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On the Cauchy problem for semilinear elliptic equations

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Abstract

We study the Cauchy problem for non-linear (semilinear) elliptic partial differential equations in Hilbert spaces. The problem is severely ill-posed in the sense of Hadamard. Under a weak *a priori* assumption on the exact solution, we propose a new regularization method for stabilising the ill-posed problem. These new results extend some earlier works on Cauchy problems for nonlinear elliptic equations. Numerical results are presented and discussed.

Keywords and phrases: Cauchy problem; Nonlinear elliptic equation; Ill-posed problem; Error estimates.
Mathematics subject Classification 2000: 35K05, 35K99, 47J06, 47H10

1. Introduction

Let H be a Hilbert space with the inner product $\langle ., . \rangle$ and the norm $\|.\|$, and let $\mathbb{L} : D(\mathbb{L}) \subset H \to H$ be a positive-definite, self-adjoint operator with compact inverse on H. Let M be a positive number, and consider finding a function $u : [0, M] \to H$ satisfying the Cauchy problem

$$\begin{cases} u_{zz} = \mathbb{L}u + f(z, u(z)), \quad z \in (0, M) \\ u(0) = \varphi, \\ u_z(0) = 0, \end{cases} \tag{1.1}$$

where the data φ is given in H and the source function f will be defined later. The Neumann condition in (1.1) need not to be necessarily homogeneous. In practice, the data $\varphi \in H$ is noisy and is represented by the perturbed data $\varphi^\epsilon \in H$ satisfying

$$\|\varphi^\epsilon - \varphi\| \le \epsilon, \tag{1.2}$$

where the constant $\epsilon > 0$ represents an upper bound on the measurement error. Such problem is not well-posed because its solution may not exist and, even if it exists, it does not depend continuously on the ”noisy” Cauchy data φ^ϵ. Hence, a regularization process is required in order to obtain a stable solution.

Equation (1.1) is an abstract version which generalizes many well-known equations. For a simple example, if $\mathbb{L} = -\Delta$ (negative of Laplace’s operator) and $f(z, u(z)) = -k^2 u(z)$ with k real or purely imaginary, then the equation (1.1) becomes the Helmholtz or modified Helmholtz equation, respectively, which arises in many engineering applications related to propagating waves in different environments or heat transfer in fins. More generally, for $\mathbb{L} = -\Delta$ and f a nonlinear function of u, equation (1.1) becomes the nonlinear Poisson equation which is encountered in numerous applications in heat and mass transfer, chemical reactions, gas dynamics and fluid flow in porous media, [2].

Nevertheless, there exist many studies on the linear problem, i.e. $f(z, u(z)) = a(z)u(z) + b(z)$, where a and b are some given functions (usually taken to be zero) in Eq. (1.1), see e.g. [3, 4, 5, 6, 7, 9, 10, 13, 14, 16, 17] to mention only a few. On the other hand, the Cauchy problem for nonlinear elliptic equations has been much less investigated, [11, 21], and it is the purpose of this study to make advances into the semi-linear problem (1.1).

2. Mathematical analysis

We assume that $\mathbb{L}$ admits an orthonormal eigenbasis $\{\phi_n\}_{n\geq 1}$ in H, associated with the eigenvalues such that

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ... \lim_{n\to\infty} \lambda_n = \infty.$$

and f satisfies the global Lipschitz condition

$$\|f(z, v_1) - f(z, v_2)\| \leq K\|v_1 - v_2\| \tag{2.3}$$

for some constant K independent of z, v_1, v_2 with

$$0 \leq K < \frac{1}{MC}, \quad C = \max\left\{\frac{1}{\sqrt{\lambda_1}}, 1\right\}. \tag{2.4}$$

More general local Lipschitz nonlinearities can also be considered, [19]. As shown in [18], the solution $u \in C([0, M]; H)$ is a weak solution of (1.1) if u satisfies the integral equation

$$u(z) = \sum_{n=1}^{\infty}\left[\cosh\left(\sqrt{\lambda_n}z\right)\varphi_n + \int_0^z \frac{\sinh\left(\sqrt{\lambda_n}(z-s)\right)}{\sqrt{\lambda_n}} f_n(u)(s)ds\right]\phi_n, \tag{2.5}$$

where $\varphi_n = \left\langle \varphi, \phi_n \right\rangle$ and $f_n(u)(s) = \left\langle f(s, u(s)), \phi_n \right\rangle$. Since $z > 0$, we know from (2.5) that, when n becomes large, the terms $\cosh\left(\sqrt{\lambda_n}z\right)$ and $\sinh\left(\sqrt{\lambda_n}(z-s)\right)$ increase rather quickly. Thus, these terms are causes for instability. Hence, to regularize the problem, we have to replace these terms by some stability terms. In the present paper, the unstable solution (2.5) is regularized by the solution U^ϵ defined as

$$\begin{aligned} U^\epsilon(z) = \sum_{n=1}^{\infty}\Bigg[&\cosh_\epsilon(\sqrt{\lambda_n}z)\varphi_n^\epsilon + \int_0^z \frac{\sinh_\epsilon(\sqrt{\lambda_n}(z-s))}{\sqrt{\lambda_n}} f_n(U^\epsilon)(s)ds \\ &- \int_z^M \frac{\beta(\epsilon)e^{-\sqrt{\lambda_n}(s-z)}}{\sqrt{\lambda_n}\left(\beta(\epsilon) + e^{-\sqrt{\lambda_n}M}\right)} f_n(U^\epsilon)(s)ds\Bigg]\phi_n, \quad z \in [0, M], \end{aligned} \tag{2.6}$$

where $\varphi_n^\epsilon = \left\langle \varphi^\epsilon, \phi_n \right\rangle$, $f_n(U^\epsilon)(s) = \left\langle f(s, U^\epsilon(s)), \phi_n \right\rangle$ and

$$\cosh_\epsilon(\sqrt{\lambda_n}z) := \frac{1}{2}\left(\frac{e^{-\sqrt{\lambda_n}(M-z)}}{\beta(\epsilon) + e^{-\sqrt{\lambda_n}M}} + e^{-\sqrt{\lambda_n}z}\right), \quad \sinh_\epsilon(\sqrt{\lambda_n}z) := \frac{1}{2}\left(\frac{e^{-\sqrt{\lambda_n}(M-z)}}{\beta(\epsilon) + e^{-\sqrt{\lambda_n}M}} - e^{-\sqrt{\lambda_n}z}\right).$$

Here $\beta(\epsilon) \geq 0$ plays the role as the regularization parameter which has to be chosen depending on the noise ϵ. Under the *a priori* assumption

$$\|u(M)\| + \|u_z(M)\| \leq E \tag{2.7}$$

where $E > 0$ is some known given positive number, we will obtain the error estimate between the exact solution u and the regularized solution U^ϵ.

To our knowledge, there has not been a regularization method for nonlinear elliptic equations which provides a convergence rate under the weak condition (2.7). We also mention that, previously, in order to get a stability estimate, Zhang and Wei [21] assumed the stronger condition on the exact solution u:

$$\sum_{n=1}^{\infty} e^{2\sqrt{\lambda_n}(M+r)} \langle u(z), \phi_n \rangle^2 \leq E_1^2, \quad z \in [0, M], \tag{2.8}$$

whilst Tuan *et al.* [18] assumed that

$$\sum_{n=1}^{\infty} e^{2\sqrt{\lambda_n}(M-z)} \left(\langle u(z), \phi_n \rangle + \frac{\langle u_z(z), \phi_n \rangle}{\sqrt{\lambda_n}} \right)^2 \leq E_2^2, \quad z \in [0, M]. \tag{2.9}$$

One can further remark that there are not too many functions u which satisfy conditions (2.8) or (2.9) and moreover, in practice, these conditions are difficult to be checked. Therefore, in our study we develop a new regularization method to obtain the error estimate under the weaker assumption (2.7).

Our main results are stated in the following theorem:

Theorem 2.1. *The integral equation* (2.6) *has a unique solution* $U^\epsilon \in C([0; M]; H)$. *Suppose that problem* (1.1) *has a weak solution u which satisfies* (2.7). *Let* $\varphi^\epsilon \in H$ *be measured data such that* (1.2) *holds. Choose* $\beta(\epsilon) > 0$ *such that* $\lim_{\epsilon\to 0} \beta(\epsilon) = \lim_{\epsilon \to 0} \frac{\epsilon}{\beta(\epsilon)} = 0$. *Then, we have the following estimate:*

$$\|U^\epsilon(z) - u(z)\| \leq Q(\epsilon; m)\beta(\epsilon)^{1-\frac{z}{M}}, \quad z \in [0, M], \tag{2.10}$$

for any $m \in \left(0, \frac{4}{K^2M^2C^2} - 1\right)$, *where*

$$Q(\epsilon; m) = \sqrt{\frac{\left(1 + \frac{1}{m}\right)\left(C^4E^2 + 2\beta(\epsilon)^{-2}\epsilon^2\right)}{1 - \frac{1}{4}\left(1 + m\right)K^2C^2M^2}}. \tag{2.11}$$

Moreover, there exists $z_\epsilon \in (0, M)$ *such that* $\lim_{\epsilon\to 0} z_\epsilon = M$ *and*

$$\|u(M) - U^\epsilon(z_\epsilon)\| \leq \left(Q(\epsilon; m) + \sup_{0 \leq z \leq M} \|u_z(z)\| \right) \sqrt{\frac{M}{\ln\left(\frac{1}{\beta(\epsilon)}\right)}}. \tag{2.12}$$

Remark 2.1. *(i) If we choose* $\beta(\epsilon) = \epsilon^\alpha$ *with* $\alpha \in (0, 1]$ *in* (2.10) *then, we get*

$$\|U^\epsilon(z) - u(z)\| \leq \sqrt{\frac{\left(1 + \frac{1}{m}\right)\left(C^4E^2 + 2\epsilon^{2-2\alpha}\right)}{1 - \frac{1}{4}\left(1 + m\right)K^2C^2M^2}}\epsilon^{\alpha - \frac{\alpha z}{M}}, \quad z \in [0, M]. \tag{2.13}$$

(ii) In order to obtain the tightest upper bound in (2.10) *we can minimize with respect to* $m \in \left(0, \frac{4}{K^2M^2C^2} - 1\right)$ *the function* $Q(\epsilon; m)$ *defined in* (2.11)*. Noticing that*

$$\lim_{m \searrow 0} Q(\epsilon; m) = \lim_{m \nearrow \left(\frac{4}{K^2M^2C^2}-1\right)} Q(\epsilon; m) = \infty,$$

and solving $\frac{\partial Q}{\partial m}(\epsilon; m) = 0$ *we obtain the minimum point* $m_{min} = \frac{2}{KMC} - 1$*. Then the estimate* (2.10) *becomes*

$$\|U^\epsilon(z) - u(z)\| \le Q(\epsilon; m_{min})\beta(\epsilon)^{1-\frac{z}{M}}, \quad z \in [0, M], \tag{2.14}$$

where

$$Q(\epsilon; m_{min}) = \frac{2\sqrt{C^4E^2 + 2\beta(\epsilon)^{-2}\epsilon^2}}{2 - KMC}. \tag{2.15}$$

3. Proof of Theorem 2.1

First we have the following lemma which will be useful in the proof of the theorem.

Lemma 3.1. *The following inequalities hold (for* $\epsilon > 0$ *small):*

$$\cosh_\epsilon(\sqrt{\lambda_n}z) \le \beta(\epsilon)^{-\frac{z}{M}}, \quad \frac{|\sinh_\epsilon(\sqrt{\lambda_n}z)|}{\sqrt{\lambda_n}} \le \frac{\beta(\epsilon)^{-\frac{z}{M}}}{2\sqrt{\lambda_1}}, \quad z \in [0, M], \tag{3.16}$$

$$\frac{|\sinh_\epsilon(\sqrt{\lambda_n}(z-s))|}{\sqrt{\lambda_n}} \le \frac{C}{2}\beta(\epsilon)^{\frac{s-z}{M}}, \quad 0 \le s \le z \le M, \tag{3.17}$$

$$\frac{\beta(\epsilon)e^{-\sqrt{\lambda_n}(s-z)}}{\sqrt{\lambda_n}\left(\beta(\epsilon) + e^{-\sqrt{\lambda_n}M}\right)} \le C\beta(\epsilon)^{\frac{s-z}{M}}, \quad 0 \le z \le s \le M. \tag{3.18}$$

Proof. First, we can deduce the following inequality:

$$\frac{e^{-\sqrt{\lambda_n}(M-z)}}{\beta(\epsilon) + e^{-\sqrt{\lambda_n}M}} = \frac{e^{-\sqrt{\lambda_n}(M-z)}}{\left(\beta(\epsilon) + e^{-\sqrt{\lambda_n}M}\right)^{1-\frac{z}{M}}\left(\beta(\epsilon) + e^{-\sqrt{\lambda_n}M}\right)^{\frac{z}{M}}} \le \left(\beta(\epsilon) + e^{-\sqrt{\lambda_n}M}\right)^{\frac{-z}{M}} \le \beta(\epsilon)^{-\frac{z}{M}}. \tag{3.19}$$

This implies that

$$\cosh_\epsilon(\sqrt{\lambda_n}z) = \frac{1}{2}\left(\frac{e^{-\sqrt{\lambda_n}(M-z)}}{\beta(\epsilon) + e^{-\sqrt{\lambda_n}M}} + e^{-\sqrt{\lambda_n}z}\right) \le \frac{1}{2}\left(\beta(\epsilon)^{-\frac{z}{M}} + 1\right) \le \beta(\epsilon)^{-\frac{z}{M}}$$

and

$$\frac{|\sinh_\epsilon(\sqrt{\lambda_n}z)|}{\sqrt{\lambda_n}} = \frac{1}{2\sqrt{\lambda_n}}\left|\frac{e^{-\sqrt{\lambda_n}(M-z)}}{\beta(\epsilon) + e^{-\sqrt{\lambda_n}M}} - e^{-\sqrt{\lambda_n}z}\right| \le \frac{1}{2\sqrt{\lambda_n}}\left(\frac{e^{-\sqrt{\lambda_n}(M-z)}}{\beta(\epsilon) + e^{-\sqrt{\lambda_n}M}}\right) \le \frac{\beta(\epsilon)^{\frac{-z}{M}}}{2\sqrt{\lambda_1}},$$

where we have used (3.19) and that $\lambda_n \ge \lambda_1$.

The inequality (3.17) is obtained immediately by replacing z with $z-s$ in the second inequality in (3.16) and using that $C \ge 1/\sqrt{\lambda_1}$, whilst the inequality (3.18) is obtained as in (3.19) by employing the inequality

$$\frac{\beta(\epsilon)e^{-\sqrt{\lambda_n}(s-z)}}{\beta(\epsilon) + e^{-\sqrt{\lambda_n}M}} = \frac{\beta(\epsilon)e^{-\sqrt{\lambda_n}(s-z)}}{\left(\beta(\epsilon) + e^{-\sqrt{\lambda_n}M}\right)^{\frac{s-z}{M}}\left(\beta(\epsilon) + e^{-\sqrt{\lambda_n}M}\right)^{1-\frac{s-z}{M}}} \le \beta(\epsilon)\left(\beta(\epsilon) + e^{-\sqrt{\lambda_n}M}\right)^{\frac{s-z}{M}-1} \le \beta(\epsilon)^{\frac{s-z}{M}}.$$

□

The proof of Theorem 2.1 consists of two steps.

Step 1. The existence and the uniqueness of a solution to (2.6).

Let us define the following norm on $C([0;M];H)$:

$$\|h\|_1 = \sup_{0\le z\le M} \beta(\epsilon)^{\frac{z}{M}} \|h(z)\|, \quad \forall h \in C([0;M];H).$$

It is easy to show that $\|.\|_1$ is a norm on $C([0;M];H)$. For any $w \in C([0;M];H)$, we define

$$J(w)(z) := \sum_{n=1}^{\infty} \Bigg[\cosh_\epsilon(\sqrt{\lambda_n} z)\varphi_n + \int_0^z \frac{\sinh_\epsilon(\sqrt{\lambda_n}(z-s))}{\sqrt{\lambda_n}} f_n(w)(s)ds$$
$$- \int_z^M \frac{\beta(\epsilon)e^{-\sqrt{\lambda_n}(s-z)}}{\sqrt{\lambda_n}\Big(\beta(\epsilon)+e^{-\sqrt{\lambda_n}M}\Big)} f_n(w)(s)ds\Bigg]\phi_n, \quad z\in[0,M].$$

We claim that, for every $w_1, w_2 \in C([0,M];H)$ we have

$$\|J(w_1) - J(w_2)\|_1 \le KCM\|w_1 - w_2\|_1. \tag{3.20}$$

First, using Lemma 2.1 we have two following estimates for all $z\in[0,M]$:

$$\sum_{n=1}^{\infty}\left(\int_0^z \frac{\sinh_\epsilon(\sqrt{\lambda_n}(z-s))}{\sqrt{\lambda_n}}(f_n(w_1)(s) - f_n(w_2)(s))ds\right)^2$$
$$\le z\sum_{n=1}^{\infty}\int_0^z \left|\frac{\sinh_\epsilon(\sqrt{\lambda_n}(z-s))}{\sqrt{\lambda_n}}(f_n(w_1)(s) - f_n(w_2)(s))\right|^2 ds$$
$$\le z\sum_{n=1}^{\infty}\int_0^z C^2\beta(\epsilon)^{\frac{2s-2z}{M}}\left|f_n(w_1)(s) - f_n(w_2)(s)\right|^2 ds$$
$$\le K^2C^2 z\int_0^z \beta(\epsilon)^{\frac{2s-2z}{M}}\|w_1(s) - w_2(s)\|^2 ds$$
$$\le \beta(\epsilon)^{\frac{-2z}{M}}K^2C^2z^2 \sup_{0\le s\le M}\left\{\beta(\epsilon)^{\frac{2s}{M}}\|w_1(s) - w_2(s)\|^2\right\} = \beta(\epsilon)^{\frac{-2z}{M}}K^2C^2z^2\|w_1 - w_2\|_1^2 \tag{3.21}$$

and

$$\sum_{n=1}^{\infty}\left(\int_z^M \frac{\beta(\epsilon)e^{-\sqrt{\lambda_n}(s-z)}}{\sqrt{\lambda_n}\Big(\beta(\epsilon)+e^{-\sqrt{\lambda_n}M}\Big)}(f_n(w_1)(s) - f_n(w_2)(s))ds\right)^2$$
$$\le (M-z)\sum_{n=1}^{\infty}\int_z^M \left|\frac{\beta(\epsilon)e^{-\sqrt{\lambda_n}(s-z)}}{\sqrt{\lambda_n}\Big(\beta(\epsilon)+e^{-\sqrt{\lambda_n}M}\Big)}(f_n(w_1)(s) - f_n(w_2)(s))\right|^2 ds$$
$$\le (M-z)\sum_{n=1}^{\infty}\int_z^M C^2\beta(\epsilon)^{\frac{2s-2z}{M}}\left|f_n(w_1)(s) - f_n(w_2)(s)\right|^2 ds$$
$$\le K^2C^2(M-z)\int_z^M \beta(\epsilon)^{\frac{2s-2z}{M}}\|w_1(s) - w_2(s)\|^2 ds$$
$$\le \beta(\epsilon)^{\frac{-2z}{M}}K^2C^2(M-z)^2\|w_1 - w_2\|_1^2. \tag{3.22}$$

Then, for $0 < z < M$, using the inequality $(a+b)^2 \le (1+p)a^2 + \left(1+\frac{1}{p}\right)b^2$ for any real numbers a and b and $p > 0$, we have

$$\begin{aligned}\|J(w_1)(z) - J(w_2)(z)\|^2 \le \beta(\epsilon)^{\frac{-2z}{M}} K^2C^2(1+p)z^2\|w_1 - w_2\|_1^2 \\ +\beta(\epsilon)^{\frac{-2z}{M}} K^2C^2\left(1+\frac{1}{p}\right)(M-z)^2\|w_1 - w_2\|_1^2.\end{aligned}$$

By choosing $p = \frac{M-z}{z}$, we obtain

$$\beta(\epsilon)^{\frac{2z}{M}}\|J(w_1)(z) - J(w_2)(z)\|^2 \le K^2C^2M^2\|w_1 - w_2\|_1^2, \quad \forall z \in (0, M). \tag{3.23}$$

On other hand, letting $z = M$ in (3.21), we have

$$\beta^2(\epsilon)\|J(w_1)(M) - J(w_2)(M)\|^2 \le K^2C^2M^2\|w_1 - w_2\|_1^2 \tag{3.24}$$

and letting $z = 0$ in (3.22), we have

$$\|J(w_1)(0) - J(w_2)(0)\|^2 \le K^2C^2M^2\|w_1 - w_2\|_1^2. \tag{3.25}$$

Combining (3.23) - (3.25), we obtain

$$\beta(\epsilon)^{\frac{z}{M}}\|J(w_1)(z) - J(w_2)(z)\| \le KCM\|w_1 - w_2\|_1, \quad \forall z \in [0, M]$$

which leads to (3.20). Since $KCM < 1$, it means that J is a contraction. It follows that the equation $J(w) = w$ has a unique solution $w \in C([0; M]; H)$.

Step 2. Estimate the error $\|U^\epsilon(z) - u(z)\|$.

Differentiating (2.5) with respect to z, adding the result obtained to (2.5) and taking the inner product with ϕ_n, we get

$$\varphi_n + \int_0^z \frac{e^{-\sqrt{\lambda_n}s}}{\sqrt{\lambda_n}} f_n(u)(s)ds = e^{-M\sqrt{\lambda_n}}\left[\left\langle u(M), \phi_n\right\rangle + \frac{\left\langle u_z(M), \phi_n\right\rangle}{\sqrt{\lambda_n}}\right] - \int_z^M \frac{e^{-\sqrt{\lambda_n}s}}{\sqrt{\lambda_n}} f_n(u)(s)ds.$$

This implies that

$$\begin{aligned}u_n(z) := \left\langle u(z), \phi_n\right\rangle &= \cosh\left(\sqrt{\lambda_n}z\right)\varphi_n + \int_0^z \frac{\sinh\left(\sqrt{\lambda_n}z\right)}{\sqrt{\lambda_n}} f_n(u)(s)ds \\ &= \cosh_\epsilon(\sqrt{\lambda_n}z)\varphi_n + \int_0^z \frac{\sinh_\epsilon(\sqrt{\lambda_n}(z-s))}{\sqrt{\lambda_n}} f_n(u)(s)ds \\ +\left[\cosh\left(\sqrt{\lambda_n}z\right) - \cosh_\epsilon(\sqrt{\lambda_n}z)\right]\varphi_n &+ \int_0^z \left[\frac{\sinh\left(\sqrt{\lambda_n}z\right)}{\sqrt{\lambda_n}} - \frac{\sinh_\epsilon(\sqrt{\lambda_n}(z-s))}{\sqrt{\lambda_n}}\right] f_n(u)(s)ds \\ &= \cosh_\epsilon(\sqrt{\lambda_n}z)\varphi_n + \int_0^z \frac{\sinh_\epsilon(\sqrt{\lambda_n}(z-s))}{\sqrt{\lambda_n}} f_n(u)(s)ds \\ &+\frac{\beta(\epsilon)e^{\sqrt{\lambda_n}z}}{2(\beta(\epsilon) + e^{-\sqrt{\lambda_n}M})}\left(\varphi_n + \int_0^z \frac{e^{-\sqrt{\lambda_n}s}}{\sqrt{\lambda_n}} f_n(u)(s)ds\right) \\ &= \cosh_\epsilon(\sqrt{\lambda_n}z)\varphi_n + \int_0^z \frac{\sinh_\epsilon(\sqrt{\lambda_n}(z-s))}{\sqrt{\lambda_n}} f_n(u)(s)ds \\ +\frac{\beta(\epsilon)e^{\sqrt{\lambda_n}(z-M)}}{2(\beta(\epsilon) + e^{-\sqrt{\lambda_n}M})}&\left[\left\langle u(M), \phi_n\right\rangle + \frac{\left\langle u_z(M), \phi_n\right\rangle}{\sqrt{\lambda_n}}\right] - \int_z^M \frac{\beta(\epsilon)e^{-\sqrt{\lambda_n}(s-z)}}{2\sqrt{\lambda_n}\left(\beta(\epsilon) + e^{-\sqrt{\lambda_n}M}\right)} f_n(u)(s)ds.\end{aligned}$$

Using that

$$\left|\left\langle u(M),\phi_n\right\rangle + \frac{\left\langle u_z(M),\phi_n\right\rangle}{\sqrt{\lambda_n}}\right| \le C\left(\left|\left\langle u(M),\phi_n\right\rangle\right| + \left|\left\langle u_z(M),\phi_n\right\rangle\right|\right),$$

expression (2.6) and Lemma 2.1 we obtain

$$\begin{aligned}
&\left|U_n^\epsilon(z) - u_n(z)\right| \\
&\le \cosh_\epsilon(\sqrt{\lambda_n}z)\left|\varphi_n^\epsilon - \varphi_n\right| + \frac{\beta(\epsilon)e^{\sqrt{\lambda_n}(z-M)}}{2(\beta(\epsilon)+e^{-\sqrt{\lambda_n}M})}\left|\left\langle u(M),\phi_n\right\rangle + \frac{\left\langle u_z(M),\phi_n\right\rangle}{\sqrt{\lambda_n}}\right| \\
&\quad + \int_0^z \frac{|\sinh_\epsilon(\sqrt{\lambda_n}(z-s))|}{\sqrt{\lambda_n}}\left|f_n(U^\epsilon)(s) - f_n(u)(s)\right|ds \\
&\quad + \int_z^M \frac{\beta(\epsilon)e^{-\sqrt{\lambda_n}(s-z)}}{2\sqrt{\lambda_n}\left(\beta(\epsilon)+e^{-\sqrt{\lambda_n}M}\right)}\left|f_n(U^\epsilon)(s) - f_n(u)(s)\right|ds \\
&\le \beta(\epsilon)^{-\frac{z}{M}}\left|\varphi_n^\epsilon - \varphi_n\right| + \frac{1}{2}C^2\beta(\epsilon)^{1-\frac{z}{M}}\left(\left|\left\langle u(M),\phi_n\right\rangle\right| + \left|\left\langle u_z(M),\phi_n\right\rangle\right|\right) \\
&\quad + \frac{C}{2}\int_0^z \beta(\epsilon)^{\frac{s-z}{M}}\left|f_n(U^\epsilon)(s) - f_n(u)(s)\right|ds + \frac{C}{2}\int_z^M \beta(\epsilon)^{\frac{s-z}{M}}\left|f_n(U^\epsilon)(s) - f_n(u)(s)\right|ds \\
&\le \beta(\epsilon)^{-\frac{z}{M}}\left|\varphi_n^\epsilon - \varphi_n\right| + \frac{1}{2}C^2\beta(\epsilon)^{1-\frac{z}{M}}\left(\left|\left\langle u(M),\phi_n\right\rangle\right| + \left|\left\langle u_z(M),\phi_n\right\rangle\right|\right) \\
&\quad + \frac{C}{2}\int_0^M \beta(\epsilon)^{\frac{s-z}{M}}\left|f_n(U^\epsilon)(s) - f_n(u)(s)\right|ds.
\end{aligned}$$

From the inequality

$$(a_1+a_2+a_3)^2 \le 2\left(1+\frac{1}{m}\right)a_1^2 + 2\left(1+\frac{1}{m}\right)a_2^2 + (1+m)a_3^2$$

for any real numbers a_1, a_2, a_3 and $m > 0$, we obtain

$$\begin{aligned}
&\|U^\epsilon(z) - u(z)\|^2 = \sum_{n=1}^{\infty}\left|U_n^\epsilon(z) - u_n(z)\right|^2 \\
&\le 2\left(1+\frac{1}{m}\right)\beta(\epsilon)^{-\frac{2z}{M}}\sum_{n=1}^{\infty}\left|\varphi_n^\epsilon - \varphi_n\right|^2 + \frac{1}{2}\left(1+\frac{1}{m}\right)\sum_{n=1}^{\infty}C^4\beta(\epsilon)^{2-\frac{2z}{M}}\left|\left\langle u(M),\phi_n\right\rangle + \left\langle u_z(M),\phi_n\right\rangle\right|^2 \\
&\quad + \frac{(1+m)}{4}\sum_{n=1}^{\infty}C^2\beta(\epsilon)^{2-\frac{2z}{M}}\left[\int_0^M \beta(\epsilon)^{\frac{s}{M}-1}\left|f_n(U^\epsilon)(s) - f_n(u)(s)\right|ds\right]^2 \\
&\le 2\left(1+\frac{1}{m}\right)\beta(\epsilon)^{-\frac{2z}{M}}\epsilon^2 + \left(1+\frac{1}{m}\right)C^4\beta(\epsilon)^{2-\frac{2z}{M}}\left(\|u(M)\|^2 + \|u_z(M)\|^2\right) \\
&\quad + \frac{(1+m)}{4}C^2M\beta(\epsilon)^{2-\frac{2z}{M}}\int_0^M \beta(\epsilon)^{\frac{2s}{M}-2}\|f(s,U^\epsilon(s)) - f(s,u(s))\|^2 ds,
\end{aligned}$$

where we have applied the Holder inequality

$$\left[\int_0^M \beta(\epsilon)^{\frac{s}{M}-1}\left|f_n(U^\epsilon)(s)-f_n(u)(s)\right|ds\right]^2 \le \int_0^M 1^2 ds \int_0^M \beta(\epsilon)^{\frac{2s}{M}-2}\left|f_n(U^\epsilon)(s)-f_n(u)(s)\right|^2 ds$$
$$= M\int_0^M \beta(\epsilon)^{\frac{2s}{M}-2}\left|f_n(U^\epsilon)(s)-f_n(u)(s)\right|^2 ds.$$

This leads to

$$\beta(\epsilon)^{\frac{2z}{M}-2}\|U^\epsilon(z)-u(z)\|^2 \le 2\left(1+\frac{1}{m}\right)\beta(\epsilon)^{-2}\epsilon^2+\left(1+\frac{1}{m}\right)C^4E^2$$
$$+\frac{(1+m)}{4}K^2C^2M\int_0^M \beta(\epsilon)^{\frac{2s}{M}-2}\|U^\epsilon(s)-u(s)\|^2 ds. \tag{3.26}$$

Set $I(z) := \beta(\epsilon)^{\frac{2z}{M}-2}\|U^\epsilon(z)-u(z)\|^2$ for all $z\in[0,M]$. Since $U^\epsilon,\ u \in C([0;M];H)$, the function I is continuous on $[0,M]$ and attains over there its maximum P at some point $z_0\in[0,M]$. Therefore, (3.26) yields

$$\beta(\epsilon)^{\frac{2z}{M}-2}\|U^\epsilon(z)-u(z)\|^2 \le 2\left(1+\frac{1}{m}\right)\beta(\epsilon)^{-2}\epsilon^2+\left(1+\frac{1}{m}\right)C^4E^2+\frac{(1+m)}{4}K^2C^2M^2P$$

Choosing $z=z_0$ on the left-hand side of this inequality, we get

$$P \le 2\left(1+\frac{1}{m}\right)\beta(\epsilon)^{-2}\epsilon^2+\left(1+\frac{1}{m}\right)C^4E^2+\frac{(1+m)}{4}K^2C^2M^2P$$

or,

$$\left[1-\frac{(1+m)}{4}K^2C^2M^2\right]P \le 2\left(1+\frac{1}{m}\right)\beta(\epsilon)^{-2}\epsilon^2+\left(1+\frac{1}{m}\right)C^4E^2 = \left(1+\frac{1}{m}\right)(C^4E^2+2\beta(\epsilon)^{-2}\epsilon^2).$$

Since $m\in\left(0,\frac{4}{K^2M^2C^2}-1\right)$ it follows that the left hand-side bracket is positive. This implies that for all $z\in[0,M]$ we have

$$\beta(\epsilon)^{\frac{2z}{M}-2}\|U^\epsilon(z)-u(z)\|^2 \le P \le Q^2(\epsilon;m).$$

Thus (2.10) holds.

Finally, in order to get the estimate (2.12) at $z=M$, we use that

$$\|u(M)-U^\epsilon(z)\| \le \|u(M)-u(z)\|+\|u(z)-U^\epsilon(z)\| \le \left(\sup_{0\le z\le M}\|u_z(z)\|\right)(M-z)+Q(\epsilon;m)\beta(\epsilon)^{1-\frac{z}{M}}.$$

For every $\epsilon>0$, there exists a unique $z_\epsilon\in(0,M)$ such that $M-z_\epsilon=\beta(\epsilon)^{1-\frac{z_\epsilon}{M}}$. This implies that $\frac{\ln(M-z_\epsilon)}{M-z_\epsilon}=\frac{\ln(\beta(\epsilon))}{M}$. Using the inequality $\ln y > -\frac{1}{y}$ for every $y>0$, we obtain $M-z_\epsilon<\sqrt{\frac{M}{\ln(\frac{1}{\beta(\epsilon)})}}$. This leads to (2.12). Theorem 1.1 has been proved.

4. Numerical experiments

Let $\Omega = (a,b)\times(c,d) \subset \mathbb{R}^2$ be a rectangle and let $M > 0$ be a constant. Consider the following Cauchy problem for the three-dimensional sine-Gordon elliptic equation:

$$\Delta u = f(x,y,z,u) = \frac{1}{2}\sin(u) + R(x,y,z), \quad (x,y,z) \in \Omega\times(0,M), \tag{4.27}$$
$$u(x,y,0) = \varphi(x,y), \quad (x,y) \in \Omega, \tag{4.28}$$
$$u_z(x,y,0) = 0, \quad (x,y) \in \Omega, \tag{4.29}$$
$$u(x,y,z) = 0, \quad (x,y,z) \in \partial\Omega\times(0,M). \tag{4.30}$$

where Δ is the three-dimensional Laplace operator. We take $R(x,y,z) = \Delta\chi(x,y,z) - \frac{1}{2}\sin(\chi(x,y,z))$, where

$$\chi(x,y,z) = \frac{\sin\left[qz^2(x-a)(b-x)(y-c)(d-y)\right]}{(x-x_0)^2+(y-y_0)^2+1} \tag{4.31}$$

plays the role of the exact solution of the above problem, for any contants x_0, y_0 and q. In addition, we can check that $\chi_z(x,y,0) = 0$ and that $\varphi(x,y) = \chi(x,y,0) = 0$ is the exact Cauchy data of the problem.

Using a uniform rectangular grid with a resolution of $I\times J$ in the xy-plane, which is defined by nodal interior points (x_i, y_j) as

$$x_i = i\delta_x + a, \quad \delta_x = \frac{b-a}{I+1}, \quad i = \overline{1,I}, \quad I \in \mathbb{N}^*, \tag{4.32}$$
$$y_j = j\delta_y + c, \quad \delta_y = \frac{d-c}{J+1}, \quad j = \overline{1,J}, \quad J \in \mathbb{N}^*, \tag{4.33}$$

we define the data input

$$\varphi^{\epsilon}_{ij} = \chi(x_i,y_j,0) + \epsilon\,\mathrm{rand}(x_i,y_j) = \epsilon\,\mathrm{rand}(x_i,y_j), \tag{4.34}$$

which is disturbed by the pseudo-random $\mathrm{rand}(\cdot,\cdot)$ function determined uniformly on $[-1,1]$ and $\epsilon \geq 0$ denotes the amplitude of noise.

Then, for the rectangle $\Omega = (a,b)\times(c,d)$ and homogeneous Dirichlet boundary conditions (4.30) on $\partial\Omega$, the regularized integral equation (2.6) can be rewritten as follows:

$$u^{\beta}(z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Bigg[\cosh_{\epsilon}(z\sqrt{\lambda_{mn}})\Big\langle\varphi^{\epsilon},\phi_{mn}\Big\rangle + \int_0^z \frac{\sinh_{\epsilon}((z-s)\sqrt{\lambda_{mn}})}{\sqrt{\lambda_{mn}}}\Big\langle f(s,u^{\beta}(s)),\phi_{mn}\Big\rangle ds - \int_z^M \frac{\beta e^{-(s-z)\sqrt{\lambda_{mn}}}}{\sqrt{\lambda_{mn}}\left(\beta+e^{-M\sqrt{\lambda_{mn}}}\right)}\Big\langle f(s,u^{\beta}(s)),\phi_{mn}\Big\rangle ds\Bigg]\phi_{mn}, \tag{4.35}$$

where $\beta = \beta(\epsilon)$ and

$$\phi_{mn}(x,y) = \sin\left(\frac{m\pi(x-a)}{b-a}\right)\sin\left(\frac{n\pi(y-c)}{d-c}\right), \quad \lambda_{mn} = \left(\frac{m\pi}{b-a}\right)^2 + \left(\frac{n\pi}{d-c}\right)^2. \tag{4.36}$$

Denote the Fourier coefficients of a function $v(x,y)$ by

$$\langle v,\,\phi_{mn}\rangle = \hat{v}_{mn} = \frac{2}{(b-a)}\frac{2}{(d-c)}\int_a^b\int_c^d v(x,y)\phi_{mn}(x,y)\,dxdy.$$

Next part explains the numerical procedures for solving Eq.(4.35).

4.1. Calculation procedures

In order to solve Eq.(4.35) numerically, we shall adopt the Picard iteration. For a given discrete data $\{\varphi^{\epsilon}_{ij}\}$ from Eq.(4.34), to obtain the left-hand-side of Eq. (4.35), we need to approximate both of the Fourier coefficients, the double summation and the integrals included in the right-hand-side. The main idea is to use trigonometric polynomials, see [8], Chapter 2, which then leads us to benefit of using the Fast Fourier Transform technique (FFT). First, we model a data function from its discrete values so that the calculation of the Fourier coefficients and double summation can be performed using the FFT, and then we numerically evaluate the integrals involved.

Firstly, using the trigonometric polynomial approximation (4.36) the data $\varphi^{\epsilon}(x, y)$ is modeled from $\{\varphi^{\epsilon}_{ij}\}$ as follows:

$$\varphi^{\epsilon}(x,y) = \sum_{m=1}^{I}\sum_{n=1}^{J} \hat{\varphi}^{\epsilon}_{mn} \sin\left(\frac{m\pi(x-a)}{b-a}\right)\sin\left(\frac{n\pi(y-c)}{d-c}\right), \tag{4.37}$$

where

$$\hat{\varphi}^{\epsilon}_{mn} := \frac{2}{I+1}\frac{2}{J+1}\sum_{i=1}^{I}\sum_{j=1}^{J} \varphi^{\epsilon}_{ij} \sin\left(\frac{m\pi i}{I+1}\right)\sin\left(\frac{n\pi j}{J+1}\right), \quad m = \overline{1, I},\ n = \overline{1, J} \tag{4.38}$$

is the so-called two-dimensional sine transform, with its inverse transformation given by

$$\varphi^{\epsilon}_{ij} = \sum_{m=1}^{I}\sum_{n=1}^{J} \hat{\varphi}^{\epsilon}_{mn} \sin\left(\frac{m\pi i}{I+1}\right)\sin\left(\frac{n\pi j}{J+1}\right). \tag{4.39}$$

The relationships between $\hat{\varphi}^{\epsilon}_{mn}$ and φ^{ϵ}_{ij} given in Eqs. (4.38) and (4.39) can also be found in [12], Chapter 12. So far, Eqs. (4.37) - (4.39) give $\varphi^{\epsilon}(x_i, y_j) = \varphi^{\epsilon}_{ij}$ (the double summation) and $\langle \varphi^{\epsilon}, \phi_{mn}\rangle = \hat{\varphi}^{\epsilon}_{mn}$ (the Fourier coefficients) precisely. In addition, one has the discrete form of Parseval's identity

$$\sum_{m=1}^{I}\sum_{n=1}^{J} \left|\hat{\varphi}^{\epsilon}_{mn}\right|^2 = \frac{2}{I+1}\frac{2}{J+1}\sum_{i=1}^{I}\sum_{j=1}^{J} \left|\varphi^{\epsilon}_{ij}\right|^2 .$$

Combining the latter identity with the triangle inequality, one can obtain, see [8], Chapter 2,

$$||\varphi^{\epsilon} - \varphi|| \le \epsilon_0, \tag{4.40}$$

where $\epsilon_0 = \epsilon\sqrt{(b-a)(d-c)} + C_1(\delta_x^2 + \delta_y^2)\sqrt{||\partial_x^2\varphi||^2 + ||\partial_y^2\varphi||^2}$ and C_1 is some positive constant independent of φ, δ_x and δ_y.

The calculations in Eqs. (4.38) and (4.39) are performed in a natural way. For instance, the sine transform $\{\varphi^{\epsilon}_{ij}\} \mapsto \{\hat{\varphi}^{\epsilon}_{mn}\}$ (Eq. (4.38)) can be computed in two steps:

Step 1: Loop for $i = \overline{1, I}$,

$$w_{ni} := \frac{2}{J+1}\sum_{j=1}^{J} \varphi^{\epsilon}_{ij} \sin\left(\frac{n\pi j}{J+1}\right), \quad n = \overline{1, J}. \tag{4.41}$$

Step 2: Loop for $n = \overline{1, J}$,

$$\hat{\varphi}^{\epsilon}_{mn} = \frac{2}{I+1}\sum_{i=1}^{I} w_{ni} \sin\left(\frac{m\pi i}{I+1}\right), \quad m = \overline{1, I}. \tag{4.42}$$

Here, the subroutine sint1f of FFTPACK5, [15], is adopted for these calculations. The total computational burden in both i- and n-loops (Eqs. (4.41) and (4.42)) is of order

$$\underbrace{I * \mathrm{O}(J \log J)}_{i\text{-loop}} + \underbrace{J * \mathrm{O}(I \log I)}_{n\text{-loop}} \sim \mathrm{O}(IJ \log(IJ)),$$

which is equal to the number of operations on a one-dimenstional vector with $I * J$ components. Similarly, calculation of the inverse transform $\{\hat{\varphi}^{\epsilon}_{mn}\} \mapsto \{\varphi^{\epsilon}_{ij}\}$ (Eq.(4.39)) is performed using the subroutine sint1b in the same manner.

Secondly, as mentioned before, a numerical solution to Eq.(4.35) can be found by a fixed-point convergent iteration. To calculate a u^{β}-profile, we need to compute the integrals inside the RHS of Eq. (4.35) from a prior u^{β}-profile. Therefore, the computation is performed on a fixed mesh in z-direction, namely,

$$z_k = (k-1)\delta_z, \quad \delta_z = \frac{M}{K+1}, \quad k = \overline{1, K}, \quad K \in \mathbb{N}^*. \tag{4.43}$$

Using Fubini's theorem, the integrals can be formed as

$$\int \langle \Phi, \phi_{mn} \rangle \, ds = \left\langle \int \Phi \, ds, \phi_{mn} \right\rangle$$

for each $m = \overline{1, I}$, $n = \overline{1, J}$ and $z \in [0, M]$, where the function $\Phi = \Phi(x, y, z, s, m, n)$ has only discrete values for variables z and s. For simplicity, we are going to approximate the integral

$$\int_{s_1}^{s_p} \Phi(s) \, ds \tag{4.44}$$

from the values of $\Phi_l = \Phi(s_l)$, $l = \overline{1, p}$. Note that $\Delta s = s_{l+1} - s_l = \delta_z$ and interval in (4.44) belongs to two cases: $s_1 = 0$, $s_p = z_k$ or $s_1 = z_k$, $s_p = M$, for each $z_k \in [0, M]$ given in Eq. (4.43). Now using Newton-Cotes formulas (closed-typed), we have

$$\int_{s_1}^{s_p} \Phi(s) \, ds \simeq \delta_z \sum_{i=1}^{p} H_{p,i} \, \Phi_i, \tag{4.45}$$

where (see [1], p. 886) the coefficients $H_{p,i}$ are given by in Table 1 for $p = \overline{2, 8}$, $i = \overline{1, p}$. For $p > 8$, we also have that Eq. (4.45) can be written as

$$\begin{aligned} \int_{s_1}^{s_p} \Phi(s) \, ds \simeq \delta_z \Bigg(& \frac{17}{48}\Phi_1 + \frac{59}{48}\Phi_2 + \frac{43}{48}\Phi_3 + \frac{49}{48}\Phi_4 + \Phi_5 + \cdots + \Phi_{p-4} \\ & + \frac{49}{48}\Phi_{p-3} + \frac{43}{48}\Phi_{p-2} + \frac{59}{48}\Phi_{p-1} + \frac{17}{48}\Phi_p \Bigg), \end{aligned} \tag{4.46}$$

with the leading error proportional to $|\Delta s|^4$.

We also approximate the function f by its own trigonometric polynomials, thus,

$$\langle f(z_l, u(z_l)), \phi_{mn} \rangle = \frac{4}{(I+1)(J+1)} \sum_{i=1}^{I} \sum_{j=1}^{J} f(u(x_i, y_j, z_l), x_i, y_j, z_l) \sin\left(\frac{m\pi i}{I+1}\right) \sin\left(\frac{n\pi j}{J+1}\right).$$

Note that the double summation in Eq. (4.35) is now finite.

Equation (4.40) indicates that the quality of data function φ^{ϵ} modeled by the trigonometric polynomials is dependent on both the noise amplitude ϵ, mesh resolution (δ_x and δ_y), and smoothness of the approximated function φ (i.e. $\|\partial_x^2 \varphi\|$ and $\|\partial_y^2 \varphi\|$). The following test cases illustrate such dependencies.

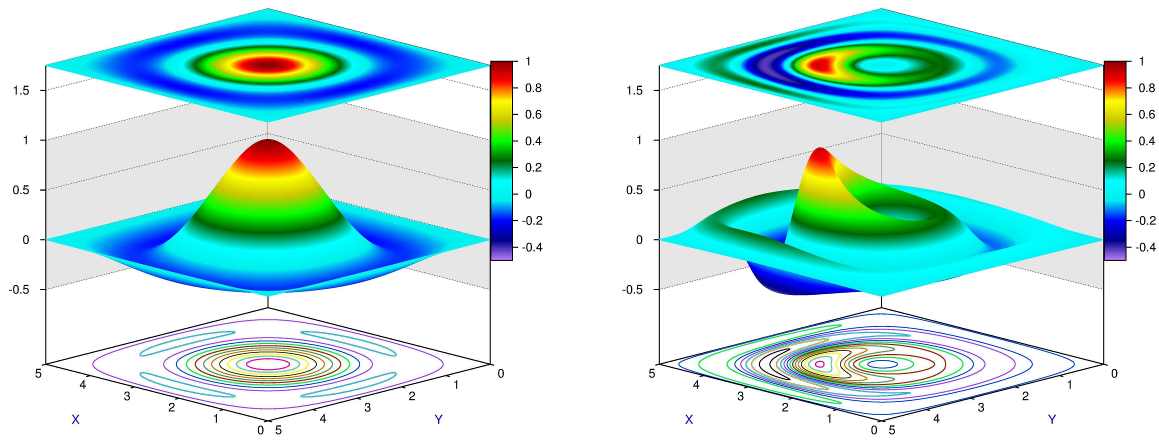
Table 1: The coefficients $H_{p,i}$ for $p = \overline{2,8}$, $i = \overline{1,p}$.

p	i								Error
	1	2	3	4	5	6	7	8	
2	$\frac{1}{2}$	$\frac{1}{2}$							$-\frac{\Phi^{(2)}(\xi)}{12}\lvert\Delta s\rvert^3$
3	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{1}{3}$						$-\frac{\Phi^{(4)}(\xi)}{90}\lvert\Delta s\rvert^5$
4	$\frac{3}{8}$	$\frac{9}{8}$	$\frac{9}{8}$	$\frac{3}{8}$					$-\frac{3\Phi^{(4)}(\xi)}{80}\lvert\Delta s\rvert^5$
5	$\frac{14}{45}$	$\frac{64}{45}$	$\frac{24}{45}$	$\frac{64}{45}$	$\frac{14}{45}$				$-\frac{8\Phi^{(6)}(\xi)}{945}\lvert\Delta s\rvert^7$
6	$\frac{95}{288}$	$\frac{375}{288}$	$\frac{250}{288}$	$\frac{250}{288}$	$\frac{375}{288}$	$\frac{95}{288}$			$-\frac{275\Phi^{(6)}(\xi)}{12096}\lvert\Delta s\rvert^7$
7	$\frac{41}{140}$	$\frac{216}{140}$	$\frac{27}{140}$	$\frac{272}{140}$	$\frac{27}{140}$	$\frac{216}{140}$	$\frac{41}{140}$		$-\frac{9\Phi^{(8)}(\xi)}{1400}\lvert\Delta s\rvert^9$
8	$\frac{5257}{17280}$	$\frac{25039}{17280}$	$\frac{9261}{17280}$	$\frac{20923}{17280}$	$\frac{20923}{17280}$	$\frac{9261}{17280}$	$\frac{25039}{17280}$	$\frac{5257}{17280}$	$-\frac{8183\Phi^{(8)}(\xi)}{518400}\lvert\Delta s\rvert^9$

4.2. Test cases

We introduce two examples based on the test function (4.31).

- **Example 1:** Choose $a = c = 0$, $b = d = 5$, $M = 1.1$, $x_0 = y_0 = 2.5$, $q = -0.1$. The graph of the exact solution is shown in Fig. 1(a).
- **Example 2:** Choose $a = c = 0$, $b = d = 5$, $M = 1.1$, $x_0 = y_0 = 3$, $q = 0.2$. The graph of the exact solution is shown in Fig. 1(b).



(a) Example 1

(b) Example 2

Figure 1: The analytical test functions $\chi(x, y, M)$ (Eq.(4.31)) for Examples 1 and 2.

The aim of the numerical experiments is to observe the relative error given by

$$\delta^{\epsilon,\beta}(M) := \sqrt{\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\left|u^{\beta}(x_i,y_j,M)-U(x_i,y_j,M)\right|^2}{\sum_{i=1}^{I}\sum_{j=1}^{J}\left|U(x_i,y_j,M)\right|^2}}, \tag{4.47}$$

as β tends to zero, in two following cases:

1. $\epsilon = 0$: $\{\varphi_{ij}^{0}\}$ represents exact data.
2. $\epsilon > 0$: $\{\varphi_{ij}^{\epsilon}\}$ represents measured data with random noise.

Here the computation domain $\Omega \times [0, M]$ is meshed with resolutions $I = J = K = 2^l - 1$ for $l = \overline{6, 9}$.

In the numerical practice of our study, the process of Picard iteration was terminated when the relative errors between two sequent solutions were less than 10^{-9}. Based on this, the number of iterations was around 8 for all of test cases. The numerical solution of the integral equation (4.35) in three-dimensions is time consuming, particularly to obtain a desired accuracy, we need to refine the three-dimensional mesh up to billions of grid points. Therefore, the numerical code has been parallelized by OpenMP [20] in Fortran90.

Tables 2 and 3 show the relative error $\delta^{\epsilon,\beta}(M)$ (Eq. (4.47)) for Examples 1 and 2, respectively. The computations were performed on a three-dimensional mesh with four resolutions $I = J = K = 2^l - 1$ for $l = \overline{6, 9}$, for exact data with $\epsilon = 0$ and for noisy data with $\epsilon > 0$. As shown in these tables, the magnitude of the relative error $\delta^{\epsilon,\beta}(M)$ depends on both of the mesh resolutions and the noise amplitude ϵ.

In case $\epsilon = 0$, convergence of numerical solution is improved with finer mesh as β decreases until $\beta = 10^{-5}$. However, for $\beta = 10^{-6}$ the error could not be decreased further with the finest mesh ($I = J = K = 511$), hence, a higher mesh resolution should be adopted if we want to obtain a higher accuracy. In addition, Figure 2 shows the graphs of $u^{\beta}(x, y, M)$ for Examples 1 and 2 with the exact data $\{\varphi_{ij}^{0}\}$ for the coarse mesh resolution $I \times J \times K = 63^3$. For β too small such as 10^{-6}, the instability phenomenon is manifested by the strongly oscillating contour lines.

Table 2: Example 1, relative error $\delta^{\epsilon,\beta}(M)$ defined by Eq. (4.47). The computations were performed with mesh resolutions $I = J = K = 2^l - 1$ for $l = \overline{6, 9}$.

β	$K = 63$	$K = 127$	$K = 255$	$K = 511$	$K = 511$				
	$\epsilon = 0$				$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-5}$	$\epsilon = 10^{-6}$
1.0E-1	7.3E-1	7.3E-1	7.3E-1	7.3E-1	7.4E-1	7.3E-1	7.3E-1	7.3E-1	7.3E-1
1.0E-2	1.3E-1	1.3E-1	1.3E-1	1.3E-1	1.1E+0	1.7E-1	1.3E-1	1.3E-1	1.3E-1
1.0E-3	1.7E-2	1.8E-2	1.7E-2	1.7E-2	1.1E+1	1.1E+0	1.1E-1	2.0E-2	1.8E-2
1.0E-4	6.5E-3	2.4E-3	1.9E-3	1.9E-3	Diverged	1.1E+1	1.1E+0	1.1E-1	1.1E-2
1.0E-5	3.7E-2	1.0E-2	3.1E-3	1.4E-3	Diverged	Diverged	1.1E+1	1.1E+0	1.1E-1
1.0E-6	2.3E-1	6.3E-2	1.9E-2	7.3E-3	Diverged	Diverged	Diverged	1.1E+1	1.1E+0

In case of noisy data with $\epsilon > 0$, to show the sensitivity of the computational accuracy to noise of the data, we repeated calculations with a variety of noise amplitudes $\epsilon = 10^{-l}$ for $l = \overline{2, 6}$, and illustrated the numerical results only with the finest mesh $I \times J \times K = 511^3$, so that errors from mesh resolution do not contribute to $\delta^{\epsilon,\beta}$. These results are shown in Tables 2 and 3 and Figures

Table 3: Example 2, relative error $\delta^{\epsilon,\beta}(M)$ defined by Eq. (4.47). The computations were performed with mesh resolutions $I = J = K = 2^l - 1$ for $l = \overline{6, 9}$.

β	$K = 63$	$K = 127$	$K = 255$	$K = 511$	$K = 511$				
	$\epsilon = 0$				$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-5}$	$\epsilon = 10^{-6}$
1.0E-1	1.7E+0	1.7E+0	1.7E+0	1.7E+0	1.7E+0	1.7E+0	1.7E+0	1.7E+0	1.7E+0
1.0E-2	6.6E-1	6.6E-1	6.6E-1	6.6E-1	1.4E+0	6.7E-1	6.6E-1	6.6E-1	6.6E-1
1.0E-3	1.9E-1	1.9E-1	1.9E-1	1.9E-1	1.2E+1	1.3E+0	2.3E-1	1.9E-1	1.9E-1
1.0E-4	5.3E-2	4.3E-2	4.1E-2	4.0E-2	Diverged	1.2E+1	1.2E+0	1.3E-1	4.2E-2
1.0E-5	1.1E-1	3.2E-2	1.3E-2	8.7E-3	Diverged	Diverged	1.2E+1	1.2E+0	1.2E-1
1.0E-6	6.6E-1	1.8E-1	5.4E-2	2.1E-2	Diverged	Diverged	Diverged	1.2E+1	1.2E+0

3 and 4 for Examples 1 and 2, respectively. As β tends to zero but its value is still greater than 10ϵ, the approximated solution u^β is still convergent in most cases, however, when β is smaller than $\leq 10\epsilon$ the numerical solutions start to diverge and become unstable. This is signaled by the contour lines becoming non-smooth. As justified by Theorem 2.1, for noisy data with $\epsilon > 0$, the value of $\beta(\epsilon)$ should be chosen according to Remark 2.1 such that the stability estimate (2.13) is ensured.

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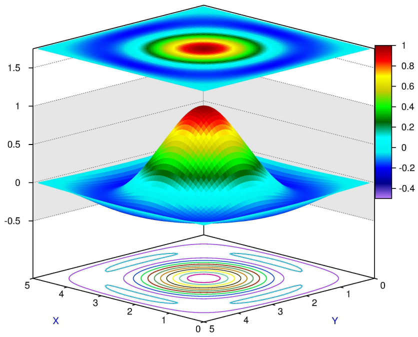
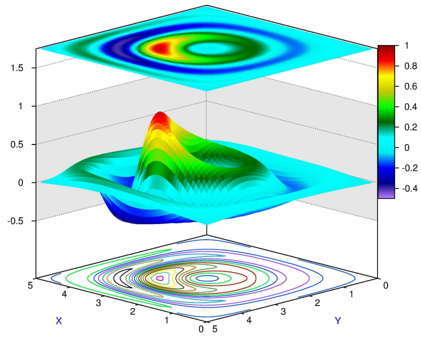
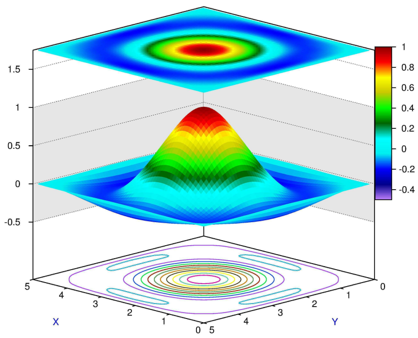


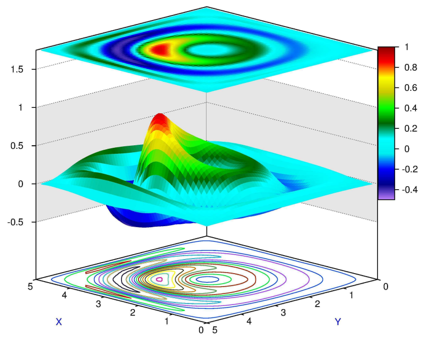
(a) Example 1, $\beta = 10^{-4}$



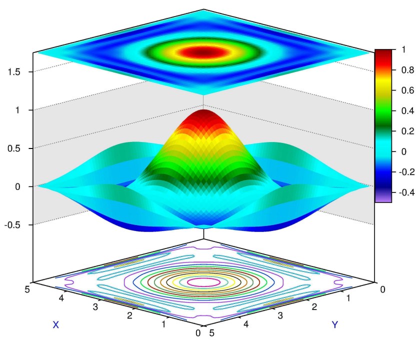
(b) Example 2, $\beta = 10^{-4}$



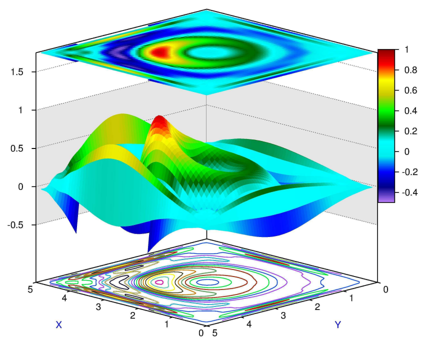
(c) Example 1, $\beta = 10^{-5}$



(d) Example 2, $\beta = 10^{-5}$

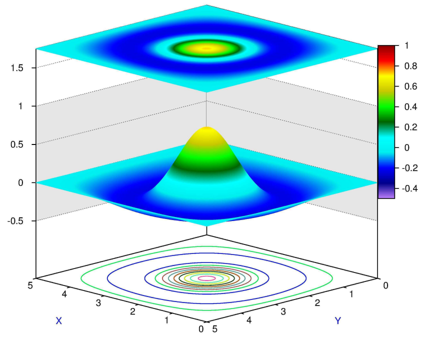


(e) Example 1, $\beta = 10^{-6}$

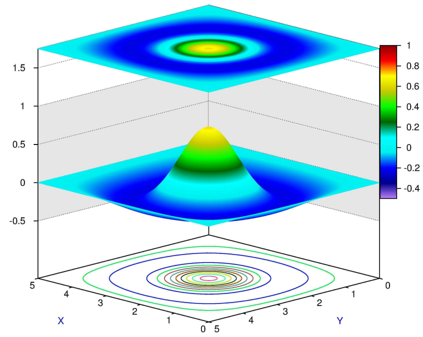


(f) Example 2, $\beta = 10^{-6}$

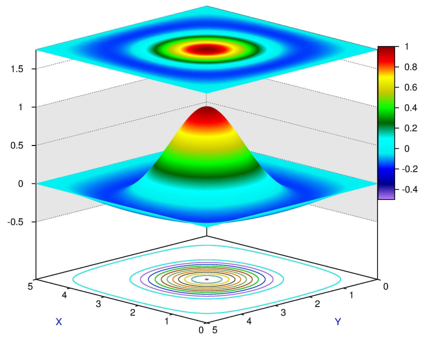
Figure 2: Graphs of $u^{\beta}(x, y, M)$ for Examples 1 and 2 with exact data φ^0_{ij}.



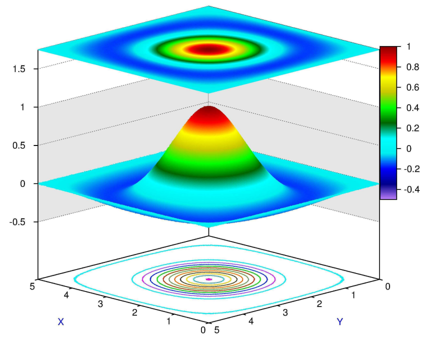
(a) $\epsilon = 0, \beta = 10^{-1}$



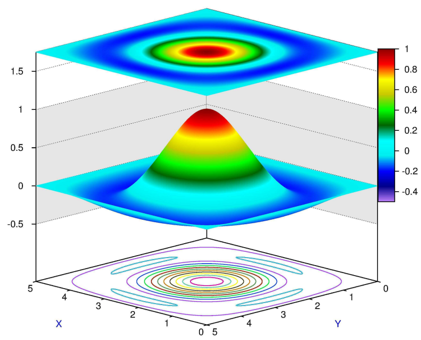
(b) $\epsilon = 10^{-4}, \beta = 10^{-1}$



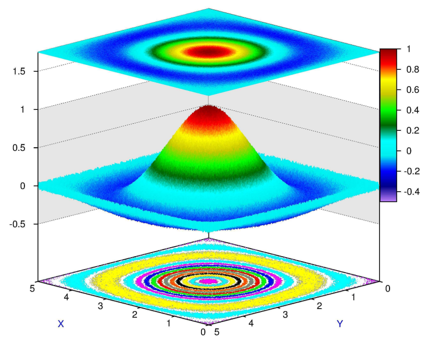
(c) $\epsilon = 0, \beta = 10^{-2}$



(d) $\epsilon = 10^{-4}, \beta = 10^{-2}$

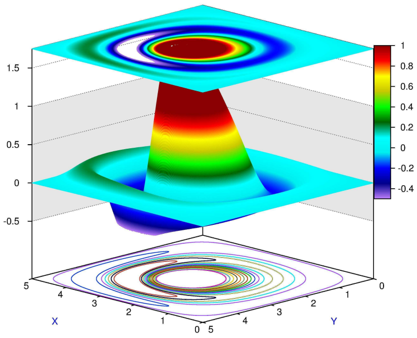


(e) $\epsilon = 0, \beta = 10^{-3}$

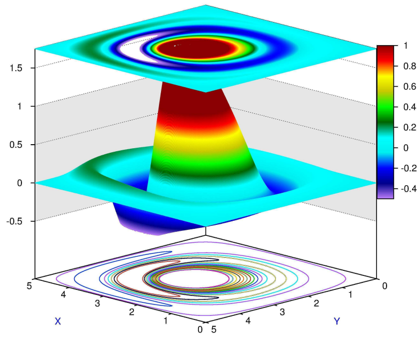


(f) $\epsilon = 10^{-4}, \beta = 10^{-3}$

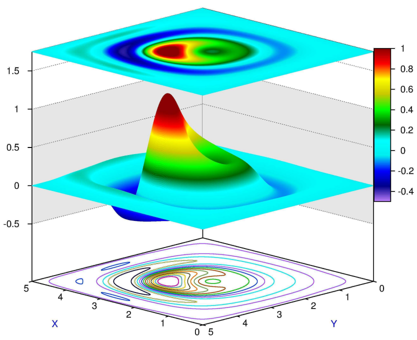
Figure 3: Graphs of $u^{\beta}(x, y, M)$ for Example 1 with data φ_{ij}^{ϵ}, $\epsilon \geq 0$.



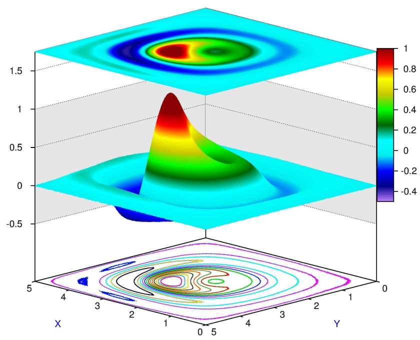
(a) $\epsilon = 0, \beta = 10^{-1}$



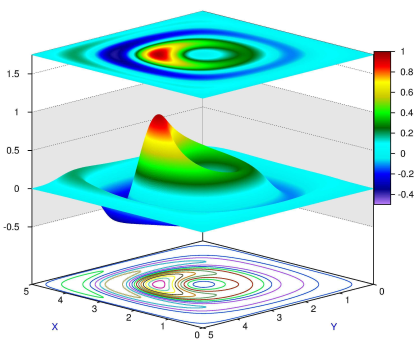
(b) $\epsilon = 10^{-4}, \beta = 10^{-1}$



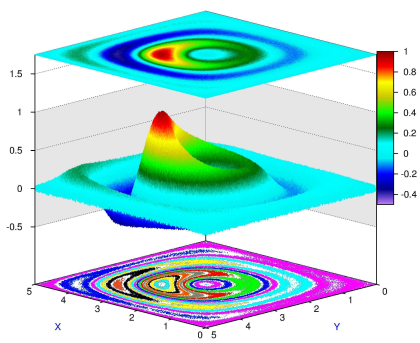
(c) $\epsilon = 0, \beta = 10^{-2}$



(d) $\epsilon = 10^{-4}, \beta = 10^{-2}$



(e) $\epsilon = 0, \beta = 10^{-3}$



(f) $\epsilon = 10^{-4}, \beta = 10^{-3}$

Figure 4: Graphs of $u^\beta(x, y, M)$ for Example 2 with data φ_{ij}^ϵ, $\epsilon \geq 0$.