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DISCRETE FIRST-ORDER MODELS FOR MULTIVARIABLE PROCESS CONTROL

by

D.H. Owens B.Sc., A.R.C.S., Ph.D., A.F.I.M.A.

Lecturer in the Department of Control Engineering,
University of Sheffield,
Mappin Street,
Sheffield S1 3JD.

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Abstract

The concept of an mxm invertible continuous first order lag is extended to define an equivalent formulation for multivariable sampled-data-systems. A large class of proportional plus summation output feedback controllers is constructed. Each controller guarantees the stability of the closed-loop system and also low-closed-loop interaction effects if the sampling rate is high enough. The results are extended to show that a multivariable discrete first order lag is, in many cases of practical interest, a quite adequate approximation for the purpose of controller design provided that the plant is minimum-phase and satisfies a contraction-mapping condition. In particular, any discrete model of a minimum-phase, continuous, linear, time-invariant plant with CB nonsingular will satisfy the contraction condition provided the sampling rate is high enough.

1. Introduction

It is well-known in classical feedback control that many large-scale continuous linear, time-invariant systems can be approximated, for the purpose of feedback design, by a low-order state-space model due to the presence of approximately cancelling pole-zero pairs in the system transfer function. Furthermore, the validity of such approximations can improve in the closed-loop situation due to the attraction of poles to system zeros. As might be expected intuitively, these ideas can be generalized to the multivariable case. Consider a unity negative feedback system for the control of the m-input, m-output plant described by the mxm transfer function matrix $G(s)$ and let $K(s)$ be the mxm forward path controller. The generalization of the classical concepts requires the solution of the following problems if it is to have any direct relevance to design studies:

(a) Relationships must be derived between $G(s)$, $K(s)$ and the reduced plant $G_{\Lambda}(s)$, ensuring that the stability of the unity feedback system with forward path transfer function matrix $G_{\Lambda}(s)\ K(s)$ guarantees that the original feedback system is stable. In this situation $G_{\Lambda}(s)$ can be confidently used as a basis for the design of $K(s)$.

(b) A large class of reduced model structures must be constructed whose parameters are easily identified in terms of plant dynamics. They should be general enough to enable the systematic and useful approximation of a large class of engineering systems but simple enough to enable direct analytic techniques to be used in the search for an appropriate control structure $K(s)$.

A useful solution to (a) has been suggested[1,2] based on the technical machinery of contraction mapping algorithms on Banach spaces of matrix-valued functions of the complex variable, analytic and bounded on a defined open set in the complex plane. The solution to (b) is not so apparent. A partial solution has been suggested based on the use of reduced models with polynomial matrix inverses(3-6). For example, if $G(s)$ takes the form

$$G^{-1}(s) = s\,A_o + A_1 + A_o\,H_o(s) \quad , \quad |A_o| \neq 0 \ , \ H(o) = 0 \qquad (1)$$

where $H_o(s)$ is proper and asymptotically stable (i.e. $G(s)$ is minimum-phase), then the multivariable first-order lag(3-6)

$$G_A^{-1}(s) = s\,A_o + A_1 \qquad (2)$$

can be a suitable approximation for feedback design studies(6), and matches both the high and low frequency characteristics of the original plant.

This paper considers the natural generalization of these ideas to the case of unity negative feedback control of controllable and observable m-input, m-output linear, time invariant discrete systems with state space models $S(\Phi,\Delta,C)$ of the form

$$x_{k+1} = \Phi\,x_k + \Delta u_k \quad , \quad x_k \in R^n$$

$$y_k = C\,x_k \ , \ y_k \in R^m \ , \ u_k \in R^m \quad , \quad k \geq 0 \qquad (3)$$

with mxm z-transfer function matrix

$$G(z) = C(z\,I_n - \Phi)^{-1}\Delta \qquad (4)$$

The results should have application, for example, to the design of simple computer control schemes for multivariable process plant using periodically sampled data and piecewise-constant control.

The fundamental approximation theorem is derived in section 2. In section 3 the concept of a multivariable discrete first-order lag is introduced and the stability of the closed-loop system ensured by the choice of suitable proportional or proportional plus summation forward path controllers. It is demonstrated that low interaction properties of the closed-loop system in response to unit step demands is only achieved if the sampling rate is fast when compared with the open-loop poles of the underlying continuous system.

In section 4 the results of sections 2 and 3 are combined It is shown that a large class of mxm minimum-phase discrete plant can be successfully approximated by a discrete first-order lag provided that the underlying continuous system possesses certain simple structural properties and that the sampling rate is high enough.

2. Feedback Stability and Fundamental Approximation Theorem

Consider the mxm discrete system $Q(z)$ subjected to unity negative feedback. For simplicity, suppose that $Q(z)$ is derived from the discrete model $S(\Phi,\Delta,C)$. The open-loop characteristic polynomial

$$\rho_o(z) = |z\, I_n - \Phi| \tag{5}$$

and the closed-loop characteristic polynomial

$$\rho_c(z) = |z\, I_n - \Phi + \Delta C| \tag{6}$$

are related by the well-known formula

$$\frac{\rho_c(z)}{\rho_o(z)} = |I_m + Q(z)| \tag{7}$$

The closed loop system is asymptotically stable if, and only if, the roots of $\rho_c(z)$ lie in the open unit circle in the complex plane.

The zero polynomial of $S(\Phi, \Delta, C)$ can be defined, by analogy with continuous systems[3], to be the polynomial in complex variable z,

$$\phi(z) \triangleq \rho_o(z)\ |Q(z)| = \begin{vmatrix} z\, I_n - \Phi & -\Delta \\ C & 0 \end{vmatrix} \tag{8}$$

The discrete system will be termed <u>invertible</u> if $\phi(z) \not\equiv 0$ or, equivalently $|Q(z)| \not\equiv 0$ when $Q^{-1}(z)$ exists and is termed the <u>inverse system</u>. If $Q(z)$ is invertible then the polynomial $\phi(z)$ has only a finite number of roots. These roots (including multiplicities) will be termed the <u>zeros</u> of $S(\Phi,\Delta,C)$. The system will be termed 'minimum-phase' if, and only if, all the zeros lie in the open unit circle in the complex plane. An important observation is that the inverse system is analytic and bounded in the region $|z| > 1$ if, and only if, $Q(z)$ is minimum phase.

We are now in a position to state the fundamental theorem of this section.

<u>Theorem 2.1.</u>

Suppose that the controllable and observable mxm invertible, minimum-phase discrete system $Q(z)$ is to be approximated by the mxm invertible, minimum-phase discrete system $Q_A(z)$. Suppose that $Q_A(z)$ is stable in the presence of unity negative feedback and that the poles of the closed-loop system generated by $Q(z)$ (subject to unity negative feedback) lie in the open ball $|z| < R$ where $R > 1$. Then the system $Q(z)$ is stable in the presence of unity negative feedback if

$$||(I_m + Q_A^{-1}(z))^{-1}(Q_A^{-1}(z) - Q^{-1}(z))|| < 1$$

(the contraction condition) (9)

where, if $L(z)$ is any mxm matrix function of z,

$$||L(z)|| \triangleq \max_{1\leq j\leq m} \max_{\substack{|z|=1\\|z|=R}} \sum_{i=1}^{m} |L_{ji}(z)| \tag{10}$$

<u>Proof</u>

The proof is a direct parallel of the constructions in references 1 and 2 for continuous feedback systems. The controllability and observability assumption ensures that the closed-loop system is asymptotically stable if, and only if, the matrix $(I_m + Q(z))^{-1}Q(z) \equiv$

$(I_m + Q^{-1}(z))^{-1}$ is analytic and bounded in the region $1 < |z| < R$ i.e. if, and only if, the solution of

$$y(z) = - Q^{-1} y(z) + \alpha \tag{11}$$

is analytic and bounded in $1 < |z| < R$ independent of the choice of $\alpha \in R^m$. Writing equation (11) in the form,

$$y = (I + Q_A^{-1})^{-1}((Q_A^{-1} - Q^{-1})y + \alpha) \tag{12}$$

the remainder of the proof is identical to previous methods[2] with the Nyquist contour and its interior replaced by the closed ring $1 \leq |z| \leq R$.

Q.E.D.

In practice R is, of course, unknown. This does not limit the applicability of the result as we can always take $R >> 1$.

Given Q and Q_A, the contraction condition (equation (9)) can, in principle, be checked numerically. The primary value of equation (9) in this paper is, however, that of a theoretical tool for closed-loop stability analysis based on some approximation Q_A to Q. In practice $Q(z) = G(z)K(z)$ where G and K are the z-transfer function matrices of the plant and forward path controllers respectively. We will use the definition $Q_A(z) = G_A(z)K(z)$ where $G_A(z)$ is some approximation to the plant dynamics. In the case of G, G_A and K all minimum phase and supposing that K is designed on the basis of the approximate plant G_A to ensure closed-loop stability, then the contraction condition (equation (9)) provides sufficient conditions for the design K to generate a stable closed-loop system for the real plant G.

Unfortunately, the results described above provide no guidelines for the choice of G_A. In the next section a useful class of approximate plant is analysed in detail.

3. Multivariable Discrete First-order Lags

By an obvious extension of previous results [3-5], we define an mxm discrete first order lag to be a controllable and observable m-input, m-output discrete time system with inverse transfer function matrix

$$G^{-1}(z) = (z-1)\, B_o + B_1 \tag{13}$$

where B_o, B_1 are real mxm matrices and $|B_o| \neq 0$. An equivalent definition is that of an m-input, m-output discrete model of state dimension n = m and $|C\Delta| \neq 0$ as is easily verified by noting that G(z) has a minimal realization of the form

$$C = I_m \quad , \quad \Phi = I_m - B_o^{-1} B_1 \quad , \quad \Delta = B_o^{-1} \tag{14}$$

Proportional Control:

Consider a unity negative-feedback system for the control of the first-order lag G(z) with forward path proportional controller of the general parametric form,

$$K(z) = B_o \operatorname{diag}\{1 - k_j\}_{1 \leq j \leq m} - B_1 \tag{15}$$

Noting that the open-loop characteristic polynomial

$$\rho_o(z) = |zI_m - \Phi| \equiv |B_o^{-1}|\,|G^{-1}(z)| \tag{16}$$

it follows directly that the closed-loop characteristic polynomial

$$\rho_c(z) = \rho_o(s)\ |I_m + G(z)\ K(z)|$$

$$= |B_o^{-1}|\ |G^{-1}(z) + K(z)|$$

$$= \prod_{j=1}^{m} (z - k_j) \qquad (17)$$

i.e. the closed-loop system is asymptotically stable if, and only if, $-1 < k_j < 1 \quad , \quad 1 \leqslant j \leqslant m.$

The closed-loop transient performance can be assessed by evaluation of the closed-loop transfer function matrix

$$H_c(z) = (I_m + G(z)K(z))^{-1}\ G(z)K(z)$$

$$= (G^{-1}(z) + K(z))^{-1}\ K(z)$$

$$= \mathrm{diag}\{\frac{1}{z - k_j}\}_{1 \leqslant j \leqslant m} \quad (\mathrm{diag}\ \{1-k_j\}_{1 \leqslant j \leqslant m} - B_o^{-1}B_1) \qquad (18)$$

In particular the closed-loop system possesses small steady state errors and small interaction effects in response to unit step demands only if the elements of the matrix $B_o^{-1}B_1$ are 'small enough'. This is not a severe restriction on the practical application of the results if it is remembered that, a priori or a postiori, the discrete model defined by equation (14) can be regarded as being derived from a continuous time model S(A,B,C) of the form,

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad , \quad x(t) \in R^n$$

$$y(t) = Cx(t)\ ,\ y(t) \in R^m \qquad , \quad u(t) \in R^m \qquad (19)$$

with sampling period h,

$$x_k \triangleq x(kh) \quad k \geqslant o \quad , \qquad (20)$$

and piecewise constant input

$$u(t) = u_k \quad , \quad kh \leqslant t < (k+1)\ h \qquad (21)$$

if such a plant is subject to the constraints $n = m$ and $|CB| \neq 0$.

Without loss of generality, suppose that $C = I_m$ and note that

$$\Phi = e^{Ah} \quad , \quad \Delta = \Phi \int_0^h e^{-At} B dt \tag{22}$$

Substituting into equation (14) it follows that

$$\lim_{h \to 0+} B_0^{-1} B_1 = \lim_{h \to 0+} \{I_m - e^{Ah}\} = 0 \tag{23}$$

and hence that the closed-loop system represented by equation (18) will possess small interaction effects and steady state errors in response to unit step demands if the sampling rate is fast enough. Assuming, for simplicity, that A has a diagonal canonical form with eigenvalues λ_j, $1 \leq j \leq m$, and eigenvector matrix E, then

$$\Phi = E \ \mathrm{diag}\ \{e^{\lambda_j h}\}_{1 \leq j \leq m} E^{-1} \tag{24}$$

suggesting that a necessary condition for $B_0^{-1} B_1$ to be small is that

$$|\lambda_j h| << 1 \quad , \quad 1 \leq j \leq m \tag{25}$$

Equivalently the sampling rate must be fast in comparison to the poles of the underlying continuous <u>open-loop</u> plant.

<u>Proportional plus Summation Control:</u>

The above analysis is easily extended to the case of controllable and observable proportional plus summation controllers of the general parametric form

$$K(z) = B_0 \ \mathrm{diag} \left\{ 1 - k_j c_j + \frac{(1-k_j)(1-c_j)z}{(z-1)} \right\}_{1 \leq j \leq m} - B_1 \tag{26}$$

Suppose that only ℓ of the parameters $c_1, c_2, \ldots, c_m$ are not equal to unity. It follows that the controller has minimal realization of state dimension ℓ and hence that the overall open-loop characteristic polynomial is

$$\rho_0(z) = (z-1)^{\ell} |zI_m - \Phi| = (z-1)^{\ell} |B_0^{-1}| \, |G^{-1}(z)| \tag{27}$$

Note, in particular, that this analysis reduces to the case of proportional control if $\ell = 0$. The stability of the closed-loop system is described by the characteristic polynomial

$$\begin{aligned}\rho_c(z) &= \rho_o(z) \left| I_m + G(z) K(z) \right| \\ &= (z-1)^{\ell} \left| B_o^{-1} \right| \cdot \left| G^{-1}(z) + K(z) \right| \\ &= \{ \prod_{j=1}^{m} (z-k_j) \} \{ \prod_{c_j \neq 1} (z-c_j) \} \qquad (28)\end{aligned}$$

i.e.the closedloop system is asymptotically stable if, and only if, $-1 < k_j < 1$, $1 \leq j \leq m$ and $-1 < c_j \leq 1$, $1 \leq j \leq m$.

The inclusion of summation action would normally be considered only in cases where the proportional design displayed large steady state errors. Also the reset times will normally be considerably longer than the rise times. In mathematical terms it is likely that

$$|1 - c_j| << |1 - k_j| \quad , \quad 1 \leq j \leq m \qquad (29)$$

and hence that equation (18) is still a good representation of closed-loop dynamics with the exception that steady state errors in response to step demands will be zero in all outputs $y_j(t)$ corresponding to parameters $c_j \neq 1$ as is easily verified by noting that

$$\begin{aligned}\lim_{z \to 1} H_c(z) &= \lim_{z \to 1} (G^{-1}(z) + K(z))^{-1} K(z) \\ &= I_m - M B_o^{-1} B_1 \qquad (30)\end{aligned}$$

where

$$M = \text{diag}\,\{m_j\} \quad 1 \leq j \leq m$$

$$m_j = \begin{cases} 0 & , \quad c_j \neq 1 \\ \dfrac{1}{1 - k_j} & , \quad c_j = 1 \end{cases} \qquad (31)$$

On the Effect of Delays:

The above analysis represents a rather idealized concept of control action. In practice there may be significant delays in the control loops due, for example, to measurement or actuator delays in the system. Intuitively these delays will not adversely affect system performance provided that the sampling rate is fast compared with the closed-loop system responses. This fact can be illustrated by considering the presence of measurement delays represented by the simple diagonal matrix

$$F(z) = z^{-p} I_m \quad (p \geq o) \tag{32}$$

in the feedback path. Using, for simplicity, the proportional controller of equation (15) the closed-loop characteristic polynomial is given by

$$\begin{aligned} \rho_c(z) &= z^{mp} |B_o^{-1}| |G^{-1}(z)| \cdot |I_m + G(z)K(z)F(z)| \\ &= z^{mp} |B_o^{-1}| \cdot |G^{-1}(z) + K(z)F(z)| \\ &= z^{mp} |(z-1)I_m + B_o^{-1}B_1 + z^{-p}\{diag\{1-k_j\}_{1 \leq j \leq m} - B_o^{-1}B_1\}| \\ &= |diag\{z^p(z-1)+1-k_j\}_{1 \leq j \leq m} + \{z^p-1\}B_o^{-1}B_1| \end{aligned} \tag{33}$$

Equation (23) indicates that, at all high enough sampling rates, we can neglect the term in $B_o^{-1}B_1$ yielding

$$\rho_c(z) \simeq \prod_{j=1}^{m} (z^p(z-1)+(1-k_j)) \qquad (h \to 0+) \tag{34}$$

Considering the solutions of $z^p(z-1) + 1 - k_j = 0$ in the vicinity of $z = k_j = 1$, the relation

$$\left.\frac{dz}{dk_j}\right|_{z = k_j = 1} = 1 \tag{35}$$

guarantees the existence of positive real numbers $\varepsilon_j > 0$, $1 \leq j \leq m$, such that the roots of $\rho_c(z)$ lie in the open unit circle for

$1 - \varepsilon_j < k_j < 1$, $1 \leq j \leq m$, and all high enough sampling rates.

4. First-order Approximations for Feedback Design

It is the purpose of this section to bring the analyses of sections 2 and 3 together in a systematic approach to design based on first order approximations to open-loop plant dynamics.

Consider a unity negative feedback system for the control of the mxm invertible, minimum-phase discrete plant $S(\Phi,\Delta,C)$ having inverse z-transfer function matrix

$$G^{-1}(z) = B_o(z-1) + B_1 + B_o H(z)$$

$$H(z) \text{ proper, , } H(1) = 0 \quad , \quad |B_o| \neq 0 \tag{36}$$

and suppose that the mxm first order lag $G_A(z)$ defined by

$$G_A^{-1}(z) = (z-1)B_o + B_1 \tag{37}$$

is to be used as a reduced model for the purposes of controller design. Note that G_A is a good approximation to the high frequency and steady state behaviour of the plant, but contains no information on its zero structure. In fact, G_A has no zeros!

Suppose that the analysis of section 3 is used to design a proportional plus summation controller of the form given in equation (26) for the approximate plant G_A. The contraction condition (equation (9)) can now be used to assess the closed-loop stability of the original plant G with the designed control system. The relevant matrix is

$$L(z) \triangleq (I_m + Q_A^{-1}(z))^{-1}(Q_A^{-1}(z)-Q^{-1}(z)) = (K(z)+G_A^{-1}(z))^{-1}(G_A^{-1}(z)-G^{-1}(z))$$

$$= (-1)\ \text{diag}\ \{\frac{(z-1)}{(z-k_j)(z-c_j)}\}_{1\leq j\leq m}\ H(z) \tag{38}$$

and a sufficient condition for closed-loop stability is that $||L(z)|| < 1$. This condition could be directly investigated if the inverse system is computed from the matrix triple (Φ,Δ,C) using known techniques[7], noting that the condition $|B_0| \neq 0$ is equivalent to the condition $|C\Delta| \neq 0$. Note also that the contraction condition can be replaced by the simpler condition,

$$\max_{1\leq j\leq m} \max_{|z|=1} \sum_{i=1}^{m} |L_{ji}(z)| < 1 \tag{39}$$

by letting R tend to $+\infty$ and observing that $L(z) = O(z^{-1})$. In practice, however, it is probably more efficient simply to simulate the closed-loop system responses to a number of unit step demands, say.

The remaining question is whether or not the contraction condition can be satisfied for a large class of process plant. The identification of the important parameters governing the behaviour of $||L||$ is also of particular importance if a detailed state vector model is not available. These questions are answered in the following analysis.

Suppose now that the model $S(\Phi,\Delta,C)$ can be regarded as a discrete representation of the continuous plant $S(A,B,C)$ (see equations (19)-(21)) with sampling period $h > 0$. The following theorem is proven in appendix 8:

<u>Theorem 4.1</u>

Let the continuous system $S(A,B,C)$ be minimum phase and $|CB| \neq 0$. Then for each choice of parameters k_j, c_j, $1 \leq j \leq m$, in the interval $(-1,1)$, there exists a real positive number h^* such that, for $0 < h < h^*$, the discrete system $S(\Phi,\Delta,C)$ is minimum phase with inverse of the form given in (36) and the contraction condition is satisfied.

In effect, the theorem states that the contraction condition is always satisfied (and hence the closed-loop system is stable) if the underlying continuous system is minimum phase, uniform rank one[3] and the sampling rate is fast enough. These conditions are only sufficient, however, and do not necessarily prevent application of the technique if they are not satisfied.

The technique is very easily applied in a step by step manner:

STEP ONE:

Evaluate the matrices B_o, B_1. This is easily done from the matrice Φ,Δ,C using known formulae[7] for the inverse system given in appendix 8. Alternative, if the system is open-loop stable, the matrices can be deduced by simulation or experimental recording of the system response to step inputs with zero initial conditions. More precisely, the output responses from zero initial conditions to the vector step demand α at $k=1$ and $k=+\infty$ are

$$y_1 = B_o^{-1}\alpha \quad , \quad y_\infty = B_1^{-1}\alpha \tag{40}$$

B_o and B_1 are easily evaluated from this data by taking m linearly independent step demands $\alpha_1,\alpha_2,\ldots,\alpha_m$.

STEP TWO:

Choose a proportional plus summation controller of the form discussed in section 3 for the approximate plant obtained from B_o and B_1 (equation (37)) to ensure the required response speed, steady state errors and, if possible, interaction effects.

STEP THREE:

Check the stability of the unity feedback system for the original plant G using K by numerical check of the contraction condition (equations (38),(39)) or, more generally, by simulation or experimental studies. If the system is unstable, consider the possibility of increasing the sampling rate.

The above theory will remain essentially the same in the presence of sensor or actuator delays, provided that the sampling rate is fast relative to the designed closed-loop responses. Also the requirement that H(1) = 0 can be dropped from the specification of the system in equation (36) without altering the results as is verified in Appendix 9. Although this can simplify the analysis of the first-order approximation and the structure of the controller (for example, choose H(z) such that B_1 = 0) it does mean that the reduced model no longer represents the steady state characteristics of the plant.

5. Illustrative Examples

5.1 Control of a First-order Process:

Suppose that the continuous, invertible system specified by the matrix triple

$$A = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad , \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{41}$$

is to be controlled by discrete feedback with sampling period h. Note that the eigenvalues of A are 0 and -3 and hence that the open-loop system real time-constants are all greater than 1/3.

It is easily verified that the derived discrete plant is described by the triple

$$\Phi = \frac{1}{3} \begin{pmatrix} 2 + e^{-3h} & 1 - e^{-3h} \\ 2 - 2e^{-3h} & 1 + 2e^{-3h} \end{pmatrix}$$

$$\Delta = \frac{1}{9} \Phi \begin{pmatrix} 6h + e^{3h} - 1 & 6h + 2 - 2e^{3h} \\ 6h + 2 - 2e^{3h} & 6h + 4e^{3h} - 4 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{42}$$

and is a discrete 2x2 first-order lag with B_o, B_1 obtained from equation (14). Consider the application of the results of section 3 to this system. Glancing at equations (18) and (14) the important matrix in feedback performance is

$$B_o^{-1} B_1 = I - \Phi = \frac{1}{3} \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} \left(1 - e^{-3h}\right) \tag{43}$$

which is small if, and only if, 3h << 1 indicating the need for fast sampling if low closed-loop interaction is desired.

Suppose that the sampling period $h = \frac{1}{30}$ and consider the case of proportional control with forward path controller specified by equation (15). Assuming, for simplicity, that similar response speeds are desired from each loop, then equation (18) suggests the choice of $k_1 = k_2 \triangleq k$ when the closed-loop transfer function matrix takes the form

$$H_c(z) = \frac{(1-k)}{(z-k)} \left\{ I_2 - \frac{1}{(1-k)} \begin{bmatrix} 0.032 & -0.032 \\ -0.064 & 0.064 \end{bmatrix} \right\} \qquad (44)$$

An examination of this identity suggests that the smallest attainable steady state errors and transient interaction effects in response to unit step demands are approximately 0.06 (obtained by examining the case of a dead-beat closed-loop system with k=o) when the closed-loop system real-time response time is of the order of the sampling period. If, however, it is required that the closed-loop response time is much longer than the sampling period, it is immediately apparent (by letting $k \rightarrow 1-$) from equation (44) that interaction effects and steady state errors can be large.

Choosing, for simplicity, $k = 0.5$ the closed-loop responses to unit step demands are shown in Fig. 1. Note that interaction and steady state errors are of peak magnitude 0.13. The steady state errors can be eliminated by the introduction of a summation term into the controller (see equation (26)) with parameters $c_1 = c_2 \triangleq c = 0.9$, say. The resulting closed-loop unit step responses are also shown in Fig. 1.

The final design is highly satisfactory and can be improved by reducing k (i.e. increasing response speeds). The design is ultimately limited, however by the assumed sampling rate.

Finally the forward path controller has z-transfer function matrix

$$K(z) = \{1 - kc + \frac{(1-k)(1-c)z}{(z-1)}\} \quad B_o - B_1$$

$$= \{0.55 + \frac{0.05z}{(z-1)}\} \begin{bmatrix} 30.5 & -0.5 \\ -0.5 & 15.5 \end{bmatrix} - \begin{bmatrix} 1.0 & -1.0 \\ -1.015 & 1.015 \end{bmatrix}$$

$$= \begin{bmatrix} 15.775 + \frac{1.525z}{(z-1)} & 0.725 - \frac{0.025z}{(z-1)} \\ 0.725 - \frac{0.025z}{(z-1)} & 7.51 + \frac{0.775z}{(z-1)} \end{bmatrix} \qquad (45)$$

which could be simplified for implementation by neglecting the small off-diagonal terms.

5.2 First-order Approximation:

Consider a system with discrete model

$$x_{k+1} = \begin{bmatrix} 0.908 & 0.002 & 0.04 \\ -0.026 & 0.906 & 0.06 \\ 0.15 & 0.05 & 0.9 \end{bmatrix} x_k + \begin{bmatrix} 0.0328 & 0.0011 \\ 0.0011 & 0.0646 \\ 0 & 0 \end{bmatrix} u_k$$

$$y_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x_k \quad , \qquad k \geqslant 0 \qquad (46)$$

which is minimum-phase (with one zero at $z = 0.9$) and invertible with inverse system of the form (after some elementary manipulation)

$$G^{-1}(z) = (z-1) \begin{bmatrix} 30.5 & -0.5 \\ -0.5 & 15.5 \end{bmatrix} + \begin{bmatrix} 1.0 & -1.0 \\ -1.015 & 1.015 \end{bmatrix}$$

$$+ \begin{bmatrix} 30.5 & -0.5 \\ -0.5 & 15.5 \end{bmatrix} \frac{(z-1)}{(z-0.9)} \begin{bmatrix} 0.06 & 0.02 \\ 0.09 & 0.03 \end{bmatrix} \qquad (47)$$

so that G^{-1} has the form given in equation (36) with

$$B_o = \begin{bmatrix} 30.5 & -0.5 \\ -0.5 & 15.5 \end{bmatrix} \quad , \quad B_1 = \begin{bmatrix} 1.0 & -1.0 \\ -1.015 & 1.015 \end{bmatrix}$$

$$H(z) = \frac{(z-1)}{(z-0.9)} \begin{bmatrix} 0.06 & 0.02 \\ 0.09 & 0.03 \end{bmatrix} \qquad (48)$$

Consider the application of the approximation method of section (4) to this system. The first order approximation to plant dynamics is

$$G_A^{-1}(z) = (z-1)\begin{bmatrix} 30.5 & -0.5 \\ -0.5 & 15.5 \end{bmatrix} + \begin{bmatrix} 1.0 & -1.0 \\ -1.015 & 1.015 \end{bmatrix} \qquad (49)$$

which is identical to the system considered in section 5.1. Assuming, for simplicity, the case of proportional control with $k_1 = k_2 = 0.5$ (as in section 5.1), the forward path controller (equation (15))

$$K(z) = \begin{bmatrix} 30.5 & -0.5 \\ -0.5 & 15.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} - \begin{bmatrix} 1.0 & -1.0 \\ -1.015 & 1.015 \end{bmatrix}$$

$$= \begin{bmatrix} 14.25 & 0.75 \\ 0.765 & 6.735 \end{bmatrix} \qquad (50)$$

yields the approximate closed-loop system with closed-loop responses as obtained in section 5.1 and illustrated in Fig. 1.

The closed-loop responses of the real system (equation (46)) with this controller to unit step demands are illustrated in Fig. 2. As can be seen the responses are stable and very similar to those obtained from the first-order approximation. The stability can also be predicted theoretically by evaluating (equations (39)-(39))

$$||L(z)|| = 0.12 \max_{|z|=1} \left| \frac{(z-1)}{(z-0.9)(z-0.5)} \right| \leq 0.2526 < 1 \qquad (51)$$

so that the contraction condition is satisfied and hence the closed-loop system is stable.

6. Conclusions

The material presented in this paper is a complete generalization of the concepts and approximation procedures previously published by the author[2-6] for the design of unity negative feedback systems for mxm invertible, minimum-phase continuous systems using first order approximations, to the discrete case. In particular a detailed analysis of proportional and proportional plus summation controllers for the defined discrete first order lags has been provided. Previous contraction mapping results[2] have been generalized to provide conditions for assessing the usefulness of plant approximations in feedback design and it has been demonstrated that, under well-defined conditions, discrete first-order lags are a useful model for design studies. These conditions relate to the properties of the underlying continuous-time system and, in particular, suggest that the results are only strictly applicable if the sampling rate is fast enough (although it does not seem to be possible to obtain explicit results concerning the required sampling rate). These restrictions are, however, typical of contraction mapping type results and directly parallel the need to use 'sufficiently high gain' in the continuous case[6]. Certainly they should not prevent application of the concepts to a large class of process plant.

7. References

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APPENDIXES

8. Proof of Theorem 4.1

(a) The discrete system $S(\Phi,\Delta,C)$ has an inverse of the form given in equation (36) if, and only if, $C\Delta (= B_o^{-1})$ is nonsingular. Using equation (22),

$$\lim_{h\to o} h^{-1} C\Delta = CB \tag{52}$$

which, by assumption, is nonsingular. It follows that equation (36) is the correct parametric representation of the inverse system at all high enough sampling rates.

(b) Applying the work of Kouvaritakis[7], note that

$$G^{-1}(z) = (z-1)(C\Delta)^{-1} - (C\Delta)^{-1}C\{\Phi-I_n\}\Delta(C\Delta)^{-1}$$
$$- (C\Delta)^{-1}C\{\Phi - I_n\}M(zI_n - N\Phi M)^{-1} N\{\Phi - I_n\}\Delta(C\Delta)^{-1} \tag{53}$$

where N,M are $(n-m)\times m$ and $n\times(n-m)$ full rank matrices respectively satisfying the relations

$$CM = 0 \quad , \quad N\Delta = 0 \quad , \quad NM = I_{n-m} \tag{54}$$

It follows directly that

$$H(z) = C\{\Phi - I_n\} M \left\{ (I_n - N\Phi M)^{-1} - (zI_n - N\Phi M)^{-1} \right\} N\{\Phi - I_n\}\Delta(C\Delta)^{-1}$$

$$= C\{\Phi - I_n\} M(z-1)(I_n - N\Phi M)^{-1}(zI_n - N\Phi M)^{-1}N\{\Phi - I_n\}\Delta(C\Delta)^{-1} \tag{55}$$

Consider the behaviour of $H(z)$ as $h\to o+$ on the unit circle $|z| = 1$. The eigenvalues $\lambda_1,\ldots,\lambda_{n-m}$ of $N\Phi M$ are simply the n-m zeros[7] of $S(\Phi,\Delta,C)$. Equivalently, they are the n-m solutions of the relation

$$z(\lambda) = \begin{vmatrix} \lambda I_n - \Phi & -\Delta \\ C & 0 \end{vmatrix} = 0 \tag{56}$$

Noting that

$$\Phi = I_n + Ah + o(h^2)$$
$$\Delta = hB + o(h^2) \tag{57}$$

it follows that

$$\lambda_j = 1 + z_j h + o(h^2) \quad , \quad 1 \leqslant j \leqslant n-m \tag{58}$$

where $z_1,\ldots,z_{n-m}$ are the zeros of the continuous system $S(A,B,C)$. By assumption $z_1,\ldots,z_{n-m}$ have strictly negative real parts and hence $\{\lambda_j\}$ lie in the open unit circle for all fast enough sampling rates i.e. $S(\Phi,\Delta,C)$ is minimum phase for all fast enough sampling rates.

(c) Without loss of generality, it is possible to assume that M is constant. The assumption that $|CB| \neq 0$ ensures that the limit $N_o \triangleq \lim_{h\to o} N$ exists, that $N_o M = I_{n-m}$ and that $N_o B = 0$. The minimum phase nature of $S(A,B,C)$ ensures(7) that $N_o A M$ is nonsingular. This, together with the equations (52) and (57) and the identity

$$(zI_n - N\,\Phi\,M)^{-1} = ((z-1)I_n - N_o\,A\,M\,h + o(h^2))^{-1} \tag{59}$$

guarantees the existence of $h^* > o$ and $k > o$ such that

$$\sup_{o<h<h^*} ||(z-1)^{-1}H(z)|| < k \tag{60}$$

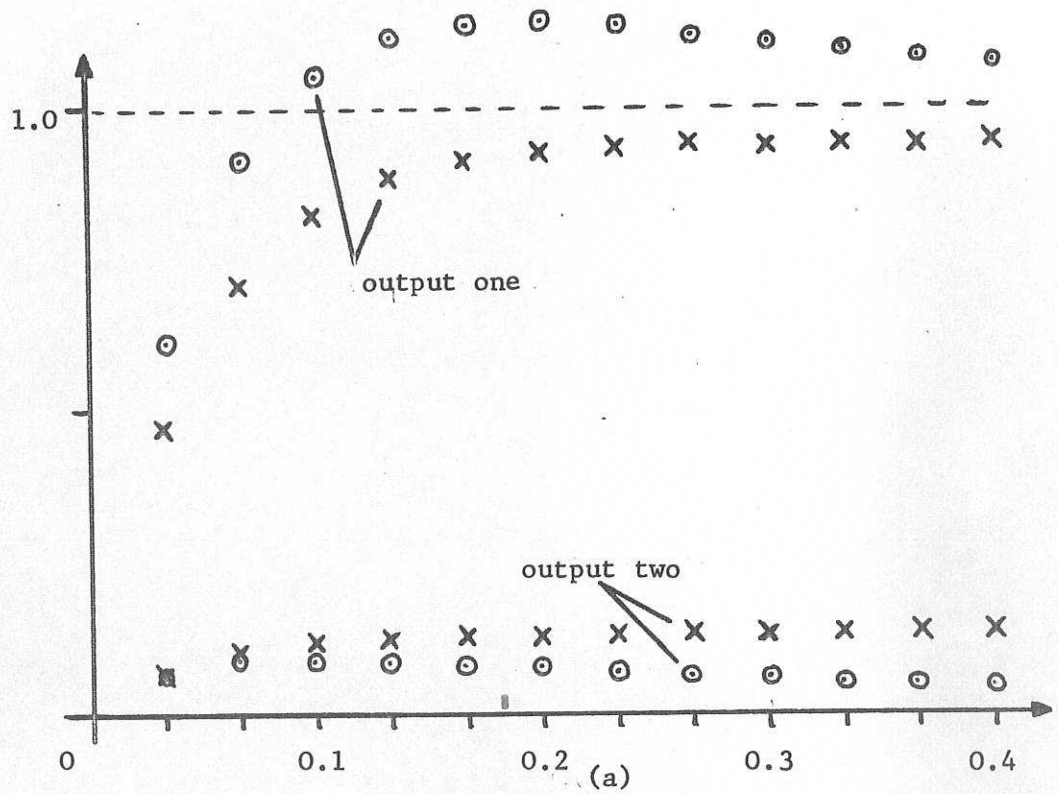
In particular, $(z-1)^{-1}H(z)$ converges uniformly to zero as $h\to o+$ on any relatively open subset T of the unit circle not containing the point $z = 1$/ from which $H(z)$ (and hence $L(z)$) converges uniformly to zero on $|z| = 1$ as $h \to o+$. This proves the desired result as suitable choice of h^* ensures that $||L|| < 1$, $o < h < h^*$.

9. Effect of the Choice of B_1

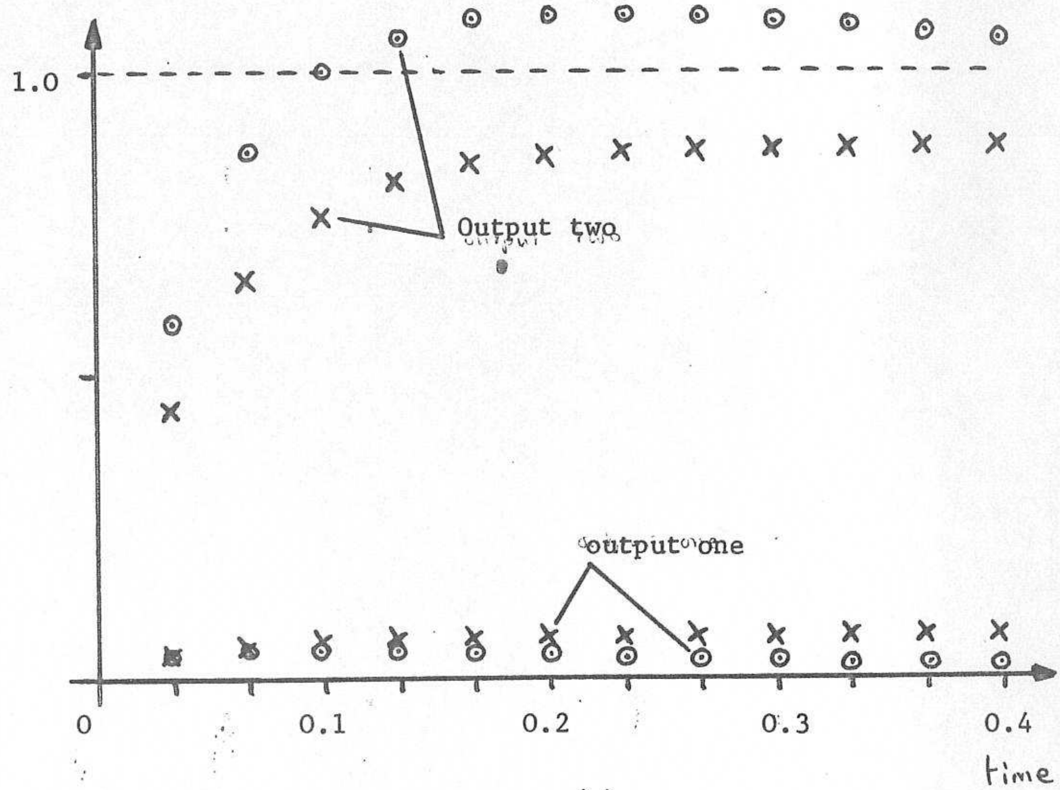
As B_1 represents steady state response characteristics, it follows that it is independent of sampling rate. Let $\tilde{B}_1$ be some other constant matrix used in place of B_1. Noting that the theoretical development is, with the exception of theorem 4.1., independent of the assumption $H(1) = 0$, the analysis follows through with $H(z)$ replaced by $H(z) + B_o^{-1}(B_1 - \tilde{B}_1)$. But $B_o^{-1} = C\Delta = O(h)$ yields

$$\lim_{h \to o+} B_o^{-1}(B_1 - \tilde{B}_1) = 0 \tag{61}$$

It follows that theorem 4.1 is valid if B_1 is replaced by any constant matrix.



(a)



(b)

Fig. 1 Closed loop responses

(a) to a unit step demand in output one

(b) to a unit step demand in output two

x = proportion control, $k_1 = k_2 = 0.5$

⊙ = proportional plus summation control $c_1 = c_2 = 0.9$

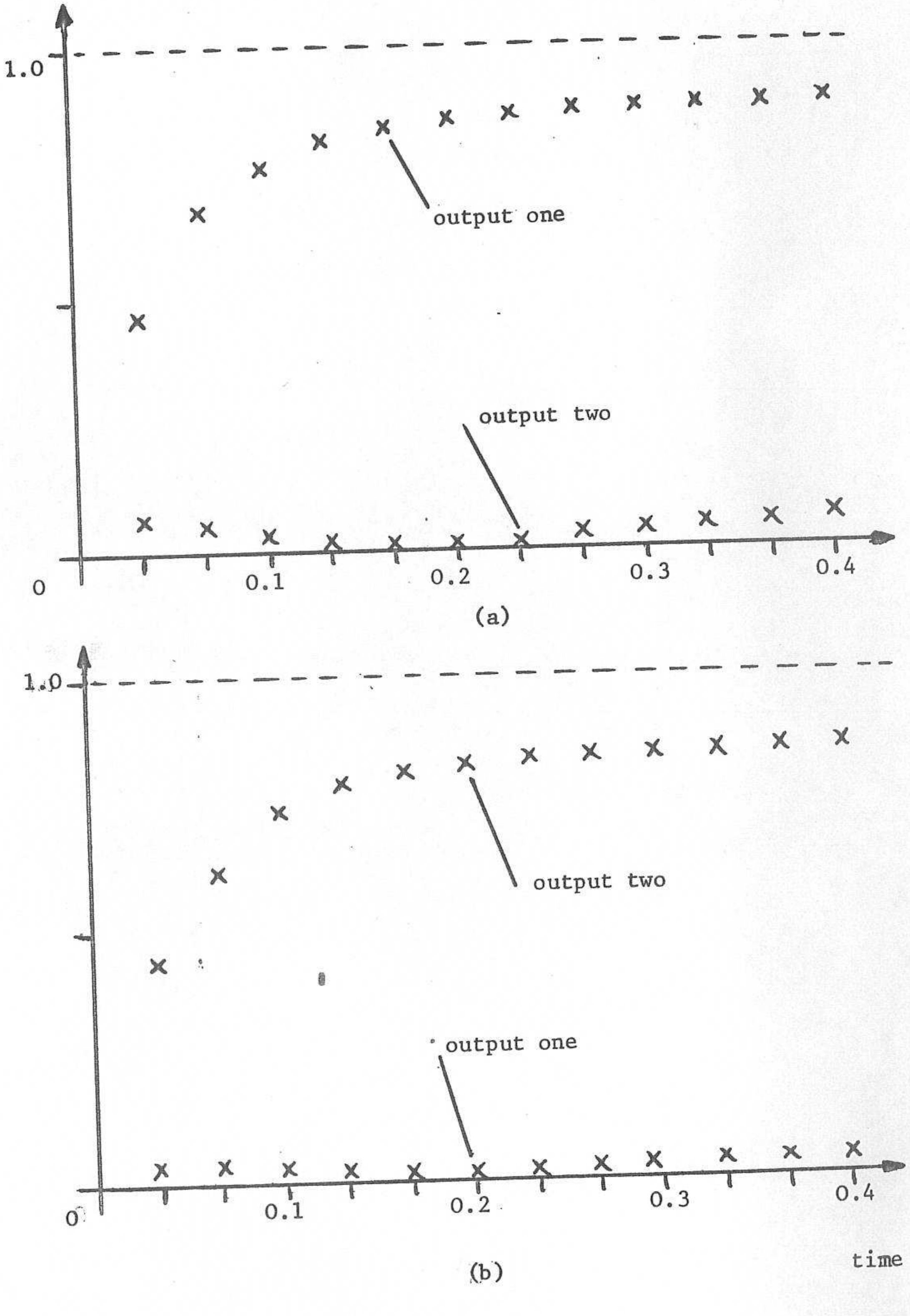


Fig. 2 Closed-loop response

(a) to a unit step demand in output one

(b) to a unit step demand in output two