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TWISTS OF PLÜCKER COORDINATES AS DIMER PARTITION FUNCTIONS

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Abstract. The homogeneous coordinate ring of the Grassmannian $\mathrm{Gr}_{k,n}$ has a cluster structure defined in terms of planar diagrams known as Postnikov diagrams. The cluster corresponding to such a diagram consists entirely of Plücker coordinates. We introduce a twist map on $\mathrm{Gr}_{k,n}$, related to the Berenstein-Fomin-Zelevinsky-twist, and give an explicit Laurent expansion for the twist of an arbitrary Plücker coordinate in terms of the cluster variables associated with a fixed Postnikov diagram. The expansion arises as a (scaled) dimer partition function of a weighted version of the bipartite graph dual to the Postnikov diagram, modified by a boundary condition determined by the Plücker coordinate. We also relate the twist map to a maximal green sequence.

1. Introduction

For positive integers $k \leq n$ let $\mathrm{Gr}_{k,n}$ denote the Grassmannian of all k-dimensional vector subspaces of $\mathbb{C}^n$. The results of [39] (see also [17, 18]) prove that its homogeneous coordinate ring $\mathbb{C}[\mathrm{Gr}_{k,n}]$ has the structure of a cluster algebra which possesses a distinguished finite family of seeds $(\widetilde{\mathbf{x}}_P, \widetilde{Q}_P)$ constructed from certain planar diagrams P, known as *alternating strand diagrams* or *Postnikov diagrams*.

The extended cluster $\widetilde{\mathbf{x}}_P$ of each seed of this kind consists entirely of Plücker coordinates which, in addition to the associated quiver $\widetilde{Q}_P$, can be read off directly from the Postnikov diagram. Moreover, every Plücker coordinate occurs as an element of $\widetilde{\mathbf{x}}_P$ for some Postnikov diagram P and thus every Plücker coordinate is either a cluster variable or a coefficient. When $k = 2$ every seed is of this form and consequently every cluster variable is a Plücker coordinate. In general the homogeneous coordinate ring is of wild type — possessing infinitely many seeds and infinitely many cluster variables, which in general will not be Plücker coordinates and which are, at present, unclassified.

In this paper we consider a certain rational map from the Grassmannian to itself which we call the *twist*. This map may be pre-composed with any regular function f in $\mathbb{C}[\mathrm{Gr}_{k,n}]$ to form a twisted version $\overleftarrow{f}$; here, we consider *twisted* Plücker coordinates. Up to coefficients[1], the twist of any cluster variable is a cluster variable (see Proposition 8.10).[2]

By the Laurent Phenomenon [15], each cluster variable in $\mathbb{C}[\mathrm{Gr}_{k,n}]$ can be expressed uniquely as a Laurent polynomial in the extended cluster of any seed. In view of this, we compute Laurent

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[1] Equal, upon specializing all coefficients to 1; see Definition 8.8.

[2] We thank David Speyer for pointing this out.

expansions for twisted Plücker coordinates in the extended cluster $\widetilde{\mathbf{x}}_P$ of a seed attached to any fixed Postnikov diagram P. We express these Laurent expansions in terms of *dimer configurations* (also known as perfect matchings) for a weighted bipartite graph G subject to boundary conditions determined by the Plücker coordinate. The graph G_P is dual (in an appropriate sense) to the Postnikov diagram, and its edges are weighted by monomials taken from the extended cluster $\widetilde{\mathbf{x}}_P$; see Definition 7.1 in Section 6.

Recall that a bipartite graph is a graph whose vertices are partitioned into two types, or colours (black and white), where edges join vertices of a different colour. Such a graph is said to be *balanced* if there is an equal number of black and white vertices. A *dimer configuration* δ of a balanced bipartite graph, G, is a collection of edges of G such that each vertex of G is incident with precisely one edge in the collection. If each edge e in G is assigned a weight w_e, then we define the weight w_δ of the dimer configuration δ to be the product

$$\mathrm{w}_\delta = \prod_{e\in\delta} \mathrm{w}_e.$$

The *dimer partition function* (also known as the matching polynomial) of G is given by:

$$Đ_G = \sum_\delta \mathrm{w}_\delta,$$

where the sum is over all dimer configurations δ of G.

If P is a Postnikov diagram, the dual bipartite graph G_P is naturally embedded in a disk, with boundary vertices labelled $1, \ldots, n$. If I is a k-subset of $\{1, \ldots, n\}$, then the induced bipartite subgraph $G_P(I)$ of G obtained by removing the boundary vertices labelled by the elements of I is balanced.

Our main result is:

Theorem 1.1. *Let P be a Postnikov diagram, with corresponding seed $(\widetilde{\mathbf{x}}_P, \widetilde{Q}_P)$ and let G_P be its weighted bipartite dual graph. For any k-subset I of $\{1, \ldots, n\}$, we have:*

$$\frac{Đ_{G_P(I)}}{\prod_{x\in\mathbf{x}_P} x} = \overleftarrow{[I]},$$

where $[I]$ denotes the Plücker coordinate of $\mathrm{Gr}_{k,n}$ associated to I, $\overleftarrow{[I]}$ denotes its twist, and $\mathbf{x}_P \subseteq \widetilde{\mathbf{x}}_P$ is the (non-extended) cluster corresponding to P.

In particular, as an element of the rational function field of the Grassmannian, the ratio on the left hand side of this formula does not depend on P; this is a key step in the proof. This rational expression is in fact the Laurent expansion of the twisted Plücker coordinate $\overleftarrow{[I]}$ for the seed $(\widetilde{\mathbf{x}}_P, \widetilde{Q}_P)$, since the edge-weights which contribute to the dimer partition function in the numerator are monomials in the extended cluster $\widetilde{\mathbf{x}}_P$.

We remark that it follows from the formula in Theorem 1.1 that the twist map preserves the positive Grassmannian (see Corollary 9.5). We note that the cluster algebra structure on $\mathbb{C}[\mathrm{Gr}_{k,n}]$ and the positive Grassmannian play an important rôle in the study of scattering amplitudes via on-shell diagrams in [2].

Dimer configurations have also been used as a method for computing Laurent expansions for cluster variables for cluster algebras of finite classical type [10, 31], and for cluster algebras associated to triangulations of surfaces [9, 32, 33, 34]. Both cases involve cluster algebras of finite mutation type, the homogeneous coordinate ring of the Grassmannian $\mathrm{Gr}_{2,n}$ (up to coefficients) being common to both cases. Note that the case considered here, i.e. $\mathbb{C}[\mathrm{Gr}_{k,n}]$, is not of finite mutation type in general.

Our interest in the twist map stems partly from its close relationship with the BFZ-twist automorphism [5, 6] defined on a unipotent cell N^w in the group N of all complex $n \times n$ unipotent matrices, where w is the Grassmann permutation. Note that the BFZ-twist has a representation-theoretic formulation [21, Theorem 6] in terms of the Auslander-Reiten translate on a categorification of the cluster algebra structure on N^w using the type A_{n-1} preprojective algebra. The cell N^w is birationally equivalent to $\mathrm{Gr}_{k,n}$, so the BFZ-twist can be transported to $\mathrm{Gr}_{k,n}$. We show in Section 5 that the (transported) BFZ-twist and the twist defined in Section 2 coincide, up to coefficients.

The results of [4], together with the BFZ-ansatz [5, 6], can be used to compute Laurent expansions of BFZ-twisted minors on N^w in terms of any seed associated to a reduced expression for w. The cluster algebra structures on N^w and $\mathrm{Gr}_{k,n}$ are identified, up to coefficients, through the birational equivalence. In particular, every seed associated to a reduced expression for w corresponds to a seed attached to a Postnikov diagram. However, not every Postnikov diagram is of this form, and, in view of this, our results can be seen as a proper extension of the combined results of [4, 6]. On a combinatorial level, the approach taken by [4] is different from the approach here: specifically, the formulas in [4] involve an analysis of families of noncrossing paths in planar diagrams, while our formulas use dimer configurations. An interesting question is how to reconcile these two approaches. It is also interesting to compare the formulas obtained in this paper with the formulas in [40, §1], also given in terms of families of paths.

We also show that, up to coefficients, the twist can be implemented by a maximal green sequence (Theorem 11.17). Such sequences occur both in cluster theory [24] and in the analysis of BPS states; see [1, 11, 12]. The maximal green sequence we give here can be regarded as a two-dimensional version of the sequence given in [8] for quivers of type A, and should be compared with the mutation sequence $\tau_-\tau_+$ occurring in the Zamolodchikov periodicity conjecture [25, 36, 42] which was proved in the case $A_{k-1} \times A_{n-k-1}$ by [41]. Note that the cluster algebra of type $A_{k-1} \times A_{n-k-1}$ coincides with $\mathbb{C}[\mathrm{Gr}_{k,n}]_1$, the cluster algebra obtained from $\mathbb{C}[\mathrm{Gr}_{k,n}]$ by specializing all of the coefficients to 1 (see Definition 8.6).

The general case of the Zamolodchikov periodicity conjecture was proved by B. Keller in [25]: see the introduction to [25] for many further references relating to this conjecture. We note that a modified version of the mutation sequence appearing in [25] in the type $A_{k-1} \times A_{n-k-1}$ case corresponds to an operator product $\widehat{m}_\square$ studied in [12, §8] in the context of a $4d$ $\mathcal{N} = 2$ field theory whose quantum monodromy is related to $(\widehat{m}_\square)^k$.

We note that the quiver $\widetilde{Q}_{k,n}$ used in [39] in the analysis of the cluster algebra structure of the Grassmannian coincides with Keller's box quiver $Q\square Q'$ (where Q, Q' are alternating orientations of A_{k-1} and A_{n-k-1} respectively), together with a rule for attaching the coefficient vertices along the boundary. In this case τ_- (respectively, τ_+) is the composition of mutations at the odd (respectively, even) internal vertices of $Q\square Q'$. One can check directly that the effect of $\tau_-\tau_+$ is to shift the coefficient vertices of $\widetilde{Q}_{k,n}$ counter-clockwise by one step, while leaving the internal part, $Q\square Q'$, unchanged. As $\widetilde{Q}_{k,n}$ is the quiver of a Postnikov diagram, one concludes that the effect of $\tau_-\tau_+$ on a Plücker coordinate $[I]$ in the initial seed is to send it to $[\sigma^{-1}(I)]$, where σ is anticlockwise rotation of its indices (see Definition 11.4). Consequently, $\tau_-\tau_+$ coincides with the automorphism of $\mathbb{C}[Gr_{k,n}]$ sending $[I]$ to $[\sigma^{-1}(I)]$ for any Plücker coordinate $[I]$. This, together with Theorem 11.17, gives us the identity $(\tau^-\tau^+)^k = \gamma^2$ up to coefficients (Corollary 11.11), where γ is an automorphism of $\mathbb{C}[\mathrm{Gr}_{k,n}]_1$ corresponding to a maximal green sequence.

In [20, §13], a mutation sequence for the categorification of a unipotent cell corresponding to any Weyl group element is given; according to Keller [24, §5] this is a maximal green sequence. In the case of the Grassmann permutation this mutation sequence should also give a maximal green sequence for the Grassmannian. As well as relating our choice of maximal green sequence to the twist, we show that it passes only through seeds given by Postnikov diagrams.

The paper is organized as follows. In Section 2, we introduce the twist map on the Grassmannian. This is defined in terms of generalized cross products of the columns of the $k \times n$ matrix representing a point in the Grassmannian.

In Section 3, we show (in Proposition 3.5) that the twist of a Plücker coordinate $[I]$ given by a k-subset I which is a disjoint union of two cyclic intervals is, up to coefficients, a Plücker coordinate of the same kind, using Turnbull's identity (as expressed in the article [28]). We suspect that these are the only Plücker coordinates with this property. For example, it can be verified by hand, in the case of the Grassmannian $\mathrm{Gr}_{3,n}$, that a twisted Plücker coordinate $\overleftarrow{[I]}$, where I is not a disjoint union of two cyclic intervals, is a cluster variable which is not a Plücker coordinate (even up to coefficients).

In Section 4, we compute the double twist of a Plücker coordinate and use this to show that the twist map is periodic, up to coefficients (Proposition 4.3; see also Proposition 8.9). We then explain the relationship to the BFZ-twist in Proposition 5.7 in Section 5.

In Section 6, we recall the cluster structure of the Grassmannian as described in [39] in terms of Postnikov diagrams [35].

In Section 7 we review the definition of the bipartite graph dual to a Postnikov diagram and the blow-up and blow-down equivalences inherited from the oriented lens creation and annihilation moves for Postnikov diagrams. We introduce a scheme for weighting the edges of such a bipartite graph, with the property that the induced weighting on dimer configurations is invariant under blow-ups and blow-downs. We then fix a Postnikov diagram, P, and a k-subset I. We show, in Proposition 7.6, that the dimer partition function of $G_P(I)$, divided by the product of the elements in $\mathbf{x}_P$, is invariant under quadrilateral moves. Since any Postnikov diagram can be reached from any other by a sequence of such moves, it follows that this scaled dimer partition function is independent of the choice of Postnikov diagram.

In Section 8, we show that the main result is true for the Plücker coordinates in $\widetilde{\mathbf{x}}_{\mathcal{R}_{k,n}}$ for a regular Postnikov diagram $\mathcal{R}_{k,n}$ whose dual bipartite graph, $G_{\mathcal{R}_{k,n}}$, is, up to some boundary edges, part of a hexagonal tiling of the plane. The diagram obtained from $\mathcal{R}_{n-k,n}$ by reversing its strands (and adding crossings at the boundary) is again a Postnikov diagram, which we denote by $\mathcal{R}^*_{n-k,n}$. We prove, in Proposition 8.11, that $G_{\mathcal{R}^*_{n-k,n}}(I)$ has a unique dimer configuration whenever $[I]$ lies in $\widetilde{\mathbf{x}}_{\mathcal{R}_{k,n}}$. This k-subset, I, is a disjoint union of two cyclic intervals in $\{1, \ldots, n\}$, which allows us to compare the dimer partition function $Đ_{G_P(I)}$, where $P = \mathcal{R}^*_{n-k,n}$, with the formula for $\overleftarrow{[I]}$ given by Proposition 3.5 in Section 2.

The main result, Theorem 1.1, is shown in Section 9, using the fact that twists of Plücker coordinates and the scaled dimer partition functions both satisfy the short Plücker relations. In Section 10, we give an example. In Section 11 we consider the relationship between the twist and maximal green sequences and in Section 12 we discuss generalization to the surface case.

2. Twist

For a positive integer r, an *r-subset* of a set S is a subset of S of cardinality r.

Let $M_{k,n}(\mathbb{C})$ denote the set of complex $k \times n$ matrices. Recall that an element of the Grassmannian $\mathrm{Gr}_{k,n}$ of k-dimensional subspaces of $\mathbb{C}^n$ can be regarded as a rank k matrix $p = (p_{ij}) \in M_{k,n}$ up to left multiplication by an element of $\mathrm{GL}_k(\mathbb{C})$. The rows correspond to a choice of basis of the subspace of $\mathbb{C}^n$ and the action of $\mathrm{GL}_k(\mathbb{C})$ corresponds to a change of basis.

Each k-subset $I = \{i_1 < i_2 < \ldots < i_k\}$ of $\{1, \ldots, n\}$ defines a minor of p associated to the row-set $\{1, \ldots, k\}$ and the column-set I (written in increasing order). We denote this minor by $[I]$. Then the map taking $p \in M_{k,n}$ to the tuple consisting of all of the minors of p of this form is a well-defined map (the Plücker embedding) from $\mathrm{Gr}_{k,n}$ to the projective space $\mathbb{P}^{\binom{n}{k}-1}$, identifying $\mathrm{Gr}_{k,n}$ with a projective subvariety defined by the Plücker relations.

Let $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be the map taking i to $i-1$ reduced modulo n. Then σ induces a map on the set of k-subsets of $\{1, \ldots, n\}$ which we also denote by σ.

For $i \in \{1, \ldots, n\}$, we write $\mathbf{i}$ for the k-subset $\{\sigma^{k-1}(i), \ldots, \sigma(i), i\}$; the corresponding Plücker coordinate is denoted $[\mathbf{i}]$.

Given vectors $v_1, \ldots, v_{k-1}$ in $\mathbb{C}^k$, the *generalized cross-product* $v_1 \times \cdots \times v_{k-1}$ is the unique vector in $\mathbb{C}^k$ satisfying the constraint:

$$\Big\langle v_1 \times \cdots \times v_{k-1} \Big\| v \Big\rangle = \det(v_1, \ldots, v_{k-1}, v)$$

for all $v \in \mathbb{C}^k$, where $\langle v \,\|\, v' \rangle := v^T v'$ is the standard scalar product for $v, v' \in \mathbb{C}^k$. We interpret an empty cross product (the case $k = 1$) as 1. It follows from basic multi-linear algebra that the cross product satisfies the *contraction formula*:

$$\Big\langle v_1 \times \cdots \times v_{k-1} \Big\| v'_1 \times \cdots \times v'_{k-1} \Big\rangle = \det \begin{pmatrix} \langle v_1 \,\|\, v'_1 \rangle & \cdots & \langle v_1 \,\|\, v'_{k-1} \rangle \\ \vdots & & \vdots \\ \langle v_{k-1} \,\|\, v'_1 \rangle & \cdots & \langle v_{k-1} \,\|\, v'_{k-1} \rangle \end{pmatrix}.$$

Let $p \in M_{k,n}$ be a $k \times n$ matrix with column vectors $p_1, \ldots, p_n \in \mathbb{C}^k$.

Definition 2.1. The (left) *twist* $\overleftarrow{p} \in M_{k,n}$ is defined to be the $k \times n$ matrix whose ith column vector is:

$$\left(\overleftarrow{p}\right)_i = \varepsilon_i \cdot (p_{\sigma^{k-1}(i)} \times p_{\sigma^{k-2}(i)} \times \cdots p_{\sigma(i)}),$$

where

$$\varepsilon_i = \begin{cases} (-1)^{i(k-i)} & \text{if } i \leq k-1; \\ 1 & \text{if } i \geq k. \end{cases}$$

This may also be written:

$$\left(\overleftarrow{p}\right)_i = \begin{cases} (-1)^{k-i} p_1 \times \cdots \times p_{i-1} \times p_{i-k+1+n} \times \cdots \times p_n & \text{if } i \leq k-1, \\ p_{i-k+1} \times \cdots \times p_{i-1} & \text{if } i \geq k. \end{cases}$$

Note that $\left(\overleftarrow{p}\right)_{ri}$ is the determinant of the submatrix of p with column set $\{\sigma^{k-1}(i), \ldots, \sigma(i), i\}$ (appearing in numerical order), in which the column where the column i of p appears is replaced with the vector e_r with a 1 in its rth position and zeros everywhere else. For example, if $k = 3$ and $n = 5$, if $p = (p_{ri})_{1 \leq r \leq 3,\, 1 \leq i \leq 5}$, then

$$\overleftarrow{p} = \begin{pmatrix} \Delta_{23}^{45}(p) & -\Delta_{23}^{15}(p) & \Delta_{23}^{12}(p) & \Delta_{23}^{23}(p) & \Delta_{23}^{34}(p) \\ -\Delta_{13}^{45}(p) & \Delta_{13}^{15}(p) & -\Delta_{13}^{12}(p) & -\Delta_{13}^{23}(p) & -\Delta_{13}^{34}(p) \\ \Delta_{12}^{45}(p) & -\Delta_{12}^{15}(p) & \Delta_{12}^{12}(p) & \Delta_{12}^{23}(p) & \Delta_{12}^{34}(p) \end{pmatrix} =$$

$$\begin{pmatrix} p_{24}p_{35} - p_{25}p_{34} & -p_{21}p_{35} + p_{25}p_{31} & p_{21}p_{32} - p_{22}p_{31} & p_{22}p_{33} - p_{23}p_{32} & p_{23}p_{34} - p_{24}p_{33} \\ -p_{14}p_{35} + p_{15}p_{34} & p_{11}p_{35} - p_{15}p_{31} & -p_{11}p_{32} + p_{12}p_{31} & -p_{12}p_{33} + p_{13}p_{32} & -p_{13}p_{34} + p_{14}p_{33} \\ p_{14}p_{25} - p_{15}p_{24} & -p_{11}p_{25} + p_{15}p_{21} & p_{11}p_{22} - p_{12}p_{21} & p_{12}p_{23} - p_{13}p_{22} & p_{13}p_{24} - p_{14}p_{23} \end{pmatrix},$$

where $\Delta_X^Y(p)$ stands for the minor of p with rows X and columns Y.

Remark 2.2. Note that, if $k = 1$, then $\overleftarrow{p}$ is always the $1 \times n$ matrix whose entries are all equal to 1.

We shall use the notation $\overleftarrow{\overleftarrow{p}}$ to denote the result of applying the twist twice to p.

Lemma 2.3. *The map $p \mapsto \overleftarrow{p}$ from $M_{k,n}$ to itself induces a well-defined rational map from $\mathrm{Gr}_{k,n}$ to itself.*

Proof. We note first that for any vectors $v, v_1, \ldots, v_{k-1}$ in $\mathbb{C}^k$ and $g \in GL_k(\mathbb{C})$,

$$\begin{aligned}\Big\langle (gv_1) \times \cdots \times (gv_{k-1}) \,\Big\|\, v \Big\rangle &= \det\left(gv_1, \ldots, gv_{k-1}, v\right) \\ &= \det(g) \det\left(v_1, \ldots, v_{k-1}, g^{-1}v\right) \\ &= \det(g)\Big\langle v_1 \times \cdots \times v_{k-1} \,\Big\|\, g^{-1}v \Big\rangle \\ &= \det(g)\Big\langle (g^{-1})^T (v_1 \times \cdots \times v_{k-1}) \,\Big\|\, v \Big\rangle.\end{aligned}$$

Hence,

$$(gv_1) \times \cdots \times (gv_{k-1}) = \det(g)(g^{-1})^T (v_1 \times \cdots \times v_{k-1}).$$

Let p be a maximal rank $k \times n$ matrix and $g \in \mathrm{GL}_k(\mathbb{C})$. Then the ith column of $\overleftarrow{gp}$ is equal to

$$\begin{aligned}\varepsilon_i \cdot (gp_{\sigma^{k-1}(i)}) \times \cdots \times (gp_{\sigma(i)}) &= \varepsilon_i \det(g)(g^{-1})^T (p_{\sigma_{k-1}(i)} \times \cdots \times p_{\sigma(i)}) \\ &= \det(g)(g^{-1})^T \left(\overleftarrow{p}\right)_i.\end{aligned}$$

so the twist preserves maximal rank and does not depend on a choice of representative $p \in M_{k,n}$.

If I is a k-subset of $\{1, \ldots, n\}$, then we have, up to sign:

$$\begin{aligned}[I]\left(\overleftarrow{p}\right) &= \det(\left(\overleftarrow{p}\right)_{i_1}, \ldots, \left(\overleftarrow{p}\right)_{i_k}) \\ &= \det(p_{\sigma^{k-1}(i_1)} \times \cdots \times p_{\sigma(i_1)}, \ldots, p_{\sigma^{k-1}(i_k)} \times \cdots \times p_{\sigma(i_k)}) \\ &= \Big\langle (p_{\sigma^{k-1}(i_1)} \times \cdots \times p_{\sigma(i_1)}) \times \cdots \times (p_{\sigma^{k-1}(i_{k-1})} \times \cdots \times p_{\sigma(i_{k-1})}) \,\Big\|\, p_{\sigma^{k-1}(i_k)} \times \cdots \times p_{\sigma(i_k)} \Big\rangle.\end{aligned}$$

Applying the contraction formula, we see that this is a determinant whose entries are Plücker coordinates. This is a homogeneous polynomial map of degree independent of the choice of I. It follows that the twist is a rational map from $\mathrm{Gr}_{k,n}$ to itself. $\square$

For example, if $k = 3$ and $n = 5$, we have:

$$\begin{aligned}[124]\left(\overleftarrow{p}\right) &= \Big\langle (p_4 \times p_5) \times (p_5 \times p_1) \,\Big\|\, p_2 \times \cdots \times p_3 \Big\rangle \\ &= \begin{vmatrix} [245] & [345] \\ [125] & [135] \end{vmatrix} \\ &= [245][135] - [125][345] = [145][235],\end{aligned} \tag{2.1}$$

using a Plücker relation (and the fact that determinants change sign when columns are interchanged).

We shall see later (see Corollary 9.5) that the twist preserves the totally positive Grassmannian and the totally nonnegative Grassmannian.

The above shows that the rational map $p \mapsto \overleftarrow{p}$ induces a regular map from the affine cone of $\mathrm{Gr}_{k,n}$ to itself (given by the same polynomials). We denote the induced homomorphism from the homogeneous coordinate ring $\mathbb{C}[\mathrm{Gr}_{k,n}]$ to itself by $f \mapsto \overleftarrow{f}$.

3. Twists of Plücker coordinates

We adopt a notation similar to that of [28]. Given vectors $v_1, \ldots, v_k \in \mathbb{C}^k$, we write the determinant of the matrix whose columns are $v_1, \ldots, v_k$ by the rectangular $1 \times k$ tableau:

$$\boxed{v_1 \quad v_2 \quad \cdots \quad v_k}$$

Given $p \in \mathrm{Gr}_{k,n}$ and $i \in \{1,\dots,n\}$, let p_i denote the ith column of p. We shall sometimes just denote this by i when using the above notation. So, for example, the Plücker coordinate $[I] = [\{i_1,\dots i_k\}]$ is given by:

$$[I] = \begin{array}{|cccc|}\hline i_1 & i_2 & \cdots & i_k \\ \hline\end{array}$$

A tableau with several rows denotes the product of minors corresponding to the rows; thus, for example:

$$[I]\cdot[J] = \begin{array}{|cccc|}\hline i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \\ \hline\end{array}$$

As in [28, §1.1], we use the box notation to denote an alternating sum of products of minors, i.e. if τ is a pair consisting of a tableau T as above together with a subset A of the entries (indicated by drawing boxes around the elements of A), then τ represents the element:

$$\tau := \sum_w \mathrm{sign}(w)\cdot w(T),$$

where the sum is over cosets w in the symmetric group on the elements of A (of degree $|A|$) of the subgroup preserving the boxed elements in each row of T, with each $w(T)$ interpreted as a product of minors as above.

For example, we have:

$$\begin{array}{|ccc|}\hline \boxed{a} & b & \boxed{c} \\ d & e & f \\ \hline\end{array} = \begin{array}{|ccc|}\hline a & b & c \\ d & e & f \\ \hline\end{array};$$

$$\begin{array}{|ccc|}\hline \boxed{a} & \boxed{b} & c \\ d & e & \boxed{f} \\ \hline\end{array} = \begin{array}{|ccc|}\hline a & b & c \\ d & e & f \\ \hline\end{array} - \begin{array}{|ccc|}\hline f & b & c \\ d & e & a \\ \hline\end{array} - \begin{array}{|ccc|}\hline a & f & c \\ d & e & b \\ \hline\end{array};$$

$$\begin{array}{|ccc|}\hline \boxed{a} & \boxed{b} & c \\ \boxed{d} & e & \boxed{f} \\ \hline\end{array} = \begin{array}{|ccc|}\hline a & b & c \\ d & e & f \\ \hline\end{array} - \begin{array}{|ccc|}\hline a & d & c \\ b & e & f \\ \hline\end{array} + \begin{array}{|ccc|}\hline a & f & c \\ b & e & d \\ \hline\end{array} + \begin{array}{|ccc|}\hline b & d & c \\ a & e & f \\ \hline\end{array} - \begin{array}{|ccc|}\hline b & f & c \\ a & e & d \\ \hline\end{array} + \begin{array}{|ccc|}\hline d & f & c \\ a & e & b \\ \hline\end{array}.$$

Remark 3.1. By [28, Prop. 1.2.1], a permutation of the vectors (boxed or otherwise) lying in a single row of a tableau changes its value by the sign of the permutation. A permutation of the boxed vectors (possibly in several rows) changes the value by the sign of the permutation. A permutation of the rows of a tableau does not change its value.

We shall need Turnbull's identity, as stated in [28, Prop. 1.2.2].

Proposition 3.2. ***(Turnbull's Identity)*** *Let τ be a rectangular tableau with k columns. If the number of boxed entries in τ is greater than k then $\tau = 0$. If not, fix a row r of τ. Let A be the set of boxed entries in row r of τ and let D be the set of boxed entries in the remaining rows of τ. Let B be a subset of the unboxed entries in row r of τ of cardinality $|D|$. Let C be the set of entries in row r not in A or B. Let ν the tableau obtained from τ by carrying out the following operations:*

(a) *Exchanging the entries in B with the entries in D, but not the boxes. The boxes originally around the entries in D are not moved and box the entries of B after the move.*
(b) *Boxing the elements of C;*
(c) *Removing the boxes from the entries in A.*

Then $\tau = \nu$.

Note that the value of the tableau thus created is independent of the choice of exchange in part (a), since any two such exchanges are related by a permutation α of B followed by a permutation of D which is identical to α under an identification of B and D.

An example involving the first row is shown below. The elements of A are denoted a_1, a_2, and similarly for B, C and D. Note however that the entries in A do not have to be adjacent in the chosen row (similarly for B and C), and that the entries in D can occur anywhere in the remaining rows.

$$\boxed{\begin{array}{cccccccccc} \boxed{a_1} & \boxed{a_2} & b_1 & b_2 & b_3 & b_4 & b_5 & c_1 & c_2 & c_3 \\ \boxed{d_1} & \boxed{d_2} & \boxed{d_3} & e & f & g & h & i & j & k \\ \boxed{d_4} & \boxed{d_5} & l & m & n & p & q & r & s & t \end{array}} = \boxed{\begin{array}{cccccccccc} a_1 & a_2 & d_1 & d_2 & d_3 & d_4 & d_5 & \boxed{c_1} & \boxed{c_2} & \boxed{c_3} \\ \boxed{b_1} & \boxed{b_2} & \boxed{b_3} & e & f & g & h & i & j & k \\ \boxed{b_4} & \boxed{b_5} & l & m & n & p & q & r & s & t \end{array}}$$

The twist of a $k \times n$ matrix p in the notation defined above is the $k \times n$ matrix with entries:

$$(\overleftarrow{p})_{ri} = \begin{cases} \boxed{\begin{array}{cccccccc} 1 & 2 & \cdots & i-1 & e_r & n+i-k+1 & n+i-k+2 & \cdots & n \end{array}} & 1 \leq i \leq k; \\ \boxed{\begin{array}{ccccc} i-k+1 & i-k+2 & \cdots & i-1 & e_r \end{array}} & k+1 \leq i \leq n, \end{cases}$$

for $1 \leq r \leq k$ and $1 \leq i \leq n$.

Definition 3.3. The permutation σ (see the start of Section 2) induces a well-defined automorphism of $\mathrm{Gr}_{k,n}$:

$$\sigma(p)_{ri} = \begin{cases} p_{r,\sigma(i)}, & i \neq 1; \\ (-1)^{k-1} p_{r,\sigma(i)}, & i = 1. \end{cases}$$

This induces an automorphism of $\mathbb{C}[\mathrm{Gr}_{k,n}]$, also denoted σ.

Lemma 3.4. *Let I be a k-subset of $\{1, \ldots, n\}$ and $p \in Gr_{k,n}$. Then we have:*

(a) $\sigma([I]) = [\sigma(I)]$;

(b) $\overleftarrow{\sigma(p)} = \sigma(\overleftarrow{p})$;

(c) $\sigma\left(\overleftarrow{[I]}\right) = \overleftarrow{\sigma([I])}$.

Proof. We first prove (a). If $1 \notin I$, then

$$\sigma([I])(p) = \boxed{\begin{array}{cccc} \sigma(i_1) & \sigma(i_2) & \cdots & \sigma(i_k) \end{array}} = [\sigma(I)](p).$$

If $1 \in I$ then $i_1 = 1$, so

$$\begin{aligned} \sigma([I])(p) &= (-1)^{k-1} \boxed{\begin{array}{cccc} n & \sigma(i_2) & \cdots & \sigma(i_k) \end{array}} \\ &= \boxed{\begin{array}{cccc} \sigma(i_2) & \cdots & \sigma(i_k) & n \end{array}} \\ &= [\sigma(I)](p), \end{aligned}$$

giving (a).

For (b), we have, by Defnition 2.1, that

$$\begin{aligned} \left(\overleftarrow{\sigma(p)}\right)_i &= \varepsilon_i \cdot \sigma(p)_{\sigma^{k-1}(i)} \times \sigma(p)_{\sigma^{k-2}(i)} \times \cdots \sigma(p)_{\sigma(i)} \\ &= \begin{cases} \varepsilon_i \cdot p_{\sigma^k(i)} \times p_{\sigma^{k-1}(i)} \times \cdots p_{\sigma^2(i)}, & i \notin \{2, \ldots, k\}; \\ (-1)^{k-1} \varepsilon_i \cdot p_{\sigma^k(i)} \times p_{\sigma^{k-1}(i)} \times \cdots p_{\sigma^2(i)}, & i \in \{2, \ldots, k\}. \end{cases} \end{aligned} \tag{3.1}$$

On the other hand, we have

$$\begin{aligned}(\sigma(\overleftarrow{p}))_i &= \begin{cases}(\overleftarrow{p})_{\sigma(i)}, & i \neq 1;\\ (-1)^{k-1}(\overleftarrow{p})_{\sigma(i)}, & i = 1\end{cases}\\ &= \begin{cases}\varepsilon_{\sigma(i)} p_{\sigma^k(i)} \times p_{\sigma^{k-1}(i)} \times \cdots p_{\sigma^2(i)}, & i \neq 1;\\ (-1)^{k-1}\varepsilon_{\sigma(i)} p_{\sigma^k(i)} \times p_{\sigma^{k-1}(i)} \times \cdots p_{\sigma^2(i)}, & i = 1.\end{cases}\end{aligned}$$

If $i = 1$, the sign appearing here is $(-1)^{k-1}\varepsilon_{\sigma(1)} = (-1)^{k-1}\varepsilon_n = (-1)^{k-1} = \varepsilon_1$. If $i \in [2, k-1]$, the sign is $\varepsilon_{\sigma(i)} = (-1)^{(i-1)(k-i+1)} = (-1)^{i(k-i)}(-1)^{k-1} = (-1)^{k-1}\varepsilon_i$. If $i = k$, the sign is $\varepsilon_{\sigma(k)} = (-1)^{k-1} = (-1)^{k-1}\varepsilon_k$. If $i \notin \{1, \ldots, k\}$, the sign is $\varepsilon_{\sigma(i)} = \varepsilon_{i-1} = 1 = \varepsilon_i$. Hence, in all cases, we obtain the same sign as in (3.1). The result follows.

For (c), we have:

$$\sigma\left(\overleftarrow{[I]}\right)(p) = [I](\overleftarrow{\sigma(p)}) = [I](\sigma(\overleftarrow{p})) = \sigma([I])(\overleftarrow{p}) = \overleftarrow{\sigma([I])}(p).$$

□

We note in passing that it follows from Lemma 3.4 that σ preserves the totally positive part of $\mathrm{Gr}_{k,n}$.

Proposition 3.5. *Let I be a k-subset of $\{1, \ldots, n\}$ expressed as a disjoint union of the form $I_1 \cup I_2$, where*

$$I_1 = \{\sigma^p(i), \ldots, \sigma(i), i\}$$

and

$$I_2 = \{\sigma^q(j), \ldots, \sigma(j), j\},$$

with $p \geq 0$, $q \geq 0$ and $p + q + 2 = k$. Let J be the k-subset:

$$\{\sigma^{p+q+1}(i), \ldots, \sigma^{p+1}(i)\} \cup \{\sigma^{p+q+1}(j), \ldots, \sigma^{q+1}(j)\}.$$

Then

$$\overleftarrow{[I]} = [J] \prod_{r=1}^{p} [\sigma^r(\mathbf{i})] \prod_{r=1}^{q} [\sigma^r(\mathbf{j})]. \tag{3.2}$$

For example, we have seen (equation (2.1)) that when $k = 3$ and $n = 5$,

$$[124](\overleftarrow{p}) = [145][235].$$

Note that formula (3.2) has the interesting property that it is multiplicity-free, i.e. no Plücker coordinate appears more than once in the product.

In order to prove Proposition 3.5, we recall the following way of expressing compound determinants:

Proposition 3.6. [28, §3] *Let $\tau = (\tau_{ij})$ be an $m \times n$ rectangular tableau with entries τ_{ij}. Suppose that τ_{i,r_i} is boxed for $1 \leq i \leq m$, where $1 \leq r_i \leq n$ for all i. Let X be the $m \times m$ matrix with X_{ij} given by the single row tableau whose entries are the ith row of τ with τ_{i,r_i} replaced with τ_{j,r_j}. Then $\tau = \det(X)$.*

In [28, §3], the author gives the following examples:

$$\boxed{\begin{array}{cccc} \boxed{a} & b & c & d \\ \boxed{e} & f & g & h \end{array}} = \begin{vmatrix} \boxed{\mathbf{a} \ b \ c \ d} & \boxed{\mathbf{e} \ b \ c \ d} \\ \boxed{\mathbf{a} \ f \ g \ h} & \boxed{\mathbf{e} \ f \ g \ h} \end{vmatrix};$$

$$\boxed{\begin{array}{cccc} \boxed{a} & b & c & d \\ e & \boxed{f} & g & h \\ i & j & \boxed{k} & l \end{array}} = \begin{vmatrix} \boxed{\mathbf{a} \ b \ c \ d} & \boxed{\mathbf{f} \ b \ c \ d} & \boxed{\mathbf{k} \ b \ c \ d} \\ \boxed{e \ \mathbf{a} \ g \ h} & \boxed{e \ \mathbf{f} \ g \ h} & \boxed{e \ \mathbf{k} \ g \ h} \\ \boxed{i \ j \ \mathbf{a} \ l} & \boxed{i \ j \ \mathbf{f} \ l} & \boxed{i \ j \ \mathbf{k} \ l} \end{vmatrix}.$$

Proof of Proposition 3.5. Let I, I_1, I_2, J be as in the statement of Proposition 3.5. By Lemma 3.4, if Proposition 3.5 holds for I, I_1, I_2, J, then it also holds for $\sigma(I), \sigma(I_1), \sigma(I_2), \sigma(J)$. Hence, we may assume that $I_1 = \{1, \ldots, p+1\}$ and $I_2 = \{j-q, \ldots, j\}$ where $1 \leq p+1 < \sigma^q(j) = j-q \leq j \leq n$. We consider two possible cases.

Case I: $j - q > k$.

By Proposition 3.6:

$$\overleftarrow{[I]} = \boxed{\begin{array}{ccccccc} \boxed{e_1} & n-k+2 & n-k+3 & & \cdots & n-1 & n \\ 1 & \boxed{e_2} & n-k+3 & & \cdots & n-1 & n \\ 1 & 2 & \boxed{e_3} & & \cdots & n-1 & n \\ \vdots & & \ddots & & & \vdots & \\ 1 & \cdots & p & \boxed{e_{p+1}} & \cdots & n-1 & n \\ j-q-k+1 & & \cdots & & & j-q-1 & \boxed{e_{p+2}} \\ j-q-k+2 & & \cdots & & & j-q & \boxed{e_{p+3}} \\ \vdots & & & & & \vdots & \\ j-k+1 & & \cdots & & & j-1 & \boxed{e_k} \end{array}}$$

Applying Turnbull's identity (Proposition 3.2) for the first row with B given by the set of all unboxed entries in the first row, we have:

$$\overleftarrow{[I]} = \boxed{\begin{array}{ccccccc} e_1 & e_2 & e_3 & & \cdots & e_{k-1} & e_k \\ 1 & \boxed{n-k+2} & n-k+3 & & \cdots & n-1 & n \\ 1 & 2 & \boxed{n-k+3} & & \cdots & n-1 & n \\ \vdots & & \ddots & & & \vdots & \\ 1 & \cdots & p & \boxed{n-k+p+1} & \cdots & n-1 & n \\ j-q-k+1 & & \cdots & & & j-q-1 & \boxed{n-k+p+2} \\ j-q-k+2 & & \cdots & & & j-q & \boxed{n-k+p+3} \\ \vdots & & & & & \vdots & \\ j-k+1 & & \cdots & & & j-1 & \boxed{n} \end{array}}$$

Since rows 2 to $p+1$ contain the entries $n-k+p+2, n-k+p+3, \ldots, n$ (not in boxes), any permutation of the entries in the boxes giving rise to a non-zero product of minors must insert $n-k+2, n-k+3, \ldots, n-k+p+1$ into the boxes in rows 2 to $p+1$. Since $n-k+3, n-k+4, \ldots, n$ are non-boxed entries in row 2, $n-k+2$ must go in row 2 (to get a non-zero term). Using similar arguments for $n-k+3, \ldots, n-k+p+1$, we see that any permutation giving rise to a non-zero term must fix the boxed elements in rows $2, \dots, p+1$, so we can remove those boxes. We can also remove the first row, as it is equal to 1. Hence,

$$\overleftarrow{[I]} = \boxed{\begin{array}{cccccccc}
1 & n{-}k{+}2 & n{-}k{+}3 & & \cdots & & n{-}1 & n \\
1 & 2 & n{-}k{+}3 & & \cdots & & n{-}1 & n \\
\vdots & & \ddots & & & & \vdots & \\
1 & \cdots & p & n{-}k{+}p{+}1 & \cdots & & n{-}1 & n \\
j{-}q{-}k{+}1 & & \cdots & & & & j{-}q{-}1 & \boxed{n{-}k{+}p{+}2} \\
j{-}q{-}k{+}2 & & \cdots & & & & j{-}q & \boxed{n{-}k{+}p{+}3} \\
\vdots & & & & & & \vdots & \\
j{-}k{+}1 & & \cdots & & & & j{-}1 & \boxed{n}
\end{array}}$$

Applying Turnbull's identity for row $p+1$ (the first row containing boxes), taking B to be the first q entries in row $p+1$, we obtain:

$$\overleftarrow{[I]} = \boxed{\begin{array}{cccccccccc}
1 & n{-}k{+}2 & n{-}k{+}3 & & & \cdots & & & n{-}1 & n \\
1 & 2 & n{-}k{+}3 & & & \cdots & & & n{-}1 & n \\
\vdots & & \ddots & & & & & & & \vdots \\
1 & \cdots & p & n{-}k{+}p{+}1 & & \cdots & & & n{-}1 & n \\
n{-}k{+}p{+}3 & \cdots & & & n & \boxed{j{-}k{+}1} & \cdots & & \boxed{j{-}q{-}1} & n{-}k{+}p{+}2 \\
j{-}q{-}k{+}2 & & & \cdots & & & & & j{-}q & \boxed{j{-}q{-}k{+}1} \\
\vdots & & & & & & & & & \vdots \\
j{-}k{+}1 & & & \cdots & & & & & j{-}1 & \boxed{j{-}k}
\end{array}},$$

noting that the n in the row with first entry $n-k+p+3$ could be to the left of the p in the row above.

Note that $j-k+1, \dots, j-q-1$ occur in the last q rows, so if the boxed entries in the row beginning $n-k+p+3$ are permuted into one of these rows, we get a zero term. Hence the boxed entries in the last column must be permuted within this column. But each such entry cannot be permuted into an earlier row (without getting zero), so they must be fixed, and we can remove the boxes in the last column. But then all the remaining boxes lie in a single row, and we are left only with the identity permutation (as we ignore permutations of boxes in a single row), and thus we can remove all of the boxes to obtain:

$$\overleftarrow{[I]} = \boxed{\begin{array}{cccccccccc}
1 & n{-}k{+}2 & n{-}k{+}3 & & & \cdots & & & n{-}1 & n \\
1 & 2 & n{-}k{+}3 & & & \cdots & & & n{-}1 & n \\
\vdots & & \ddots & & & & & & & \vdots \\
1 & \cdots & p & n{-}k{+}p{+}1 & & \cdots & & & n{-}1 & n \\
n{-}k{+}p{+}3 & & \cdots & & n & j{-}k{+}1 & \cdots & & j{-}q{-}1 & n{-}k{+}p{+}2 \\
j{-}q{-}k{+}2 & & & \cdots & & & & & j{-}q & j{-}q{-}k{+}1 \\
\vdots & & & & & & & & & \vdots \\
j{-}k{+}1 & & & \cdots & & & & & j{-}1 & j{-}k
\end{array}}.$$

We may then permute the entries in the last rows, applying a cyclic permutation once to each of the last q rows and a cyclic permutation q times to the row immediately above them. This gives us a sign contribution, $(-1)^{(k-1)(q+q)} = 1$, and we obtain, noting that $n-k+p+2 = n-q$:

$$\overleftarrow{[I]} = \boxed{\begin{array}{cccccccccc}
1 & n-k+2 & n-k+3 & & & & \cdots & & n-1 & n \\
1 & 2 & n-k+3 & & & & \cdots & & n-1 & n \\
\vdots & & \ddots & & & & & & & \vdots \\
1 & \cdots & p & n-k+p+1 & & & \cdots & & n-1 & n \\
j-k+1 & & \cdots & & j-q-1 & n-q & \cdots & & n-1 & n \\
j-q-k+1 & & & \cdots & & & & & & j-q \\
\vdots & & & & & & & & & \vdots \\
j-k & & & \cdots & & & & & & j-1
\end{array}}$$

$$= [J]\prod_{r=1}^{p}[\mathbf{r}]\prod_{r=j-q}^{j-1}[\mathbf{r}],$$

as required.

Case II: $j-q \leq k \leq j$.

By Proposition 3.6:

$$\overleftarrow{[I]} = \boxed{\begin{array}{cccccccccccc}
\boxed{e_1} & n-k+2 & n-k+3 & & & & \cdots & & & & n-1 & n \\
1 & \boxed{e_2} & n-k+3 & & & & \cdots & & & & n-1 & n \\
1 & 2 & \boxed{e_3} & & & & \cdots & & & & n-1 & n \\
\vdots & & \ddots & & & \vdots & & & & & & \\
1 & \cdots & p & \boxed{e_{p+1}} & & & \cdots & & & & n-1 & n \\
1 & & \cdots & & j-q-2 & j-q-1 & \boxed{e_{p+2}} & j-q-k+1+n & & \cdots & n-1 & n \\
1 & & \cdots & & & j-q-1 & j-q & \boxed{e_{p+3}} & j-q-k+2+n & \cdots & n-1 & n \\
\vdots & & & & & & & & \ddots & & & \vdots \\
1 & & & & & \cdots & & & & & k-1 & \boxed{2k-j} \\
2 & & & & & \cdots & & & & & k & \boxed{2k+1-j} \\
\vdots & & & & & & & & & & & \vdots \\
j-k+1 & & & & & \cdots & & & & & j-1 & \boxed{k}
\end{array}}$$

Applying Turnbull's identity for the first row, taking B to be the set of all unboxed entries in the first row, we obtain that $\overleftarrow{[I]}$ is given by:

$$\boxed{\begin{array}{cccccccccccc}
e_1 & e_2 & e_3 & & & & \cdots & & & & e_{n-1} & e_n \\
1 & \boxed{n-k+2} & n-k+3 & & & & \cdots & & & & n-1 & n \\
1 & 2 & \boxed{n-k+3} & & & & \cdots & & & & n-1 & n \\
\vdots & & \ddots & & & \vdots & & & & & & \\
1 & \cdots & p & \boxed{n-k+p+1} & & & \cdots & & & & n-1 & n \\
1 & & \cdots & & j-q-2 & j-q-1 & \boxed{n-k+p+2} & j-q-k+1+n & & \cdots & n-1 & n \\
1 & & \cdots & & & j-q-1 & j-q & \boxed{n-k+p+3} & j-q-k+2+n & \cdots & n-1 & n \\
\vdots & & & & & & & & \ddots & & & \vdots \\
1 & & & & & \cdots & & & & & k-1 & \boxed{n+k-j} \\
2 & & & & & \cdots & & & & & k & \boxed{n+k+1-j} \\
\vdots & & & & & & & & & & & \vdots \\
j-k+1 & & & & & \cdots & & & & & j-1 & \boxed{n}
\end{array}}$$

As in Case I, we can remove row 1 and the boxes in rows 2 to $p+1$ to obtain that $\overleftarrow{[I]}$ is given by:

$$
\boxed{\begin{array}{ccccccccc}
1 & n{-}k{+}2 & n{-}k{+}3 & & \cdots & & & n{-}1 & n \\
1 & 2 & n{-}k{+}3 & & \cdots & & & n{-}1 & n \\
\vdots & & \ddots & \vdots & & & & & \\
1 & \cdots & p & n{-}k{+}p{+}1 & \cdots & & & n{-}1 & n \\
1 & \cdots & j{-}q{-}2 & j{-}q{-}1 & \boxed{n{-}k{+}p{+}2} & j{-}q{-}k{+}1{+}n & \cdots & n{-}1 & n \\
1 & \cdots & & j{-}q{-}1 & j{-}q & \boxed{n{-}k{+}p{+}3} & j{-}q{-}k{+}2{+}n\ \cdots & n{-}1 & n \\
1 & \cdots & & & j{-}q & j{-}q{+}1 & \boxed{n{-}k{+}p{+}4}\ \cdots & n{-}1 & n \\
\vdots & & & & & & \ddots & & \vdots \\
1 & & & \cdots & & & & k{-}1 & \boxed{n{+}k{-}j} \\
2 & & & \cdots & & & & k & \boxed{n{+}k{+}1{-}j} \\
\vdots & & & & & & & & \vdots \\
j{-}k{+}1 & & & \cdots & & & & j{-}1 & \boxed{n}
\end{array}}
$$

Applying an appropriate cyclic permutation to row $p+1$, we obtain that $\overleftarrow{[I]}$ is $(-1)^{(k-1)(j-q-1)}$ times the following:

$$
\boxed{\begin{array}{ccccccccccc}
1 & n{-}k{+}2 & n{-}k{+}3 & & & & \cdots & & & n{-}1 & n \\
1 & 2 & n{-}k{+}3 & & & & \cdots & & & n{-}1 & n \\
\vdots & & \ddots & & & \vdots & & & & & \\
1 & \cdots & p & n{-}k{+}p{+}1 & & & \cdots & & & n{-}1 & n \\
\boxed{n{-}k{+}p{+}2} & j{-}q{-}k{+}1{+}n & \cdots & n{-}1 & n & 1 & \cdots & & & j{-}q{-}2 & j{-}q{-}1 \\
1 & & \cdots & & & j{-}q & \boxed{n{-}k{+}p{+}3} & j{-}q{-}k{+}2{+}n\ \cdots & & n{-}1 & n \\
1 & & \cdots & & & j{-}q & j{-}q{+}1 & \boxed{n{-}k{+}p{+}4}\ \cdots & & n{-}1 & n \\
\vdots & & & & & & & \ddots & & & \vdots \\
1 & & & & & \cdots & & & & k{-}1 & \boxed{n{+}k{-}j} \\
2 & & & & & \cdots & & & & k & \boxed{n{+}k{+}1{-}j} \\
\vdots & & & & & & & & & & \vdots \\
j{-}k{+}1 & & & & & \cdots & & & & j{-}1 & \boxed{n}
\end{array}}
$$

Applying Turnbull's identity to row $p+1$ (the first row containing boxed elements) with B given by the entries in row $p+1$ in columns $2,\dots,q+1$, we obtain that $\overleftarrow{[I]}$ is $(-1)^{(k-1)(j-q-1)}$ times the following:

$$
\begin{array}{|cccccccccccc|}
\hline
1 & n-k+2 & n-k+3 & & & & \cdots & & & & n-1 & n \\
1 & 2 & n-k+3 & & & & \cdots & & & & n-1 & n \\
\vdots & & \ddots & & & \vdots & & & & & & \\
1 & \cdots & p & n-k+p+1 & & & \cdots & & & & n-1 & n \\
n-k+p+2 & n-k+p+3 & \cdots & & n & \boxed{j-k+1} & & \cdots & & & \boxed{j-q-2} & \boxed{j-q-1} \\
1 & & \cdots & & & & j-q & \boxed{j-q-k+1+n} & j-q-k+2+n & \cdots & n-1 & n \\
1 & & \cdots & & & & j-q & j-q+1 & \boxed{j-q-k+2+n} & \cdots & n-1 & n \\
\vdots & & & & & & & & \ddots & & & \vdots \\
1 & & & & & \cdots & & & & & k-1 & \boxed{n} \\
2 & & & & & \cdots & & & & & k & \boxed{1} \\
\vdots & & & & & & & & & & & \vdots \\
j-k+1 & & & & & \cdots & & & & & j-1 & \boxed{j-k} \\
\hline
\end{array}
$$

As in Case (I), we can remove all the boxes, to obtain that $\overleftarrow{[I]}$ is $(-1)^{(k-1)(j-q-1)}$ times the following:

$$
\begin{array}{|cccccccccccc|}
\hline
1 & n-k+2 & n-k+3 & & & & \cdots & & & & n-1 & n \\
1 & 2 & n-k+3 & & & & \cdots & & & & n-1 & n \\
\vdots & & \ddots & & & \vdots & & & & & & \\
1 & \cdots & p & n-k+p+1 & & & \cdots & & & & n-1 & n \\
n-k+p+2 & n-k+p+3 & \cdots & & n & j-k+1 & & \cdots & & & j-q-2 & j-q-1 \\
1 & & \cdots & & & & j-q & j-q-k+1+n & j-q-k+2+n & \cdots & n-1 & n \\
1 & & \cdots & & & & j-q & j-q+1 & j-q-k+2+n & \cdots & n-1 & n \\
\vdots & & & & & & & & \ddots & & & \vdots \\
1 & & & & & \cdots & & & & & k-1 & n \\
2 & & & & & \cdots & & & & & k & 1 \\
\vdots & & & & & & & & & & & \vdots \\
j-k+1 & & & & & \cdots & & & & & j-1 & j-k \\
\hline
\end{array}
$$

We may then permute the entries in the last rows, applying a cyclic permutation once to each of the last $j-k$ rows and a cyclic permutation $q+1$ times to the row immediately above them. This gives us a sign contribution, $(-1)^{(k-1)(q+1+j-k)}$. Combined with the sign $(-1)^{(k-1)(j-q-1)}$ we already have, this becomes 1. We obtain, noting that $n-k+p+2 = n-q$:

$$
\begin{aligned}
\overleftarrow{[I]} &= \begin{array}{|cccccccccccc|}
\hline
1 & n-k+2 & n-k+3 & & & & \cdots & & & & n-1 & n \\
1 & 2 & n-k+3 & & & & \cdots & & & & n-1 & n \\
\vdots & & \ddots & & & \vdots & & & & & & \\
1 & \cdots & p & n-k+p+1 & & & \cdots & & & & n-1 & n \\
j-k+1 & \cdots & j-q-2 & j-q-1 & n-q & n-q+1 & & & \cdots & & n-1 & n \\
1 & & \cdots & & & & j-q & j-q-k+1+n & j-q-k+2+n & \cdots & n-1 & n \\
1 & & \cdots & & & & j-q & j-q+1 & j-q-k+2+n & \cdots & n-1 & n \\
\vdots & & & & & & & & \ddots & & & \vdots \\
1 & & & & & \cdots & & & & & k-1 & n \\
1 & & & & & \cdots & & & & & k-1 & k \\
\vdots & & & & & & & & & & & \vdots \\
j-k & j-k+1 & & & & \cdots & & & & & j-2 & j-1 \\
\hline
\end{array} \\
&= [J] \prod_{r=1}^{p} [\mathbf{r}] \prod_{r=j-q}^{j-1} [\mathbf{r}],
\end{aligned}
$$

as required. The proposition is proved. $\square$

Remark 3.7. If $k > 1$, we can take I and J in Proposition 3.5 to be non-empty disjoint subsets whose union is $[\mathbf{i}]$. We obtain:

$$\overleftarrow{[\mathbf{i}]} = [\sigma^{k-1}(\mathbf{i})] \cdots [\sigma(\mathbf{i})]. \tag{3.3}$$

Note that it is easy to check this directly in the case $k = 1$: the left hand side evaluates to 1 and we view the product on the right-hand-side as an empty product.

By (3.3), the domain of the twist contains the subvariety $\mathrm{Gr}^*_{k,n}$ consisting of matrices p for which $[\mathbf{i}](p) \neq 0$ for $1 \leq i \leq n$; furthermore, this subvariety is stable under the twist. Note that this variety is the largest *open positroid subvariety* of $\mathrm{Gr}_{k,n}$ (see [30, §1]).

4. Periodicity

In this section, we use the cross product formulation to show that, if $k > 1$, then applying the twist $2n$ times to a Plücker coordinate gives back the same Plücker coordinate multiplied by a monomial in the Plücker coordinates $[\mathbf{i}]$, $i \in \{1, 2, \dots, n\}$.

Lemma 4.1. *Suppose that $k > 1$, and let $p \in M_{k,n}$ be a $k \times n$ matrix with $k > 1$. Then, for $1 \leq i \leq n$, we have:*

$$\left(\overleftarrow{p}\right)_i = (-1)^{k-1} \varepsilon_{\sigma(i)} \varepsilon_i [\sigma^2(\mathbf{i})](p) \cdots [\sigma^{k-1}(\mathbf{i})](p) \cdot p_{\sigma^k(i)}. \tag{4.1}$$

Proof. The formula (4.1) is a polynomial identity in the matrix entries of p and so it is enough to verify this formula when p varies over any fixed non-empty Zariski-open subset of the variety of all $k \times n$ matrices. We restrict attention to $k \times n$ matrices p satisfying the (open) determinantal conditions $[\mathbf{i}](p) \neq 0$ for all $i \in \{1, \dots, n\}$. In particular, this implies that each column vector $p_i \neq 0$ for $i \in \{1, \dots, n\}$.

We begin by computing the scalar product of the ith column $\left(\overleftarrow{p}\right)_i$ of $\overleftarrow{p}$ with $p_{\sigma^k(i)}$. We have:

(4.2)
$$\begin{aligned}
\left\langle \left(\overleftarrow{p}\right)_i \,\middle\|\, p_{\sigma^k(i)} \right\rangle &= \varepsilon_i \left\langle \left(\overleftarrow{p}\right)_{\sigma^{k-1}(i)} \times \cdots \times \left(\overleftarrow{p}\right)_{\sigma(i)} \,\middle\|\, p_{\sigma^k(i)} \right\rangle \\
&= \varepsilon_i \det\left(\left(\overleftarrow{p}\right)_{\sigma^{k-1}(i)}, \dots, \left(\overleftarrow{p}\right)_{\sigma(i)}, p_{\sigma^k(i)} \right) \\
&= (-1)^{k-1} \varepsilon_i \left\langle p_{\sigma^k(i)} \times \left(\overleftarrow{p}\right)_{\sigma^{k-1}(i)} \times \cdots \times \left(\overleftarrow{p}\right)_{\sigma^2(i)} \,\middle\|\, \left(\overleftarrow{p}\right)_{\sigma(i)} \right\rangle \\
&= (-1)^{k-1} \varepsilon_{\sigma(i)} \varepsilon_i \left\langle p_{\sigma^k(i)} \times \left(\overleftarrow{p}\right)_{\sigma^{k-1}(i)} \times \cdots \times \left(\overleftarrow{p}\right)_{\sigma^2(i)} \,\middle\|\, p_{\sigma^k(i)} \times \cdots \times p_{\sigma^2(i)} \right\rangle.
\end{aligned}$$

For $1 \leq s, t \leq k-1$, we have:

$$\begin{aligned}
\left\langle \left(\overleftarrow{p}\right)_{\sigma^s(i)} \,\middle\|\, p_{\sigma^t(i)} \right\rangle &= \varepsilon_{\sigma^s(i)} \left\langle p_{\sigma^{s+k-1}(i)} \times \cdots \times p_{\sigma^{s+1}(i)} \,\middle\|\, p_{\sigma^t(i)} \right\rangle \\
&= \varepsilon_{\sigma^s(i)} \det\left(p_{\sigma^{s+k-1}(i)}, \dots, p_{\sigma^{s+1}(i)}, p_{\sigma^t(i)} \right).
\end{aligned}$$

If $2 \leq s < t \leq k-1$, then this is zero since the column $p_{\sigma^t(i)}$ is repeated. If $s = t$, we obtain

$$\begin{aligned}
\left\langle \left(\overleftarrow{p}\right)_{\sigma^s(i)} \,\middle\|\, p_{\sigma^s(i)} \right\rangle &= \varepsilon_{\sigma^s(i)} \left\langle p_{\sigma^{s+k-1}(i)} \times \cdots \times p_{\sigma^{s+1}(i)} \,\middle\|\, p_{\sigma^s(i)} \right\rangle \\
&= [\sigma^s(\mathbf{i})](p).
\end{aligned}$$

Consequently, after applying the contraction formula, we obtain

$$\left\langle \left(\overset{\leftleftarrows}{p}\right)_i \,\middle\|\, p_{\sigma^k(i)} \right\rangle = (-1)^{k-1}\varepsilon_{\sigma(i)}\varepsilon_i \det \begin{pmatrix} |p_{\sigma^k(i)}|^2 & \cdots & & * \\ \vdots & [\sigma^{k-1}(\mathbf{i})](p) & & \\ & & \ddots & \\ 0 & & & [\sigma^2(\mathbf{i})](p) \end{pmatrix}.$$

It follows that

$$\left\langle \left(\overset{\leftleftarrows}{p}\right)_i \,\middle\|\, p_{\sigma^k(i)} \right\rangle = (-1)^{k-1}\varepsilon_{\sigma(i)}\varepsilon_i \cdot [\sigma^2(\mathbf{i})](p) \cdots [\sigma^{k-1}(\mathbf{i})](p) \cdot |p_{\sigma^k(i)}|^2,$$

which, in view of our assumptions, is non-zero. From this and the second line of (4.2) we may conclude that the vectors

$$\left(\overleftarrow{p}\right)_{\sigma^{k-1}(i)}, \ldots, \left(\overleftarrow{p}\right)_{\sigma(i)}, p_{\sigma^k(i)}$$

form a basis for $\mathbb{C}^k$. Since

$$\left(\overset{\leftleftarrows}{p}\right)_i = \varepsilon_i \left(\left(\overleftarrow{p}\right)_{\sigma^{k-1}(i)} \times \cdots \times \left(\overleftarrow{p}\right)_{\sigma(i)} \right)$$

is clearly orthogonal to each of the basis vectors

$$\left(\overleftarrow{p}\right)_{\sigma^{k-1}(i)}, \ldots, \left(\overleftarrow{p}\right)_{\sigma(i)}$$

(as is $p_{\sigma^k(i)}$) it follows that

$$\left(\overset{\leftleftarrows}{p}\right)_i = (-1)^{k-1}\varepsilon_{\sigma(i)}\varepsilon_i [\sigma^2(\mathbf{i})](p) \cdots [\sigma^{k-1}(\mathbf{i})](p) \cdot p_{\sigma^k(i)},$$

and we are done. □

Corollary 4.2. *Let I be a k-subset of $\{1, \ldots, n\}$, with $k > 1$. Then we have:*

$$\overset{\leftleftarrows}{[I]} = [\sigma^k(I)] \cdot \prod_{i \in I} [\sigma^2(\mathbf{i})] \cdots [\sigma^{k-1}(\mathbf{i})].$$

Proof. By Lemma 4.1, we have:

$$\overset{\leftleftarrows}{[I]} = (-1)^{s(k-s)} [\sigma^k(I)] \cdot \prod_{i \in I} (-1)^{k-1}\varepsilon_{\sigma(i)}\varepsilon_i [\sigma^2(\mathbf{i})] \cdots [\sigma^{k-1}(\mathbf{i})],$$

where

$$s = |\{i \in I \, : i \leq k\}|.$$

An easy computation shows that:

$$(-1)^{k-1}\varepsilon_{\sigma(i)}\varepsilon_i = \begin{cases} 1, & \text{if } i \leq k; \\ (-1)^{k-1}, & \text{otherwise.} \end{cases}$$

Hence

$$(-1)^{s(k-s)} \prod_{i \in I} (-1)^{k-1}\varepsilon_{\sigma(i)}\varepsilon_i = (-1)^{s(k-s)}(-1)^{(k-s)(k-1)} = (-1)^{k(k-1)-s(s-1)} = 1,$$

and the result follows. □

Proposition 4.3. *Suppose that $k > 1$ and I is a k-subset of $\{1, \ldots, n\}$. Then, applying the twist $2n$ times to $[I] \in \mathbb{C}[\mathrm{Gr}_{k,n}]$ gives $[I]$ multiplied by a monomial in the Plücker coordinates $[\mathbf{i}]$, $i \in \{1, \ldots, n\}$.*

Proof. The result follows from Corollary 4.2 and the fact that the twist of a Plücker coordinate of the form $[\mathbf{i}]$ with $1 \leq i \leq n$ is a monomial in the Plücker coordinates of the same form (see Remark 3.7). $\square$

We will give an cluster algebra-theoretic periodicity statement for the twist in Proposition 8.9.

5. Relationship to the BFZ-twist

Our aim in this section is to the explain the relationship between the twist discussed in Section 2 and the BFZ-twist map [5, 6]. We consider the quotient map from $SL_n(\mathbb{C})$ to the Grassmannian, restricted to the appropriate unipotent cell. The image of this map is the open positroid variety $\mathrm{Gr}^*_{k,n-k}$, and we give an explicit inverse from $\mathrm{Gr}^*_{k,n-k}$ to the unipotent cell (Theorem 5.5). This allows us to give a formula (Proposition 5.7) relating the twist defined in Section 2 and the BFZ-twist.

Let $G = \mathrm{SL}_n(\mathbb{C})$, and let N denote the unipotent group of all complex $n \times n$ unipotent matrices in G. The BFZ-*twist* is a regular automorphism defined on a stratum of N known as a *unipotent cell*, defined as follows.

Let T denote the torus of diagonal matrices in $SL_n(\mathbb{C})$. The Weyl group $W = N_{SL_n(\mathbb{C})}(T)/T$ of $SL_n(\mathbb{C})$ is the symmetric group S_n. For each $w \in W$ we shall denote by $\dot{w}$ a lifting of w to $N_{SL_n(\mathbb{C})}(T)$. We shall often write this just as w if there is no ambiguity.

Given $w \in S_n$, the unipotent cell N^w associated to w is defined as the intersection $N \cap B_- w B_-$; here B_- is the group of all invertible complex $n \times n$ lower triangular matrices. Each unipotent cell is a quasi-affine complex algebraic variety whose dimension is $\ell(w)$. Given an element $g \in N^w$ and its transpose $g^T \in B_-$ the intersection $N \cap B_- w g^T$ always consists a single element, denoted $\eta(g)$. The map $g \mapsto \eta(g)$ defines a regular automorphism of N^w which is called the *BFZ-twist*.

The concept was introduced as a tool for evaluating factorizations of unipotent matrices within a fixed cell N^w into elementary *Jacobi matrices*. More specifically, a combinatorial ansatz expresses the complex parameters associated to a unique factorization of a (generic) element in a fixed unipotent cell as a Laurent monomial in matrix minors of the corresponding (inverse) twisted element.

The Grassmannian $\mathrm{Gr}_{k,n}$ inherits a birational version of the twist from the unipotent cell N^w associated to the Grassmann permutation w given by $w(i) = i + k \mod n$ for $i \in \{1 \ldots n\}$ via the restriction q of the natural projection from G to $\mathrm{Gr}_{k,n}$, defined by mapping an element $g \in \mathrm{SL}_n(\mathbb{C})$ to the $k \times n$ submatrix given by its first k rows, as we shall now explain. Note that q can be identified with the quotient map $G \to P_- \backslash G$ for a maximal parabolic subgroup P_-.

For subsets I, J of $\{1, \ldots, n\}$ of the same cardinality, let $\Delta_{I,J}$ denote the corresponding minor, as a function on $SL_n(\mathbb{C})$, with row-set I and column-set J. We first check that the image of q is contained in the open positroid variety $\mathrm{Gr}^*_{k,n}$ in $\mathrm{Gr}_{k,n}$ defined by the non-vanishing of the Plücker coordinates $[\mathbf{i}]$ for $i \in \{1, \ldots, n\}$.

Lemma 5.1. *Let $x \in N^w$. Then $q(x) \in \mathrm{Gr}^*_{k,n}$.*

Proof. Let $x \in N^w$. We can write x in the form $y_1 w y_2$, where $y_1, y_2 \in B_-$. We must show that $\Delta_{\{1,\ldots,k\},\mathbf{r}}(x) \neq 0$, for $1 \leq r \leq n$. Suppose first that $1 \leq r \leq k-1$. Then, first using the fact that $x \in N$ and then using the Cauchy-Binet formula, we have:

$$\begin{aligned}
\Delta_{\{1,\ldots,k\},\{1,\ldots,r\}\cup\{r-k+1+n,\ldots,n\}]}(x) &= \Delta_{\{r+1,\ldots,k\},\{r-k+1+n,\ldots,n\}]}(x) \\
&= \sum_{I,J} \Delta_{\{r+1,\ldots,k\}],I}(y_1)\Delta_{I,J}(w)\Delta_{J,\{r-k+1+n,\ldots,n\}]}(y_2),
\end{aligned}$$

where the sum is over all $(k-r)$-subsets I, J of $\{1, \ldots, n\}$. Since $y_2 \in B_-$, $\Delta_{J,\{r-k+1+n,\ldots n\}}(y_2)$ is only non-zero if $J = \{r-k+1+n, \ldots, n\}$. Furthermore, $\Delta_{I,J}(w)$ is non-zero if and only if

$I = w(J) = \{r+1, \ldots, k\}$. We also have $\Delta_{\{r+1,\ldots,k\},\{r+1,\ldots,k\}}(y_1) \neq 0$, since $y_1 \in B_-$. It follows that $\Delta_{\{1,\ldots,k\},\{1,\ldots,r\}\cup\{r-k+1+n,\ldots,n\}}(x) \neq 0$, as required.

Suppose next that $k \leq r \leq n$. Then, first using the fact that $x \in N$ and then using the Cauchy-Binet formula, we have:

$$\begin{aligned}\Delta_{\{1,\ldots k\},\{r-k+1,\ldots r\}}(x) &= \Delta_{\{1,\ldots,k\}\cup\{r+1,\ldots,n\},\{r-k+1,\ldots,n\}}(x) \\ &= \sum_{I,J} \Delta_{\{1,\ldots,k\}\cup\{r+1,\ldots,n\},I}(y_1)\Delta_{I,J}(w)\Delta_{J,\{r-k+1,\ldots,n\}}(y_2),\end{aligned}$$

where the sum is over all $(k+n-r)$-subsets I, J of $\{1, \ldots, n\}$. Since $y_2 \in B_-$, $\Delta_{J,\{r-k+1,\ldots,n\}}(y_2)$ is only non-zero if $J = \{r-k+1, \ldots, n\}$. Furthermore, $\Delta_{I,J}(w)$ is non-zero if and only if $I = w(J) = \{1, \ldots, k\} \cup \{r+1, \ldots, n\}$. We also have $\Delta_{\{1,\ldots,k\}\cup\{r+1,\ldots,n\},\{1,\ldots,k\}\cup\{r+1,\ldots,n\}}(y_1) \neq 0$, since $y_1 \in B_-$. It follows that $\Delta_{\{1,\ldots,k\},\{r-k+1,\ldots,r\}}(x) \neq 0$, as required. We are done. □

Let $\varphi : \mathrm{Gr}^*_{k,n} \to N$ be the map defined by

$$\varphi(p)_{ij} \;=\; \begin{cases} \dfrac{\big[\mathbf{i} - \{i\} \cup \{j\}\big](p)}{[\mathbf{i}](p)}, & i \leq j+k-1; \\ 0, & i \geq j+k. \end{cases}$$

We will show that φ is the inverse of q restricted to N^w. We first need the following.

Lemma 5.2. *Let $J = \{j_1 < j_2 < \cdots < j_k\}$ be a k-subset of $\{1, \ldots, n\}$ and $0 \leq r \leq j_1 - 1$. Let $p \in \mathrm{Gr}^*_{k,n}$. Then, we have:*

(a)

$$\Delta_{\{r+1,\ldots,r+k\},J}\Big(\varphi(p)\Big) = \frac{[J](p)}{[\mathbf{s}](p)},$$

where $s = r+k$, and

(b)

$$\Delta_{\{r,\ldots,r+k\},J}\Big(\varphi(p)\Big) = 0.$$

Proof. Note that, by the assumptions, $0 \leq r \leq n-k$. We first consider the proof of (a). Since $r \leq j_1 - 1$, we have $r + a \leq r + k \leq j_1 + k - 1 < j_b + k - 1$ for $1 \leq a \leq k$ and $1 \leq b \leq k$. It follows that for all $(i, j) \in \{r+1, \ldots, r+k\} \times J$, we have

$$\varphi(p)_{ij} = \frac{\big[\mathbf{i} - \{i\} \cup \{j\}\big](p)}{[\mathbf{i}](p)}.$$

Hence, $\Delta_{\{r+1,\ldots,r+k\},J}\Big(\varphi(p)\Big)$ is equal to the following (dropping the notation (p) to save space):

$$\frac{1}{\prod_{s=r+1}^{r+k}[\mathbf{s}]}\begin{vmatrix} [1,\ldots,r,j_1,r-k+2+n,\ldots,n] & [1,\ldots,r,j_2,r-k+2+n,\ldots,n] & \cdots & [1,\ldots,r,j_k,r-k+2+n,\ldots,n] \\ [1,\ldots,r+1,j_1,r-k+3+n,\ldots,n] & [1,\ldots,r+1,j_2,r-k+3+n,\ldots,n] & \cdots & [1,\ldots,r+1,j_k,r-k+3+n,\ldots,n] \\ \vdots & \vdots & & \vdots \\ [1,\ldots,k-1,j_1] & [1,\ldots,k-1,j_2] & \cdots & [1,\ldots,k-1,j_k] \\ \vdots & \vdots & & \vdots \\ [r+1,\ldots,r+k-1,j_1] & [r+1,\ldots,r+k-1,j_2] & \cdots & [r+1,\ldots,r+k-1,j_k] \end{vmatrix}, \quad r < k-1$$

$$\frac{1}{\prod_{s=r+1}^{r+k}[\mathbf{s}]}\begin{vmatrix} [r-k+2,\ldots,r,j_1] & [r-k+2,\ldots,r,j_2] & \cdots & [r-k+2,\ldots,r,j_k] \\ [r-k+3,\ldots,r+1,j_1] & [r-k+3,\ldots,r+1,j_2] & \cdots & [r-k+3,\ldots,r+1,j_k] \\ \vdots & \vdots & & \vdots \\ [r+1,\ldots,r+k-1,j_1] & [r+1,\ldots,r+k-1,j_2] & \cdots & [r+1,\ldots,r+k-1,j_k] \end{vmatrix}, \qquad r \geq k-1.$$

We can rewrite this (by Proposition 3.6) as:

$$\Delta_{\{r+1,\dots,r+k\},J}\Big(\varphi(p)\Big) = \begin{cases} \dfrac{1}{\prod_{s=r+1}^{r+k}[\mathbf{s}]} \boxed{\begin{array}{ccccccc} 1 & \cdots\ r & \boxed{j_1} & r-k+2+n & r-k+3+n & \cdots & n \\ 1 & \cdots\ r & r+1 & \boxed{j_2} & r-k+3+n & \cdots & n \\ \vdots & & & & & & \vdots \\ 1 & & & \cdots & & k-1 & \boxed{j_{k-r}} \\ \vdots & & & & & & \vdots \\ r+1 & & & \cdots & & r+k-1 & \boxed{j_k} \end{array}}, & r < k-1; \\ \\ \dfrac{1}{\prod_{s=r+1}^{r+k}[\mathbf{s}]} \boxed{\begin{array}{ccccc} r-k+2 & r-k+3 & \cdots & r & \boxed{j_1} \\ r-k+3 & r-k+4 & \cdots & r+1 & \boxed{j_2} \\ \vdots & \vdots & & \vdots & \\ r+1 & r+2 & \cdots & r+k-1 & \boxed{j_k} \end{array}}, & r \geq k-1. \end{cases}$$

We apply Turnbull's identity for the first row with B given by the set of all unboxed entries in the first row, obtaining:

$$\Delta_{\{r+1,\dots,r+k\},J}\Big(\varphi(p)\Big) = \begin{cases} \dfrac{1}{\prod_{s=r+1}^{r+k}[\mathbf{s}]} \boxed{\begin{array}{ccccccc} j_{k+1-r} & \cdots\ j_k & j_1 & j_2 & \cdots & & j_{k-r} \\ 1 & \cdots & r+1 & \boxed{r-k+2+n} & r-k+3+n & \cdots & n \\ 1 & \cdots & & r+2 & \boxed{r-k+3+n} & \cdots & n \\ \vdots & & & & & & \vdots \\ 1 & & & \cdots & & k-1 & \boxed{n} \\ 2 & & & \cdots & & k & \boxed{1} \\ \vdots & & & & & & \vdots \\ r+1 & & & \cdots & & r+k-1 & \boxed{r} \end{array}}, & r < k-1; \\ \\ \dfrac{1}{\prod_{s=r+1}^{r+k}[\mathbf{s}]} \boxed{\begin{array}{ccccc} j_2 & j_3 & \cdots & j_k & j_1 \\ r-k+3 & r-k+4 & \cdots & r+1 & \boxed{r-k+2} \\ \vdots & \vdots & & \vdots & \vdots \\ r+1 & r+2 & \cdots & r+k-1 & \boxed{r} \end{array}}, & r \geq k-1. \end{cases}$$

We can remove the boxes in the tableau (in either case), since every other ordering of the boxed elements evaluates to zero. Hence,

$$\Delta_{\{r+1,\dots,r+k\},J}\Big(\varphi(p)\Big) = \begin{cases} \dfrac{(-1)^{(k-1)(k-r)+(k-1)r}[J]\prod_{s=r+1}^{r+k-1}[\mathbf{s}]}{\prod_{s=r+1}^{r+k}[\mathbf{s}]}, & r < k-1; \\ \dfrac{(-1)^{k(k-1)}[J]\prod_{s=r+1}^{r+k-1}[\mathbf{s}]}{\prod_{s=r+1}^{r+k}[\mathbf{s}]}, & r \geq k-1; \end{cases}$$

$$= \frac{[J]\prod_{s=r+1}^{r+k-1}[\mathbf{s}]}{\prod_{s=r+1}^{r+k}[\mathbf{s}]},$$

and the proof of (a) is complete.

The proof of (b) is similar: $\Delta_{\{r,\dots,r+k\},J}(\varphi(p))$ can be written in terms of a $(k+1)\times(k+1)$ tableau in which $k+1$ elements are boxed. Since this is greater than k, it follows that this is zero by [28, Prop. 1.2.2(i)] (see Proposition 3.2). $\square$

By definition, $\varphi(\mathrm{Gr}^*_{k,n})\subseteq N$. In order to check that the image of φ is contained in N^w, we use the following result, which can easily be deduced from [14, 4.1].

Proposition 5.3. *Let $x\in SL_n(\mathbb{C})$ and $w\in W$. Then $x\in B_-wB_-$ if and only if the following hold:*

(a) $\Delta_{\{1,\dots,i\},w^{-1}(\{1,\dots,i\})}(x)\neq 0$ *for* $i=1,2,\dots,n-1$.

(b) $\Delta_{\{1,\dots,i\},w^{-1}(\{1,\dots,i-1\}\cup\{j\})}(x)=0$ *for all* $1\leq i<j\leq n$ *satisfying* $w^{-1}(i)<w^{-1}(j)$.

Lemma 5.4. *Let $p\in\mathrm{Gr}^*_{k,n}$ and $x=\varphi(p)$. Then $x\in N^w$.*

Proof. Let $p\in\mathrm{Gr}^*_{k,n}$ and consider $x=\varphi(p)$. As indicated above, $x\in N$. We must show that (a) and (b) in Proposition 5.3 both hold, so that we can conclude that $x\in B_-wB_-$ also. We first claim that, for $1\leq i<j\leq k$, the following hold:

$$x_{i,i+n-k}\neq 0, \text{ if } 1\leq i\leq k; \tag{5.1}$$

$$x_{i,j+n-k}=0, \text{ if } 1\leq i<j\leq k. \tag{5.2}$$

Fix $1\leq i\leq k$. Then we have:

$$x_{i,i+n-k}=\frac{[\mathbf{i}\setminus\{i\}\cup\{i+n-k\}](p)}{[\mathbf{i}](p)}=\frac{[\sigma(\mathbf{i})](p)}{[\mathbf{i}](p)}\neq 0,$$

since $p\in\mathrm{Gr}^*_{k,n}$. This shows (5.1). Fix $1\leq i<j\leq k$. Then we have:

$$x_{i,j+n-k}=\frac{[\mathbf{i}\setminus\{i\}\cup\{j+n-k\}](p)}{[\mathbf{i}](p)}=0,$$

since $j+n-k\in\mathbf{i}\setminus\{i\}$. This shows (5.2).

It follows that the submatrix of x with rows $\{1,\dots,k\}$ and columns $\{n-k+1,\dots,n\}$ has non-zero entries along its diagonal and zero entries above its diagonal. Hence, for $1\leq i\leq k$, the submatrix of x with rows $\{1\dots,i\}$ and columns $w^{-1}(\{1,\dots,i\})=\{n-k+1,\dots,n-k+i\}$ has the same property and thus non-zero determinant, showing (a) for $1\leq i\leq k$. And, for $1\leq i<j\leq k$, the submatrix of x with rows $\{1,\dots,i\}$ and columns $w^{-1}(\{1,\dots,i-1\}\cup\{j\})$ has non-zero entries along its diagonal, except the bottom right entry with is zero, and zero entries above the diagonal. Hence it has zero determinant, and (b) is shown for $1\leq i<j\leq k$.

It remains to show (a) for $k+1\leq i\leq n$ and (b) for $k+1\leq i<j\leq n$.

We first fix $k+1\leq i\leq n$ and write $i=k+r$ where $1\leq r\leq n-k$. We have

$$\begin{aligned}\Delta_{\{1,\dots,i\},w^{-1}(\{1,\dots,i\})}(x)&=\Delta_{\{1,\dots,k+r\},\{n-k+1,\dots,n+r\}}(x)\\&=\Delta_{\{1,\dots,k+r\},\{1,\dots,r\}\cup\{n-k+1,\dots,n\}}(x)\\&=\Delta_{\{r+1,\dots,r+k\},\{n-k+1,\dots,n\}}(x),\end{aligned}$$

where in the last step we have used the fact that $x\in N$.

Since $r\leq n-k$, we have by Lemma 5.2(a) that

$$\Delta_{\{r+1,\dots,r+k\},\{n-k+1,\dots,n\}}(x)=\frac{[\mathbf{n}](p)}{[\mathbf{s}](p)},$$

where $s=r+k$. This is non-zero, since $p\in\mathrm{Gr}^*_{k,n}$.

Next we suppose that $i=k+r$ and $j=k+t$, where $1\leq r<t\leq n-k$. We have that

$$\begin{aligned}\Delta_{\{1,\dots,i\},w^{-1}(\{1,\dots,i-1\}\cup\{j\})}(x)&=\Delta_{\{1,\dots,k+r\},\{n-k+1,\dots,n\}\cup\{1,\dots,r-1\}\cup\{t\}}(x)\\&=\Delta_{\{r,\dots k+r\},\{t\}\cup\{n-k+1,\dots,n\}}(x),\end{aligned}$$

using the fact that $x \in N$. We have assumed that $r \leq t-1$, so this is zero by Lemma 5.2(b). $\square$

Let $G_0 = N_- HN$, where N^- denotes the set of lower unitriangular matrices in $SL_n(\mathbb{C})$. The *Gaussian decomposition* of an element $x \in G_0$ is

$$x = [x]_-[x]_0[x]_+,$$

where $[x]_- \in N_-$, $[x]_0 \in H$ and $[x]_+ \in N$.

We set

$$N_+(w) = N \cap wN_-w^{-1}; \tag{5.3}$$

$$N_-(w) = N_- \cap w^{-1}Nw. \tag{5.4}$$

Then $N_+(w)$ coincides with the unipotent radical of the maximal parabolic subgroup P associated to the complement of the root $\boldsymbol{\alpha}_k$.

By [14, Props. 2.10, 2.17], for $x \in N^w$, $xw^{-1} \in G_0$ and, moreover, the map $x \mapsto [xw^{-1}]_+$ defines a biregular isomorphism

$$\beta : N^w \to N_+(w) \cap G_0 w.$$

Theorem 5.5. *The maps $q : N^w \to \mathrm{Gr}^*_{k,n}$ and $\varphi : \mathrm{Gr}^*_{k,n} \to N^w$ are mutual inverses (and hence biregular).*

Proof. By Lemmas 5.1 and 5.4, the image of q is contained in $\mathrm{Gr}^*_{k,n}$ and the image of φ is contained in N^w. For $x \in N^w$,

$$q(x) = P_-x = P_-xw^{-1}w = P_-[xw^{-1}]_+w = P_-\beta(x)w,$$

using the definition of the Gaussian decomposition. So if $q(x) = q(y)$, then $P_-\beta(x)w = P_-\beta(y)w$, so $P_-\beta(x) = P_-\beta(y)$. Since $\beta(x), \beta(y) \in N_+(w)$, it follows that $\beta(x) = \beta(y)$, as the projection map q restricted to $N_+(w)$ is injective by [7, Prop. 14.21] (as reformulated in [19, §2.2]), so $x = y$, since β is a bijection. Therefore q is injective.

By Lemma 5.2(a) (in the case $r = 0$), the composition $q\varphi$ is equal to the identity on $\mathrm{Gr}^*_{k,n}$. Hence q is surjective and therefore a bijection. It follows that φ is its inverse and we are done. $\square$

Remark 5.6. It is easy to see that the image of N^w under q is the same as the image of $B_-B_+ \cap B_-wB_-$ and is thus an open Richardson variety, part of the stratification of $P_-\backslash G$ introduced by Lusztig [29]. The natural map $\pi : G \to B_-\backslash G$ is injective on restriction to N, and hence on restriction to N^w, and the projection $B_-\backslash G \to P_-\backslash G$ is injective on $\pi(N^w)$ (see [37, 7.1]). This gives an alternative way to see the injectivity of q on N^w.

The BFZ-twist for the open positroid variety $\mathrm{Gr}^*_{k,n}$ is defined by transporting the BFZ-twist η for N^w to $\mathrm{Gr}^*_{k,n}$ by setting: $\widehat{p} := \varphi^{-1} \circ \eta \circ \varphi(p)$ for $p \in \mathrm{Gr}^*_{k,n}$, so that we have the following commutative diagram:

$$\begin{tikzcd}
N^w \arrow[r, "q"] \arrow[d, "\eta"'] & \mathrm{Gr}^*_{k,n} \arrow[d, "p \mapsto \widehat{p}"] \\
N^w \arrow[r, "q"] & \mathrm{Gr}^*_{k,n}
\end{tikzcd}$$

For a k-subset I, we denote by $\widehat{[I]}$ the BFZ-twisted Plücker coordinate defined by the composition $p \mapsto [I](\widehat{p})$. We recall that the domain of the twist $p \mapsto \overleftarrow{p}$ contains $\mathrm{Gr}^*_{k,n}$ (see Remark 3.7).

Proposition 5.7. *For any k-subset I the twisted Plücker coordinate and the BFZ-twisted Plücker coordinate are related by the following formula:*

$$\frac{\widehat{[I]}}{\widehat{[\mathbf{k}]}} = \frac{\overleftarrow{[I]} \cdot [\mathbf{k}]}{\prod_{i \in I} [\mathbf{i}]}$$

as elements of the coordinate ring $\mathbb{C}[\mathrm{Gr}^*_{k,n}]$.

Proof. Firstly, let $\overline{w}$ be the matrix whose entries are $\overline{w}_{i+k,i} = (-1)^k$ for $1 \leq i \leq n-k$, $\overline{w}_{i+k-n,i} = 1$ for $n-k+1 \leq i \leq n$ and zero otherwise. Then $\overline{w}$ is a particular choice of representative for w, and we can write our arbitrary choice of representative in the form $w = t\overline{w}$, where $t = \mathrm{diag}(t_1, \ldots, t_n) \in T$.

Notice that $q(bg) = q(g)$ for any $g \in \mathrm{SL}_n(\mathbb{C})$ and any $b \in B_-$. By definition the twist $\eta(g)$ for $g \in N^w$ can be written in the form bwg^T for some (unique) element $b \in B_-$ and therefore $q(\eta(g))$ coincides with $q(wg^T)$. Hence

$$q\left(w\varphi(p)^T\right) = q\left(\eta\left(\varphi(p)\right)\right) = q\varphi(\widehat{p}) = \widehat{p}$$

for any point $p \in \mathrm{Gr}^*_{k,n}$. It is easy to see that, for $1 \leq i \leq k$, the (i,j)-matrix entry of $w\varphi(p)^T = t\overline{w}\varphi(p)^T$ is

$$t_i \cdot \frac{[\mathbf{j} - \{j\} \cup \sigma^k(i)](p)}{[\mathbf{j}](p)}.$$

On the other hand, for $1 \leq i \leq k$, the (i,j)-entry of $\varphi(\overleftarrow{p})$ is

$$\frac{\overleftarrow{[\mathbf{i} - \{i\} \cup \{j\}]}(p)}{\overleftarrow{[\mathbf{i}]}(p)},$$

which after employing Proposition 3.5 and performing cancellations may be re-expressed as

$$\frac{[\mathbf{j} - \{j\} \cup \{\sigma^k(i)\}](p)}{[\sigma(\mathbf{i})](p)}.$$

So we may deduce that

$$\Delta_{\{1,\ldots,k\},I}\left(\varphi(\overleftarrow{p})\right) = t_1^{-1} \cdots t_k^{-1} \cdot \frac{\prod_{i \in I} [\mathbf{i}](p)}{\prod_{j=1}^k [\sigma(\mathbf{j})](p)} \cdot \Delta_{\{1,\ldots k\},I}\left(w\varphi(p)^T\right). \tag{5.5}$$

By the definition of q, we have that

$$\frac{\Delta_{\{1,\ldots k\},I}\left(w\varphi(p)^T\right)}{\Delta_{\{1,\ldots k\},\{1,\ldots k\}}\left(w\varphi(p)^T\right)} = \frac{[I]\left(q\left(w\varphi(p)^T\right)\right)}{[\mathbf{k}]\left(q\left(w\varphi(p)^T\right)\right)} = \frac{\widehat{[I]}(p)}{\widehat{[\mathbf{k}]}(p)}. \tag{5.6}$$

Using the Cauchy-Binet formula, we have (summing over all k-subsets J of $\{1, \ldots, n\}$):

$$\begin{aligned}
\Delta_{\{1,\ldots,k\},\{1,\ldots k\}}\left(w\varphi(p)^T\right) &= \sum_J \Delta_{\{1,\ldots,k\},J}(w)\Delta_{J,\{1,\ldots,k\}}\left(\varphi(p)^T\right) \\
&= \Delta_{\{1,\ldots,k\},\{n-k+1,\ldots,n\}}(w)\Delta_{\{n-k+1,\ldots n\},\{1,\ldots,k\}}\left(\varphi(p)^T\right) \\
&= \Delta_{\{1,\ldots,k\},\{n-k+1,\ldots,n\}}(t\overline{w})\Delta_{\{n-k+1,\ldots n\},\{1,\ldots,k\}}\left(\varphi(p)^T\right) \\
&= t_1 \cdots t_k \cdot \Delta_{\{1,\ldots k\},\{n-k+1,\ldots,n\}}(\varphi(p)).
\end{aligned}$$

Hence, by Lemma 5.2(a),

$$\Delta_{\{1,\dots k\},\{1,\dots k\}}\left(w\varphi(p)^T\right) \;=\; t_1\cdots t_k\cdot\frac{[\mathbf{n}](p)}{[\mathbf{k}](p)}. \tag{5.7}$$

Combining equations (5.5), (5.6) and (5.7), we may conclude that:

$$\Delta_{\{1,\dots,k\},I}\left(\varphi(\overleftarrow{p})\right) \;=\; \frac{\prod_{i\in I}[\mathbf{i}](p)}{\prod_{j=1}^{k}[\sigma(\mathbf{j})](p)}\cdot\frac{\widehat{[I]}(p)}{\widehat{[\mathbf{k}]}(p)}\cdot\frac{[\mathbf{n}](p)}{[\mathbf{k}](p)}.$$

Since we have:

$$\Delta_{\{1,\dots,k\},I}\left(\varphi(\overleftarrow{p})\right) = \frac{\overleftarrow{[I]}(p)}{\overleftarrow{[\mathbf{k}]}(p)}\quad\text{and}\quad\frac{\prod_{j=1}^{k-1}[\mathbf{j}](p)}{\overleftarrow{[\mathbf{k}]}(p)}=1,$$

we may perform cancellations and obtain the asserted formula. □

6. Cluster structure of the Grassmannian

In this section we recall the description of the cluster structure on the Grassmannian from [39]. We firstly recall the definition of a skew-symmetric cluster algebra of geometric type. Fix $l,m\in\mathbb{N}$ and let $\mathbb{F}$ denote the field of rational functions in indeterminates $u_1,\dots,u_l,\dots,u_{l+m}$ over $\mathbb{Q}$. We consider *seeds* $(\widetilde{\mathbf{x}},\widetilde{Q})$ consisting of a free generating set $\widetilde{\mathbf{x}}=\{x_1,x_2,\dots,x_{l+m}\}$ (known as an *extended cluster*) of $\mathbb{F}$ over $\mathbb{Q}$ and a quiver $\widetilde{Q}$ on vertices $1,2,\dots,l+m$ (known as the exchange quiver), with no arrows between vertices labelled $l+1,l+2,\dots,l+m$ (the coefficients) and no two-cycles or loops. The tuple $\mathbf{x}=(x_1,x_2,\dots,x_l)$ is called a *cluster*, with the remaining elements of $\widetilde{\mathbf{x}}$ known as *coefficients*. The subquiver Q of $\widetilde{Q}$ on vertices $1,2,\dots,l$ is called the *principal part* of $\widetilde{Q}$. Vertices in $\widetilde{Q}$ but not Q are referred to as *frozen vertices*.

Fix $r\in\{1,\dots,l\}$. The *mutation* of $(\widetilde{\mathbf{x}},\widetilde{Q})$ at r is the pair $(\widetilde{\mathbf{x}}',\widetilde{Q}')$, where

$$\widetilde{\mathbf{x}}'=(x_1,\dots,x_{r-1},x_r',x_{r+1},\dots,x_{l+m}),$$

with x_r' defined by the *exchange relation*:

$$x_rx_r'=\prod_{j\to r}x_j+\prod_{r\to j}x_j,$$

with the first product taken over all arrows in $\widetilde{Q}$ ending at r, and the second over all arrows in $\widetilde{Q}$ starting at r. The quiver $\widetilde{Q}'$ is obtained from $\widetilde{Q}$ via quiver mutation. Firstly, an arrow $i\to j$ is added to $\widetilde{Q}$ for every path of length two from i to j passing through r, then the arrows incident with r are reversed, and finally a maximal collection of two-cycles is removed.

The *cluster algebra* associated to $(\widetilde{\mathbf{x}},\widetilde{Q})$ is the $\mathbb{C}$-subalgebra of $\mathbb{F}$ generated by the free generating sets occurring in the seeds obtained from $(\widetilde{\mathbf{x}},\widetilde{Q})$ by iterated mutation.

In [39] (see also [17, 18]) it is shown that $\mathbb{C}[\mathrm{Gr}_{k,n}]$ is a cluster algebra, using certain diagrams in a disk known as Postnikov diagrams [35] (in [35] they are referred to as alternating strand diagrams), which we now recall.

Definition 6.1. Label the vertices of a convex $2n$-sided polygon $1',1,2',2,\dots,n',n$ clockwise around its boundary. A *Postnikov diagram* consists of n oriented strands inside the polygon, satisfying the conditions:

(a) No strand intersects itself.
(b) All intersections of strands are transversal.
(c) There are finitely many such intersections.
(d) Strand i starts at i and ends at $(i+k)'$.

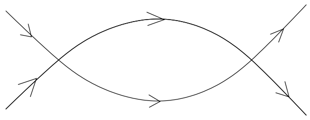


Figure 1. Forbidden configuration

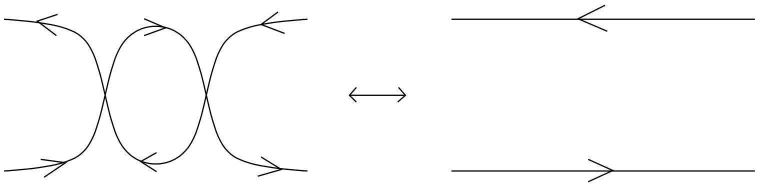


Figure 2. Creation or annihilation of a local oriented lens

(e) Travelling along strand i, the directions of the strands crossing it alternate, starting with left to right and ending with right to left.
(f) No two strands, regardless of the rest of diagram, are allowed to form an unoriented lens, as depicted in Figure 1.

Postnikov diagrams are considered up to isotopy and the creation/annihilation of a local oriented lens, as in Figure 2 (or with the orientation in this figure reversed). Note that we specify the conditions on the boundary precisely.

For an example of a Postnikov diagram, see Figure 3.

We orient the boundary of the polygon clockwise. Then each face (i.e. connected component of the complement of the strands inside the polygon) of a Postnikov diagram either has an alternating boundary or an oriented boundary. We refer to the former as *alternating faces* and the latter as *oriented faces*. Each alternating face F is labelled with the subset I_F of $\{1, \ldots, n\}$ consisting of the strands which have F on their left (as they are traversed).

Proposition 6.2. *(Postnikov; see* [39, Prop. 5]*) Let P be a Postnikov diagram. Then we have:*

(1) For any alternating face F of P, the cardinality of its labelling set I_F is exactly k.
(2) There are exactly $k(n-k)+1$ alternating faces in P, with n of them situated along the boundary. The remaining $(k-1)(n-k-1)$ alternating faces are internal.
(3) The labels of the alternating boundary faces are the cyclic intervals $[\mathbf{1}], \ldots, [\mathbf{n}]$.

Remark 6.3. The label on the alternating boundary face immediately clockwise of i is $\mathbf{i}$.

Remark 6.4. Note that the set of labels on a Postnikov diagram is not changed by the creation or annihilation of oriented lenses.

Definition 6.5. The *quadrilateral move* on a Postnikov diagram (called geometric exchange in [39]) is the local move depicted in Figure 4.

We recall the following result of Postnikov, referred to in [39, Prop. 6].

Proposition 6.6. *Any two Postnikov diagrams are connected by a sequence of quadrilateral moves.*

Scott [39, Sect. 5], associates a quiver $\widetilde{Q}(P)$ to a Postnikov diagram P. The vertices correspond to the alternating faces and the arrows are as shown in Figure 5 (with the thick arrow indicating the arrow in the quiver), cancelling any two-cycles. Note that cancelling two-cycles corresponds to applying the maximal annihilation of oriented lenses as in Figure 2 (from left to right).

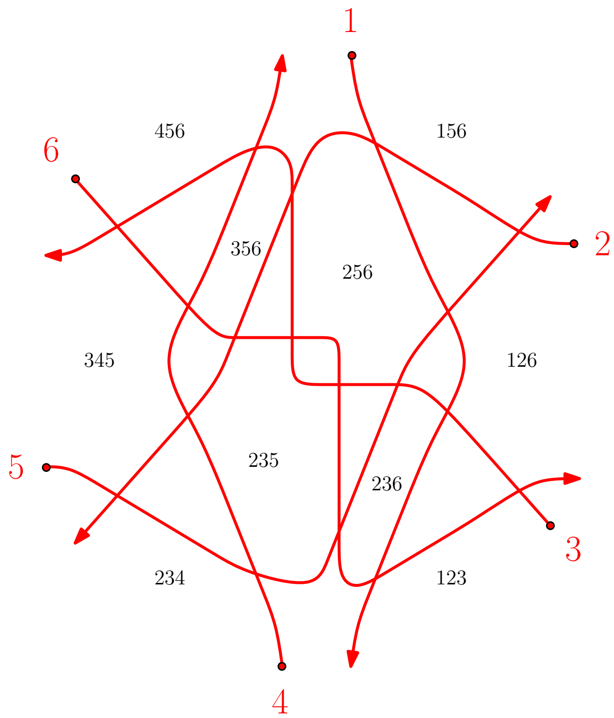


FIGURE 3. A Postnikov diagram (in the case $k = 3, n = 6$).

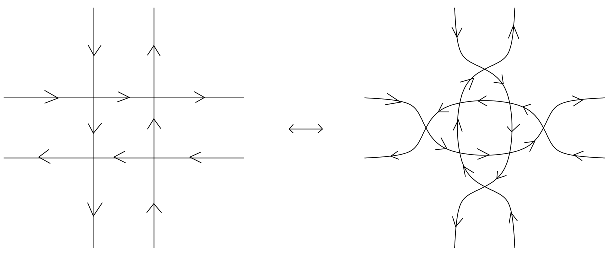


FIGURE 4. The quadrilateral move

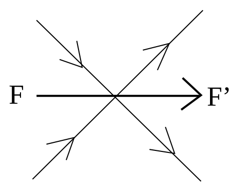


FIGURE 5. Arrow in the quiver of a Postnikov diagram

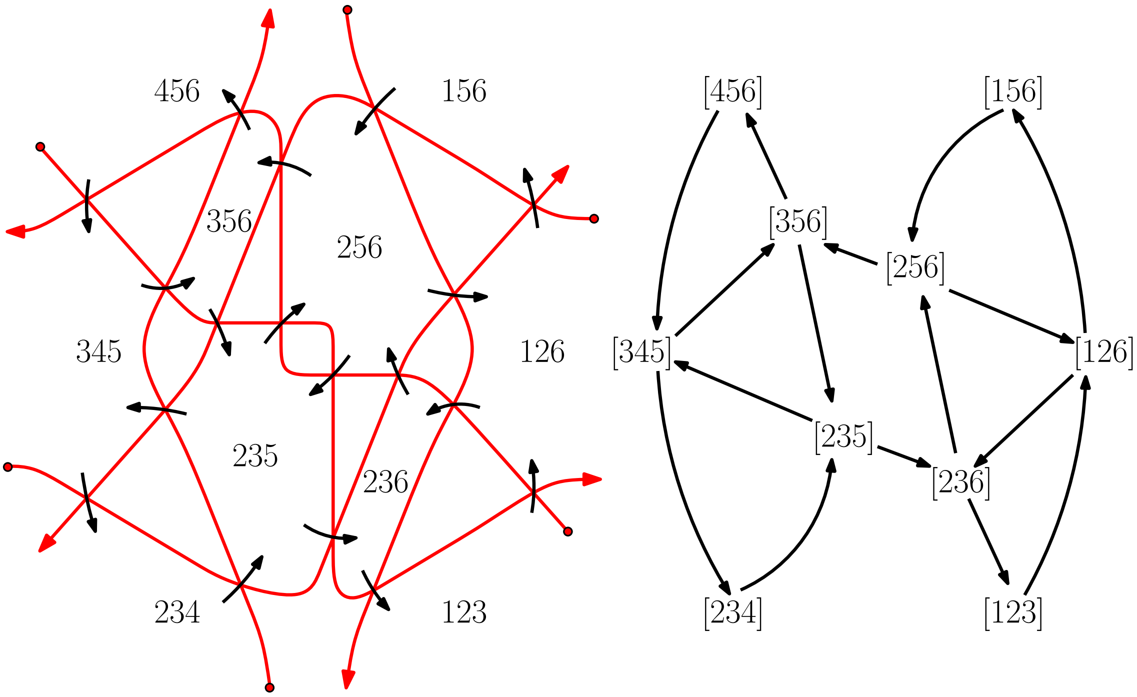


FIGURE 6. The quiver associated to the Postnikov diagram in Figure 3.

Figure 6 shows an example of the quiver of a Postnikov diagram.

Theorem 6.7. [39] *Let P be a Postnikov diagram. Let*

$$\widetilde{\mathbf{x}}(P) = \{[I_F] \,:\, F \text{ an alternating face of } P\},$$

$$\mathbf{x}(P) = \{[I_F] \,:\, F \text{ an internal alternating face of } P\}.$$

Then:

(a) The pair $(\widetilde{\mathbf{x}}(P), \widetilde{Q}(P))$ is a seed in the rational function field $\mathbb{C}(\mathrm{Gr}_{k,n})$, with coefficients $[\mathbf{1}], \ldots, [\mathbf{n}]$.

(b) If P, P' are related by a single quadrilateral move, then $(\widetilde{\mathbf{x}}(P'), \widetilde{Q}(P'))$ can be obtained from $(\widetilde{\mathbf{x}}(P), \widetilde{Q}(P))$ by a single mutation. The exchange relation in this case is a short Plücker relation (see Remark 6.8, below).

(c) The cluster algebra determined by $(\widetilde{\mathbf{x}}(P), \widetilde{Q}(P))$ (for any P) is $\mathbb{C}[\mathrm{Gr}_{k,n}] \subseteq \mathbb{C}(\mathrm{Gr}_{k,n})$.

Remark 6.8. If F is an alternating internal face about which a quadrilateral move can be performed, then there are four indices $a < b < c < d$ in $\{1, \ldots, n\}$ and a $(k-2)$-subset J with $J \cap \{a, b, c, d\} = \phi$ such that the exchange relation corresponding to the quadrilateral move is the short Plücker relation:

$$[Jac][Jbd] = [Jab][Jcd] + [Jad][Jbc],$$

where $Jac = J \cup \{a, c\}$, etc.

7. DIMER PARTITION FUNCTIONS

In this section, we consider the dual bipartite graph of a Postnikov diagram. We give a weighting to the edges of this graph and define a dimer partition function, for each k-subset I of $\{1, \ldots, n\}$,

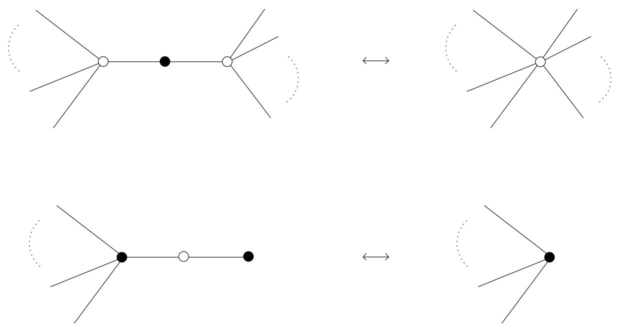


FIGURE 7. The blow-down/blow-up move on the bipartite graph. In the lower figure, the black vertices on the right hand side of the diagrams must be boundary vertices.

in terms of perfect matchings of a subgraph. We show that this function is invariant under blow-up or blow-down moves. We then show that a scaled version of each dimer partition function is, in addition, independent of the choice of Postnikov diagram P. We then use this to associate a polynomial to each k-subset I.

Given a Postnikov diagram, P, a dual graph $G = G_P$ is defined as follows. The vertices of G_P are in bijection with the oriented faces of P, and the edges correspond to points of intersection of the boundaries of the corresponding faces. The internal alternating faces of P correspond to internal faces of G_P, and the latter inherits a k-subset label from P, which we replace with the associated minor. For each $i \in \{1, \ldots, n\}$, the boundary vertices i and i' of P lie on the boundary of an oriented face of P that corresponds to a boundary vertex of G_P, which we label with i. Thus the graph G_P lies inside a polygon with vertices $1, 2, \ldots, n$. Furthermore, for each i, the boundary vertices i and $(i+1)'$ of P lie on the boundary of an alternating face of P which corresponds to part of the boundary of G_P. We regard the part of the plane bounded by this and the side of the above polygon between vertices i and $i+1$ as a boundary face of G_P. Thus G_P has n boundary faces in total.

The graph G_P is bipartite: a vertex corresponding to a clockwise face (respectively, anticlockwise face) of P is coloured black (respectively, white). The graph G_P is considered up to local moves, shown in Figure 7, corresponding to the creation or annihilation of oriented lenses as in Figure 2 (we also allow blow-ups and blow-downs for black vertices). A blow-up at a vertex v involves a partition of the edges incident with the vertex into two subsets; we restrict to the case where both subsets are nonempty, unless v is a boundary black vertex (see Figure 7 for an example).

We refer to the move corresponding to the annihilation of an oriented lens as a *blow-down*, and the move corresponding to the creation of an oriented lens as a *blow-up*.

For an example of the dual bipartite graph of the Postnikov diagram in Figure 3, see Figure 8. The labels of the faces are shown in Figure 9.

Definition 7.1. We give weights to the edges of G_P as follows. Let e be an edge of G_P. Then we label e with the product w_e of the Plücker coordinates labelling the faces of G_P which are incident with the white vertex incident with e but not with e itself. See Figure 10.

Remark 7.2. Given an arbitary edge weighting (κ_e) of G_P define the shear weight y_F of a face F of G_P to be the following ratio of edge weights:

$$y_F = \prod_{e \in \partial F} m_e^{\varepsilon_F(e)},$$

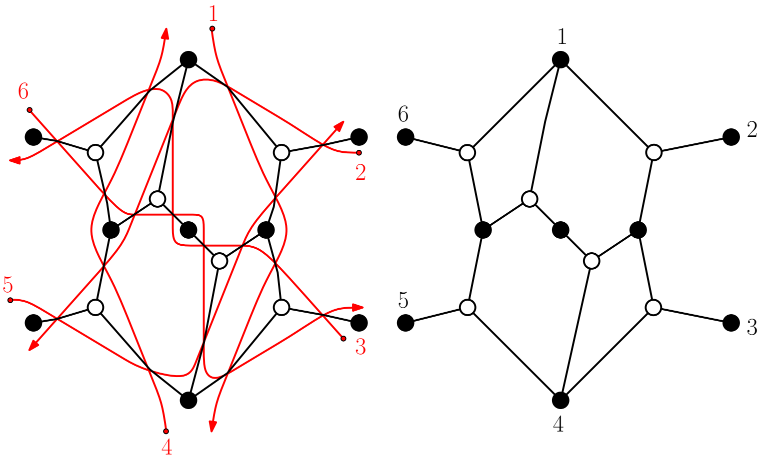


FIGURE 8. The dual bipartite graph associated to the Postnikov diagram in Figure 3.

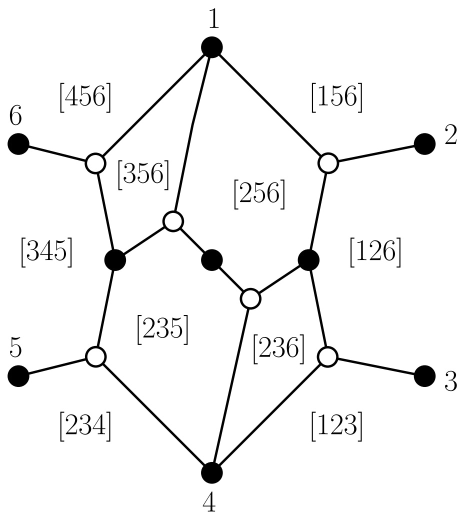


FIGURE 9. The face labels on dual bipartite graph associated to the Postnikov diagram in Figure 3.

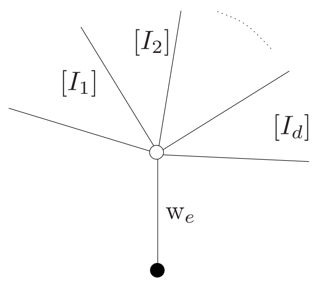


FIGURE 10. Weighting of an edge in the bipartite graph: $\mathrm{w}_e = [I_1] \cdot [I_2] \cdots [I_d]$.

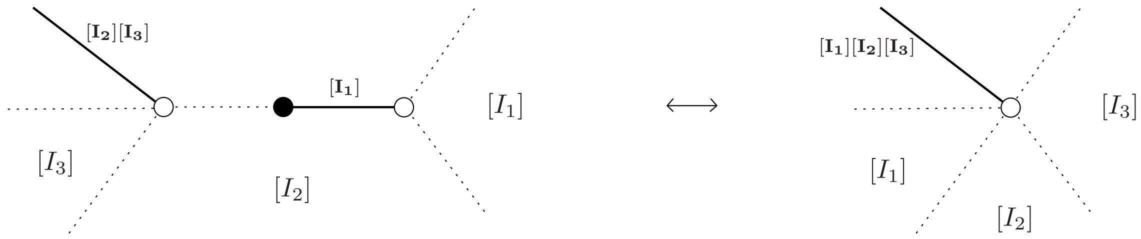


FIGURE 11. Proof of Lemma 7.3.

where $\varepsilon_F(e)$ is $+1$ if the orientation of e induced by the counter-clockwise orientation of ∂F points from black to white, and is -1 otherwise. In the case where $\kappa_e = w_e$ for all edges e, it is easy to check that the shear weights coincide with the y-variables (as defined in [16])

$$y_F = \prod_{\substack{F' \to F \\ \text{in } \widetilde{Q}(P)}} [I_F] \cdot \prod_{\substack{F \to F' \\ \text{in } \widetilde{Q}(P)}} [I_F]^{-1}.$$

Let P be a Postnikov diagram with dual graph G_P. We write $G_P(I)$ for the graph G with the vertices labelled by elements of I removed from the boundary. We recall the following (as discussed in the introduction).

The *dimer partition function* of $G_P(I)$ is the sum:

$$Đ_{G_P(I)} = \sum_{\delta} \mathrm{w}_\delta = \sum_{\delta} \prod_{e \in \delta} \mathrm{w}_e,$$

where δ varies over all dimer configurations of $G_P(I)$. Since $G_P(I)$ is only defined up to blow-up and blow-down moves, we need to check that the dimer partition function is invariant under these moves.

Lemma 7.3. *Let P be a Postnikov diagram and I a k-subset of $\{1, \ldots, n\}$. Suppose that G' is a graph obtained from $G = G_P(I)$ by applying the blow-down at the top Figure 7. Then a dimer configuration δ on G induces a dimer configuration δ' on G' by removing the edge e incident with the black vertex in the middle of the diagram. Then the weights of δ and δ' coincide. A similar result holds if the vertex in the middle is white.*

Proof. Let f denote the unique edge incident with one of the white vertices in the part of G shown, and let f' be the corresponding edge in G'. Then it is easy to see that $\mathrm{w}_e \mathrm{w}_f = \mathrm{w}_{f'}$ (see Figure 11), and it follows that $\mathrm{w}_\delta = \mathrm{w}_{\delta'}$. A similar argument applies in the case where the vertex in the middle is white. □

As a corollary, we see that the dimer partition function $Đ_{G_P}(I)$ is invariant under blow-ups and blow-downs:

Corollary 7.4. *Let P be a Postnikov diagram and I a k-subset of $\{1, \ldots, n\}$. Then $Đ_{G_P}(I)$ is invariant under a blow-up or blow-down move applied to G_P (see Figure 7).*

Proof. This follows from Lemma 7.3, noting that the map $\delta \mapsto \delta'$ defined there is a bijection between the set of dimer configurations on G and the set of dimer configurations on G'. □

Definition 7.5. Given a Postnikov diagram P and a k-subset I of $\{1, \ldots, n\}$, we set

$$\widetilde{Đ}_{G_P}(I) := \frac{Đ_{G_P}(I)}{\prod_{x \in \mathbf{x}(P)} x}.$$

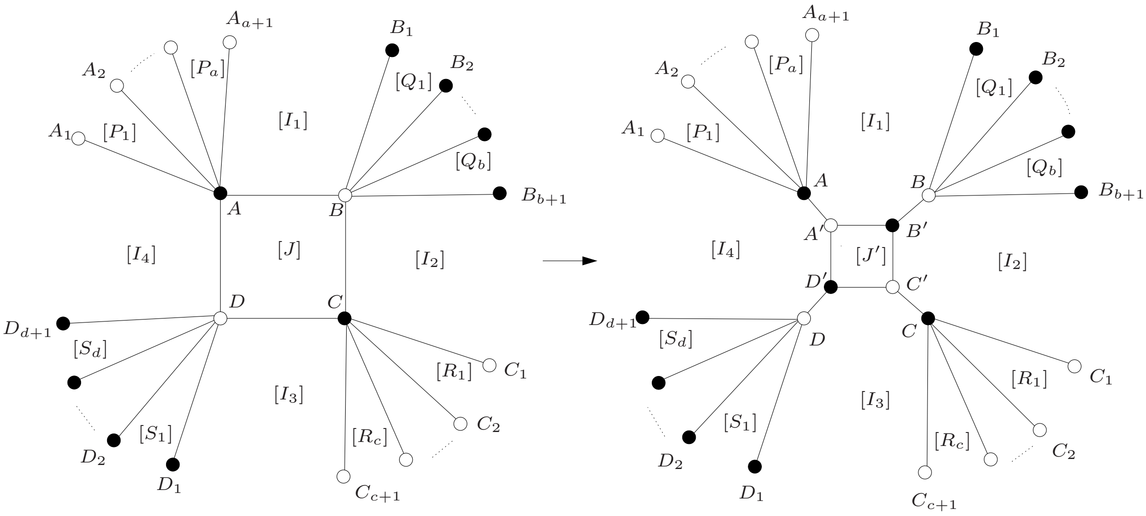


FIGURE 12. The effect of a quadrilateral move on the associated bipartite graph.

Recall that $\mathbf{x}(P)$, the set of variables occuring in the denominator, is the set of cluster variables corresponding to the internal alternating faces of P (see Theorem 6.7).

By Corollary 7.4, $\widetilde{Đ}_{G_P}(I)$ is also invariant under blow-up and blow-down moves. We shall next show that the scaled dimer partition function $\widetilde{Đ}_{G_P}(I)$ is invariant under the quadrilateral move (see Definition 6.5) as well.

Note that the effect of the quadrilateral move on the associated bipartite graph is sometimes known as *urban renewal*; see Figure 12 (see [13, 26]; the latter reference also mentioning G. Kuperberg). The proof of Proposition 7.6 below uses the condensation principle discussed in [13, 27].

Proposition 7.6. *Let P be a Postnikov diagram, and I a k-subset of $\{1,\dots,n\}$. Suppose that P' is obtained from P by applying a quadrilateral move to P. Then $\widetilde{Đ}_{G_P}(I)=\widetilde{Đ}_{G_{P'}}(I)$.*

Proof. Suppose that P' is obtained from P by applying a quadrilateral move. Then $G'=G_{P'}(I)$ is obtained from $G=G_P(I)$ by applying the urban renewal move dual to a quadrilateral move, as illustrated in Figure 12.

For a dimer configuration δ of G, we shall write, as above, w_δ for the weight of δ, and similarly $\mathrm{w}'_{\delta'}$ for a dimer configuration δ' of G'.

Let G_0 be the full subgraph of G on vertices A,B,C,D and let G_0' be the full subgraph of G' on vertices A,B,C,D,A',B',C',D'. For a dimer configuration δ of G, let δ_0 be its restriction to G_0, and for a dimer configuration δ' of G', let δ_0' be its restriction to G_0'. Since the edges in $G\setminus G_0$ can be identified with those of $G'\setminus G_0'$, a dimer configuration on $G\setminus G_0$ can be regarded as a dimer configuration on $G'\setminus G_0'$.

In order to carry out the proof, we temporarily define an equivalence relation on dimer configurations of G by stating that two dimer configurations in which the edges AB, CD are replaced with AD, BC are equivalent (i.e. the difference between the dimer configurations, when regarded as elements in the $\mathbb{Z}_2$-vector space on the edges of G, is the cycle ABCD in the graph G).

Similarly, on δ', we stipulate that two dimer configurations are equivalent when the edges $A'B', C'D'$ are replaced by $A'D', B'C'$. The equivalence class of a dimer configuration δ on G (respectively, δ' on G') will be denoted $\overline{\delta}$ (respectively, $\overline{\delta'}$).

We define a map φ from equivalence classes of dimer configurations on G to equivalence classes of dimer configurations on G' as follows. Let $\overline{\delta}$ be an equivalence class, where δ is a dimer configuration on G.

(a) If $\delta_0 = \{AB, CD\}$ or $\{AD, BC\}$, let δ' be δ with δ_0 replaced by $\delta'_0 = \{AA', BB', CC', DD'\}$ and set $\varphi(\overline{\delta}) = \overline{\delta}'$.

(b) If δ_0 consists of a single edge XY where $X, Y \in \{A, B, C, D\}$, then let δ' be δ with δ_0 replaced by the edge between the vertices in $\{A', B', C', D'\} \setminus \{X', Y'\}$, together with the two edges XX' and YY'. Set $\varphi(\overline{\delta}) = \overline{\delta}'$.

(c) If δ_0 is empty, then set $\varphi(\overline{\delta}) = \overline{\delta}'$ where $\delta'_0 = \{A'B', C'D'\}$. Note that the other element of $\overline{\delta}'$ contains $\{A'D', B'C'\}$ instead.

Then the following holds:

Claim: Let C be an equivalence class of dimer configurations on G. Then

$$\frac{\sum_{\delta \in C} \mathrm{w}_\delta}{\prod_{x \in \mathbf{x}(P)} x} = \frac{\sum_{\delta' \in \varphi(C)} \mathrm{w}'_{\delta'}}{\prod_{x \in \mathbf{x}(P')} x}.$$

Proof of claim: By the definition of φ, it is enough to consider the contribution to w_δ from edges incident with vertices in G_0 (respectively, G'_0). Also, since $\mathbf{x}(P)$ and $\mathbf{x}(P')$ coincide apart from $[J]$ and $[J']$, it is enough to replace the denominator on the left (respectively, right) hand side with $[J]$ (respectively, $[J']$).

Case (a): On the left hand side we have:

$$\begin{aligned} &\frac{[Q_1]\cdots[Q_b][I_2]\cdot[S_1]\cdots[S_d][I_4] + [Q_1]\cdots[Q_b][I_1]\cdot[S_1]\cdots[S_d][I_3]}{[J]} \\ &= \frac{[Q_1]\cdots[Q_b][S_1]\cdots[S_d]([I_2][I_4] + [I_1][I_3])}{[J]} \\ &= \frac{[Q_1]\cdots[Q_b][S_1]\cdots[S_d]([J][J'])}{[J]} \\ &= [Q_1]\cdots[Q_b][S_1]\cdots[S_d][J']. \end{aligned}$$

On the right hand side we have:

$$\frac{[J'][Q_1]\cdots[Q_b][J'][S_1]\cdots[S_d]}{[J']} = [Q_1]\cdots[Q_b][S_1]\cdots[S_d][J'],$$

as required.

Case (b): Suppose that $\delta_0 = \{CD\}$ so that $\delta'_0 = \{A'B', CC', DD'\}$. Assume that the edge in δ (and thus also in δ') incident with A is e_A and the edge in δ incident with B is e_B. On the left hand side we have:

$$\frac{\mathrm{w}_{e_A}\mathrm{w}_{e_B}\mathrm{w}_{CD}}{[J]} = \frac{\mathrm{w}_{e_A}\mathrm{w}_{e_B}[S_1]\cdots[S_d][I_4]}{[J]}.$$

Noting that $\mathrm{w}'_{e_A} = \mathrm{w}_{e_A}$ and $\mathrm{w}'_{e_B} = \mathrm{w}_{e_B}/[J]$, on the right hand side we obtain:

$$\frac{\mathrm{w}'_{e_A}\mathrm{w}'_{e_B}\mathrm{w}'_{A'B'}\mathrm{w}'_{CC'}\mathrm{w}'_{DD'}}{[J']} = \frac{\mathrm{w}_{e_A}\mathrm{w}_{e_B}[I_4][J'][S_1]\cdots[S_d]}{[J][J']}.$$

The other possibilities are similar.

Case (c): We suppose that the edges incident with A, B, C, D in δ (and thus in either element of $\varphi(\overline{\delta})$) are e_A, e_B, e_C and e_D respectively. On the left hand side we have

$$\frac{\mathrm{w}_{e_A}\mathrm{w}_{e_B}\mathrm{w}_{e_C}\mathrm{w}_{e_D}}{[J]}.$$

Note that for either dimer configuration δ' in $\varphi(\overline{\delta})$ we have $\mathrm{w}'_{e_A} = \mathrm{w}_{e_A}$, $\mathrm{w}'_{e_B} = \mathrm{w}_{e_B}/[J]$, $\mathrm{w}'_{e_C} = \mathrm{w}_{e_C}$ and $\mathrm{w}'_{e_D} = \mathrm{w}_{e_D}/[J]$, so on the right hand side we have:

$$\frac{\mathrm{w}_{e_A}\mathrm{w}_{e_B}\mathrm{w}_{e_C}\mathrm{w}_{e_D}([I_4][I_2]+[I_3][I_1])}{[J]^2[J']} = \frac{\mathrm{w}_{e_A}\mathrm{w}_{e_B}\mathrm{w}_{e_C}\mathrm{w}_{e_D}[J][J']}{[J]^2[J']}$$
$$= \frac{\mathrm{w}_{e_A}\mathrm{w}_{e_B}\mathrm{w}_{e_C}\mathrm{w}_{e_D}}{[J]}.$$

The claim is proved and the proposition follows. □

We note that a similar correspondence between perfect matchings under mutation is used independently in [23, Thm. 4.7].

Corollary 7.7. *Let I be a k-subset of $\{1,\ldots,n\}$. Then $\widetilde{\text{Đ}}_{G_P}(I)$ does not depend on the choice of Postnikov diagram P.*

Proof. This follows from Proposition 7.6 and Proposition 6.6. □

We may therefore write $\widetilde{\text{Đ}}(I)$ for the polynomial $\widetilde{\text{Đ}}_{G_P}(I)$ for any choice of Postnikov diagram P.

8. Some regular Postnikov diagrams

In this section, we introduce a regular Postnikov diagram, $\mathcal{R}_{k,n}$, for which the dual bipartite graph is, apart from a few extra edges, part of a hexagonal tiling of the plane. We assume in this section that $k \neq 1, n-1$. Reversing the strands in the corresponding diagram for $\mathrm{Gr}(n-k,n)$ gives rise to another regular Postnikov diagram for $\mathrm{Gr}_{k,n}$, which we denote by $\mathcal{R}^*_{n-k,n}$. The k-subsets labelling $\mathcal{R}_{k,n}$ are disjoint unions of two cyclic intervals (or coefficients, which consist of just one cyclic interval), and we show that the k-subsets labelling $\mathcal{R}^*_{n-k,n}$ also have this form, and in fact that the corresponding Plücker coordinates are exactly the twists of the Plücker coordinates corresponding to the labels of $\mathcal{R}_{k,n}$, up to a product of coefficient Plücker coordinates.

The regular form of $\mathcal{R}^*_{n-k,n}$ means that, given any k-subset I labelling $\mathcal{R}_{k,n}$, there is a unique dimer configuration on the bipartite graph $G_{\mathcal{R}^*_{n-k,n}}(I)$. This allows us to compute the corresponding scaled dimer partition function $\widetilde{\text{Đ}}_{\mathcal{R}^*_{n-k,n}}(I)$ explicitly and thus to show that for k-subsets I labelling $\mathcal{R}_{k,n}$, the twist of $[I]$ coincides with $\widetilde{\text{Đ}}_{\mathcal{R}^*_{n-k,n}}(I)$. The main result will then be shown in Section 9, using the fact that the twists of Plücker coordinates and the scaled dimer partition functions both satisfy the Plücker relations.

Let $\mathcal{R}_{k,n}$ be the diagram defined as follows. We take a tiling of the plane by regular hexagons and equilateral triangles in which each hexagon has 6 triangles adjacent to it and each triangle has 3 adjacent hexagons. The edges in the tiling are assumed to be horizontal or at an angle of $\pm\pi/3$ to the horizontal.

We consider the subset of the tiling obtained by taking $n-k-1$ rows of $k-1$ hexagons, with each row above and to the right of the previous row. We also include all of the triangles in the tiling adjacent to the hexagons.

The boundary triangles on the left of the diagram are labelled $T_{k+1}, T_{k+2}, \ldots, T_n$ from bottom to top, and the boundary triangles at the top are labelled $T_1, T_2, \ldots, T_k$ from left to right. We label the hexagon which is i hexagons across in the jth row from the bottom of the diagram by $H_{k,n}(i,j)$, for $1 \leq i \leq k-1$ and $1 \leq j \leq n-k-1$. We label the incomplete hexagons along the left of the diagram by $H_{k,n}(0,j)$, with $H_{k,n}(0,j)$ to the left of $H_{k,n}(1,j)$. We label the incomplete hexagons along the top of the diagram by $H_{k,n}(i,n-k)$, with $H_{k,n}(i,n-k)$ above $H_{k,n}(i,n-k-1)$.

For an example ($k=4, n=9$), see Figure 13.

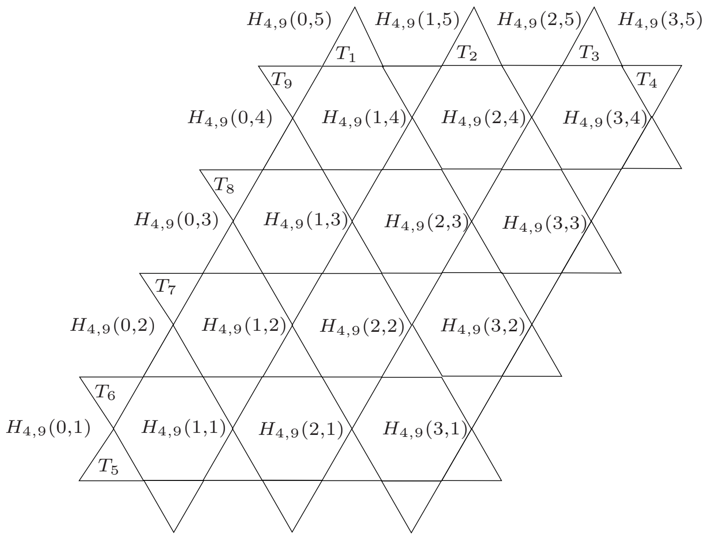


FIGURE 13. Subset of the planar tiling for $k = 4$, $n = 9$, used in construction of $\mathcal{R}_{4,9}$.

Next, the edges in the tiling are oriented by stipulating that each horizontal edge is oriented left to right, that each triangle is oriented (clockwise or anticlockwise), and that each hexagon has an alternating orientation.

Boundary triangles which are oriented clockwise (i.e. $T_k, T_{k+2}, T_{k+3}, \dots, T_n$) are split at the boundary vertex which is not incident with any hexagon, while for the remaining boundary triangles the two edges meeting at the boundary vertex are extended beyond it. The strand starting at triangle T_i is labelled i, for $i \in \{1, \dots, n\}$. See Figure 14 for $\mathcal{R}_{4,9}$. The following is easy to check.

Lemma 8.1. *The diagram $\mathcal{R}_{k,n}$ is a Postnikov diagram.*

We use $H_{k,n}(0,0)$ to denote the bottom right boundary face.

The bipartite dual $G_{k,n}$ of $\mathcal{R}_{k,n}$ consists of a subset of a hexagonal tiling of the plane in which the edges are either vertical or at $\pm\pi/3$ to the horizontal. There are $n-k-1$ adjacent horizontal rows of $k-1$ hexagons, together with extra vertical edges attached to the topmost $k-1$ vertices and a single extra edge at an angle of $\pi/3$ to the horizontal attached to the bottom left hexagon. See Figure 15 for $G_{4,9}$.

Lemma 8.2. *Fix $0 \le i \le k-1$ and $1 \le j \le n-k$. Then the strands (which exist) on the boundary of hexagon $H_{k,n}(i,j)$ in $\mathcal{R}_{k,n}$ are labelled as in Figure 16.*

Lemma 8.3. *Fix $0 \le i \le k-1$ and $1 \le j \le n-k$. Then the k-subset labelling $H_{k,n}(i,j)$ in $\mathcal{R}_{k,n}$ is*

$$M_{k,n}(i,j) = \{1, 2, \dots, i\} \cup \{i+j+1, i+j+2, ..., j+k\}.$$

The k-subset labelling $H_{k,n}(0,0)$ is $\{1, 2, \dots, k\}$.

Proof. The statement for $i = 0$ or $j = n-k$ follows from Remark 6.3. The result for all i, j then follows by induction on i using Lemma 8.2. Assume that the result is true for some $H_{k,n}(i,j)$. Then, since strands $i+j+1$ and $i+1$ cross between $H_{k,n}(i,j)$ and $H_{k,n}(i+1,j)$ to its right (point X in Figure 16), with $i+1$ going down and $i+j+1$ going up, it follows that

$$M_{k,n}(i+1,j) = M_{k,n}(i,j) \setminus \{i+j+1\}) \cup \{i+1\},$$

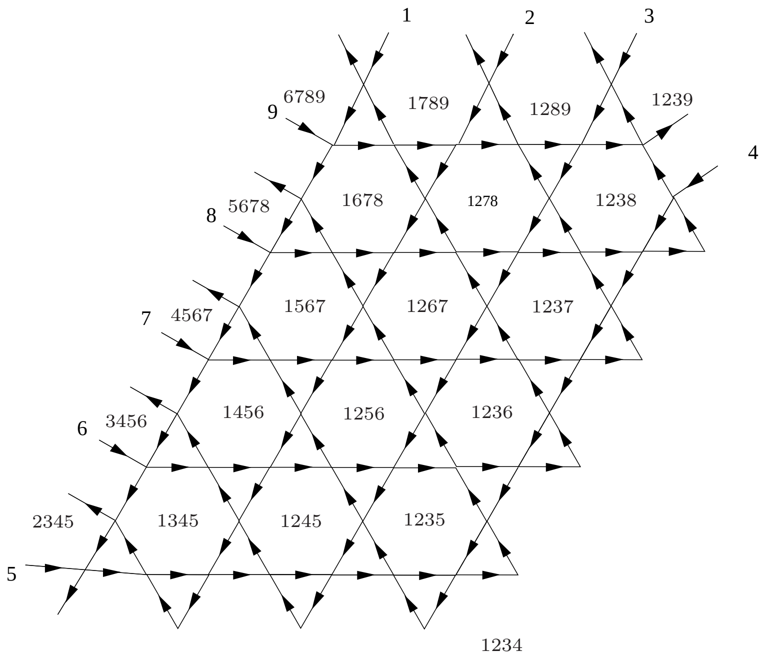


FIGURE 14. The Postnikov diagram $\mathcal{R}_{4,9}$.

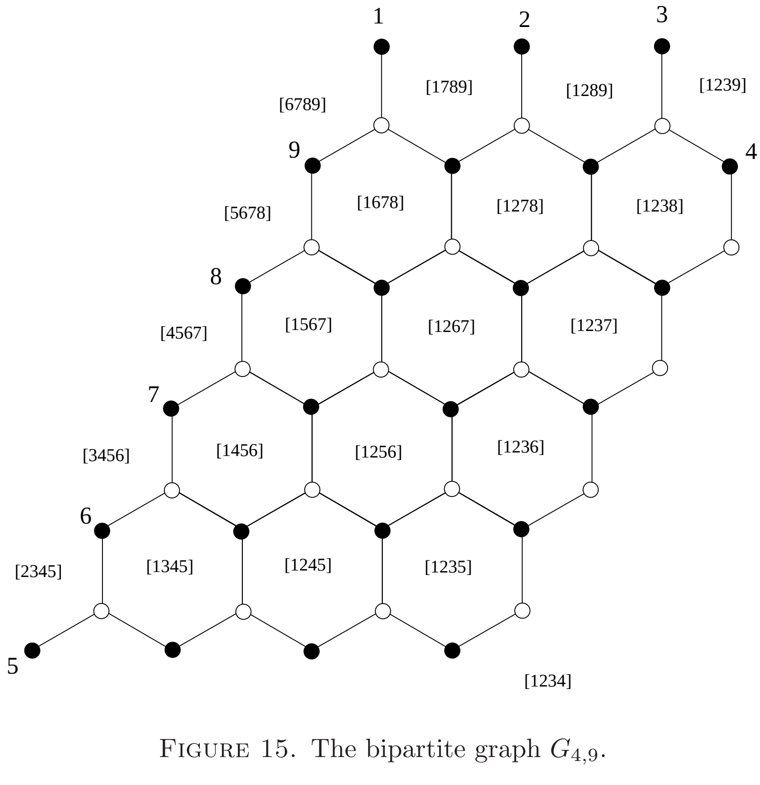


FIGURE 15. The bipartite graph $G_{4,9}$.

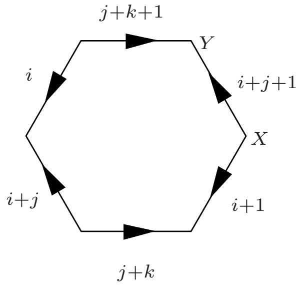


FIGURE 16. A hexagon $H_{k,n}(i,j)$ in $\mathcal{R}_{k,n}$.

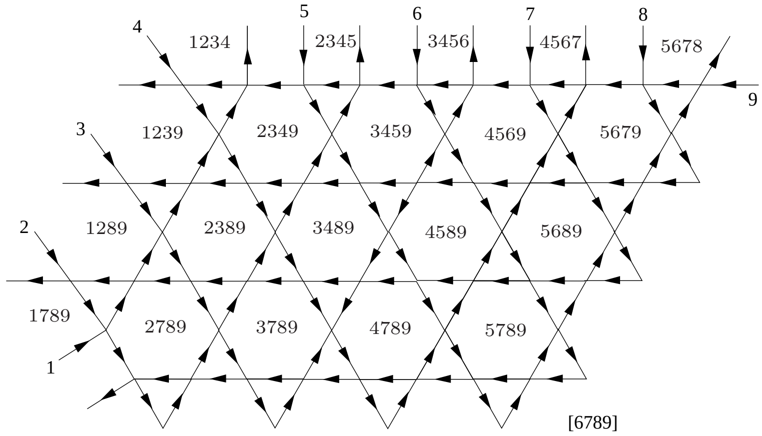


FIGURE 17. The Postnikov diagram $\mathcal{R}^*_{5,9}$.

as required. $\square$

For a Postnikov diagram P for $\mathrm{Gr}_{k,n}$ let P^* denote the diagram obtained from P by reversing the orientation of each of the strands in P. Each strand retains the same label. We also add an extra crossing at the boundary between the strand starting at i and the strand ending at i', for each i. We then annihilate any local oriented lenses produced by this procedure. The following can be seen by checking that the appropriate conditions in Definition 6.1 are satisfied.

Lemma 8.4. *For any Postnikov diagram P for $\mathrm{Gr}_{k,n}$, the diagram P^* is a Postnikov diagram for $\mathrm{Gr}_{n-k,n}$.*

It follows that $\mathcal{R}^*_{n-k,n}$ is a Postnikov diagram for $\mathrm{Gr}_{k,n}$. The labels of its alternating faces are the complements (in $\{1,2,\ldots,n\}$) of the labels of the corresponding faces in $\mathcal{R}_{n-k,n}$. For example, see $\mathcal{R}^*_{5,9}$ in Figure 17. The bipartite graph $G_{\mathcal{R}^*_{5,9}}$ is shown in Figure 18.

The diagram $\mathcal{R}_{n-k,n}$ has faces $H_{n-k,n}(j,i)$, with $1 \le i \le k$ and $0 \le j \le n-k-1$ (and $i=j=0$). The label of $H_{n-k,n}(j,i)$ is:

$$M_{n-k,n}(j,i) = \{1,2,\ldots,j\} \cup \{i+j+1, i+j+2, \ldots, i+n-k\}.$$

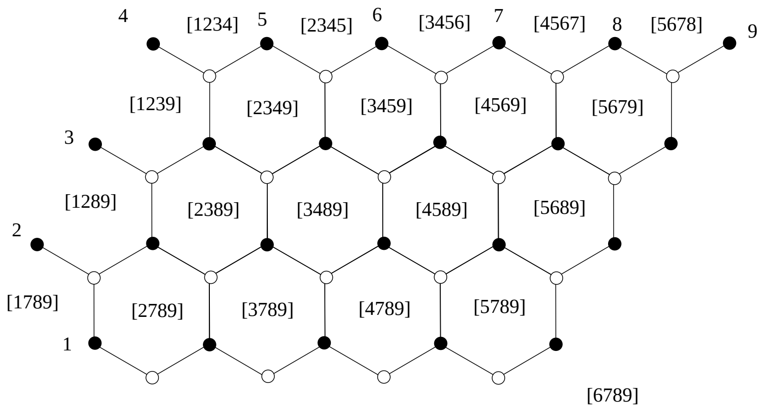


FIGURE 18. The bipartite graph $G_{\mathcal{R}^*_{5,9}}$.

We denote the corresponding faces in $\mathcal{R}^*_{n-k,n}$ by $H^*_{n-k,n}(j,i)$. The label of $H^*_{n-k,n}(j,i)$ is the complement of $M_{n-k,n}(j,i)$ in $\{1,\ldots,n\}$, which is:

$$M^*_{n-k,n}(j,i) = \{j+1, j+2, \ldots, i+j\} \cup \{i+n-k+1, i+n-k+2, \ldots, n\}.$$

Lemma 8.5. *Fix $0 \le i \le k-1$ and $1 \le j \le n-k$, or $(i,j) = (0,0)$. Then we have*

$$\overleftarrow{[M_{k,n}(i,j)]} = [M^*_{n-k,n}(j,i)] \cdot \prod_{r=1}^{i-1} [\mathbf{r}] \cdot \prod_{r=i+j+1}^{j+k-1} [\mathbf{r}].$$

Hence the Plücker coordinates whose k-subsets label the alternating faces of $\mathcal{R}^*_{n-k,n}$ are (up to multiplication by coefficients) exactly the twists of the Plücker coordinates whose k-subsets label the alternating faces of $\mathcal{R}_{k,n}$.

Proof. Fix $0 \le i \le k-1$ and $1 \le j \le n-k$, or $(i,j) = (0,0)$. Recall that, by Lemma 8.3,

$$M_{k,n}(i,j) := \{1, 2, \ldots, i\} \cup \{i+j+1, i+j+2, ..., j+k\}.$$

Hence, by Proposition 3.5,

$$\overleftarrow{[M_{k,n}(i,j)]} = [J] \cdot \prod_{r=1}^{i-1} [\mathbf{r}] \cdot \prod_{r=i+j+1}^{j+k-1} [\mathbf{r}],$$

where

$$J = \{n+1+i-k, n+2+i-k, \ldots, n\} \cup \{j+1, \ldots, i+j\} = M^*_{n-k,n}(j,i),$$

and the result follows. □

By Lemmas 8.1 and 8.3, the pair $(\widetilde{\mathbf{x}}(\mathcal{R}_{k,n}), \widetilde{Q}(\mathcal{R}_{k,n}))$ is a seed of $\mathbb{C}[\mathrm{Gr}_{k,n}]$, where

$$\widetilde{\mathbf{x}}(\mathcal{R}_{k,n}) = \{[M_{k,n}(i,j)] \,:\, 0 \le i \le k-1, 1 \le j \le n-k \text{ or } i = j = 0\}.$$

Recall that, for a quiver $\widetilde{Q}$ in a seed of $\mathbb{C}[\mathrm{Gr}_{k,n}]$, Q denotes the principal part of $\widetilde{Q}$. In particular, $Q(\mathcal{R}_{k,n})$ is the principal part of $\widetilde{Q}(\mathcal{R}_{k,n})$.

Definition 8.6. We define $\mathbb{C}[\mathrm{Gr}_{k,n}]_1$ to be the cluster algebra in $\mathbb{C}(\mathbf{x}_{\mathcal{R}_{k,n}})$ with initial seed $(\mathbf{x}_{\mathcal{R}_{k,n}}, Q(\mathcal{R}_{k,n}))$.

Note that using any other seed of $\mathbb{C}[\mathrm{Gr}_{k,n}]$ in Definition 8.6 would give the same result.

Remark 8.7. By [16, §12] (see also [3, §6]) there is a surjective ring homomorphism $\pi : \mathbb{C}[\mathrm{Gr}_{k,n}] \to \mathbb{C}[\mathrm{Gr}_{k,n}]_1$ mapping the coefficients in $\mathbb{C}[\mathrm{Gr}_{k,n}]$ to 1 and cluster variables to cluster variables (surjectively) such that whenever $S = (\widetilde{\mathbf{x}}, \widetilde{Q})$ is a seed in $\mathbb{C}[\mathrm{Gr}_{k,n}]$, $\pi(S) = (\pi(\mathbf{x}), Q)$ is a seed in $\mathbb{C}[\mathrm{Gr}_{k,n}]_1$. This map respects mutation in the following sense. If $S = (\widetilde{\mathbf{x}}, \widetilde{Q})$ and $S' = (\widetilde{\mathbf{x}}', \widetilde{Q}')$ are seeds in $\mathbb{C}[\mathrm{Gr}_{k,n}]$ and S' is obtained from S by mutation at $x \in \mathbf{x}$, then $\pi(S')$ can be obtained from $\pi(S)$ by mutation at $\pi(x)$.

Definition 8.8. We shall say that two elements $f, g \in \mathbb{C}[\mathrm{Gr}_{k,n}]$ are *equal up to coefficients* if $\pi(f) = \pi(g)$, and that two homomorphisms $\varphi_1, \varphi_2 : \mathbb{C}[Gr_{k,n}] \to \mathbb{C}[\mathrm{Gr}_{k,n}]$ are *equal up to coefficients* if $\pi \circ \varphi_1 = \pi \circ \varphi_2$.

We have the following:

Proposition 8.9. *Up to coefficients, the $(2n)$th power of the twist on $\mathbb{C}[\mathrm{Gr}_{k,n}]$ is equal to the identity.*

Proof. Suppose that $k > 1$ and I is a k-subset of $\{1, \ldots, n\}$. By Proposition 4.3, applying the twist $2n$ times to $[I] \in \mathbb{C}[\mathrm{Gr}_{k,n}]$ gives $[I]$ multiplied by a monomial in the Plücker coordinates $[\mathbf{i}]$, $i \in \{1, \ldots, n\}$. The images of this element and $[I]$ under π are equal. The result then follows from the fact that the Plücker coordinates generate $\mathbb{C}[\mathrm{Gr}_{k,n}]$. If $k = 1$, then the twist of any Plücker coordinate is 1 by Remark 3.7, giving the result in this case. □

We can also show that the twist of a cluster variable is again a cluster variable, up to coefficients. We thank David Speyer for communicating this proof to us.

Proposition 8.10. *Let $(\widetilde{\mathbf{x}}, \widetilde{Q})$ be a seed in $\mathbb{C}[\mathrm{Gr}_{k,n}]$. Then $\left(\pi\left(\overleftarrow{\mathbf{x}}\right), Q\right)$ is a seed in $\mathbb{C}[\mathrm{Gr}_{k,n}]_1$. In particular, if x is a cluster variable in $\mathbb{C}[\mathrm{Gr}_{k,n}]$ then $\pi\left(\overleftarrow{x}\right)$ is a cluster variable in $\mathbb{C}[\mathrm{Gr}_{k,n}]_1$, and hence of the form $\pi(y)$ for some cluster variable y in $\mathbb{C}[\mathrm{Gr}_{k,n}]$.*

Proof. Since the principal parts of $\mathcal{R}_{k,n}$ and $\mathcal{R}^*_{n-k,n}$ are isomorphic, the statement holds for the seed $(\widetilde{\mathbf{x}}(\mathcal{R}_{k,n}), \widetilde{Q}(\mathcal{R}_{k,n}))$ by Lemma 8.5. Let $(\widetilde{\mathbf{x}}, \widetilde{Q})$ be a seed of $\mathbb{C}[\mathrm{Gr}_{k,n}]$. We shall prove the result by induction on the length of the shortest path in the exchange graph from $(\widetilde{\mathbf{x}}_{\mathcal{R}_{k,n}}, \widetilde{Q}(\mathcal{R}_{k,n}))$ to $(\widetilde{\mathbf{x}}, \widetilde{Q})$.

Suppose that the result holds for $(\widetilde{\mathbf{x}}, \widetilde{Q})$ and that the seed $S' = (\widetilde{\mathbf{x}}', \widetilde{Q}')$ is obtained from $S = (\widetilde{\mathbf{x}}, \widetilde{Q})$ by mutating at a cluster variable x_r, replacing it with x'_r. The corresponding exchange relation is:

$$x_r x'_r = \prod_{j \to r} x_j + \prod_{r \to j} x_j, \tag{8.1}$$

with the first product taken over all arrows in $\widetilde{Q}$ ending at r, and the second over all arrows in $\widetilde{Q}$ starting at r. We apply the twist and then π to 8.1. Note that both of these maps are ring homomorphisms and the twist of a coefficient is a product of coefficients by Remark 3.7. We therefore obtain:

$$\pi\left(\overleftarrow{x_r}\right) \pi\left(\overleftarrow{x'_r}\right) = \prod_{j \to r} \pi\left(\overleftarrow{x_j}\right) + \prod_{r \to j} \pi\left(\overleftarrow{x_j}\right), \tag{8.2}$$

with the first product taken over all arrows in Q ending at r, and the second over all arrows in Q starting at r.

By our assumption, $\left(\pi\left(\overleftarrow{\mathbf{x}}\right), Q\right)$ is a seed in $\mathbb{C}[\mathrm{Gr}_{k,n}]_1$. Hence, (8.2) is the exchange relation for $\pi\left(\overleftarrow{x_r}\right)$ in the seed $\left(\pi\left(\overleftarrow{\mathbf{x}}\right), Q\right)$ of $\mathbb{C}[\mathrm{Gr}_{k,n}]_1$. It follows that the seed in $\mathbb{C}[\mathrm{Gr}_{k,n}]_1$ obtained from $\pi\left(\overleftarrow{\mathbf{x}}\right)$ via (8.2) is $\left(\pi\left(\overleftarrow{\mathbf{x}'}\right), Q'\right)$, since the mutation of Q at r is Q', the principal part of $\widetilde{Q}'$. In particular, this pair is a seed in $\mathbb{C}[\mathrm{Gr}_{k,n}]_1$ as required. The result follows by induction. □

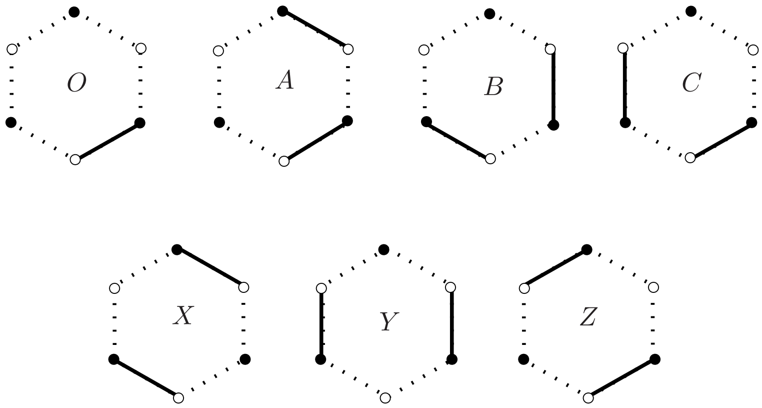


Figure 19. Edge choices in a hexagon

Next, we consider seven different choices of edges of a hexagon, O, A, B, C, X, Y, Z, displayed in Figure 19. Dashed lines are not in the dimer configuration, while full lines are in the dimer configuration.

If i is a boundary vertex which is incident to a unique edge in the dual bipartite graph G_P of a Postnikov diagram P, we say that i is on a *stalk*. For any k-subset I of $\{1,\dots,n\}$, we modify $G_{\mathcal{R}^*_{n-k,n}}$ by applying blow-up moves to all boundary vertices in I which are not on stalks. This ensures that each boundary vertex in $G_{\mathcal{R}^*_{n-k,n}}(I)$ is on a stalk. We draw all edges on the boundary of the diagram in such a way as to continue the hexagonal tiling.

Proposition 8.11. *Fix $0 \le i \le k-1$ and $1 \le j \le n-k$ or $(i,j) = (0,0)$. Then the bipartite graph $G_{\mathcal{R}^*_{n-k,n}}(M_{k,n}(i,j))$ has a unique dimer configuration, in which a hexagon $H^*_{n-k,n}(a,b)$ for $0 \le a \le n-k-1$ and $1 \le b \le k$ has type given by the following table:*

Type of hexagon	*Restriction on a, b*
O	$(a,b) = (j,i)$
A	$a < j,\ b = i$
B	$a + b = i + j,\ a < j$ *(equivalent to $b > i$)*
C	$a = j,\ b > i$
X	$a + b < i + j,\ b > i$
Y	$a + b > i + j,\ a < j$
Z	*otherwise.*

Here we regard an incomplete hexagon as having the appropriate type if, for the edges that do appear, the edges in the dimer configuration correspond to the choice of edges in the type (note that this may not be unique).

Proof. Let δ be a dimer configuration on $G = G_{\mathcal{R}^*_{n-k,n}}(M_{k,n}(i,j))$. Denote the set of hexagons appearing in the table next to letter A by $\mathcal{H}(A)$, and similarly for A, B, C, X, Y and Z. We may write $M_{k,n}(i,j)$ in the form $I_1 \cup I_2$ where I_1 and I_2 are disjoint integer intervals (i.e. in the usual sense of intervals), where I_2 may be empty.

Note that any edge in a stalk in $G_{\mathcal{R}^*_{n-k,n}}(M_{k,n}(i,j))$ must be in a perfect matching on this graph. We use this fact repeatedly in the following.

Case I: Firstly, assume that $1 \leq i \leq k-1$, $1 \leq j \leq n-k-1$. Then $I_1 = \{1,\dots,i\}$ and $I_2 = \{i+j+1,\dots,j+k\}$, and both are nonempty. Write $\{1,\dots,n\} = I_1 \cup J_1 \cup I_2 \cup J_2$ as disjoint union of intervals (not cyclic). Let a_1 (respectively, a_2, b_1, b_2) be the largest element of I_1 (respectively, I_2, J_1, J_2). Note that $b_2 = n$.
Case I(a): Suppose first that $i+j \leq k$. In this case, the boundary face labelled $[\mathbf{a}]$ in G must be of type Z if $a \in I_1 \setminus \{a_1\}$, of type A if $a = a_1$, of type X if $a \in J_1 \setminus \{b_1\}$, of type B if $a = b_1$, of type Y if $a \in I_2 \setminus \{a_2\}$, of type C if $a = a_2$ and of type Z if $a \in J_2 \setminus \{b_2\}$. See Figure 20 for an example.
Case I(b): Secondly, assume that $i+j > k$. In this case, the boundary face labelled $[\mathbf{a}]$ in G must be of type Z if $a \in I_1 \setminus \{a_1\}$, of type A if $a = a_1$, of type X if $a \in J_1 \setminus \{b_1\}$, of type B if $a = b_1$, of type Y if $a \in I_2 \setminus \{a_2\}$, of type C if $a = a_2$, and of type Z if $a \in J_2 \setminus \{b_2\}$. See Figure 21 for an example.
Case II: Next, we assume that $i = 0$ and $1 \leq j \leq n-k$. Then $I_1 = \{j+1,\dots,j+k\}$ and I_2 is empty. Write $\{1,\dots,n\}$ as a disjoint union $J_1 \cup I_1 \cup J_2$ of intervals, defining a_1, b_1, b_2 as above. Note that $b_2 = n$. Then a boundary face labelled $[\mathbf{a}]$ in G must be of type X if $a \in J_1 \setminus \{b_1\}$, of type B if $a = b_1$, of type Y if $a \in I_1 \setminus \{a_1\}$, of type C if $a = a_1$, and of type Z if $a \in J_2 \setminus \{b_2\}$. See Figure 22 for an example.
Case III: Next, we assume that $1 \leq i \leq k-1$ and $j = n-k$. Then $I_1 = \{1,\dots,i\}$ and $I_2 = \{i+n-k+1,\dots,n\}$, and both are non-empty. Write $\{1,\dots,n\} = I_1 \cup J_1 \cup I_2$ as a disjoint union of intervals, defining a_1, a_2, b_1 as before. Note that $a_2 = n$. Then a boundary face labelled $[\mathbf{a}]$ in G must be of type Z if $a \in I_1 \setminus \{a_1\}$, of type A if $a = a_1$, of type X if $a \in J_1 \setminus \{b_1\}$, of type B if $a = b_1$, of type Y if $a \in I_2 \setminus \{a_2\}$. See Figure 23 for an example.
Case IV: The final case is $i = j = 0$. Then $I_1 = \{1,\dots,k\}$ and I_2 is empty. Then a boundary face labelled $[\mathbf{a}]$ in G for any $a \geq \{1,\dots,n\}$ must be of type Z.

In any of these cases, this forces the hexagons in $\mathcal{H}(X)$, $\mathcal{H}(Y)$ and $\mathcal{H}(Z)$ to be of type X, Y or Z, respectively, using an easy induction argument starting from the boundary. This also forces the hexagons in the boundary strips $\mathcal{H}(A)$, $\mathcal{H}(B)$ and $\mathcal{H}(C)$ inbetween to be of the appropriate types. If $1 \leq i \leq k-1$ and $1 \leq j \leq n-k-1$, i.e. if $M_{k,n}(i,j)$ is not a coefficient, a single hexagon remains, i.e. the unique element of $\mathcal{H}(O)$, which is forced to be of type O. Otherwise, there are no hexagons remaining and type O does not occur.

Note that the regions $\mathcal{H}(X)$ and $\mathcal{H}(Y)$ are roughly triangular (or truncated triangular), with $\mathcal{H}(Z)$ forming the remainder (apart from $\mathcal{H}(O)$ and the boundary strips $\mathcal{H}(A), \mathcal{H}(B)$ and $\mathcal{H}(C)$). □

Remark 8.12. If $i = 0$ or $j = n-k$, then the restrictions $a < j$ for A and B and $b > i$ for C are automatically satisfied and thus can be omitted. Furthermore, if if $i = 0$ and $0 \leq j \leq n-k-1$ there are no hexagons of type A, and if if $1 \leq i \leq k-1$ and $j = n-k$, there are no hexagons of type C. If $i = j = 0$ then every hexagon is of type Z.

The example in Figure 20 shows a case where $\mathcal{H}(Y)$ forms a truncated triangle, while in Figure 21, $\mathcal{H}(X)$ forms a truncated triangle. Figure 22 is a case where I_2 is empty (and $\mathcal{H}(Y)$ is a truncated triangle), while Figure 23 illustrates a case where $\mathcal{H}(X)$ is a truncated triangle. In the examples in Figures 22 and 23, $[M_{k,n}(i,j)]$ is a coefficient.

In each case, the edges in the dimer configuration are drawn as full lines, and the other edges as dotted lines). The type of each hexagon is also indicated.

Corollary 8.13. *For any Postnikov diagram P and k-subset I, the dual bipartite graph $G_P(I)$ is balanced.*

Proof. By Proposition 8.11, $G_{\mathcal{R}^*_{n-k,n}}(I)$ has a dimer configuration for any k-subset I labelling $\mathcal{R}_{k,n}$, so it is balanced. It follows that $G_{\mathcal{R}^*_{n-k,n}}(I)$ is balanced for any k-subset I, since all the boundary vertices are black. Since the blow-up, blow-down and quadrilateral moves preserve the difference

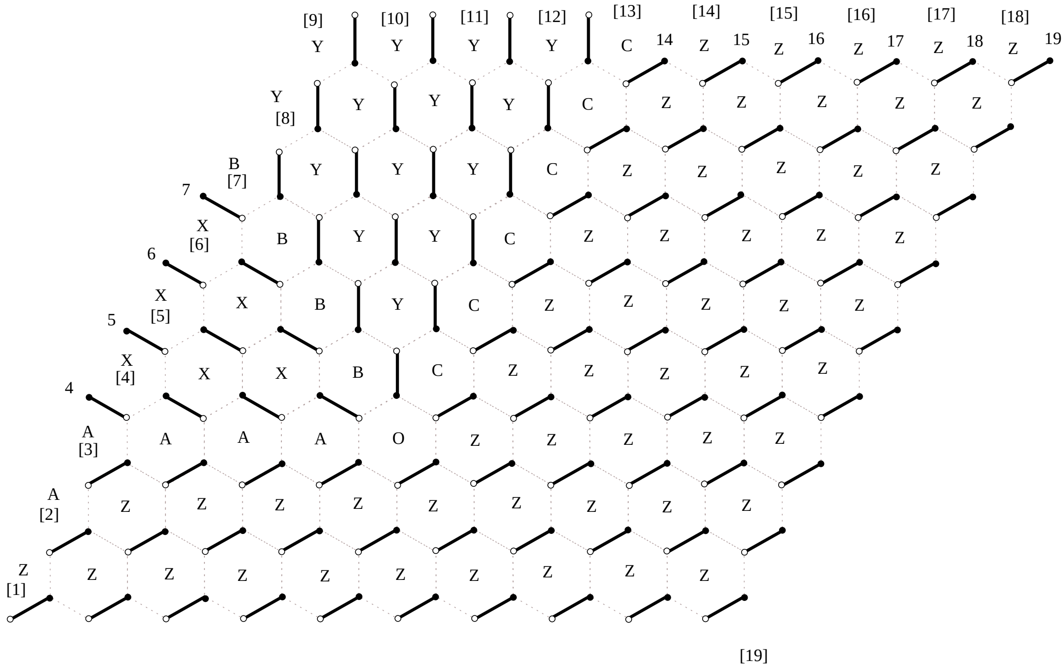


Figure 20. The unique dimer configuration in $G_{\mathcal{R}^*_{10,19}}(M_{9,19}(3,4)) = G_{\mathcal{R}^*_{10,19}}(\{1,2,3\} \cup \{8,9,10,11,12,13\})$. As per definition, $G_{\mathcal{R}^*_{10,19}}(M_{9,19}(3,4))$ is constructed from $G_{\mathcal{R}^*_{10,19}}$ by deleting the boundary vertices in $M_{9,19}(3,4) = \{1,2,3\} \cup \{8,9,10,11,12,13\}$ (and their labels).

between the number of white vertices and the number of black vertices, the result follows, using Proposition 6.6. □

Fix $0 \leq i \leq k-1$ and $1 \leq j \leq n-k$ or $(i,j) = (0,0)$. Then we have the scaled dimer partition function $\widetilde{\text{Đ}}_{\mathcal{R}^*_{n-k,n}}(M_{k,n}(i,j))$, which is the dimer partition function of $G_{\mathcal{R}^*_{n-k,n}}(M_{k,n}(i,j))$ divided by the product of Plücker coordinates labelling the interior faces of $G_{\mathcal{R}^*_{n-k,n}}(M_{k,n}(i,j))$ (i.e. the Plücker coordinates lying in the corresponding cluster which are not coefficients).

Proposition 8.14. *Fix $0 \leq i \leq k-1$ and $1 \leq j \leq n-k$ or $(i,j) = (0,0)$. Then*

$$\overleftarrow{[M_{k,n}(i,j)]} = \widetilde{\text{Đ}}_{\mathcal{R}^*_{n-k,n}}(M_{k,n}(i,j)).$$

Proof. Recall that, by Lemma 8.5, we have:

$$\overleftarrow{[M_{k,n}(i,j)]} = [M^*_{n-k,n}(j,i)] \cdot \prod_{r=1}^{i-1}[\mathbf{r}] \cdot \prod_{r=i+j+1}^{j+k-1}[\mathbf{r}].$$

By Proposition 8.11, $\widetilde{\text{Đ}}_{\mathcal{R}^*_{n-k,n}}(M_{k,n}(i,j))$ is the weight w_δ associated to the unique dimer configuration δ on $G_{\mathcal{R}^*_{n-k,n}}(M_{k,n}(i,j))$ given in the lemma, divided by the product of the Plücker coordinates labelling non-boundary faces in $\mathcal{R}^*_{n-k,n}(M_{k,n}(i,j))$. By the definition of the weighting on $G_{\mathcal{R}^*_{n-k,n}}(M_{k,n}(i,j))$, the exponent of a Plücker coordinate in w_δ associated to a face of $G_{\mathcal{R}^*_{n-k,n}}(M_{k,n}(i,j))$ is equal to the number of edges in δ for which only the white vertex of the

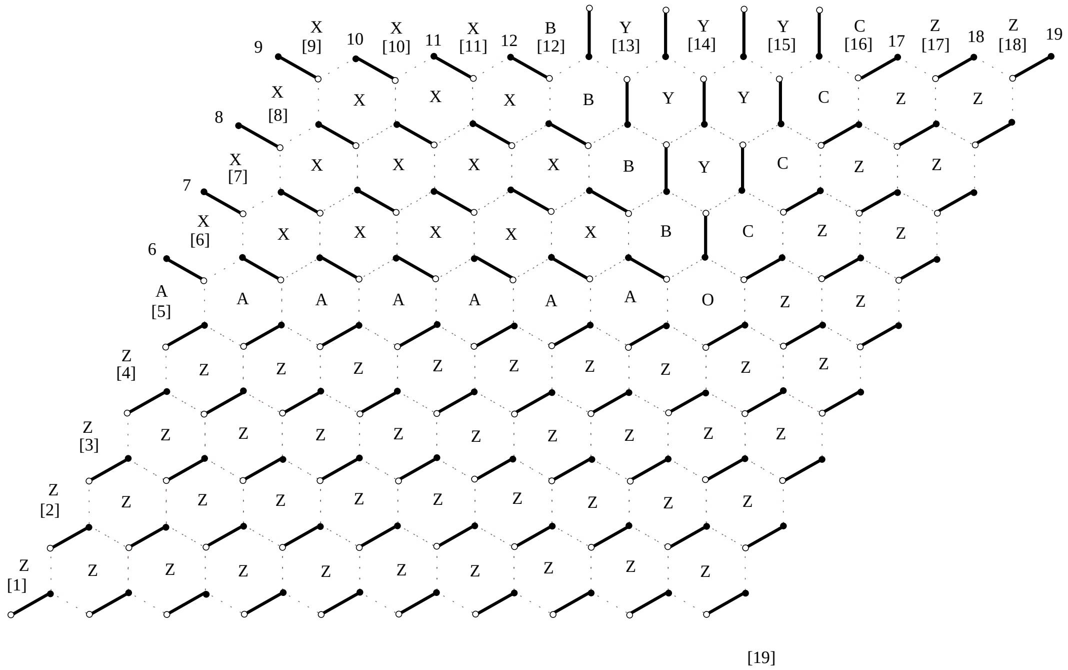


FIGURE 21. The unique dimer configuration in $G_{\mathcal{R}^*_{10,19}}(M_{9,19}(5,7)) = G_{\mathcal{R}^*_{10,19}}(\{1,2,3,4,5\} \cup \{13,14,15,16\})$.

edge is incident with the face. Therefore (since δ is a dimer configuration), the exponent in w_δ of a Plücker coordinate corresponding to a face of $G_{\mathcal{R}^*_{n-k,n}}(M_{k,n}(i,j))$ coincides with the number of white vertices on the boundary of the face which are not incident with an edge incident with two vertices on the boundary.

It follows (see the description of the hexagon types in Figure 19) that the exponent of a Plücker coordinate corresponding to a non-boundary face is exactly one for all cases except the Plücker coordinate corresponding to the hexagon labelled O (if one exists) in which case the exponent is 2. Note that this hexagon, if it exists, is labelled with the Plücker coordinate $[M^*_{n-k,n}(j,i)]$.

We can now check that, in each of the cases of the proof of Proposition 8.11, and using the notation there, the exponent of a coefficient $[\mathbf{a}]$ in $\widetilde{\text{Đ}}_{\mathcal{R}^*_{n-k,n}}(M_{k,n}(i,j))$ is 1 if $a \in I_1 \setminus \{a_1\}$ or $a \in I_2$, and zero otherwise. For example, in Case I(a), if $a \in I_1 \setminus \{a_1\}$, then the part of $G_{\mathcal{R}^*_{n-k,n}}(M_{k,n}(i,j))$ adjacent to the boundary face labelled $[\mathbf{a}]$ must be as in Figure 24, so the exponent of $[\mathbf{a}]$ must be 1. The other cases are similar. The result follows. □

9. Proof of the main result

In this section we prove our main result. Recall that the short Plücker relations are the relations of the form

$$[Jac][Jbd] = [Jab][Jcd] + [Jad][Jbc],$$

where J is a $(k-2)$-subset with $J \cap \{a,b,c,d\} = \phi$.

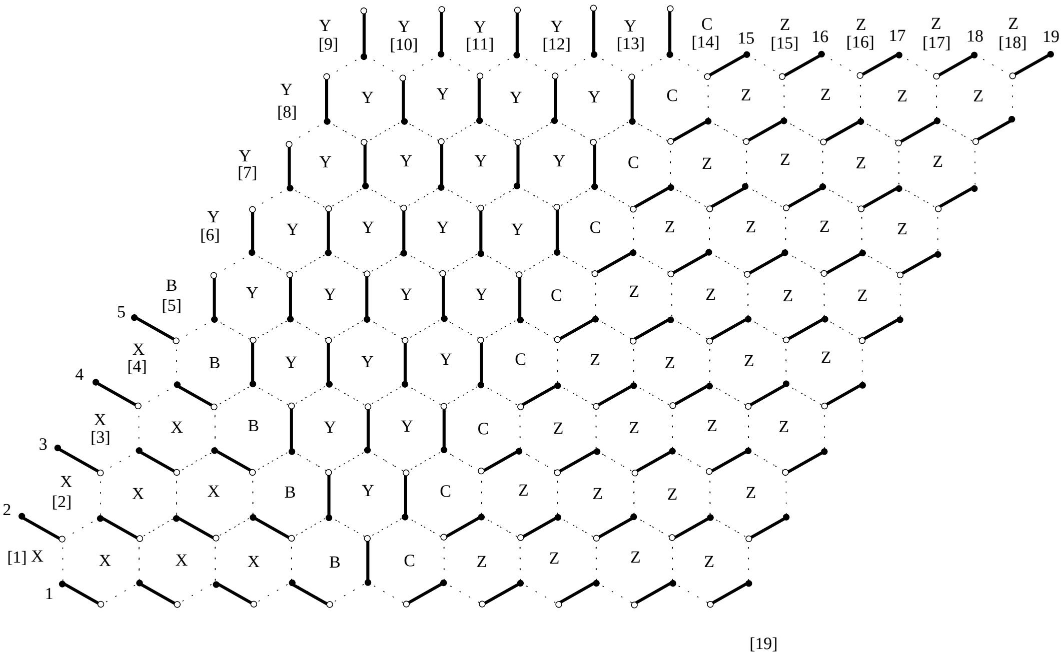


FIGURE 22. The unique dimer configuration in $G_{\mathcal{R}^*_{10,19}}(M_{9,19}(0,5)) = G_{\mathcal{R}^*_{10,19}}(\{6,7,8,9,10,11,12,13,14\})$.

Proposition 9.1. *Let P be a Postnikov diagram. Then the elements $\widetilde{\text{Đ}}_{G_P}(I)$, for I a k-subset of $\{1,\ldots,n\}$, satisfy the short Plücker relations.*

Proof. The proposition for the $\text{Đ}_{G_P}(I)$ follows from [27, Thm. 2.1], noting that the proof given there carries over to weighted graphs as well as the enumeration result stated. But then the result follows, as the $\widetilde{\text{Đ}}_{G_P}(I)$ are obtained from the $\text{Đ}_{G_P}(I)$ by scaling by a constant monomial. □

Lemma 9.2. *The elements $\overleftarrow{[I]}$, for I a k-subset of $\{1,\ldots,n\}$, satisfy the short Plücker relations.*

Proof. This follows from the facts that the Plücker coordinates $[I]$ satisfy the short Plücker relations and that the twist is a homomorphism of algebras. □

Proof of Theorem 1.1. If $k=1$ or $n-1$ there is, up to blow-up and blow-down moves, a unique Postnikov diagram, and it is easy to check that the result holds directly. Otherwise, if I is the label of an alternating face of $\mathcal{R}_{k,n}$, then the theorem holds for I by Proposition 8.14. For the general case, we prove the result by induction on the number e of quadrilateral moves required to get from $\mathcal{R}_{k,n}$ to a Postnikov diagram P which has I labelling one of its alternating faces. The above deals with the case $e=0$. Suppose the result is known for all k-subsets labelling Postnikov diagrams which can be obtained by a sequence of fewer than e quadrilateral moves starting from $\mathcal{R}_{k,n}$. Then there is a Postnikov diagram P', related to P by a quadrilateral move, for which the result is known for all k-subsets labelling P'. So if I labels an alternating face of P', we are done. But if not, $[I]$ is related to the Plücker coordinates of P' by a short Plücker relation. That the result then holds for I follows from Proposition 9.1 and Lemma 9.2. The result then follows by induction, using Proposition 6.6. □

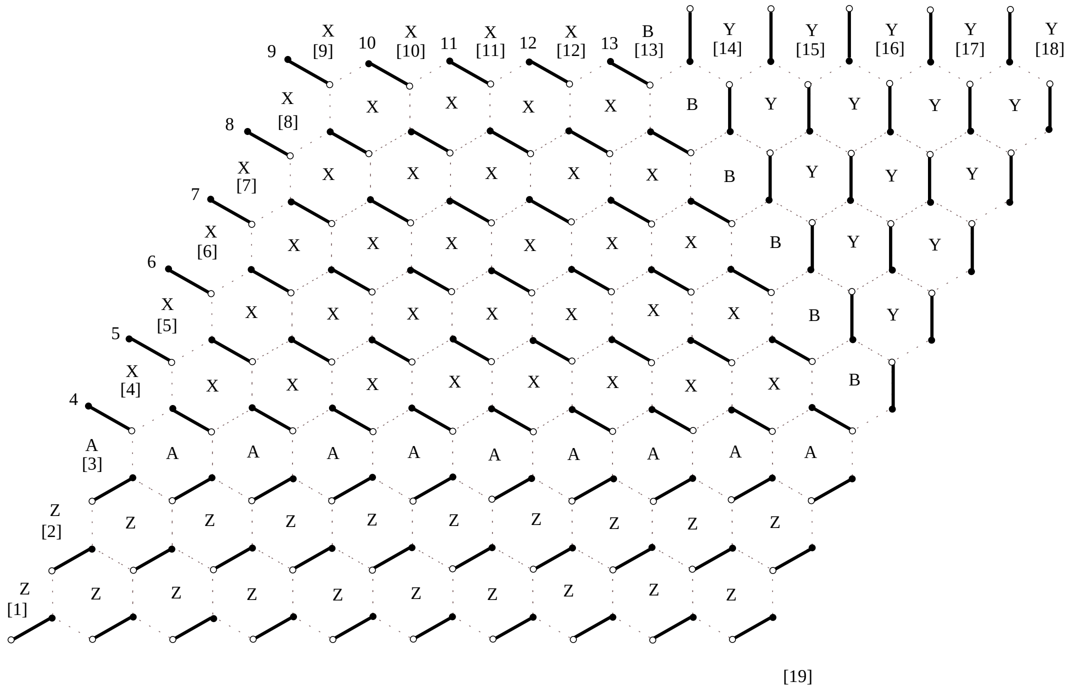


FIGURE 23. The unique dimer configuration in $G_{\mathcal{R}^*_{10,19}}(M_{9,19}(3,10)) = G_{\mathcal{R}^*_{10,19}}(\{1,2,3\} \cup \{14,15,16,17,18,19\})$.

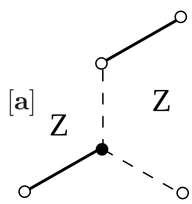


FIGURE 24. Proof of Proposition 8.14.

Remark 9.3. If $k = 2$ then, by Proposition 3.5, the twist of an arbitrary Plücker coordinate $[ab]$ is $\overleftarrow{[ab]} = [\sigma(a)\sigma(b)]$ and we see that every Plücker coordinate is the twist of a Plücker coordinate, so Theorem 1.1 gives a formula for any Plücker coordinate as a positive Laurent polynomial in terms of any initial cluster. We note that in this case, such a formula has already been given in [38].

Remark 9.4. Note also that it follows from Theorem 1.1 that for every Postnikov diagram and k-subset I, the bipartite graph $G_P(I)$ admits at least one dimer configuration, i.e. it is *factorizable*.

Recall that the *totally positive* part of the real Grassmannian $Gr_{k,n}(\mathbb{R})$ is the subset

$$(\mathrm{Gr}_{k,n})_{>0} = \{p \in Gr_{k,n}(\mathbb{R}) \,:\, [I](p) > 0 \text{ for all } k\text{-subsets } I \text{ of } \{1,\ldots,k\}\},$$

while the totally nonnegative part is

$$(\mathrm{Gr}_{k,n})_{\geq 0} = \{p \in Gr_{k,n}(\mathbb{R}) \,:\, [I](p) \geq 0 \text{ for all } k\text{-subsets } I \text{ of } \{1,\ldots,k\}\}.$$

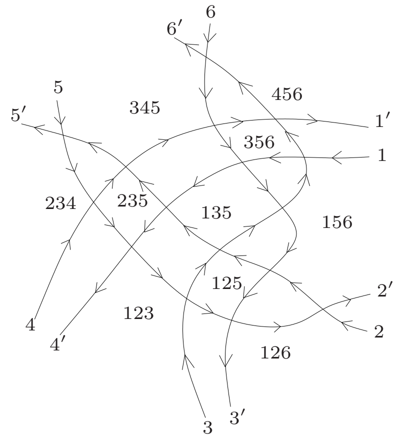


Figure 25. A Postnikov diagram for $\mathrm{Gr}_{3,6}$.

Corollary 9.5. *The twist preserves the totally positive part of the Grassmannian and the totally nonnegative part of the Grassmannian.*

Proof. This follows from Theorem 1.1 and Remark 9.4. □

10. An example

We give an example of the main result, taking $k = 3$ and $n = 6$. Consider the Postnikov diagram P for $\mathrm{Gr}_{3,6}$ shown in Figure 25. The weighted bipartite graph G_P is shown in Figure 26, and $G_P(\{2,5,6\})$ can be obtained from G_P by removing the black boundary vertices labelled $2, 5$ and 6. There are six dimer configurations $\delta_1, \ldots, \delta_6$ on $G_P(\{2,5,6\})$, and the corresponding monomials w_{δ_i} are shown in Figure 27.

The corresponding dimer partition function, $\text{Đ}_{G_P}(\{2,5,6\})$, is the sum of these. Applying short Plücker relations we obtain:

$$\begin{aligned}
\mathrm{w}_{\delta_1} + \mathrm{w}_{\delta_2} &= [123][145][156][235][345][356];\\
\mathrm{w}_{\delta_3} + \mathrm{w}_{\delta_4} &= [125][134][156][235][345][356];\\
\mathrm{w}_{\delta_5} + \mathrm{w}_{\delta_6} &= [126][135][145][235][345][356].
\end{aligned}$$

Applying a short Plücker relation to $(\mathrm{w}_{\delta_1} + \mathrm{w}_{\delta_2}) + (\mathrm{w}_{\delta_3} + \mathrm{w}_{\delta_4})$ and to the sum of this and $\mathrm{w}_{\delta_5} + \mathrm{w}_{\delta_6}$, we obtain:

$$\begin{aligned}
\mathrm{w}_{\delta_1} + \mathrm{w}_{\delta_2} + \mathrm{w}_{\delta_3} + \mathrm{w}_{\delta_4} &= [124][135][156][235][345][356];\\
\text{Đ}_{G_P}(\{2,5,6\}) = \mathrm{w}_{\delta_1} + \mathrm{w}_{\delta_2} + \mathrm{w}_{\delta_3} + \mathrm{w}_{\delta_4} + \mathrm{w}_{\delta_5} + \mathrm{w}_{\delta_6} &= [125][135][146][235][345][356].
\end{aligned}$$

and hence

$$\widetilde{\text{Đ}}_{G_P}(\{2,5,6\}) = \frac{\text{Đ}_{G_P}(\{2,5,6\})}{[125][135][235][356]} = [146][345],$$

which coincides with $\overleftarrow{[256]}$ as stated by Proposition 3.5. We have thus demonstrated Theorem 1.1 in this case.

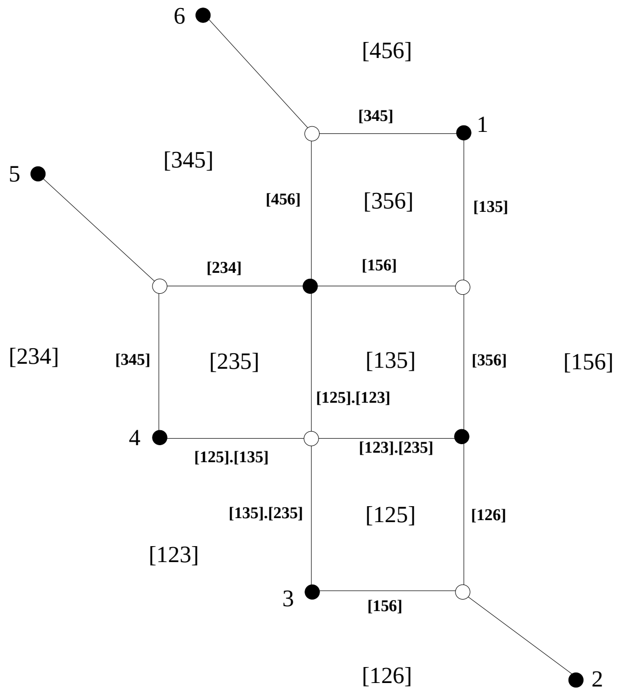


Figure 26. The weighted bipartite graph of the arrangement in Figure 25.

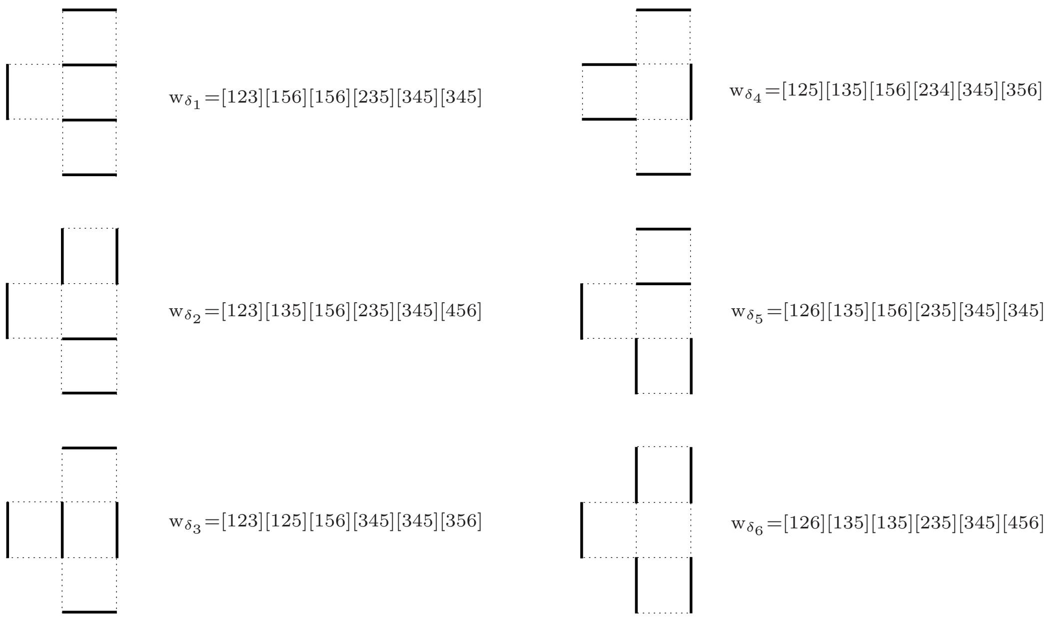


Figure 27. The dimer configurations and corresponding monomials.

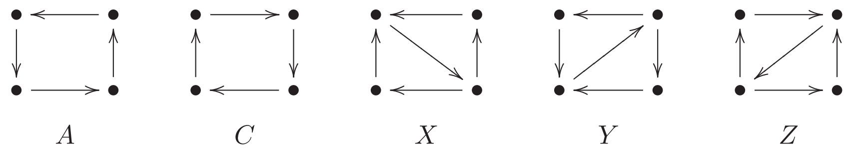


Figure 28. Four-vertex quivers

11. Maximal green sequences

We have seen (Lemma 8.5) that the twist of the cluster in the seed corresponding to $\mathcal{R}_{k,n}$ coincides with the cluster in the seed corresponding to $\mathcal{R}^*_{n-k,n}$, up to coefficients. In this section we will show that up to coefficients and a power of σ, this twisting can be realised via a maximal green sequence. Such sequences arose independently in [24], in the context of cluster algebras and in [1, 11] (see also [8, §3]), in the context of the spectrum of Bogomolńyi–Prasad–Sommerfield (BPS) states. We shall assume first that $k \neq 1, 2, n-1, n-2$; see the proof of Theorem 11.17 below for a discussion of these cases. For convenience, we shall denote the inverse of the map σ (see Section 2) by ϱ.

Our aim will be to define a maximal green sequence which starts at the seed corresponding to $\mathcal{R}_{k,n}$ and ends at a seed whose quiver is isomorphic to $\mathcal{R}^*_{n-k,n}$ and has the property that applying σ^k to its cluster gives the cluster associated to $\mathcal{R}_{k,n}$. To do this, we will first introduce a compact notation for the quivers appearing in the maximal green sequence.

The lattice $\mathbb{Z}^2$ induces a tiling of $\mathbb{R}^2$ by unit square tiles. We denote the tile with $(x, y) \in \mathbb{Z}^2$ in its lower left corner by $U_{x,y}$. Let R be the rectangle in $\mathbb{R}^2$ with corners $(1, 1)$, $(k-1, 1)$, $(1, n-k-1)$ and $(k-1, n-k-1)$. Then R contains the tiles $U_{x,y}$ for $1 \leq x \leq k-2$ and $1 \leq y \leq n-k-2$. Let $\widetilde{R}$ be the larger rectangle whose vertices are $(0, 0)$, $(k, 0)$, $(0, n-k)$ and $(k, n-k)$. We call the tiles in $\widetilde{R} \setminus R$ *boundary tiles*. We define $R_{\mathbb{Z}} = R \cap \mathbb{Z}^2$ and $\widetilde{R}_{\mathbb{Z}} = \widetilde{R} \cap \mathbb{Z}^2$.

Next, we consider the four-vertex quivers shown in Figure 28.

Definition 11.1. We will consider labellings L of the unit square tiles in $\widetilde{R}$ satisfying the following conditions:

(a) Each tile in R is labelled by one A, C, X, Y, Z.
(b) For any pair of tiles in R which share an edge, the corresponding four-vertex quivers (with vertices at the corners of the tile) have arrows in the same direction along the common boundary.
(c) n of the boundary tiles are labelled $1, 2, \ldots, n$, in order clockwise around the boundary.
(d) The four corner boundary tiles are always labelled.

See Figure 29 for an example of a labelling.

We will use the notation $\mathcal{T}_i$ for the boundary tile labelled i. For each $i \in \{1, \ldots, n\}$ let v_i be the external corner of $\widetilde{R}$ on the boundary of $\mathcal{T}_i$ if $\mathcal{T}_i$ is a corner tile of $\widetilde{R}$; otherwise we take v_i to be the midpoint of the boundary edge of $\mathcal{T}_i$ which is part of the boundary of $\widetilde{R}$.

Definition 11.2. If L is a labelling as above, we set $Q(L)$ to be the quiver with vertices $R_{\mathbb{Z}} \cup \{v_1, \ldots, v_n\}$ and arrows given by the following:

(a) The full subquiver of $Q(L)$ on the vertices of any tile in R is given by its label as in Figure 28.
(b) For each corner tile $\mathcal{T}_i$, there is an arrow between v_i and the unique vertex in $\mathcal{T}_i \cap R$, oriented towards v_i if $v_i = (0, 0)$ or $(k, n-k)$, and away from v_i otherwise.
(c) For each non-corner boundary tile $\mathcal{T}_i$, there are arrows between v_i and the two vertices in $\mathcal{T}_i \cap R_{\mathbb{Z}}$, oriented in such a way as to create an oriented 3-cycle on v_i and these two vertices.

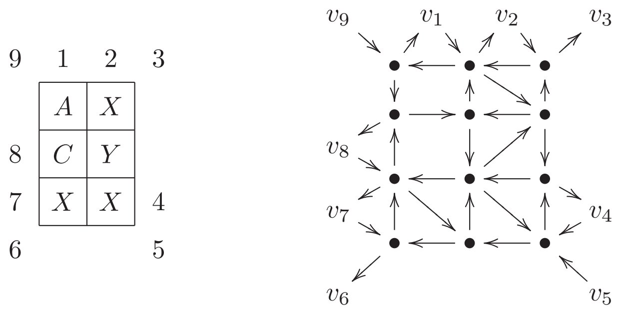


FIGURE 29. Example of a labelling, L (left) and the corresponding quiver $Q(L)$ (right), in the case $k = 4$, $n = 9$.

See Figure 29 for an example of the quiver $Q(L)$ associated to a labelling.

Next, we give a collection of local mutation rules which will allow us to mutate quivers given by labellings directly, without decoding.

Lemma 11.3. *Let L be a labelling and $v \in R_{\mathbb{Z}}$ a vertex with the property that the labels of the tiles around v are as in one of the initial diagrams in a rule in Figure 30. Then $\mu_v(Q(L)) = Q(L')$ where L' is the labelling L in which the tiles around v have been changed according to the rule in Figure 30. Furthermore, the vertices $v_1, \ldots, v_n$ in $Q(L)$ correspond to the vertices $v_1, \ldots, v_n$ in $Q(L')$.*

Proof. This is a simple case-by-case check, noting that a path of length 2 through v in $Q(L)$ must start and end at vertices of tiles incident with v. $\square$

Let $<$ denote the reverse lexicographic ordering on $R_{\mathbb{Z}}$, i.e. we set $(x, y) < (x', y')$ if $y < y'$ or if $y = y'$ and $x < x'$. Note that $(1, 1)$ is the unique minimal element in this ordering and $(k-1, n-k-1)$ is the unique maximal element. If $(x, y) \neq (k-1, n-k-1)$, we write $(x, y)^+$ for the successor of (x, y) in this ordering. We have:

$$(x, y)^+ = \begin{cases} (x + 1, y), & \text{if } 1 \le x \le k - 2; \\ (1, y + 1), & \text{if } x = k - 1. \end{cases}$$

Let

$$\Omega = \{(x, y; l) \in \mathbb{Z}^2 \times \mathbb{Z} \,:\, 1 \le l \le k - 1,\ 1 \le x \le k - l,\ 1 \le y \le n - k - 1\} \sqcup \{\top\}.$$

We shall use this set to parametrize the seeds appearing in the maximal green sequence. Note that the pair (x, y) appearing in an element $(x, y; l)$ of Ω always lies in $R_{\mathbb{Z}}$.

We denote the reverse lexicographic ordering on Ω by $<$, extended by setting $\top$ to be a maximum element. We shall denote the successor of $\omega \in \Omega \setminus \{\top\}$ in this ordering by ω^+. We have, for $\omega = (x, y; l) \in \Omega \setminus \{\top\}$:

$$\omega^+ = \begin{cases} ((x, y)^+; l) & \text{if } (x, y) \neq (k - l, n - k - 1); \\ (1, 1; l + 1) & \text{if } (x, y) = (k - l, n - k - 1),\ l \neq k - 1; \\ \top & \text{if } (x, y; l) = (1, n - k - 1; k - 1). \end{cases}$$

We denote the smallest element, $(1, 1; 1)$, of Ω, by $\perp$. We will associate a seed to each element of Ω with the property that the mutation at (x, y) of the seed associated to $(x, y; l)$ is the seed associated to $(x, y; l)^+$.

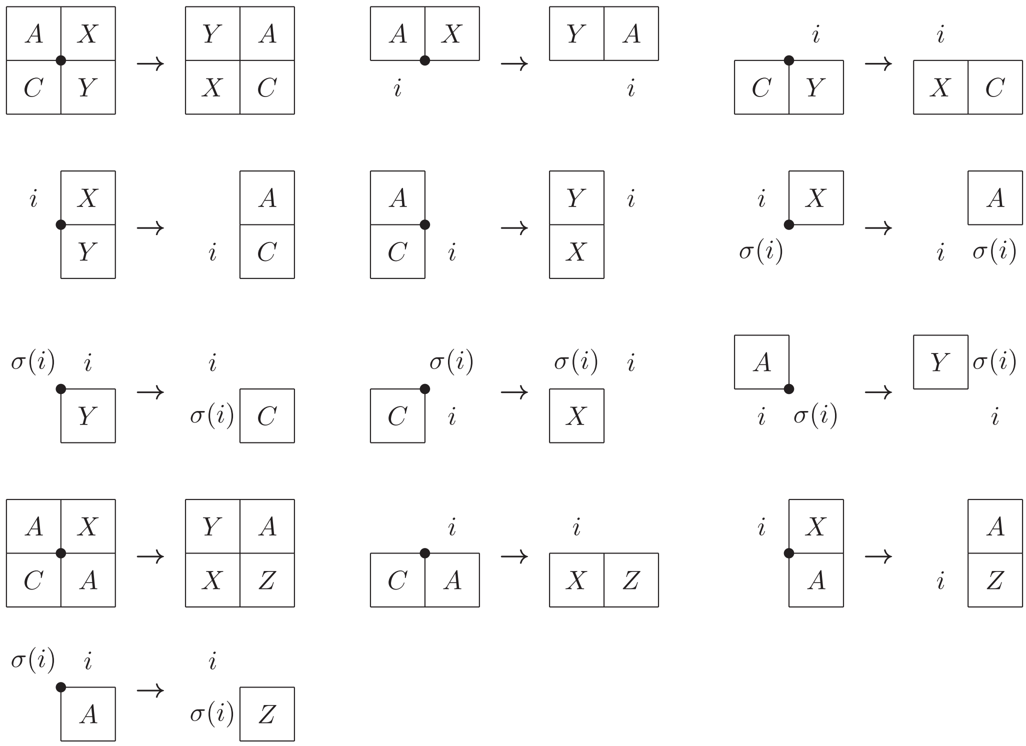


FIGURE 30. Local mutation rules for labellings

Definition 11.4. Let pr denote the map from $\Omega \setminus \{\top\}$ to $R_{\mathbb{Z}}$ taking $(x, y; l)$ to (x, y). We define $\boldsymbol{\alpha}$ to be the sequence of elements of $R_{\mathbb{Z}}$ obtained by applying pr to each of the elements of $\Omega \setminus \{\top\}$, written in the total ordering $<$.

The sequence $\boldsymbol{\alpha}$ can be regarded as scanning the rows of vertices in $R_{\mathbb{Z}}$ from left to right, starting at the bottom left and ending at the top right (giving a first 'page' of mutations), then a second page repeating the process but omitting the last vertex in each row, then repeating this but omitting the last two vertices in each row, and so on, until a final $(k-1)$st page in which only the first vertex of each row is mutated. The index l indicates the page number.

For example, if $k = 4$ and $n = 9$, then the set $\Omega \setminus \{\top\}$ written in order is:

$$(1,1;1),(2,1;1),(3,1;1),(1,2;1),(2,2;1),(3,2;1),(1,3;1),(2,3;1),(3,3;1),$$
$$(1,4;1),(2,4;1),(3,4;1),(1,1;2),(2,1;2),(1,2;2),(2,2;2),(1,3;2),(2,3;2),$$
$$(1,4;2),(2,4;2),(1,1;3),(1,2;3),(1,3;3),(1,4;3).$$

Hence

$$\boldsymbol{\alpha} = (1,1),(2,1),(3,1),(1,2),(2,2),(3,2),(1,3),(2,3),(3,3),(1,4),(2,4),(3,4),(1,1),(2,1),(1,2),$$
$$(2,2),(1,3),(2,3),(1,4),(2,4),(1,1),(1,2),(1,3),(1,4).$$

Definition 11.5. Fix $\omega \in \Omega$. We will associate to ω a seed $(\widetilde{\mathbf{x}}_\omega, \widetilde{Q}_\omega)$, Firstly, we define a labelling L_ω as follows. If $\omega = (x, y; l)$, then for $1 \leq i \leq k-2$ and $1 \leq j \leq n-k-2$, we decorate the unit square U_{ij} in R according to the following rules:

(a) A if $(i,j) = (x-1, y)$ or $(i,j) = (k-l, y-1)$;
(b) C if $(i,j) = (x-1, y-1)$;
(c) Y if either:
  (i) $1 \leq i \leq x-2$ and $j = y$, or
  (i) $x \leq i \leq k-l-1$ and $j = y-1$;
(d) Z if either:
  (i) $i = k-l$ and $1 \leq j \leq y-2$, or
  (ii) $i > k-l$;
(e) X, otherwise.

If $\omega = \top$, we decorate each unit square U_{ij} in R with a Z.

The following table determines a collection of non-corner boundary unit squares according to their location on the boundary of $\widetilde{R}$ (i.e. left hand side, right hand side, top or bottom) and according to the label of their unique adjacent square in R (a check mark indicates membership in the collection):

	left hand side	right hand side	top	bottom
A	$\times$	$\times$	✓	✓
C	✓	✓	$\times$	$\times$
X	✓	$\times$	✓	$\times$
Y	$\times$	✓	✓	$\times$
Z	✓	$\times$	$\times$	✓

It is easy to check that there are $n-4$ squares in the above collection. We label the four corner squares and the boundary squares in this collection $1, 2, \ldots, n$ clockwise around the boundary in such a way that the top right corner square of $\widetilde{R}$ is labelled $k-1$ if $\omega = (x, y; l)$ with $l = 1$, or is labelled k if $\omega = (x, y; l)$ with $l > 1$ or if $\omega = \top$. This completes the definition of L_ω.

Then, for each $\omega \in \Omega$, we set $\widetilde{Q}_\omega = Q(L_\omega)$ and we set Q_ω to be the full subquiver of $\widetilde{Q}_\omega$ on the vertices $R_{\mathbb{Z}}$.

Next, suppose that $\omega = (x, y; l)$ where $1 \leq l \leq k-1$, $1 \leq x \leq k-l$, and $1 \leq y \leq n-k-1$. For $(i,j) \in R_{\mathbb{Z}}$ we define:

$$r_\omega(i,j) = \begin{cases} l, & i \leq k-l, \ (i,j) < (x,y); \\ l-1, & i \leq k-l, \ (i,j) \not< (x,y); \\ k-i, & i > k-l. \end{cases} \tag{11.1}$$

We also define $r_\top(i,j) = k-i$ for all $(i,j) \in R_{\mathbb{Z}}$.

We associate Plücker coordinates to the vertices of $\widetilde{Q}_\omega$ by associating $[\mathbf{i}]$ to the vertex v_i for $1 \leq i \leq n$ and $[\varrho^{r_\omega(i,j)}(M_{k,n}(i,j))]$ to each vertex $(i,j) \in R_{\mathbb{Z}}$. We then define $\widetilde{\mathbf{x}}_\omega$ (respectively, $\mathbf{x}_\omega$) to be the set of Plücker coordinates associated to the vertices of $\widetilde{Q}_\omega$ (respectively, the vertices of Q_ω).

For an example of a labelling L_ω, see Figure 31. We also show the labelling L_ω for each $\omega \in \Omega$ in the case $\mathrm{Gr}_{4,9}$ below, with the vertex (x,y) indicated by a dot when $\omega = (x, y; l)$. See Figure 32 for the quiver $Q_\perp$, and Figure 33 for the quiver $Q_\top$, in the case $k = 4$, $n = 9$.

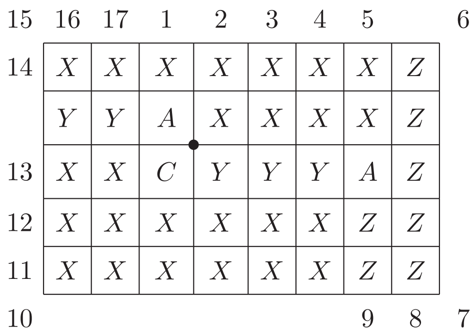
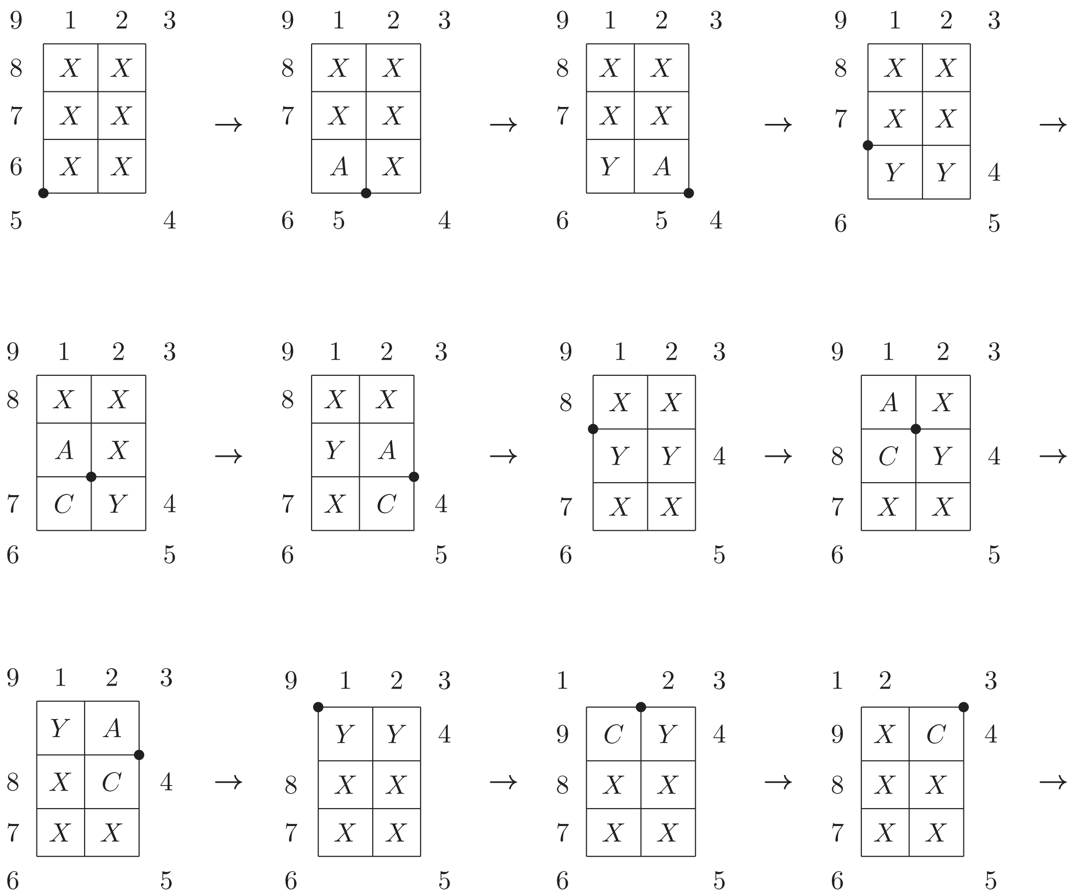
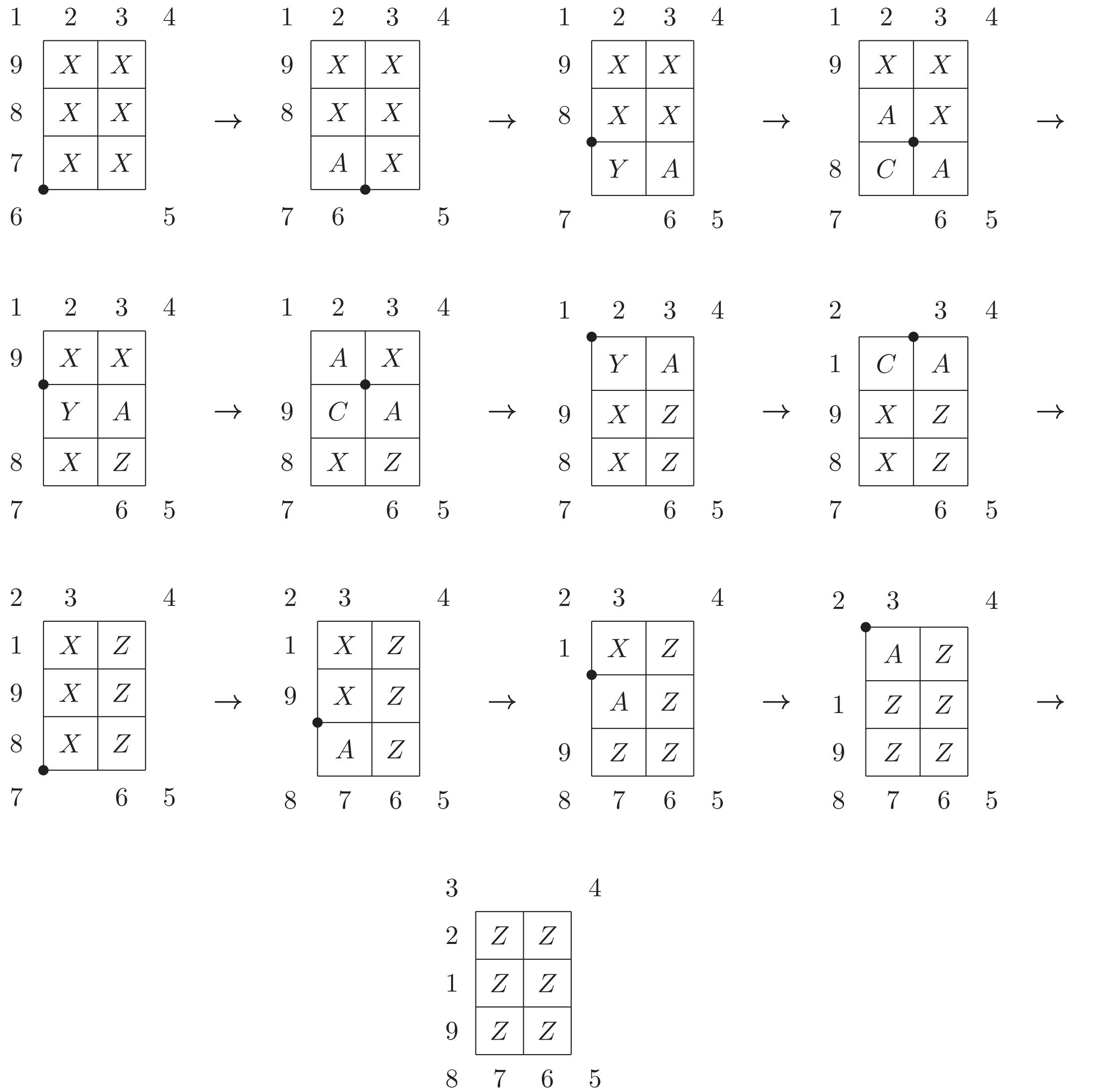


FIGURE 31. The labelling $L_{(4,4;3)}$ for $\mathrm{Gr}_{10,17}$.





If $\omega = (x, y; l)$, recall that the vertex (x, y) is labelled with the Plücker coordinate $[\varrho^{l-1}(M_{k,n}(x, y))]$ in the seed $(\widetilde{\mathbf{x}}_\omega, \widetilde{Q}_\omega)$. The following is easy to check.

Lemma 11.6. *Let $\omega = (x, y; l) \in \Omega \setminus \{\top\}$. Then the arrows incident with the vertex (x, y) in the seed $(\widetilde{\mathbf{x}}_\omega, \widetilde{Q}_\omega)$ are as follows, allowing the cases $i = k$ or $j = 0$ in the formula for $M_{k,n}(x, y)$ in Lemma 8.3.* $\square$

$$\begin{tikzcd} & \varrho^{l-1}(M_{k,n}(x,y+1)) & \\ \varrho^{l}(M_{k,n}(x-1,y)) \arrow[r] & \varrho^{l-1}(M_{k,n}(x,y)) \arrow[u] \arrow[d] & \varrho^{l-1}(M_{k,n}(x+1,y)) \arrow[l] \\ & \varrho^{l}(M_{k,n}(x,y-1)) & \end{tikzcd}$$

Lemma 11.7. *Let $\omega \in \Omega \setminus \{\top\}$, so $\omega = (x,y;l)$ where $1 \leq l \leq k-1$, $1 \leq x \leq k-l$ and $1 \leq y \leq n-k-1$. Then*
$$\mu_{(x,y)}(\widetilde{\mathbf{x}}_\omega, \widetilde{Q}_\omega) = (\widetilde{\mathbf{x}}_{\omega^+}, \widetilde{Q}_{\omega^+}).$$

Proof. Using Lemma 11.3, it is easy to check in each case that $\mu_{(x,y)}(Q(L_\omega)) = Q(L_{\omega^+})$. The last four rules in Figure 30 occur on pages $l > 1$ only: at the end of a row, at the end of the top row, on the last page, and in the final mutation on the last page, respectively.

Mutating $(\widetilde{\mathbf{x}}_\omega, \widetilde{Q}_\omega)$ at (x,y) replaces the Plücker coordinate $[\varrho^{l-1}(M_{k,n}(x,y))]$ with a new element of $\mathbb{C}[\mathrm{Gr}_{k,n}]$. We claim that this new element is $[\varrho^l(M_{k,n}(x,y))]$.

By Lemma 11.6, the cluster variable $[\varrho^{l-1}(M_{k,n}(x,y))]^*$ associated to (x,y) after mutation at (x,y) is
$$\frac{[\varrho^l(M_{k,n}(x-1,y))][\varrho^{l-1}(M_{k,n}(x+1,y))] + [\varrho^{l-1}(M_{k,n}(x,y+1))][\varrho^l(M_{k,n}(x,y-1))]}{[\varrho^{l-1}(M_{k,n}(x,y))]}.$$

Suppose first that $l = 1$, so we have:
$$[M_{k,n}(x,y)]^* = \frac{[\varrho(M_{k,n}(x-1,y))][M_{k,n}(x+1,y)] + [M_{k,n}(x,y+1)][\varrho(M_{k,n}(x,y-1))]}{[M_{k,n}(x,y)]}.$$

Recall (Lemma 8.3) that $M_{k,n}(i,j) = \{1,\ldots,i\} \cup \{i+j+1,\ldots,j+k\}$. Let $A = \{2,\ldots,x\} \cup \{x+y+2,\ldots,y+k\}$. Then we have, using a Plücker relation (and noting that $y \leq n-k-1$, so $y+k+1 \leq n$ and $1 < x+1 < x+y+1 < y+k+1$), that $[M_{k,n}(x,y)]^*$ is given by:
$$\begin{aligned}
&\frac{[A \cup \{x+y+1, y+k+1\}][A \cup \{1, x+1\}] + [A \cup \{1, y+k+1\}][A \cup \{x+1, x+y+1\}]}{[M_{k,n}(x,y)]} \\
&= \frac{[A \cup \{1, x+y+1\}][A \cup \{x+1, y+k+1\}]}{[M_{k,n}(x,y)]} \\
&= \frac{[M_{k,n}(x,y)][\varrho(M_{k,n}(x,y))]}{[M_{k,n}(x,y)]} = [\varrho(M_{k,n}(x,y))].
\end{aligned}$$

It follows that, for any l, we have
$$[\varrho^{l-1}(M_{k,n}(x,y))]^* = [\varrho^l(M_{k,n}(x,y))], \tag{11.2}$$
as claimed.

Suppose first that $(x,y) \neq (k-l, n-k-1)$. Then $(x,y;l)^+ = ((x,y)^+, l)$. Then the Plücker coordinate associated to (i,j) in $(\widetilde{\mathbf{x}}_\omega, \widetilde{Q}_\omega)$ is $[\varrho^{r_\omega(i,j)}(M_{k,n}(i,j))]$, where $r_\omega(i,j)$ is given by (11.1). The coordinate associated to (i,j) in $(\widetilde{\mathbf{x}}_{\omega^+}, \widetilde{Q}_\omega)$ is $[\varrho^{r_{\omega^+}(i,j)}(M_{k,n}(i,j))]$, where
$$r_{\omega^+}(i,j) = \begin{cases} l, & i \leq k-l,\ (i,j) < (x,y)^+; \\ l-1, & i \leq k-l,\ (i,j) \not< (x,y)^+; \\ k-i, & i > k-l. \end{cases}$$

We see that $r_{\omega^+}(i,j) = r_\omega(i,j)$ for all $(i,j) \in R_\mathbb{Z}$ except for $(i,j) = (x,y)$: we have $r_\omega(x,y) = l-1$ while $r_{\omega^+}(x,y) = l$, so the result holds in this case by (11.2).

Suppose that $(x,y) = (k-l, n-k-1)$. Then $(x,y;l)^+ = (1,1;l+1)$ (or $\top$ if $l = k-1$). In either case, the coordinate associated to (i,j) in $(\widetilde{\mathbf{x}}_{\omega^+}, \widetilde{Q}_{\omega^+})$ is $[\varrho^{r_{\omega^+}(i,j)}(M_{k,n}(i,j))]$, where
$$r_{\omega^+}(i,j) = \begin{cases} l, & i \leq k-l, \\ k-i, & i > k-l. \end{cases}$$

We see that $r_{\omega^+}(i,j) = r_\omega(i,j)$ for all (i,j) except that $r_\omega(x,y) = l-1$ while $r_{\omega^+}(x,y) = l$, and we see that the lemma also holds in this case. $\square$

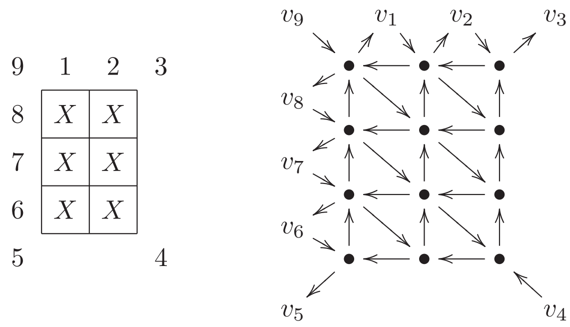


FIGURE 32. The labelling, $L_\perp$ (left) and the corresponding quiver $Q_\perp$, isomorphic to $\widetilde{Q}(\mathcal{R}_{4,9})$ (right), in the case $k = 4$, $n = 9$.

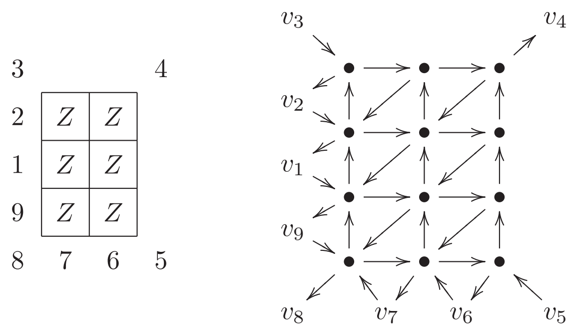


FIGURE 33. The labelling $L_\top$ (left) and corresponding quiver $\widetilde{Q}_\top$, isomorphic to $\mathcal{R}^*_{5,9}$ (right), in the case $k = 4$, $n = 9$.

Lemma 11.8. (a) *The seed of* $\mathbb{C}[\mathrm{Gr}_{k,n}]$ *corresponding to the Postnikov diagram* $\mathcal{R}_{k,n}$ *is* $(\widetilde{\mathbf{x}}_\perp, \widetilde{Q}_\perp)$.

(b) *For all* $\omega \in \Omega$, $(\widetilde{\mathbf{x}}_\omega, \widetilde{Q}_\omega)$ *is a seed of* $\mathbb{C}[\mathrm{Gr}_{k,n}]$.

(c) *The quiver* $\widetilde{Q}_\top$ *is isomorphic to* $\widetilde{Q}(\mathcal{R}^*_{n-k,n})$.

Proof. We first prove part (a). Note that $r_\perp(i,j) = 0$ for all $(i,j) \in R_\mathbb{Z}$, so the coordinate attached to (i,j) in $(\widetilde{\mathbf{x}}_\perp, \widetilde{Q}_\perp)$ is $[M_{k,n}(i,j)]$. It is easy to check that the quiver $\widetilde{Q}_\perp$ is isomorphic to the quiver of $\mathcal{R}_{k,n}$ (Figure 32 gives an example, in the case $k = 4$, $n = 9$: compare with Figure 14).

Part (b) follows from part (a) and Lemma 11.7.

Part (c) is an easy check (note that the embedding in the plane of $\widetilde{Q}(\mathcal{R}^*_{n-k,n})$ is different from that of $\widetilde{Q}_\top$). (Figure 33 shows $L_\top$ and the corresponding quiver in the case $k = 4$, $n = 9$: compare with Figure 17). □

For $(i,j) \in R_\mathbb{Z}$, let $[\widetilde{M}_{k,n}(i,j)]$ be the Plücker coordinate in $\widetilde{\mathbf{x}}_\top$ associated to the vertex (i,j) of $\widetilde{Q}_\top$.

Lemma 11.9. *For all $(i,j) \in R_{\mathbb{Z}}$, we have that $\overleftarrow{[M_{k,n}(k-i,j)]}$ is equal to a monomial in the coefficients multiplied by $[\sigma^k(\widetilde{M}_{k,n}(i,j))]$. Hence, up to coefficients,*

$$\overleftarrow{[M_{k,n}(k-i,j)]} = [\sigma^k(\widetilde{M}_{k,n}(i,j))].$$

Proof. We have

$$\begin{aligned} \widetilde{M}_{k,n}(i,j) &= \varrho^{k-i}(M_{k,n}(i,j)) \\ &= \varrho^{k-i}(\{1,\ldots,i\} \cup [i+j+1, j+k]). \end{aligned} \tag{11.3}$$

By Lemma 8.5 we have that $\overleftarrow{[M_{k,n}(k-i,j)]}$ is equal to $[I]$ multiplied by a monomial in the coefficients, where

$$\begin{aligned} I &= [j+1, k-i+j] \cup [n-i+1, n] \\ &= [n+1-i, n] \cup [j+1, j+k-i] \\ &= \varrho^{-i}(\{1,\ldots,i\} \cup [i+j+1, j+k]) \\ &= \varrho^{-k}(\widetilde{M}_{k,n}(i,j)), \end{aligned}$$

by (11.3), giving the result. □

Remark 11.10. It follows from Lemma 11.8, parts (a) and (c), that there is an isomorphism from $Q_\top$ to $Q_\perp$ taking (i,j) to $(k-i,j)$ for $1 \le i \le k-1$ and $1 \le j \le n-k-1$. Since $(Q_\perp, \pi(\mathbf{x}_\perp))$ and $(Q_\top, \pi(\mathbf{x}_\top))$ are both seeds of $\mathbb{C}[\mathrm{Gr}_{k,n}]_1$ (by Lemma 11.8(b) and Remark 8.7) there is an automorphism γ of $\mathbb{C}[\mathrm{Gr}_{k,n}]_1$ sending $\pi([M_{k,n}(i,j)])$ to $\pi([\widetilde{M}_{k,n}(k-i,j)])$ for all i,j.

Corollary 11.11. *The following identity holds:*

$$\gamma^2\pi = \pi\sigma^{-k}.$$

Proof. By Lemma 11.9, $\sigma^{-k}\left(\overleftarrow{[M_{k,n}(i,j)]}\right)$ is equal to a monomial in the coefficients multiplied by $[\widetilde{M}_{k,n}(k-i,j)]$. So, using Remark 11.10,

$$\begin{aligned} \pi\sigma^{-k}\left(\overleftarrow{[M_{k,n}(i,j)]}\right) &= \pi[\widetilde{M}_{k,n}(k-i,j)] \\ &= \gamma\pi[M_{k,n}(i,j)]. \end{aligned}$$

Hence, for any $x \in \mathbb{C}[\mathrm{Gr}_{k,n}]$, we have $\pi\sigma^{-k}\overleftarrow{x} = \gamma\pi(x)$. By Corollary 4.2, $\pi\left(\overleftarrow{\overleftarrow{x}}\right) = \pi\sigma^k(x)$ for any $x \in \mathbb{C}[\mathrm{Gr}_{k,n}]$. So, using Lemma 3.4, we have:

$$\begin{aligned} \gamma^2\pi(x) &= \gamma\pi\sigma^{-k}\left(\overleftarrow{x}\right) \\ &= \pi\sigma^{-k}\left(\overleftarrow{\sigma^{-k}\left(\overleftarrow{x}\right)}\right) \\ &= \pi\left(\overleftarrow{\overleftarrow{\sigma^{-2k}(x)}}\right) \\ &= \pi\sigma^{-k}(x), \end{aligned}$$

giving the result. □

Let $\mu_{\boldsymbol{\alpha}}$ be the sequence of mutations coming from the sequence $\boldsymbol{\alpha}$ (see Definition 11.4). We have the following result.

Proposition 11.12. (a) *Up to coefficients, $\sigma^k(\mathbf{x}_\top) = \overleftarrow{\mathbf{x}(\mathcal{R}_{k,n})}$.*

(b) $\mu_{\boldsymbol{\alpha}}(\widetilde{\mathbf{x}}(\mathcal{R}_{k,n}), \widetilde{Q}(\mathcal{R}_{k,n})) = (\widetilde{\mathbf{x}}_\top, \widetilde{Q}_\top)$.

(c) *The sequence $\mu_{\boldsymbol{\alpha}}$ of mutations, when applied to $(\widetilde{\mathbf{x}}(\mathcal{R}_{k,n}), \widetilde{Q}(\mathcal{R}_{k,n}))$, passes only through seeds corresponding to Postnikov diagrams.*

Proof. Part (a) follows from Lemmas 11.9 and 8.5, noting Lemma 3.4. Part (b) follows from Lemma 11.7. For part (c), we note that, for each mutation in $\mu_{\boldsymbol{\alpha}}$, there are two arrows pointing towards the mutation vertex and two arrows pointing away, by Lemma 11.6. It follows that the mutation corresponds to a quadrilateral move in the Postnikov diagram (see Figure 4). □

It remains to check that $\boldsymbol{\alpha}$ is a maximal green sequence. We first recall the definition.

Let Q be a quiver with no loops or 2-cycles, with vertices $Q_0 = \{1, \dots, l\}$ and arrows Q_1. The *framing* $\widehat{Q}$ of Q is the quiver with vertices $\widehat{Q}_0 = \{1, \dots, 2l\}$. Writing $r' = r + l$ for $r = 1, \dots, n$, the arrows are

$$\widehat{Q}_1 = Q_1 \sqcup \{r \to r' \,:\, r \in Q_0\}.$$

Then $\widehat{Q}$ is an exchange quiver (see the start of Section 6). Let $\mathrm{Mut}(\widehat{Q})$ denote the mutation class of $\widehat{Q}$, i.e. the set of quivers which can be obtained from $\widehat{Q}$ by iterated mutation. Let $R \in \mathrm{Mut}(\widehat{Q})$. Then a vertex $r \in Q_0$ (thus non-frozen) is said to be *green* (respectively, *red*) if every arrow between r and a vertex s', $s \in Q_0$, points towards (respectively, away from) s'.

A sequence $r_1, \dots, r_m$ of vertices of Q_0 is said to be a *green* sequence for Q if r_j is green in $\mu_{r_{j-1}} \cdots \mu_{r_1}(\widehat{Q})$ for $j = 1, \dots, m$. If every non-frozen vertex in $\mu_{r_m} \cdots \mu_{r_1}(\widehat{Q})$ is red then $r_1, \dots, r_m$ is said to be a *maximal green sequence for Q*.

We recall the following result:

Theorem 11.13. [8, Thm. 4.4, §9.1] *Let Q be a Dynkin quiver with vertices Q_0. Let $r_1, \dots, r_N$ be a sequence of vertices of Q_0 such that:*

(a) *$r_1, \dots, r_N$ is an admissible sequence of sinks in Q (i.e. r_j is a sink in $\mu_{r_{j-1}} \cdots \mu_{r_1}(Q)$ for $j = 1, \dots, N$), and*

(b) *$s_{r_1} \cdots s_{r_N}$ is a reduced expression for the longest element w_0 of the Weyl group W of Q, where s_r is the simple reflection in W corresponding to $r \in Q_0$.*

Then $r_1, \dots, r_N$ is a maximal green sequence for Q.

For $1 \le j \le n - k - 1$, let

$$V_j = \{(i, j) \,:\, 1 \le i \le k - 1\} \subseteq R_{\mathbb{Z}}.$$

Note that we can regard V_j as a subset of the set of vertices of Q_ω for any $\omega \in \Omega$. Let $\widehat{Q}_\perp$ be the framing of $Q_\perp$, with arrows $(i, j) \to (i, j)'$ for all $(i, j) \in R_{\mathbb{Z}}$.

For any $\omega \in \Omega$, let $\boldsymbol{\alpha}(\omega)$ be the sequence of elements of $R_{\mathbb{Z}}$ obtained by applying pr to the elements of $\{\omega' \in \Omega \,:\, \omega' < \omega\}$ written in the order $<$. Thus $\boldsymbol{\alpha}(\top) = \boldsymbol{\alpha}$ (see Definition 11.4). By Lemma 11.7, we have $\mu_{\boldsymbol{\alpha}(\omega)} Q_\perp = Q_\omega$. We define $\Gamma_\omega = \mu_{\boldsymbol{\alpha}(\omega)} \widehat{Q}_\perp$. In particular, $\Gamma_\perp = \widehat{Q}_\perp$.

For $1 \le j \le n-k-1$, let $\boldsymbol{\alpha}^j(\omega)$ be the subsequence of $\boldsymbol{\alpha}(\omega)$ consisting of only those entries which are vertices in V_j. For $1 \le j \le n - k - 1$, let Γ^j_ω be the induced subquiver of Γ_ω on the vertices $V_j \cup V'_j$ and let Q^j_ω be the induced subquiver of Q_ω on the vertices V_j. Note that $\Gamma^j_\perp = \widehat{Q}^j_\perp$, the framing of $Q^j_\perp$.

Lemma 11.14. *Fix $1 \le j \le n - k - 1$. Then $\boldsymbol{\alpha}^j$ is a maximal green sequence for $Q^j_\perp$.*

Proof. The sequence $\boldsymbol{\alpha}^j$ is an admissible sequence of sinks for $Q^j_\perp$ giving a reduced expression for w_0, so this follows from Theorem 11.13. □

We can now prove the following:

Lemma 11.15. *Let $\omega \in \Omega$. Then, for any $1 \le j \le n - k - 1$, we have:*

(a)
$$\Gamma^j_\omega = \mu_{\boldsymbol{\alpha}^j(\omega)} \widehat{Q}^j_\perp$$

(b) *Let a be an arrow in Γ_ω with an endpoint in V'_j. Then the other endpoint of a lies in V_j.*
(c) *If $\omega = (x, y; l)$ then the vertex (x, y) in Γ_ω is green.*

Proof. We prove this by induction on the the length of $\boldsymbol{\alpha}(\omega)$. The result clearly holds when $\omega = \perp$, since $\Gamma_\perp = \widehat{Q}_\perp$. Suppose that the result holds for $\omega = (x, y; l) \in \Omega \setminus \{\top\}$. We will show that it holds for ω^+.

Note that $\Gamma_{\omega^+} = \mu_{(x,y)}\Gamma_\omega$. By Lemma 11.6, we have that
(i) An arrow in Γ_ω between (x, y) and a non-frozen vertex of Γ_ω in V_j for some $j \neq y$ must point away from (x, y).

Hence:
(ii) If $j \neq y$, there is no path of length 2 in Γ_ω starting at a vertex of V_j and ending at a vertex of V_j and passing through (x, y), i.e. no path of the form:

$$(i, j) \longrightarrow (x, y) \longrightarrow (i', j)$$

where $1 \leq i, i' \leq k - 1$.

By (c) for ω, we have:
(iii) Any arrow in Γ_ω between (x, y) and a vertex of V'_y must point away from (x, y).

By (b) for ω, there can be no arrow in Γ_ω between (x, y) and a vertex of V'_j with $j \neq y$. So:
(iv) For $j \neq y$, there can be no path in Γ_ω of length 2 (in either direction) between a vertex in V'_j and a vertex in V_l, for any l, passing through (x, y), i.e. no path of the form:

$$(i_1, j)' \longrightarrow (x, y) \longrightarrow (i_2, l) \qquad (i_2, l) \longrightarrow (x, y) \longrightarrow (i_1, j)'$$

where $1 \leq i_1, i_2 \leq k - 1$ and $1 \leq l \leq n - k - 1$.

It follows from (ii) and (iv) (with $l = j$) that:
(v) For $j \neq y$, applying $\mu_{(x,y)}$ to Γ_ω has no effect on the induced subquiver of Γ_ω on the vertices $V_j \cup V'_j$. In other words, the induced subquiver of $\mu_{(x,y)}\Gamma_\omega$ on $V_j \cup V'_j$ coincides with $\mu_{(x,y)}\Gamma^j_\omega$.

It follows from (iv) that applying $\mu_{(x,y)}$ to Γ_ω does not introduce or delete any arrows between a vertex in V'_j, $j \neq y$, and V_l for any l. By (i) and (iii), there are no paths of length 2 in Γ_ω (passing through (x, y)) between a vertex of V'_y and a vertex of V_l, $l \neq y$, i.e. no paths of the form:

$$(i, y)' \longrightarrow (x, y) \longrightarrow (i', l) \qquad (i', l) \longrightarrow (x, y) \longrightarrow (i, y)'$$

where $1 \leq i, i' \leq k - 1$ and $1 \leq l \leq n - k - 1$. Hence, applying $\mu_{(x,y)}$ to Γ_ω does not introduce any arrows between vertices in V'_y and vertices of V_l, $l \neq y$. We have shown that (b) holds for ω^+.

Note that if Q' is a full subquiver of a quiver Q, then mutating Q at a vertex v of Q' has the same effect on Q' as mutating Q' at v. Hence

$$\begin{aligned} \Gamma^y_{\omega^+} &= \mu_{(x,y)}\Gamma^y_\omega && (\text{since } \Gamma_{\omega^+} = \mu_{(x,y)}\Gamma_\omega) \\ &= \mu_{(x,y)}\mu_{\boldsymbol{\alpha}^y}(\omega)\widehat{Q}^j_\perp && (\text{by part (a) for } \omega) \\ &= \mu_{\boldsymbol{\alpha}^y}(\omega^+)\widehat{Q}^j_\perp, \end{aligned}$$

giving part (a) for ω^+ in the case $j = y$.

If $j \neq y$ then we have

$$\begin{aligned}\Gamma^j_{\omega^+} &= \Gamma^j_{\omega} \text{ (by (v))}\\ &= \mu_{\boldsymbol{\alpha}^j(\omega)}\widehat{Q}^j_{\perp} \text{ (by part (a) for } \omega)\\ &= \mu_{\boldsymbol{\alpha}^j(\omega^+)}\widehat{Q}^j_{\perp} \text{ (as } j \neq y).\end{aligned}$$

It follows that part (a) for ω^+ holds for $j \neq y$ and hence for all j. If $\omega^+ \neq \top$ then part (c) for ω^+ follows from part (a) for ω^+ (with $j = y$), part (b) for ω^+ and the fact that $\boldsymbol{\alpha}^j$ is a maximal green sequence for $Q^j_{\perp}$ (Lemma 11.14).

By induction, the result holds for all $\omega \in \Omega$. □

Proposition 11.16. *The sequence $\boldsymbol{\alpha}$ is a maximal green sequence for $Q(\mathcal{R}_{k,n}) = Q_{\perp}$.*

Proof. By Lemma 11.15(c), the vertex (x, y) in Γ_ω is green for all $\omega \in \Omega \setminus \{\top\}$, from which it follows that $\boldsymbol{\alpha}$ is a green sequence for $Q_{\perp}$.

Suppose that $1 \leq j \leq n-k-1$. Then, by Lemma 11.14, every non-frozen vertex in $\mu_{\boldsymbol{\alpha}^j}\widehat{Q}^j_{\perp}$ is red. By Lemma 11.15(a) with $\omega = \top$, $\Gamma^j_{\top} = \mu_{\boldsymbol{\alpha}^j}\widehat{Q}^j_{\perp}$. Hence, by Lemma 11.15(b) with $\omega = \top$, every non-frozen vertex in $\Gamma_{\top}$ is red. It follows that $\boldsymbol{\alpha}$ is a maximal green sequence for $Q(\mathcal{R}_{k,n}) = Q_{\perp}$. □

We can now sum up our results in this section as follows:

Theorem 11.17. *Suppose that $k \neq 1, n-1$. Then there is a maximal green sequence, passing only through seeds corresponding to Postnikov diagrams, taking the seed $(\widetilde{\mathbf{x}}(\mathcal{R}_{k,n}), \widetilde{Q}(\mathcal{R}_{k,n}))$ for $\mathbb{C}[\mathrm{Gr}_{k,n}]$ to a seed $(\widetilde{\mathbf{x}}_{\top}, \widetilde{Q}_{\top})$ whose quiver is isomorphic to $\widetilde{Q}(\mathcal{R}^*_{n-k,n})$. The image of the cluster in $\mathbf{x}_{\top}$ under σ^k coincides, up to coefficients, with $\overleftarrow{\mathbf{x}(\mathcal{R}_{k,n})}$.*

Proof. Note that we may ignore the cases $k = 1$ or $n-1$, since then all cluster variables are coefficients. If $k \neq 2, n-2$ then the result follows from Lemma 11.8, Propositions 11.12 and 11.16. So, suppose that $k = 2$ (the case $k = n-2$ is similar). Then the seed $(\widetilde{\mathbf{x}}(\mathcal{R}_{2,n}), \widetilde{Q}(\mathcal{R}_{2,n}))$ is as shown on the left hand side of Figure 34, where we have labelled each vertex by the corresponding cluster variable or coefficient in $\widetilde{\mathbf{x}}(\mathcal{R}_{2,n})$. We see that:

$$\mathbf{x}(\mathcal{R}_{2,n}) = \{[1j] \,:\, 3 \leq j \leq n-1\}.$$

By Remark 9.3, we have that $\overleftarrow{[1,j]} = [j-1, n]$ for $j = 3, \ldots, n-1$. It is easy to check directly that mutating at the vertices initially labelled with $[1,3], \ldots, [1, n-1]$, in this ordering, transforms the seed $(\widetilde{\mathbf{x}}(\mathcal{R}_{2,n}), \widetilde{Q}(\mathcal{R}_{2,n}))$ to the seed shown on the right hand side of Figure 34. It is also easy to check that the quiver in this seed is isomorphic to $\widetilde{Q}(\mathcal{R}^*_{n-2,n})$. The cluster in this seed is

$$\mathbf{x} = \{[2, j+1] \,:\, 3 \leq j \leq n-1\}.$$

Since $\sigma^2([2, j+1]) = [j-1, n]$ for $3 \leq j \leq n-1$, we see that $\sigma^2(\mathbf{x}) = \overleftarrow{\mathbf{x}(\mathcal{R}_{2,n})}$. It is clear that the mutation sequence above passes only through Postnikov diagrams (indeed, in this case, every seed arises from a Postnikov diagram [39, Cor. 2]). Furthermore, the sequence above is a maximal green sequence for $Q(\mathcal{R}_{2,n})$ by [8, Lemma 2.20]. □

12. Surfaces and future directions

The approach developed here can be generalised in a straightforward way to a *surface graph* — a bipartite graph G equipped with an embedding into a surface Σ with (or without) boundary $\partial\Sigma$, in such a way that no two edges of G cross within Σ and each face of G is homeomorphic to a disk;

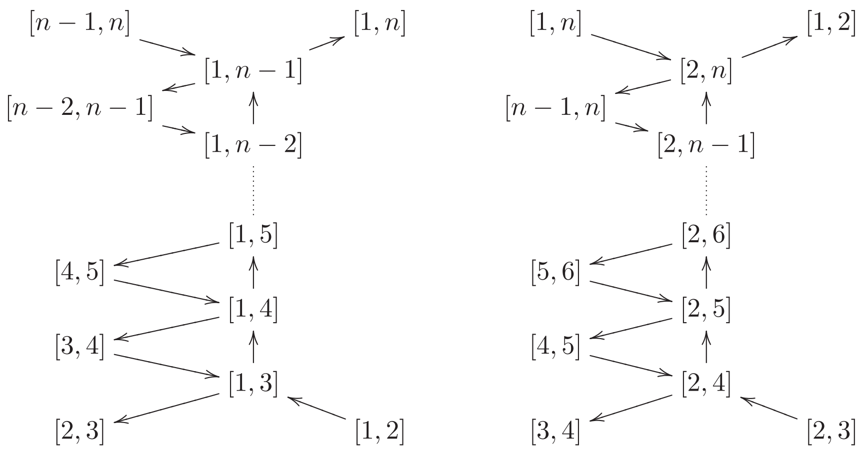


FIGURE 34. The seed $(\widetilde{x}(\mathcal{R}_{2,n}), \widetilde{Q}(\mathcal{R}_{2,n}))$ (left) and its image after mutating at the maximal green sequence in the part of the proof of Theorem 11.17 for $k = 2$ (right).

by definition faces are the connected components of the complement of G in Σ. An example of a surface graph is shown in Figure 35.

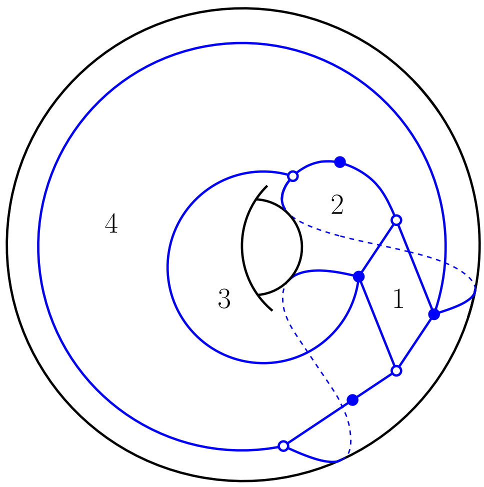


FIGURE 35. Bipartite graph G embedded on a torus Σ (drawn in black). In this case $\partial\Sigma$ is empty and $n = k = 0$ and there are 4 faces.

By choosing a transcendence basis — whose elements label the faces of G — we obtain a seed (and a cluster algebra) whose quiver Q is the *face dual* graph of G. The vertices of Q correspond to faces of G. For each common edge separating a pair of faces E and F, an arrow is drawn from the vertex corresponding to face E to the vertex corresponding to face F, in such a way that the white vertex lies to the left when crossing the edge from E to F; as usual oriented 2-cycles are annihilated afterwards. See Figure 36 for an example.

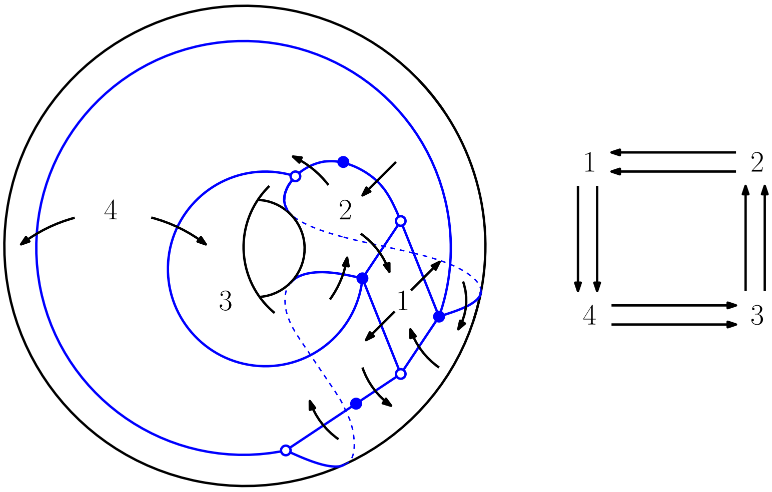


FIGURE 36. The quiver Q of an embedded bipartite graph $G \hookrightarrow \Sigma$: initial stage of the quiver construction (superimposed in black on the left) and then redrawn (on the right) after removing all oriented two cycles.

The concepts explained in Sections 6 and 7 — the definition of edge weights, the blow-up and blow-down equivalences, quadrilateral mutation — are all local notions, i.e. defined and/or constructed with a neighbourhood of the participating edges and vertices, and for this reason can be used unambiguously in the surface case. (See Figure 37 for an example of a quadrilateral move performed on a torus.) In addition, the weight of a dimer configuration (as in Lemma 7.3) will remain invariant under blow-ups and blow-downs, and therefore so will the dimer partition function (as in Corollary 7.4). Similarly, Proposition 7.6 generalizes to the surface setting, ensuring that the $\widetilde{\text{Đ}}_G(I)$ are invariant under quadrilateral moves. In the remainder of this section we formulate some questions concerning the general surface case.

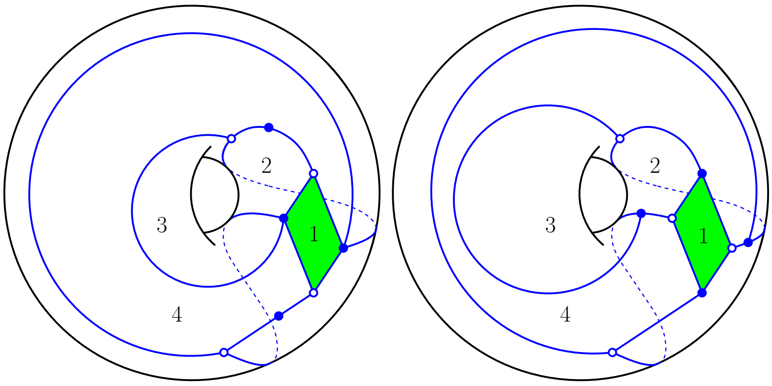


FIGURE 37. Mutation of a embedded bipartite graph $G \hookrightarrow \Sigma$ associated to a quadrilateral face (shaded in green)

• Can the dimer partition functions for specific boundary conditions be on $G \cap \partial\Sigma$ be identified with specific cluster variables? If so, which cluster variables are these? Can this be done with reference to the faces of G? A satisfactory answer to this question ought to explain how to define and express an analogue of the BFZ-*twist* [5, 6] for the cluster algebra associated to G in local coordinates. On a combinatorial level this question is related to determining those boundary conditions which admit precisely one *dimer configuration* for G.

• How to view *FZ-mutation* in this context? The simplest type of mutation, the so-called quadrilateral move, corresponds to a special kind of local rotation in G which fixes Σ and $G \cap \partial\Sigma$ and which conserves the dimer partition functions. Higher order mutation, however, not only changes G but also Σ; for instance hexagonal mutation increases the genus by adding a handle. Since the topology changes some care is needed in analysing how the dimer partition functions are transformed.

• In cases where such a cluster algebra arising from a surface graph can be realised as (or is related to) the coordinate ring $\mathbb{C}[V]$ of some reasonable quasi-projective algebraic variety V (e.g. [22]), or in the phase spaces considered in [23], what is the geometric meaning of the BFZ-twist analogue?

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