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The approximation property implies that convolvers are pseudo-measures

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Abstract

This paper (not for formal publication) grew out of the authors' attempts to understand Cowling's argument that for a locally compact group G with the approximation property, we have that $PM_p(G) = CV_p(G)$ ("all convolvers are pseudo-measures".) We have ended up giving a somewhat self-contained survey of Cowling's construction of a predual for $CV_p(G)$, together with a survey of old ideas of Herz relating to Herz-Schur multipliers. Thus none of the results are new, but we make some claim to originality of presentation. We hope this account may help other researchers, and in particular, that this might spur others to study this problem.

1 Introduction

For a locally compact group G equipped with the left Haar measure, we let $\lambda_p : G \to \mathcal{B}(L^p(G))$ be the left regular representation of G on $L^p(G)$:

$$\big(\lambda_p(s)(f)\big)(t) = f(s^{-1}t) \qquad (s, t \in G, f \in L^p(G)).$$

Letting $p' = p/(p-1)$ be the conjugate index, there is a contractive bilinear map $L^p(G) \times L^{p'}(G) \to C_0(G)$ given by $f \otimes g \mapsto g * \check{f}$, the convolution of g with $\check{f}$, the latter defined by $\check{f}(t) = f(t^{-1})$. Notice that for $t \in G$, we have that $(g * \check{f})(t) = \langle \lambda_p(t)(f), g\rangle$, using the usual dual pairing between $L^p(G)$ and $L^{p'}(G)$. Thus we have a contractive map from the projective tensor product $L^p(G)\widehat{\otimes}L^{p'}(G) \to C_0(G)$. We let the coimage be $A_p(G)$, the Figa-Talamanca–Herz algebra of G (see [7]) which becomes a Banach algebra when equipped with the norm making $A_p(G)$ isometrically a quotient of $L^p(G)\widehat{\otimes}L^{p'}(G)$. Thus every $a \in A_p(G)$ has the form

$$a = \sum_{n=1}^{\infty} g_n * \check{f}_n \quad \text{with} \quad \|a\|_{A_p} = \inf\Big\{ \sum_{n=1}^{\infty} \|f_n\|_p \|g_n\|_{p'} \Big\}, \tag{1}$$

where the infimum is taken over all such representations.

Herz showed in [8, Proposition 3] that $A_p(G)$ is a regular, tauberian algebra of functions on G. In particular, the elements of $A_p(G)$ with compact support are dense in $A_p(G)$. This raises the following question:

> **Compact Approximation:** Given $a \in A_p(G)$ with compact support, can we write a in the form of (1) with every f_n, g_n having compact support? Can we compute the norm of a using such "compact representations"?

There are of course two questions here: the first always has a positive answer, as shown by Cowling in [1, Lemma 1], using an argument of Herz. Thus the question is really about norm control of compactly supported elements.

Recall that $\mathcal{B}(L^p(G))$, the space of bounded linear maps on $L^p(G)$, is the dual space of $L^p(G)\widehat{\otimes}L^{p'}(G)$ for the pairing $\langle T, f\otimes g\rangle = \langle T(f), g\rangle$. The kernel of the map $L^p(G)\widehat{\otimes}L^{p'}(G) \to A_p(G)$ is precisely the set I of those $\tau \in L^p(G)\widehat{\otimes}L^{p'}(G)$ such that $\langle \lambda_p(s), \tau\rangle = 0$ for all $s\in G$. Hence if we denote by $PM_p(G)$ the weak*-closed linear span of the operators $\lambda_p(s)$, then $I = {}^{\perp}PM_p(G)$ and so

$$A_p(G)^* = I^{\perp} = PM_p(G).$$

Notice that clearly $PM_p(G)$ is an algebra– it is the algebra of *p-pseudo-measures*. Relatedly, let $\rho_p : G \to \mathcal{B}(L^p(G))$ be the right-regular representation

$$\big(\rho_p(s)(f)\big)(t) = f(ts)\Delta(s)^{1/p} \qquad (s,t\in G, f\in L^p(G)),$$

where Δ is the modular function. Let $CV_p(G)$ be the commutant of $\{\rho_p(s) : s\in G\}$; so also $CV_p(G)$ is the bicommutant of $PM_p(G)$. It is easy to see that $PM_p(G)\subseteq CV_p(G)$. Herz showed in [8, Theorem 5] that when G is amenable, $CV_p(G) = PM_p(G)$. This leads to the question:

Convolvers are Pseudo-Measures: Does it hold that $CV_p(G) = PM_p(G)$?

Cowling showed implicitly in [1] that these two questions are equivalent– stated as they are here, this is perhaps somewhat surprising. We quickly sketch Cowling's argument below in Section 2 (see Theorem 2.3). Some von Neumann algebra theory (see Section 5 below for why we need more than the bicommutant theorem!) shows that $CV_2(G) = PM_2(G)$ for any G, and so the "compact approximation" problem has a positive answer for $p = 2$, that is, for the Fourier algebra.

In Section 3 we quickly explore Herz's theory of p-spaces, and show that $CV_p(G)$ is a module over a certain multiplier algebra. Again, such ideas are already implicit in [8, 9], but we give an essentially self-contained account. Then we show a result claimed without proof in [1]: if G has the approximation property, then $CV_p(G) = PM_p(G)$ for all p. We finish with some comments about the philosophy of why this proof works, and what is different in the $p = 2$ case.

2 A predual of the convolvers

It is obvious that $CV_p(G)$ is weak*-closed in $\mathcal{B}(L^p(G))$ because $T\in CV_p(G)$ if and only if T annihilates the space

$$cv_p(G) = \overline{\operatorname{lin}}\{\rho_p(s)f\otimes g - f\otimes\rho_p(s)^*g : s\in G, f\in L^p(G), g\in L^{p'}(G)\}\subseteq L^p(G)\widehat{\otimes}L^{p'}(G).$$

Here $\rho_p(s)^* \in \mathcal{B}(L^{p'}(G))$ denotes the Banach space adjoint of the operator $\rho_p(s)$. A simple calculation shows that $\rho_p(s)^* = \rho_{p'}(s^{-1})$.

In [1], Cowling constructs a "function space" predual of $CV_p(G)$, which we now sketch. Let K be a compact neighbourhood of e in G, and let $L^p(K)$ be the subspace of $L^p(G)$ consisting of those functions supported on K. Let $\check{A}_{p,K}(G)$ be the space of (necessarily continuous) functions a on G of the form

$$a = \sum_{n=1}^{\infty} g_n * \check{f}_n$$

where $(f_n)\subseteq L^p(K), (g_n)\subseteq L^{p'}(K)$, and with $\sum_n \|f_n\|_p\|g_n\|_{p'} < \infty$. Again give $\check{A}_{p,K}(G)$ the norm given by the infimum of all such sums. Now let $\check{A}_p(G)$ be the union of such spaces, as K varies, equipped with the norm

$$\|a\|_{\check{A}_p} = \inf\Big\{\|a\|_{\check{A}_{p,K}} : K \text{ a compact n'hood of } e,\ a\in\check{A}_{p,K}\Big\}.$$

Cowling shows that $\check{A}_p(G)$ is a normed algebra of functions on G. Letting $\overline{A_p}(G)$ be the Banach space completion, we have that $\overline{A_p}(G)$ is a commutative Banach algebra whose spectrum is G.

The main result of [1] is that the dual space of $\check{A}_p(G)$ may be identified (isometrically) with $CV_p(G)$ in the following way: $T \in CV_p(G)$ corresponds to $\Phi_T \in \check{A}_p(G)^*$ where

$$\langle \Phi_T, a \rangle = \sum_{n=1}^{\infty} \langle T(f_n), g_n \rangle \text{ whenever } a = \sum_{n=1}^{\infty} g_n * \check{f}_n.$$

We shall not prove this result here, instead referring the reader to Cowling's paper.

Let us now draw some further conclusions from this. We first claim that we may define a map

$$\phi : \check{A}_p(G) \to L^p(G) \widehat{\otimes} L^{p'}(G)/cv_p(G); \quad g * \check{f} \mapsto f \otimes g + cv_p(G),$$

and using linearity and continuity to extend to all of $\check{A}_p(G)$. This is indeed well-defined, for if

$$a = \sum_{n=1}^{\infty} g_n * \check{f}_n = \sum_{n=1}^{\infty} g'_n * \check{f}'_n,$$

for suitable families $(f_n), (g_n), (f'_n), (g'_n)$, then by Cowling's result, for any $T \in CV_p(G)$,

$$\sum_{n=1}^{\infty} \langle T(f_n), g_n \rangle - \sum_{n=1}^{\infty} \langle T(f'_n), g'_n \rangle = \langle \Phi_T, a - a \rangle = 0.$$

As $cv_p(G)$ is the pre-annihilator of $CV_p(G)$, this shows that

$$\sum_{n=1}^{\infty} f_n \otimes g_n - \sum_{n=1}^{\infty} f'_n \otimes g'_n \in cv_p(G),$$

as required to show that ϕ is well-defined. Thus ϕ extends by continuity to a contraction $\overline{A_p}(G) \to L^p(G) \widehat{\otimes} L^{p'}(G)/cv_p(G)$.

Proposition 2.1. *The map ϕ is an isometric isomorphism.*

Proof. Just observe that $\phi^* : CV_p(G) \to \overline{A_p}(G)^* = \check{A}_p(G)^*$ is just the map $T \mapsto \Phi_T$ which by Cowling's result is an isometric isomorphism. ☐

As the adjoint of the map $\overline{A_p}(G) \to A_p(G)$ is the isometric inclusion $PM_p(G) \to CV_p(G)$, it follows that $\overline{A_p}(G)$ quotients onto $A_p(G)$. We could also see this by observing that $cv_p(G)$ is a subset of the pre-annihilator ${}^{\perp}PM_p(G)$, and so we have a quotient map

$$\overline{A_p}(G) \cong L^p(G) \widehat{\otimes} L^{p'}(G)/cv_p \to L^p(G) \widehat{\otimes} L^{p'}(G)/{}^{\perp}PM_p(G) \cong A_p(G).$$

Hence the Gelfand transform of $\overline{A_p}(G)$ "is" this quotient map $\overline{A_p}(G) \to A_p(G)$, if we think of $A_p(G)$ as an algebra of functions on G. In particular, we have the following.

Proposition 2.2. *$\overline{A_p}(G)$ is semi-simple if and only if $\overline{A_p}(G) = A_p(G)$.*

We can now prove that the two questions asked in the introduction are indeed equivalent.

Theorem 2.3. *For any locally compact group G and any p, we have that $A_p(G)$ satisfies "compact approximation" if and only if $CV_p(G) = PM_p(G)$.*

Proof. Suppose that $CV_p(G) = PM_p(G)$. Let $a \in A_p(G)$ have compact support. By [1, Lemma 1], $a \in \check{A}_p(G)$, though maybe *a priori* with a different norm. However, then

$$\begin{aligned}\|a\|_{\check{A}_p} &= \sup\{|\langle T, a\rangle| : T \in CV_p(G), \|T\| \leq 1\} \\ &= \sup\{|\langle T, a\rangle| : T \in PM_p(G), \|T\| \leq 1\} = \|a\|_{A_p}.\end{aligned}$$

So we can compute the norm of a in $\check{A}_p(G)$ by using elements with common compact support in $L^p(G)$ and $L^{p'}(G)$, which is exactly "compact approximation".

Conversely, if we have compact approximation, then identification of functions gives an isometry $\check{A}_p(G) \to A_p(G)$ which again by [1, Lemma 1] has dense range. Thus $\overline{A_p}(G) = A_p(G)$, and so by duality, $CV_p(G) = PM_p(G)$. □

3 Herz's Theory of p-spaces

In [7] Herz defines a Banach space E to be a *p-space* if there is a contraction γ_E such that for any G, the following diagram commutes:

$$\begin{tikzcd} L^p(G;E)\widehat{\otimes}L^{p'}(G;E^*) \arrow[r, "\gamma_E"] & L^p(G)\widehat{\otimes}L^{p'}(G) \\ (L^p(G)\widehat{\otimes}E)\widehat{\otimes}(L^{p'}(G)\widehat{\otimes}E^*) \arrow[u] \arrow[ur, "c_E"] & \end{tikzcd}$$

where $c_E(f \otimes x \otimes g \otimes \mu) = \langle \mu, x\rangle f \otimes g$. This is equivalent to, for each $T \in \mathcal{B}(L^p(G))$, there existing $S \in \mathcal{B}(L^p(G;E))$ with $\|S\| = \|T\|$, and with $S(f \otimes x) = T(f) \otimes x$ on elementary tensors.

These days we recognise that E is a p-space if and only if it is an SQ_p-space, that is, E is (isometrically isomorphic to) a subspace of a quotient of an L^p space, see [10, Theorem 2].

Herz's interest in such spaces was because they are a natural setting in which to study the fact that $A_p(G)$ is an algebra. In fact, in [9] he introduced the notion we now call a *Herz–Schur* multiplier. Let $\mathcal{M}_p(G)$ be the space of continuous functions φ on G such that there is a p-space E and continuous functions $\alpha : G \to E$ and $\beta : G \to E^*$ such that $\varphi(ts^{-1}) = \langle \alpha(s), \beta(t)\rangle$ for all $s, t \in G$. We take $\|\varphi\|$ to be the infimum of $\|\alpha\|_\infty\|\beta\|_\infty$ over all such representations of φ. Then $\mathcal{M}_p$ is a Banach algebra, and is an algebra of functions on G which multiples $A_p(G)$ into itself. Within the framework of p-operator spaces, these are precisely the multipliers of $A_p(G)$ which are p-completely bounded, see [2] and references therein.

In [8, Section 1] some p-space ideas are deployed to show that $L^p(G)\widehat{\otimes}L^{p'}(G)$ is an $A_p(G)$ module. Let us give the details of this argument, in a more general setting. Note that we regard $L^p(G)\widehat{\otimes}L^{p'}(G)$ as a space of (equivalence classes of) functions on $G \times G$, by, for example, using the contractive injective map $L^p(G)\widehat{\otimes}L^{p'}(G) \to L^p(G; L^{p'}(G))$.

Proposition 3.1. *$L^p(G)\widehat{\otimes}L^{p'}(G)$ is a contractive $\mathcal{M}_p(G)$ module for the module action given by*

$$(\varphi \cdot \tau)(s,t) = \varphi(ts^{-1})\tau(s,t) \qquad (\varphi \in \mathcal{M}_p(G), \tau \in L^p(G)\widehat{\otimes}L^{p'}(G), s, t \in G).$$

Proof. It suffices to establish that $\varphi \cdot \tau$ really is in $L^p(G)\widehat{\otimes}L^{p'}(G)$ with $\|\varphi \cdot \tau\| \leq \|\varphi\|\|\tau\|$. By the definition of the projective tensor product, it's enough to do this when $\tau = f \otimes g$. Let $\varphi(ts^{-1}) = \langle \alpha(s), \beta(t)\rangle$ for suitable α, β as defined above. Define

$$F \in L^p(G;E);\ F(s) = f(s)\alpha(s), \qquad G \in L^{p'}(G;E^*);\ G(t) = g(t)\beta(t).$$

Then, for $s, t \in G$,

$$\gamma_E(F \otimes G)(s,t) = f(s)g(t)\langle \alpha(s), \beta(t)\rangle = \varphi(ts^{-1})\tau(s,t).$$

Thus $\varphi \cdot \tau = \gamma_E(F \otimes G) \in L^p(G)\widehat{\otimes}L^{p'}(G)$ with $\|\varphi \cdot \tau\| \leq \|F\|\|G\| \leq \|f\|\|\alpha\|_\infty\|g\|\|\beta\|_\infty$ as required. $\square$

Notice that all the hard work was carried by the existence of the map γ_E.

Proposition 3.2. *This action of $\mathcal{M}_p(G)$ maps $cv_p(G)$ into itself, and hence drops to a contractive action on $L^p(G)\widehat{\otimes}L^{p'}(G)/cv_p(G)$. This space is isomorphic to $\overline{A_p}(G)$, and under this isomorphism, the action of $\mathcal{M}_p(G)$ is just (pointwise) multiplication of functions. By duality, we hence turn $CV_p(G)$ into a contractive $\mathcal{M}_p(G)$ module.*

Proof. We continue with the notation of the previous proof. Denote $F = f\alpha$, and similarly $G = g\beta$. For $s, t \in G$, let $\alpha^t(s) = \alpha(st)$. Then, as E is a p-space, for any $t \in G$ the operator $\rho_p(t) \otimes I_E$ is an invertible isometry on $L^p(G; E)$. Then

$$(\rho_p(t) \otimes I_E)(f\alpha)(s) = (f\alpha)(st)\Delta(t)^{1/p} = (\rho_p(t)f)(s)\alpha^t(s) \qquad (s \in G)$$

and so $(\rho_p(t) \otimes I_E)(f\alpha) = (\rho_p(t)f)\alpha^t$. However, then notice that for $r \in G$,

$$\begin{aligned}&\gamma_E\big((\rho_p(r)f)\alpha \otimes g\beta - f\alpha \otimes (\rho_{p'}(r^{-1})g)\beta\big)\\ &\qquad = \gamma_E\big((\rho_p(r) \otimes I_E)(f\alpha^{r^{-1}}) \otimes g\beta - f\alpha \otimes (\rho_{p'}(r^{-1}) \otimes I_{E^*})(g\beta^r)\big).\end{aligned}$$

Now, for $s, t \in G$,

$$\langle \alpha^{r^{-1}}(s), \beta(t)\rangle = \langle \alpha(sr^{-1}), \beta(t)\rangle = \varphi(trs^{-1}) = \langle \alpha(s), \beta(tr)\rangle = \langle \alpha(s), \beta^r(t)\rangle.$$

So the pairs $(\alpha^{r^{-1}}, \beta)$ and (α, β^r) define the same element of $\mathcal{M}_p(G)$. Thus by the previous proof, and using an obvious property of γ_E, we see that

$$\begin{aligned}\gamma_E\big((\rho_p(r) \otimes I_E)(f\alpha^{r^{-1}}) \otimes g\beta\big) &= (\rho_p(r) \otimes I_{L^{p'}(G)})\gamma_E\big(f\alpha^{r^{-1}} \otimes g\beta\big)\\ &= (\rho_p(r) \otimes I_{L^{p'}(G)})\gamma_E\big(f\alpha \otimes g\beta^r\big) = \gamma_E\big((\rho_p(r) \otimes I_E)(f\alpha) \otimes g\beta^r\big).\end{aligned}$$

Putting these together, we see that

$$\begin{aligned}&\gamma_E\big((\rho_p(r)f)\alpha \otimes g\beta - f\alpha \otimes (\rho_{p'}(r^{-1})g)\beta\big)\\ &\qquad = \gamma_E\big((\rho_p(r) \otimes I_E)(f\alpha) \otimes g\beta^r - f\alpha \otimes (\rho_{p'}(r^{-1}) \otimes I_{E^*})(g\beta^r)\big)\\ &\qquad = (\rho_p(r) \otimes I_{L^{p'}(G)})\gamma_E\big(f\alpha \otimes g\beta^r\big) - (I_{L^p(G)} \otimes \rho_{p'}(r^{-1}))\gamma_E\big(f\alpha \otimes g\beta^r\big),\end{aligned}$$

which by definition is a member of $cv_p(G)$.

We have hence established that the module action of $\mathcal{M}_p(G)$ maps $cv_p(G)$ into itself. Then using the map $\phi : \overline{A_p}(G) \to L^p(G)\widehat{\otimes}L^{p'}(G)/cv_p(G)$, we see that

$$\begin{aligned}\phi^{-1}\big(\varphi \cdot \phi(g * \check{f})\big)(r) &= \int_G (\varphi \cdot (f \otimes g))(r^{-1}s, s)\ ds\\ &= \int_G \varphi(s(r^{-1}s)^{-1})f(r^{-1}s)g(s)\ ds = \varphi(r)(g * \check{f})(r),\end{aligned}$$

so the induced action of $\mathcal{M}_p(G)$ on $\overline{A_p}(G)$ is just function multiplication. $\square$

Recall the quotient map $\overline{A_p}(G) \to A_p(G)$, and let the kernel of this map be I_p. As this is also the kernel of the Gelfand map, it follows that the module action of $\mathcal{M}_p$ annihilates I_p. Furthermore, the quotient map $\overline{A_p}(G) \to A_p(G)$ is an $\mathcal{M}_p$-module homomorphism.

3.1 Support of convolvers

In [7], Herz defines a notion of "support" for a member of $CV_p(G)$ which generalises the notion of support for members of $PM_p(G)$. It is quite easy to combine a number of results of [7] to prove the following, but we choose instead to give a short, direct proof. In particular, compare with [7, Proposition 9].

Proposition 3.3. *Let $T \in CV_p(G)$ be arbitrary, and let $\varphi \in \mathcal{M}_p(G)$ have compact support (thinking of φ as a continuous functions on G). Then $S = \varphi \cdot T \in PM_p(G)$.*

Proof. Let K be any compact neighbourhood of the identity in G, and let $k \in C_{00}(G)$ with $\operatorname{supp}(k) \subseteq K$. We claim that $S(k) \in L^p(G)$ has support contained in the compact set $\operatorname{supp}(\varphi)K$. Indeed, if $g \in C_{00}(G)$ vanishes on $\operatorname{supp}(\varphi)K$, then $g(st) = 0$ for all $t \in K$ and s with $\varphi(s) \neq 0$. Thus, for such s,

$$(g * \check{k})(s) = \int_G k(t)g(st)\ dt = \int_K k(t)g(st)\ dt = 0,$$

and so $\varphi(g * \check{k}) = 0$ identically, and hence

$$\langle S(k), g\rangle = \langle \varphi \cdot T, g * \check{k}\rangle = \langle T, \varphi(g * \check{k})\rangle = 0,$$

as required.

So then $S(k) \in L^p(G)$ has compact support, and so by Hölder's inequality, $S(k) \in L^1(G)$. As $S \in CV_p(G)$, for $u \in C_{00}(G)$ we have that $S(k * u) = S(k) * u$, and then a limiting argument (and using that $S(k) \in L^1(G)$) shows that this holds for all $u \in L^p(G)$. Letting k run through a bounded approximate identity for convolution, it follows that

$$S(u) = \lim_k S(k * u) = \lim_k S(k) * u \qquad (u \in L^p(G)).$$

As left convolution by $S(k) \in L^1(G)$ is a member of $PM_p(G)$, this shows that the operator S is the weak*-limit of elements of $PM_p(G)$; as $PM_p(G)$ is weak*-closed, the result follows. ☐

4 Main result

We finally come to the study of groups G with the *approximation property*, for which we follow the original paper [5]. Recall that $\mathcal{M}_2(G)$ is a dual space with predual $Q(G)$, which is the closure of $L^1(G)$ inside $\mathcal{M}_2(G)^*$ given by the obvious "action by integration" of $L^1(G)$ on $\mathcal{M}_2(G)$ (this being first studied in [3, Proposition 1.10].) Then G has the approximation property when 1 is in the weak*-closure of $A(G)$ in $\mathcal{M}_2(G)$.

We now adapt [5, Proposition 1.3]. Let $f \in \check{A}_p(G)$ be pointwise positive with $\int_G f = 1$. It is easily seen that $\mathcal{M}_p(G)$ is translation invariant, and so for any $\varphi \in \mathcal{M}_p(G)$, also $f * \varphi \in \mathcal{M}_p(G)$ with $\|f * \varphi\| \leq \|\varphi\|$. For $\tau \in L^p(G)\widehat{\otimes}L^{p'}(G)$, by Proposition 3.1, we have that $(f * \varphi) \cdot \tau \in L^p(G)\widehat{\otimes}L^{p'}(G)$. Then, for $T \in CV_p(G) \subseteq \mathcal{B}(L^p(G))$, we see that

$$\mathcal{M}_p(G) \to \mathbb{C}; \quad \varphi \mapsto \langle T, (f * \varphi) \cdot \tau\rangle$$

is a bounded linear map, with norm at most $\|\tau\|\|T\|$, say giving $\mu \in \mathcal{M}_p(G)^*$.

Finally, recall that 2-spaces are precisely the Hilbert spaces, and that every Hilbert space is a p-space (see [7]). Hence identification of functions gives a contraction $\mathcal{M}_2(G) \to \mathcal{M}_p(G)$.

Proposition 4.1. *Let $\tau \in C_{00}(G) \otimes C_{00}(G) \subseteq L^p(G)\widehat{\otimes}L^{p'}(G)$, let f, T be as above, and form $\mu \in \mathcal{M}_p(G)^*$. The composition of $\mathcal{M}_2(G) \to \mathcal{M}_p(G)$ followed by μ gives a functional on $\mathcal{M}_2(G)$ which is weak* continuous, that is, which is a member of $Q(G)$.*

Proof. We shall construct $g \in L^1(G)$ with $\langle \mu, \varphi \rangle = \int_G \varphi g$ for all $\varphi \in \mathcal{M}_2(G)$. Let $a \in \check{A}_p(G)$ be the element induced by τ and let $S = \operatorname{supp}(f)^{-1} \operatorname{supp}(a)$, a compact subset of G. For $s \in \operatorname{supp}(a)$,

$$(f * \varphi)(s) = \int_G f(t)\varphi(t^{-1}s) \, dt = \int_G f(t)\chi_S(t^{-1}s)\varphi(t^{-1}s) \, dt$$

as $f(t) \neq 0 \implies t^{-1}s \in S$ as $s \in \operatorname{supp}(a)$. Thus

$$(f * \varphi)(s)a(s) = (f * \chi_S\varphi)(s)a(s)$$

for all $s \in G$, as if $s \notin \operatorname{supp}(a)$ then both sides are 0. Define $f_t(s) = f(st)$, and then

$$\big((f * \chi_S\varphi)a\big)(s) = \int_G f_t(s)(\chi_S\varphi)(t^{-1})a(s) \, dt.$$

Now, the map $\Phi : G \to \check{A}_p(G); t \mapsto f_t a$ is bounded and continuous, because $\check{A}_p(G)$ is a normed algebra and $t \mapsto f_t$ is a continuous function.[1] Then set

$$g(t) = \chi_S(t)\Delta(t^{-1})\langle T, \Phi(t^{-1})\rangle \qquad (t \in G).$$

Then g is the restriction of a continuous function to the compact set S, and hence $g \in L^1(G)$. Then, using the above calculations,

$$\begin{aligned}\int_G g(t)\varphi(t) \, dt &= \int_G \chi_S(t)\Delta(t^{-1})\langle T, \Phi(t^{-1})\rangle\varphi(t) \, dt = \int_G \chi_S(t^{-1})\langle T, f_t a\rangle\varphi(t^{-1}) \, dt \\ &= \langle T, \int_G f_t a(\chi_S\varphi)(t^{-1}) \, dt\rangle = \langle T, (f * \chi_S\varphi)a\rangle = \langle T, (f * \varphi)a\rangle = \langle \mu, \varphi\rangle,\end{aligned}$$

as required. Here we used Proposition 3.2 which allows us to identify $(f \star \varphi)a \in \overline{A_p}(G)$ with the image of $(f \star \varphi) \cdot \tau$ in the quotient $L^p(G)\widehat{\otimes}L^{p'}(G)/cv_p(G)$, which is all we care about, as $T \in CV_p(G)$. ☐

Theorem 4.2. *If G has the approximation property, then $CV_p(G) = PM_p(G)$ for all p.*

Proof. By hypothesis, there is a net (φ_i) of elements of $A(G)$ with compact support (compare [5, Remark 1.2] as to why we can assume compact support) such that $\varphi_i \to 1$ weak* in $\mathcal{M}_2(G)$.

Fix $f \in \check{A}_p(G)$ as above, and let $T \in CV_p(G)$. As $C_{00}(G)\otimes C_{00}(G)$ is dense in $L^p(G)\widehat{\otimes}L^{p'}(G)$, by continuity and the previous proposition, for any $\tau \in L^p(G)\widehat{\otimes}L^{p'}(G)$ there is $\mu \in Q(G)$ such that $\langle \varphi, \mu\rangle = \langle T, (f * \varphi) \cdot \tau\rangle$ for all $\varphi \in \mathcal{M}_2(G)$. Thus

$$\lim_i \langle T, (f * \varphi_i) \cdot \tau\rangle = \lim_i \langle \varphi_i, \mu\rangle = \langle 1, \mu\rangle = \langle T, (f * 1) \cdot \tau\rangle = \langle T, 1 \cdot \tau\rangle = \langle T, \tau\rangle.$$

As τ was arbitrary, it follows that $(f * \varphi_i) \cdot T \to T$ weak* in $\mathcal{B}(L^p(G))$. As f and φ_i have compact support, so does $f * \varphi_i$, and so Proposition 3.3 shows that $(f * \varphi_i) \cdot T \in PM_p(G)$ for all i. Again, as $PM_p(G)$ is weak*-closed, it follows that $T \in PM_p(G)$, as required. ☐

5 Closing remarks

We remark that the ideas of Cowling, as expounded in this note, and Herz's proof [7, Theorem 5], are all "localisation" results: we use some weak notion of a "compactly supported approximate identity" to multiply elements of $CV_p(G)$ into $PM_p(G)$, and then take weak*-limits. The class

[1] Indeed, if $f = g * \check{h}$ then $f_t(s) = \int_G h((st)^{-1}r)g(r) \, dr = \int_G (\lambda_p(t)h)(s^{-1}r)g(r) \, dr = h * \check{k}$ if $k = \lambda_p(t)h$. Continuity then follows because $t \mapsto \lambda_p(t)h$ is continuous.

of groups with the approximation is large, but does not include all locally compact groups, due to recent counter-examples [6, 11].

When $p = 2$, it is tempting to think that the bicommutant theorem immediately implies that $PM_2(G) = CV_2(G)$; however, some care is required. For X a subset of $\mathcal{B}(L^p(G))$, let $\mathrm{alg}(X)$ be the algebra generated by X, and let $\mathrm{alg}^{w*}(X)$ be the weak*-closure of $\mathrm{alg}(X)$. Then it's easy to see that $X' = \mathrm{alg}(X)' = \mathrm{alg}^{w*}(X)'$. By definition, $PM_p(G) = \mathrm{alg}^{w*}(\lambda_p(G))$; define also $PM_p^r(G) = \mathrm{alg}^{w*}(\rho_p(G))$. Then by definition $CV_p(G) = \rho_p(G)' = PM_p^r(G)'$; define also $CV_p^r(G) = \lambda_p(G)' = PM_p(G)'$. When $p = 2$, von Neumann's bicommutant theorem tells us that $PM_2(G) = \mathrm{alg}^{w*}(\lambda_2(G)) = \lambda_2(G)'' = CV_2^r(G)'$. However, $CV_2(G) = PM_2^r(G)'$, so it is not immediately clear why we would have $PM_2(G) = CV_2(G)$? Clearly $\lambda_2(G) \subseteq \rho_2(G)'$ and so $\lambda_2(G)' = CV_2^r(G) \supseteq \rho_2(G)'' = PM_2^r(G)$. Thus the problem is to prove that if $T \in \lambda_2(G)'$, then why is $T \in PM_2^r(G) = \mathrm{alg}^{w*}(\rho_2(G)) = \rho_2(G)''$?

Herz claims in [8, Section 8] that the argument given in [4, Part 1, Chapter 5, Exercises] works for any p to show that $PM_p(G) = CV_p(G)$. This reference is to Dixmier's book, where he studies (quasi)-Hilbert algebras. We do not immediately see why Herz's claim, that such ideas work for $p \neq 2$, holds.[2] For a modern approach, which certainly shows that $PM_2(G) = CV_2(G)$, we may follow Tomita-Takesaki theory, see in particular [12, Chapter VII, Proposition 3.1].

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