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THE STABILITY OF LINEAR MULTI-PASS PROCESSES

by

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Abstract

The recent contribution by Edwards to the stability analysis of multi-pass processes using the familiar inverse-Nyquist method is discussed using the techniques of functional analysis. It is noted that the modelling procedure suggested by Edwards neglects the finite pass length nature of the processes and takes no accout of the initial conditions for each pass. A natural and physically meaningful definition of multi-pass stability is proposed and characterized by conditions on the system operator. Application of the results to a cogging process and a class of linear, time-invariant system indicates that previous results are highly pessimistic. The anomaly is explained in terms of a defined notion of stability along the pass.

1. INTRODUCTION

In a recent paper[1], the general characteristics of multipass processes were discussed and an approach to stability analysis was suggested using the well-known inverse Nyquist method. It was claimed that the processes suffer from stability problems arising from their multi-pass nature and stability criteria were suggested for several situations including long-wall coal-cutting, ploughing and metal rolling.

The essential characteristic of a multi-pass process can be illustrated[1] by consideration of machining operations where the material, or workpiece, involved is processed by a sequence of passes of the processing tool. Assuming that the pass-length α is constant[1], the output vector $y_k(x)$, $o \leqslant x \leqslant \alpha$ (x = 'distance' variable), generated during the k^{th} pass acts as a forcing function on the next pass and hence contributes to the dynamics of the new output $y_{k+1}(x)$, $o \leqslant x \leqslant \alpha$. For example, a simplified scalar model of a cogging process[1] with control action takes the form

$$y_{k+1}(x) = - k_1 \, y_{k+1}(x - X) + k_2 \, y_k(x) + k_1 \, r(x)$$
$$X > o \quad , \quad o \leqslant x \leqslant \alpha \quad , \quad k \geqslant o \qquad (1)$$

where X is the sensor delay, k_1, k_2 are real parameters and $r(x)$ is the desired system output. The process is assumed to be subject to the initial conditions

$$y_k(x) = o \quad , \quad -X \leqslant x \leqslant o \qquad (2)$$

The essence of the stability analysis previously suggested[1] is to convert the system into an infinite single-pass process by expressing the relationships between the process by a single coordinate $\hat{x} = k\alpha + x$ and identify $y_k(x)$ as a function $y(\hat{x})$ defined for , $o \leqslant \hat{x} < + \infty$. The variable $\hat{x}$ can be interpreted as the total distance traversed. Equation (1) is then expressed in the form,

$$y(\hat{x}) = -k_1 y(\hat{x} - X) + k_2\, y(\hat{x} - \alpha) + k_1 r(\hat{x}) \qquad (3)$$

and the multipass process of equations (1) - (2) is said to be stable if, and only if, the system of equation (3) is asymptotically stable. It is disturbing to note, however, that the initial conditions for each pass (equation (2)) are totally neglected in this approach. Intuitively the 'resetting' action of the initial conditions on each pass could be a form of 'stabilizing' action preventing the growth of disturbances which suggests that it is necessary in these cases to recognise the true multipass nature of the problem. In the authors opinion, this observation requires that previous work(1) should be subjected to careful scrutiny and interpretation to highlight what is meant by the term multi-pass stability.

The analysis presented in this paper considers the problem of the definition of stability of a linear multipass process and the characterisation of stability in terms of properties of the system operators. A distinction is made between the concept of multi-pass stability and the idea of stability along the pass, and application of the results is illustrated by a consideration of the cogging process of equations (1)-(2) and a general form of linear, time-invariant multi-pass process. The approach used is that of functional analysis(2) which has the advantage of revealing the essential structural properties of the process and enables the discussion of a large class of processes using the same mathematical results and intuitive concepts.

2. THE STABILITY OF LINEAR MULTI-PASS PROCESSES

2.1 An Abstract Model of a Linear Multi-pass Process.

The multipass process of equations (1) - (2) can be regarded as a linear relation between elements of a Banach space E_α of continuous, complex-valued functions on the interval $o \leqslant x \leqslant \alpha$ satisfying the constraint $y(o) = 0$. The norm on E_α is taken to be

$$||y||_\alpha \triangleq \sup_{o \leq x \leq \alpha} |y(x)| \tag{4}$$

The system equations can be written in the form

$$y_\alpha(k + 1) = L_\alpha y_\alpha(k) + b_\alpha \quad , \quad k \geq o \tag{5}$$

where $y_\alpha(k) \in E_\alpha$ represents the pass profile on the k^{th} pass, $b_\alpha \in E_\alpha$ is the term representing the effect of reference (or known disturbance) signals and L_α is a linear operator in E_α , bounded in the sense that[(2)]

$$||L\alpha||_\alpha \triangleq \sup_{||y||_\alpha \leq 1} \frac{||L_\alpha y||_\alpha}{||y||_\alpha} < +\infty \tag{6}$$

With this motivation, the following definition is taken as a characterisation of a general linear multipass process.

Definition 2.1

A linear multi-pass process $S(L_\alpha)$ of pass length α consists of a Banach space E_α, a linear subspace W_α of E_α and a bounded linear operator L_α of E_α into itself. The dynamics of the process are described by the recursion relations of equation (5) where the initial profile $y_\alpha(o) \in E_\alpha$ and the disturbance term $b_\alpha \in W_\alpha$.

This definition is of a general nature, including the process (1) - (2) as a special case and, as will be seen later, many other processes of practical interest.

2.2 Multi-pass Stability

The simplest and most natural definition of the stability of the linear multipass process defined by equation (5) is to demand that, given any initial profile $y_\alpha(o) \in E_\alpha$ and known disturbance $b_\alpha \in W_\alpha$, the sequence of pass profiles $y_\alpha(k)$, $k \geq o$ 'settles down' to an equilibrium profile $y_\alpha(\infty)$ as $k \to +\infty$, satisfying

$$y_\alpha(\infty) = L_\alpha y_\alpha(\infty) + b_\alpha \tag{7}$$

More precisely, the sequence $y_\alpha(k)$, $k \geqslant o$ is required to converge to $y_\alpha(\infty)$ in the sense of the norm on E_α:

$$\lim_{k\to\infty} ||y_\alpha(k) - y_\alpha(\infty)||_\alpha = 0 \tag{8}$$

(when, in fact, $y_\alpha(\infty)$ does satisfy equation (7)).

Although the above definition has a strong intuitive motivation, it is recognized that, in practical applications, the effect of modelling errors and uncertainties will produce more uncertainty in the structure of L_α and hence the following definition of stability is preferred.

Definition 2.2

A linear multipass process $S(L_\alpha)$ is said to be uniformly asymptotically stable if there exists a real number $\delta > o$ such that, given any initial profile $y_\alpha(o) \in E_\alpha$ and disturbance $b_\alpha \in W_\alpha$, the sequence $y_\alpha(k)$, $k \geqslant o$, generated by the recursion relations

$$y_\alpha(k + 1) = (L_\alpha + \gamma)\, y_\alpha(k) + b_\alpha \tag{9}$$

converges strongly (i.e. with respect to the norm) for all bounded linear operators γ mapping E_α into itself and satisfying $||\gamma||_\alpha \leqslant \delta$.

Intuitively, the definition requires that the process settles down to an equilibrium profile despite the prescence of small modelling errors. The use of the term asymptotically stable can be justified by considering the case when $b_\alpha = 0$ and $\gamma = L_\alpha \delta/||L_\alpha||_\alpha$. By definition, the sequence $(L_\alpha + \gamma)^k y_\alpha(o) = y_\alpha(k)$ has a strong limit as $k \to +\infty$ for all $y_\alpha(o) \in E_\alpha$. The sequence is hence bounded for all $y_\alpha(o) \in E_\alpha$. Applying the Banach-Steinhaus theorem[(2)] there exists a real number $M_\alpha > 0$ such that $||(L_\alpha + \gamma)^k||_\alpha \leqslant M_\alpha$, $k \geqslant o$, i.e.

$$\left(1 + \frac{\delta}{||L_\alpha||_\alpha}\right)^k ||L_\alpha^k||_\alpha \leqslant M_\alpha \quad , \quad k \geqslant o \tag{10}$$

and hence, returning to the real system, when $\gamma = o$, taking $b_\alpha = o$, and defining $\lambda_\alpha = (1 + \frac{\delta}{||L_\alpha||_\alpha})^{-1} < 1$,

$$
\begin{aligned}
||y_\alpha(k)||_\alpha &= ||L_\alpha^k \, y_\alpha(o)||_\alpha \\
&\leqslant ||L_\alpha^k||_\alpha \cdot ||y_\alpha(o)||_\alpha \\
&\leqslant M_\alpha \lambda_\alpha^k \, ||y_\alpha(o)||_\alpha
\end{aligned}
\tag{11}
$$

so that, in the absence of disturbance terms, $y_\alpha(k)$ converges strongly to zero for all initial profiles. In physical terms the effect of the initial profile is attenuated after a large number of passes.

A superficial consideration of equation (5) indicates a similarity between the structure of the multi-pass process $S(L_\alpha)$ and the well-known linear, time-invariant discrete time-system. In this sense, the stability of the process can be expected to depend explicitly on the spectrum[2] of L_α. The spectrum $\sigma(L_\alpha)$ can be defined[2] to be the smallest subset of the complex plane such that $\lambda \notin \sigma(L_\alpha)$ implies that the bounded linear operator $\lambda I - L_\alpha$ (I = identity map in E_α) has range dense in E_α and a bounded inverse $(\lambda I - L_\alpha)^{-1}$. The spectral radius $r_\infty(L_\alpha)$ is defined[2] by

$$
r_\infty(L_\alpha) = \sup_{\lambda \in \sigma(L_\alpha)} |\lambda| \tag{12}
$$

and is positive and finite. An equivalent expression for the spectral radius is[2]

$$
r_\infty(L_\alpha) = \lim_{k \to \infty} ||L_\alpha^k||_\alpha^{1/k} \tag{13}
$$

The following theorem provides an explicit characterisation of the uniform asymptotic stability of $S(L_\alpha)$.

Theorem 2.1

$S(L_\alpha)$ is uniformly asymptotically stable if, and only if, $r_\infty(L_\alpha) < 1$.

Proof

If $S(L_\alpha)$ is uniformly asymptotically stable then equation (10) with $\lambda_\alpha = (1 + \frac{\delta}{||L_\alpha||_\alpha})^{-1}$ indicates that

$$r_\infty(L_\alpha) = \lim_{k\to\infty} ||L_\alpha^k||_\alpha^{1/k} \leqslant \lim_{k\to\infty} M_\alpha^{1/k}\lambda_\alpha = \lambda_\alpha < 1 \qquad (14)$$

Conversely, if $r_\infty(L_\alpha) < 1$, then there exists a real number $\varepsilon > 0$ such that $r_\infty(L_\alpha) + \varepsilon < 1$, and a real number $\delta > 0$ such that $||\gamma||_\alpha < \delta$ implies that

$$r_\infty(L_\alpha + \gamma) < r_\infty(L_\alpha) + \varepsilon < 1 \qquad (15)$$

Using (13) it follows directly that there exists an integer $N_\gamma(\varepsilon) \geqslant 1$ such that $||(L_\alpha + \gamma)^k||_\alpha \leqslant (r_\infty(L_\alpha) + \varepsilon)^k$, $k \geqslant N_\gamma(\varepsilon)$. Considering the solution of equation (9)

$$y_\alpha(k) = (L_\alpha + \gamma)^k\, y_\alpha(o) + \sum_{i=1}^{k} (L_\alpha + \gamma)^{i-1}\, b_\alpha \qquad (16)$$

then

$$||y_\alpha(k)|| < ||(L_\alpha + \gamma)^k||_\alpha \cdot ||y_\alpha(o)||_\alpha + \sum_{i=1}^{k} ||(L_\alpha + \gamma)^{i-1}||_\alpha \cdot ||b||_\alpha \qquad (17)$$

which converges absolutely as $k \to +\infty$ i.e. $y_\alpha(k)$, $k \geqslant o$, is a Cauchy sequence in E_α and hence has a strong limit $y_\alpha(\infty) \in E_\alpha$. This completes the proof of the result.

The above result is obvious of E_α is finite dimensional when L_α can be represented by a complex square matrix and $r_\infty(L_\alpha)$ is simply the maximum of the moduli of the eigenvalues of L_α. In this case the stability criterion requires that all the eigenvalue of L_α lie in the open unit disc in the complex plane.

2.3 Uniform Asymptotic Stability of a Cogging Process[1]

Consider the multipass process by equations (1),(2), with $W_\alpha = E_\alpha$ as defined in section 2.1. The operator L_α is defined by the relation

$$y_{k+1}(x) = -k_1\, y_{k+1}(x-X) + k_2 y_k(x)$$

$$y_k(x) = o \quad , \quad -X \leqslant x \leqslant o \quad . \qquad (18)$$

It is easily checked that L_α maps E_α into itself and is bounded. The calculation of the spectral values of L_α is undertaken by considering solutions (if they exist) of the relations

$$(\lambda I - L_\alpha)y = y_o \quad , y_o \in E_\alpha \tag{19}$$

Noting that $X > 0$ by assumption, there exists an integer $n \geq 1$ such that $(n-1)X < \alpha \leq nX$ i.e. using equation (18), equation (19) takes the form,

$$(\lambda I_n - K)\begin{bmatrix} y(x) \\ y(X+x) \\ \vdots \\ y((n-1)X+x) \end{bmatrix} = \begin{bmatrix} y_o(x) \\ y_o(X+x) \\ \vdots \\ y_o((n-1)X+x) \end{bmatrix} \quad , o \leq x < X \tag{20}$$

where I_n is the nxn unit matrix, and the nxn matrix K is lower triangular of the form,

$$\begin{aligned} K_{ii} &= k_2 \quad &, 1 \leq i \leq n \\ K_{i+1,j+1} &= K_{i,j} \quad &, 1 \leq i,j \leq n-1 \end{aligned} \tag{21}$$

If $\lambda \neq k_2$ then $(\lambda I_n - K)$ is invertible so that equation (20) specifies $y(x)$ uniquely point by point, $o \leq x \leq \alpha$. In particular

$$(\lambda - k_2)\, y(x) = y_o(x) \quad , o \leq x < X \tag{22}$$

so that $y(o) = y_o(o) = 0$. Equation (21) can also be used to show that $y(x)$ is continuous on $o \leq x \leq \alpha$ i.e. $y \in E_\alpha$ and $(\lambda I - L_\alpha)$ has range dense in E_α and a bounded inverse. The only remaining candidate for a spectral value of L_α is $\lambda = k_2$. In this case $(\lambda I_n - K)$ is singular and hence $(\lambda I - L_\alpha)$ cannot have range dense in E_α i.e.

$$r_\infty(L_\alpha) = |k_2|$$

Applying theorem 2.1., the system is uniformly asymptotically stable if, and only if,

$$|k_2| < 1 \tag{23}$$

It is noted that the stability is independent of pass length α (provided it is finite) and the stability conditions differ significantly from that predicted previously[1] (equation 48). A more detailed discussion of this discrepancy is given in section 3. At this stage it is sufficient to state that the difference arises from the inclusion of the initial conditions at the beginning of each pass and the natural definition 2.1, of the concept of uniform asymptotic stability.

2.4 Linear Time-invariant Multi-pass Processes

Consider a multipass process described by the linear, time-invariant model[1]

$$\dot{x}_k(t) = Ax_k(t) + B_1 y_k(t) + B_2 r(t) \quad , \quad x_k(o) = x_o$$
$$y_{k+1}(t) = C\, x_k(t) + Dy_k(t) \tag{24}$$

where $x(t) \in R^n$, $y_k(t) \in R^m$, $k \geq o$, and $o \leq t \leq \alpha < +\infty$. An equivalent formulation is,

$$y_{k+1}(t) = C\, e^{At} x_o + Dy_k(t) + \int_o^t C\, e^{A(t-s)} \{B_1 y_k(s) + B_2 r(s)\} ds \tag{25}$$

Considering the problem in the context of the Banach space $E_\alpha = C_m(o,\alpha)$ of continuous mappings from the interval , $o \leq t \leq \alpha$, into the vector space C^m of complex m-vectors, with norm

$$||y||_\alpha \triangleq \sup_{o \leq t \leq \alpha} ||y(t)||^1 \tag{26}$$

($||\cdot||^1$ is any suitable norm in C^m), then L_α is defined by the relation

$$(L_\alpha y)\ (t) = D\, y(t) + \int_o^t C\, e^{A(t-s)}\, B_1 y(s) ds \quad , \quad o \leq t \leq \alpha \tag{27}$$

Calculation of the spectrum of L_α is undertaken by consideration of the solutions of equation (19). Let λ_j , $1 \leq j \leq m$, be the eigenvalues of D and write equation (19) in the form

$$(\lambda I_m - D)\, y(t) - \int_o^t C\, e^{A(t-s)}\, B_1\, y(s) ds = y_o(t) \quad , \quad o \leq t \leq \alpha \tag{28}$$

If $\lambda \neq \lambda_j$, $1 \leq j \leq m$, then $(\lambda I_m - D)$ is invertible and equation (28) can be written in the form

$$\dot{z}(t) = Az(t) + B_1 y(t) \quad , \quad z(o) = o$$
$$y(t) = (\lambda I_m - D)^{-1} \{Cz(t) + y_o(t)\} \quad , \quad o \leq t \leq \alpha \tag{29}$$

After some rearrangement,

$$\dot{z}(t) = \{A + B_1(\lambda I_m - D)^{-1} C\} z(t) + B_1(\lambda I_m - D)^{-1}\, y_o(t)$$
$$y(t) = (\lambda I_m - D)^{-1} \{Cz(t) + y_o(t)\} \tag{30}$$

i.e. $(\lambda I - L_\alpha)$ has range equal to E_α and a bounded inverse and hence

$\sigma(L_\alpha) \subset \{\lambda_j\}_{1 \leq j \leq m}$. It follows directly that a sufficient condition for uniform asymptotic stability is that $\max_{1\leq j\leq m} |\lambda_j| < 1$. To prove this is necessary, consider the sequence defined by equation (24) for $x_o = o$, $t = o$

$$y_k(o) = D^k y_o(o) \quad , \quad k \geq o \tag{31}$$

If the process is uniformly asymptotically stable, this sequence should have a strong limit independent of any 'small' perturbation to D i.e. it is necessary that $\max_{1\leq j\leq m} |\lambda_j| < 1$. Combining the results with theorem 2.1 it follows that a necessary and sufficient condition for the multi-pass process of equation (24) to be uniformly asymptotically stable is that

$$r_\infty(L_\alpha) = \max_{1\leq j\leq m} |\lambda_j| < 1 \tag{32}$$

It is interesting to note that uniform asymptotic stability is independent of A, B_1, B_2, C and, in particular, independent of the eigenvalues of A. In fact, if D = 0, the process is <u>always</u> uniformly asymptotically stable. The equilibrium profile is obtained by setting $y_{k+1}(t) = y_k(t) = y_\infty(t)$ in equation (24) and rewriting the expression in the form

$$\begin{aligned} \dot{x}(t) &= \{A + B_1(I_m - D)^{-1}C\}\, x(t) + B_2 r(t) \;, \; x(o) = x_o \\ y_\infty(t) &= (I_m - D)^{-1} C x(t) \end{aligned} \tag{33}$$

3. ALONG-PASS STABILITY

The notion of uniform asymptotic stability discussed in section 2 is, in essence, a mathematical formulation of the idea that the process settles down to some equilibrium profile satisfying equation (7), convergence being guaranteed despite the presence of small modelling or simulation errors. For example, considering the linear time-invariant multi-pass process of equation (24) then, if the eigenvalue of D lie in the open unit disc in the complex plane, the process settles down to an equilibrium profile described by equation (33). The equilibrium profile can be used to obtain information on the along pass dynamics and to reveal control difficulties or the need for control action. For example, consider the simple multi-pass process described by the relations

$$\dot{x}_k(t) = -x_k(t) + (1+\beta)y_k(t) + r(t) \quad , \; x_k(o) = 0$$

$$y_{k+1}(t) = x_k(t) \quad , \; o \leq t \leq \alpha < +\infty \qquad (34)$$

with reference signal $r(t) \equiv 1$. Using the initial profile $y_o(t) = o$, $o \leq t \leq \alpha$, the profile along the first pass is given by

$$y_1(t) = 1 - e^{-t} \quad , \quad o \leq t \leq \alpha \qquad (35)$$

i.e. the process appears to be following the demand signal. The process is uniformly asymptotically stable ($D = o$) and has limit profile satisfying,

$$\dot{y}_\infty(t) = \beta\, y_\infty(t) + 1 \qquad y_\infty(o) = 0$$

i.e.
$$y_\infty(t) = \beta^{-1}\{ e^{\beta t} - 1\} \quad , \; o \leq t \leq \alpha \qquad (36)$$

If β is large and positive then, although the process settles down to a bounded limit profile on $o \leq t \leq \alpha$, the limit profile does not track, the demand signal. The process can hence be regarded as being 'unstable along the pass' in some sense. An intuitive approach to the notion of along-pass stability for the general linear, time-invariant process of equation (24) is to demand that the limit profile (equation (33)) be stable as $\alpha \to +\infty$ i.e. the eigenvalues of $A + B_1(I_m - D)^{-1}C$ all have negative real parts. In general, however, multi-pass processes can possess a form of longitudinal interaction(3) due, for example, to smoothing effects between the interpass dynamics. In such cases, this intuitive approach can no longer be applied. The following discussion suggests a definition of along-pass stability based on the rate of disturbance rejection as the pass lengths $\alpha \to +\infty$.

Consider for simplicity, the multi-pass process,

$$\dot{y}_{k+1}(t) = y_k(t) \quad , \quad o \leq t \leq \alpha < +\infty \quad , \alpha \geq 1$$
$$y_o(t) \equiv 1 \quad , \quad y_k(o) = o \quad , \quad k \geq 1 \tag{37}$$

in the complex Banach space $E_\alpha = C_1(o,\alpha)$. It is easily shown that $y_k(t) = t^k/k!$, $o \leq t \leq \alpha$ and hence $||y_k||_\alpha = \alpha^k/k! \to o \ (k\to+\infty)$. The process is uniformly asymptotically stable but, if the pass length α is increased, the rate of rejection of the initial disturbance $y_o(t)$ is reduced. For the general multi-pass process, with $b_\alpha = o$,

$$y_\alpha(k+1) = L_\alpha y_\alpha(k)$$
i.e $$y_\alpha(k) = L_\alpha^k y_\alpha(o) \quad , \quad k \geq o \quad , \tag{38}$$

If the system is uniformly asymptotically stable then (equation (10)) there exists $M_\alpha > 0$ and $0 < \lambda_\alpha < 1$ such that $||L_\alpha^k||_\alpha \leq M_\alpha \lambda_\alpha^k$ and hence

$$||y_\alpha(k)||_\alpha \leq M_\alpha \lambda_\alpha^k ||y_\alpha(o)||_\alpha \quad , \quad k \geq o \tag{39}$$

Equivalently, the process rejects the initial disturbance at a rate described by a geometric progression of upper bounds.

Considering a family $\{S(L_\alpha)\}_{\alpha \geq \alpha_o}$ of models of the process (termed an extended multi-pass process) obtained by modelling the dynamics over a physical range of pass lengths $\alpha_o \leq \alpha < +\infty$, then each model will be characterized by real numbers M_α, λ_α.

Definition 3.1

The extended multi-pass process $\{S(L_\alpha)\}_{\alpha \geq \alpha_o}$ is said to be stable along the pass if there exists real numbers $M_\infty > 0$, $0 < \lambda_\infty < 1$, such that

$$||L_\alpha^k||_\alpha \leq M_\infty \lambda_\infty^k \quad , \quad k \geq o \quad , \alpha \geq \alpha_o \tag{40}$$

In effect the definition demands that the disturbance rejection rate for each element of the extended process has a guaranteed geometric upper bound independent of the length of the process. Note, in particular that

$$r_\infty(L_\alpha) = \lim_{k\to\infty} ||L_\alpha^k||_\alpha^{1/k} \leqslant \lim_{k\to\infty} M_\infty^{1/k}\lambda_\infty = \lambda_\infty < 1 \quad (41)$$

and hence a necessary condition for the system to be stable along the pass is that $S(L_\alpha)$ is uniformly asymptotically stable for all $\alpha \geqslant \alpha_o$.

Theorem 3.1

The extended multi-pass process $\{S(L_\alpha)\}_{\alpha \geqslant \alpha_o}$ is stable along the pass if, and only if,

(a) $r_\infty \triangleq \sup_{\alpha\geqslant\alpha_o} r_\infty(L_\alpha) < 1$

(b) $M \triangleq \sup_{\alpha\geqslant\alpha_o} \sup_{|\eta|=\lambda} ||(\eta I - L_\alpha)^{-1}||_\alpha < +\infty$ for some real number λ in the range $r_\infty < \lambda < 1$

Proof

To prove necessity, note that equation (41) implies that $r_\infty \leqslant \lambda_\infty < 1$. Also

$$||(\eta I - L_\alpha)^{-1}||_\alpha = |\eta|^{-1}||\sum_{i=o}^{\infty} \frac{1}{\eta^i} L_\alpha^i ||$$

$$\leqslant \frac{1}{|\eta|} \sum_{i=o}^{\infty} \frac{1}{|\eta|^i} ||L_\alpha^i||_\alpha$$

$$\leqslant \frac{1}{|\eta|} M_\infty \sum_{i=o}^{\infty} \frac{\lambda_\infty^i}{|\eta|^i} = \frac{M_\infty}{\lambda(1-\lambda_\infty/\lambda)} \quad (42)$$

if we choose $|\eta| = \lambda$, $\lambda > \lambda_\infty$. To prove sufficiency, suppose that $r_\infty < 1$ and consider the contour C in the complex plane defined by the relation $|\eta| = \lambda$, $r_\infty < \lambda < 1$. Writing[2]

$$L_\alpha^k = \frac{1}{2\pi i} \int_c \eta^k (\eta I - L_\alpha)^{-1}\, d\eta \quad (43)$$

where $\eta=\lambda e^{i\theta}$ and hence, taking norms, $||L_\alpha^k||_\alpha \leqslant (M\lambda)\lambda^k$ which proves the result.

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The proof of necessity immediately indicates that

Corollary 3.1

A necessary condition for $\{S(L_\alpha)\}_{\alpha\geq\alpha_o}$ to be stable along the pass is that

$$\sup_{\alpha\geq\alpha_o} \sup_{|\eta|\geq 1} ||(\eta I - L_\alpha)^{-1}||_\alpha < +\infty$$

To illustrate the application of the result consider the process defined by equation (1)-(2) and discussed in detail in section 2.3. Note that $r_\infty(L_\alpha) = |k_2|$, $\alpha \geq \alpha_o$ so that $r_\infty = |k_2|$ and $|k_2| < 1$ is required for stability along the pass. Consider the solution of the relation,

$$(\eta I - L_\alpha)y_\alpha = r_\alpha \quad , \quad r_\alpha \in E_\alpha \tag{44}$$

or, equivalently,

$$\begin{aligned} \eta y_\alpha(x) - \hat{y}_\alpha(x) &= r_\alpha(x) \\ \hat{y}_\alpha(x) &= -k_1\, \hat{y}_\alpha(x-X) + k_2 y_\alpha(x) \quad , \quad o \leq x \leq \alpha \\ \hat{y}_\alpha(x) &= r_\alpha(x) = y_\alpha(x) = o, \quad -X \leq x \leq o \end{aligned} \tag{45}$$

After some rearranging

$$y_\alpha(x) = -\frac{\eta k_1}{\eta - k_2}\, y_\alpha(x-X) + \frac{\{k_1 r_\alpha(x-X) + r_\alpha(x)\}}{\eta - k_2} \tag{46}$$

A necessary and sufficient condition for condition (b) of theorem 3.1 to hold is hence that there exists a real number λ in the range $|k_2| < \lambda < 1$ such that

$$\sup_{|\eta|=\lambda} \left| \frac{-\eta k_1}{\eta - k_2} \right| < 1 \tag{47}$$

or, equivalently, the process is stable along the pass if, and only if,

$$|k_2| < 1 \qquad\qquad |k_1| < 1 - |k_2| \tag{48}$$

Equation (48) is the stability criterion obtained by Edwards[1]. It is seen therefore that the modelling and stability analysis techniques suggested by Edwards here, in fact, produced a stability criterion involving the idea of along-pass stability and the notion of disturbance rejection. In the authors opinion, this type of stability is very strong and may, in many practical applications, be unnecessary and possibly unobtainable. In most cases the uniform asymptotic stability of the system, coupled perhaps with control action to produce an acceptable limit profile will produce acceptable dynamic characteristics.

Finally, consider the problem of characterising the stability along the pass of the linear time-invariable multi-pass process of equation (24). It is easily seen that $r_\infty = \max_{1 \leq j \leq m} |\lambda_j|$ which is necessarily less than unity. The solution of (44) in this case can be represented by the differential equations

$$\dot{z}(t) = (A + B_1(\eta I_m - D)^{-1}C)\, z(t) + B_1(\eta I_m - D)^{-1}\, r_\alpha(t) \quad , \; z(o) = o$$

$$y_\alpha(t) = (\eta I_m - D)^{-1}\, \{Cz(t) + r_\alpha(t)\} \;, \; o \leq t \leq \alpha \qquad (49)$$

and hence, a necessary and sufficient condition for condition (b) of theorem 3.1 to hold is that there exists a real number λ, $r_\infty < \lambda < 1$, such that the eigenvalues of $A + B_1(\eta I_m - D)^{-1}C$ have strictly negative real parts for all $|\eta| = \lambda$. In particular, applying corollary 3.1 and letting $|\eta| \to \infty$, it is necessary that A be a stability matrix.

The stability along the pass can hence be represented by the characteristic polynomial

$$\begin{aligned} |sI_n - A - B_1\,(\eta I_m - D)^{-1}C| &= |sI_n - A| \,.\, |I_n - (sI_n - A)^{-1}B_1(\eta I_m - D)^{-1}C| \\ &= |sI_n - A| \,.\, |I_m - C(sI-A)^{-1}\, B_1\,(\eta I_m - D)^{-1}| \\ &= \frac{|sI_n - A|}{|\eta I_m - D|} \,.\, |\eta I_m - \{D + C(sI_n - A)^{-1}B_1\}| \end{aligned} \qquad (50)$$

Equivalently, defining the transfer function matrix

$$G(s) = D + C\,(sI_n - A)^{-1}B_1 \qquad (51)$$

then, a necessary and sufficient condition for stability along the pass is that A is a stability matrix and there exists a real number λ, $r_\infty < \lambda < 1$ such that the zeros of the numerator of $|\eta I_m - G(s)|$ have negative real parts for all $|\eta| = \lambda$. A convenient technique for checking this criterion is to calculate the characteristic loci[4] $q_i(s)$, $1 \leq j \leq m$, of G(s) and note that

$$|\eta I_m - G(s)| = \prod_{j=1}^{m} (\eta - q_i(s)) \tag{52}$$

In particular, if m = 1, it follows directly from the Nyquist stability criterion that the frequency response $G(i\omega)$, $o \leq \omega < +\infty$ should not intersect the circle $|\eta| = \lambda$. That is, if m = 1, a necessary and sufficient condition for stability along the pass is that A is a stability matrix and

$$|G(j\omega)| < 1 \quad , \quad o \leq \omega < +\infty \tag{53}$$

The physical interpretation of this criterion can be obtained by taking the Laplace transform of equation (24) with zero initial conditions and $r(t) \equiv 0$, when , $y_{k+1}(s) = G(s)y_k(s)$ i.e. the process can only be stable along the pass if each frequency component is attenuated from pass to pass. Edwards(1) has suggested a similar result on intuitive grounds only. The above analysis provides a rigorous treatment of the problem, and provides a rigorous mathematical and intuitive formulation of the result.

4. CONCLUSIONS

Following the work by Edwards(1) on the stability of linear multipass process, it has been noted that the modelling approach and stability analysis suggested(1) suffers from the neglect of pass initial conditions and neglects the essential finite length nature of the processes by the use of anapproximate single pass model of infinite length . A natural definition (Definition 2.1) of multipass stability has been proposed by demanding that the system settles down to an equilibrium profile for all possible initial pass profiles and reference signals and independent of small system modelling errors. Necessary and sufficient conditions for multi-pass stability have been derived in terms of the spectral radius of the system operator and the results illustrated by application to a cogging process considered by Edwards[1] and a general class of linear, time-invariant multipass processes. Some surprising results are obtained:

(a) In the case of the multi-pass cogging process, the results differ significantly from those obtained by Edwards[1] indicating that the concept of stability used by Edwards differs from the natural definition used in this paper. In particular, the results obtained by Edwards require much stronger conditions indicating that his results are pesimistic.

(b) In the case of the linear, time-invariant multi-pass process, stability depends only upon the high frequency component of the system dynamics. This suggests in particular that, a multi-pass process is stable from pass to pass if it can be represented by a low pass filter. (i.e. $D = o$).

In an attempt to explain these features, a concept of stability along the pass (Definition 3.1) has been introduced based on the idea of disturbance rejection as the pass length $\alpha \to +\infty$. Necessary and sufficient conditions for stability along the pass have been derived. In the case of the cogging process it is seen that Edwards results[1] are equivalent to a condition for along pass stability. In the case of linear time-invariant multi-pass processes, a necessary and sufficient condition for stability along the pass is that the process is stable along the first pass as $\alpha \to +\infty$ and that the frequency response of the system lies in the open unit circle in the complex plane.

Finally, as the results are presented in the abstract language of functional analysis, the theory can be applied to multipass processes of more complex type[3] involving interpass smoothing effects and delay-differential models.