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On bounded continuous solutions of the archetypal equation with rescaling

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The 'archetypal' equation with rescaling is given by $y(x) = \iint_{\mathbb{R}^2} y(a(x-b))\,\mu(\mathrm{d}a,\mathrm{d}b)$ $(x \in \mathbb{R})$, where μ is a probability measure; equivalently, $y(x) = \mathbb{E}\{y(\alpha(x-\beta))\}$, with random α, β and $\mathbb{E}$ denoting expectation. Examples include: (i) functional equation $y(x) = \sum_i p_i y(a_i(x-b_i))$; (ii) functional-differential ('pantograph') equation $y'(x) + y(x) = \sum_i p_i y(a_i(x - c_i))$ $(p_i > 0,\ \sum_i p_i = 1)$. Interpreting solutions $y(x)$ as harmonic functions of the associated Markov chain (X_n), we obtain Liouville-type results asserting that any bounded continuous solution is constant. In particular, in the 'critical' case $\mathbb{E}\{\ln|\alpha|\} = 0$ such a theorem holds subject to uniform continuity of $y(x)$; the latter is guaranteed under mild regularity assumptions on β, satisfied e.g. for the pantograph equation (ii). For equation (i) with $a_i = q^{m_i}$ $(m_i \in \mathbb{Z},\ \sum_i p_i m_i = 0)$, the result can be proved without the uniform continuity assumption. The proofs utilize the iterated equation $y(x) = \mathbb{E}\{y(X_\tau)\,|\,X_0 = x\}$ (with a suitable stopping time τ) due to Doob's optional stopping theorem applied to the martingale $y(X_n)$.

1. Introduction

(a) The archetypal equation

This paper concerns the equation with rescaling (referred to as 'archetypal') of the form

$$y(x) = \iint_{\mathbb{R}^2} y(a(x-b))\,\mu(\mathrm{d}a,\mathrm{d}b), \qquad x \in \mathbb{R}, \qquad (1.1)$$

where $\mu(\mathrm{d}a,\mathrm{d}b)$ is a probability measure on $\mathbb{R}^2$. The integral in (1.1) has the meaning of expectation with

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respect to a random pair (α, β) with distribution $\mathbb{P}\{(\alpha,\beta) \in \mathrm{d}a \times \mathrm{d}b\} = \mu(\mathrm{d}a, \mathrm{d}b)$; thus, equation (1.1) can be written in compact form as

$$y(x) = \mathbb{E}\{y(\alpha(x-\beta))\}, \qquad x \in \mathbb{R}. \tag{1.2}$$

As has been observed by Derfel [1] (and will be illustrated below in §1(b)), this equation is a rich source of various functional and functional-differential equations with rescaling, specified by a suitable choice of the probability measure μ (i.e. the distribution of (α,β)). It is for that reason that we propose to call (1.2) (as well as its integral counterpart (1.1)) the *archetypal equation* (*AE*). The study of this equation allows one to enhance and unify earlier results for particular subclasses of equations with rescaling, while making the analysis more transparent and efficient (cf. [2]).

Noting that any function $y(x) \equiv \mathrm{const}$ satisfies the AE (1.2), it is natural to investigate whether there are any *non-trivial* (i.e. non-constant) bounded continuous (b.c.) solutions. Such a question naturally arises in the context of functional and functional-differential equations with rescaling, where the possible existence of bounded solutions (e.g. periodic, almost periodic, compactly supported, etc.) is of major interest in physical and other applications (see e.g. [3–6]). Solutions under study may also be bounded by nature, e.g. representing a ruin probability as a function of initial capital [7,8]. On the other hand, confining oneself to bounded solutions may be considered as a first step towards a full description of the asymptotic behaviour of solutions.

Thus, the goal of the present paper is to give conditions on the distribution μ of the random coefficients (α,β), under which any b.c.-solution of equation (1.2) is constant on $\mathbb{R}$. For shorthand, we refer to statements of this kind as *Liouville-type* theorems by analogy with similar results in complex analysis and harmonic function theory, bearing in mind that $y(x)$ in (1.2) is a weighted average of other values, thus resembling the usual harmonic function. More details highlighting the pertinence of 'harmonicity' in the context of equation (1.2) are provided in §1(d).

Remark 1.1. Continuity of $y(x)$ (or some other regularity assumption) is needed to avoid pathological solutions, as is well known in the theory of functional equations (cf. [9, Ch. 2]). For example, all b.c.-solutions of the equation $y(x) = py\big(\tfrac{1}{2}(x+1)\big) + (1-p)y\big(\tfrac{1}{2}(x-1)\big)$ $(0 \le p \le 1)$ are constant by Theorem 1.1(a) stated below, but if the continuity requirement is dropped then one can easily construct other bounded solutions, e.g. the Dirichlet function $y(x) = \mathbb{1}_{\mathbb{Q}}(x)$ (i.e. $y(x) = 1$ if x is rational and $y(x) = 0$ otherwise), which is everywhere discontinuous.

(b) Some subclasses of the archetypal equation; historical remarks

Before outlining our results, we illustrate the remarkable capacity of equation (1.2) justifying the name 'archetypal'. General surveys of functional and functional-differential equations with rescaling are found in Derfel [10] and Baron & Jarczyk [11], both with extensive bibliographies.

(i) Functional equations and self-similar measures

To start with, in the simplest case $\alpha \equiv 1$ equations (1.1), (1.2) are reduced to

$$y(x) = \int_{\mathbb{R}} y(x-t)\,\mu_\beta(\mathrm{d}t) \quad \Longleftrightarrow \quad y(x) = \mathbb{E}\{y(x-\beta)\}, \tag{1.3}$$

where $\mu_\beta(\mathrm{d}t) := \mathbb{P}(\beta \in \mathrm{d}t)$. This equation (sometimes called the *integrated Cauchy functional equation* [12]) plays a central role in potential theory and harmonic analysis on groups [13,14], and is also prominent in probability theory in relation to renewal theorems [15, §XI.9], Markov chains [16, Ch. 5], queues [17, §III.6], characterization of probability distributions [12, Ch. 2], etc. A Liouville-type result in this case is rendered by the celebrated *Choquet–Deny theorem* [18] (see also [12] and references therein).

Note that equation (1.3) can be written in the convolution[1] form $y = y \star \mu_\beta$. More generally, if α has a discrete distribution (with atoms a_i and masses p_i) then, denoting by μ_β^i the conditional

[1]The convolution between function $y(x)$ and measure σ in $\mathbb{R}$ is defined as $y \star \sigma(x) := \int_{\mathbb{R}} y(x-t)\,\sigma(\mathrm{d}t)$.

distribution of β given $\alpha = a_i$, the AE (1.2) is conveniently expressed in convolutions,

$$y(x) = \sum\nolimits_i p_i y(a_i x) \star \mu_\beta^i.$$

For a purely discrete measure μ, with atoms (a_i, b_i) and masses $p_i = \mathbb{P}(\alpha = a_i, \beta = b_i)$, the AE (1.2) specializes to

$$y(x) = \sum\nolimits_i p_i y(a_i(x - b_i)). \tag{1.4}$$

If all $a_i > 1$ then (1.4) is an example of *Hutchinson's equation* [19] for the distribution function of a self-similar probability measure which is invariant under a family of contractions (here, affine transformations $x \mapsto b_i + x/a_i$). An important subclass of (1.4), with $a_i \equiv a > 1$, is exemplified by

$$y(x) = \tfrac{1}{2} y(a(x+1)) + \tfrac{1}{2} y(a(x-1)).$$

This equation describes the (self-similar) distribution function of the random series $\sum_{n=0}^{\infty} \pm a^{-n}$, where the signs are chosen independently with probability $\frac{1}{2}$. Characterization of this distribution for different $a > 1$ is the topical *Bernoulli convolutions problem* [20].

Returning to equation (1.4) with $a_i \equiv a > 1$, the density $z(x) := y'(x)$ (if it exists) satisfies

$$z(x) = a \sum\nolimits_i p_i z(a(x - b_i)), \tag{1.5}$$

often called the *two-scale difference equation* or *refinement equation* [21]. Construction of compactly supported continuous solutions of (1.5) plays a crucial role in wavelet theory [22,23] and also in subdivision schemes and curve design [4,24], which is a rapidly growing branch of approximation theory. A special version of (1.5) known as *Schilling's equation*

$$z(x) = \alpha \big[\tfrac{1}{4} z(\alpha x + 1) + \tfrac{1}{2} z(\alpha x) + \tfrac{1}{4} z(\alpha x - 1)\big]$$

arises in solid state physics in relation to spatially chaotic structures in amorphous materials [5, p. 230], where the existence of compactly supported continuous solutions is again of major interest; see [25] for a full characterization of this problem in terms of arithmetical properties of α.

(ii) Functional-differential equations

Let us now turn to the situation where the distribution of β conditioned on α is absolutely continuous (i.e. has a density). It appears that for certain simple densities the AE (1.2) produces some well-known functional-differential equations. An important example is the celebrated *pantograph equation*, introduced by Ockendon & Tayler [26] as a mathematical model of the overhead current collection system on an electric locomotive.[2] In its classical (one-dimensional) form the pantograph equation reads

$$y'(x) = c_0 y(x) + c_1 y(\alpha x). \tag{1.6}$$

This equation and its ramifications have emerged in a striking range of applications, including number theory [7], astrophysics [28], queues & risk theory [29], stochastic games [8], quantum theory [6], and population dynamics [30]. The common feature of all such examples is some self-similarity of the system under study. Thorough asymptotic analysis of equation (1.6) was given by Kato & McLeod [31]. A more general first-order pantograph equation (with matrix coefficients, and also allowing for a term with a rescaled derivative) was studied by Iserles [27], where a fine geometric structure of almost-periodic solutions was also described. Further developments include analysis in the complex domain [32], higher-order equations [27,33], and stochastic versions [34]. Among recent important analytic results is a proof by da Costa *et al.* [35] of the unimodality of solutions which plays a significant role in medical imaging of tumours [36].

A *balanced* version of the pantograph equation is given by (see [1,2])

$$y'(x) + y(x) = \sum\nolimits_i p_i y(a_i(x - c_i)), \qquad p_i > 0, \quad \sum\nolimits_i p_i = 1. \tag{1.7}$$

As explained in §3(c), equation (1.7) is essentially equivalent to the AE (1.2) whereby α is discrete, with $\mathbb{P}(\alpha = a_i) = p_i$, and β conditioned on $\alpha = a_i$ has the unit exponential distribution on (c_i, ∞),

[2] The term 'pantograph equation' was coined by Iserles [27].

with the density function $\mathrm{e}^{c_i - t}\mathbb{1}_{(c_i,\infty)}(t)$. The discreteness of α is not significant here, and a similar connection with the AE holds for more general integro-differential equations (cf. [37])

$$y'(x) + y(x) = \mathbb{E}\{y(\alpha(x-\gamma)\} \equiv \iint_{\mathbb{R}^2} y(a(x-c))\,\mu_{\alpha,\gamma}(\mathrm{d}a, \mathrm{d}c), \tag{1.8}$$

where γ is a random variable and $\mu_{\alpha,\gamma}(\mathrm{d}a,\mathrm{d}c) = \mathbb{P}(\alpha \in \mathrm{d}a, \gamma \in \mathrm{d}c)$ is the distribution of (α,γ). Higher-order pantograph equations can also be deduced from the AE, e.g.

$$-y''(x) + y(x) = \sum\nolimits_i p_i y(a_i(x-c_i)), \qquad p_i > 0, \quad \sum\nolimits_i p_i = 1,$$

and more generally (cf. [2])

$$C_2 y''(x) + C_1 y'(x) + y(x) = \mathbb{E}\{y(\alpha(x-\gamma)\} \qquad (C_1, C_2 \in \mathbb{R},\ \ C_1^2 - 4C_2 \geq 0).$$

For an example of a different kind, take $\alpha \equiv 2$ and assume that β has the unform distribution on $[-\frac{1}{2}, \frac{1}{2}]$, then equation (1.2) is reduced to

$$y(x) = \int_{x-1/2}^{x+1/2} y(2u)\,\mathrm{d}u. \tag{1.9}$$

Differentiating (1.9), for $z(x) := y'(x)$ we obtain *Rvachev's equation* [3]

$$z'(x) = 2\big[z(2x+1) - z(2x-1)\big]. \tag{1.10}$$

A compactly supported solution of (1.10) (called the '*up*-function') and its generalizations (unified under the name *atomic functions*) have extensive applications in approximation theory (see [3,24] and references therein); all such functions can be obtained as solutions of suitable versions of the AE (1.2) (see [1]).

(c) Main results

Let us summarize our results. First, certain degenerate cases warrant a separate analysis but need to be excluded in general theory, namely: (i) $\alpha = 0$ with positive probability; (ii) $|\alpha| \equiv 1$; and (iii) $\alpha(c-\beta) \equiv c$ for some $c \in \mathbb{R}$ (*resonance*). Note that (ii) includes the case $\alpha \equiv 1$ settled in the Choquet–Deny theorem mentioned in §1(b)-i; in §2(b) we generalize this result (Theorem 2.3). As for cases (i) and (iii), a Liouville theorem holds here unconditionally, which is easy to prove for (i), analytically and probabilistically alike (see Theorem 2.1). In the resonance case (iii), the proof is more involved relying heavily on the Choquet–Deny theorem (see §2(c)), but the result itself is quite lucid and appealing.

In the non-degenerate situation, existence of non-trivial b.c.-solutions is essentially governed by the sign of $K := \iint_{\mathbb{R}^2} \ln|a|\,\mu(\mathrm{d}a\times\mathrm{d}b) = \mathbb{E}\{\ln|\alpha|\}$. More precisely, one can prove (see [38]) the following dichotomy between the *subcritical* ($K<0$) and *supercritical* ($K>0$) regimes.

Theorem 1.1. *Suppose that $K = \mathbb{E}\{\ln|\alpha|\}$ is finite and $\mathbb{E}\{\ln\max(|\beta|,1)\} < \infty$.*

(a) *If $K<0$ then any b.c.-solution of the AE (1.2) is constant.*

(b) *If $K>0$ and $\alpha>0$, then there is a b.c.-solution of (1.2) given by the distribution function $F_\Upsilon(x) := \mathbb{P}(\Upsilon \leq x)$, where $\Upsilon := \sum_{n=1}^{\infty} \beta_n \prod_{j=1}^{n-1} \alpha_j^{-1}$ and $\{(\alpha_n,\beta_n)\}$ is a sequence of independent identically distributed (i.i.d.) random pairs with the same distribution μ as (α,β).*

Remark 1.2. Almost sure (a.s.) convergence of the random series Υ (for $\alpha \neq 0$) and continuity of $F_\Upsilon(x)$ on $\mathbb{R}$ were proved by Grintsevichyus [39].

Remark 1.3. The result of Theorem 1.1 was obtained by Derfel [1] under a stronger moment condition $\mathbb{E}\{|\beta|\} < \infty$ and only for $\alpha > 0$ (which is essential in (b) but not in (a)); however, his arguments hold in the general case with minor changes.

Remark 1.4. In contrast with the subcritical case $K<0$, which is insensitive to the sign of α (see Theorem 1.1(a)), the supercritical case $K>0$ is more delicate: if $\mathbb{P}(\alpha<0)>0$ then $y = F_\Upsilon(x)$ is no longer a solution of the AE (1.2); e.g. if $\alpha < 0$ (a.s.) then this function satisfies the equation $y(x) = 1 - \mathbb{E}\{y(\alpha(x-\beta))\}$ (cf. [39, Eq. (5)]). Moreover, one can prove [38] that any bounded solution of (1.2) with limits at $\pm\infty$ is constant; thus, any non-trivial solution must be oscillating, which is drastically different from the case $\alpha > 0$ (a.s.).

The *critical case* $K = 0$ is much more challenging, and it has remained open since [1]. More recently, for a pantograph equation (1.8) without shift (i.e. $\gamma \equiv 0$) and some second-order extensions, a Liouville theorem in the case $K = 0$ was established by Bogachev *et al.* [2]. In the present paper, we prove the following general result (cf. Theorem 3.1 below).

Theorem 1.2. *Assume that* $\mathbb{P}(|\alpha| \neq 1) > 0$ *and* $K = \mathbb{E}\{\ln|\alpha|\} = 0$. *If* $y(x)$ *is a bounded solution of the AE* (1.2) *which is uniformly continuous on* $\mathbb{R}$, *then it is constant.*

Although an unwanted restriction, the uniform continuity assumption can be shown to be satisfied provided there exists the probability density of β conditioned on α (Theorem 3.3). An alternative criterion tailored to the model $\beta = \gamma + \xi$ with ξ independent of (α, γ) (Theorem 3.4) is applicable to a large class of examples including the pantograph equation (1.8) and its generalizations (§3(c)). As a consequence, we obtain a Liouville theorem for the general (balanced) pantograph equation in the critical case (cf. Theorem 3.8), significantly extending the result of Bogachev *et al.* [2]. In particular, for the first-order pantograph equation (1.8) we have

Theorem 1.3. *If* $\mathbb{P}(|\alpha| \neq 1) > 0$ *and* $K = \mathbb{E}\{\ln|\alpha|\} = 0$, *then any bounded solution of* (1.8) *is constant.*

Remark 1.5. As explained in §3(d) below, Theorem 1.3 extends to the case $|\alpha| = 1$ (a.s.) by virtue of the generalized Choquet–Deny theorem proved in §2(b).

On the other hand, for a subclass of functional equations (1.4) with multiplicatively commensurable coefficients $\{a_i\}$ (i.e. $a_i = q^{m_i}$, $q > 1$, $m_i \in \mathbb{Z}$), where the random shift β is discrete and therefore the criteria of Theorems 3.3, 3.4 do not apply, the Liouville theorem in the critical case $K = 0$ (i.e. $\sum_i p_i m_i = 0$) can be proved by a different method that circumvents the hypothesis of uniform continuity (Theorem 3.9).

(d) The method: associated Markov chain and iterations

In this subsection, we describe the probabilistic approach to the AE based on Markov chains and martingales, and introduce some basic notation and definitions.

Consider a Markov chain (X_n) on $\mathbb{R}$ defined recursively by

$$X_n = \alpha_n (X_{n-1} - \beta_n) \quad (n \in \mathbb{N}), \qquad X_0 = x, \tag{1.11}$$

where $\{(\alpha_n, \beta_n)\}_{n\in\mathbb{N}}$ is a sequence of i.i.d. random pairs with the same distribution as (α, β) (see (1.2)). Note that the AE (1.2) can then be expressed as

$$y(x) = \mathbb{E}_x\{y(X_1)\}, \qquad x \in \mathbb{R}, \tag{1.12}$$

where index x in the expectation refers to the initial condition in (1.11). That is to say, any solution of the AE (1.2) is a *harmonic function* of the Markov chain (X_n) (cf. [16, p. 40]).

Remark 1.6. Bounded harmonic functions play a paramount role in the general theory of Markov chains (see more details and some references in appendix B.

Remark 1.7. Stochastic recursion (1.11) is well known in the literature as the *random difference equation* (see e.g. [40–43] and further references therein).

Note that equation (1.12) propagates along the Markov chain (X_n), i.e. for any $n \in \mathbb{N}$

$$y(x) = \mathbb{E}_x\{y(X_n)\}, \qquad x \in \mathbb{R}. \tag{1.13}$$

Equivalently, an integral form of equation (1.13) can be obtained by iterating forward the AE (1.1). The recursion (1.11) can also be iterated to give explicitly

$$X_n = A_n x - D_n \qquad (n \in \mathbb{N}), \tag{1.14}$$

where

$$A_n := \prod_{k=1}^{n} \alpha_k, \qquad D_n := \sum_{k=1}^{n} \beta_k \prod_{j=k}^{n} \alpha_j. \tag{1.15}$$

Recall that $K = \mathbb{E}\{\ln|\alpha|\}$. In the subcritical case ($K < 0$), for the proof of Theorem 1.1(a) it suffices to consider iterations (1.13) as $n \to \infty$. Indeed, Kolmogorov's strong law of large numbers implies that $S_n := \sum_{k=1}^{n} \ln|\alpha_k| \to -\infty$ and hence $|A_n| = \exp(S_n) \to 0$ a.s.; in view of (1.14), this indicates that the right-hand side of (1.13) eventually becomes x-free (see more details in [38]). In the critical case ($K = 0$), the random walk (S_n) is recurrent but none the less $\liminf_{n\to\infty} S_n = -\infty$ (a.s.); hence, at some random times τ_ϵ we have $|A_{\tau_\epsilon}| = \exp(S_{\tau_\epsilon}) < \epsilon$ (for any $\epsilon > 0$), which can be used to infer that $y(x) \equiv \mathrm{const}$ in a similar fashion as before.

Expanding this idea, our approach to the analysis of equation (1.2), first probed in [2], is based on replacing a fixed n in the iterated equation (1.13) by a suitable *stopping time* τ, defined as a random (integer-valued) variable such that for any $n \in \mathbb{N}$ the event $\{\tau \le n\}$ is determined by $(\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)$. It suffices for our purposes to work with 'hitting times' $\tau_B := \inf\{n \ge 1\colon A_n \in B\} \le \infty$,[3] where $A_n = \alpha_1 \cdots \alpha_n$ (see (1.15)) and $B \subset \mathbb{R}$ is an interval or a single point.

We shall routinely use the following central lemma (where continuity of $y(x)$ is not required).

Lemma 1.4. *Let (X_n) be the associated Markov chain (1.11), and τ a stopping time such that $\tau < \infty$ a.s. If $y(x)$ is a bounded solution of the AE (1.2) then it satisfies the 'stopped' equation*

$$y(x) = \mathbb{E}_x\{y(X_\tau)\}, \qquad x \in \mathbb{R}. \tag{1.16}$$

The crucial fact is that $y(X_n)$ is a *martingale* (cf. [16, p. 43, Proposition 1.8]); indeed, by (1.11)

$$\mathbb{E}_x\{y(X_n)|(\alpha_k, \beta_k),\, k < n\} = \mathbb{E}_x\{y(\alpha_n(X_{n-1} - \beta_n))|X_{n-1}\} = y(X_{n-1}) \quad \text{(a.s.)}, \tag{1.17}$$

which verifies the martingale property [44, §10.3, p. 94]. The lemma then readily follows by Doob's optional stopping theorem (e.g. [44, p. 100, Theorem 10.10(b)]). For the sake of a more self-contained exposition, a direct proof of Lemma 1.4 is included as appendix A.

Layout

The rest of the paper is organized as follows. In §2 we work out the degenerate cases mentioned at the beginning of §1(c), namely: $\mathbb{P}(\alpha = 0) > 0$ (§2(a)), $|\alpha| \equiv 1$ (§2(b)), and $\alpha(c - \beta) \equiv c$ (§2(c)). In §3(a) we prove our main result for the critical case (Theorem 3.1; cf. Theorem 1.2), backed up in §3(b) by simple sufficient conditions for the uniform continuity of solutions (Theorems 3.3 and 3.4). In §3(c) we explain in detail the remarkable link between the pantograph equations and the AE, which enables us to prove a Liouville theorem for the general (integro-differential) pantograph equation of any order (Theorem 3.8). This is complemented in §3(d) by a Liouville theorem for the functional equation (1.4) with $a_i = q^{m_i}$ (Theorem 3.9). Appendix A provides an elementary proof of Lemma 1.4, and appendix B gives a brief compendium of basic facts illuminating the fundamental role of bounded harmonic functions in the general theory of Markov chains.

[3] Here and below, we adopt the convention that $\inf \emptyset := \infty$.

2. Three degenerate cases

Before embarking on a general discussion of the AE, we need to study the problem of bounded solutions in certain special cases of possible values of α and β, which will be excluded from consideration thereafter. Recall the notation $A_n := \prod_{k=1}^{n} \alpha_k$ (see (1.15)).

(a) Vanishing of the scaling coefficient

Let us consider the case where the scaling coefficient α may take the value zero. Note that continuity of solutions $y(x)$ is not assumed *a priori*.

Theorem 2.1. *Suppose $p_0 := \mathbb{P}(\alpha = 0) > 0$. Then any bounded solution of the AE* (1.2) *is constant on* $\mathbb{R}$.

We first give an elementary 'analytic' proof of this simple theorem and then present another proof to illustrate the method based on Lemma 1.4.

Proof of Theorem 2.1. With $y_0(x) := y(x) - y(0)$, equation (1.1) may be written in the form

$$y_0(x) = (1 - p_0) \iint_{\mathbb{R}^2 \setminus \{a=0\}} y_0(a(x-b))\, \tilde{\mu}(\mathrm{d}a, \mathrm{d}b), \qquad x \in \mathbb{R}, \tag{2.1}$$

where $\tilde{\mu} := (1-p_0)^{-1}\mu$, so that $\tilde{\mu}(\mathbb{R}^2 \setminus \{a=0\}) = 1$. Denoting by $\|f\| := \sup_{x\in\mathbb{R}} |f(x)|$ the sup-norm on $\mathbb{R}$, from (2.1) we obtain

$$|y_0(x)| \le (1-p_0)\|y_0\| \quad (x \in \mathbb{R}) \quad \Longrightarrow \quad \|y_0\| \le (1-p_0)\|y_0\|. \tag{2.2}$$

Since $1 - p_0 < 1$, the second inequality in (2.2) immediately implies that $\|y_0\| = 0$, and then the first inequality gives $y_0(x) \equiv 0$, i.e. $y(x) \equiv y(0)$, as claimed. □

Alternative proof of Theorem 2.1. Consider the stopping time $\tau_0 := \inf\{n \ge 1 \colon A_n = 0\}$. Note that

$$\mathbb{P}(\tau_0 > n) = \mathbb{P}(A_n \ne 0) = \mathbb{P}(\alpha_1 \ne 0, \dots, \alpha_n \ne 0) = (1-p_0)^n \to 0 \qquad (n \to \infty),$$

hence $\tau_0 < \infty$ a.s. Now, using the iteration formulas (1.14), (1.15), and noting that $A_{\tau_0} = 0$ a.s., by Lemma 1.4 we obtain

$$y(x) = \mathbb{E}_x\{y(X_{\tau_0})\} = \mathbb{E}\{y(xA_{\tau_0} - D_{\tau_0})\} = \mathbb{E}\{y(-D_{\tau_0})\}, \qquad x \in \mathbb{R}. \tag{2.3}$$

Since the right-hand side of (2.3) does not depend on x, it follows that $y(x) = \mathrm{const}$. □

(b) An extension of the Choquet–Deny theorem

(i) The classical case $\alpha \equiv 1$

As mentioned in §1(b), the AE (1.2) with $\alpha \equiv 1$ is reduced to

$$y(x) = \mathbb{E}\{y(x-\beta)\}, \qquad x \in \mathbb{R}. \tag{2.4}$$

The famous Choquet–Deny theorem [18] (cf. [15, p. 382, Corollary] or [16, p. 161, Theorem 1.3]) asserts that any b.c.-solution of (2.4) is constant provided that (the distribution of) the shift β is non-arithmetic, i.e. not supported on any set $\lambda\mathbb{Z} = \{\lambda k,\ k \in \mathbb{Z}\}$ (with *span* $\lambda \in \mathbb{R}$).

Remark 2.1. In connection with the uniform continuity condition in Theorem 1.2, it may be of interest to note that some proofs of the Choquet–Deny theorem (e.g. [15, p. 382]) deploy the convolution $\tilde{y}(x) = y \star \varphi_{0,\sigma^2}(x)$ of a b.c.-solution $y(x)$ with the density function $\varphi_{0,\sigma^2}(x)$ of the normal distribution with zero mean and variance σ^2, whereby the function $\tilde{y}(x)$ is uniformly continuous and still satisfies equation (2.4). Once it has been proved that $\tilde{y}(x)$ is constant, this is extended to the original solution $y(x)$ by taking the limit as $\sigma \to 0$.

A *discrete* version of the Choquet–Deny theorem [18] (see also [45, §XIII.11] or [46, p. 276, Theorem T1]) refers to the case where equation (2.4) is considered on $\mathbb{Z}$ and β is integer-valued. Namely, assume that the smallest additive group containing the set $\{x\in\mathbb{Z}\colon\ \mathbb{P}(\beta=x)>0\}$ coincides with $\mathbb{Z}$; then the theorem asserts that $y(x)\equiv\mathrm{const}$ for all $x\in\mathbb{Z}$.

In the context of equation (2.4) on the whole line, this enables one to give a full description of b.c.-solutions in the arithmetic case (excluding the degenerate case $\beta\equiv 0$). The next result is essentially well known in folklore; we give its proof for the sake of completeness.

Theorem 2.2. *Assume that the distribution of β is arithmetic, i.e. its support $\varLambda$ is contained in the set $\lambda\mathbb{Z}$ with maximal span $\lambda>0$. Then the general b.c.-solution of equation (2.4) is given by $y(x)=\mathrm{g}(x/\lambda)$, where $\mathrm{g}(\cdot)$ is any continuous periodic function of period 1.*

Proof. We start by showing that the smallest additive subgroup $\mathcal{G}\subset\lambda\mathbb{Z}$ generated by $\varLambda$ coincides with $\lambda\mathbb{Z}$. Indeed, for $n\in\mathbb{N}$ let $d_n\in\lambda\mathbb{N}$ be the greatest common divisor of the (finite) set $\varLambda_n:=\{s\in\varLambda\colon\ |s|\le\lambda n\}\subset\mathcal{G}$. By Bézout's identity (see e.g. [47, §1.2]) we have $d_n=\sum_{s_i\in\varLambda_n}m_i s_i$ with some integers m_i, and it follows that $d_n\in\mathcal{G}$ $(n\in\mathbb{N})$. Since the sequence $d_n/\lambda\in\mathbb{N}$ is non-increasing, there exists the limit $k^*:=\lim_{n\to\infty}d_n/\lambda=d_{n^*}/\lambda\in\mathbb{N}$ and so $\varLambda\subset k^*\lambda\mathbb{Z}$. But $\lambda>0$ is the maximal span, hence $k^*=1$ and thus $\lambda=d_{n^*}\in\mathcal{G}$, which implies $\mathcal{G}=\lambda\mathbb{Z}$, as claimed.

Now, it is easy to see that equation (2.4) splits into separate discrete equations $\tilde{z}(k)=\mathbb{E}\{\tilde{z}(k-\beta/\lambda)\}$ on every coset $x_0+\lambda\mathbb{Z}$ $(x_0\in[0,\lambda))$, where $\tilde{z}(k):=z(x_0+k\lambda)$ $(k\in\mathbb{Z})$. The discrete Choquet–Deny theorem shows that $\tilde{z}(k)$ is constant on $\mathbb{Z}$; in other words, any bounded solution of (2.13) on $\mathbb{R}$ is λ-periodic, and the claim of the theorem easily follows. ☐

Remark 2.2. It is evident that any function $y(x)=\mathrm{g}(x/\lambda)$ satisfies equation (2.4); the main point of Theorem 2.2 is that *there are no other* b.c.-solutions.

Remark 2.3. Laczkovich [48] gives a full characterization of non-negative measurable solutions of the equation $y(x)=\sum_{i=1}^{\ell}C_i y(x-b_i)$ (with arbitrary coefficients $C_1,\dots,C_\ell>0$) in terms of the real roots of the characteristic equation $\sum_{i=1}^{\ell}C_i\,\mathrm{e}^{-b_i s}=1$.

Remark 2.4. The original proof by Choquet & Deny [18] (as well as many subsequent proofs and extensions) is based on a reduction to a uniformly continuous solution (cf. Remark 2.1) and establishing that the latter must reach its maximum at a finite point $x_0\in\mathbb{R}$. In view of the martingale techniques used in the present paper for a general AE, it is of interest to point out an elegant martingale proof found by Székely & Zeng [49] (cf. Rao & Shanbhag [12, Ch. 3]).

(ii) Case $|\alpha|\equiv 1$

We prove here an extension of the Choquet–Deny theorem for α taking the values ±1; to the best of our knowledge, such a result has not yet been mentioned in the literature.

Theorem 2.3. *Suppose that $|\alpha|\equiv 1$ and $\mathbb{P}(\alpha=1)<1$. Let β^+, β^- have the distribution of β conditioned on $\alpha=1$ and $\alpha=-1$, respectively (in case $\alpha\equiv-1$, set $\beta^+\equiv 0$).*

(a) *If β^+ is non-arithmetic then every b.c.-solution of equation (1.2) is constant.*

(b) *Let β^+ be arithmetic with span $\lambda\ne 0$.*

  (b-i) *If the distribution of β^- is not supported on any set $\lambda_0+\lambda\mathbb{Z}$ $(\lambda_0\in\mathbb{R})$, then every b.c.-solution of equation (1.2) is constant.*

  (b-ii) *Otherwise, the general b.c.-solution of equation (1.2) is of the form $y(x)=\mathrm{g}(x/\lambda)$, where $\mathrm{g}(\cdot)$ is a continuous 1-periodic function symmetric about point $x_0=\tfrac12\lambda_0/\lambda$, i.e. $\mathrm{g}(x_0-x)=\mathrm{g}(x_0+x)$ $(x\in\mathbb{R})$.*

Remark 2.5. In part (b-ii), all functions with the required symmetry property may be represented (though not uniquely) as $\mathrm{g}(x)=\mathrm{g}_0(x-x_0)+\mathrm{g}_0(-x+x_0)$, where $\mathrm{g}_0(\cdot)$ is an arbitrary (continuous) 1-periodic function. It is straightforward to check that so constructed functions $y(x)=\mathrm{g}(x/\lambda)$ satisfy equation (1.2) (with $|\alpha|\equiv 1$), but part (b-ii) asserts that *all* b.c.-solutions are contained in this class.

Proof of Theorem 2.3. Consider the stopping time $\tau_1 := \inf\{n \geq 1\colon A_n = 1\}$. With $p_1 := \mathbb{P}(\alpha = 1) < 1$ and $q_1 := \mathbb{P}(\alpha = -1) = 1 - p_1 > 0$, the distribution of τ_1 is given by[4]

$$\mathbb{P}(\tau_1 = 1) = p_1, \qquad \mathbb{P}(\tau_1 = n) = q_1^2 p_1^{n-2} \quad (n \geq 2), \tag{2.5}$$

hence $\tau_1 < \infty$ a.s. Since $A_{\tau_1} = 1$ (a.s.), from (1.14) we have $X_{\tau_1} = x - D_{\tau_1}$, and by Lemma 1.4

$$y(x) = \mathbb{E}\{y(x - D_{\tau_1})\}, \qquad x \in \mathbb{R}. \tag{2.6}$$

Now, in view of the Choquet–Deny theorem, we need to investigate whether the random shift D_{τ_1} is non-arithmetic, i.e. $\mathbb{P}(D_{\tau_1} \in \lambda\mathbb{Z}) < 1$ for any $\lambda \in \mathbb{R}$.

Let (β_n^+) and (β_n^-) be two sequences of i.i.d. random variables each, with the same distribution as β^+ and β^-, respectively. Conditioning on τ_1 and using (1.15) and (2.5), we obtain

$$\begin{aligned}\mathbb{P}(D_{\tau_1} \in \lambda\mathbb{Z}) &= p_1\, \mathbb{P}(\beta_1^+ \in \lambda\mathbb{Z}) + \sum_{n=3}^{\infty} q_1^2 p_1^{n-2}\, \mathbb{P}(\beta_1^- - \beta_2^+ - \cdots - \beta_{n-1}^+ - \beta_n^- \in \lambda\mathbb{Z})\\ &\leq p_1 + q_1^2 \sum_{n=2}^{\infty} p_1^{n-2} = 1.\end{aligned} \tag{2.7}$$

If $p_1 > 0$ then (2.7) implies that $\mathbb{P}(D_{\tau_1} \in \lambda\mathbb{Z}) < 1$ unless $\beta_1^+ \in \lambda\mathbb{Z}$ (a.s.) and for all $n \geq 2$

$$\beta_1^- - \beta_2^+ - \cdots - \beta_{n-1}^+ - \beta_n^- \in \lambda\mathbb{Z} \quad \text{(a.s.)}. \tag{2.8}$$

Since all β_i^+ are i.i.d., the first of these inclusions implies that $\beta_2^+ + \cdots + \beta_{n-1}^+ \in \lambda\mathbb{Z}$ a.s. for all $n \geq 2$; hence, conditions (2.8) are reduced to $\beta_1^- - \beta_2^- \in \lambda\mathbb{Z}$ (a.s.). In turn, the last condition is equivalent to $\beta^- \in \lambda_0 + \lambda\mathbb{Z}$ for some $\lambda_0 \in \mathbb{R}$. Indeed, applying Lebesgue's decomposition theorem [15, p. 142] to each of the i.i.d. random variables β_1^- and β_2^-, it is evident that the continuous part of their common distribution must vanish, so that this distribution is purely discrete; furthermore, its (countable) support $\{b_i\}$ satisfies the condition $b_i - b_j \in \lambda\mathbb{Z}$ for all i, j, and the claim follows.

A similar argument is also valid for $p_1 = 0$, whereby $\beta^+ = 0$, $\beta^- = \beta$, and (2.7) simplifies to

$$\mathbb{P}(D_{\tau_1} \in \lambda\mathbb{Z}) = \mathbb{P}(\beta_1 - \beta_2 \in \lambda\mathbb{Z}) \leq 1.$$

This completes the proof of parts (a) and (b-i).

Finally, we prove part (b-ii), whereby $\beta^+ \in \lambda\mathbb{Z}$, $\beta^- \in \lambda_0 + \lambda\mathbb{Z}$ and $D_{\tau_1} \in \lambda\mathbb{Z}$ (a.s.). By Theorem 2.2, any b.c.-solution of equation (2.6) must be of the form $y(x) = \mathrm{g}(x/\lambda)$, with some 1-periodic function $\mathrm{g}(\cdot)$. Substituting this into the original equation (2.6) (with $|\alpha| \equiv 1$) we get

$$\begin{aligned}\mathrm{g}(x/\lambda) &= p_1\, \mathbb{E}\{\mathrm{g}((x - \beta^+)/\lambda)\} + q_1\, \mathbb{E}\{\mathrm{g}((-x + \beta^-)/\lambda)\}\\ &= p_1 \mathrm{g}(x/\lambda) + q_1 \mathrm{g}((-x + \lambda_0)/\lambda),\end{aligned} \tag{2.9}$$

and since $q_1 \neq 0$ it follows that $\mathrm{g}(\cdot)$ satisfies the functional equation $\mathrm{g}(x) = \mathrm{g}(-x + \lambda_0/\lambda)$ $(x \in \mathbb{R})$, which is equivalent to the symmetry condition stated in the theorem. □

Remark 2.6. If $\alpha \equiv 1$ then $\beta^+ \equiv \beta$, $\beta^- \equiv 0$, and parts (a) and (b-ii) of Theorem 2.3 formally recover the Choquet–Deny theorem. No extra requirement on $\mathrm{g}(\cdot)$ arises from (2.9), since $q_1 = 0$.

Remark 2.7. Note that the values $\alpha = 1$ and $\alpha = -1$ (and the corresponding conditional distributions of β represented by β^+ and β^-, respectively) feature in Theorem 2.3 in a non-symmetric way: e.g. if β^+ has a non-arithmetic distribution then, according to part (a), there are no b.c.-solutions except constants, irrespectively of β^-; however, if β^+ is arithmetic with span $\lambda \neq 0$ whilst β^- is non-arithmetic but supported on a set $\lambda_0 + \lambda\mathbb{Z}$ (i.e. with $\lambda_0 \neq 0$ incommensurable with λ) then there exist non-trivial b.c.-solutions, according to part (b-ii).

Example 2.1. Theorem 2.3 with $\alpha \equiv -1$ is exemplified by the equations

(a) $y(x) = \int_0^\infty y(t - x)\, \mathrm{e}^{-t}\, \mathrm{d}t$ (equivalent to the pantograph equation $y'(x) + y(x) = y(-x)$, cf. (1.7)), which by part (b-i) has only constant b.c.-solutions;

[4]These formulas include the case $p_1 = 0$ (under the convention $0^0 := 1$), whereby $\tau_1 = 2$ a.s.

(b) $y(x) = \frac{1}{3}y(-x+1) + \frac{2}{3}y(-x-1)$, which has periodic solutions of the form $y(x) = \mathrm{g}(x) + \mathrm{g}(-x)$, in accordance with part (b-ii).

(c) The resonance case

Definition 2.1. The random coefficients α, β of the AE (1.2) are said to be *in resonance* if there is a non-random constant $c \in \mathbb{R}$ such that $\alpha(c-\beta) = c$ (a.s.).

The special role of resonance is clear from the observation that if $X_0 = c$ then by recursion (1.11) we have $X_n = c$ (a.s.) for all $n \geq 0$. It turns out that a Liouville-type theorem is always true in the resonance case. Recall that we assume $\alpha \neq 0$ a.s.

Theorem 2.4. *Let $\mathbb{P}(|\alpha| \neq 1) > 0$, and suppose that α, β are in resonance. Then any b.c.-solution of the AE (1.2) is constant.*

Proof. Let $c \in \mathbb{R}$ be such that $\alpha(c-\beta) \equiv c$. Observe that the substitution $\tilde{y}(x) := y(x+c)$ eliminates the random shift in equation (1.2),

$$\tilde{y}(x) = y(x+c) = \mathbb{E}\{\alpha x + \alpha(c-\beta))\} = \mathbb{E}\{y(\alpha x + c)\} = \mathbb{E}\{\tilde{y}(\alpha x)\}.$$

Thus, without loss of generality, we can consider the equation $y(x) = \mathbb{E}\{y(\alpha x)\}$. Denote $\tau_+ := \inf\{n \geq 1\colon A_n > 0\}$ and $p := \mathbb{P}(\alpha > 0)$, then

$$\mathbb{P}(\tau_+ > n) = (1-p)\, p^{n-1} \to 0 \qquad (n \to \infty),$$

so that $\tau_+ < \infty$ a.s. Hence, by virtue of Lemma 1.4 the equation $y(x) = \mathbb{E}\{y(\alpha x)\}$ is reduced to $y(x) = \mathbb{E}\{y(\tilde{\alpha} x)\}$ with $\tilde{\alpha} := A_{\tau_+} > 0$ a.s.

We first check that $\mathbb{P}(\tilde{\alpha} = 1) < 1$. Indeed, with $p = \mathbb{P}(\alpha > 0)$ as above we have

$$\mathbb{P}(\tilde{\alpha} = 1) = p\,\mathbb{P}(A_1 = 1) + \sum_{n=2}^{\infty} (1-p)^2 p^{n-2}\, \mathbb{P}(A_n = 1) \leq p + (1-p)^2 \sum_{n=2}^{\infty} p^{n-2} = 1. \tag{2.10}$$

If the probability on the left-hand side of (2.10) equals 1 and $p > 0$, then we must have $\mathbb{P}(A_1 = 1) = \mathbb{P}(\alpha = 1) = 1$, which contradicts the theorem's hypothesis; similarly, if $p = 0$ then the inequality (2.10) implies that $A_2 = \alpha_1 \alpha_2 = 1$ (a.s.), and since α_1, α_2 are i.i.d. the latter equality is possible only if $|\alpha| = 1$ a.s., which is again a contradiction.

Now, the equation $y(x) = \mathbb{E}\{y(\tilde{\alpha} x)\}$ on $\mathbb{R}$ (with $\tilde{\alpha} > 0$) splits into two separate equations, for $x \geq 0$ and $x \leq 0$, linked by the continuity condition at zero. For instance, consider the equation

$$y(x) = \mathbb{E}\{y(\tilde{\alpha} x)\} \quad (x > 0), \qquad y(0) = \lim_{x \to 0+} y(x). \tag{2.11}$$

Our aim is to show that $y(x) \equiv \mathrm{const}$ for all $x \geq 0$. By the change of variables

$$t = -\ln x \in \mathbb{R}, \qquad z(t) = y(\mathrm{e}^{-t}), \tag{2.12}$$

the initial value problem (2.11) is transformed into

$$z(t) = \mathbb{E}\{z(t - \tilde{\beta})\} \quad (t \in \mathbb{R}), \qquad \lim_{t \to +\infty} z(t) = y(0), \tag{2.13}$$

which is an archetypal equation (1.2) with the unit rescaling coefficient and random shift $\tilde{\beta} := \ln \tilde{\alpha}$ (such that $\mathbb{P}(\tilde{\beta} \neq 0) > 0$), subject to an additional limiting condition at $+\infty$.

If $\tilde{\beta}$ is non-arithmetic then the Choquet–Deny theorem readily implies that all b.c.-solutions of equation (2.13) are constant (even without the limit condition at $+\infty$). In the arithmetic case, by Theorem 2.2 any b.c.-solution of (2.13) is λ-periodic, but due to the limit in (2.13) it must be constant. Returning to (2.11) via the substitution (2.12), we conclude that in all cases $y(x) \equiv \mathrm{const}$ for $x \geq 0$. By symmetry, the same is true for $x \leq 0$, and the proof is completed by invoking continuity of $y(x)$ at $x = 0$. □

Assumption 2.1. Henceforth, unless explicitly stated otherwise, we assume that

$$\text{(i) } \mathbb{P}(\alpha \neq 0) = 1; \quad \text{(ii) } \mathbb{P}(|\alpha| \neq 1) > 0; \quad \text{(iii) } \alpha, \beta \text{ are not in resonance.}$$

3. The critical case

(a) Liouville theorem subject to uniform continuity

Recall the notation $K := \mathbb{E}\{\ln|\alpha|\}$. The next theorem deals with the case $K \le 0$ (including the critical case $K = 0$) under an additional *a priori* hypothesis of uniform continuity of the solution; on the other hand, in contrast to Theorem 1.1 no moment conditions are imposed on β. Note that conditions (i) and (ii) of Assumption 2.1 are in force, but (iii) is not needed.

Theorem 3.1. *Assume that* $\mathbb{P}(|\alpha| \neq 1) > 0$ *and* $\mathbb{E}\{\ln|\alpha|\} \le 0$. *Let* $y(x)$ *be a bounded solution of the AE* (1.2) *which is uniformly continuous on* $\mathbb{R}$. *Then* $y(x) \equiv \text{const}$.

Proof. By uniform continuity, for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that if $|x_1 - x_2| < \delta$ then $|y(x_1) - y(x_2)| < \varepsilon$ ($x_1, x_2 \in \mathbb{R}$). Furthermore, for a given $x \in \mathbb{R}$ choose $M = M(\delta, x) > 0$ such that $|x|\,\mathrm{e}^{-M} < \delta$, and define the stopping time

$$\tau_M := \inf\bigl\{n \ge 1 \colon |A_n| \le \mathrm{e}^{-M}\bigr\}.$$

Since $\mathbb{E}\{\ln|\alpha|\} \le 0$, it follows (see e.g. [15, pp. 395–397]) that

$$\liminf_{n\to\infty} \ln|A_n| = \liminf_{n\to\infty} \sum_{k=1}^{n} \ln|\alpha_k| = -\infty \qquad \text{(a.s.)},$$

implying that $\tau_M < \infty$ a.s. Hence, by Lemma 1.4 we have $y(x) = \mathbb{E}_x\{y(X_{\tau_M})\}$ or, more explicitly (using the iteration formulas (1.14), (1.15)),

$$y(x) = \mathbb{E}\{y(xA_{\tau_M} - D_{\tau_M})\}, \qquad x \in \mathbb{R}. \tag{3.1}$$

In particular, (3.1) with $x = 0$ gives $y(0) = \mathbb{E}\{y(-D_{\tau_M})\}$. Hence, from (3.1) we obtain

$$|y(x) - y(0)| \le \mathbb{E}\bigl|y(xA_{\tau_M} - D_{\tau_M}) - y(-D_{\tau_M})\bigr|. \tag{3.2}$$

But according to the definition of the stopping time τ_M and the choice of M, we have

$$\bigl|(xA_{\tau_M} - D_{\tau_M}) - (-D_{\tau_M})\bigr| = |x| \cdot |A_{\tau_M}| \le |x|\,\mathrm{e}^{-M} < \delta.$$

Due to uniform continuity of $y(x)$ (see above), this implies

$$\bigl|y(xA_{\tau_M} - D_{\tau_M}) - y(-D_{\tau_M})\bigr| < \varepsilon,$$

and from (3.2) we readily obtain $|y(x) - y(0)| \le \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $y(x) \equiv y(0)$, which completes the proof. □

(b) Sufficient conditions for the uniform continuity of solutions

Consider the general AE (1.2) with no restriction on the value $K = \mathbb{E}\{\ln|\alpha|\}$. Simple sufficient conditions for the uniform continuity of its solutions are based on the following well-known fact from real analysis (see e.g. [50, p. 74, Proposition 2.5]).

Lemma 3.2. *Suppose that* $f \in L^1(\mathbb{R})$. *Then*

$$\lim_{h\to 0} \int_{\mathbb{R}} |f(t+h) - f(t)|\,\mathrm{d}t = 0.$$

For continuous functions with compact support the lemma holds by dominated convergence; the general case follows since such functions are dense in $L^1(\mathbb{R})$ [50, p. 71, Theorem 2.4(iii)].

Theorem 3.3. *Assume that for each $a \in \mathbb{R}$ in the support of the random variable α there exists the conditional density function $f_\beta(t|a) = \mathbb{P}(\beta \in \mathrm{d}t | \alpha = a)/\mathrm{d}t$. Then any bounded solution of* (1.2) *is uniformly continuous on $\mathbb{R}$.*

Proof. Let $\mu_\alpha(\mathrm{d}a) := \mathbb{P}(\alpha \in \mathrm{d}a)$. By Fubini's theorem, equation (1.1) becomes

$$y(x) = \int_{\mathbb{R}} \mu_\alpha(\mathrm{d}a) \int_{\mathbb{R}} y(a(x-t))\, f_\beta(t|a)\,\mathrm{d}u \equiv \int_{\mathbb{R}} \mu_\alpha(\mathrm{d}a) \int_{\mathbb{R}} y(-au)\, f_\beta(x+u|a)\,\mathrm{d}u,$$

where we used the change of variables $u = t - x$. Hence, uniformly in $x \in \mathbb{R}$

$$\begin{aligned} |y(x+h) - y(x)| &\le \int_{\mathbb{R}} \mu_\alpha(\mathrm{d}a) \int_{\mathbb{R}} |y(-au)| \cdot \big| f_\beta(x+h+u|a) - f_\beta(x+u|a) \big|\,\mathrm{d}u \\ &\le \|y\| \int_{\mathbb{R}} \mu_\alpha(\mathrm{d}a) \int_{\mathbb{R}} \big| f_\beta(t+h|a) - f_\beta(t|a) \big| \mathrm{d}t \to 0 \qquad (h \to 0), \end{aligned}$$

due to the bound $\|y\| < \infty$ and also using Lemma 3.2 (applied to $f_\beta(\cdot|a)$ for each a) and Lebesgue's dominated convergence theorem. □

In many cases, another sufficient condition is more suitable. Let the random variable β in the AE (1.2) be of the form $\beta = \gamma + \xi$, where ξ is independent of the random pair (α, γ) and has the density function, $f_\xi(t)$. Set

$$\varphi(x) := \mathbb{E}\{y(\alpha(x-\gamma))\} \equiv \iint_{\mathbb{R}^2} y(a(x-c))\, \mu_{\alpha,\gamma}(\mathrm{d}a, \mathrm{d}c), \tag{3.3}$$

where $\mu_{\alpha,\gamma}(\mathrm{d}a, \mathrm{d}c) := \mathbb{P}(\alpha \in \mathrm{d}a, \gamma \in \mathrm{d}c)$. For example, if the measure $\mu_{\alpha,\gamma}$ is discrete, with atoms (a_i, c_i) and respective masses p_i, then

$$\varphi(x) = \sum\nolimits_i p_i y(a_i(x - c_i)), \qquad x \in \mathbb{R}.$$

Observe that, by independence of ξ, we have

$$\begin{aligned} \mathbb{E}\{y(\alpha(x-\beta))\} &= \mathbb{E}\{y(\alpha(x-\xi-\gamma))\} \\ &= \int_{\mathbb{R}} \mathbb{E}\{y(\alpha(x-t-\gamma))\}\, f_\xi(t)\,\mathrm{d}t = \int_{\mathbb{R}} \varphi(x-t)\, f_\xi(t)\,\mathrm{d}t, \end{aligned}$$

according to the definition (3.3). Hence, the AE (1.2) becomes

$$y(x) = \int_{\mathbb{R}} \varphi(x-t)\, f_\xi(t)\,\mathrm{d}t, \qquad x \in \mathbb{R}. \tag{3.4}$$

Theorem 3.4. *Any bounded solution of equation* (3.4) *is uniformly continuous on $\mathbb{R}$.*

Proof. By the substitution $u = x - t$ equation (3.4) is rewritten as

$$y(x) = \int_{\mathbb{R}} \varphi(u)\, f_\xi(x-u)\,\mathrm{d}u, \qquad x \in \mathbb{R}.$$

Hence, uniformly in $x \in \mathbb{R}$

$$\begin{aligned} |y(x+h) - y(x)| &\le \|\varphi\| \int_{\mathbb{R}} \big| f_\xi(x+h-u) - f_\xi(x-u) \big| \mathrm{d}u \\ &= \|\varphi\| \int_{\mathbb{R}} \big| f_\xi(t+h) - f_\xi(t) \big|\,\mathrm{d}t \to 0 \qquad (h \to 0), \end{aligned}$$

according to Lemma 3.2, and the claim follows. □

Remark 3.1. In both Theorems 3.3 and 3.4, continuity of solutions is not assumed *a priori*.

(c) Pantograph equation

In this subsection, we explain the link pointed out in §1(b)-ii between the AE (1.2) and a class of functional-differential 'pantograph' equations. We begin with an elementary proof for a first-order pantograph equation (§3(c)-i), and then treat the general case (§3(c)-ii). In turn, this allows us to establish the uniform continuity of solutions by virtue of Theorem 3.4, and hence to prove a Liouville theorem using Theorem 3.1 (§3(c)-iii).

(i) First-order pantograph equation

Assume that the random variable β in (1.2) is independent of α and has the unit exponential distribution, with the density function $\mathrm{e}^{-t}\mathbb{1}_{(0,\infty)}(t)$. Then equation (1.2) specializes to (cf. (3.4))

$$y(x)=\int_0^\infty \varphi_0(x-t)\,\mathrm{e}^{-t}\mathrm{d}t,\qquad x\in\mathbb{R},\tag{3.5}$$

where

$$\varphi_0(x):=\mathbb{E}\{y(\alpha x)\}\equiv\int_{\mathbb{R}} y(ax)\,\mu_\alpha(\mathrm{d}a),\qquad \mu_\alpha(\mathrm{d}a):=\mathbb{P}(\alpha\in\mathrm{d}a).$$

Proposition 3.5. *Every b.c.-solution of (3.5) satisfies the pantograph equation*

$$y'(x)+y(x)=\varphi_0(x),\qquad x\in\mathbb{R}.\tag{3.6}$$

Conversely, any bounded solution of (3.6) satisfies equation (3.5).

Proof. The substitution $u=x-t$ transforms equation (3.5) into $y(x)=\mathrm{e}^{-x}\int_{-\infty}^x \varphi_0(u)\,\mathrm{e}^u\mathrm{d}u$, and it is now evident that the right-hand side is continuous and, moreover, differentiable in $x\in\mathbb{R}$. Hence, by the Newton–Leibniz theorem we readily obtain equation (3.6).

Conversely, let $y(x)$ be a bounded solution of (3.6). Then by variation of constants

$$y(x)=y(0)\,\mathrm{e}^{-x}+\int_0^x \varphi_0(u)\,\mathrm{e}^{u-x}\mathrm{d}u.\tag{3.7}$$

Since $y(x)$ is bounded, we have $y(x)\,\mathrm{e}^x\to 0$ as $x\to-\infty$, and it follows from (3.7) that $y(0)=\int_{-\infty}^0 \varphi_0(u)\,\mathrm{e}^u\mathrm{d}u$. Substituting this back into (3.7) and combining the integrals, we obtain

$$y(x)=\int_{-\infty}^x \varphi_0(u)\,\mathrm{e}^{u-x}\mathrm{d}u=\int_0^\infty \varphi_0(x-t)\,\mathrm{e}^{-t}\mathrm{d}t,$$

which is exactly equation (3.5). □

(ii) Higher-order pantograph equations

The correspondence demonstrated in §3(c)-i can be extended to more general equations, including higher orders. Like in Theorem 3.4, suppose that $\beta=\gamma+\xi$, where ξ is independent of (α,γ) and has density $f_\xi(t)$. Following [1], fix $r\in\mathbb{N}$ and (real) constants $\kappa_1,\dots,\kappa_r\ne 0$ (some or all of which may coincide), and let $f_\xi(t)$ be given by

$$f_\xi(t)=g_1\star\cdots\star g_r(t),\qquad g_j(t):=|\kappa_j|\,g_0(\kappa_j t)\quad (i=1,\dots,r),$$

where $\star$ denotes convolution and $g_0(t):=\mathrm{e}^{-t}\mathbb{1}_{(0,\infty)}(t)$. On the other hand, consider the pantograph equation of order r

$$\prod_{j=1}^r\left(1+\frac{\mathcal{D}}{\kappa_j}\right)y(x)=\varphi(x),\qquad \mathcal{D}:=\frac{\mathrm{d}}{\mathrm{d}x},\tag{3.8}$$

where (see (3.3))

$$\varphi(x)=\mathbb{E}\{y(\alpha(x-\gamma))\},\qquad x\in\mathbb{R}.\tag{3.9}$$

Finally, recall from (3.4) that the AE (1.2) is equivalently rewritten as

$$y(x)=\int_{\mathbb{R}}\varphi(x-t)\,f_\xi(t)\,\mathrm{d}t,\qquad x\in\mathbb{R}.\tag{3.10}$$

Proposition 3.6. *Every b.c.-solution of* (3.10) *satisfies the pantograph equation* (3.8). *Conversely, any bounded solution of* (3.8) *satisfies equation* (3.10).

Proof. Let $y(x)$ be a b.c.-solution of (3.10). Then the function $\varphi(x)$ defined in (3.9) is bounded, $\|\varphi\| \le \|y\| < \infty$; furthermore, it is continuous by virtue of dominated convergence. Therefore, one can apply the convolution inversion formula (see [51, p. 28, Theorem 7.1]), which readily yields that $y(x)$ satisfies equation (3.8).

If $y(x)$ is a bounded (continuous) solution of (3.8), then $\varphi(x)$ is again b.c. and it follows that $y(x)$ satisfies equation (3.10) thanks to [51, p. 34, Theorem 9.3]. □

Example 3.1. The following are three simple examples illustrating Proposition 3.6.

(a) Take $r=1$ and $\kappa_1=1$, then equation (3.8) is reduced to $y'(x)+y(x)=\varphi(x)$ (cf. (3.6)). Here, ξ has the unit exponential density, $f_\xi(t)=g_0(t)$. Thus, Proposition 3.6 extends Proposition 3.5 to pantograph equations with the right-hand side $\varphi(x)$ given by (3.9).

(b) For $r=2$ and $\kappa_1=1$, $\kappa_2=-1$, equation (3.8) specializes to $-y''(x)+y(x)=\varphi(x)$. This corresponds to a symmetric (two-sided) exponential density $f_\xi(t)=g_0(t)\star g_0(-t)=\frac{1}{2}\mathrm{e}^{-|t|}$ $(t\in\mathbb{R})$.

(c) Likewise, with $r=2$ and $\kappa_1=\kappa_2=1$ we have $y''(x)+2y'(x)+y(x)=\varphi(x)$. Here, $f_\xi(t)=g_0\star g_0(t)=t\,\mathrm{e}^{-t}$ $(t>0)$.

(iii) Liouville theorem for the pantograph equation

By virtue of Proposition 3.6, Theorem 3.4 implies the following

Proposition 3.7. *Any bounded solution of the pantograph equation* (3.8) *is uniformly continuous.*

The next result settles a Liouville theorem for the general class of the pantograph equations (3.8) in the critical and subcritical cases (cf. Theorem 1.3 stated in §1(c)).

Theorem 3.8. *If* $K=\mathbb{E}\{\ln|\alpha|\}\le 0$ *then any bounded solution of equation* (3.8) *is constant.*

Proof. By Proposition 3.7, bounded solutions of equation (3.8) are uniformly continuous; hence, if $\mathbb{P}(|\alpha|=1)<1$ then the claim readily follows by Theorem 3.1. If $|\alpha|=1$ a.s. then we can apply the generalized Choquet–Deny theorem (Theorem 2.3), noting that either β^+ or β^- (i.e. β conditioned on $\alpha=1$ or $\alpha=-1$, respectively) must have a continuous distribution, because $\beta=\gamma+\xi$ and ξ is exponentially distributed independently of (α,γ). Thus, the theorem is proved. □

(d) q-Difference equations with shifts

For the purely functional equation (cf. (1.4))

$$y(x)=\sum_{i=1}^{\ell} p_i y(a_i(x-b_i)) \qquad (x\in\mathbb{R}), \qquad p_i>0, \quad \sum_{i=1}^{\ell} p_i=1, \tag{3.11}$$

where the above criteria of uniform continuity (Theorems 3.3, 3.4) are not usable, no general results of the Liouville type are currently available in the critical case, except for equations with $\alpha\equiv 1$ treated by the Choquet–Deny theorem (see §§1(a)-i, 2(b)) and its generalization to the case $|\alpha|\equiv 1$ (Theorem 2.3), and also equations without shifts (i.e. $\beta\equiv 0$) covered by Theorem 2.4.

In this subsection, we consider an important subclass of functional equations (3.11), for which a Liouville theorem (in the critical case) can be proved without the *a priori* hypothesis of uniform continuity. Namely, assume that the coefficients $a_i>0$ in (3.11) are multiplicatively commensurable, i.e. $a_i=q^{m_i}$ with some $q>1$ and $m_i\in\mathbb{Z}$ $(i=1,\dots,\ell)$. The resulting equations $y(x)=\sum_{i=1}^{\ell} p_i y(q^{m_i}(x-b_i))$ are known as *q-difference equations*; the general theory of such equations (albeit without shifts b_i) was developed by Birkhoff [52] and Adams [53].

To avoid trivialities, we assume in (3.11) that $\ell\ge 2$ and $(a_i,b_i)\ne(1,0)$ for all $i=1,\dots,\ell$. We also exclude the case $a_1=\dots=a_\ell=1$, which is covered by the Choquet–Deny theorem (see (1.3)). The theorem below handles the critical case, $K=\sum_{i=1}^{\ell} p_i\ln a_i=0$.

Theorem 3.9. *Assume that* $\sum_{i=1}^{\ell} p_i m_i = 0$*. Then any b.c.-solution of equation* (3.11) *is constant.*

Proof. Set

$$\rho_i := \frac{b_i}{1 - a_i^{-1}}, \qquad i = 1, \dots, \ell. \tag{3.12}$$

If $a_i = 1$ (but $b_i \neq 0$) then (3.12) is understood as $\rho_i := \infty$. Note that if $\rho_1 = \cdots = \rho_\ell = c \in \mathbb{R}$ then equations (3.12) are combined as $\alpha(c - \beta) \equiv c$; that is to say, α and β are in resonance (see Definition 2.1), and the desired result readily follows by Theorem 2.4.

Assuming now that not all ρ_i are the same, let us follow a similar martingale strategy as in the proof of Theorems 2.1, 2.3 and 3.1, but based on the stopping time $\tau_0 := \inf\{n \geq 1 \colon S_n = 0\}$ of the random walk $S_n := \sum_{j=1}^{n} \log_q \alpha_j$ $(S_0 = 0)$. Since $\mathbb{E}\{\log_q \alpha\} = \sum_{i=1}^{\ell} p_i m_i = 0$, the random walk S_n is recurrent (see e.g. [46, p. 33, Theorem T1]), hence $\tau_0 < \infty$ a.s. Thus, by Lemma 1.4

$$\mathbb{E}\{y(x - D_{\tau_0})\} = y(x), \qquad x \in \mathbb{R}, \tag{3.13}$$

where the sequence (D_n) is defined in (1.15). By the Choquet–Deny theorem (see §2(b)), every b.c.-solution of (3.13) is constant on $\mathbb{R}$ provided that *the distribution of the shift* D_{τ_0} *is non-arithmetic*. In the rest of the proof, our aim is to verify the last condition.

Due to recurrence of the random walk (S_n), there are integers $k_1, \dots, k_\ell \geq 0$ such that

$$k_1 m_1 + \cdots + k_\ell m_\ell = 0, \qquad k_1|m_1| + \cdots + k_\ell |m_\ell| > 0. \tag{3.14}$$

This corresponds to paths $\{S_j \not\equiv 0,\ 0 \leq j \leq k^* := k_1 + \cdots + k_\ell\}$ with exactly k_i steps $\log_q \alpha = m_i$ $(i = 1, \dots, \ell)$, so that $S_{k^*} = 0$. Let us choose a sequence of these steps so that $S_j > 0$ for all $j = 1, \dots, k^* - 1$, which would ensure that k^* is a possible value of the return time τ_0 occurring with probability $\mathbb{P}(\tau_0 = k^*) = p_1^{k_1} \cdots p_\ell^{k_\ell} > 0$. To this end, split the indices $i = 1, \dots, \ell$ into disjoint groups with the same value ρ_i in each group. Due to the balance condition in (3.14), the integers m_i cannot all have the same sign, and recalling that not all ρ_i coincide, it is easy to see that one can find two indices i^* and j^* such that

$$\rho_{i^*} \neq \rho_{j^*}, \qquad m_{i^*} > 0, \qquad m_{j^*} < 0,$$

which also implies that ρ_{i^*}, ρ_{j^*} are finite. Hence, by a suitable relabelling of $a_1, \dots, a_\ell$ (so that i^* and j^* become 1 and ℓ, respectively), we can assume without loss of generality that $\rho_1 \neq \rho_\ell$ and

$$m_1 > 0, \qquad m_2, \dots, m_r \geq 0, \qquad m_{r+1}, \dots, m_{\ell-1} \leq 0, \qquad m_\ell < 0.$$

In particular, it follows that

$$s_i := k_1 m_1 + \cdots + k_i m_i > 0, \qquad i = 1, \dots, \ell - 1. \tag{3.15}$$

Now, recalling that $a_1^{k_1} \cdots a_\ell^{k_\ell} = q^{k_1 m_1 + \cdots + k_\ell m_\ell} = q^0 = 1$, we obtain

$$\theta_n := \sum_{i=1}^{\ell} \rho_i (1 - a_i^{-nk_i}) \prod_{j=i}^{\ell} a_j^{nk_j} = \sum_{i=1}^{\ell} \rho_i (1 - a_i^{-nk_i}) \prod_{j=1}^{i-1} a_j^{-nk_j}, \qquad n \in \mathbb{N}. \tag{3.16}$$

Equation (3.16) means that θ_n belongs to the support of the random variable D_{τ_0}. Expanding and rearranging (3.16), and using conditions (3.15), we have

$$\begin{aligned}
\theta_n &= \sum_{i=1}^{\ell} \rho_i \prod_{j=1}^{i-1} a_j^{-nk_j} - \sum_{i=1}^{\ell} \rho_i \prod_{j=1}^{i} a_j^{-nk_j} \\
&= \rho_1 + \sum_{i=1}^{\ell-1} (\rho_{i+1} - \rho_i) \prod_{j=1}^{i} a_j^{-nk_j} - \rho_\ell \prod_{j=1}^{\ell} a_j^{-nk_j} \\
&= \rho_1 + \sum_{i=1}^{\ell-1} (\rho_{i+1} - \rho_i)\, q^{-ns_i} - \rho_\ell \to \rho_1 - \rho_\ell \neq 0 \qquad (n \to \infty).
\end{aligned}$$

Hence, for any $\varepsilon > 0$ there are $n, n' \in \mathbb{N}$ such that $0 \neq |\theta_n - \theta_{n'}| < \varepsilon$, which implies that the distribution of D_{τ_0} is non-arithmetic, as required. $\square$

Appendix A. Direct proof of Lemma 1.4

The idea of the proof of the identity (1.16) is to propagate equation (1.12) according to the (random) value of stopping time τ, resulting in the relation

$$y(x) = \sum_{i=1}^{n} \mathbb{E}_x\big\{y(X_i) I_{\{\tau=i\}}\big\} + \mathbb{E}_x\big\{y(X_n) I_{\{\tau>n\}}\big\}, \qquad n \in \mathbb{N}, \tag{A 1}$$

where I_A is the indicator of event A (i.e. $I_A = 1$ if A occurs and $I_A = 0$ otherwise).

We prove formula (A 1) by mathematical induction. For $n = 1$ it is reduced to (1.12). Suppose now that (A 1) holds for some $n \in \mathbb{N}$. For shorthand, denote $\mathcal{F}_n := \sigma\{(\alpha_k, \beta_k),\ k \le n\}$, i.e. the smallest σ-algebra containing all events $\{\alpha_k \le a, \beta_k \le b\}$ ($a, b \in \mathbb{R}$, $k = 1, \dots, n$). The assumption of the lemma that τ is a stopping time (relative to $(\mathcal{F}_n)$) means that $\{\tau \le n\} \in \mathcal{F}_n$ ($n \in \mathbb{N}$). By the total expectation formula [44, §9.7(a)] we can write

$$\begin{aligned}\mathbb{E}_x\{y(X_{n+1}) I_{\{\tau>n\}}\} &= \mathbb{E}_x\big\{\mathbb{E}_x\big[y(X_{n+1}) I_{\{\tau>n\}} | \mathcal{F}_n\big]\big\} \\ &= \mathbb{E}_x\big\{I_{\{\tau>n\}}\, \mathbb{E}_x\big[y(X_{n+1}) | \mathcal{F}_n\big]\big\} = \mathbb{E}_x\big\{y(X_n) I_{\{\tau>n\}}\big\},\end{aligned} \tag{A 2}$$

thanks to the martingale property (1.17); note that $I_{\{\tau>n\}} = 1 - I_{\{\tau\le n\}}$ is $\mathcal{F}_n$-measurable, so can be taken outside the conditional expectation $\mathbb{E}_x(\cdot|\mathcal{F}_n)$ [44, §9.7(j)]. Now, substituting equality (A 2) into (A 1) and noting that

$$\mathbb{E}_x\{y(X_{n+1}) I_{\{\tau>n\}}\} = \mathbb{E}_x\big\{y(X_{n+1}) I_{\{\tau=n+1\}}\big\} + \mathbb{E}_x\big\{y(X_{n+1}) I_{\{\tau>n+1\}}\big\},$$

we conclude that the induction step is complete, and hence (A 1) holds for all $n \in \mathbb{N}$.

Finally, since $\|y\| := \sup_{x\in\mathbb{R}} |y(x)| < \infty$ and $\tau < \infty$ a.s., we have

$$\big|\mathbb{E}_x\big\{y(X_n) I_{\{\tau>n\}}\big\}\big| \le \|y\|\, \mathbb{P}_x(\tau > n) \to 0, \qquad n \to \infty.$$

Hence, passing to the limit in (A 1) as $n \to \infty$ yields

$$y(x) = \sum_{i=1}^{\infty} \mathbb{E}_x\{y(X_i) I_{\{\tau=i\}}\} = \mathbb{E}_x\{y(X_\tau) I_{\{\tau<\infty\}}\} = \mathbb{E}_x\{y(X_\tau)\},$$

and the identity (1.16) is proved.

Appendix B. Harmonic functions and asymptotics of Markov chains

Here, we provide a brief summary of some facts pertaining to the role of harmonic functions in the asymptotic classification of general Markov chains. For a more systematic exposition, we refer the reader to monographs by Revuz [16] and Meyn & Tweedie [54].

Consider a general (time-homogeneous) Markov chain (X_n) with state space $\mathbb{R}$. Denote by $\mathbb{P}_x$ the probability law of (X_n) started from $X_0 = x \in \mathbb{R}$, and by $\mathbb{E}_x$ the corresponding expectation. Let $\mathbb{R}^\infty$ be the space of real sequences $\bar{x} = (x_0, x_1, \dots)$ and $\mathcal{B}^\infty$ the smallest σ-algebra containing all cylinder sets $\{\bar{x} \in \mathbb{R}^\infty \colon (x_0, \dots, x_m) \in B_m\}$ with a Borel base $B_m \subset \mathbb{R}^{m+1}$. Event A is said to be *invariant* if there exists $\bar{B} \in \mathcal{B}^\infty$ such that $A = \{(X_m, X_{m+1}, \dots) \in \bar{B}\}$ for every $m \ge 0$. For example, the event $\{\lim_{n\to\infty} X_n = \infty\}$ is clearly invariant because it is not affected by the shifts $(X_n) \mapsto (X_{n+m})$ ($m \in \mathbb{N}$). The class of all invariant events is a σ-algebra denoted $\mathcal{I}$; a random variable Y is called *invariant* if it is $\mathcal{I}$-measurable.

Recall that a function $y(x)$ is called *harmonic* if $y(x) = \mathbb{E}_x\{y(X_1)\}$ for all $x \in \mathbb{R}$. A fundamental result (see e.g. [16, p. 56, Proposition 3.2] or [54, p. 425, Theorem 17.1.3]) is that there is a one-to-one correspondence between bounded harmonic functions and (equivalence classes of) bounded invariant random variables, expressed by the pair of relations

$$Y = \lim_{n\to\infty} y(X_n) \quad \text{(a.s.)}, \qquad y(x) = \mathbb{E}_x\{Y\} \quad (x \in \mathbb{R}). \tag{B 1}$$

This result highlights the significance of information about bounded harmonic functions in the asymptotic characterization of Markov chains, especially with regard to transience *vs.* recurrence. For a Borel set $B \subset \mathbb{R}$, put $N_B := \sum_{n=0}^{\infty} \mathbb{1}_B(X_n)$ ($=$ the total number of visits to B). Since $\{N_B = \infty\} \in \mathfrak{I}$, by (B 1) the function $h_B(x) := \mathbb{P}_x(N_B = \infty)$ is (bounded) harmonic. We say [16, §2.3] that B is *transient* if $h_B(x) \equiv 0$ and *recurrent* (or *Harris recurrent* [54, Ch. 9]) if $h_B(x) \equiv 1$. The following criterion is valid [16, p. 58, Proposition 3.8]: in order that *every (Borel) set be either transient or recurrent*, it is necessary and sufficient that either of the two equivalent conditions hold: (i) $\mathfrak{I}$ is trivial (up to a.s.-equivalence); (ii) all bounded harmonic functions are constant.

This criterion can be illustrated by our result on the existence of a non-constant (continuous) solution of the AE (1.2) in the supercritical case $K > 0$, described in Theorem 1.1(b). Namely, take $B = (b, \infty)$ ($b \in \mathbb{R}$), then by inspection of the proof (see details in [38]) it is evident that the function $h_B(x)$ coincides with the solution $F_\Upsilon(x) = \mathbb{P}(\Upsilon \le x)$, where the random series Υ is defined in the theorem; thus, any such B is neither transient nor recurrent.

Ethics statement. The paper does not include any contents regarding animals or human subjects.

Data accessibility. This paper is theoretical and contains no data.

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