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REGULARITY OF THE SOLUTIONS TO SPDES IN METRIC MEASURE SPACES

ELENA ISSOGLIO AND MARTINA ZÄHLE

ABSTRACT. In this paper we study the regularity of non-linear parabolic PDEs and stochastic PDEs on metric measure spaces admitting heat kernel estimates. In particular we consider mild function solutions to abstract Cauchy problems and show that the unique solution is Hölder continuous in time with values in a suitable fractional Sobolev space. As this analysis is done via a-priori estimates, we can apply this result to stochastic PDEs on metric measure spaces and solve the equation in a pathwise sense for almost all paths. The main example of noise term is of fractional Brownian type and the metric measure spaces can be classical as well as given by various fractal structures. The whole approach is low dimensional and works for spectral dimensions less than 4.

1. INTRODUCTION

In this paper the following non-linear Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} = -Au + F(u) + G(u) \cdot \dot{z}, & t \in (0, t_0] \\ u(0) = f \end{cases} \quad (1)$$

is considered on σ -finite metric measure spaces (X, μ, d) . Here $t_0 > 0$ is arbitrary, $-A$ is the generator of a *Markovian strongly continuous symmetric semigroup* $\{T(t), t \geq 0\}$ on $L_2(\mu)$, F and G are sufficiently regular functions. The term \dot{z} denotes a fractional space-time perturbation which will be made more precise later on. In the case of linear spaces it can be interpreted as a formal time derivative of a spatial distribution. Solutions to (1) are considered in the mild form, formally given by

$$u(t) = T(t)f + \int_0^t T(t-s)F(u(s))ds + \int_0^t T(t-s)G(u(s))dz(s). \quad (2)$$

This formulation is, in a first place, only formal. We will give it a proper mathematical meaning as it is done in [24], in particular the last term involving the noise z is defined by means of fractional derivatives and it

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is shown to be indeed well defined using the notion of pointwise product of functions and “distributions”. The spaces which we will use to describe the space-regularity of the solution (2) are fractional Sobolev spaces defined on metric measure spaces by means of the associated semigroups.

The main aim of this paper is to show that the mild solution u of the Cauchy problem (1) given by (2) is γ -Hölder continuous in time with respect to the $H^\delta(\mu)$ -norm in space (Theorem 3.4). This result is achieved under the same assumptions as in Theorem 1.2 in [24], where it was shown that a unique solution exists and belongs to the space $W^\gamma([0, t_0], H_\infty^\delta(\mu))$. Moreover, under slightly stricter conditions, we can show that the solution in fact belongs to any space $W^\gamma([0, t_0], H_\infty^\delta(\mu)) \cap C^\gamma([0, t_0], H^\delta(\mu))$ for all γ and δ smaller than certain parameters determined by the regularity properties of the distributional noise z and the initial function f (Corollary 3.6). In the case of noises of fractional Brownian type these parameters are determined by means of the Hurst exponents in space and in time. Note that the fractional time regularity is always greater than $1/2$.

In Remark 3.7 we outline an extension of the results to an appropriate parameter condition for the case of spectral dimensions ≤ 1 , which has not been considered in [24]. In particular, white noise in space can be treated in this low dimensional case.

Deterministic elliptic equations and some parabolic equations without noises on classes of fractals have been studied, e.g. in [1, 7, 8, 13, 36, 9].

Abstract problems with Brownian and fractional Brownian noises have been considered in many papers with various approaches, in particular, in [38, 6, 12, 29, 37, 16, 31, 11]. None of these covers the results of the present paper.

Some relationships have been discussed in [24], see also [21]. To these two references, we only add a brief comparison between the present paper and the rough path approach developed in recent years. For example in a series of papers [18, 19, 17] the rough path approach has been applied to study a stochastic Burger-type equation with multiplicative white noise. Even though the equation considered there is of a different kind than the one studied in this paper (the setting is Euclidean and not metric measure space, the noise is white and not coloured and there is a non-linear product term of the form $g(u)\partial_x u$ which we do not have), it is interesting to notice that one of the main difficulties is to give a meaning to the solution, in particular to the non-linear product term $g(u)\partial_x u$, and this is done using the notion of *paraproduct*.

2. PRELIMINARIES

2.1. Semigroups and potential spaces. Throughout the paper we use the letter c for a general finite constant which might change value

from line to line.

We first recall some basic notions and relationships which are known from the literature. In the case of metric measure spaces the analogues of the classical fractional Sobolev (or Bessel potential) spaces in the literature are introduced by means of the given semigroup $\{T(t), t \geq 0\}$, i.e., of its generator $-A$:

The *generalized Bessel potential operator* on $L_2(\mu)$ is defined for $\sigma \geq 0$ as

$$J^\sigma(\mu) := (A + \text{Id})^{-\sigma/2}.$$

To each operator there corresponds a *potential space* defined as

$$H^\sigma(\mu) := J^\sigma(L_2(\mu))$$

and equipped with the norm $\|u\|_{H^\sigma(\mu)} := \|u\|_{L_2(\mu)} + \|A^{\sigma/2}u\|_{L_2(\mu)}$, which is equivalent to $\|(A + \text{Id})^{\sigma/2}u\|_{L_2(\mu)}$. In fact these spaces correspond to the domains of fractional powers of A , i.e., $D((A + \text{Id})^{\sigma/2}) = D(A^{\sigma/2}) = H^\sigma(\mu)$. In particular, for any $\alpha \geq 0$ the operator J^α acts as an isomorphism between $H^\sigma(\mu)$ and $H^{\alpha+\sigma}(\mu)$. Analogously one can define the potential spaces corresponding to the generators $-A_p, 1 < p < \infty$, of Markovian semigroups on $L_p(\mu)$. They are denoted by $H_p^\sigma(\mu)$ and clearly $H_2^\sigma(\mu) = H^\sigma(\mu)$. We will also consider the spaces

$$H_\infty^\sigma(\mu) := H^\sigma(\mu) \cap L_\infty(\mu)$$

normed by $\|\cdot\|_{H^\sigma(\mu)} + \|\cdot\|_\infty$, with slight abuse of notation. Here the norm $\|\cdot\|_\infty$ in $L_\infty(\mu)$ is given by the essential supremum.

The dual spaces of $H_p^\sigma(\mu)$ will be used in the sequel: for $1 < p < \infty, \sigma \geq 0$ they are denoted by

$$H_{p'}^{-\sigma}(\mu) := (H_p^\sigma(\mu))^*,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. In case $p = 2$ we do not write p explicitly. Note that $H^{-\sigma}(\mu) \subseteq (H_\infty^\sigma(\mu))^*$ often being a strict inclusion.

For the regularity in time of the solution we consider the following spaces frequently used in the literature: Let $0 < \eta < 1$ and $(X, \|\cdot\|_X)$ be a normed space. Then $W^\eta([0, t_0], X)$ denotes the space of functions $v : [0, t_0] \rightarrow X$ such that $\|v\|_{\eta, X} < \infty$, where

$$\|v\|_{\eta, X} := \sup_{0 \leq t \leq t_0} \left(\|v(t)\|_X + \int_0^t \frac{\|v(t) - v(s)\|_X}{(t-s)^{\eta+1}} ds \right)$$

is the norm in $W^\eta([0, t_0], X)$.

In the sequel we assume that 0 is not an eigenvalue of A . This is guaranteed under the additional condition (HKE(β)) below. Then in the definition of the above potential spaces the operators J^σ may be replaced by $A^{-\sigma/2}$.

We will use the *short notations* for the following norms:

$$\|\cdot\|_{\alpha, \infty} := \|\cdot\|_{H_\infty^\alpha(\mu)} \text{ and } \|\cdot\|_\alpha := \|\cdot\|_{H^\alpha(\mu)} \text{ for each } \alpha \in \mathbb{R}.$$

Then we recall that for $\nu \geq 0$ and $t > 0$ the operators $T(t)$ and A^ν commute on $D(A^\nu)$ and satisfy the following well-known estimates (see e.g. [30]) for $u, v \in D(A^\nu)$:

$$\|T(t)v\|_\nu \leq ct^{-\frac{\nu}{2}}\|v\|_0, \quad (3)$$

and for $0 < \nu < 1$,

$$\|T(t)u - u\|_0 \leq ct^\nu\|u\|_{2\nu}. \quad (4)$$

The symmetry of the semigroup $\{T(t), t \geq 0\}$ has been used in order to extend it to elements from the dual spaces. If $w \in H^{-\beta}(\mu)$ then $T(t)w$ is the element of $L_2(\mu)$ determined by the duality relation

$$(v, T(t)w) := (T(t)v, w), \quad v \in L_2(\mu).$$

Then we get

$$|(v, T(t)w)| = |(T(t)v, w)| \leq \|T(t)v\|_\beta\|w\|_{-\beta}$$

and hence,

$$\|T(t)w\|_0 \leq ct^{-\frac{\beta}{2}}\|w\|_{-\beta}$$

in view of (3). Applying the latter again and using $T(t) = T(\frac{t}{2}) \circ T(\frac{t}{2})$ we infer

$$\|T(t)w\|_\delta \leq ct^{-\frac{\delta}{2}-\frac{\beta}{2}}\|w\|_{-\beta} \quad (5)$$

for any $\delta, \beta > 0$.

Similarly one obtains from (4)

$$\|T(t)w - w\|_{-\beta-2\nu} \leq ct^\nu\|w\|_{-\beta}, \quad (6)$$

for any $\beta > 0$, $w \in H^{-\beta}(\mu)$ and $0 < \nu < 1$.

Note that the constants in the estimates depend on the related parameters.

Throughout the paper we make the following standing assumptions which are the same as in [24]:

Assumption (MMS): (X, d) is a locally compact separable metric space. We consider the Borel σ -field on X and a Radon measure μ on (X, d) .

Assumption (HKE(β)): The transition kernel $P_t(x, dy)$ associated with the semigroup $T(t), t \geq 0$ admits a transition density $P_t(x, dy) = p(t, x, y)\mu(dy)$ which satisfies for almost all $x, y \in X$ the following *heat kernel estimate*

$$t^{-\frac{d_H}{w}}\Phi_1(t^{-\frac{1}{w}}d(x, y)) \leq p(t, x, y) \leq t^{-\frac{d_H}{w}}\Phi_2(t^{-\frac{1}{w}}d(x, y))$$

if $0 < t < R_0$ for some constants $R_0 > 0$, $w \geq 2$ and nonnegative bounded decreasing functions Φ_i on $[0, \infty)$, where d_H is the *Hausdorff dimension* of (X, d) . For $t \geq R_0$,

$$p(t, x, y) \leq p_t$$

and p_t decreases in t . (In this case the semigroup is *ultracontractive*, i.e.,

$$\|T(t)\|_{L^\infty(\mu)} \leq p_t \|f\|_{L_2(\mu)},$$

where $p_t := ct^{-d_S/4}$, if $t < R_0$, and the value $d_S = \frac{2d_H}{w}$ agrees with its *spectral dimension*. w is called *walk dimension* of the semigroup.) For a given $\beta > 0$ we further assume the *integrability condition*

$$\int_0^\infty s^{d_H + \beta\omega/2 - 1} \Phi_2(s) ds < \infty.^1$$

Heat kernels of this type have been studied in Grigor'yan and Kumagai [15] and related references therein. Further relationships are presented in the recent survey [14] of Grigor'yan, Hu and Lau.

In order to make the integral in (2) precise we need *pointwise products* of functions and dual elements from the potential spaces. In [24] the following is proved which also extends related results for the Euclidean case.

Proposition 2.1. [24, Corollary 4.1]

Suppose (MMS) and (HKE(β)) for $0 < \beta < \delta < \min(\frac{d_S}{2}, 1)$. Then for $q = \frac{d_S}{\delta}$ the product gh of $g \in H^\delta(\mu)$ and $h \in H_q^{-\beta}(\mu)$ is well defined in $H^{-\beta}(\mu)$ by the duality relation $(f, gh) := (fg, h)$, $f \in H^\delta(\mu)$, and the following estimate holds true:

$$\|gh\|_{-\beta} \leq c \|g\|_\delta \|h\|_{H_q^{-\beta}(\mu)}.$$

2.2. The integral equation and mild solution. A rigorous definition for the integral and a contraction principle for the solution to equation (2) are given in [24] by means of fractional calculus in Banach spaces, in particular, under the following additional conditions.

Assumption (FG): The nonlinear functions F and G are such that $F \in C^1(\mathbb{R}^n)$, $F(0) = 0$ and F has bounded Lipschitz derivative F' and $G \in C^2(\mathbb{R}^n)$, $G(0) = 0$ and G has bounded Lipschitz second derivative G'' .

For the parameters we here consider the case II from [24].

Assumption (P): $0 < \alpha < \gamma$, $0 < \beta < \delta < \min(\frac{d_S}{2}, 1)$, $\gamma < 1 - \alpha - \frac{\beta}{2} - \frac{d_S}{4}$, where β and d_S are from (HKE(β)), and $q = \frac{d_S}{\delta}$.

We now will briefly summarize the construction.

¹In [24] the ω is missing in the exponent.

If $u \in W^\gamma([0, t_0], H_\infty^\delta(\mu))$ the operator $U(t; s) : H_q^{-\beta}(\mu) \rightarrow H_\infty^\delta(\mu)$ is defined as

$$U(t; s)w := T(t - s) (G(u(s))w) \quad (7)$$

for $w \in H_q^{-\beta}(\mu)$. Then under the above assumptions on the function G and the parameters (P) for any $0 < \eta < \gamma$ the *left-sided Weyl-Marchaud fractional derivative* of order η is determined by

$$D_{0+}^\eta U(t; s) := \frac{\mathbb{1}_{(0,t)}(s)}{\Gamma(1-\eta)} \left(\frac{U(t; s)}{s^\eta} + \eta \int_0^s \frac{U(t; s) - U(t; \tau)}{(s - \tau)^{\eta+1}} d\tau \right)$$

as an element of $L_1([0, t], L(H_q^{-\beta}(\mu), H^\delta(\mu)))$ (in the sense of Bochner integration). This is shown in [24, Lemma 5.2, (ii)]².

Let us now consider the regulated version of $z \in C^{1-\alpha}([0, t_0], H_q^{-\beta})$ on $[0, t]$ given by $z_t(s) := \mathbb{1}_{(0,t)}(s)(z(s) - z(t))$. If additionally $1 - \eta < 1 - \alpha$, which is always possible in view of (P), one can define the *right-sided Weyl-Marchaud fractional derivative* of z_t of order $1 - \eta$ by

$$D_{t-}^{1-\eta} z_t(s) := \frac{(-1)^{1-\eta} \mathbb{1}_{(0,t)}(s)}{\Gamma(\eta)} \left(\frac{z(s) - z(t)}{(t - s)^{1-\eta}} + (1 - \eta) \int_s^t \frac{z(s) - z(\tau)}{(\tau - s)^{(1-\eta)+1}} d\tau \right)$$

as an element of $L_\infty([0, t], H_q^{-\beta}(\mu))$.

For more details on these fractional derivative we refer the reader to [23], [34], [39] and [40]. They are used for one of the results in [24]:

Proposition 2.2. *Suppose (MMS), (HKE(β)), (FG), the parameter conditions (P) and $z \in C^{1-\alpha}([0, t_0], H_q^{-\beta}(\mu))$. Then we have the following.*

- (a) [24, Lemma 5.1] *For the operator $U(t; x) = T(t - x)(G(u(x))\cdot)$ as in (7) with $u \in W^\gamma([0, t_0], H_\infty^\delta(\mu))$ the integral $\int_s^t U(t; x) dz(x)$ is well defined by*

$$\int_s^t U(t; x) dz(x) := (-1)^\eta \int_s^t D_{s+}^\eta U(t; x) D_{t-}^{1-\eta} z_t(x) dx, \quad (8)$$

independently of the choice of η with $\eta < \gamma$ and $1 - \eta < 1 - \alpha$. (In particular, the integrand on the right side is a Lebesgue integrable real function.)

- (b) [24, Theorem 1.2] *For any initial function $f \in H^{2\gamma+\delta+\varepsilon}(\mu)$ with some $\varepsilon > 0$ there exists a unique solution u to equation (2) for the definition (8) of the integral such that $u \in W^\gamma([0, t_0], H_\infty^\delta(\mu))$.*

²We remark that there is a typo in [24, Lemma 5.2], namely in (ii) and (iii) the right hand side of the main condition on the parameters should read $2 - 2\eta - \beta$ instead of $2 - 2\eta - (\beta \vee \frac{d_S}{2})$.

3. THE MAIN RESULT

3.1. Regularity of the solution. The main results are stated in Theorem 3.4 and Corollary 3.6.

In this section we use *short notations* for the following norms:

$$\|\cdot\|_L := \|\cdot\|_{L(H_q^{-\beta}(\mu), H^\delta(\mu))}, \quad \|\cdot\|_{W^\gamma} := \|\cdot\|_{W^\gamma([0, t_0], H_\infty^\delta(\mu))}$$

with the specified parameters as in Assumption (P).

First recall that under Assumption (FG) the nonlinear operators F and G are bounded from $H_\infty^\delta(\mu)$ into itself (see [24, Proposition 3.1]).

Lemma 3.1. *Suppose (MMS), (HKE(β)), (FG), the parameter conditions (P) and let $z \in C^{1-\alpha}([0, t_0], H_q^{-\beta}(\mu))$. Then there is a positive constant c such that*

$$\left\| \int_s^t T(t-x)F(u(x))dx \right\|_\delta \leq c\|u\|_{W^\gamma}(t-s). \quad (9)$$

Proof. Since the semigroup $T(t)$ is a contraction on $H^\delta(\mu)$ we have

$$\begin{aligned} \left\| \int_s^t T(t-x)F(u(x))dx \right\|_\delta &\leq \int_s^t \|T(t-x)F(u(x))\|_\delta dx \\ &\leq \int_s^t \|F(u(x))\|_\delta dx \\ &\leq c \int_s^t \|u(x)\|_\delta dx \leq c\|u\|_{W^\gamma}(t-s), \end{aligned}$$

where the latter bound follows from the definition of the W^γ -norm. \square

Lemma 3.2. *Under the same conditions as in Lemma 3.1 we have*

$$\begin{aligned} &\|T(t-x)(G(u(x))) - T(t-y)(G(u(x)))\|_L \\ &\leq c\|u\|_{W^\gamma}(t-x)^{-\frac{\delta}{2}-\frac{\beta}{2}-\nu}(x-y)^\nu \end{aligned}$$

for any $0 < \nu < 1$.

Proof. Let h be an arbitrary element of $H_q^{-\beta}(\mu)$ with $\|h\|_{H_q^{-\beta}(\mu)} \leq 1$. Then Proposition 2.1 implies

$$\|G(u(x))h\|_{-\beta} \leq c\|u\|_{W^\gamma}$$

uniformly in the time argument $x \in [0, t_0]$ by the mapping property of G and the definition of the W^γ -norm. Using (5), (6) and the last estimate we infer

$$\begin{aligned} &\|T(t-x)(\text{Id} - T(x-y))G(u(x))h\|_\delta \\ &\leq c(t-x)^{-\delta/2-\beta/2-\nu} \|(\text{Id} - T(x-y))G(u(x))h\|_{-\beta-2\nu} \\ &\leq c(t-x)^{-\delta/2-\beta/2-\nu}(x-y)^\nu \|G(u(x))h\|_{-\beta} \\ &\leq c(t-x)^{-\delta/2-\beta/2-\nu}(x-y)^\nu \|u\|_{W^\gamma}. \end{aligned}$$

This bound together with the definition of the L -norm

$$\begin{aligned} & \|T(t-x)(\text{Id} - T(x-y))G(u(x))\|_L \\ &= \sup_{\|h\|_{H_q^{-\beta}} \leq 1} \|T(t-x)(\text{Id} - T(x-y))(G(u(x))h)\|_\delta \end{aligned}$$

completes the proof. \square

Lemma 3.3. *Suppose (MMS), (HKE(β)), (FG), the parameter conditions (P) and let $z \in C^{1-\alpha}([0, t_0], H_q^{-\beta}(\mu))$. Then there is a positive constant c such that*

$$\left\| \int_s^t T(t-x)G(u(x))dz(x) \right\|_\delta \leq c(t-s)^\gamma. \quad (10)$$

Proof. Since by assumption $z \in C^{1-\alpha}([0, t_0], H_q^{-\beta}(\mu))$ then for any η such that $1 - \eta < 1 - \alpha$ we have

$$\sup_{t \in [0, t_0]} \sup_{x \in [0, t]} \|D_{t-}^{1-\eta} z_t(x)\|_{H_q^{-\beta}(\mu)} \leq c < \infty.$$

Let us fix η throughout the proof as some number slightly bigger than α such that $\alpha < \eta < \gamma$ and at the same time $\gamma < 1 - \eta - \frac{\delta}{2} - \frac{\beta}{2}$ which is always possible in view of (P). We then get

$$\begin{aligned} & \left\| \int_s^t U(t; x) dz(x) \right\|_\delta \\ &= \left\| \int_s^t D_{s+}^\eta U(t; x) D_{t-}^{1-\eta} z_t(x) dx \right\|_\delta \\ &\leq \sup_{t \in [0, t_0]} \sup_{x \in [0, t]} \|D_{t-}^{1-\eta} z_t(x)\|_{H_q^{-\beta}(\mu)} \int_s^t \|D_{s+}^\eta U(t; x)\|_L dx \\ &\leq c \int_s^t \|D_{s+}^\eta T(t-x)G(u(x))\|_L dx \\ &\leq c \int_s^t \frac{\|T(t-x)G(u(x))\|_L}{(x-s)^\eta} dx \\ &\quad + c \int_s^t \int_s^x \frac{\|T(t-x)G(u(x)) - T(t-y)G(u(y))\|_L}{(x-y)^{1+\eta}} dy dx \\ &=: S_1 + S_2. \end{aligned}$$

Consider S_1 first. Using (5) and Proposition 2.1 we obtain

$$\begin{aligned}
S_1 &\leq c \int_s^t \sup_{\|w\|_{H_q^{-\beta}(\mu)} \leq 1} \frac{\|T(t-x)(G(u(x))w)\|_\delta}{(x-s)^\eta} dx \\
&\leq c \int_s^t (t-x)^{-\frac{\delta}{2}-\frac{\beta}{2}} \sup_{\|w\|_{H_q^{-\beta}(\mu)} \leq 1} \frac{\|G(u(x))w\|_{-\beta}}{(x-s)^\eta} dx \\
&\leq c \int_s^t (t-x)^{-\frac{\delta}{2}-\frac{\beta}{2}} (x-s)^{-\eta} \|G(u(x))\|_\delta \sup_{\|w\|_{H_q^{-\beta}(\mu)} \leq 1} \|w\|_{H_q^{-\beta}(\mu)} dx \\
&\leq c \int_s^t (t-x)^{-\frac{\delta}{2}-\frac{\beta}{2}} (x-s)^{-\eta} \|u\|_{W^\gamma} dx \\
&\leq c(t-s)^{1-\frac{\delta}{2}-\frac{\beta}{2}-\eta} \leq c(t-s)^\gamma,
\end{aligned}$$

the latter following from $1 - \frac{\delta}{2} - \frac{\beta}{2} - \eta > \gamma$ by construction. Moreover the integral is finite since $\gamma > 0$.

Consider S_2 . The numerator inside the integral can be bounded as follows

$$\begin{aligned}
&\|T(t-x)G(u(x)) - T(t-y)G(u(y))\|_L \\
&\leq \|T(t-x)G(u(x)) - T(t-y)G(u(x))\|_L \\
&\quad + \|T(t-y)G(u(x)) - T(t-y)G(u(y))\|_L,
\end{aligned} \tag{11}$$

so that we have

$$\begin{aligned}
S_2 &= c \int_s^t \int_s^x \frac{\|T(t-x)G(u(x)) - T(t-y)G(u(y))\|_L}{(x-y)^{1+\eta}} dy dx \\
&\leq c \int_s^t \int_s^x \frac{\|T(t-x)G(u(x)) - T(t-y)G(u(x))\|_L}{(x-y)^{1+\eta}} dy dx \\
&\quad + c \int_s^t \int_s^x \frac{\|T(t-y)G(u(x)) - T(t-y)G(u(y))\|_L}{(x-y)^{1+\eta}} dy dx \\
&=: S_3 + S_4.
\end{aligned}$$

Let us consider the term S_4 first. We have (with similar computations as for S_1)

$$\begin{aligned}
S_4 &= c \int_s^t \int_s^x \frac{\|T(t-y)G(u(x)) - T(t-y)G(u(y))\|_L}{(x-y)^{1+\eta}} dy dx \\
&\leq c \int_s^t \int_s^x \sup_{\|w\|_{H_q^{-\beta}(\mu)} \leq 1} \frac{\|T(t-y)([G(u(x)) - G(u(y))]w)\|_\delta}{(x-y)^{1+\eta}} dy dx \\
&\leq c \int_s^t (t-x)^{-\frac{\delta}{2}-\frac{\beta}{2}} \int_s^x \frac{\|G(u(x)) - G(u(y))\|_\delta}{(x-y)^{1+\eta}} dy dx
\end{aligned}$$

recall that $\eta < \gamma$ by definition of η , thus

$$\begin{aligned} &\leq c \int_s^t (t-x)^{-\frac{\delta}{2}-\frac{\beta}{2}} \|u\|_{W^\gamma} dx \\ &\leq c(t-s)^{1-\frac{\delta}{2}-\frac{\beta}{2}} \leq c(t-s)^\gamma, \end{aligned}$$

the latter being true as $\gamma < 1 - \frac{\delta}{2} - \frac{\beta}{2}$ by assumption. Regarding the term S_3 , we apply Lemma 3.2 with $\nu > \eta$ to the numerator inside the integral of S_3 , so that S_3 can be bounded by

$$\begin{aligned} S_3 &\leq c \int_s^t \int_s^x \frac{\|T(t-x)G(u(x)) - T(t-y)G(u(x))\|_L}{(x-y)^{1+\eta}} dy dx \\ &\leq c \int_s^t \int_s^x \frac{(t-x)^{-\frac{\delta}{2}-\frac{\beta}{2}-\nu} (x-y)^\nu}{(x-y)^{1+\eta}} dy dx \\ &\leq c \int_s^t (t-x)^{-\frac{\delta}{2}-\frac{\beta}{2}-\nu} \int_s^x (x-y)^{\nu-1-\eta} dy dx \\ &\leq c \int_s^t (t-x)^{-\frac{\delta}{2}-\frac{\beta}{2}-\nu} (x-s)^{\nu-\eta} dx \\ &\leq c(t-s)^{1-\frac{\delta}{2}-\frac{\beta}{2}-\nu+\nu-\eta} \leq c(t-s)^\gamma, \end{aligned}$$

the latter bound being true as $\gamma < 1 - \eta - \frac{\delta}{2} - \frac{\beta}{2}$. The proof is complete. \square

We are now about to state and prove the main regularity property of the solution u under the Assumptions (MMS), (HKE(β)), (FG) and (P).

Theorem 3.4. *Suppose (MMS), (HKE(β)) and (FG). Let $0 < \alpha < \gamma$, $0 < \beta < \delta < \min(\frac{d_S}{2}, 1)$ and $\gamma < 1 - \alpha - \frac{\beta}{2} - \frac{d_S}{4}$ and set $q = \frac{d_S}{\delta}$. If $z \in C^{1-\alpha}([0, t_0], H_q^{-\beta}(\mu))$ and the initial condition f is an element of $H^{\delta+2\gamma+\varepsilon}(\mu)$ for some $\varepsilon > 0$, then the unique solution $u \in W^\gamma([0, t_0], H_\infty^\delta(\mu))$ for (2) is also an element of $C^\gamma([0, t_0], H^\delta(\mu))$.*

Remark 3.5. For $u \in W^\gamma([0, t_0], H_\infty^\delta(\mu))$ the second integral in equation (2) has to be interpreted as above. For u considered as element of $C^\gamma([0, t_0], H^\delta(\mu))$ this integral agrees with the corresponding Riemann-Stieltjes integral with values in the Banach space $H^\delta(\mu)$. The latter has been used in Gubinelly, Lejay and Tindel [16] in an abstract setting in the sense of Young.

Proof. Let $0 \leq s < t \leq t_0$. We consider the solution at time t as the evolution of u according to (1) with initial condition at time s being $u(s)$, that is

$$u(t) = T(t-s)u(s) + \int_s^t T(t-r)F(u(r))dr + \int_s^t T(t-r)G(u(r))dz(r),$$

so that

$$\begin{aligned} u(t) - u(s) &= (T(t-s) - \text{Id})u(s) \\ &\quad + \int_s^t T(t-r)F(u(r))dr + \int_s^t T(t-r)G(u(r))dz(r) \end{aligned} \quad (12)$$

The $H^\delta(\mu)$ -norms of the two integrals are bounded by $c(t-s)$ and $c(t-s)^\gamma$ according to Lemma 3.1 and Lemma 3.3, respectively. If we show that $\|u(t)\|_{\delta+2\gamma} < c$ uniformly in t for $t \in [0, t_0]$, then the $H^\delta(\mu)$ -norm of the term involving the initial condition can be easily bounded. In fact note that in this case $(A + \text{Id})^{\frac{\delta}{2}}u \in D(A^\gamma)$ thus we can apply (4) to get

$$\begin{aligned} \|(T(t-s) - \text{Id})u(s)\|_\delta &\leq c\|(A + \text{Id})^{\delta/2}(T(t-s) - \text{Id})u(s)\|_0 \\ &\leq c\|(T(t-s) - \text{Id})(A + \text{Id})^{\delta/2}u(s)\|_0 \\ &\leq c(t-s)^\gamma\|(A + \text{Id})^{\delta/2}u(s)\|_{2\gamma} \\ &\leq c(t-s)^\gamma\|u(s)\|_{\delta+2\gamma} \leq c(t-s)^\gamma \end{aligned}$$

as wanted. It remains to prove that $\|u(t)\|_{\delta+2\gamma} < c$ uniformly in t for $t \in [0, t_0]$. Recall that

$$u(t) = T(t)f + \int_0^t T(t-r)F(u(r))dr + \int_0^t T(t-r)G(u(r))dz(r),$$

thus

$$\begin{aligned} \|u(t)\|_{\delta+2\gamma} &\leq \|T(t)f\|_{\delta+2\gamma} \\ &\quad + \int_0^t \|T(t-r)F(u(r))\|_{\delta+2\gamma}dr \\ &\quad + \int_0^t \|D_{0+}^\eta U(t; r)D_{t-}^{1-\eta}z_t(r)\|_{\delta+2\gamma}dr \\ &=: S_1 + S_2 + S_3. \end{aligned}$$

The term S_1 is easily bounded by $c\|f\|_{\delta+2\gamma}$. The term S_2 is bounded recalling that $\|T(t-r)F(u(r))\|_{\delta+2\gamma} \leq (t-r)^{-\gamma}\|F(u(r))\|_\delta$ because of the smoothing action of the semigroup. Thus $S_2 \leq ct^{1-\gamma}$. The last term can be treated in a similar way as the proof of Lemma 3.3 with the difference that the space $H^\delta(\mu)$ is replaced by $H^{\delta+2\gamma}(\mu)$. All computations for the $H^{\delta+2\gamma}(\mu)$ -norm term carry out in the same way, except that the exponent $-\frac{\delta}{2} - \frac{\beta}{2}$ is replaced by $-\frac{\delta}{2} - \frac{\beta}{2} - \gamma$ so that $S_3 \leq ct^{1-\eta-\frac{\delta}{2}-\frac{\beta}{2}-\gamma}$ with $1-\eta-\frac{\delta}{2}-\frac{\beta}{2}-\gamma > 0$ by construction of η . Clipping the result together we have

$$\|u(t)\|_{\delta+2\gamma} \leq c + ct^{1-\gamma} + ct^{1-\eta-\frac{\delta}{2}-\frac{\beta}{2}-\gamma},$$

and finally taking the supremum over $t \in [0, t_0]$ we get the uniform bound. \square

With slightly more restrictive assumptions on the noise we can show that the unique solution u belongs to the spaces $W^\gamma([0, t_0], H_\infty^\delta(\mu))$, and thus to $C^\gamma([0, t_0], H^\delta(\mu))$, for all (γ, δ) such that $0 < \gamma < 1 - \alpha - \frac{\beta}{2} - \frac{d_S}{4}$ and $\beta < \delta < \min(\frac{d_S}{2}, 1)$.

Corollary 3.6. *Suppose (MMS), (HKE(β)) and (FG).*

- (a) *Let $0 < \alpha < \frac{1}{2}$ and $0 < \beta < \min(\frac{d_S}{2}, 1 - 2\alpha, 2(1 - \alpha) - \frac{d_S}{2})$ be given. Suppose that $z \in C^{1-\alpha}([0, t_0], H_q^{-\beta}(\mu))$ for any $1 < q < \frac{d_S}{\beta}$ and $f \in H^{2(1-\alpha)-\beta}(\mu)$. Then for any $\beta < \delta < \min(\frac{d_S}{2}, 1)$ and $0 < \gamma < 1 - \alpha - \frac{\beta}{2} - \frac{d_S}{4}$ Equation (2) has a unique solution in the space $W^\gamma([0, t_0], H_\infty^\delta(\mu))$ and hence, it has a unique solution belonging to all these spaces.*
- (b) *Moreover, this solution is an element of $C^\gamma([0, t_0], H^\delta(\mu))$ for any γ and δ as before.*

Proof. Part (a). Take δ and γ as in the assumption. Then $2(1-\alpha)-\beta > 2\gamma + \frac{d_S}{2} > 2\gamma + \delta + \varepsilon$ for some $\varepsilon > 0$ implies that $f \in H^{2\gamma+\delta+\varepsilon}(\mu)$. Moreover $H_q^{-\beta}(\mu) \subset H_{\frac{d_S}{\delta}}^{-\beta}(\mu)$ for $1 < q < \frac{d_S}{\beta}$ since $\beta < \delta < d_S$, thus $z \in C^{1-\alpha}([0, t_0], H_{\frac{d_S}{\delta}}^{-\beta}(\mu))$. Then we are under the assumptions of Theorem 3.4 and thus there exists a unique solution to (1) which belongs to $W^\gamma([0, t_0], H_\infty^\delta(\mu))$. Because of the embedding of the spaces involved, clearly $u \in W^{\gamma'}([0, t_0], H_\infty^{\delta'}(\mu))$ for any $0 < \delta' \leq \delta$ and $0 < \gamma' \leq \gamma$, too. We also know that for δ' and γ' satisfying the assumptions there exists a unique solution u' to (1) which is in $W^{\gamma'}([0, t_0], H_\infty^{\delta'}(\mu))$. As the initial condition f and the noise term z are the same, then by uniqueness we must have $u = u'$ in the larger space $W^{\gamma'}([0, t_0], H_\infty^{\delta'}(\mu))$.

Part (b). It follows directly from part (a) and Theorem 3.4. Obviously, due to the embedding of the fractional Sobolev spaces $H^\delta(\mu) \subset H^{\delta'}(\mu)$ for $\delta' < \delta$ and of the Hölder spaces $C^\gamma \subset C^{\gamma'}$ for $\gamma' < \gamma$, we have that $u \in C^{\gamma'}([0, t_0], H^{\delta'}(\mu))$ for all $0 < \delta' < \delta$ and $0 < \gamma' < \gamma$. \square

Remark 3.7. The parameter condition (P), in particular $\delta < d_S/2$, and the integrability condition on the function Φ_2 in the heat kernel estimate (HKE) have been used only for the product estimate in Proposition 2.1. An analysis of the proofs, in particular those of [24], shows that the assertions of Theorem 3.4 and Corollary 3.6 remain valid under the parameter condition

$$0 < \frac{d_S}{2} \leq \beta \leq \delta < 1, \quad 0 < \alpha < \gamma < 1 - \alpha - \frac{\beta}{2} - \frac{\delta}{2} \quad (13)$$

provided $z \in C^{1-\alpha}([0, t_0], H_q^{-\beta}(\mu))$ for some $q > 2$ such that the multiplication property

$$\|vz\|_{-\beta} \leq c\|v\|_\delta \|z\|_{H_q^{-\beta}(\mu)} \quad (14)$$

holds true for any $v \in H^\delta(\mu)$.

Then one obtains a complement to the former assertions for the case $d_S \leq 1$: Here we can choose $\frac{1}{2} \leq \beta < 1 - 2\alpha$, which implies $1 - \alpha > \frac{3}{4}$, in order to get a Hölder continuous solution to equation (2). In particular, in the Gaussian setting (see below) such a \dot{z} may be interpreted as a noise white in space and colored in time.

3.2. Applications and extensions. With similar techniques as in the previous section it is possible to treat equation (1) with A replaced by a fractional power of A , that is for instance A^θ for $0 < \theta \leq 1$. This is done in [24]. In this case the semigroup associated with the fractional power $-A^\theta$ will be the subordinated semigroup $T^{(\theta)}(t)$ and the power θ must be taken into account in Assumption (P) and all following theorems accordingly. Then similar regularity properties as shown for the case $\theta = 1$ follow. These results go in the direction of [32] where the authors study a stochastic equation with fractional dissipation (that is with a term $A^\theta, 0 < \theta \leq 1$) but with the difference that in the present paper the noise is coloured, whereas in [32] the noise is white in time and solutions are strong and local.

Clearly, the special case of linear F and G can be considered. In this case the L_∞ -boundedness of the solution is no longer needed (see [24, Theorem 1.3]) and the conditions on the parameters are weaker, in particular in Assumption (P) it is sufficient that $\gamma < 1 - \alpha - \frac{\beta}{2} - \frac{\delta}{2}$. Moreover the spectral dimension restriction $d_S < 4$ can be lifted. Using the aforementioned corresponding results from [24] and the present methods one can get similar results as in the previous section, the proofs being completely analogous.

Moreover, one can easily consider linear combinations of noise terms such as

$$\sum_{i=1}^N G_i(u) \cdot \dot{z}_i$$

for any finite integer N and G_i and z_i as G and z in this paper.

Here we list a few examples. The results of Section 3.1 can be applied to the same kind of stochastic equations considered in the paper by Hinz and Zähle [24, Section 7]. In particular, we can consider stochastic partial differential equations driven by fractional Brownian noises on metric measure spaces. The equations are studied in the pathwise sense and the results are valid \mathbb{P} -a.s.

Example 3.8. A classical example is the nonlinear heat equation on a smooth bounded domain $D \subset \mathbb{R}^n$, $n \leq 3$, provided with the Lebesgue measure μ , driven by fractional Brownian field, see e.g. [23, Section 6]. We consider a real valued fractional Brownian sheet $\{B^{H,K}(t, x), [0, t_0] \times$

\mathbb{R}^n with Hurst indices $0 < H < 1$ and $0 < K < 1$ for time and space respectively. This is a centered Gaussian field on $[0, t_0] \times \mathbb{R}^n$ with stationary rectangular increments satisfying

$$E(B^{H,K}(s, x) - B^{H,K}(s, y) - B^{H,K}(t, x) + B^{H,K}(t, y))^2 = c|s-t|^{2H}|x-y|^{2K}.$$

It can be shown [23, Section 6] that there exists a version such that for almost all trajectories $\omega \in \Omega$ one has $B^{H,K}(\omega) \in C^{1-\alpha}([0, t_0], H_q^\sigma(\mu))$ for $0 < 1 - \alpha < H$, $0 < \sigma < K$ and $1 < q < \infty$. Thus the distributional spatial partial derivatives $\frac{\partial}{\partial x_i} B^{H,K}$ for $i = 1, \dots, n$ belong to $C^{1-\alpha}([0, t_0], H_q^{-\beta}(\mu))$, where $-\beta = \sigma - 1$. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}$ and $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that F and each component G_i satisfy Assumption (FG). Then Theorem 3.4 and Corollary 3.6 can be applied in the pathwise sense to

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_D u + F(u) + \langle G(u), \frac{\partial}{\partial t} \nabla B^{H,K} \rangle \\ u(0, x) = 0, \text{ for } t \in (0, t_0) \\ u(t, x) = 0, \text{ for } x \in \partial D \end{cases} \quad (15)$$

for almost all paths. Here Δ_D is the classical Dirichlet Laplacian, i.e., $-\Delta_D$ generates a heat semigroup on $L_2(D)$ with Gaussian estimates. $\nabla B^{H,K}$ denotes the distributional gradient of the fractional Brownian sheet. The term $\langle G(u), \frac{\partial}{\partial t} \nabla B^{H,K} \rangle$ in (15) is given by

$$\sum_{i=1}^n G_i(u) \frac{\partial^2}{\partial t \partial x_i} B^{H,K},$$

where $\frac{\partial^2}{\partial t \partial x_i} B^{H,K}$ is interpreted as \dot{z}_i in the above sense.

Since the spectral dimension of \mathbb{R}^n is $d_S = n$, for almost all sample paths the unique solution is an element of

$$W^\gamma([0, t_0], H_\infty^\delta(\mu)) \cap C^\gamma([0, t_0], H^\delta(\mu))$$

for all γ and δ such that $1 - H < \gamma < H - \frac{1-K}{2} - \frac{n}{4}$ and $1 - K < \delta < \min(\frac{n}{2}, 1)$. This can be satisfied only if $n \leq 3$.

According to Remark 3.7 for $d_S = n = 1$ the result remains valid under the parameter condition (13), since the multiplication property (14) is fulfilled in this Euclidean case. The latter can be seen as follows. Theorem 4.5.2 in Runst and Sickel [33] and restriction to functions vanishing on ∂D lead to the estimate

$$\|uv\|_{H_p^\beta(\mu)} \leq c\|u\|_\delta \|v\|_\beta, \quad 1 < p < 2, \quad \frac{1}{2} \leq \beta < \delta < 1.$$

Then for these parameters (14) follows by duality arguments. Hence, for $n = 1$ and $K \leq \frac{1}{2}$ Hölder continuous solutions to equation (2) can be obtained if $H > \frac{2-K}{2}$. Note that $K = \frac{1}{2}$ means white noise in space.

Example 3.9. A more sophisticated case mentioned in [24, Example 2]) is the following. Let (X, d, μ) be a compact metric measure space satisfying Assumption (MMS) and admitting a semigroup $\{T(t), t \geq 0\}$

generated by a (fractal) Laplacian Δ associated to a local regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on X , i.e., $-A = \Delta$. For various classes of fractals the corresponding heat kernels exist and satisfy Assumption (HKE(β)) for any $\beta > 0$ (see, e.g., Barlow and Bass [2] and [3], Barlow, Bass, Kumagai and Teplyaev [4], Fitzsimmons, Hambly and Kumagai [10], Hambly and Kumagai [20], Kigami [28], Barlow, Grigor'yan and Kumagai [5] and the references therein).

A standard example for the noise process z , modified for our situation, is the following: Let $\{e_i\}_{i \in \mathbb{N}}$ be a complete orthonormal system of eigenfunctions of A in $L_2(\mu)$ and λ_i be the corresponding eigenvalues, $\{B_i^H(t)\}_{i \in \mathbb{N}}$ are i.i.d. fractional Brownian motions in \mathbb{R} with Hurst exponent $\frac{1}{2} < H < 1$, and consider the formal series

$$b^H(t) = \sum_{i=1}^{\infty} B_i^H(t) q_i e_i$$

for real coefficients q_i . Then we get a modification b^H such that a.s.

$$z := b^H \in C^{1-\alpha}([0, t_0], H_q^{-\beta}(\mu))$$

(with convergence of the series in these spaces) for any $0 < 1 - \alpha < H$, $\beta_* < \beta < 1$ and $q > 2$ under the following conditions on the measure μ , the parameter β_* , the eigenfunctions e_i and the coefficients q_i :

(a) $\|e_i\|_{\infty} \leq c_1 \lambda_i^{a_1/2}$ and

(b) $|e_i(x) - e_i(y)| \leq c_2 \lambda_i^{a_2/2} d(x, y)^b$ (up to an exceptional set)

for some positive constants $a_1, a_2, b, c_0, c_1, c_2$, and for $a := \max(a_1, a_2)$,

$$\sum_{i=1}^{\infty} q_i^2 \lambda_i^{-\beta_* + a - b2/w} < \infty.$$

(Note that in the case $q = 2$, which is not relevant for our purposes, Conditions (a) and (b) are not needed and the convergence $\sum_{i=1}^{\infty} q_i^2 \lambda_i^{-\beta_*} < \infty$ would be sufficient for the above property of z .)

(Idea of proof: By the mapping properties of the resolvent operators $J^\sigma(\mu)$, which may be replaced here by $I^\sigma(\mu) := (-\Delta)^{\sigma/2}$ since the eigenvalues are strictly positive (cf. Section 2.1), it is equivalent to get a modification, which satisfies a.s.

$$I^{\beta_* + b2/w}(\mu) b^H \in C^{1-\alpha}([0, t_0], H_q^{-\beta + \beta_* + b2/w}(\mu))$$

for any q and $\beta > \beta_*$. For $0 < \delta' < \delta$ the embedding of the Hölder space $C^\delta(X)$ into $H_q^{\delta'/2/w}(\mu)$ can be seen, e.g., from the arguments in the proof of [26, Proposition 5.6] (using there only the upper estimates taking into regard that the lower heat kernel estimates imply $\mu(B(x, r)) \leq c_0 r^{d_H}$ for any ball with centre x and radius r). Therefore a sufficient condition

for the above one is that

$$I^{\beta_*+b2/w}(\mu)b^H \in C^{1-\alpha}([0, t_0], C^\delta(X))$$

for any $\delta < b$, and the latter can be proved by means of the Kolmogorov principle for the random function

$$Y(t, x) := J^{\beta_*+b2/w}b^H(t, x) = \sum_{i=1}^{\infty} B_i^H(t) q_i \lambda_i^{-(\beta_*+b2/w)/2} e_i(x).$$

We get

$$\begin{aligned} & \mathbb{E}(Y(t_1, x_1) - Y(t_2, x_1) - (Y(t_1, x_2) - Y(t_2, x_2)))^2 \\ &= \sum_{i=1}^{\infty} q_i^2 \lambda_i^{-(\beta_*+b2/w)} \mathbb{E}(B_i^H(t_1) - B_i^H(t_2))^2 (e_i(x_1) - e_i(x_2))^2 \\ &\leq \sum_{i=1}^{\infty} q_i^2 \lambda_i^{-(\beta_*+b2/w)} |t_1 - t_2|^{2H} c_2 \lambda_i^a d(x_1, x_2)^{2b} \\ &\leq c |t_1 - t_2|^{2H} d(x_1, x_2)^{2b}. \end{aligned}$$

Moreover,

$$\begin{aligned} & \mathbb{E}(Y(t_1, x) - Y(t_2, x))^2 \\ &= \sum_{i=1}^{\infty} q_i^2 \lambda_i^{-(\beta_*+b2/w)} \mathbb{E}(B_i^H(t_1) - B_i^H(t_2))^2 |e_i(x)|^2 \\ &\leq \sum_{i=1}^{\infty} q_i^2 \lambda_i^{-(\beta_*+b2/w)} |t_1 - t_2|^{2H} c_1^2 \lambda_i^a \\ &\leq c |t_1 - t_2|^{2H}. \end{aligned}$$

Using that the higher moments of centered Gaussian random variables are powers of the second moments this ensures the usual construction of a modification of Y with the desired Hölder regularity by means of an extension of the values on a countable dense subset of $[0, t_0] \times X$.)

Note that because of the above ultracontractivity of the semigroup Condition (a) is always fulfilled for $a_1 := \frac{d_S}{2}$. Furthermore, if we work with the resistance metric $R(x, y)$ w.r.t. the Dirichlet form \mathcal{E} then Condition (b) is satisfied for $a_2 = b = 1$.

For p.c.f. fractals with regular harmonic structures we have $d_S = \frac{2d_H}{d_H+1} < 2$, see Kigami [27]. Moreover, under some mild additional assumptions on such fractals in Euclidean spaces the resistance metric R satisfies $R(x, y) \leq |x - y|^b$ for some $b > 0$, see Hu and Wang [25]. Hence, in this case Condition (b) is also fulfilled for the Euclidean metric.

Examples with spectral dimension greater than 2 are provided by generalized Sierpinski carpets, see Barlow and Bass [3], or by certain products of fractals, see Strichartz [35].

According to Theorem 3.4 function solutions to Equation (2) which are Hölder regular in time can be found for $d_S = \frac{2d_H}{w} < 4$ (recall that d_H denotes the Hausdorff dimension of X and w the walk dimension of the semigroup) and Hurst exponent $H > \frac{1}{2} + \frac{d_S}{8}$. Recall that in this case we have $\beta < \frac{d_S}{2}$.

If $\frac{d_S}{2} \leq 1$ then Remark 3.7 provides the alternative parameter condition $\frac{1}{2} \leq \beta < 2H - 1$ for existence of Hölder continuous solutions. In particular, for $\beta = \frac{1}{2}$ the noise is “white” in space.

For the classical case of the Dirichlet Laplace operator on the unit interval this example has been treated in Gubinelly, Lejay and Tindel [16] with different methods.

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