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Smith, MJ, Hazelton, ML, Lo, H et al. (2 more authors) (2013) The Long Term Behaviour of Day-to-Day Traffic Assignment Models. Transportmetrica A: Transport Science, 10 (7). 647 - 660. ISSN 2324-9943

https://doi.org/10.1080/18128602.2012.751683

The Long Term Behaviour of Day-to-Day Traffic Assignment Models

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Abstract The dynamical behaviour of deterministic process, day-to-day traffic assignment models is sometimes characterized by convergence to a variety of different fixed equilibrium points dependent upon the initial flow pattern, even though individual trajectories are unique for a given start point. This non-uniqueness is seemingly in sharp contrast to the evolution of stochastic process, day-to-day models; under certain assumptions these converge in law to a unique stationary distribution, irrespective of the start point. In this paper we show how models may be constructed which exhibit characteristics of both deterministic models and stochastic models, and illustrate the ideas by using a simple example network.

Keywords: control, deterministic, dynamics, stochastic

1 Introduction

1.1 The Basic Idea

Bie and Lo [2010] and Watling [1999] give examples of smooth, deterministic process, day-to-day dynamical systems where different start points lead to different basins of attraction and ultimately different equilibria. Sets of unstable states separate the different basins of attraction. These sepatrices are thin; i.e. not of full dimension. Such properties may be expected regardless of the kind of point equilibrium reached, notably whether Wardrop user equilibrium or stochastic user equilibrium or some other kind of fixed point. A quite different kind of approach is the stochastic process approach to day-to-day dynamical systems, whereby the trajectory (and ultimate states) can only be determined in terms of a probability law. When the same examples that produced non-uniqueness in the deterministic process case are analysed using some forms of stochastic process model, there is a unique equilibrium limiting probability distribution spread over the whole state space, whatever the starting point; Cascetta [1989] and Watling [1996] provide a series of examples illustrating this point. Separating sets do not exist in such stochastic models; the only irreducible stable set is the whole space.

This paper demonstrates different ways of combining deterministic process and stochastic process notions within a single stochastic process, day-to-day dynamic model. It is shown that in such stochastic process models, depending on the particular model assumptions made, there may naturally be distinct,

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non-communicating, basins of attraction (as in standard deterministic process models); or thick/full dimension separating sets of transient states (as in standard stochastic process models). Thus the new stochastic process models introduced here sometimes have at least some of the properties normally associated with deterministic process models while retaining some of the properties of previously-proposed stochastic process models.

To illustrate the concepts clearly the paper considers variations on one simple modal split model already considered in the literature.

1.2 A Control Implication

The difference between deterministic and stochastic process model characteristics above has a variety of implications. One implication is as follows. With a deterministic model having two or more basins of attraction, if an intervention moves the flow vector from one basin of attraction to another and the intervention is then removed then this may result in a different long run future [e.g. Bie and Lo, 2010]. However, with the usual stochastic process models the limiting or long-run future is unchanged by any temporary intervention as there is only one basin of attraction and so no possibility of moving to another. This is shown, for example, in the control example in Watling [1996], where we see that the explanation is that the stochastic process model instead 'moves' its prediction in respect to control by rebalancing the probabilities of system states arising. However, even though the process may almost appear to separate (the probability of some states communicating can be almost vanishingly small, but still non-zero) irreducibility is maintained in theory and so no theoretical separation occurs.

In contrast, in the stochastic process model constructed in this paper (which inherits some deterministic process properties as well as some standard stochastic process characteristics), we allow some state communication probabilities to actually reach zero, rather than just be very small. Thus the process is no longer irreducible, and so the pseudo-basins separated by small communication probabilities which can be seen in existing stochastic process models now separate into multiple, distinct basins of attraction. Thus the long-run future of the stochastic process may be changed by a temporary intervention in a way more analogous to deterministic process models, because separate basins of attraction may now exist. A remark that should be made is that here and throughout this paper we are here discussing the theoretical, infinite-time properties of these processes; as shown in the examples in Watling [1996], the theoretical properties of existing, irreducible stochastic process models over extremely long but finite time-horizons may typically exhibit 'pseudo-stable' behaviour, a behaviour that (over finite time) mimics multiple basins of attraction; furthermore, this is particularly evident if, as is common, Monte Carlo simulation is used to estimate the stochastic process model, whereby apparent multiple basins may exist. However, the point about the new form of model present here is that even in theory, in infinite time, the stochastic process model presented can give rise to actually distinct basins of attraction.

1.3 A Very Brief Summary of Previous Work

Many authors have considered *deterministic process* models for representing day-to-day dynamics in a traffic assignment context. Perhaps somewhat confusingly for those unfamiliar with the field, these deterministic process models are associated with two traditional forms of equilibria with which we are familiar in transport, namely the Wardrop or Deterministic User Equilibrium (DUE) [Wardrop, 1952] and the Stochastic User Equilibrium (SUE) [Daganzo, 1983]. Smith [1984*c*], for example, considered a deterministic process model and its relation to DUE, as did He et al. [2010] in their recent link-based study of day-to-day assignment. Watling [1999] and Bie and Lo [2010] also considered deterministic process models, but in their case in relation to SUE fixed points. There have been many other studies of deterministic process, day-to-day models, the references to which may be found in these source papers. The important taxonomy to bear in mind for the present paper is that while these papers may be interested in either DUE or SUE, the type of process they adopt is fully determined (with probability 1) given the initial conditions, i.e. the process is deterministic, whatever name we might traditionally associate with its endpoint (with SUE being the most obvious potential source of confusion here). A fuller discussion of this point, with examples, is provided in Watling and Hazelton [2003]. It is also noted in passing that some of these studies have interesting relationships with the problem of devising iterative algorithms to calculate equilibrium points (as opposed to represent deterministic dynamics); see, for example, the works of Smith [1984*b*,*a*].

The study of *stochastic process* models of day-to-day traffic assignment has attracted relatively little attention, although a growing body of work exists in the literature, examples including those papers by Cascetta [1989], Davis and Nihan [1993], Cantarella and Cascetta [1995], Watling [1996] and Hazelton and Watling [2004]. The counterpart of a state of DUE/SUE that may be found in deterministic process studies is now that of a stationary or equilibrium probability distribution of flow states. This does not have a counterpart in traditional transport studies. The equilibrium probability distribution may vary according to the behavioural and traffic assumptions assumed in the stochastic process, and so studying the equilibrium distributions of stochastic process models generates a whole new range of predictive models for us to use in transportation analysis.

1.4 Organization of the Paper

In section 2 we introduce a deterministic process model following Smith [1984*c*]. We apply this model to a simple two route example in section 3. We analyse the day-to-day dynamics of the system and look at equilibria. We generalize this model in section 4 by incorporating a random term into the dynamical system, thus obtaining a stochastic process model, but one of a rather different form to those previously proposed for studying transportation networks. By restricting the support of this random term we obtain a system for which the long term dynamics (in terms of basins of attraction) can be regarded as a 'softened' version of those for the purely deterministic process model. In section 5 we examine an alternative type of stochastic process model, based on the Markov process specifications proposed by Cascetta [1989] and Cantarella and Cascetta [1995]. Again working with the simple two route example, we show that the long term behaviour depends critically on whether or not the random variation in (perceived) travel costs is bounded.

2 Deterministic Dynamics – One Possibility

To be specific here, we assume given a fixed demand model with one OD pair joined by several routes. Let $\boldsymbol{x}(t)$ be the vector of route flows on day t. The components of $\boldsymbol{x}(t)$ always add to the fixed total origin-destination (OD) flow, which we denote N.

The initial deterministic day-to-day dynamical assumption in this paper is that

$$\text{the flow vector tomorrow} = \text{flow vector today} + \boldsymbol{d}$$

or, more precisely:

$$\boldsymbol{x}(t+1) = \boldsymbol{x}(t) + \boldsymbol{d}(\boldsymbol{x}(t)) \text{ for } t = 0, 1, 2, \ldots. \tag{1}$$

To obtain $\boldsymbol{x}(t+1)$ here we just need to know $\boldsymbol{x}(t)$ and $\boldsymbol{d}(\boldsymbol{x}(t))$.

2.1 A Possibility for $\boldsymbol{d}(\boldsymbol{x})$ with one OD pair

Let Δ_{rs} have -1 in the rth place and +1 in the sth place. It is the swap vector from route r to route s. For $k > 0$ small let

$$\boldsymbol{d}(\boldsymbol{x}) = \sum_{r,s} k[c_r(\boldsymbol{x}) - c_s(\boldsymbol{x})]_+ x_r \Delta_{rs}. \tag{2}$$

Here, in (2), the c_r is costs of traversing route r, and $u_+ = \max(0, u)$. The $\boldsymbol{d}$ in this equation is the assumption in Smith [1984c]. We assume here that the costs arising in equation (2) depend continuously on only $\boldsymbol{x}$.

It is clear that $\sum_r x_r$ stays constant. That is:

$$\sum_r x_r(t+1) = \sum_r x_r(t) \text{ for all } t = 0, 1, 2, 3, \ldots.$$

So if the dynamical system (1) starts within the set $\{\boldsymbol{x} : \sum_r x_r = N\}$ then it remains within that set. Further, $\boldsymbol{d}$ is a continuous function of flows and costs and so, for any given continuous cost flow function $c = c(\boldsymbol{x})$, $\boldsymbol{d}$ becomes a continuous function of the current flow vector. It may then readily be shown that, under natural conditions and provided k is sufficiently small,

$$\boldsymbol{x} \geq \boldsymbol{0} \Rightarrow \boldsymbol{x} + \boldsymbol{d}(\boldsymbol{x}) \geq \boldsymbol{0}$$

where the vector inequality should be interpreted componentwise.

The two previous paragraphs imply that if the assignment $\boldsymbol{x}$ is feasible then $\boldsymbol{x} + \boldsymbol{d}(\boldsymbol{x})$ is also feasible. Thus if we denote the set of feasible route-flows by F so that

$$F = \{\boldsymbol{x} : \sum_r x_r = N, \boldsymbol{x} \geq \boldsymbol{0}\}$$

then for all $t = 0, 1, 2, 3, \ldots$:

$$\boldsymbol{x}(t) \in F \Rightarrow \boldsymbol{x}(t+1) \in F.$$

Thus if $\boldsymbol{x}(0) \in F$ then $\boldsymbol{x}(t) \in F$ for all $t > 0$. In this paper we regard (2) as one possible reasonable representation of day to day deterministic behaviour.

3 A Deterministic Process Day-to-Day Example

This simple example is based on a simple network considered by Daganzo [1983], Watling [1996], Watling and Hazelton [2003] and Bie and Lo [2010]. The network has two routes as shown in figure 1.

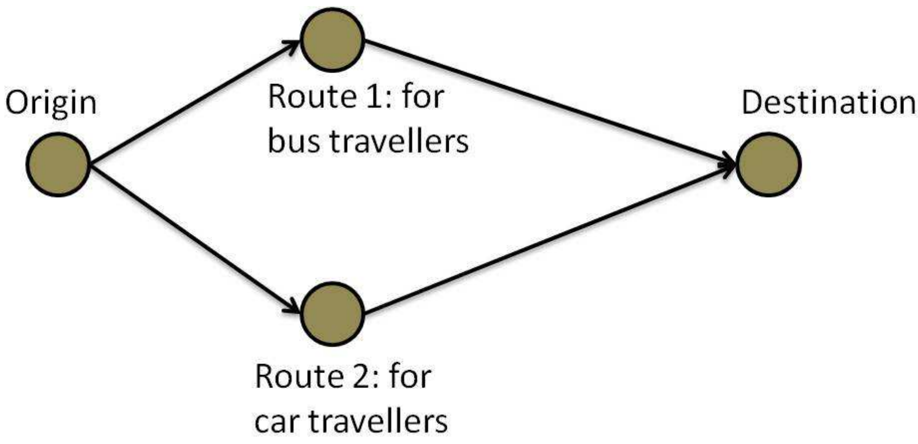

Figure 1: A two route network.

Here we let $x_1 \geq 0$ be the number or flows of bus travellers; $x_2 \geq 0$ be the number or flow of car travellers; and set the total travel demand to be $x_1 + x_2 = N$. Travel costs in this model are to be as follows:

$$c_1(\boldsymbol{x}) = \frac{8x_2}{N}$$

while that for car travel is

$$c_2(\boldsymbol{x}) = 2 + \frac{4x_2}{N}.$$

Thus the costs of bus and car journeys depend only on the volume of cars. We may make costs of bus and car travel depend only on the volume of bus and car travel respectively by utilising the condition that $x_1 + x_2 = N$. Doing this we obtain the separable cost functions

$$c_1(x_1) = 8 - \frac{8x_1}{N}$$

and

$$c_2(x_2) = 2 + \frac{4x_2}{N}.$$

Clearly

$$c_2(x_2) - c_1(x_1) = 2 + \frac{4x_2}{N}\left(8 - \frac{8x_1}{N}\right) = \frac{4x_2}{N} + \frac{8x_1}{N} - 6$$

and so

$$c_2(x_2) - c_1(x_1) = 0 \Leftrightarrow x_2 = x_1 = N/2.$$

Thus following Wardrop [1952] we say that $(N/2, N/2)^{\mathsf{T}}$ is an equilibrium: no traveller in this case has a route with a cheaper cost and so there is an equilibrium.

Looking now at unsymmetrical traffic distributions,

$$\begin{aligned} c_2(x_2) - c_1(x_1) > 0 \quad &\Leftrightarrow \quad x_1 > N/2 \\ c_2(x_2) - c_1(x_1) < 0 \quad &\Leftrightarrow \quad x_1 < N/2. \end{aligned}$$

Following (2), we have $\Delta_{12} = (-1, 1)^{\mathsf{T}}$ and $\Delta_{21} = (1, -1)^{\mathsf{T}}$ so that for small $k > 0$,

$$\boldsymbol{d} = k\left\{[c_1 - c_2]_+ x_1 \Delta_{12} + [c_2 - c_1]_+ x_2 \Delta_{21}\right\}. \tag{3}$$

With this particular formula for $\boldsymbol{d}$,

$$\begin{aligned}
x_1 = N/2 \quad &\Rightarrow \quad c_2 - c_1 = 0 \Rightarrow \boldsymbol{d}(\boldsymbol{x}) = \boldsymbol{0}, \\
x_1 > N/2 \quad &\Rightarrow \quad c_2 - c_1 > 0 \Rightarrow \boldsymbol{d}(\boldsymbol{x}) = k(c_2 - c_1)x_2\Delta_{21}, \\
x_1 < N/2 \quad &\Rightarrow \quad c_2 - c_1 < 0 \Rightarrow \boldsymbol{d}(\boldsymbol{x}) = k(c_1 - c_2)x_1\Delta_{12}.
\end{aligned}$$

where we have suppressed the dependence of the costs c_1 and c_2 on the flows x_1 and x_2.

Thus there is more than just the symmetrical Wardrop equilibrium here. It is clear from the previous paragraph that also

$$\begin{aligned}
x_1 = 0 \quad &\Rightarrow \quad \boldsymbol{d}(\boldsymbol{x}) = \boldsymbol{0} \\
x_2 = 0 \quad &\Rightarrow \quad \boldsymbol{d}(\boldsymbol{x}) = \boldsymbol{0}.
\end{aligned}$$

It follows that there are three equilibria (where $\boldsymbol{d} = 0$) and also that the directions of motion, $\boldsymbol{d}$, at non-equilibria are as shown in Figure 2 below. (As indicated above, for k small enough $\boldsymbol{x} + \boldsymbol{d}$ is feasible for each feasible $\boldsymbol{x}$.) An alternative representation of the dynamical properties of the system is displayed in Figure 3

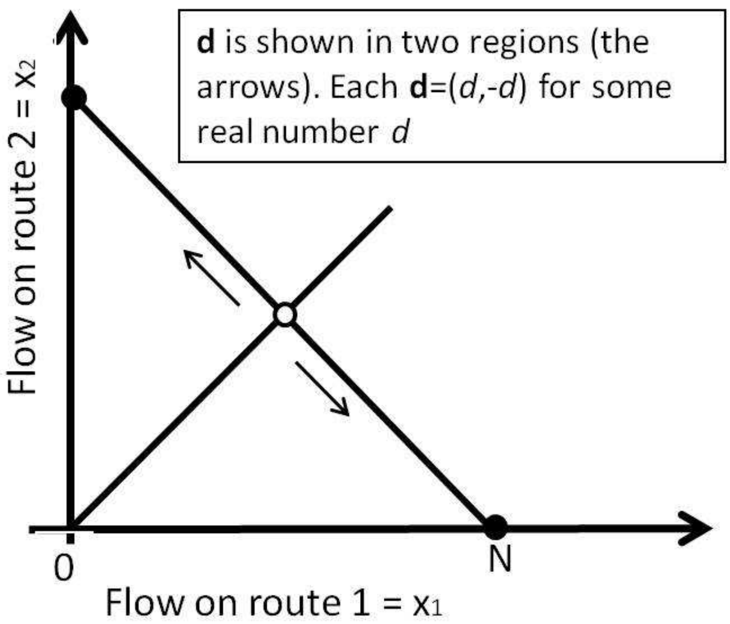

Figure 2: Deterministic dynamics following $\boldsymbol{d}(\boldsymbol{x})$ is indicated by the arrows for one value of N. Three equilibria are shown as dots (solid for unstable; open for stable). The two solid dots form the only irreducible stable or absorbing sets. All other states (or sets of states) are either transient non-equilibria or unstable equilibria.

It should be noted that we may allow time t in the system (1) to be made continuous rather than discrete. So we may also consider the continuous dynamical system

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{d}(\boldsymbol{x}(t)). \tag{4}$$

In the figures below we may think in terms of either (1) or (4).

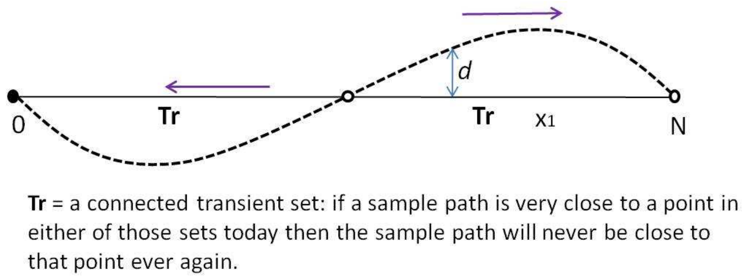

Figure 3: Another representation of the deterministic dynamics in Figure 2. The horizontal arrows correspond to the arrows in Figure 2. The dotted line is the graph of d (the first component of $\boldsymbol{d}$) against x_1. The slope of d is always between -1 and +1 here.

4 A Possible Stochastic Process Version of This Day-to-Day Model

One simple way of transforming the above deterministic process model into a stochastic process one is to add a stochastic component $\boldsymbol{\varepsilon} = (\varepsilon, -\varepsilon)^{\mathsf{T}}$ to (1). Then we obtain:

$$\boldsymbol{x}(t+1) = \boldsymbol{x}(t) + \boldsymbol{d}(\boldsymbol{x}(t)) + \boldsymbol{\varepsilon}. \tag{5}$$

Here we assume that each random component ε is uniformly distributed over $[-m, m]$ for some $m > 0$. We remark, as noted earlier, that this is a rather different way to introducing stochasticity into day-to-day dynamics than has previously been considered in the transportation field.

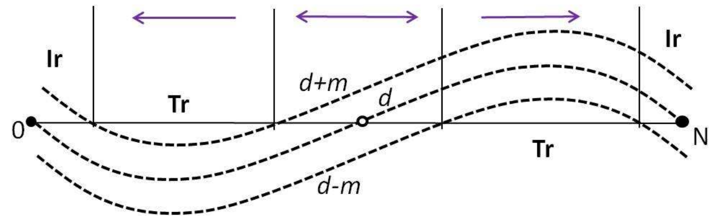

Figure 4: Representation of stochastic/deterministic dynamics. The double-headed arrow marks a thick set separating two basins of attraction. Points in the transient sets marked Tr can only exit one way. Ir marks sets which are stable but no subset is stable. They are irreducible stable sets.

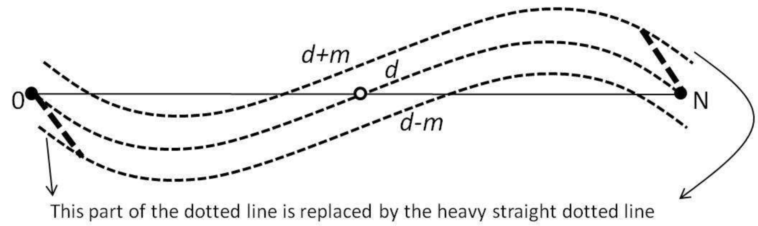

Figure 5: Stochastic/deterministic dynamics with edge modification. The edge modifications on ε are specified diagrammatically here.

In the dynamical system (5) and in Figure 4 as drawn, if $x_1(t)$ is very close to N today then $x_1(t+1) = x_1(t) + d + m$ will exceed N. To avoid this $d + m$ must be modified on the right and $d - m$ must be modified on the left of Figure 5 as shown. The heavier dotted lines in Figure 5 are at 45 degrees. These two modifications ensure (for example) that, close to N, $d + m$ cannot exceed $N - x_1(t)$. Hence

$$x_1(t) \le N \Rightarrow x_1(t+1) = x_1(t) + d + m \le x_1(t) + N - x_1(t) = N.$$

5 An alternative stochastic process specification: Markov Models with Stochastic Costs

The Markov models of Cascetta [1989] and Cantarella and Cascetta [1995] provide an alternative stochastic version of (1). Using a simple Markov model, each traveller on the system will independently review their mode choice on each day based upon the costs experienced the day before. As a result, x_1^{t+1} will follow a binomial $\mathsf{Binom}(N, q^{t+1})$ distribution where the probability of any given individual choosing bus travel (i.e. selecting route 1 in our two route example) on day $t+1$ is given by

$$q^{t+1} = \phi(c_2^t - c_1^t) \tag{6}$$

for some function $\phi(\cdot)$. Note that we have employed superscript t (as opposed to t in brackets) to denote time here (and henceforth in this section) to produce tidier notation.

The function ϕ may be defined implicitly if we focus on the distribution of perceived travel costs. Let us denote by C_i^t the travel cost for route i on day t as perceived by a randomly selected traveller, and let c_i^t be the corresponding mean (or 'measured') cost. A common type of model is

$$C_i^t = c_i(\boldsymbol{x}^t) + \varepsilon_i^t$$

where ε_i^t is a random variable with zero mean, independent of ε_j^s for all pairs $(j, s) \neq (i, t)$. We assume that the distribution of ε_i^t does not depend on c_i. We then have

$$\phi(c_2^t - c_1^t) = \mathbb{P}(C_1^t - C_2^t < 0).$$

It follows that q^{t+1} can take the values in the set $\{0, 1\}$ if and only if the distribution of ε has finite support.

In order to analyse such a process we require an analogous notion of steady-state equilibrium (and the possible convergence to multiple such equilibria) to that we have adopted for deterministic systems in sections 2.1 and 3; for a more in-depth discussion of these issues the reader is referred to Watling and Cantarella [2012]. We note first that in the model we consider in the present section, we have made a subtle change to focus on discrete, integer flow states (as generated by the Binomial transitions), and so the only possible states that the process may occupy are for x_1 in the set $S = \{0, 1, 2, ..., N\}$. Since the process is 1-dependent (only depends on the previous day's state), it follows that on any day we may fully define the probabilities of the process occupying these different states by a vector of length $N+1$. Let us consider any one such possible vector of probabilities, which we shall denote as $\boldsymbol{p}$, which clearly must have elements on [0,1] and which must sum to 1. Define $E \subseteq S$ to be an *ergodic subset* of S if it is a minimal subset (no proper subset of it exists with this property) such that there is zero probability of the process leaving it once entered. Suppose now that we initialise the process within one such ergodic subset, with initial state probability distribution $\boldsymbol{p}$. Then if the process transforms (after one day) into the same probability distribution $\boldsymbol{p}$ then we say that $\boldsymbol{p}$ is a stationary distribution over ergodic subset E. Thus, our analogy with the study of multiple equilibria in the deterministic process world is an interest in combinations of $(\boldsymbol{p}, E)$ that gives us multiple stationary distributions $\boldsymbol{p}$ over ergodic subsets of S; in both cases/notions we can consider the potential for convergence of the process to these steady-states, in the deterministic case to a single point and in the stochastic case to a vector of state probabilities. In the deterministic limit, our transitions are single valued and map to a single state, and so this definition also extends to the deterministic world, where our ergodic subsets are singletons, and stationarity maps to the deterministic notion of equilibrium. Thus, with our proposed model we will aim to bridge the deterministic process and stochastic process worlds.

Now, the limiting dynamical properties of our stochastic model depend critically on whether q^{t+1} can take the values zero or one. If q^{t+1} is restricted to the open interval $(0, 1)$ then the overall process will be ergodic (exactly one stationary distribution will exist), and will converge in law to a unique stationary distribution. In that case every flow pattern will be visited infinitely often in the long run. If, on the other hand, q^{t+1} can take the value 0 and/or 1, then the states may split into separate transient and persistent classes, with states $x_1 = 0$ and $x_1 = N$ possibly being absorbing. When the states do split in this way there is no unique stationary distribution over the whole state space, while there may or may not be multiple stationary distributions over ergodic subsets.

Consider our bus/car example with $N = 10$ travellers. We will look first at a truncated linear probability model where

$$\phi(c_2^t - c_1^t) = \begin{cases} 1 & \frac{1}{2} - \frac{\beta}{4}(c_2^t - c_1^t) > 1 \\ 0 & \frac{1}{2} - \frac{\beta}{4}(c_2^t - c_1^t) < 0 \\ \frac{1}{2} - \frac{\beta}{4}(c_2^t - c_1^t) & \text{otherwise.} \end{cases}$$

in which β is a parameter controlling travellers' sensitivity to cost differences. (The reason for the factor of 4 in the denominator in the definition of ϕ is to make this parameter directly comparable with β in a logit model that we shall examine shortly.) In this model the probability q^{t+1} will take values zero and one respectively when $c_2^t - c_1^t \geq 2/\beta$ and $c_2^t - c_1^t \leq -2/\beta$. If we use the cost functions given in the introduction then these inequalities cannot hold if $\beta < 1$, and hence q^{t+1} can never take the value zero or one. When $\beta = 1$ the inequalities hold only when $x_1^t = 0$ (so that $c_2^t - c_1^t = 2$) and $x_1^t = N$ (so that $c_2^t - c_1^t = -2$) respectively. For any $\beta \geq 1$ the states $x_1 = 0$ and $x_1 = N$ are absorbing and all other states are transient.

In some senses the model for $\beta \geq 1$ mirrors the deterministic process one, in that a unique fixed point equilibrium will be attained in the log run. However, unlike the deterministic process case, we generally cannot be sure which equilibrium state will eventuate if the result of an intervention is to move the system to one of the transient states.

To illustrate this behaviour we compute (by Monte Carlo simulation) the probability $p_i(\beta)$ that, starting from initial state $x_1 = i$, the system reaches $x_1 = 10$ (universal bus usage) before hitting $x_1 = 0$. For $\beta \geq 1$ the system will remain at $x_1 = 10$ thereafter, while for $\beta < 1$ the system will continue to move around according to its stationary distribution in the long run. In Figure 6 we plot p_i for $i \in \{0, 1, \ldots, 10\}$ for $\beta = 1/2, 1, 2$ and 4. We superimpose the stationary distribution (when it exists) on these plots, using thick vertical grey lines to indicate *relative* probabilities.

As one would expect, when travellers are insensitive to cost differences ($\beta = 1/2$) the stationary distribution is concentrated around $x_1 = 5$. Any attempt to force the system towards $x_1 = 10$ by nudging x_1 towards 7 or 8 is by no means certain of success. For large values of β, on the other hand, we can be certain of converging to $x_1 = 10$ by a sufficiently large nudge.

A more common probability model is the logit model (equivalent to assuming that the negative of the random term, $-\varepsilon$, follows a Gumbel distribution centred on zero). In that case

$$\phi(c_2^t - c_1^t) = \frac{1}{1 + \exp\{\beta(c_2^t - c_1^t)\}}$$

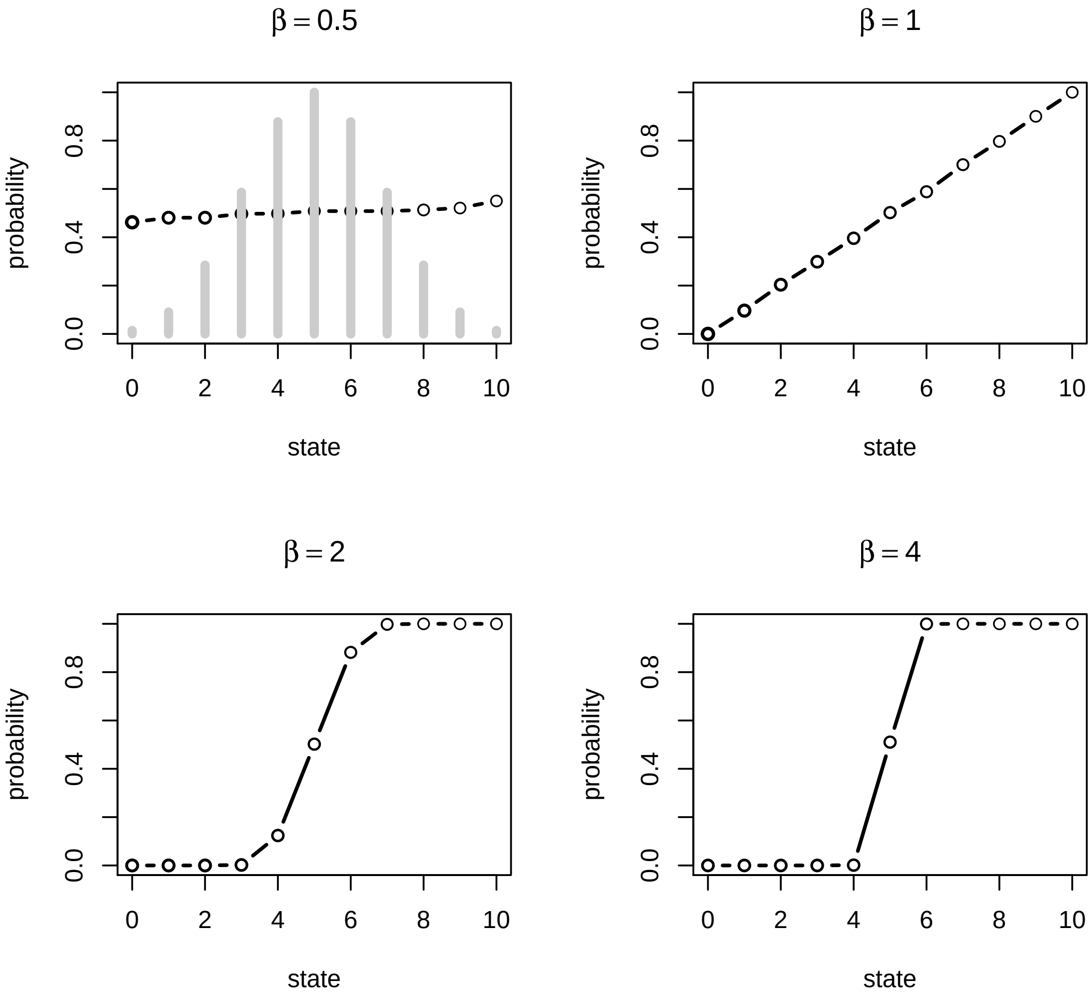

Figure 6: Plots of the probability of reaching universal bus usage before universal car usage for a stochastic process day-to-day model with truncated linear probability function. Vertical grey lines represent relative probabilities from the stationary distribution for the system (where this exists).

where β is again a cost sensitivity parameter. The truncated linear model earlier can be consider an approximation to the logit model as illustrated in Figure 7 (where the value of β is common in both functional forms due to the scaling mentioned earlier).

We repeated the analysis of the probability of attaining universal bus usage prior to universal car usage. The resulting plots (in corresponding format to Figure 6) are displayed in Figure 8. These results are qualitatively similar to those for the truncated linear model, as one might expect given the similarity of the underlying probability functions. However, the probabilities for reaching universal bus usage never quite reach zero or one, so that in theory at least we can never be certain of achieving the desired goal of universal bus usage.

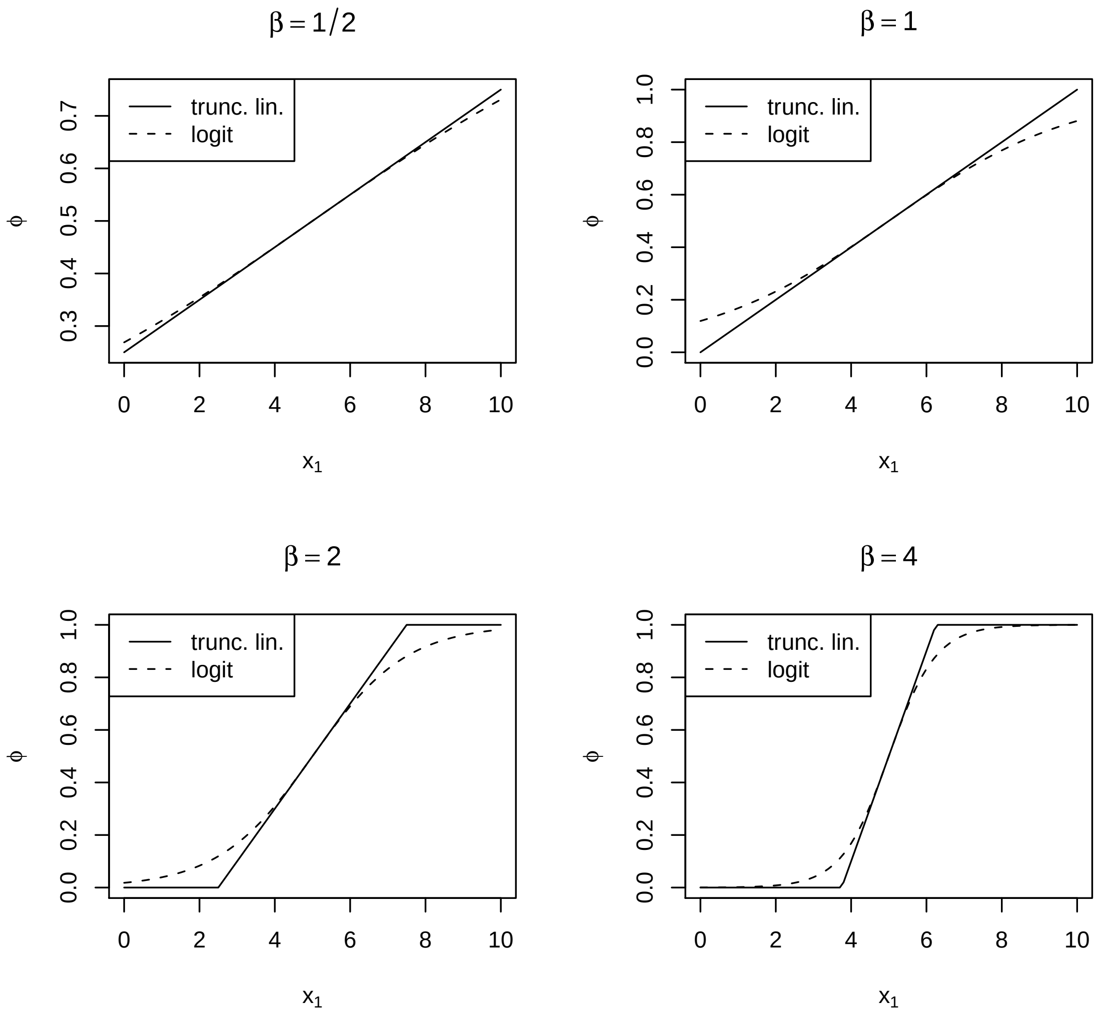

Figure 7: Graphical comparison of truncated linear and logit probability functions.

6 Discussion

Usually, for a given initial or start point, deterministic process, day to day route choice models generate unique trajectories which are confined to that basin of attraction containing the start point. These models are usually designed to converge under suitable conditions to a single (locally stable) equilibrium point within that single basin. This may then be regarded as a point estimate or forecast of the likely long run behaviour of the given network for the particular start point. However, different forecasts may arise from different start points.

On the other hand, the stochastic process models previously proposed in the transportation literature are usually specified so as to yield a single limiting probability distribution which is spread over the whole state space, irrespective of the particular state chosen as a start point. For example, this is true of Markov models with finite memory in which conditional choice probabilities are given by a logit-based model. So stochastic process models yield a unique probability distribution describing the likely long run behaviour of the given network, from which we can calculate unique summary measures for prediction. Thus, with the stochastic process model our long-term predictions of the system will not

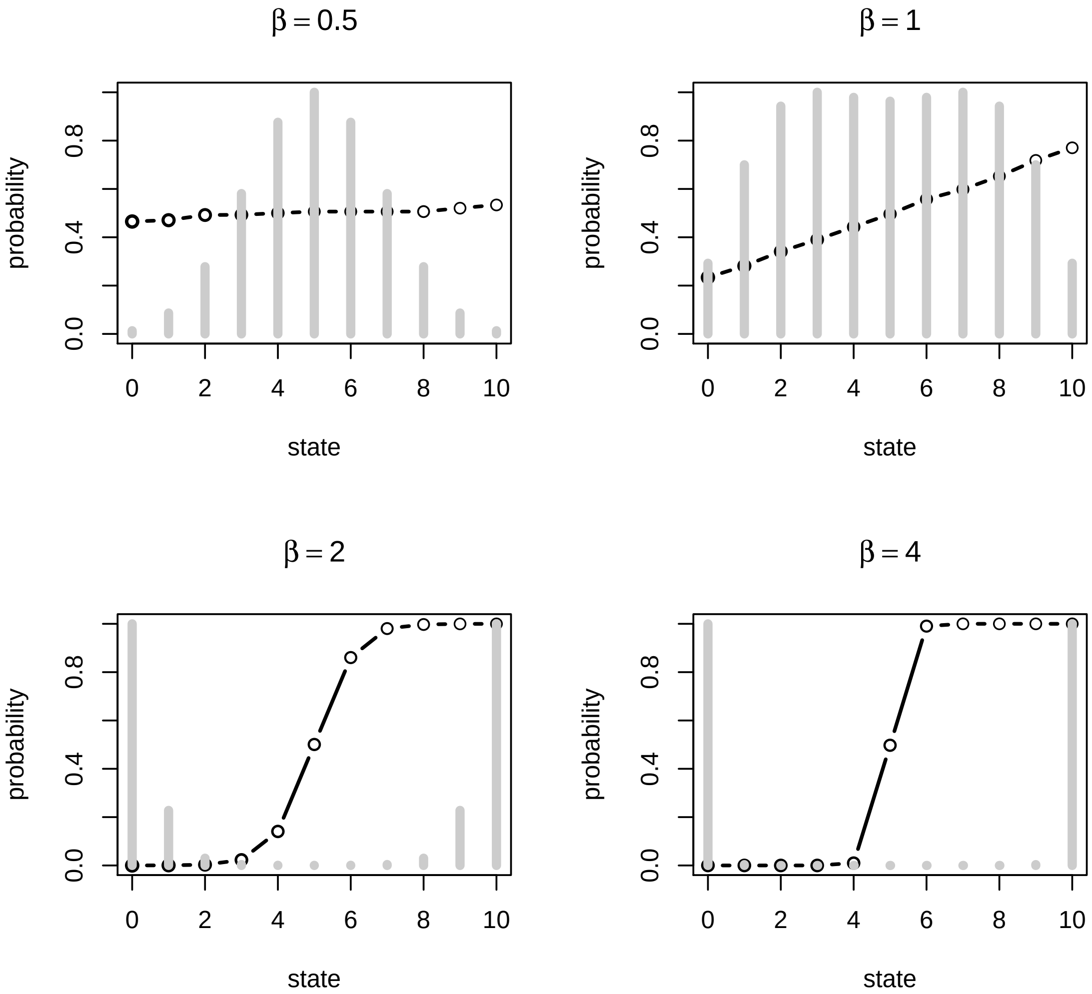

Figure 8: Plots of the probability of reaching universal bus usage before universal car usage for a stochastic day-to-day model with logit probability function. Vertical grey lines represent relative probabilities from the stationary distribution for the system.

depend on the initial conditions, whereas the deterministic process model predicts multiple trajectories depending on the initial conditions. This difference may have a profound effect on predictions of the network response to an intervention. For instance, in the bus/car example above an attempt to nudge the system towards universal bus usage (say by imposing a temporary charge or a temporary access restriction) leading to a pattern where $x_1 = 8$ (and then removing the charge or access restriction) will lead to swift convergence to the desired state according to the deterministic process model. However in the logit-based stochastic process model the mean first hitting time at $x_1 = 10$ can be very large or even infinite because there is a small (but non-zero) probability that the system will visit states close to $x_1 = 0$ and spend a long spell in the vicinity of that state. At the moment, we do not offer a suggestion as to which of these approaches is more *realistic*, the point is that qualitatively different system behaviour may occur in the *model*, and so for the benefit of model-users it seems important that we better understand these differences.

It is therefore natural, we believe, to explore whether dynamical models exist which lie between these two extremes. In this paper we have been able to construct two such models. The first is obtained by adding a not-too-large stochastic element to a deterministic process, day-to-day model. With this model day to day trajectories become stochastic and forecasts become more blurred; different sample paths or model trajectories, eventually entering different basins of attraction, may arise even from the same start point. The second model starts with a standard stochastic process model and then restricts the stochasticity. With this model, if certain states are reached then the process cannot then visit certain other states; in the case here (where we have chosen the truncated linear rather than the logit for representing the dependence of choice probabilities on predicted costs) the system is bound to remain in that state.

Generally, perhaps the long-run dynamical properties of the model types studied in this note can be thought of as forming a continuum, with convergence to a specific fixed-point present in the deterministic model at one end, and the unique limiting stationary distribution spread over the whole state space at the other end. The two intermediate stochastic models designed here lie somewhere in between, in that they both possess two distinct basins of attraction (and so in this respect are 'like' a deterministic model), but behaviour in the transient states before a basin of attraction is entered is a random process. Furthermore, behaviour *within* a basin of attraction may be random too.

The long-term dynamics we have investigated in the present paper focuses on the 'unforced' response of a dynamical system due to the initial condition. However, as we suggested in section 1.2, our reason for understanding such dynamics may be a desire to control or influence the system trajectory in some way. For example, given multiple future trajectories, then a planner may wish to do influence the dynamics toward the most desirable trajectory based on some normative measure (sustainability, congestion, etc.). In this case, the way of influencing the system may itself be dynamic (rather than a one-off policy measure), responding to the changing demands over time. Such a study could, for example, be a stochastic process counterpart to the study of responsive signal-settings in a deterministic process environment as reported in Cantarella, Velonà and Vitetta [2012].

Finally, the question of which of the approaches considered in the paper would better approximate reality, we believe, depends on the problem context and objectives, the day-to-day dynamics adopted and the associated adjustment parameters. This is an important future research direction, which may be partially informed by the empirical analysis of actual observations.

Acknowledgements

We would like to thank the anonymous referees for useful suggestions that have helped to clarify the exposition of the paper.

Mike Smith acknowledges financial support provided by a research fund of Giulio Cantarella at the University of Salerno and the Engineering and Physical Sciences Research Council, UK, via the FREEFLOW project.

Martin Hazelton acknowledges financial support from the Marsden Fund administered by the Royal Society of New Zealand.

Hong Lo acknowledges the financial support provided by Competitive Earmarked Research Grant #617209 from the Research Grants Council of the Hong Kong Special Administrative Region and Special Research Funding Initiative #SRFI11EG15.

Giulio E. Cantarella acknowledges the financial support provided by PRIN national grant n. 2007R9CSXY_004 financial year 2007 and UNISA local grant n. ORSA091208 financial year 2009.

David Watling acknowledges the financial support provided to him by a research fund of Giulio Cantarella at the University of Salerno.

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