of each type. The system consists of three subsystems in series configuration.

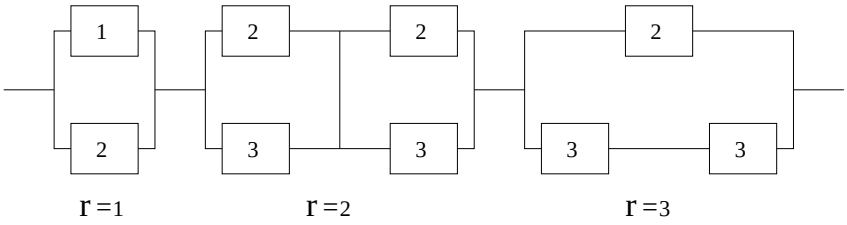

Figure 4.6: System with 3 types of components

The survival signatures for the subsystems are easily derived and given in Tables 4.11, 4.12 and 4.13. The survival signature for this full system can be obtained using Equation 4.15 to first combine subsystems 1 and 2, resulting in the combined survival signature as shown in Table 4.14. Then, using the result in Table 4.14 together with the survival signature for subsystem 3, these can be combined by Equation 4.15, leading to the overall system's survival sugnature, which is given in Table 4.15, where apart from $\Phi(0,0,0)$=0 all not presnted $\Phi_s(l_1,l_2,l_3)$ with $l_1 \in \{0,1\}$, $l_2 \in \{0,1,2,3,4\}$ and $l_1, l_2 \in \{0,1,2,3,4\}$ are equal to 1. Let us briefly explain some of the values in Table 4.15. Consider $\Phi(0,1,4)$ and $\Phi(0,2,2)$ in detail. For $\Phi(0,1,4)$ the component of Type 1 does not function while precisely 1 component of Type 2 and 4 components of Type 3 function. There are 4 such vectors, for only one of these, namely $(0,1,0,0,0,1,1,1,1)$, the system functions, so indeed $\Phi(0,1,4) = 1/4$. For $\Phi(0,2,2)$ we need to check $\binom{4}{2}\ \binom{4}{2} = 36$ vectors, the system functions for only one of these, namely $(0,1,0,0,1,1,1,0,0)$, so indeed $\Phi(0,2,2) = 1/36$.

To illustrate the use of this system's survival signature, suppose that $n_1 = 2$ components exchangeable with those of Type 1 were tested, $n_2 = 2$ components exchangeable with those of Type 2 and also $n_3 = 2$ components exchangeable with

l_1	l_2	l_3	$\Phi(l_1, l_2, l_3)$
0	0	0	0
1	0	0	1
0	1	0	1
1	1	0	1

Table 4.11: Survival signature for subsystem 1

l_1	l_2	l_3	$\Phi(l_1, l_2, l_3)$
0	0	0	0
0	0	1	0
0	0	2	1
0	1	0	0
0	1	1	1/2
0	1	2	1
0	2	0	1
0	2	1	1
0	2	2	1

Table 4.12: Survival signature for subsystem 2

l_1	l_2	l_3	$\Phi(l_1, l_2, l_3)$
0	0	0	0
0	0	1	0
0	0	2	1
0	1	0	1
0	1	1	1
0	1	2	1

Table 4.13: Survival signature for subsystem 3

l_1	l_2	l_3	$\Phi(l_1, l_2, l_3)$	l_1	l_2	l_3	$\Phi(l_1, l_2, l_3)$
0	0	0	0	1	0	0	0
0	0	1	0	1	0	1	0
0	0	2	0	1	0	2	1
0	1	0	0	1	1	0	0
0	1	1	0	1	1	1	1/3
0	1	2	1/3	1	1	2	1
0	2	0	0	1	2	0	1/3
0	2	1	1/3	1	2	1	4/6
0	2	2	2/3	1	2	2	1
0	3	0	1	1	3	0	1
0	3	1	1	1	3	1	1
0	3	2	1	1	3	2	1

Table 4.14: Survival signature for combined subsystem 1, 2

those of Type 3. Suppose that the failure times of these tested components were ordered as $t_1^1 < t_1^2 < t_2^1 < t_1^3 < t_2^2 < t_2^3$. Table 4.16 presents the NPI lower and upper survival functions $\underline{S}_{T_S}(t)$ and $\overline{S}_{T_S}(t)$ for the system failure time T_S, for the system presented in Figure 4.6, with these ordered test data.

l_1	l_2	l_3	$\Phi(l_1,l_2,l_3)$	l_1	l_2	l_3	$\Phi(l_1,l_2,l_3)$	l_1	l_2	l_3	$\Phi(l_1,l_2,l_3)$
0	0	0	0	0	3	2	7/24	1	1	4	1
0	0	1	0	0	3	3	8/16	1	2	0	0
0	0	2	0	0	3	4	3/4	1	2	1	2/24
0	0	3	0	0	4	0	1	1	2	2	8/36
0	0	4	0	1	4	1	1	1	2	3	1/2
0	1	0	0	1	4	2	1	1	2	4	1
0	1	1	0	1	4	3	1	1	3	0	1/4
0	1	2	0	1	4	4	1	1	3	1	6/16
0	1	3	0	1	0	0	0	1	3	2	13/24
0	1	4	1/4	1	0	1	0	1	3	3	12/16
0	2	0	0	1	0	2	0	1	3	4	1
0	2	1	0	1	0	3	0	1	4	0	1
0	2	2	1/36	1	0	4	1	1	4	1	1
0	2	3	1/6	1	1	0	0	1	4	2	1
0	2	4	1/2	1	1	1	0	1	4	3	1
0	3	0	0	1	1	2	1/24	1	4	4	1
0	3	1	2/16	1	1	3	4/16				

Table 4.15: Survival signature of system in Figure 4.6

$t_1^1 < t_1^2 < t_2^1 < t_1^3 < t_2^2 < t_2^3$		
$t \in$	$\underline{S}_{T_S}(t)$	$\overline{S}_{T_S}(t)$
$(0, t_1^1)$	0.639	1
(t_1^1, t_1^2)	0.574	1
(t_1^2, t_2^1)	0.291	0.889
(t_2^1, t_1^3)	0.193	0.778
(t_1^3, t_2^2)	0.111	0.574
(t_2^2, t_2^3)	0	0.291
(t_2^3, ∞)	0	0.147

Table 4.16: $\underline{S}_{T_S}(t)$ and $\overline{S}_{T_S}(t)$ for the system in Figure 4.6

Example 4.7

Consider the system presented in Figure 4.7. This system is made up of the same three subsystems as the system considered in the previous example, but now in a different configuration. Using the results in Table 4.14 together with the survival signature for subsystem 3 as given in Table 4.13, we use Equation 4.16 to derive the overall system's survival sugnature, which is given in Table 4.17. Let us briefly explain some of the values in Table 4.17. Consider $\Phi(0,2,0)$ and $\Phi(1,0,2)$ in detail. For $\Phi(0,2,0)$ precisely 2 of the 4 components of Type 2 function, so we need to check 6 such vectors. For only three of these, namely $(0,1,0,0,1,0,0,0,0)$, $(0,0,1,0,1,0,0,0,0)$ and $(0,0,0,1,1,0,0,0,0)$, the system functions, so $\Phi(0,2,0) = 3/6$. For $\Phi(1,0,2)$, we also need to check 6 vectors, for two of these vectors the system functions, namely $(1,0,0,0,0,1,1,0,0)$ and $(1,0,0,0,0,0,0,1,1)$, so $\Phi(1,0,2) = 2/6$.

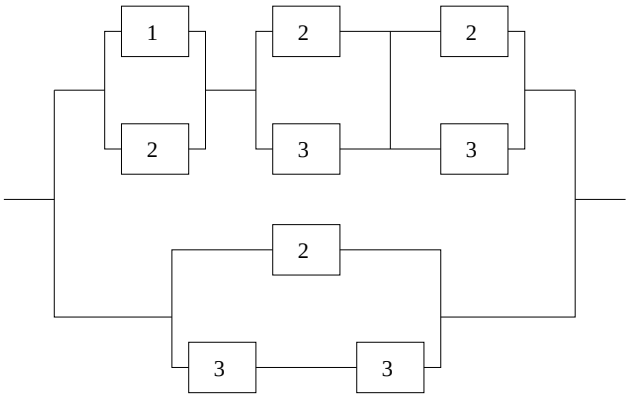

Figure 4.7: System with 3 types of components

To illustrate the use of this system's survival signature, suppose again that $n_1 = 2$ components exchangeable with those of Type 1 were tested, $n_2 = 2$ components exchangeable with those of Type 2 and also $n_3 = 2$ components exchangeable with those of Type 3. Suppose that the failure times of these tested components were ordered as $t_1^1 < t_1^2 < t_2^1 < t_1^3 < t_2^2 < t_2^3$. Table 4.18 presents the NPI lower and upper survival functions $\underline{S}_{T_S}(t)$ and $\overline{S}_{T_S}(t)$ for the system failure time T_S, for the system presented in Figure 4.7, with these ordered test data. Table 4.18 shows that no component of Type 3 has failed up to the fourth interval, so the NPI upper survival function is equal to 1 as in this system all components of Type 3 functioning is

l_1	l_2	l_3	$\Phi(l_1, l_2, l_3)$	l_1	l_2	l_3	$\Phi(l_1, l_2, l_3)$
0	0	0	0	1	0	0	0
0	0	1	0	1	0	1	0
0	0	2	1/6	1	0	2	2/6
0	0	3	1/2	1	0	3	1
0	0	4	1	1	0	4	1
0	1	0	1/4	1	1	0	1/4
0	1	1	1/4	1	1	1	6/16
0	1	2	10/24	1	1	2	16/24
0	1	3	12/16	1	1	3	1
0	1	4	1	1	1	4	1
0	2	0	3/6	1	2	0	4/6
0	2	1	14/24	1	2	1	18/24
0	2	2	27/36	1	2	2	32/36
0	2	3	22/24	1	2	3	1

Table 4.17: Survival signature of system in Figure 4.7

sufficient for the system to function. In the last interval, we are beyond the failure times of all the tested components, so we no longer have evidence in favour of the system to function, which is reflected by $\underline{S}_{T_S}(t) = 0$, but the system might of course still function, as represented by $\overline{S}_{T_S}(t) = 0.525$.

Furthermore, Table 4.19 also presents the NPI lower and upper survival functions $\underline{S}_{T_S}(t)$ and $\overline{S}_{T_S}(t)$ for the system failure time T_S, for the same system presented in Figure 4.7 but with differently ordered failure times. In the first interval, we have not yet seen a single failure in the test data, so the NPI upper probability that the system will function is equal to one. In the second interval, one failure of Type 3 has occurred before, but we do not have any evidence from the data against the possibility that a component of Type 1 and Type 2 will certainly function at times in this interval, so the upper probability remains one. In the third interval one failure of Type 2 and one failure of Type 3 have occurred, but due to insufficiency of only having a component of Type 1 function, the NPI upper survival function leads to

$t_1^1 < t_1^2 < t_2^1 < t_1^3 < t_2^2 < t_2^3$		
$t \in$	$\underline{S}_{T_S}(t)$	$\overline{S}_{T_S}(t)$
$(0, t_1^1)$	0.920	1
(t_1^1, t_1^2)	0.905	1
(t_1^2, t_2^1)	0.748	1
(t_2^1, t_1^3)	0.712	1
(t_1^3, t_2^2)	0.494	0.905
(t_2^2, t_2^3)	0.167	0.748
(t_2^3, ∞)	0	0.525

Table 4.18: $\underline{S}_{T_S}(t)$ and $\overline{S}_{T_S}(t)$ of system in Figure 4.7

be less than one, which in this case is 0.936.

$t_1^3 < t_1^2 < t_2^3 < t_1^1 < t_2^2 < t_2^1$		
$t \in$	$\underline{S}_{T_S}(t)$	$\overline{S}_{T_S}(t)$
$(0, t_1^3)$	0.920	1
(t_1^3, t_1^2)	0.830	1
(t_1^2, t_2^3)	0.556	0.936
(t_2^3, t_1^1)	0.389	0.847
(t_1^1, t_2^2)	0.378	0.830
(t_2^2, t_2^1)	0	0.556
(t_2^1, ∞)	0	0.525

Table 4.19: $\underline{S}_{T_S}(t)$ and $\overline{S}_{T_S}(t)$ of system in Figure 4.7

4.6 Bounds on survival signatures

In Chapter 3 we considered the use of bounds for the signature, for example as may result if its computation is stopped before the exact signature is derived. In this section we follow the same approach for the survival signature. The survival signature is in general difficult to compute exactly if the system consists of a large

number of components, but any algorithm will provide bounds if the computation stops early. However, based on partial information, e.g. following from checking only a subset of all combinations of functioning and not functioning components, one can derive bounds for quantities of interest. If such bounds suffice for a specific inferential question, then of course one would not need to compute the survival signature further. This is a straightforward idea, which was also presented In Chapter 3 for system signatures. Given the monotonicity of the survival signature in each of its components for coherent systems, working with bounds for it is straightforward as long as the inference of interest is monotone as function of the survival signature. If interest is in (NPI lower and upper) probabilities for the event that the system functions at time t, as considered in this chapter, then indeed it is straigthforward to use the information about the survival signature, consisting of bounds for its values. Before presenting the general results for any number K of component types, the case of a system consisting of $K = 2$ types of components is considered.

4.6.1 Bounds on survival signatures with 2 component types

Consider a system consisting of 2 types of components. Its survival signature $\Phi(l_1, l_2)$ is an increasing function of (l_1, l_2), we assume that $\Phi(0, 0)$ =0 and $\Phi(m_1, m_2) = 1$. Let $\mathcal{V}$ be the set

$$\mathcal{V} = \{(v_1, v_2) \quad | \quad 0 \leq v_k \leq m_k, \quad \forall k \in \{1, 2\}\}$$

and let $\mathcal{V}_c$ be a subset of $\mathcal{V}$ consisting of all points in $\mathcal{V}$ at which either the survival signature has been computed or bounds for it have been calculated. We show how to derive optimal lower and upper bounds for the survival signature, denoted by $\Phi_l(l_1, l_2)$ and $\Phi_u(l_1, l_2)$ respectively, based on such limited information about the survival signature. We call $\Phi_l(v_1, v_2)$ the lower survival signature, and $\Phi_u(v_1, v_2)$ the upper survival signature. These are the maximum lower bound and minimum upper bound, respectively, for the survival signature $\Phi(v_1, v_2)$ based on the available calculations. The upper survival signature is

$$\Phi_u(l_1, l_2) = \min \{\Phi(v_1, v_2) | (v_1, v_2) \in \mathcal{V}_c^u(l_1, l_2)\} \tag{4.17}$$

with

$$\mathcal{V}_c^u(l_1, l_2) = \{(v_1, v_2) \in \mathcal{V}_c : v_1 \geq l_1, v_2 \geq l_2\}$$

First, suppose that $\Phi(v_1, v_2)$ is precisely known for all $(v_1, v_2) \in \mathcal{V}_c$. All points $(v_1, v_2) \in \mathcal{V}_c^u$ give us upper bounds for $\Phi(l_1, l_2)$, because

$$\Phi(l_1, l_2) \leq \Phi(v_1, v_2) \quad \forall (v_1, v_2) \in \mathcal{V}_c^u(l_1, l_2)$$

More generally, based on lower bounds $\Phi_l(v_1, v_2)$ and upper bounds $\Phi_u(v_1, v_2)$ for all $(v_1, v_2) \in \mathcal{V}_c^u(l_1, l_2)$, the upper survival signature at (l_1, l_2) is

$$\Phi_u(l_1, l_2) = \min \{\Phi_u(v_1, v_2) | (v_1, v_2) \in \mathcal{V}_c^u(l_1, l_2)\} \tag{4.18}$$

The corresponding lower survival signature can be determined similarly. First, if $\Phi(v_1, v_2)$ is precisely known for all $(v_1, v_2) \in \mathcal{V}_c$, then we take the subset $\mathcal{V}_c^l$ of $\mathcal{V}_c$,

$$\mathcal{V}_c^l(l_1, l_2) = \{(v_1, v_2) \in \mathcal{V}_c : v_1 \leq l_1, v_2 \leq l_2\}$$

All points $(v_1, v_2) \in \mathcal{V}_c^l$ give lower bounds for $\Phi(l_1, l_2)$, because

$$\Phi(l_1, l_2) \geq \Phi(v_1, v_2) \quad \forall (v_1, v_2) \in \mathcal{V}_c^l(l_1, l_2)$$

So, we can define

$$\Phi_l(l_1, l_2) = \max \left\{\Phi(v_1, v_2) | (v_1, v_2) \in \mathcal{V}_c^l(l_1, l_2)\right\} \tag{4.19}$$

If lower bounds $\Phi_l(v_1, v_2)$ and upper bounds $\Phi_u(v_1, v_2)$ are known for all $(v_1, v_2) \in \mathcal{V}_c^l(l_1, l_2)$, then the lower survival signature at (l_1, l_2) is

$$\Phi_l(l_1, l_2) = \max \left\{\Phi_l(v_1, v_2) | (v_1, v_2) \in \mathcal{V}_c^l(l_1, l_2)\right\} \tag{4.20}$$

The corresponding optimal lower and upper bounds for the NPI lower survival function for T_S follow from Equation 4.9, and by the monotonicity of the survival signature these are easily seen to be

$$\underline{S}_{T_S}^l(t) = \underline{P}^l(T_S > t) = \sum_{l_1=0}^{m_1} \sum_{l_K=0}^{m_2} \Phi_l(l_1, l_2) \prod_{k=1}^{2} \overline{D}(C_k(t) = l_k) \tag{4.21}$$

$$\underline{S}_{T_S}^u(t) = \underline{P}^u(T_S > t) = \sum_{l_1=0}^{m_1} \sum_{l_K=0}^{m_2} \Phi_u(l_1, l_2) \prod_{k=1}^{2} \overline{D}(C_k(t) = l_k) \tag{4.22}$$

Similarly, the corresponding optimal lower and upper bounds for the NPI upper survival function for T_S follow from Equation 4.10,

$$\overline{S}^l_{T_S}(t) = \overline{P}^l(T_S > t) = \sum_{l_1=0}^{m_1} \sum_{l_K=0}^{m_2} \Phi_l(l_1, l_2) \prod_{k=1}^{2} \underline{D}(C_k(t) = l_k) \tag{4.23}$$

$$\overline{S}^u_{T_S}(t) = \overline{P}^u(T_S > t) = \sum_{l_1=0}^{m_1} \sum_{l_K=0}^{m_2} \Phi_u(l_1, l_2) \prod_{k=1}^{2} \underline{D}(C_k(t) = l_k) \tag{4.24}$$

In the following subsection, the above arguments will be extended to the general case with K component types, the main ideas are the same as for this case with $K = 2$.

4.6.2 Bounds on survival signatures with $K \geq 2$ component types

We now consider a system with $K \geq 2$ types of components, with m_k components of type k for each $k = 1, \ldots, K$, so $m = m_1 + \cdots + m_K$ components in total. Let $\mathcal{V}$ be the set

$$\mathcal{V} = \{(v_1, \ldots, v_K) \quad | \quad 0 \leq v_k \leq m_k, \quad \forall k \in \{1, \ldots, K\}\}$$

and let $\mathcal{V}_c$ be a subset of $\mathcal{V}$ consisting of all points in $\mathcal{V}$ at which either the survival signature has been computed or bounds for it have been calculated. We show how to derive optimal lower and upper bounds for the survival signature, denoted by $\Phi_l(l_1, \ldots, l_K)$ and $\Phi_u(l_1, \ldots, l_K)$ respectively, based on such limited information about the survival signature. We call $\Phi_l(v_1, \ldots, v_K)$ the lower survival signature, and $\Phi_u(v_1, \ldots, v_K)$ the upper survival signature. These are the maximum lower bound and minimum upper bound, respectively, for the survival signature $\Phi(v_1, \ldots, v_K)$ based on the available calculations. So, in this case we define

$$\Phi_u(l_1, \ldots, l_K) = \min\{\Phi(v_1, \ldots, v_K) | (v_1, \ldots, v_K) \in \mathcal{V}^u_c(l_1, \ldots, l_K)\} \tag{4.25}$$

First, suppose that $\Phi(v_1, \ldots, v_K)$ is precisely known for all $(v_1, \ldots, v_K) \in \mathcal{V}_c$. We take the subset

$$\mathcal{V}^u_c(l_1, \ldots, l_K) = \{(v_1, \ldots, v_K) \in \mathcal{V} : v_k \geq l_k, \quad \forall k \in \{1, \ldots, K\}\}$$

All points $(v_1, \ldots, v_K) \in \mathcal{V}_c^u$ give upper bounds for $\Phi(l_1, \ldots, l_K)$, because

$$\Phi(l_1, \ldots, l_K) \leq \Phi(v_1, \ldots, v_K) \quad \forall (v_1, \ldots, v_K) \in \mathcal{V}_c^u(l_1, \ldots, l_K)$$

If only lower bounds $\Phi_l(v_1, \ldots, v_K)$ and upper bounds $\Phi_u(v_1, \ldots, v_K)$ are known for all $(v_1, \ldots, v_K) \in \mathcal{V}_c^u(l_1, \ldots, l_K)$, then the upper survival signature at $(l_1, \ldots, l_K)$ is

$$\Phi_u(l_1, \ldots, l_K) = \min \{\Phi_u(v_1, \ldots, v_K) | (v_1, \ldots, v_K) \in \mathcal{V}_c^u(l_1, \ldots, l_K)\} \tag{4.26}$$

The corresponding lower survival signature can be determined similarly. First, if $\Phi(v_1, \ldots, v_K)$ is precisely known for all $(v_1, \ldots, v_K) \in \mathcal{V}_c$, then we take the subset

$$\mathcal{V}_c^l(l_1, \ldots, l_K) = \{(v_1, \ldots, v_K) \in \mathcal{V} : v_k \leq l_k, \quad \forall k \in \{1, \ldots, K\}\}$$

All points $(v_1, \ldots, v_K) \in \mathcal{V}_c^l$ give lower bounds for $\Phi(l_1, \ldots, l_K)$, because

$$\Phi(l_1, \ldots, l_K) \geq \Phi(v_1, \ldots, v_K) \quad \forall (v_1, \ldots, v_K) \in \mathcal{V}_c^l(l_1, \ldots, l_K)$$

So, we define

$$\Phi_l(l_1, \ldots, l_k) = \max \left\{\Phi(v_1, \ldots, v_K) | (v_1, \ldots, v_K) \in \mathcal{V}_c^l(l_1, \ldots, l_k)\right\} \tag{4.27}$$

Similarly, if only bounds $\Phi_l(v_1, \ldots, v_K)$ and upper bounds $\Phi_u(v_1, \ldots, v_K)$ are known for all $(v_1, \ldots, v_K) \in \mathcal{V}_c^l(l_1, \ldots, l_K)$ then, the lower survival signature at $(l_1, \ldots, l_K)$ is

$$\Phi_l(l_1, \ldots, l_k) = \max \left\{\Phi_l(v_1, \ldots, v_K) | (v_1, \ldots, v_K) \in \mathcal{V}_c^l(l_1, \ldots, l_k)\right\} \tag{4.28}$$

The corresponding optimal lower and upper bounds for the NPI lower survival function for T_S follow from Equation 4.9,

$$\underline{S}_{T_S}^l(t) = \underline{P}^l(T_S > t) = \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi_l(l_1, \cdots, l_K) \prod_{k=1}^{K} \overline{D}(C_k(t) = l_k) \tag{4.29}$$

$$\underline{S}_{T_S}^u(t) = \underline{P}^u(T_S > t) = \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi_u(l_1, \cdots, l_K) \prod_{k=1}^{K} \overline{D}(C_k(t) = l_k) \tag{4.30}$$

Similarly, the corresponding optimal lower and upper bounds for the NPI upper survival function for T_S follow from Equation 4.10,

$$\overline{S}_{T_S}^l(t) = \overline{P}^l(T_S > t) = \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi_l(l_1, \cdots, l_K) \prod_{k=1}^{K} \underline{D}(C_k(t) = l_k) \tag{4.31}$$

$$\overline{S}_{T_S}^u(t) = \overline{P}^u(T_S > t) = \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi_u(l_1, \cdots, l_K) \prod_{k=1}^{K} \underline{D}(C_k(t) = l_k) \tag{4.32}$$

We present two examples to illustrate such bounds to obtain optimal lower and upper bounds for the NPI lower and upper survival functions.

Example 4.8

Consider again the system in Figure 4.2, with survival signature given in Table 4.2. As it is assumed that $\Phi(0,0,0)$ =0 and $\Phi(m_1,m_2,m_3) = 1$, we assume in order to illustrate the use of bounds on the survival signature, as presented in this section, that the survival signature has been derived partially for some values, with the bounds for three cases as presented in Tables 4.20, 4.21 and 4.22.

l_1 \ l_2	0	1	2	3
0	0	0	0	0
1	0	0	[0,6/9]	[0,6/9]
2	0	0	[0,6/9]	6/9
3	1	1	1	1

Table 4.20: Lower and upper survival functions for Case 1 Fig 4.2

l_1 \ l_2	0	1	2	3
0	0	0	0	0
1	0	0	1/9	[1/9,1]
2	0	0	[1/9,1]	[1/9,1]
3	1	1	1	1

Table 4.21: Lower and upper survival functions for Case 2 Fig 4.2

Suppose that $n_1 = 2$ components exchangeable with those of Type 1 were tested, and $n_2 = 2$ observed failure times for components exchangeable with those of type 2 were tested, leading to ordered failure times $t_1^2 < t_1^1 < t_2^2 < t_2^1$. Table 4.23 presents the NPI lower and upper survival functions $\underline{S}_{T_S}(t)$ and $\overline{S}_{T_S}(t)$ for the system failure time T_S, for this system and for these three cases.

l_1 \ l_2	0	1	2	3
0	0	0	0	0
1	0	0	1/9	[1/9,6/9]
2	0	0	[1/9,6/9]	6/9
3	1	1	1	1

Table 4.22: Lower and upper survival functions for Case 3 Fig 4.2

The lower bounds for the NPI lower survival function $\underline{S}^l_{T_S}(t)$, as given in Equation 4.21 and the upper bounds for the NPI lower survival function $\overline{S}^u_{T_S}(t)$, as given in Equation 4.24 are presented in Table 4.23 for three cases. This table illustrates the use of the bounds as presented in this section. It shows that the imprecision between the upper bound of the NPI upper survival function and the lower bound of the NPI lower survival function decreases if more details about the survival function become known.

	Case 1		Case 2		Case 3	
$t \in$	$\underline{S}^l_{T_S}(t)$	$\overline{S}^u_{T_S}(t)$	$\underline{S}^l_{T_S}(t)$	$\overline{S}^u_{T_S}(t)$	$\underline{S}^l_{T_S}(t)$	$\overline{S}^u_{T_S}(t)$
$(0, t_1^2)$	0.480	1	0.439	1	0.506	1
(t_1^2, t_1^1)	0.420	1	0.417	1	0.433	1
(t_1^1, t_2^2)	0.113	0.633	0.117	0.697	0.128	0.600
(t_2^2, t_2^1)	0	0.500	0	0.514	0	0.478
(t_2^1, ∞)	0	0.200	0	0.197	0	0.167

Table 4.23: $\underline{S}^l_{T_S}(t)$ and $\overline{S}^u_{T_S}(t)$ of system in Figure 4.2

Example 4.9

Consider again the system in Figure 4.7, with the survival signature given in Table 4.17. It is assumed that $\Phi(0,0,0)$ =0 and $\Phi(m_1, m_2, m_3) = 1$. It may be easy to find some (l_1, l_2, l_3) for which $\Phi(l_1, l_2, l_3) = 0$ or $\Phi(l_1, l_2, l_3) = 1$.

Table 4.24 provides bounds for the survival signature, together with the exact survival signature, for four cases, representing four possible subsequent stages of its direct computation (so not using the method presented in Section 4.5). The bounds are denoted as intervals, entries which are either 0 or 1 for all stages and where this follows by monotonicity from other entries in the table have been deleted. Case 1 only involved an initial assessment for rather trivial values of (l_1, l_2, l_3) for which the system either functions or not with certainty. Without further calculations, the survival signature is only known to be in $[0, 1]$ at all other (l_1, l_2, l_3). Case 2 shows the effect of calculating $\Phi(0, 0, 3) = 1/2$, Case 3 of the additional calculations $\Phi(0, 1, 0) = \Phi(0, 1, 1) = 1/4$, these are all pretty trivial to derive. For Case 4 we calculated, by going through all relevant combinations, the precise values of the survival signature at 6 further points (l_1, l_2, l_3), as shown in Table 4.24. Most of these precise values affect some bounds at other points due to the monotinicity of the survival signature, but not all. However, all these calculations affect the related bounds for the inferences. It is also possible to calculate the survival signature only partially at a point (l_1, l_2, l_3), leading to bounds at that point which also affect bounds at other points.

To illustrate the effect of such increased knowledge of the system's survival signature, we present its application in the NPI method using simulated failure times as given in Table 4.25, which for Type k were simulated from the Weibull distribution with shape parameter k and scale parameter 1.

The corresponding lower bounds for the NPI lower survival function, $\underline{S}^l_{T_S}(t)$ as given in Equation 4.29, and the upper bounds for the NPI upper survival function, $\overline{S}^u_{T_S}(t)$ as given in Equation 4.31, are presented in the plots in Figure 4.8 for the four cases, where in each plot also the NPI lower and upper survival functions are presented based on the exact survival signature, as given in Table 4.17. These plots illustrate the use of the bounds as presented in this section. Figure 4.9 presents the same functions of Figures 4.8.1 and 4.8.2, so for Cases 1 and 2, but on a larger scale in order to see the differences more clearly. Due to the monotonicities involved, additional calculations for the survival signatures lead to sharper bounds for the NPI lower and upper survival functions, with the effect of the rather straightforward

Case 1							
l_1	l_2	l_3	$\Phi(l_1,l_2,l_3)$	l_1	l_2	l_3	$\Phi(l_1,l_2,l_3)$
0	0	0	0	1	0	0	0
0	0	1	0	1	0	1	0
0	0	2	[0,1]	1	0	2	[0,1]
0	0	3	[0,1]	1	0	3	1
0	0	4	1	1	0	4	1
0	1	0	[0,1]	1	1	0	[0,1]
0	1	1	[0,1]	1	1	1	[0,1]
0	1	2	[0,1]	1	1	2	[0,1]
0	1	3	[0,1]	1	1	3	1
0	1	4	1	1	1	4	1
0	2	0	[0,1]	1	2	0	[0,1]
0	2	1	[0,1]	1	2	1	[0,1]
0	2	2	[0,1]	1	2	2	[0,1]
0	2	3	[0,1]	1	2	3	1

Case 2							
l_1	l_2	l_3	$\Phi(l_1,l_2,l_3)$	l_1	l_2	l_3	$\Phi(l_1,l_2,l_3)$
0	0	0	0	1	0	0	0
0	0	1	0	1	0	1	0
0	0	2	[0,1/2]	1	0	2	[0,1]
0	0	3	1/2	1	0	3	1
0	0	4	1	1	0	4	1
0	1	0	[0,1]	1	1	0	[0,1]
0	1	1	[0,1]	1	1	1	[0,1]
0	1	2	[0,1]	1	1	2	[0,1]
0	1	3	[1/2,1]	1	1	3	1
0	1	4	1	1	1	4	1
0	2	0	[0,1]	1	2	0	[0,1]
0	2	1	[0,1]	1	2	1	[0,1]
0	2	2	[0,1]	1	2	2	[0,1]
0	2	3	[1/2,1]	1	2	3	1

Case 3							
l_1	l_2	l_3	$\Phi(l_1,l_2,l_3)$	l_1	l_2	l_3	$\Phi(l_1,l_2,l_3)$
0	0	0	0	1	0	0	0
0	0	1	0	1	0	1	0
0	0	2	[0,1/2]	1	0	2	[0,1/2]
0	0	3	1/2	1	0	3	1
0	0	4	1	1	0	4	1
0	1	0	1/4	1	1	0	[1/4,1]
0	1	1	1/4	1	1	1	[1/4,1]
0	1	2	[1/4,1]	1	1	2	[1/4,1]
0	1	3	[1/2,1]	1	1	3	1
0	1	4	1	1	1	4	1
0	2	0	[1/4,1]	1	2	0	[1/4,1]
0	2	1	[1/4,1]	1	2	1	[1/4,1]
0	2	2	[1/4,1]	1	2	2	[1/4,1]
0	2	3	[1/2,1]	1	2	3	1

Case 4							
l_1	l_2	l_3	$\Phi(l_1,l_2,l_3)$	l_1	l_2	l_3	$\Phi(l_1,l_2,l_3)$
0	0	0	0	1	0	0	0
0	0	1	0	1	0	1	0
0	0	2	1/6	1	0	2	2/6
0	0	3	1/2	1	0	3	1
0	0	4	1	1	0	4	1
0	1	0	1/4	1	1	0	1/4
0	1	1	1/4	1	1	1	[1/4,32/36]
0	1	2	[1/4,32/36]	1	1	2	[1/4,32/36]
0	1	3	[1/2,1]	1	1	3	1
0	1	4	1	1	1	4	1
0	2	0	3/6	1	2	0	4/6
0	2	1	[1/2,1]	1	2	1	[4/6,32/36]
0	2	2	[1/2,32/36]	1	2	2	32/36
0	2	3	[1/2,1]	1	2	3	1

Table 4.24: Survival signature of system in Figure 4.7

Type 1		Type 2		Type 3	
0.004	0.629	0.290	1.006	0.321	0.876
0.112	0.752	0.412	1.029	0.348	0.877
0.177	0.839	0.531	1.057	0.375	0.920
0.196	0.974	0.579	1.113	0.613	0.955
0.223	1.234	0.603	1.127	0.650	0.973
0.239	1.311	0.617	1.146	0.689	1.064
0.260	1.325	0.677	1.252	0.743	1.102
0.356	1.436	0.883	1.350	0.747	1.110
0.486	3.097	0.901	1.586	0.788	1.129
0.490	4.150	0.942	1.885	0.856	1.150

Table 4.25: Component failure times

calculations in Cases 2 and 3 already quite substantial. The additional calculations in Case 4 lead to bounds that are already mostly close to the actual NPI lower and upper survival functions.

The NPI upper survival function is equal to one for $t \leq 0.321$, because no component of type 3 has failed yet at such times. The NPI lower survival function is equal to 0.99991 for $t \leq 0.004$, the NPI lower survival function is less than one even before a failure from any type of component occurs. While there are some small changes in the NPI upper and lower survival functions at $t = 3.097$ and $t = 4.150$, these values in order to see the figures more clearly are not included. Depending on the inference of interest, these bounds may already be sufficient to derive the conclusion, in which case further calculation of the survival signature would not be required.

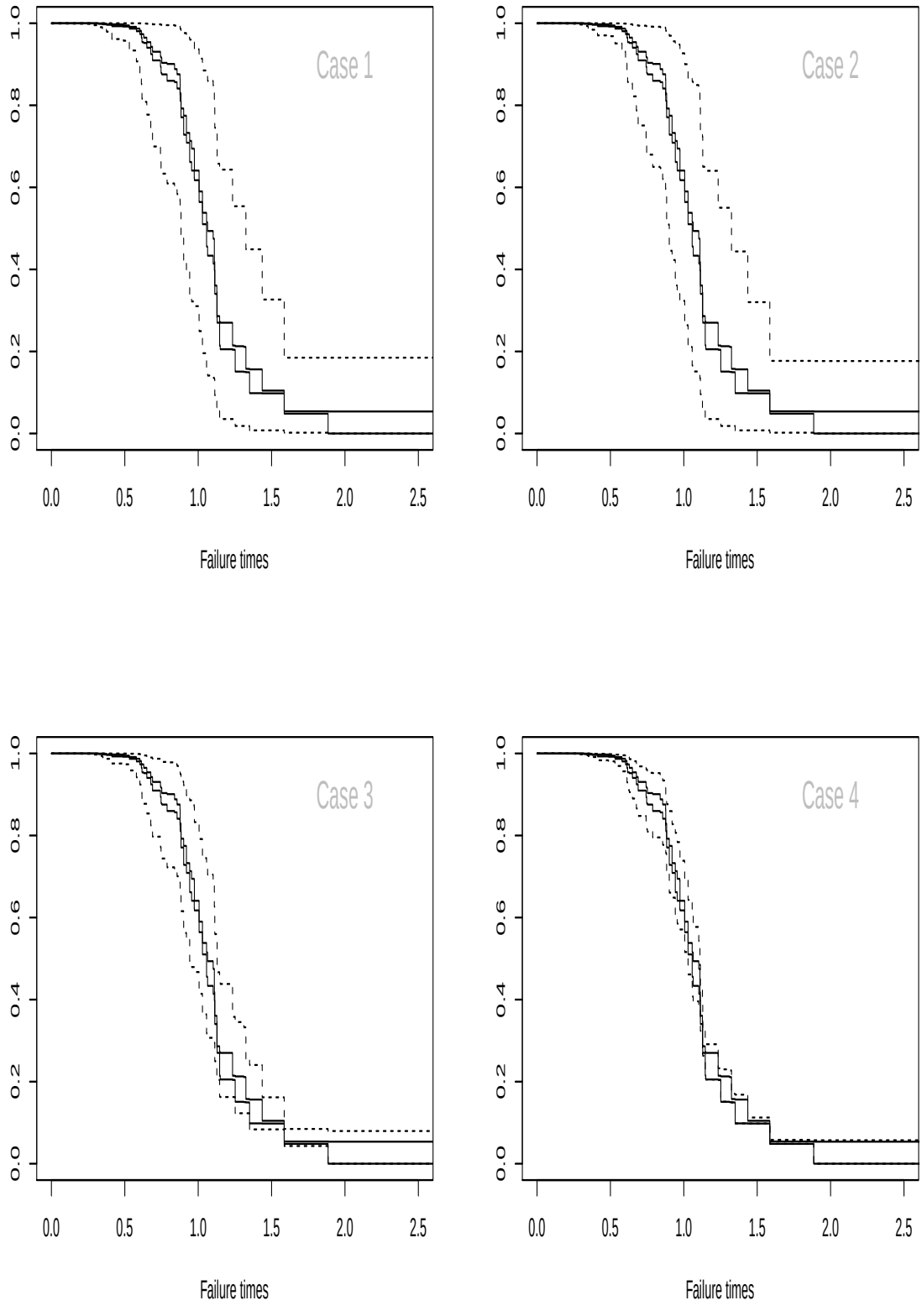

Figure 4.8: NPI lower and upper survival functions

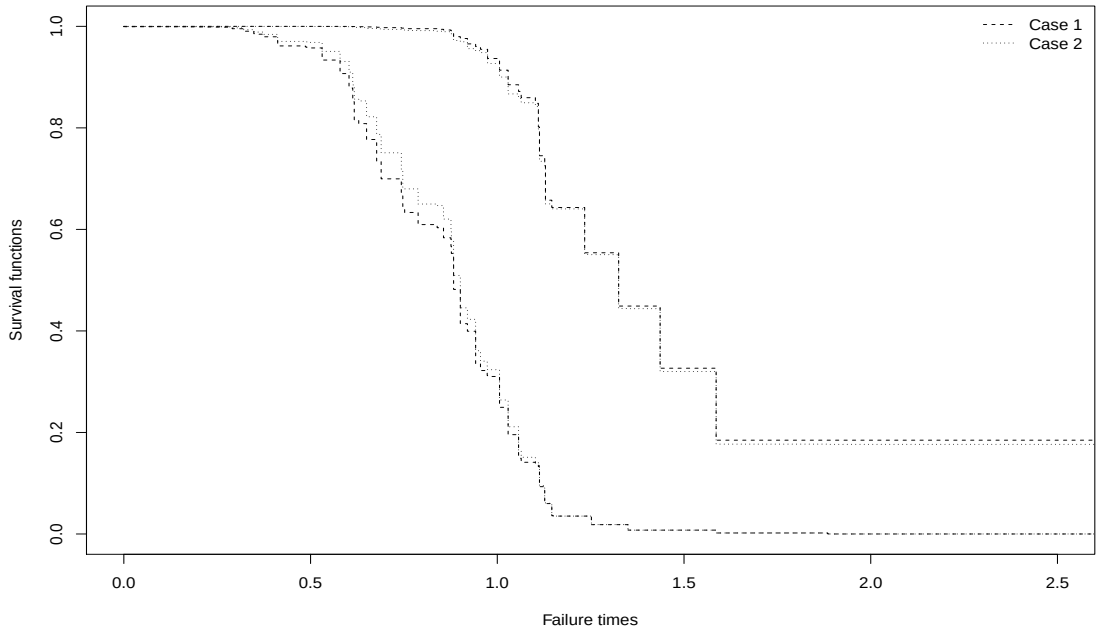

Figure 4.9: NPI lower and upper survival functions

4.7 Concluding remarks

Computation of the survival signature of a system is difficult unless the number of components is small or the system has a relatively straightforward structure. For systems with only one type of components, the signature has been derived for some specific system structures [44–46, 61]. Deriving the survival signature for specific system structures in the case of multiple component types is an interesting topic for research. Recently, Aslett [10] has created a function in the statistical software R to compute the survival signature, given a graphical presentation of the system structure. This can, in principle, be used for systems of any size, but for real-world systems with more than about 20 components computation time rapidly becomes an issue. It may be possible to implement the results in Sections 4.4 and 4.5 in this R function. It is also interesting to investigate if it is possible to benefit from established methods to quantify system reliability, for example fault trees, Bayesian networks or binary decision diagrams, to derive the corresponding survival signature.

While the emphasis in this chapter has been on system reliability, the closely related topic of reliability of networks is of great practical importance, for example in energy provision. In such networks there are typically many components of multiple types, with often quite large numbers of components of a specific type for which the assumption of exchangeable failure times may be reasonable. Developing the survival signature approach for network reliability is therefore also an important research challenge, which includes computational challenges and could lead to results with great practical impact.

There are many further research challenges related to theory and application of survival signatures. For example, one could consider the use of right-censored observations, which is likely to be possible with an adaptation of NPI for Bernoulli data in line with the corresponding NPI theory for real-valued data with right-censored observations [33], which is related to the well-known Kaplan-Meier estimator for such data. It will also be of interest to consider possible system failure due to multiple failure modes [59], where the NPI approach provides interesting opportunities to consider unobserved or even unknown competing risks [34, 54]. Topics of optimal system design in order to provide suitable levels of redundancy [1,17,23,37,53], possibly taking costs into account, also pose interesting questions for which the use of the survival signature might provide new solutions. In the NPI framework some of such issues have been considered, but only for systems with relatively limited structures, for which the combinatorial aspects in computations already became quite complex [1]. The theory presented in this chapter provides a framework in which these, and many other, problems can be studied for a wide variety of system structures. It will also be of interest to consider the use of the survival signature if failure data at the system level are available, possibly together with some component level data. As mentioned in Chapter 3, Bayesian inference for this situation where the system signature is used was recently presented by Aslett [9], who also considered inferring the signature from failure data, which may be relevant for black-box systems. It will be interesting to develop similar methods for survival signatures, particularly because it widens applicability of such learning methods to systems with multiple types of components.

Chapter 5

Concluding Remarks

5.1 Conclusions

This thesis reports the first work in which signatures are combined with theory of lower and upper probabilities. Nonparametric Predictive Inference (NPI) combined with signatures in the study of system reliability was presented. The NPI approach to system reliability provides a useful method for statistical inference on system reliability on the basis of limited information resulting from component testing. Chapter 2 presented the use of signatures to derive lower and upper survival functions for the failure time of systems with exchangeable components, given failure times of tested components that are exchangeable with those in the system. The imprecision in these inferences reflects the limited amount of data available.

Computing the system signature is not straightforward for larger systems, but any algorithm will provide bounds if stopped early. Chapter 3 is a sequel to Chapter 2, showing how limited information about the signature can be used to derive bounds on these lower and upper survival functions and related inferences. We call the signature within these bounds, which provides the maximum lower bound for the NPI lower and upper survival functions, the pessimistic signature, and the signature which provides the minimum upper bound for the NPI lower and upper survival functions the optimistic signature.

Chapter 4 presents a further step in the development of NPI for reliability of systems with multiple types of components, by considering system reliability evalu-

ation using the survival signature, which was recently introduced as an alternative to the system signature. The method of calculating survival signatures is developed further, and the NPI method for system reliability using this concept is proposed. New formulas have been derived and illustrated for the calculation of the survival signature for a system consisting of series and parallel subsystems. Also, new formulas have been derived and illustrated for the NPI approach to system reliability using survival signatures. The method is identical to the NPI method for the system signature for a system with a single type of components, as presented in Chapter 3, but the extension to systems with multiple types of components is crucial for application to real-world systems and networks.

5.2 Links to alternative methods

We briefly comment on some related topics, namely the possible alternatives to use (imprecise) Bayesian methods or bootstrap methods. Aslett [9] shows how the signature can be used for reliability quantification for systems and networks from a Bayesian perspective. Aslett et al. [11] present Bayesian inference using the survival signature. They consider the situation where test data are available on each type of component in a system or network, which is used to infer reliability of the system. Using the survival signature, the uncertainty in the reliability of multiple types of components can be propagated to uncertainty in the lifetime of an entire system comprising those types of components. Computations for the Bayesian approach may require the use of simulation-based methods, which may be harder to implement than the NPI approach. The Bayesian approach as presented in [9, 11] can be generalized to an imprecise Bayesian approach by taking a class of priors, this is an interesting topic for further research.

Following Walley [65], many of the imprecise probability-based contributions to statistics follow a generalized Bayesian approach, with models typically closely related to the usual Bayesian statistical methods, using a standard precise parametric sampling model but with a set of prior distributions instead of a single prior. The use of models from the exponential family is popular in conjunction with classes

of conjugate priors. In these models, updating to take new information into account is effectively done by updating all elements of the set of prior distributions as in Bayesian statistics with precise prior distributions, leading to a set of posterior distributions which forms the basis for inferences.

Williamson [68] presents a detailed overview of objective Bayesianism, under classical probability. He proposes an empirical norm and a logical norm for objective inference, and shows that these are not both satisfied for Bayesian methods. NPI enables both Williamson's norms to be satisfied as shown by Coolen [20], but with slightly reformulated norms to fit with theory of imprecise probability. The NPI-based lower and upper probabilities can be said to be sensible in the sense that the empirical probabilities are always in the intervals created by the corresponding NPI lower and upper probabilities, and that the length of such intervals decreases as a function of n, leading to precise probabilities for $n \to \infty$. NPI can be seen as an objective inference method, so strongly based on available data with only few further assumptions, precise Bayesian methods cannot achieve this. However, the latter are useful if one wants to take further information, e.g. expert knowledge, into account, in which case NPI should not be used.

The bootstrap method was introduced by Efron [42]. It is a resampling technique for estimating the distribution of statistics based on independent observations and it has been developed to work for many statistical inferences. Chernick [18] discussed the key ideas and applications of bootstrap, illustrated by applications to regression models, time series, confidence intervals and hypothesis tests. Davison and Hinkley [39] and Efron and Tibshirani [43] have developed bootstrap methods further for a range of applications. Good [47] provided a brief review of bootstrap methods together with computer code in order to put this method into practice.

Recently, Bin Himd [13] presented an alternative to the classical bootstrap method, within the NPI framework. This method is called Nonparametric Predictive Inference Bootstrap (NPI-B). In the classical bootstrap method [42], a bootstrap sample $t^* = (t_1^*, t_2^*, \ldots, t_n^*)$ is obtained by random sampling, n times, with replacement from the original sample $t_1, t_2, \ldots, t_n$. In NPI-B, the $n + 1$ intervals created by the n observations are used. One value is drawn from within these intervals and

added to the data set. Now the next value is sampled similarly, but using the $n+1$ data values. So each sampled data value is added to the data set to sample the next one. This is applied until n new bootstrap values have been sampled, these (so without the original data) from an NPI-B sample. The crucial difference from the classical bootstrap method is that an NPI-B sample does not consist of the observations from the original sample but of points from the whole possible data range, because the sampling in NPI-B is from the intervals in between the data values and also outside the data range [13]. The way of sampling observations of NPI-B, with values sampled from the intervals between the data points and new values dependent on each other, leads to greater variation in the NPI-B samples than in the bootstrap samples, and to accurate predictive inference [13]. NPI-B is fully in agreement with NPI for future order statistics [13], so variation in bootstrap samples is already reflected in the approach presented in this thesis, hence comparison with bootstrap methods is not very useful. Only if the data set is so large that our method leads to computational problem, use of NPI-B instead may need to be explored. However, it is more likely that computational challenges are with regard to deriving the survival signature, this would affect any inference method the same way.

5.3 Research challenges

Challenging topics for future research include generalization of the approach presented in Chapter 2 for test data including right-censored observations, as often occur for failure time data [33]. This first requires development of NPI for future order statistics with such data, which is a challenge indeed as Equation 1.14 cannot be applied in such a setting and simple counting arguments may need to be replaced by complex optimisation methods. Once the approach has been extended to include right-censored data, multiple comparisons are also of interest and can follow the same approach as presented in [35, 36].

Signatures can also be used for reliability quantification for systems for which only failure or non-failure upon request for functioning is of interest, so without explicit focus on failure time. Applying this to systems with exchangeable components

will be relatively straightforward and will generalize the results in [22].

As presented in Chapter 4, the survival signature is a suitable generalization of the signature to systems with multiple types of components. One may wish to decide on optimal testing in order to demonstrate a required level of system reliability, possibly taking costs and time required for testing, and corresponding constraints, into account [58]. It will also be of interest to consider possible system failure due to competing risks, where the NPI approach provides interesting new opportunities to consider unobserved or even unknown failure modes [34, 54]. Of course, the main challenges will result from the application of the new theory to large-scale real-world systems, which we expect to be more feasible with the new results presented in Chapter 4.

The Bayesian method with the system signature, presented by Aslett [9], offers the possibility to use data including failure times for the system, so not only for individual components. This is practical interest if only system failure data are available. It is not straightforward to develop the NPI approach for such data, this provides an interesting challenge for future research.

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