Article:
Chalendar, I, Esterle, J and Partington, JR (2015) Lower estimates near the origin for functional calculus on operator semigroups. Journal of Functional Analysis, 269 (6). pp. 1620-1635. ISSN 0022-1236

https://doi.org/10.1016/j.jfa.2015.07.005

Lower estimates near the origin for functional calculus on operator semigroups

I. Chalendar[*], J. Esterle[†] and J.R. Partington[‡]

April 9, 2015

Abstract

This paper provides sharp lower estimates near the origin for the functional calculus $F(-uA)$ of a generator A of an operator semigroup defined on the (strictly) positive real line; here F is given as the Laplace transform of a measure or distribution. The results are linked to the existence of an identity element or an exhaustive sequence of idempotents in the Banach algebra generated by the semigroup. Both the quasinilpotent and non-quasinilpotent cases are considered, and sharp results are proved extending many in the literature.

Mathematics Subject Classification (2000): Primary: 47D03, 46J40, 46H30 Secondary: 30A42, 47A60

Keywords: strongly continuous semigroup, functional calculus, Laplace transform, maximum principle.

1 Introduction

This article is concerned with estimates for $F(-uA)$ where A is the generator of a strongly continuous semigroup $(T(t))_{t>0}$ on a Banach space. Here F is an

[*]I. C. J., UFR de Mathématiques, Université Lyon 1, 43 bld. du 11/11/1918, 69622 Villeurbanne Cedex, France. chalendar@math.univ-lyon1.fr.

[†]I.M.B., UMR 5251, Université de Bordeaux, 351 cours de la Libération, 33405 Talence Cedex, France. jean.esterle@math.u-bordeaux1.fr.

[‡]School of Mathematics, University of Leeds, Leeds LS2 9JT, U.K. J.R.Partington@leeds.ac.uk.

entire function with $F(0) = 0$, given as the Laplace transform of a measure or distribution; the functional calculus defining $F(-uA)$ is given by means of an integral.

This can be seen as providing a wide generalization of results in [1, 4, 6], for example, where quantities such as $\|T(t) - T(2t)\|$ (or its spectral radius) are estimated near the origin. For example, if $\|T(t) - T(2t)\| < 1/4$ on an interval $(0, t_0)$, then, roughly speaking, $(T(t))_{t>0}$ has a bounded infinitesimal generator (see [1]).

There are two cases to consider, namely, the quasinilpotent and non-quasinilpotent cases, and the techniques used are based on strong maximum principles for analytic functions.

In Section 2, the case of quasinilpotent semigroups is considered. Then in Section 3 the non-quasinilpotent case is analysed, providing conditions to obtain either an identity in the closed algebra generated by the semigroup or else an exhaustive sequence $(P_n)_{n\geq 1}$ of idempotents such that $(P_nT(t))_t$ has a bounded generator. Here, the sharpness of the estimates is shown in Remark 3.5.

Notation:

We write $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, and similarly for $\mathbb{C}_-$.

Let $D(a, R)$ denote the complex disc $\{|z - a| < R\}$.

For a Jordan curve $\Gamma \subset \mathbb{C}$, we write $\operatorname{int}\Gamma$ (the interior of Γ) for the open set of points in $\mathbb{C}$ about which the winding number of Γ is non-zero.

For $S \subset \mathbb{C}$ let $M_c(S)$ denote the space of regular Borel measures having compact support contained in S.

2 Quasinilpotent semigroups

Suppose that $(T(t))_{t>0}$ is a nontrivial strongly continuous semigroup of quasinilpotent operators acting on a Banach space $(\mathcal{X}, \|.\|)$. Then we write $\mathcal{X}_0 = \left[\bigcup_{t>0} T(t)\mathcal{X}\right]^{-\|.\|}$ (closure in norm), and define a norm

$$\|x\|_1 = \sup_{t\geq 0} \|T(t)x\|, \qquad \text{where} \quad T(0)x = x,$$

on the subspace $\mathcal{X}_1 := \{x \in \mathcal{X}_0 : \|x\|_1 < \infty\}$, which is a Banach space under the norm $\|.\|_1$. Further, we write

$$\widetilde{\mathcal{X}}_1 := \left[\bigcup_{t>0} T(t)\mathcal{X}\right]^{-\|.\|_1} \subset \mathcal{X}_1. \tag{1}$$

The following result follows immediately from the main result of [5]. It will be used to reduce the case of a quasinilpotent semigroup to that of a contractive quasinilpotent semigroup.

Theorem 2.1 *Let $(T(t))_{t>0}$ be a nontrivial strongly continuous semigroup of quasinilpotent operators acting on a Banach space $(\mathcal{X}, \|.\|)$. Then with $(\widetilde{\mathcal{X}}_1, \|.\|_1)$ defined as in (1) the semigroup $(T(t)_{|\widetilde{\mathcal{X}}_1})_{t>0}$ is a strongly continuous semigroup of quasinilpotent contractions. Moreover for all operators R in the commutant $\{T(t) : t > 0\}'$ we have $\|R_{|\widetilde{\mathcal{X}}_1}\|_1 \leq \|R\|$.*

2.1 Some complex function theory

Theorem 2.2 *Let $f : \overline{\mathbb{C}_+} \to \mathbb{C}$ be a continuous bounded nonconstant function, holomorphic on $\mathbb{C}_+$, such that $f([0,\infty)) \subset \mathbb{R}$, $f(0) = 0$, and with $\lim_{x\to\infty, x\in\mathbb{R}} f(x) = 0$.*

Suppose that $\alpha > 0$ is such that $f(\alpha) \geq |f(x)|$ for all $x \in [0,\infty)$. Then there exist $a_1, a_2 \in \mathbb{C}_+$, $a_0 \in (\alpha, a_1)$ and $a_3 \in i\mathbb{R}$ with $\operatorname{Im} a_j > 0$ for $j = 1, 2, 3$, and $\operatorname{Im} a_2 = \operatorname{Im} a_3$, and a simple piecewise linear Jordan curve Γ_1 joining a_1 to a_2 in the upper right half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$ and $\delta > 0$ such that

(i) $|f(z)| \geq f(\alpha) + \delta|z-\alpha|^m$ *for all* $z \in [\alpha, a_1]$*, where* m *(even) is the smallest positive integer with* $f^{(m)}(\alpha) \neq 0$*;*

(ii) $|f(z)| > |f(a_0)|$ *for all* $z \in \Gamma_1 \cup [a_2, a_3]$.

Proof : Since f is holomorphic in $\mathbb{C}_+$, we have, by Taylor's theorem, constants $M > 0$ and $\eta > 0$ such that

$$\left| f(z) - f(\alpha) - \frac{(z-\alpha)^m}{m!} f^{(m)}(\alpha) \right| \leq M|z-\alpha|^{m+1},$$

whenever $|z-\alpha| < \eta$. By choosing a_1 with $|a_1 - \alpha|$ sufficiently small and with argument such that $(a_1-\alpha)^m < 0$ (e.g. $\arg(a_1-\alpha) = \pi/m$), we have condition (i) and hence $|f(a_1)| > |f(\alpha)| = f(\alpha)$; we may then choose a point $a_0 \in [\alpha, a_1]$ with

$$|f(\alpha)| < |f(a_0)| < |f(a_1)|.$$

Let $U = \{z \in \mathbb{C}_+ : |f(z)| > |f(a_0)|\}$. Since $a_1 \in U$, this is a nonempty open set; we let V denote the connected component of U containing a_1.

We claim that $\partial V \cap i\mathbb{R} \neq \emptyset$. Indeed, note that if $z \in \partial V \cap \mathbb{C}_+$, then $|f(z)| \leq |f(a_0)|$, as otherwise if $|f(z)| > |f(a_0)|$, then by the continuity of $|f|$ a neighbourhood of z is contained in V. But we cannot have $|f(z)| \leq |f(a_0)|$ on the whole of ∂V, as then by the strong maximum principle (see, e.g. [7, Thm. 9.4]) this inequality would hold for all $z \in V$ including a_1.

So there exists $a_3 \in \partial V \cap i\mathbb{R}$ with $\operatorname{Im} a_3 > 0$ and $|f(a_3)| > |f(a_0)|$. By the continuity of $|f|$, there exists $\beta > 0$ such that $|f(z)| > |f(a_0)|$ for all $z \in \overline{\mathbb{C}_+}$ with $|z - a_3| < \beta$. It follows that there is a point $a_2 \in \mathbb{C}_+$ with $\operatorname{Im} a_2 = \operatorname{Im} a_3$ and $(a_3, a_2] \subset V$.

Since V is open and connected, it is path-connected, and so we may join a_1 to a_2 by a polygonal path in V. We may also guarantee that it is simple (does not cross itself): the only difficulty arises if it crosses itself on the arc (a_0, a_1), when we may replace a_1 by the crossing point closest to a_0, or if it crosses itself on the line (a_2, a_3), when we may replace a_2 by the crossing point closest to a_3.

□

The curve constructed in the proof of Theorem 2.2 may be seen as the upper part of Figure 1.

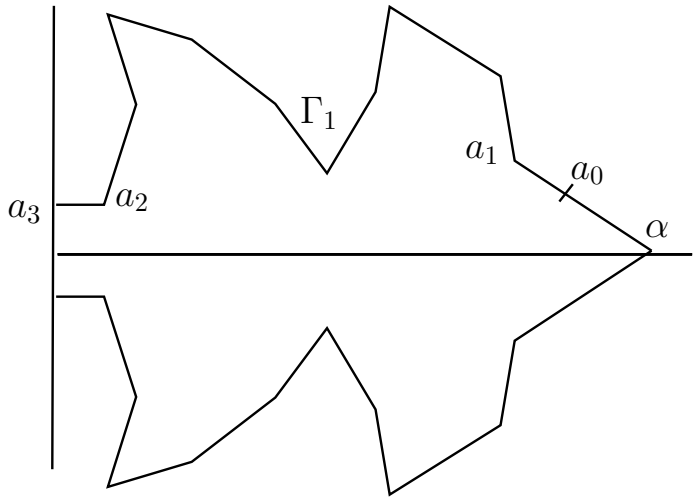

Figure 1: The curve constructed in the proof of Theorem 2.5

We shall also require the following easy result.

Lemma 2.3 *Let $f \in H(\mathbb{C})$ with $f(0) = 0$, f nonconstant. Then there is an $r > 0$ such that for all $u \in \mathbb{C} \setminus \{0\}$, with $|u| < r$, we have*

$$\sup_{|z| \leq \frac{r}{|u|}} |f(zu)| < \sup_{x \geq 0} |f(xu)|.$$

Proof : Choose $R > 0$ such that $f(z) \neq 0$ for all z with $|z| = R$. Let $\delta = \inf\{|f(z)| : |z| = R\}$ so that $\delta > 0$. Now take $r > 0$ such that $\sup_{|z| \leq r} |f(z)| < \delta$, using the continuity of f and the fact that $f(0) = 0$.

Then $\sup_{x>0} |f(xu)| \geq \delta$ for all $u \in \mathbb{C} \setminus \{0\}$, and the conclusion follows. □

2.2 The main result

Recall that if $(T(t))_{t>0}$ is a uniformly bounded strongly continuous semigroup with generator A, then

$$(A+\lambda I)^{-1} = -\int_0^\infty e^{\lambda t} T(t)\, dt,$$

for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < 0$. Here the integral is taken in the sense of Bochner with respect to the strong operator topology.

If, in addition, $(T(t))_{t>0}$ is quasinilpotent, then

$$(A+\lambda I)^{-1} = -\int_0^\infty e^{\lambda t} T(t)\, dt,$$

for all $\lambda \in \mathbb{C}$.

Similarly, if $\mu \in M_c(0,\infty)$ with Laplace transform

$$F(s) := \mathcal{L}\mu(s) = \int_0^\infty e^{-s\xi}\, d\mu(\xi), \tag{2}$$

and $(T(t))_{t>0}$ is a strongly continuous semigroup of bounded operators on $\mathcal{X}$, then we have a functional calculus for its generator A, defined by

$$F(-A) = \int_0^\infty T(\xi)\, d\mu(\xi),$$

in the sense of the strong operator topology; i.e.,

$$F(-A)x = \int_0^\infty T(\xi)x\, d\mu(\xi), \qquad (x \in \mathcal{X}),$$

which exists as a Bochner integral.

Lemma 2.4 *Let $\mu \in M_c(0,\infty)$ and $(T(t))_{t>0}$ a strongly continuous quasinilpotent semigroup of contractions. Set $F = \mathcal{L}\mu$. Then we have for $\operatorname{Re}\lambda \geq 0$,*

$$\left\|(F(-A) - F(\lambda)I)(A+\lambda I)^{-1}\right\| \leq \int_0^\infty t\, d|\mu|(t).$$

Proof : We have

$$\begin{aligned} F(-A)(A+\lambda I)^{-1} &= -\left(\int_0^\infty T(t)\,d\mu(t)\right)\left(\int_0^\infty e^{\lambda s}T(s)\,ds\right) \\ &= -\int_0^\infty e^{-\lambda t}\left[\int_0^\infty e^{\lambda(s+t)}T(s+t)\,ds\right]\,d\mu(t) \\ &= -\int_0^\infty e^{-\lambda t}\left[\int_t^\infty e^{\lambda v}T(v)\,dv\right]\,d\mu(t), \end{aligned}$$

where $v = s + t$. This in turn equals

$$\begin{aligned} &-\int_0^\infty e^{-\lambda t}\left[\int_0^\infty e^{\lambda v}T(v)\,dv\right]\,d\mu(t) + \int_0^\infty e^{-\lambda t}\left[\int_0^t e^{\lambda v}T(v)\,dv\right]\,d\mu(t) \\ = \;& F(\lambda)(A+\lambda I)^{-1} + \int_0^\infty \left[\int_0^t e^{\lambda(v-t)}T(v)\,dv\right]\,d\mu(t). \end{aligned}$$

For $\operatorname{Re}\lambda \geq 0$, we have

$$\left\|\int_0^t e^{\lambda(v-t)}T(v)\,dv\right\| \leq t,$$

and so the conclusion follows.

□

The following theorem applies to several examples studied recently in [1, 3, 4, 6]; these include $\mu = \delta_1 - \delta_2$, the difference of two Dirac measures, where $F(s) := \mathcal{L}\mu(s) = e^{-s} - e^{-2s}$ and $F(-sA) = T(s) - T(2s)$. More importantly, the theorem applies to many other examples, such as $d\mu(t) = (\chi_{[1,2]} - \chi_{[2,3]})(t)dt$ and $\mu = \delta_1 - 3\delta_2 + \delta_3 + \delta_4$, which are not accessible with the methods of [1, 3, 4, 6].

Theorem 2.5 *Let $\mu \in M_c(0,\infty)$ be a nontrivial real measure such that $\int_0^\infty d\mu(t) = 0$, and let $(T(t))_{t>0}$ be a nontrivial strongly continuous quasinilpotent semigroup of bounded operators on a Banach space $\mathcal{X}$. Set $F = \mathcal{L}\mu$. Then there is an $\eta > 0$ such that*

$$\|F(-sA)\| > \max_{x\geq 0}|F(x)| \qquad \textit{for} \quad 0 < s \leq \eta.$$

Proof : It follows from Theorem 2.1 that we may assume without loss of generality that $(T(t))_{t>0}$ is a strongly continuous quasinilpotent semigroup of contractions. Let $\alpha > 0$ be such that $|F(x)| \leq |F(\alpha)|$ for all $x \geq 0$, and let $s > 0$. By considering $-\mu$ instead of μ, if necessary, we may suppose that $F(\alpha) > 0$.

By Lemma 2.4 applied to the semigroup $(T(st))_{t>0}$, for $\operatorname{Re}\lambda \geq 0$ we obtain

$$\left\|F(-sA)(sA+\lambda I)^{-1}\right\| \geq \left\|F(\lambda)(sA+\lambda I)^{-1}\right\| - \int_0^\infty t\, d|\mu|(t).$$

It follows that

$$\|F(-sA)\| \geq |F(\lambda)| - \frac{1}{\|(sA+\lambda I)^{-1}\|}\int_0^\infty t\, d|\mu|(t)$$

for $s > 0$ and $\operatorname{Re}\lambda \geq 0$.

Suppose that there exists $s \in (0,1)$ such that $\|F(-sA)\| \leq F(\alpha)$, and consider the simple Jordan curve

$$\Gamma := [\alpha, a_1] \cup \Gamma_1 \cup [a_2, a_3] \cup [a_3, \overline{a_3}] \cup [\overline{a_3}, \overline{a_2}] \cup \overline{\Gamma_1} \cup [\overline{a}_1, \alpha],$$

where Γ_1, a_1, a_2, a_3 are defined as in Theorem 2.2, taking $f = F$ (see Figure 1).

We now make various estimates of $\|(sA+\lambda I)^{-1}\|$ for λ on three different parts of Γ.

1) For $\lambda \in [\alpha, a_1] \cup [\overline{a_1}, \alpha]$ we have

$$\begin{aligned} F(\alpha) \geq \|F(-sA)\| &\geq |F(\lambda)| - \frac{1}{\|(sA+\lambda I)^{-1}\|}\int_0^\infty t\, d|\mu|(t) \\ &\geq F(\alpha) + \delta|\lambda-\alpha|^m - \frac{1}{\|(sA+\lambda I)^{-1}\|}\int_0^\infty t\, d|\mu|(t). \end{aligned}$$

Hence we obtain

$$\left\|(sA+\lambda I)^{-1}\right\| \leq \frac{1}{\delta|\lambda-\alpha|^m}\int_0^\infty t\, d|\mu|(t). \tag{3}$$

2) For $\lambda \in \Gamma_1 \cup [a_2, a_3] \cup [\overline{a_3}, \overline{a_2}] \cup \overline{\Gamma_1}$ we have

$$\begin{aligned} F(\alpha) \geq \|F(-sA)\| &\geq |F(\lambda)| - \frac{1}{\|(sA+\lambda I)^{-1}\|} \int_0^\infty t\, d|\mu|(t) \\ &\geq |F(a_0)| - \frac{1}{\|(sA+\lambda I)^{-1}\|} \int_0^\infty t\, d|\mu|(t). \end{aligned}$$

It follows that

$$\left\|(sA+\lambda I)^{-1}\right\| \leq \frac{1}{|F(a_0)| - F(\alpha)} \int_0^\infty t\, d|\mu|(t). \tag{4}$$

3) For $x \in \mathbb{R}$,

$$\begin{aligned} \left\|(A+ixI)^{-1}\right\| &= \left\| -\int_0^\infty T(t) e^{ixt}\, dt \right\| \\ &\leq \int_0^\infty \|T(t)\|\, dt < \infty, \end{aligned}$$

since $(T(t))_{t>0}$ is quasinilpotent and contractive. Therefore

$$\left\|(sA+\lambda I)^{-1}\right\| = \frac{1}{s} \left\| \left(A + \frac{\lambda}{s} I\right)^{-1} \right\| \leq \frac{1}{s} \int_0^\infty \|T(t)\|\, dt \tag{5}$$

for all $\lambda \in [a_3, \overline{a_3}]$.

We can now provide estimates for the quantity $\left\|(\lambda-\alpha)^m \left(A + \frac{\lambda}{s} I\right)^{-1}\right\|$ for λ on Γ. Let $R = \max_{\lambda \in \Gamma} |\lambda - \alpha|$.

By (3)

$$\left\|(\lambda-\alpha)^m \left(A + \frac{\lambda}{s} I\right)^{-1}\right\| \leq \frac{s}{\delta} \int_0^\infty t\, d|\mu|(t)$$

for all $\lambda \in [\alpha, a_1] \cup [\overline{a_1}, \alpha]$.

By (4)

$$\left\|(\lambda-\alpha)^m \left(A + \frac{\lambda}{s} I\right)^{-1}\right\| \leq \frac{sR^m}{|F(a_0)| - F(\alpha)} \int_0^\infty t\, d|\mu|(t)$$

for all $\lambda \in \Gamma_1 \cup [a_2, a_3] \cup [\overline{a_3}, \overline{a_2}] \cup \overline{\Gamma_1}$.

By (5)

$$\left\|(\lambda-\alpha)^m\left(A+\frac{\lambda}{s}I\right)^{-1}\right\| \le R^m\int_0^\infty \|T(t)\|\,dt$$

for all $\lambda \in [a_3, \overline{a_3}]$.

Since $0 < s \le 1$, for all $z \in \Gamma \cup \operatorname{int}\Gamma$ we have

$$\left\|\left(A+\frac{z}{s}I\right)^{-1}\right\| \le \frac{M}{|z-\alpha|^m},$$

by the maximum modulus principle, where

$$M = \max\left(R^m\int_0^\infty \|T(t)\|\,dt, \frac{R^m}{|F(a_0)|-F(\alpha)}\int_0^\infty t\,d|\mu|(t), \frac{1}{\delta}\int_0^\infty t\,d|\mu|(t)\right).$$

Since by hypothesis $F(0) = 0$, there is an $r \in (0, \alpha)$ such that

$$\sup_{|z|\le r}|F(z)| < F(\alpha).$$

Since $\overline{D(0,r)} \cap \Gamma \cap \mathbb{C}_+ = \emptyset$, we have $\overline{D(0,r)} \cap \overline{\mathbb{C}_+} \subset \Gamma \cup \operatorname{int}\Gamma$.

Now if $z \in \overline{D(0,r)}$ with $\operatorname{Re} z > 0$, we have $|z-\alpha| \ge \alpha - r$, and thus we have

$$\left\|\left(A+\frac{z}{s}I\right)^{-1}\right\| \le \frac{M}{|z-\alpha|^m} \le \frac{M}{(\alpha-r)^m}.$$

Also

$$\sup_{\operatorname{Re} z\le 0}\left\|(A+zI)^{-1}\right\| \le \int_0^\infty \|T(t)\|\,dt < \infty.$$

Now, since by Liouville's theorem the function $z \mapsto \left\|(A+zI)^{-1}\right\|$ is unbounded on $\mathbb{C}$, it follows that for all $u > 0$ sufficiently small the inequality

$$\left\|\left(A+\frac{z}{u}I\right)^{-1}\right\| \le \frac{M}{(\alpha-r)^m}$$

fails to hold for some $z \in \overline{D(0,r)} \cap \mathbb{C}_+$, depending on u.

It follows that there is an $\eta > 0$ such that

$$\|F(-uA)\| > F(\alpha) \qquad \text{for all} \quad u \in (0, \eta].$$

□

If $\mu \in M_c(0,\infty)$ is now a complex measure, then we write $\widetilde{F}(z) = \overline{F(\overline{z})}$, which is also an entire function, indeed, the Laplace transform of $\overline{\mu}$.

Corollary 2.6 *Let $\mu \in M_c(0,\infty)$ be a nontrivial complex measure such that $\int_0^\infty d\mu(t) = 0$, and let $(T(t))_{t>0}$ be a nontrivial strongly continuous quasinilpotent semigroup of bounded operators on a Banach space $\mathcal{X}$. Set $F = \mathcal{L}\mu$. Then there is an $\eta > 0$ such that*

$$\|F(-sA)\widetilde{F}(-sA)\| > \max_{x\geq 0} |F(x)|^2 \qquad \textit{for} \quad 0 < s \leq \eta.$$

Proof : The result follows on applying Theorem 2.5 to the real measure $\nu := \mu * \overline{\mu}$, whose Laplace transform satisfies

$$\mathcal{L}\nu(s) = F(s)\widetilde{F}(s).$$

□

We now give similar results for smoother semigroups: let $p > 0$ be an integer, and write $\mathcal{U}^{(p)}$ for the class of semigroups $(T(t))_{t>0}$ such that the mapping $t \mapsto T(t)$ is p times continuous differentiable with respect to the norm topology. Let $\mathcal{E}^{(p)}$ denote the class of distributions of order p with compact support in $(0,\infty)$. For $\varphi \in \mathcal{E}^{(p)}$ its action on a C^p function f may be specified in terms of measures $\mu_0, \ldots . \mu_p$, namely,

$$\langle f, \varphi \rangle = \sum_{j=0}^{p} \int_0^\infty f^{(j)}(t)\, d\mu_j(t).$$

The Laplace transform of φ is given by

$$F(z) := \mathcal{L}\varphi(z) = \sum_{j=0}^{p} \int_0^\infty (-z)^j e^{-zt}\, d\mu_j(t).$$

We write $G_j = \mathcal{L}\mu_j$ and $F_j(z) = (-z)^j G_j(z)$ for each j. Likewise

$$F(-A) = \sum_{j=0}^{p} A^j G_j(-A) = \sum_{j=0}^{p} \int_0^\infty A^j T(t)\, d\mu_j(t). \tag{6}$$

We begin with the counterpart of Lemma 2.4.

Lemma 2.7 *Let $p \geq 1$ and $\varphi \in \mathcal{E}^{(p)}$, and let $(T(t))_{t>0}$ be a quasinilpotent $\mathcal{U}^{(p)}$ semigroup of contractions. Set $F = \mathcal{L}\varphi$. Then we have for* $\operatorname{Re}\lambda \geq 0$,

$$\begin{aligned}\left\|(F(-A) - F(\lambda)I)A^{-p}(A+\lambda I)^{-1}\right\| \;&\leq \\ &\sum_{m=0}^{p} c_m \|A^{m-p}\| \;+\; \sum_{m=0}^{p} d_m \left(\sum_{k=0}^{m-1} |\lambda|^k \|A^{m-1-k-p}\|\right),\end{aligned}$$

where

$$c_m = \int_0^\infty t\, d|\mu_m|(t) \qquad \text{and} \quad d_m = \int_0^\infty d|\mu_m|(t)$$

for $m = 0, 1, \ldots, p$.

Proof : Write $B := (F(-A) - F(\lambda)I)A^{-p}(A+\lambda I)^{-1}$. Then by (6) we have

$$B = \sum_{m=0}^{p} A^{-p}(A^m G_m(-A) - (-\lambda)^m G_m(\lambda)I)(A+\lambda I)^{-1}.$$

This can be rewritten as

$$\begin{aligned}&\sum_{m=0}^{p} A^{m-p}(G_m(-A) - G_m(\lambda))(A+\lambda I)^{-1} \\ &+ \sum_{m=0}^{p} G_m(\lambda) A^{-p}(A^m - (-\lambda)^m I)(A+\lambda I)^{-1}.\end{aligned}$$

Thus

$$\begin{aligned}B = \quad &\sum_{m=0}^{p} A^{m-p}(G_m(-A) - G_m(\lambda))(A+\lambda I)^{-1} \\ &+ \sum_{m=0}^{p} G_m(\lambda) \left[\sum_{k=0}^{m-1} A^{m-1-k}(-\lambda)^k\right] A^{-p}.\end{aligned}$$

Now the first terms can be estimated using Lemma 2.4, and for the second we use the obvious estimate $|G_m(\lambda)| \leq d_m$ for $\operatorname{Re}\lambda \geq 0$.

Theorem 2.8 *Let $p > 1$ and $\varphi \in \mathcal{E}^{(p)}$ be a nontrivial real distribution given by measures $\mu_0, \ldots, \mu_p$ such that $\int_0^\infty d\mu_0(t) = 0$, and let $(T(t))_{t>0}$ be a nontrivial quasinilpotent $\mathcal{U}^{(p)}$ semigroup of bounded operators on a Banach space $\mathcal{X}$. Set $F = \mathcal{L}\varphi$. Then there is an $\eta > 0$ such that*

$$\|F(-sA)\| > \max_{x \geq 0} |F(x)| \qquad \textit{for} \quad 0 < s \leq \eta.$$

Proof : The proof is very similar to the proof of Theorem 2.5, but using Lemma 2.7, so we indicate the changes necessary. It will be convenient to take $0 < s \leq 1$ and to write

$$K = K(s, \lambda) = \sum_{m=0}^{p} c_m \|(sA)^{m-p}\| + \sum_{m=0}^{p} d_m \left(\sum_{k=0}^{m-1} |\lambda|^k \|(sA)^{m-1-k-p}\| \right),$$

noting the dependence on s and λ. With the notation of the proof of Theorem 2.5 we have three key estimates:

1) For $\lambda \in [\alpha, a_1] \cup [\overline{a_1}, \alpha]$ we have

$$\begin{aligned} F(\alpha) \geq \|F(-sA)\| &\geq |F(\lambda)| - \frac{K}{\|(sA + \lambda I)^{-1}(sA)^{-p}\|} \\ &\geq F(\alpha) + \delta|\lambda - \alpha|^m - \frac{K}{\|(sA + \lambda I)^{-1}(sA)^{-p}\|}. \end{aligned}$$

Hence we obtain

$$\left\|(sA + \lambda I)^{-1}(sA)^{-p}\right\| \leq \frac{K}{\delta|\lambda - \alpha|^m}. \tag{7}$$

2) For $\lambda \in \Gamma_1 \cup [a_2, a_3] \cup [\overline{a_3}, \overline{a_2}] \cup \overline{\Gamma_1}$ we have

$$\begin{aligned} F(\alpha) \geq \|F(-sA)\| &\geq |F(\lambda)| - \frac{K}{\|(sA + \lambda I)^{-1}(sA)^{-p}\|} \\ &\geq |F(a_0)| - \frac{K}{\|(sA + \lambda I)^{-1}(sA)^{-p}\|}. \end{aligned}$$

It follows that

$$\left\|(sA + \lambda I)^{-1}(sA)^{-p}\right\| \leq \frac{K}{|F(a_0)| - F(\alpha)}. \tag{8}$$

3) For $x \in \mathbb{R}$,

$$\begin{aligned}\left\|(A+ixI)^{-1}A^{-p}\right\| &= \left\|-\int_0^\infty T(t)e^{ixt}\,dtA^{-p}\right\| \\ &\leq \|A^{-1}\|^p \int_0^\infty \|T(t)\|\,dt < \infty,\end{aligned}$$

since $(T(t))_{t>0}$ is quasinilpotent and contractive. Therefore

$$\left\|(sA+\lambda I)^{-1}(sA)^{-p}\right\| \leq \frac{1}{s^{p+1}}\|A^{-1}\|^p \int_0^\infty \|T(t)\|\,dt \tag{9}$$

for all $\lambda \in [a_3, \overline{a_3}]$.

We estimate $\left\|(\lambda-\alpha)^m \left(A+\frac{\lambda}{s}I\right)^{-1} A^{-p}\right\|$ for λ on Γ.

Let $R = \max_{\lambda\in\Gamma}|\lambda-\alpha|$. By (7)

$$\left\|(\lambda-\alpha)^m \left(A+\frac{\lambda}{s}I\right)^{-1} A^{-p}\right\| \leq \frac{Ks^{p+1}}{\delta}$$

for all $\lambda \in [\alpha, a_1] \cup [\overline{a_1}, \alpha]$.

By (8)

$$\left\|(\lambda-\alpha)^m \left(A+\frac{\lambda}{s}I\right)^{-1} A^{-p}\right\| \leq \frac{Ks^{p+1}R^m}{|F(a_0)| - F(\alpha)}$$

for all $\lambda \in \Gamma_1 \cup [a_2, a_3] \cup [\overline{a_3}, \overline{a_2}] \cup \overline{\Gamma_1}$.

By (9)

$$\left\|(\lambda-\alpha)^m \left(A+\frac{\lambda}{s}I\right)^{-1} A^{-p}\right\| \leq R^m\|A^{-1}\|^p \int_0^\infty \|T(t)\|\,dt$$

for all $\lambda \in [a_3, \overline{a_3}]$.

Since $0 < s \leq 1$, for all $z \in \Gamma \cup \operatorname{int}\Gamma$ with $|z| \leq r$ we have

$$\left\|\left(A+\frac{z}{s}I\right)^{-1} A^{-p}\right\| \leq \frac{M}{|z-\alpha|^m} \leq \frac{M}{(\alpha-r)^m}, \tag{10}$$

by the maximum modulus principle, where

$$M = \sup_{\substack{0<s\leq 1 \\ z\in\Gamma\cup\operatorname{int}\Gamma}} \max\left(\frac{K(s,z)s^{p+1}}{\delta}, \frac{K(s,z)s^{p+1}R^m}{|F(a_0)|-F(\alpha)}, R^m\|A^{-1}\|^p\int_0^\infty \|T(t)\|\,dt\right),$$

which is finite.

With this new choice of M, the proof is now concluded as for the proof of Theorem 2.5, using the observation that $\|(A+zI)^{-1}A^{-p}\|$ is unbounded on $\mathbb{C}$, and obtaining a contradiction from (10).

□

3 The non-quasinilpotent case

Let $(T(t))_{t>0}$ be a semigroup of non-quasinilpotent operators, and let $\mathcal{A}_T$ denote the closed (commutative) algebra generated by the semigroup. We write $\widehat{\mathcal{A}}_T$ for the maximal ideal space of $\mathcal{A}_T$. Recall that this is compact if and only if $\mathcal{A}_T/\operatorname{Rad}(\mathcal{A}_T)$ is unital; otherwise it is locally compact, and the function $\widehat{a}:\chi\mapsto\chi(a)$ is continuous on $\widehat{\mathcal{A}}_T$ for every $a\in\mathcal{A}_T$.

Recall that $\mathcal{A}_T$ is said to have an *exhaustive sequence of idempotents* $(P_n)_{n\geq 1}$ if $P_n^2 = P_nP_{n+1} = P_n$ for all n and for every $\chi\in\widehat{\mathcal{A}}_T$ there is a p such that $\chi(P_n)=1$ for all $n\geq p$.

The following result is part of the folklore of the subject, and it is partly contained in [4, Lem. 3.1] and [1, Lem. 3.1]. It enables us to regard A itself as an element of $C(\widehat{\mathcal{A}}_T)$ by defining an appropriate value $\chi(A) = -a_\chi$ for each $\chi\in\widehat{\mathcal{A}}_T$.

Lemma 3.1 *For a strongly continuous and eventually norm-continuous semigroup $(T(t))_{t>0}$ and a nontrivial character $\chi\in\widehat{\mathcal{A}}_T$ there is a unique $a_\chi\in\mathbb{C}$ such that $\chi(T(t)) = e^{-ta_\chi}$ for all $t>0$. Moreover, the mapping $\chi\mapsto a_\chi$ is continuous, and $\chi(F(-uA)) = F(ua_\chi)$ in the case that $F=\mathcal{L}\mu$, as in (2).*

Proof : The existence of a_χ is given in [4], and its uniqueness is clear since the values of e^{-ta_χ} for $t>0$ determine a_χ uniquely.

For the continuity, note that

$$\chi(T(1)) = e^{-a_\chi}$$

and

$$\chi\left(e^{\lambda}\int_1^{\infty} T(t)e^{-\lambda t}\,dt\right) = \frac{1}{a_\chi+\lambda}e^{-a_\chi}$$

if λ is taken sufficiently large that the integral converges. Thus if we have a net $\chi_\alpha \to \chi$ then $e^{-a_{\chi\alpha}} \to e^{-a\chi}$ and $\frac{1}{a_{\chi\alpha}+\lambda}e^{-a_{\chi\alpha}} \to \frac{1}{a_\chi+\lambda}e^{-a_\chi}$, which easily implies that $a_{\chi_\alpha} \to a_\chi$.

The final observation follows from an easy argument using Bochner integrals.

□

The following result will also be required.

Lemma 3.2 *Let $(T(t))_{t>0}$ be a non-quasinilpotent and eventually norm-continuous semigroup in a Banach algebra, with infinitesimal generator A; let $\mathcal{A}_T$ be the subalgebra generated by the semigroup and $\Lambda = \{a_\chi : \chi \in \widehat{\mathcal{A}}_T\}$, as in Lemma 3.1. Then the following conditions are equivalent:*

(i) $\mathcal{A}_T$ has an exhaustive sequence of idempotents.

(ii) For each integer $m \geq 1$ the set $\Lambda_m := \{\lambda \in \Lambda : \operatorname{Re}\lambda \leq m\}$ is contained in a compact relatively open subset of Λ.

Proof : $(i) \Rightarrow (ii)$: Let $\theta : \chi \to a_\chi$ be the homeomorphism given by Lemma 3.1. Now $K := \theta^{-1}(\Lambda_m)$ is a closed subset of the compact space $\widehat{\mathcal{A}}_T \cup \{0\}$, since $\operatorname{Re} a_\chi \leq m$ if and only if $|\chi(T(1))| \geq e^{-m}$. Hence K is compact. If (i) holds, then for each $\chi \in \widehat{\mathcal{A}}_T$, then there is an $n_\chi > 0$ be such that $\chi(P_n) = 1$ for $n \geq n_\chi$. So $\Big(\{\chi \in \widehat{\mathcal{A}}_T : \chi(P_n) = 1\}\Big)_n$ is an open cover of K. By compactness, there is an N such that $\chi(P_N) = 1$ for all $\chi \in K$, so $K \subset \{\chi \in \widehat{\mathcal{A}}_T : \chi(P_N) = 1\}$, and $\theta(K)$ is compact, open in Λ, and contains Λ_m.

$(ii) \Rightarrow (i)$: Conversely, if (ii) holds, then for each $m \geq 1$, the set Λ_m is contained in a compact relatively open set $\Omega_m \subset \Lambda$. So $\theta^{-1}(\Omega_m) \subset \mathcal{A}_T$ is compact and open; hence by Shilov's idempotent theorem [2, Thm. 2.4.33] there is an idempotent P_m in $\mathcal{A}_T$ such that $\chi(P_m) = 1$ for $\chi \in \theta^{-1}(\Omega_m)$ and 0 otherwise. Now $(P_m)_{m\geq 1}$ is an exhaustive sequence of idempotents in $\mathcal{A}_T$. □

Theorem 3.3 *Let $(T(t))_{t>0}$ be a nontrivial strongly continuous and eventually norm-continuous non-quasinilpotent semigroup on a Banach space $\mathcal{X}$, with generator A. Let $F = \mathcal{L}\mu$, where $\mu \in M_c(0,\infty)$ is a real measure such that $\int_0^\infty d\mu = 0$. If there exists $(u_k)_k \subset (0,\infty)$ with $u_k \to 0$ such that*

$$\rho(F(-u_k A)) < \sup_{x>0} |F(x)|, \tag{11}$$

then the algebra $\mathcal{A}_T$ possesses an exhaustive sequence of idempotents $(P_n)_{n\geq 1}$ such that each semigroup $(P_n T(t))_{t>0}$ has a bounded generator.

If, further, $\|F(-u_k A)\| < \sup_{x>0} |F(x)|$, then $\bigcup_{n\geq 1} P_n \mathcal{A}_T$ is dense in $\mathcal{A}_T$.

Proof : For $m \geq 1$ let $\Lambda_m = \{a_\chi : \operatorname{Re} a_\chi \leq m\}$, where $\chi(T(t)) = e^{-a_\chi t}$ as in Lemma 3.1.

Let $K := \theta^{-1}(\Lambda_m)$, which is a compact set, as seen in the proof of Lemma 3.2. Hence, $\Lambda_m = \theta(K)$ is compact, since $\theta : \chi \mapsto a_\chi$ is continuous by Lemma 3.1.

Therefore, there is an $R_m > 0$ such that $\Lambda_m \subset D(0, R_m)$. Note that, by the definition of Λ_m, we have

$$\Lambda \cap \mathbb{C}_- = \Lambda \cap \mathbb{C}_- \subset D(0, R_m).$$

By hypothesis there exists a $u_k > 0$ such that

$$|u_k| < \frac{r}{R_m},$$

where $r > 0$ is given by Lemma 2.3 and

$$\rho(F(-u_k A)) < \sup_{x>0} |F(x u_k)|.$$

It follows that $\Lambda_m \subset D(0, R_m) \subset D(0, r/|u_k|)$ and

$$|F(u_k a_\chi)| < \sup_{x>0} |F(u_k x)|$$

for all $a_\chi \in \Lambda_m$. Let α_k be such that $\sup_{x>0} |F(xu_k)| = |F(\alpha_k u_k)|$.

By Theorem 2.2 there exists a curve $\Gamma_{k,0}$ in $\{z : \operatorname{Re} z \geq 0, \operatorname{Im} z > 0\}$ joining $\alpha_k \in \mathbb{R}_+$ to $v_k \in i\mathbb{R}_+$ with $|v_k| > R_m$ on which $|F(u_k z)| \geq |F(\alpha_k u_k)|$.

Let $\Gamma_k = \Gamma_{k,0} \cup \{z \in \mathbb{C} : \operatorname{Re} z < 0,\ |z| = |v_k|\} \cup \overline{\Gamma_{k,0}}$ (see Figure 2).

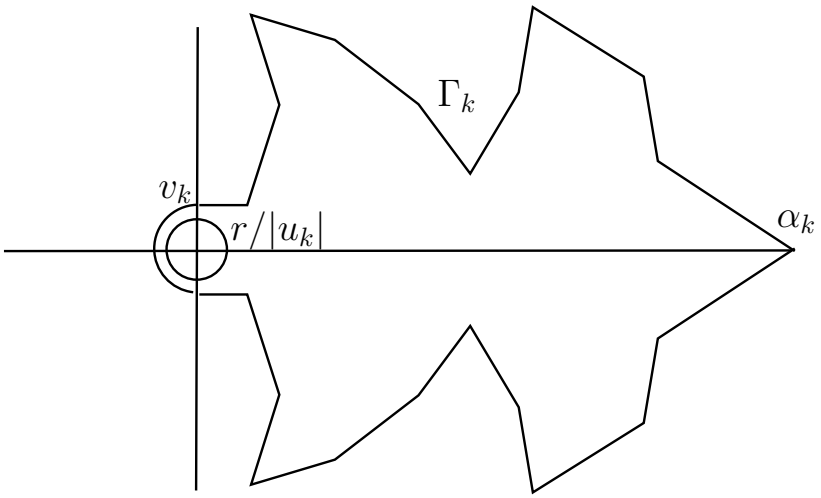

Figure 2: Diagram for the proof of Theorem 3.3

Then $\Lambda \cap \Gamma_k = \emptyset$ since $|F(u_k a_\chi)| = |\chi(F(-u_k A))| < |F(u_k \alpha_k)|$ for $a_\chi \in \Lambda$ and $|F(u_k z)| > |F(u_k \alpha_k)|$ for $z \in \Gamma_{k,0} \cup \overline{\Gamma_{k,0}}$, so $\Lambda \cap (\Gamma_{k,0} \cup \overline{\Gamma_{k,0}}) = \emptyset$. Also $\Lambda \cap \mathbb{C}_- = \Lambda_m \cap \mathbb{C}_-$ so $\Lambda \cap \{z \in \mathbb{C}_- : |z| = |v_k|\} = \emptyset$.

Now $\Lambda_m = \Lambda \cap \operatorname{int} \Gamma_k$, which is compact (since $\Lambda \cap \Gamma_k = \emptyset$) and relatively open in Λ, so we may now apply Lemma 3.2 to deduce that $\mathcal{A}_T$ has an exhaustive sequence of idempotents.

If P is an idempotent of $\mathcal{A}_T$, then $\bigcup_{t>0} PT(t)\mathcal{A}_T$ is dense in the unital Banach algebra $P\mathcal{A}_T$. Hence $P\mathcal{A}_T = \bigcup_{t>0} PT(t)\mathcal{A}_T$; also $PT(t)$ is invertible

in $P\mathcal{A}_T$ for some, and hence for all, $t > 0$, and then $\lim_{t\to 0+} \|P - PT(t)\| = 0$, since the semigroup is eventually continuous.

For the last observation, it follows from Theorem 2.5 that $\pi(T(t)) = 0$ for every $t > 0$, where $\pi : \mathcal{A}_T \to \mathcal{A}_T / \bigcup_{n\geq 1} P_n \mathcal{A}_T$ denotes the canonical surjection.

□

Remark 3.4 *A similar result holds for complex measures $\mu \in M_c(0,\infty)$; namely, we replace*

$$\rho(F(-u_k A)) < \sup_{x>0} |F(x)|,$$

by the symmetrised version

$$\rho(F(-u_k A)\widetilde{F}(-u_k A)) < \sup_{x>0} |F(x)|^2,$$

as in Corollary 2.6.

Remark 3.5 *The following example is given in [1], and shows that Theorem 3.3 is sharp. Indeed, consider $C_0[0,1]$, the Banach algebra of all continuous complex-valued functions on $[0,1]$ that vanish at 0, equipped with the supremum norm, and the semigroup $(S(t))_{t>0}$ defined by $x \mapsto x^t$ for $x \in [0,1]$.*

We see that

$$F(-uA) = \int_0^\infty S(u\xi)\, d\mu(\xi) = \int_0^\infty x^{u\xi}\, d\mu(\xi),$$

and for $s > 0$ we have

$$F(s) = \int_0^\infty e^{-s\xi}\, d\mu(\xi).$$

Thus we obtain equality in (11), choosing $x^u = e^{-s_0}$, where s_0 is defined by $|F(s_0)| = \sup_{s>0} |F(s)|$. Note that the norm and spectral radius are equal in $C_0[0,1]$ and that the algebra does not possess any non-trivial idempotents.

Acknowledgements

This work was partially supported by the ANR project ANR-09-BLAN-0058-01, the London Mathematical Society (Scheme 2), and the Institut Camille Jordan. The authors are grateful to the referee for a careful reading of the paper and some valuable comments.

References

[1] Z. Bendaoud, I. Chalendar, J. Esterle, and J. R. Partington. Distances between elements of a semigroup and estimates for derivatives. *Acta Math. Sin. (Engl. Ser.)*, 26(12):2239–2254, 2010.

[2] H. G. Dales. *Banach algebras and automatic continuity*, volume 24 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, New York, 2000. Oxford Science Publications.

[3] J. Esterle. Distance near the origin between elements of a strongly continuous semigroup. *Ark. Mat.*, 43(2):365–382, 2005.

[4] J. Esterle and A. Mokhtari. Distance entre éléments d'un semi-groupe dans une algèbre de Banach. *J. Funct. Anal.*, 195(1):167–189, 2002.

[5] W. Feller. On the generation of unbounded semi-groups of bounded linear operators. *Ann. of Math. (2)*, 58:166–174, 1953.

[6] N. Kalton, S. Montgomery-Smith, K. Oleszkiewicz, and Y. Tomilov. Power-bounded operators and related norm estimates. *J. London Math. Soc. (2)*, 70(2):463–478, 2004.

[7] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997.