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BOUNDEDNESS PROPERTIES OF NONLINEAR MULTIVARIABLE FEEDBACK SYSTEMS

by

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Abstract

A boundedness theorem is derived for a commonly encountered class of memoryless nonlinearities in multivariable feedback systems. The result can be useful when absolute stability cannot be proved or when investigating the existence of limit cycles.

The determination of the stability of nonlinear feedback systems is a fundamental problem of systems and control theory. Several exact and approximate methods are known for stability assessment for single variable systems. For applications in engineering, the techniques based on frequency domain concepts are very often the most useful. These include absolute stability methods such as the Popov and circle criteria[1] and approximate methods such as the use of describing functions[2] and the Aizerman and Kalman conjectures[3,4]. In addition, when absolute stability is not guaranteed, several criteria are known for assessing the frequency of possible limit cycles[5-6] whilst others assess the boundedness of the autonomous[7] system response. Recently, significant efforts have been made to extend these procedures to multivariable systems[2,8,9].

All mathematical models of physical systems are valid only over a limited range of system variables. It is often important, however, to be able to obtain boundedness results for specific mathematical representations. If, for example boundedness of the response (but not necessarily absolute stability) can be proved, then techniques such as the use of the describing function (really a method for the evaluation of limit cycles) are more easily justified for stability investigations. In addition, sophisticated computational techniques for the precise evaluation of periodic modes in general nonlinear[10,11] and relay systems can be invoked[12].

The contribution of this note is to produce a boundedness result for multivariable feedback systems containing a commonly encountered class of nonlinearities. The feedback system is shown in Fig. 1 with output y and

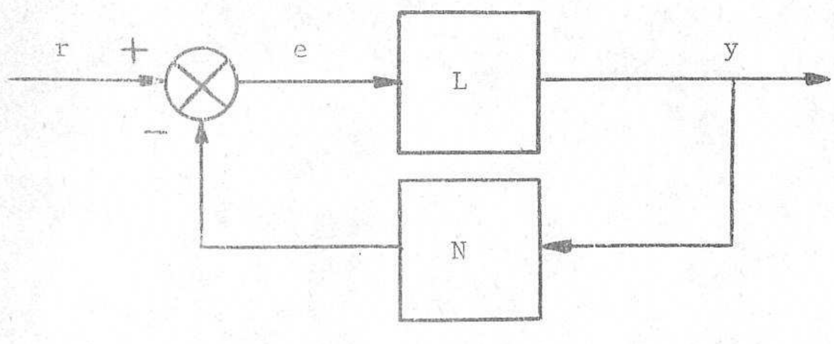

Fig. 1

demand signal r. The linear element L is assumed to take the form of a (possibly unstable) state space model S(A,B,C) of the form

$$\dot{x}(t) = A\,x(t) + B\,e(t) \quad , \quad x(t) \in R^n$$
$$y(t) = C\,x(t) \quad y(t) \in R^m, \quad e(t) \in R^m \tag{1}$$

with some arbitrary initial condition $x(o) \in R^n$. The nonlinear element N is taken to be a time-invariant, memoryless element described by the map $f : R^m \to R^m$ of the form $f(y) = [f_1(y_1,\ldots,y_m),\ldots,f_m(y_1,\ldots,y_m)]$ with the property that

$$f(y) = Ky + \varepsilon(y) \tag{2}$$

where K is a constant mxm matrix and there exists a real constant $M_1 > 0$ such that

$$\max_{1\leq j\leq m} \sup_{y \in R^m} |\varepsilon_j(y)| \leq M_1 \tag{3}$$

More compactly, if $||y||_m \triangleq \max_{1\leq j\leq m} |y_j|$ is taken as norm on R^m, the nonlinearity satisfies the condition

$$\sup_{y \in R^m} ||f(y) - Ky||_m \leq M_1 \tag{4}$$

Typical examples of such nonlinearities are when each of the f_i is a deadzone and/or measurement quantization effect. These are commonly encountered in practice and often induce limit cycle behaviour. The following result however ensures the boundedness of the response.

Theorem 1

If the linear system S(A - BKC,B,C) obtained by replacing the nonlinearity by the linear gain K is asymptotically stable, then the solutions of the nonlinear feedback system

$$\dot{x}(t) = Ax(t) + B(r(t) - f(y(t)))$$
$$y(t) = Cx(t) \tag{5}$$

are bounded for each choice of bounded input r(t) and initial condition x(o).

Proof

Suppose that $x_L(t)$ and $y_L(t)$ are the responses of the linear system $S(A - BKC, B, C)$ to the demand $r(t)$ from the initial condition. Asymptotic stability guarantees the existence of scalars $M_2 > 0$, $M_3 > 0$ such that

$$\sup_{t>o} ||y_L(t)||_m < M_2 ||x(o)||_m + M_3 \sup_{t\geq o} ||r(t)||_m \qquad (6)$$

If $x(t) \triangleq x_L(t) + \tilde{x}(t)$ and $y(t) \triangleq y_L(t) + \tilde{y}(t)$ then it is easily verified that $\tilde{x}(t)$ and $\tilde{y}(t)$ are the state and output responses of $S(A - BKC, B, C)$ from zero initial conditions to the 'input' $Ky(t) - f(y(t))$. Using (4) and (6)

$$\sup_{t\geq o} ||\tilde{y}(t)||_m \leq M_3 \sup_{t\geq o} ||Ky(t) - f(y(t))||_m$$

$$\leq M_3 M_1 < +\infty \qquad (7)$$

so that $\tilde{y}$ is bounded. It follows directly that $y = y_L + \tilde{y}$ is bounded if r is bounded which proves the result.

A particular case of interest is as follows and is a partial generalization of theorem 4 of Vogt[7] to the multivariable case.

Theorem 2

Theorem 1 is valid if f is continuous and

$$\lim_{R\to+\infty} \sup_{||y||_m \geq R} ||\varepsilon(y)||_m = 0 \qquad (8)$$

(In effect, the closed-loop responses are bounded if the nonlinearity 'looks linear' at uniformly high' input).

Proof

Choose $R > 0$ such that $\sup_{||y||_m \geq R} ||\varepsilon(y)||_m < 1$ and note that the continuity of f ensures the boundedness of $\varepsilon = f - K$ on the compact ball $||y||_m < R$.

Probably the most useful interpretation of the results is for a system with a diagonal nonlinearity of the form

$$f_i(y_1,\ldots,y_m) = k_i y_i + \varepsilon_i(y_i)$$

$$\sup_{y_i} |\varepsilon_i(y_i)| \leq M_1 \quad , \quad 1 \leq i \leq m \tag{9}$$

when $K = \mathrm{diag}\,\{k_1,k_2,\ldots,k_m\}$. In such cases it may be particularly simple to interpret theorem 1 in terms of **desirable properties of the linear element** L. Consider, for example, the case of $m = 2$ and suppose that the linear element L has been designed such that the unity negative feedback system (obtained from Fig. 1) by replacing the nonlinearity by a unity gain) is stable. Suppose also that, at implementation, the measurement of y_2 is subject to saturation. We can represent this situation by the introduction of the diagonal nonlinearity $f(y) = [y_1, f_2(y_2)]^T$ where f_2 is a unit slope element saturating at $|y_2| = M_1$, say. More precisely the condition of equation (4) is satisfied with $\varepsilon(y) = [0,\ f_2(y_2)]^T$ and

$$K = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{10}$$

when theorem 1 states that the closed-loop nonlinear system has a bounded response if the original linear feedback system is stable in the presence of sensor failure in loop two[13].

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