Large Sets in Constructive Set Theory

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Submitted in accordance with the requirements for the degree of Doctor of Philosophy

The University of Leeds

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December 2014

Acknowledgements

I would like to thank my supervisor Professor Michael Rathjen for sharing his vast experience and for providing valuable guidance through both the mathematical and the organisational challenges of life as a PhD student.

I am grateful to the Leeds Logic Group for their continued support and stimulating discussions, for organising interesting talks and providing such a friendly environment.

As a fellow of the network "From Mathematical Logic to Applications" (MALOA), I would also like to thank the European Union's Marie Curie Initial Training Networks programme for their funding and training.

I would also like to express my gratitude to my family for their love and support.

Abstract

This thesis presents an investigation into large sets and large set axioms in the context of the constructive set theory CZF.

We determine the structure of large sets by classifying their von Neumann stages and use a new modified cumulative hierarchy to characterise their arrangement in the set theoretic universe. We prove that large set axioms have good metamathematical properties, including absoluteness for the common relative model constructions of CZF and a preservation of the witness existence properties CZF enjoys. Furthermore, we use realizability to establish new results about the relative consistency of a plurality of inaccessibles versus the existence of just one inaccessible. Developing a constructive theory of clubs, we present a characterisation theorem for Mahlo sets connecting classical and constructive approaches to Mahloness and determine the amount of induction contained in the assertion of a Mahlo set. We then present a characterisation theorem for 2-strong sets which proves them to be equivalent to a logically simpler concept.

We also investigate several topics connected to elementary embeddings of the set theoretic universe into a transitive class model of CZF, where considering different equivalent classical formulations results in a rich and interconnected spectrum of measurability for the constructive case. We pay particular attention to the question of cofinality of elementary embeddings, achieving both very strong cofinality properties in the case of Reinhardt embeddings and constructing models of the failure of cofinality in the case of ordinary measurable embeddings, some of which require only surprisingly low conditions. We close with an investigation of constructive principles incompatible with elementary embeddings.

To Annika and Fiona

Contents

List of figures

List of tables

Chapter 1

Introduction

Large cardinals play a pivotal role in classical set theory, both as useful tools to be applied in other set theoretic pursuits and as worthy objects of study in their own right, and even as ways to bolster ZFC's strength in order to handle mathematical challenges outside of set theory (e.g. by supplying Grothendieck universes). There is little reason to believe that in the context of constructive set theory, similar axioms would prove less fruitful. On the contrary, due to the more modest strength claimed by predicative systems of constructive set theory such as CZF, which has achieved a prominent role as foundation for constructive mathematics since its introduction in [Acz78], it seems that mathematicians working in these systems might be forced to revert to an axiom of infinity more often than their classical colleagues if they want to achieve general results. Thus the advantage of adopting axioms of largeness might be even greater than in classical set theory. And indeed in the context of CZF, axioms like REA (which can be considered to be a principle of largeness) are commonly used e.g. in the development of formal topology or for general induction.

Furthermore, systems which strive to preserve some level of predicativity like CZF present avenues of investigation which are closed for systems of as yet unmanageable strength like ZFC. Most prominently, the much praised linearity of consistency strengths of different large cardinal axioms becomes a quantifiable commodity in constructive set theory in the sense that the proof theoretic ordinal can be exactly determined for the system of CZF enhanced by large set axioms. Indeed, the research connected to constructive analoga for large cardinals in the context of CZF has in large part been focussed on their ordinal analysis (see e.g. [Rat98, Gib02, CR02, GRT05]).

In contrast, the theory of what could roughly be called the more set theoretic properties and consequences of principles of largeness has perhaps not been quite as thoroughly developed (at least above the level of what we will call "tiny" large sets, i.e. beyond regularity and its variants), and this thesis aims at making some contributions to remedy this situation. For this, we aspire to develop the theory of large sets in CZF in as systematic a way as can reasonably be expected for such an as yet untamed field, posing and answering basic questions about their characteristic features and exploring the realm of axioms of infinity from smaller infinities like those implied by the existence of regular models of set theory towards the enormous infinities asserted by elementary embeddings.

In this endeavour, we will mostly investigate concepts of largeness which in the presence of the excluded middle correspond to well known classical large cardinal axioms. However, large cardinal axioms can not be directly lifted from classical set theory without modification, as the very concept of a cardinal number does not seem to mesh well with constructive thought, and depending on its exact formulation is either very awkward to handle in a theory like CZF or implies the excluded middle outright. Instead, it has become standard procedure to focus on sets exhibiting the appropriate closure properties which could be expected from the von Neumann stage corresponding to the large cardinal. These properties will be detailed in Chapter 2, as well as general notational conventions and preliminaries about the system of constructive set theory we use.

We start our investigation in Chapter 3 by what can be figuratively seen as slicing large sets according to von Neumann rank and examining the individual slices. This would be a futile exercise in classical set theory as every slice from an inaccessible would either be empty or identical to the corresponding slice from any other inaccessible. This is not the case in the context of CZF, although it turns out that all the variation can be captured in the third and fourth slice[1]: If those are identical, then the rest of the inhabited slices are as well. In case the Subcountability axiom holds, the third slice alone can even be used to determine the whole set.

We continue in Chapter 4 by analysing desirable properties of large set axioms, in particular absoluteness for realizability and Heyting models as well as witness existence properties like the disjunction property and numerical existence property. This involves relative model constructions for CZF, namely models based on applicative topologies for the absoluteness results and models based on realizability with truth for the witness existence properties.

The following two chapters are concerned with the number and order of inaccessible sets in the set theoretic universe. In Chapter 5, we develop a variation of the classical von Neumann hierarchy which retains its potency even in the absence of the excluded middle and the Powerset axiom. This not only allows us to deduce the arrangement of large sets as ordinally ordered, but also serves to reconnect large sets to the ordinals they correspond to in classical set theory. The number of large sets on the other hand behaves very differently than in classical set theory, where it is tied directly to consistency strength. In Chapter 6, we will use a realizability model with an inaccessible pca to demonstrate how to obtain proper classes of inaccessibles with just one inaccessible assumed to exist in the background universe.

Chapter 7 turns to what is usually considered to be the next important concept of largeness after inaccessibility and α-inaccessibility, namely Mahlo's second hierarchy of infinities. While the established approach in a constructive context is via the reflection of total relations, we first develop a constructive theory of clubs to investigate the classical definition via ordinals and then use the modified von Neumann hierarchy from Chapter 5 to establish an equivalence of the constructive and classical definitions even in the absence of the excluded middle. The chapter is rounded off by an investigation of the amount of induction contained in Mahloness and the characterisation of the claim that the universe is Mahlo by a nondeterministic induction principle.

[1] i.e. $I \cap V_2$ and $I \cap V_3$

Next in Chapter 8 we address weakly compact cardinals, which can be characterised in many different ways in ZFC. After exploring some of these formulations in a setting devoid of the excluded middle, we turn our focus to the variant which has been developed most extensively in a constructive context, namely 2-strongness. It turns out that it is possible to characterise 2-strong sets by a considerably simpler definition, which like 2-strongness itself corresponds to the indescribability facet of classical weak compactness.

The last but also most extensive chapter deals with elementary embeddings[2] and the very large sets which constitute their critical points. In Section 9.1 we revisit a distinction made by Friedman and Ščedrov in the context of IZF, namely that it makes quite a difference which large set condition one imposes on a critical point. We delve into the spectrum of measurability generated by this distinction and show how to enhance the largeness properties of critical points using induction. In Section 9.2 we unlock the power of a different weakening of measurability, namely Δ_0-elementary embeddings. We show that with care, these too can be made to imply all the usual large set principles below measurability, although even in the presence of classical logic they need cofinality as an extra assumption to imply full measurability.

The cofinality of elementary embeddings is the topic of Section 9.3, where we show that while the ordinary proof of cofinality using classical logic fails utterly in CZF, in the case $j : V \to V$ it is still possible to establish very strong cofinality properties which even transcend the ones possible in the classical case, using a newly defined ordering relation on the set theoretic universe. The next two sections however show that in the general case, elementary embeddings need not be cofinal in CZF. Chapter 9.4 constructs a relative model with a map $j : V \to M$ outright without any extra assumptions on the background universe, where however the elementarity scheme only holds as implication, not as equivalence (i.e. $\Phi(\overrightarrow{a}) \to \Phi^M(j(\overrightarrow{a}))$. While classically equivalent to a measurable cardinal, the fact that this is relatively consistent with CZF shows that constructively, it constitutes a much more contained concept. We show that this model

[2]To be precise, we investigate elementary embeddings $j : V \to M$ such that the axiom schemes of set theory also hold for formulae that contain j and M — elementary embeddings fulfilling this condition are equivalent to the existence of measurable cardinals in ZFC.

refutes cofinality. In Chapter 9.5 we construct a model refuting cofinality for a fully elementary embedding using the existence of such an embedding as assumption on the background universe. We develop some new tools to prove that certain sets lie outside a transitive class model.

This investigation closes in Section 9.6 with a result that seems limiting and yet also constitutes a bridge from the inherently constructive axiom of Subcountability back to the classical axiom $V = L$: It turns out that these two axioms from very different backgrounds have a very similar effect on elementary embeddings, namely they both deny the possibility of their existence. This has some fundamental consequences for the search of models for CZF with elementary embeddings and depending on one's branch of constructivism also has critical implications for the very status of elementary embeddings as constructively admissible concepts.

Chapter 2

Preliminaries

2.1 Notations and Conventions

Background and Metatheory

The subject theory of the investigations presented in this thesis is the constructive set theory CZF. While we will never reason directly in the metatheory, any theory capable of basic primitive recursive reasoning can serve as metatheory, e.g. PRA [Sko23].

When a result in the (possibly extended) language of set theory is presented without reference to an axiom system of set theory, this is to be read as the claim that the statement is a theorem of CZF (possibly after all used definitions have been translated back into the language of pure set theory). If the result is stated with the name of a different axiom system in brackets directly after the type of result (e.g. "Theorem (ZF)" or "Lemma (IZF)"), then this is to be read as the claim that the statement is a theorem of that axiom system. When working in an extension of CZF, we will usually write "([Axiom])" instead of "(CZF + [Axiom])".

Where the existence of a class with certain properties is claimed, this is to be read that the metatheory proves the existence of a specific class term (which can be extracted from the proof). Where something is claimed for all classes, this is to be read as a scheme of statements.

Order of Operations

In order to suppress brackets and increase readability, the order of operations for logical symbols in formulae is listed in Table 2.1.

$\forall x, \exists x, \exists! x, \forall x \in a, \exists x \in a, \exists! x \in a$	quantifiers
$\wedge, \vee$	monotone junctors
$\rightarrow, \leftarrow, \leftrightarrow$	nonmonotone junctors
$\forall x., \exists x., \exists! x., \forall x \in a., \exists x \in a., \exists! x \in a.$	quantifiers with full stops
$\Vdash, \vDash$	model relations
$:\leftrightarrow$	definitional equivalence
$\vdash$	logical implication

Table 2.1: Orders of Precedence for Logical Symbols

A colon before a equivalence ($:\leftrightarrow$) has no logical meaning except the above mentioned bearing on the order of precedence. It is used solely for readability and is meant to stress that the integral part of the left equivalent is defined by this equivalence (we will employ $:=$ with the analogous usage), e.g. defining unique existence with the formula

$$\exists! x \Phi(x) :\leftrightarrow \exists x \Phi(x) \wedge \forall x, y. \Phi(x) \wedge \Phi(y) \rightarrow x = y \tag{2.1.1}$$

A full stop after the bound variable of a quantifier has no logical meaning except the above mentioned bearing on the order of precedence. Thus $\forall x. \Phi \rightarrow \Psi$ is parsed as the same formula as $\forall x(\Phi \rightarrow \Psi)$ while $\forall x \Phi \rightarrow \Psi$ is parsed as the same formula as $(\forall x \Phi) \rightarrow \Psi$.

Bounded quantifiers are to be read as

$$\forall x \in a \Phi \quad :\leftrightarrow \quad \forall x . x \in a \rightarrow \Phi \tag{2.1.2}$$

$$\exists x \in a \Phi \quad :\leftrightarrow \quad \exists x . x \in a \wedge \Phi \tag{2.1.3}$$

$$\exists ! x \in a \Phi \quad :\leftrightarrow \quad \exists ! x . x \in a \wedge \Phi \tag{2.1.4}$$

Terms and Class Terms

Formally, the language of CZF only contains the symbols $\in$ and $=$, but we will use the usual set theoretic symbols to make up terms, in the knowledge that formulae containing them can be unwound and translated into formulae in the pure language. This will not clash with concepts such as bounded formulae, as per [AR10], Proposition 9.6.2, terms for globally definable functions can be conservatively added to CZF (and allowed to appear in the axiom schemata).

We will also use common abbreviations for formulae, e.g. $a \subseteq b :\leftrightarrow \forall x \in a . x \in b$. Such defined relation symbols can always be used in axiom schemes like Subset Collection and Strong Collection, but only in certain cases in the axiom scheme of Δ_0-Collection. Also, for any two place relation symbol $\mathcal{R}$ like $\in$ or $\subseteq$, we write $a\mathcal{R}b\mathcal{R}c$ to signify that $a\mathcal{R}b \wedge b\mathcal{R}c$. Similarly, we will use $a \ni b$ for $b \in a$ and $a \supseteq b$ for $b \subseteq a$.

Following the common traditions of set theory, we will also liberally use class terms of the form $\{t(x)|\Phi(x)\}$ which do not necessarily denote an element of the universe but which can be compared to them and to each other via the definitions listed in Table 2.2.

$\{t(x)\vert\Phi(x)\} \subseteq a$	$:\leftrightarrow$	$\forall x.\Phi(x) \to t(x) \in a$
$\{t(x)\vert\Phi(x)\} \supseteq a$	$:\leftrightarrow$	$\forall y \in a \exists x.y = t(x) \wedge \Phi(x)$
$\{t(x)\vert\Phi(x)\} = a$	$:\leftrightarrow$	$\{t(x)\vert\Phi(x)\} \subseteq a \wedge \{t(x)\vert\Phi(x)\} \supseteq a$
$\{t(x)\vert\Phi(x)\} \subseteq \{s(x)\vert\Psi(x)\}$	$:\leftrightarrow$	$\forall x.\Phi(x) \to \exists y.\Psi(y) \wedge t(x) = s(y)$
$\{t(x)\vert\Phi(x)\} = \{s(x)\vert\Psi(x)\}$	$:\leftrightarrow$	$\{t(x)\vert\Phi(x)\} \subseteq \{s(x)\vert\Psi(x)\}$
		$\wedge\{s(x)\vert\Psi(x)\} \subseteq \{t(x)\vert\Phi(x)\}$
$\{t(x)\vert\Phi(x)\} \in a$	$:\leftrightarrow$	$\exists b.b = \{t(x)\vert\Phi(x)\} \wedge b \in a$
$\{t(x)\vert\Phi(x)\} \ni a$	$:\leftrightarrow$	$\exists x.\Phi(x) \wedge t(x) = a$
$\{t(x)\vert\Phi(x)\} \in \{s(x)\vert\Psi(x)\}$	$:\leftrightarrow$	$\exists a.a = \{t(x)\vert\Phi(x)\} \wedge a \in \{s(x)\vert\Psi(x)\}$

Table 2.2: Atomic Formulae containing Class Terms

As usual, we will write $\{t(x) \in a|\Phi(x)\}$ and $\{t(x) \subseteq a|\Phi(x)\}$ as an abbreviation for $\{t(x)|x \in a \wedge \Phi(x)\}$ and $\{t(x)|x \subseteq a \wedge \Phi(x)\}$ respectively and when appropriate, we will identify an object a of the universe with the class term $\{x|x \in a\}$ which is equal to a.

In a slight abuse of notation, we will also use terms which operate on class terms. When unwinding their definitions, one arrives at a class term in each concrete instance. For example, for each class Γ, we can define its powerclass $\mathcal{P}(\Gamma)$, which means that for every concrete class term, we find a class term for its powerclass in a primitive recursive way.

Examples for such terms are presented in Table 2.3.

Symbol	Name	Definition
$\emptyset$, 0	empty set	$\{x\|\bot\}$
V	universe	$\{x\|\top\}$
$\mathcal{P}(a)$	powerclass	$\{b\|\forall x \in b.x \in a\}$
$a \cap b$	binary intersection	$\{x\|x \in a \wedge x \in b\}$
$a \cup b$	binary union	$\{x\|x \in a \vee x \in b\}$
$\bigcup a$	union	$\{x\|\exists y \in a.x \in y\}$
$\bigcup_{x \in a} t(x)$	union	$\bigcup\{t(x)\|x \in a\}$
$\bigcap a$	intersection	$\{x\|\forall y \in a.x \in y\}$
$\bigcap_{x \in a} t(x)$	intersection	$\{y\|\forall x \in a.y \in t(x)\}$
$\{a, b\}$	unordered pair	$\{x\|x = a \vee x = b\}$
$\{a\}$	singleton	$\{a, a\}$
$a + 1$	successor	$a \cup \{a\}$
ω	natural numbers	$\bigcap\{a\|0 \in a \wedge \forall x \in ax + 1 \in a\}$
(a, b)	ordered pair	$\{a, \{a, b\}\}$
$a \times b$	Cartesian product	$\{(x, y)\|x \in a, y \in b\}$
dom(f)	domain	$\{x\|\exists y.(x, y) \in f\}$
im(f)	range	$\{y\|\exists x.(x, y) \in f\}$
$f(a)$	application	$\{x\|\exists y \in f \exists b \in \bigcup\bigcup f.y = (a, b) \wedge x \in b\}$
$f''a$	function image	$\{f(x)\|x \in a \cap \text{dom}(f)\}$
$f^{-1}(a)$	function preimage	$\{x\|\exists y \in a.(x, y) \in f\}$
$f \restriction a$	restriction	$\{(x, y) \in f\|x \in a\}$

Table 2.3: Defined Terms

Total Relations and Functions

Different types of binary relations play an important role in CZF. Define the following concepts:

Symbol	Description	Definition
$R : A \rightrightarrows B$	total	$\forall a \in A \exists b \in B.(a, b) \in R$
$R : A \leftleftarrows B$		$\forall b \in B \exists a \in A.(a, b) \in R$
$R : A \leftleftarrows\!\rightrightarrows B$	bitotal	$R : A \rightrightarrows B \wedge R : A \leftleftarrows B$
$f : A \to_p B$	partial function	$\forall (a, b), (a, b') \in f \cap (A \times B).b = b'$
$f : A \to B$	functional	$f : A \rightrightarrows B \wedge f : A \to_p B$
$f : A \hookrightarrow B$	injective, 1:1	$f : A \to B$
		$\wedge \forall (a, b), (a', b) \in f \cap (A \times B).a = a'$
$f : A \twoheadrightarrow B$	surjective, onto	$f : A \to B \wedge \forall b \in B \exists a \in A.(a, b) \in f$
$f : A \hookrightarrow\!\!\!\!\rightarrow B$	bijective	$f : A \hookrightarrow B \wedge f : A \twoheadrightarrow B$

Table 2.4: Types of Relations

Note that in this definition, $f : A \to B$ does not imply that $f \subseteq A \times B$. We will however only call f a function from A to B if that is indeed the case and define the class of functions as

$$^{A}B := \{f \subseteq A \times B | f : A \to B\} \tag{2.1.5}$$

Analogously, the class of multivalued functions is defined as

$$\mathrm{mv}(A, B) := \{R \subseteq A \times B | R : A \rightrightarrows B\} \tag{2.1.6}$$

2.2 CZF, Similar Constructive Set Theories and Related Axioms

During the last century, classical set theory, in particular ZF set theory, has established itself as a good candidate for a possible framework in which to develop all of classical mathematics, adding choice principles or large cardinal axioms as needed. For constructive mathematics, set theoretic foundations seem to work similarly well, even though other avenues like type theories (e.g. [MLS84]) are also being pursued very successfully. The system CZF as developed by Aczel in [Acz78] has both the advantage of being a subsystem of classical ZF set theory and thus affording the possibility of developing large parts of mathematics in an established way and the advantage of being canonically interpretable into type theory and thus being able to draw from the philosophical justifications of type theory. It is a generalized predicative theory with a rich and interesting metatheory [AR10] and admits a host of useful model constructions [Rat03b, Gam06, Zie12].

A comprehensive reference for CZF is [AR10].

The Axioms of CZF

CZF uses intuitionistic logic with equality and a binary relation symbol $\in$. It uses the following axioms (formulated using abbreviations and conventions from the previous section), consisting of two general axioms:

Extensionality.

$$\forall a, b. a \subseteq b \subseteq a \rightarrow a = b \tag{2.2.7}$$

Set Induction. For all formulae $\Phi(x)$

$$((\forall x \in a \Phi(x)) \rightarrow \Phi(a)) \rightarrow \forall x \Phi(x) \tag{2.2.8}$$

Several basic (explicit) set existence axioms:

Pairing.

$$\forall a, b \exists c. c = \{a, b\} \tag{2.2.9}$$

Union.

$$\forall a \exists b. b = \bigcup a \tag{2.2.10}$$

Infinity.

$$\exists a.a = \omega \tag{2.2.11}$$

Emptyset.

$$\exists a.a = \emptyset \tag{2.2.12}$$

Binary Intersection.

$$\forall a, b \exists c.c = a \cap b \tag{2.2.13}$$

Instead of the axioms of Emptyset and Binary Intersection, the scheme of Δ_0-Collection could have been used as an axiom as it is equivalent to them on the basis of the other axioms of CZF [AR01]:

Δ_0-Collection. For any Δ_0 formula $\Phi(x)$ not containing b,

$$\forall a \exists b.b = \{x \in a | \Phi(x)\} \tag{2.2.14}$$

Furthermore, CZF includes two non-explicit set existence axiom schemes:

Strong Collection. For any class Γ,

$$\forall a.\Gamma : a \rightrightarrows V \to \exists b.\Gamma : a \leftleftarrows\!\!\rightrightarrows b \tag{2.2.15}$$

Subset Collection. For any class family of classes[1] $(\Gamma(u))_{u \in V}$,

$$\forall a, b \exists c \forall u.\Gamma(u) : a \rightrightarrows b \to \exists b' \in c.\Gamma(u) : a \leftleftarrows\!\!\rightrightarrows b' \tag{2.2.16}$$

[1] i.e. class with a free variable

On the basis of the other axioms of CZF, the axiom scheme of Subset Collection is equivalent to the single axiom of Fullness:

Fullness.

$$\forall a, b \exists c \forall R . R : a \rightrightarrows b \rightarrow \exists R' \in c . R' \subseteq R \wedge R' : a \rightrightarrows b \tag{2.2.17}$$

Such a c is said to be full in $\mathrm{mv}(a, b)$.

Variations of CZF

An explicit variant of Strong Collection which is only slightly weaker is Replacement:

Replacement. For any class Γ,

$$\forall a . \Gamma : a \rightarrow V \rightarrow \exists b . \Gamma : a \twoheadrightarrow b \tag{2.2.18}$$

Another slightly weaker varaint is Collection:

Collection. For any class Γ,

$$\forall a . \Gamma : a \rightrightarrows V \rightarrow \exists b . \Gamma : a \rightrightarrows b \tag{2.2.19}$$

An explicit variant of Fullness is the only slightly weaker ([Lub05], [RT06]) axiom of Exponentiation:

Exponentiation. For any class Γ,

$$\forall a, b \exists c . c = {}^{a}b \tag{2.2.20}$$

There is a host of interesting axioms between Exponentiation and Fullness ([CIS05], [ACI+06], [Zie10]).

Myhill's Constructive Set Theory CST [Myh75] can be obtained from CZF by replacing Strong Collection by Replacement, Fullness by Exponentiation and adding Dependent Choice (see below).

On the other hand, it is also interesting to strengthen some of the axioms of CZF. Fullness can be strengthened to the Powerset axiom:

Powerset.

$$\forall a \exists b. b = \mathcal{P}(a) \tag{2.2.21}$$

The Powerset axiom is equivalent to the statement that the one element set $1 := \{0\}$ has a powerset [AR01].

Bounded Separation can be strengthened to full Separation:

Separation. For any class Γ,

$$\forall a \exists b. b = a \cap \Gamma \tag{2.2.22}$$

Adding any one of these axioms to CZF increases the proof theoretic strength of the theory dramatically [AR01]. Adding them both yields the system known as IZF [Myh73], which is equiconsistent with classical ZF set theory. Adding the scheme of the excluded middle to CZF yields classical ZF set theory outright [AR01].

Choice

There are various choice principles which can be added to CZF. Of particular interest are the following, the first of which is implied by any of the two others:

Dependent Choice (DC).

$$\forall a \forall x_0 \in a \forall R : a \rightrightarrows a \exists f : \omega \to a. f(0) = x_0 \wedge \forall n \in \omega.(f(n), f(n+1)) \in R \quad (2.2.23)$$

Relativized Dependent Choice (RDC). For any classes Γ and Δ,

$$\forall x_0 \in \Delta. \Gamma : \Delta \rightrightarrows \Delta \to \exists f : \omega \to \Delta. f(0) = x_0 \wedge \forall n.(f(n), f(n+1)) \in \Gamma \quad (2.2.24)$$

Presentation Axiom (PAx).

$$\forall a \exists f, b. f : b \twoheadrightarrow a \wedge \forall R : b \rightrightarrows V \exists r : b \to V. r \subseteq R \quad (2.2.25)$$

All three are weakenings of the classical Axiom of Choice, which however implies the principle of the excluded middle for bounded formulae on the basis of the other axioms of CZF ([AR01]).

Arguably not a choice principle but often useful to circumvent choice ([Acz08]) is the following scheme designed by Aczel to be a useful consequence of RDC but still provable in ZF:

Relation Reflection Scheme (RRS). Let Γ and Δ be classes and $d_0 \in \Delta$. If

$$\Gamma : \Delta \rightrightarrows \Delta \quad (2.2.26)$$

Then there is a set $D \subseteq \Delta$ such that $d_0 \in D$ and

$$\Gamma : D \rightrightarrows D \quad (2.2.27)$$

There is the following slight variation:

Strong Relation Reflection Scheme (SRRS). Let Γ and Δ be classes and $d_0 \in \Delta$. If

$$\Gamma : \Delta \times \Delta \rightrightarrows \Delta \tag{2.2.28}$$

Then there is a set $D \subseteq \Delta$ such that $d_0 \in D$ and

$$\Gamma : D \times D \rightrightarrows D \tag{2.2.29}$$

This is also called RRS_2 in the literature, which would however clash with our naming conventions for axioms — in this thesis, RRS_2 refers to the second order variant of RRS.

It might be noted that RRS implies not only that every relation is reflected in a set but that every set of relations is reflected in a set:

Proposition 2.1. *(RRS) Let Γ_i and Δ be classes for each $i \in I$ and $d_0 \in \Delta$. If*

$$\forall i.\Gamma_i : \Delta \rightrightarrows \Delta \tag{2.2.30}$$

Then there is a set $D \subseteq \Delta$ such that $d_0 \in D$ and

$$\forall i.\Gamma_i : D \rightrightarrows D \tag{2.2.31}$$

Proof. Let $\Delta' := \mathcal{P}(\Delta)$ and $d_0' := \{d_0\} \in \Delta'$. Define Γ' as

$$(a, b) \in \Gamma' :\leftrightarrow \forall i \in I \forall x \in a \exists y \in b.(x, y) \in \Gamma_i \tag{2.2.32}$$

By Strong Collection,

$$\Gamma' : \Delta' \rightrightarrows \Delta' \tag{2.2.33}$$

So by RRS, there is a $D' \subseteq \Delta'$ with $d_0' \in D'$. Set

$$D := \bigcup D' \tag{2.2.34}$$

Then $d_0 \in D$ and for each $x \in D$ and $i \in I$, there is a with $x \in a \in D'$. So by totality of Γ' on D', there is a $b \in D'$ with a $y \in b$ and $(x, y) \in \Gamma_i$. Thus $y \in D'$. So

$$\forall i.\Gamma_i : D \rightrightarrows D \tag{2.2.35}$$

□

Subcountability

We will have occasion to use the Subcountability axiom during these investigations. This simply states that every set is subcountable, where

Definition 2.2. *A set a is called **subcountable** if there is some $A \subseteq \omega$ and $f : A \twoheadrightarrow a$.*

Adding Subcountability to CZF does not increase its proof theoretic strength ([Rat02]). However, it does contradict the Powerset axiom and thus the principle of the excluded middle, which makes it a specifically constructive axiom.

Second Order Set Theory

We will have occasion to employ second order set theories, which are theories in a two sorted language with one sort indicated by using lower case and the other sort using upper case, the former corresponding to first order variables and the latter to second order variables, which are typed with positive natural numbers to express tuples of subsets. When not omitted, the type is indicated by an upper index. We are not concerned with a calculus of inference for theories but merely with their semantics. A (full) model of such a theory is defined in the usual way as follows (all cases but the second order quantifiers are omitted):

Definition 2.3. *For any set M and formula Φ with free variables $\overrightarrow{x}$ and $\overrightarrow{X}$, and for all elements $\overrightarrow{a} \in M$ and sets $\overrightarrow{A}$ where A_n is a subset of M^m if X_n has type m, define a model relation*

$$M \vDash_{\overrightarrow{x}:=\overrightarrow{a},\overrightarrow{X}:=\overrightarrow{A}} \Phi \tag{2.2.36}$$

by structural recursion on Φ, including the cases:

If X^m is of type m, free for Φ and does not appear in $\overrightarrow{X}$, then

$$M \vDash_{\overrightarrow{x}:=\overrightarrow{a},\overrightarrow{X}:=\overrightarrow{A}} \forall X^m \Phi :\leftrightarrow \forall A \subseteq M^m . M \vDash_{\overrightarrow{x}:=\overrightarrow{a},\overrightarrow{X}:=\overrightarrow{A},X^m:=A} \Phi \tag{2.2.37}$$

$$M \vDash_{\overrightarrow{x}:=\overrightarrow{a},\overrightarrow{X}:=\overrightarrow{A}} \exists X^m \Phi :\leftrightarrow \exists A \subseteq M^m . M \vDash_{\overrightarrow{x}:=\overrightarrow{a},\overrightarrow{X}:=\overrightarrow{A},X^m:=A} \Phi \tag{2.2.38}$$

Note that the case where $\vec{a}$ and $\vec{A}$ are empty, i.e. $M \vDash \Phi$, can then only hold if there are no free variables in Φ.

All set theories under consideration in this thesis have a canonical corresponding second order set theory where single axioms are unchanged (the usual variables being interpreted as first order variables) and schemes are transformed into a single axiom, where the scheme consisting of the formulae of the form $\Phi(\Psi)$ for all formulae Ψ correspond to the second order statement $\forall X.\Phi(X)$.

We will designate the canonical second order version of a theory or an axiom scheme by a lower index 2.

Example 2.4. 1. *Set Induction$_2$, the canonical second order version of Set Induction, is equivalent to*

$$\forall X.\forall a(\forall x \in a\ X(x) \to X(a)) \to \forall a.X(a) \tag{2.2.39}$$

2. *Strong Collection$_2$, the canonical second order version of Strong Collection,, is equivalent to*

$$\begin{multline}\forall R^2 \forall a.\forall x \in a \exists y\ R(x,y) \to \\ \exists b.\forall x \in a \exists y \in b\ R(x,y) \wedge \forall y \in b \exists x \in a.R(x,y)\end{multline} \tag{2.2.40}$$

3. *The canonical second order version of CZF, i.e. CZF$_2$, is equivalent to the non-scheme axioms of CZF as given above plus Fullness and the axioms Set Induction$_2$ and Strong Collection$_2$. The scheme of Δ_0 collection does not have a canonical second order version, which is one of the reasons that we used Binary Intersection instead, which is a first order statement.*

2.3 The Concepts of Largeness under Consideration

Classically, large cardinals are certain ordinals fulfilling very strong closure properties (usually properties which the class of all ordinals can be thought to share as well) such that the claim of their existence increases the proof theoretic strength of the theory. However, while classical ZF set theory is very well versed in reasoning about properties of cardinals, they do not seem to be a concept that meshes in well with constructive set theory and it seems to be difficult to even define the concept of cardinal in any useful way in the context of CZF.

Thus, it has become established practice to instead investigate large sets, i.e. transitive sets with strong closure properties (usually properties which the class of all sets can be thought to share as well) such that the claim of their exisistence increases the proof theoretic strength of the theory. These correspond to the large cardinals of ZFC via the von Neumann hierarchy - classically if κ fulfills a certain large cardinal property, then the κ-th iteration of the powerset operator $x \mapsto \mathcal{P}(x)$, i.e. V_κ, fulfills the corresponding large set property[2]. Conversely, if A fulfills a certain large set property, then the set of ordinals it contains fulfills the corresponding large cardinal property[3].

In classical set theory, large cardinals properties are sometimes split into small large cardinal properties (those which are consistent with $V = L$) and large large cardinal properties (those which are not)[4]. In the context of CZF it makes sense to distinguish a third category of properties: Tiny large sets, those which can not be proved to be models of CZF_2.

2.3.1 Tiny Large Sets

All of the properties in this section concern transitive sets, as defined for example in [AR01] as:

[2]This relation holds at inaccessibility and larger cardinals. When analyzing the smaller concepts of regularity and similar, instead of V_κ the set H_κ needs to be employed, defined as the set of hereditarily smaller sets than κ, which is equal to V_k for inaccessible κ.

[3]This relation holds at $\bigcup$-regularity and larger concepts (so in particular at inaccessible). When analyzing the smaller concept of regularity, instead of $A \cap \mathrm{O_n}$, the set $\mathrm{rk}(A)$ needs to be employed, which is equal to $A \cap \mathrm{O_n}$ for $\bigcup$-regular A.

[4]Of course, $V = L$ is not the most common of axioms in a constructive setting. However, Theorem 9.93 suggests that its role might be taken up by another axiom instead.

Definition 2.5. *A set A is called **transitive**, if*

$$\forall a \in A \forall x \in a . x \in A \tag{2.3.41}$$

One of the most basic properties a transitive set can have is that of regularity:

Definition 2.6. *A transitive set A is called **regular** if*

$$A \vDash \textit{Strong Collection}_2 \tag{2.3.42}$$

This is equivalent to demanding

$$\forall a \in A \forall R : a \rightrightarrows A \exists b \in A . R : a \leftleftarrows\!\!\rightrightarrows b \tag{2.3.43}$$

Often it is demanded that a regular set be inhabited, but as all relevant examples are inhabited in any case, this condition would have no bearing on the following.

The main large set axiom related to regularity is the Regular Extension Axiom REA:

REA. Every set is an element of some regular set.

REA was proposed by Peter Aczel and is eminently useful when dealing with inductive definitions. In particular, bounded (monotone) inductive definitions always yield sets when assuming REA [AR01]. In fact, this is already implied by one of its several weakenings:

Definition 2.7. *A transitive set A is called **weakly regular** if*

$$A \vDash \textit{Collection}_2 \tag{2.3.44}$$

*A transitive set A is called **functionally regular** if*

$$A \vDash \textit{Replacement}_2 \tag{2.3.45}$$

wREA. Every set is an element of some weakly regular set.

fREA. Every set is an element of some functionally regular set.

Functional regularity is enough for several basic set operations. The following lemma is a reformulation of the results from [AR01]:

Lemma 2.8. *If $A \ni 2$ is functionally regular, then $A \Vdash$ Pairing. In particular, A is closed under the formation of ordered pairs.*

On the other hand, it is possible to strengthen REA by imposing stronger closure conditions:

Definition 2.9. *A regular set A is called **$\bigcup$-regular** if*

$$A \Vdash \textit{Union} \tag{2.3.46}$$

This is equivalent to demanding

$$\forall a \in A. \bigcup a \in A \tag{2.3.47}$$

*A $\bigcup$-regular set A is called **strongly regular** if*

$$A \Vdash \textit{Exponentiation} \tag{2.3.48}$$

This is equivalent to demanding

$$\forall a, b \in A.\ {}^{a}b \in A \tag{2.3.49}$$

$\bigcup$REA. Every set is an element of some $\bigcup$-regular set.

sREA. Every set is an element of some strongly regular set.

Another direction to go into is demanding some measure of relation reflection from the regular set. The following concepts are useful:

Definition 2.10. *A $\bigcup$-regular set A is called **$*$-regular** if $A \Vdash RRS_2$. It is called **$*_2$-regular** if $A \Vdash SRRS_2$.*[5]

$*$REA. Every set is an element of a $*$-regular set.

[5] These concepts are also known as RRS $\bigcup$-regular or union-closed RRS-regular, and RRS_2 $\bigcup$-regular or union-closed RRS_2-regular respectively [AR01],[AINS12].

$*_2$**REA.** Every set is an element of a $*_2$-regular set.

These concepts make sense when trying to avoid choice and while logically weaker than $\bigcup$REA+DC, they are implied by it and have been used to obtain several theorems which previously relied on choice [Acz08, AINS12, AR10].

All the tiny set axioms discussed in this subsection have the same proof theoretic strength when added to CZF, namely that of the classical theory KPi or equivalently that of the subsystem of second order arithmetic with Δ^1_2-comprehension and bar induction [GR94, Rat03a, Rat05c].

When added to classical set theory however, none of the discussed axioms increases the consistency strength at all except for sREA which is equivalent to the existence of unboundedly many inaccessibles [RL03, AR10].

2.3.2 Small Large Sets

A central large set property in this thesis is that of inaccessibility, which in the context of classical ZFC corresponds to the large cardinal property of (strong) inaccessibility as introduced by [ST30].

Definition 2.11. *A transitive set I is called* ***inaccessible*** *if*

$$A \Vdash \mathit{CZF}_2 \tag{2.3.50}$$

IEA. Every set is a member of an inaccessible set.

This definition differs slightly from others given in the literature, but it is equivalent to most.

Proposition 2.12. *Let A be a transitive set. The following are equivalent:*

1. *A is inaccessible.*

2. *A is regular and* $A \Vdash$ *CZF.*[6]

3. *A is regular, for all* $a, b \in I$,

   (a) $\omega \in I$

   (b) $\bigcup a \in I$

   (c) x *inhabited* $\to \bigcap_{x \in a} x \in I$

   (d) $\exists c.\forall R : a \rightrightarrows b \exists R' \in c.R' \subseteq R \wedge R' : a \rightrightarrows b$*, i.e.* c *full in* $\mathrm{mv}(a, b)$.[7]

Proof. $1 \to 2$ is immediate as any model of CZF$_2$ is a model of CZF. $2 \leftrightarrow 3$ is Corollary 10.27 from [AR01]. For $2 \to 1$, let A be a regular model of CZF. To show that $A \Vdash \mathrm{CZF}_2$, we need to show that the axioms of Strong Collection$_2$ and Set Induction$_2$ are fulfilled in A (noting that Subset Collection$_2$ and Δ_0 − Collection$_2$ are equivalent to the first order axioms of Fullness and Binary Intersection on the basis of the other axioms).

$A \Vdash$ Strong Collection$_2$ by definition of A being regular. Set Induction$_2$ holds in A as for any $X \subseteq A$,

$$(\forall a \in A(\forall x \in ax \in X) \to a \in X) \to A = X \tag{2.3.51}$$

by Set Induction on the outside. □

For any class of sets Γ, the following two forms of derivatives of Γ can be considered:

Definition 2.13. *Let* Γ *be a class. Then the class* $I(\Gamma)$ *is defined by*

$$A \in I(\Gamma) :\leftrightarrow \forall a \in A \exists c \in \Gamma \cap A.a \in c \tag{2.3.52}$$

The class $M(\Gamma)$ *is defined by*

$$A \in M(\Gamma) :\leftrightarrow \forall R : A \rightrightarrows A \exists c \in \Gamma \cap A.R : c \rightrightarrows c \tag{2.3.53}$$

[6]This is used as defining inaccessibility in [Gib02] or [CR02] and (with minor variations) [AR01] and [GRT05]

[7]This is considered in [AR01].

The class $M'(\Gamma)$ is defined by

$$A \in M'(\Gamma) :\leftrightarrow \forall B \subseteq A \forall b \in B \forall R : B \rightrightarrows B \exists c \in \Gamma \cap B. b \in c \wedge R : c \rightrightarrows c \quad (2.3.54)$$

In slight abuse of notation, write $V \in I(\Gamma)$ for the statement

$$\forall a \exists c \in \Gamma . a \in c \quad (2.3.55)$$

And let $V \in M(\Gamma)$ be the scheme that for all classes R,

$$R : V \rightrightarrows V \to \exists c \in \Gamma . R : c \rightrightarrows c \quad (2.3.56)$$

Also let $V \in M'(\Gamma)$ be the scheme that for all classes R, B with $b \in B$

$$R : B \rightrightarrows B \to \exists c \in \Gamma . b \in c \wedge R : c \rightrightarrows c \quad (2.3.57)$$

Example 2.14. *Many axioms can be cast using these derivatives of classes:*

1. *REA is equivalent to $V \in I(\{x|x \text{ regular}\})$.*
2. *IEA is equivalent to $V \in I(\{x|x \text{ inaccessible}\})$.*
3. *RRS is equivalent to $V \in M'(V)$.*
4. *Assuming AC(ω, ω)[8], RDC is equivalent to $V \in M'(\{x|x \text{ countable}\})$.*

Classically, these operations on classes can be iterated transfinitely by taking the intersection at limit cases. In the constructive case, this iteration can be cast as follows, leading to what corresponds to Mahlo's hierarchies of large cardinals.

Definition 2.15. *Let Γ be a class. Then define by recursion over a set a*

$$I^a(\Gamma) := \Gamma \cap \bigcap_{x \in a} I^x(\Gamma) \quad (2.3.58)$$

$$M^a(\Gamma) := \Gamma \cap \bigcap_{x \in a} M^x(\Gamma) \quad (2.3.59)$$

[8]This is the statement that ${}^{\omega}\omega$ is full in $\mathrm{mv}(\omega, \omega)$.

$$M'^{a}(\Gamma) := \Gamma \cap \bigcap_{x \in a} M'^{x}(\Gamma) \tag{2.3.60}$$

*Call a set I a-**inaccessible** if $I \in I^a(\{x|x$ inaccessible$\})$.*

*Call a set I **Mahlo** if $I \in M(\{x|x$ inaccessible$\})$.*

*Call a set I a-**Mahlo** if $I \in M^a(\{x|x$ inaccessible$\})$.*

Note that according to this definition, inaccessible is the same as 0-inaccessible (not 1-inaccessible as e.g. in [AR01]), while Mahlo is the same as 1-Mahlo (with 0-Mahlo being inaccessible again).

These concepts give rise to large cardinal axioms in a similar vein to those considered before, e.g.:

MEA. Every set is an element of some Mahlo set, or in other words,

$$V \in I(M(\{x|x \text{ inaccessible}\})) \tag{2.3.61}$$

The following slightly weaker axiom stating that the universe is Mahlo has been featured prominently in [Gib02].

(M). $V \in M(\{x|x \text{ inaccessible}\})$

Remark 2.16. *Some authors (e.g. [Gib02] or [RGP98]) require that an inaccessible set needs to model REA. This is only a small difference and disappears when using inaccessibility as a starting point to define larger concepts. In fact, if we call an inaccessible set inaccessible' if it models REA and define α-inaccessible', Mahlo' and α-Mahlo' analogously, then*

$$A \text{ inaccessible} \leftarrow A \text{ inaccessible'} \leftarrow A \text{ 1-inaccessible} \leftarrow A \text{ 1-inaccessible'} \leftarrow \ldots \tag{2.3.62}$$

At stage ω and later, the notions have caught up with each other:

$$\omega \subseteq \alpha \rightarrow (A\ \alpha\text{-inaccessible} \leftrightarrow A\ \alpha\text{-inaccessible'}) \tag{2.3.63}$$

$$A \textit{ Mahlo} \leftrightarrow A \textit{ Mahlo'} \tag{2.3.64}$$

$$\alpha \textit{ inhabited} \rightarrow (A\ \alpha\textit{-Mahlo} \leftrightarrow A\ \alpha\textit{-Mahlo'}) \tag{2.3.65}$$

Similarly, the axiom (M) is equivalent to its analogon (M') which states that every total relation on the universe is reflected in some inaccessible' set.

α-inaccessible sets or α-Mahlo sets correspond to the classical concept of (strongly) α-inaccessible cardinals and (strongly) α-Mahlo sets via the functions $I \mapsto \mathrm{rk}(I)$ and $\kappa \mapsto V_\kappa$ respectively [GRT05].

Climbing further up the ladder of large cardinal concepts, the classical notion of weak compactness can be fruitfully expressed through the concept 2-strong, which was introduced by Rathjen in [Rat98] and has the following definition capturing a version of Π_1-indescribability:

Definition 2.17. *An inaccessible set K is called **2-strong** if for any set S, if*

$$\forall R : K \rightrightarrows K \forall u \in K \exists x \in K, v \in K . x \subseteq R \wedge (x, u, v) \in S \tag{2.3.66}$$

Then there is some inaccessible $I \in K$ such that

$$\forall R : I \rightrightarrows I \forall u \in I \exists x \in I, v \in I . x \subseteq R \wedge (x, u, v) \in S \tag{2.3.67}$$

Reasoning in ZFC, a set is 2-strong if and only if it is the κth von Neumann stage V_κ for some weakly compact cardinal κ [Gib02]. This is however not the only way to cast weak compactness in CZF, as addressed in Chapter 8.

2.3.3 Large Large Sets

Measurable cardinals can be cast in different ways in classical set theory. In a constructive context, the avenue that seems most promising is presenting them as the critical point of a nontrivial elementary embedding, first considered for IZF in [FŠ84]. The existence of such an embedding (which is a proper class) can not be stated in the language of pure set theory, which is why we need to extend the language from $\{=, \in\}$ by a unary relation symbol M and a unary function symbol j.

When talking about CZF in this extended language, it should be noted that we imply that the schemes of Strong Collection and Set Induction are extended to also hold for formulae containing j and M (likewise for Subset Collection and Δ_0-Collection, but as they are equivalent to the single axioms of Fullness and Binary Intersection, this follows automatically).

This theory, call it CZF' for the remainder of this paragraph, is a conservative extension of CZF as can be seen quite easily: If $CZF' \vdash \Phi$ with j and M not appearing in Φ, then also $CZF' + \forall x.M(x) + \forall x.j(x) = x \vdash \Phi$. This latter theory can however be trivially and isomorphically interpreted into CZF by using the interpretation arising from replacing terms of the form $j^n(x)$ by x and subformulae of the form $M(x)$ by $\top$. So also $CZF \vdash \Phi$.

Thus it is reasonable to call the theory in the extended language with the axiom schemes admitting formulae containing the new symbols also CZF (in a slight abuse of notation). Also, we will write $x \in M$ instead of $M(x)$ and treat M as a class. The critical axiom is that j is an embedding from V into the class M.

$j : V \overset{\equiv}{\hookrightarrow} M$**.** M is transitive and for any formula $\Phi(\overrightarrow{x})$ which does not contain j or M and has all free variables displayed

$$\forall \overrightarrow{x}.\Phi(\overrightarrow{x}) \leftrightarrow \Phi^M(\overrightarrow{j(x)}) \tag{2.3.68}$$

Here Φ^M is the formula obtained from Φ by replacing all unbounded quantifiers of the forms $\forall x$, $\exists x$ by quantifiers $\forall x \in M$, $\exists x \in M$.

Note that this behaves well with regards to bounded quantification:

Lemma 2.18. *($j : V \overset{\equiv}{\hookrightarrow} M$) Let M be transitive. To each formula Φ assign a formulae Φ_M by recursion over Φ:*

$$
\begin{array}{lll}
\Phi_M & :\equiv & \Phi \textit{ if } \Phi \textit{ is quantifier free} \\
(\Phi J \Psi)_M & :\equiv & \Phi_M J \Psi_M \textit{ for any junctor } J \\
(\forall x \in a \Phi(x))_M & :\equiv & \forall x \in a \Phi(x)_M \\
(\exists x \in a \Phi(x))_M & :\equiv & \exists x \in a \Phi(x)_M \\
(\forall x \Phi(x))_M & :\equiv & \forall x \in M \Phi(x)_M \textit{ if} \\
 & & \Phi(x) \textit{ is not of the form } x \in a \to \Psi \\
(\exists x \Phi(x))_M & :\equiv & \exists x \in M \Phi(x)_M \textit{ if} \\
 & & \Phi(x) \textit{ is not of the form } x \in a \wedge \Psi
\end{array}
$$

Then if $\Phi(\overrightarrow{x})$ displays all free variables in Φ,

$$\forall \overrightarrow{x}.\Phi(\overrightarrow{x}) \leftrightarrow \Phi_M(\overrightarrow{j(x)}) \tag{2.3.69}$$

Proof. The proof is done by induction over Φ with the critical cases being the bounded quantifications, which will be the only ones we present.

Let $\Phi(\overrightarrow{y})$ be $\forall x \in a\Psi(x, \overrightarrow{y})$ and have all free variables displayed (with a being one of the variables in $\overrightarrow{y}$). By induction hypothesis,

$$\forall x, \overrightarrow{y}.\Psi(x, \overrightarrow{y}) \leftrightarrow \Psi_M(j(x), \overrightarrow{j(y)}) \tag{2.3.70}$$

By elementarity, this leads to

$$\forall \overrightarrow{y}.(\forall x.x \in a \to \Psi(x, \overrightarrow{y})) \leftrightarrow (\forall x \in M.x \in j(a) \to \Psi_M(j(x), \overrightarrow{j(y)})) \tag{2.3.71}$$

As $x \in j(a)$ implies $x \in M$ by transitivity, the right side of the biconditional is equivalent to $\Phi_M(\overrightarrow{j(y)})$.

Alternatively, let $\Phi(\overrightarrow{y})$ be $\exists x \in a\Psi(x, \overrightarrow{y})$ and have all free variables displayed (with a being one of the variables in $\overrightarrow{y}$). By induction hypothesis,

$$\forall x, \overrightarrow{y}.\Psi(x, \overrightarrow{y}) \leftrightarrow \Psi_M(j(x), \overrightarrow{j(y)}) \tag{2.3.72}$$

By elementarity, this leads to

$$\forall \overrightarrow{y}.(\exists x.x \in a \wedge \Psi(x, \overrightarrow{y})) \leftrightarrow (\exists x \in M.x \in j(a) \wedge \Psi_M(j(x), \overrightarrow{j(y)})) \tag{2.3.73}$$

As $x \in j(a)$ implies $x \in M$ by transitivity, the right side of the biconditional is equivalent to $\Phi_M(\overrightarrow{j(y)})$. $\square$

Similarly, elementarity behaves well with regards to recursively Δ_0-definable concepts:

Lemma 2.19. *($j : V \overset{\equiv}{\hookrightarrow} M$) Let $\Phi(R, \overrightarrow{x})$ be a Δ_0 formula (possibly containing other free variables). Then extending the language of CZF by a unary symbol A and the axiom*[9]

$$\forall \overrightarrow{a}.A(\overrightarrow{a}) \leftrightarrow \Phi(\{\overrightarrow{x} \in \overrightarrow{a} | \overrightarrow{x} \in A\}, \overrightarrow{a}) \tag{2.3.74}$$

is a conservative extension and for any formula $\Phi(\overrightarrow{x})$ which does not contain j or M, but may contain A and has all free variables displayed

$$\forall \overrightarrow{x}.\Phi(\overrightarrow{x}) \leftrightarrow \Phi^M(\overrightarrow{j(x)}) \tag{2.3.75}$$

is provable in that conservative extension.

Proof. The conservativity works just as for the corresponding results from [AR01].

To show elementarity, consider tc(x), i.e. the transitive closure of x, the smallest transitive set containing x as an element. The formula $tc(a) = b$ can be viewed as a Δ_0-formula[10] [AR01].

By set induction, we get that for all sequences of sets $\overrightarrow{a}$ the formula $A(\overrightarrow{a})$ is equivalent to

$$\begin{aligned}
&\forall B \subseteq \text{tc}(a_1) \times ... \times \text{tc}(a_n). \\
&\forall \overrightarrow{x} \in \text{tc}(a_1) \times ... \times \text{tc}(a_n)(\overrightarrow{x} \in B \leftrightarrow \Phi(\{\overrightarrow{y} \in \overrightarrow{a} | \overrightarrow{y} \in B\}, \overrightarrow{x})) \\
&\rightarrow \Phi(\{\overrightarrow{x} \in \overrightarrow{a} | \overrightarrow{x} \in B\}, \overrightarrow{a})
\end{aligned}$$

It is also equivalent to

[9] Note that this axiom refers to what appears to be a class. Using the conventions established in section 2.1 however, this is not a problem. In this particular instance, the class actually turns out to be a set in any case.

[10] To be precise, enriching the language of CZF by a relation sign for this formula and adding the instances of axiom schemes where the new sign appears yields a conservative extension, in which all relevant procedures can be carried out. It is a simple matter of induction over the complexity of the formula (with set induction in the base case) to see that tc may appear in the elementarity scheme as well.

$$\exists B \subseteq \text{tc}(a_1) \times ... \times \text{tc}(a_n).$$
$$\forall \overrightarrow{x} \in \text{tc}(a_1) \times ... \times \text{tc}(a_n)(\overrightarrow{x} \in B \leftrightarrow \Phi(\{\overrightarrow{y} \in \overrightarrow{a} | \overrightarrow{y} \in B\}, \overrightarrow{x}))$$
$$\wedge \Phi(\{\overrightarrow{x} \in \overrightarrow{a} | \overrightarrow{x} \in B\}, \overrightarrow{a})$$

Thus $A(\overrightarrow{a})$ is a Δ_1 formula[11] and thus by elementarity for appropriate Δ_0 formulae Φ, Ψ,

$$\begin{aligned}
A(\overrightarrow{a}) &\leftrightarrow \exists B.\Phi(B, \overrightarrow{a}) \\
&\leftrightarrow \exists B \in M.\Phi(B, \overrightarrow{j(a)}) \\
&\rightarrow \exists B.\Phi(B, \overrightarrow{j(a)}) \\
&\leftrightarrow A(\overrightarrow{j(a)})
\end{aligned}$$

And also

$$\begin{aligned}
A(\overrightarrow{j(a)}) &\leftrightarrow \forall B.\Psi(B, \overrightarrow{j(a)}) \\
&\rightarrow \forall B \in M.\Psi(B, \overrightarrow{j(a)}) \\
&\leftrightarrow \forall B.\Psi(B, \overrightarrow{a}) \\
&\leftrightarrow A(\overrightarrow{a})
\end{aligned}$$

This shows that the elementarity scheme holds for atomic formulae containing A, the rest follows by a direct structural recursion. $\square$

Example 2.20. *The rank function is recursively Δ_0-definable:*

$$rk(a) := \bigcup_{x \in a} rk(x) + 1 \tag{2.3.76}$$

[11]i.e. a formula equivalent to both a formula of the form $\exists B.\Phi(B)$ with Φ being Δ_0 and to a formula of the form $\forall B.\Psi(B)$ with Ψ being Δ_0.

As $rk(a) = b$ is equivalent to the Δ_0 formula

$$\forall x \in a \exists y \in b\ \boldsymbol{rk}(x) = y \wedge \forall y \in b \exists x \in a.\boldsymbol{rk}(x) = y \vee \boldsymbol{rk}(x) \ni y \tag{2.3.77}$$

On its own, $j : V \overset{\equiv}{\hookrightarrow} M$ is no large cardinal axiom and in fact carries no strength at all, as there is nothing to prevent M from being V and j from being the identity in which case the elementarity scheme becomes trivial. However, there are a number of interesting large cardinal axioms expressible with $j : V \overset{\equiv}{\hookrightarrow} M$ being a starting point:

There is a measurable set. $j : V \overset{\equiv}{\hookrightarrow} M$ and there is an inaccessible[12] set K such that

$$K \in j(K) \wedge \forall x \in K.j(x) = x \tag{2.3.78}$$

This was considered in [FŠ84] in the context of IZF, as well as:

There is a Reinhardt set. $j : V \overset{\equiv}{\hookrightarrow} M$ and there is an inaccessible set K such that

$$K \in j(K) \wedge \forall x \in K.j(x) = x \tag{2.3.79}$$

Also, $V = M$.

The latter axiom is known to be inconsistent with ZFC [Kun71]. However, there is no reason to believe that it should also be inconsistent with CZF. Nevertheless, it can be shown to be quite strong.

There is a spectrum of sensible variations of these axioms which shall be considered later in this thesis.

[12] As will transpire later, such a set then also fulfills stronger conditions like Mahloness or 2-strongness.

Chapter 3

What do Large Sets look like?

3.1 Introduction and Results

Classically, full models of second order set theory (i.e. inaccessible sets) are quite one-dimensional: They may differ in height, but even if one model contains larger ordinals than another, it is simply an extension and the two models are identical on the stages[1] that they share:

Observation 3.1. *(ZF) If I and J are inaccessibles and $\alpha \in I \cap J$, then*

$$V_\alpha \cap I = V_\alpha \cap J \tag{3.1.2}$$

This is because classically for $\alpha \in I$ we have that $V_\alpha \cap I = V_\alpha$. However, the constructive case is vastly different: There is no reason to suppose that two inaccessible sets even share the same truth values (i.e. that $V_2 \cap I = V_2 \cap J$). And even if they do share the same truth values, it is absolutely unclear whether they contain the same sets of truth

[1] The von Neumann stages V_α are recursively defined as

$$V_\alpha := \bigcup_{\beta \in \alpha} \mathcal{P}(V_\beta) \tag{3.1.1}$$

They will be introduced in more detail during the chapter adapting the von Neumann hierarchy to constructive set theory, in particular see Definition 5.4.

values (i.e. $V_3 \cap I = V_3 \cap J$) — for example, it seems plausible that one might model the powerset axiom (and thus $V_2 \cap I \in I \cap V_3$), while the other might not (which is equivalent to $V_2 \cap J \notin J \cap V_3$).

So two inaccessibles can diverge pretty quickly in CZF, starting at von Neumann stage 2. However, if they did not diverge by stage 3, then they never do:

Theorem 3.2. *Let I, J be inaccessibles. If $I \cap V_3 = J \cap V_3$, then*

$$\forall \alpha \in I \cap J.\ I \cap V_\alpha = J \cap V_\alpha \tag{3.1.3}$$

In particular, the finite stages of an inaccessible depend only on the first three stages.

3.2 Proof of Theorem 3.2

In the following, for a bounded formula Φ use the abbreviation $[\![\Phi]\!]$ defined by:

$$[\![\Phi]\!] := \{0|\Phi\} \tag{3.2.4}$$

This is a set by Δ_0-Collection and is inhabited iff Φ holds. We will revisit this construction in Definition 5.3.

We make use of the following central lemma:

Lemma 3.3. *Let I, J be inaccessibles with $I \cap V_3 = J \cap V_3$ and $X \in I$. If $\bigcup X \in J$, then also $X \in J$.*

Proof. Note that $X \in \mathcal{P}(\mathcal{P}(\bigcup X))$.

For all $y \in \bigcup X$, define

$$A_y := \{[\![y \in x]\!]|x \in X\} \in I \cap V_3 = J \cap V_3 \tag{3.2.5}$$

Recall the definition of dependent products as sets of functions and note that inaccessibles are closed under them, so that

$$\Pi_{y\in\bigcup X}A_y \in J \tag{3.2.6}$$

For every element $F \in \Pi_{y\in\bigcup X}A_y$, define

$$\vartheta_F := [\![\{y \in \bigcup X | 0 \in F(y)\} \in X]\!] \in I \cap V_2 = J \cap V_2 \tag{3.2.7}$$

Note that by regularity, the mapping $F \mapsto \vartheta_F$ with domain $\Pi_{y\in\bigcup X}A_y$ is in J, and thus so is its support, i.e. the set

$$Y := \{F \in \Pi_{y\in\bigcup X}A_y | 0 \in \vartheta_F\} \in J \tag{3.2.8}$$

Finally, we claim the equality

$$X = \{\{y \in \bigcup X | 0 \in F(y)\} | F \in Y\} \in J \tag{3.2.9}$$

To prove $\subseteq$, let $x \in X$. Let F be the function mapping $y \in \bigcup X$ to $[\![y \in x]\!]$, which is a well defined element of $\Pi_{y\in\bigcup X}A_y \in J$. Then $F \in Y$, as

$$\vartheta_F = [\![\{y \in \bigcup X | 0 \in F(y)\} \in X]\!] = 1, \tag{3.2.10}$$

which holds because the set $\{y \in \bigcup X | 0 \in F(y)\}$ is just x.

To prove $\supseteq$, let $F \in Y$. Then

$$\vartheta_F = [\![\{y \in \bigcup X | 0 \in F(y)\} \in X]\!] = 1 \tag{3.2.11}$$

And thus there is an $x \in X$ with $\{y \in \bigcup X | 0 \in F(y)\} = x$.

As X is thus proved to be equal to a set in J, it is an element of J. $\square$

Now we can prove Theorem 3.2:

Proof of Theorem 3.2. Let I, J be inaccessibles with $I \cap V_3 = J \cap V_3$. By induction on α, prove that $V_\alpha \cap I = V_\alpha \cap J$.

Let $x \in V_\alpha \cap I$. For $y \in x$, there is a $\beta \in \alpha$ with $y \in V_\beta$ (as generally, there is a $\beta \in \alpha$ with $x \subseteq V_\beta$). Descending one step further this becomes:

$$\forall z \in y \in x \exists \gamma \in \beta \in \alpha . z \in V_\gamma \tag{3.2.12}$$

So $\bigcup x$ can be split over the different stages as follows:

$$\bigcup x = \bigcup_{\gamma \in \bigcup \alpha} (V_\gamma \cap \bigcup x) \tag{3.2.13}$$

The sets $V_\gamma \cap \bigcup x$ are elements of $I \cap V_\beta$ for some $\beta \in \alpha$ by I modelling Δ_0-Seperation. By induction hypothesis, they are thus also in J, and by J being regular and noting that $\bigcup \alpha \in J$, so is their union.

So $\bigcup x \in J$, and by lemma 3.3, this implies $x \in J$. $\square$

Corollary 3.4. *If two inaccessibles contain the same sets of ordinals, then they are equal.*

This corollary is however weaker than the statement that will later be obtained using the modified von Neumann hierarchy, namely that if two inaccessibles contain the same ordinals, then they are equal.

3.3 Why 3?

It is a well known result that whether any V_α is a set for $\alpha > 1$ is equivalent to the Powerset axiom and this is equivalent to the question whether $V_2 = \mathcal{P}(1)$ is a set (or equivalently whether $V_2 \in V_3$). The main theorem in this section can be read as a logical extension of this: The set approximations of the powerclass of 1 (i.e. the contents of V_3) determine the set approximations of all powerclass operations (and thus all the V_αs).

3.4 A much stronger Classification Theorem in case of Subcountability

The previous results show that the stages of an inaccessible $V_\alpha \cap I$ for $\alpha \in I$ are uniquely determined by the fourth stage $V_3 \cap I$. In the presence of the Subcountability axiom however, it is even possible to not only determine the stages for $\alpha \in I$, but the whole of I just by fixing the set $V_2 \cap I$ of truth values in I:

Theorem 3.5. *(Subcountability) Let I, J be inaccessible. If $I \cap V_2 = J \cap V_2$, then $I = J$.*

In fact, even the following holds:

Proposition 3.6. *(Subcountability) Let A, B be regular with $\omega \in A, B$, let A be closed under the binary operation $x, y \mapsto [\![x \in y]\!]$ and B be closed under the operation $x \mapsto \bigcup x$. If $A \cap V_2 \subseteq B \cap V_2$, then $A \subseteq B$.*

To prove this, we will take advantage of the following useful lemma:

Lemma 3.7. *(Subcountability) Let A be regular with $\omega \in A$. Then A models the Subcountability axiom.*

Proof. Let $a \in A$ and by Subcountability, let $N \subseteq \omega$ and $f : N \twoheadrightarrow a$. Then for all $x \in a$, there is an $n \in \omega \in A$ such that $(n, a) \in f$. By regularity, collect such n into a set $N' \in A$. As the n are all in N, we have $N' \subseteq N$, so

$$f \upharpoonright N' : N' \to a \tag{3.4.14}$$

This function is also surjective, as for every element of a there is such an n by the instance of regularity invoked to get N'. Also it holds that the function is in A, i.e.

$$f \upharpoonright N' \in A \tag{3.4.15}$$

This is because $f \upharpoonright N'$ is the uniquely determined set whose existence is implied by the instance of regularity owing to the relation

$$\forall n \in N' \exists! z \in A. z = (n, f(n)) \tag{3.4.16}$$

So $f \upharpoonright N' : N' \twoheadrightarrow a$ and both function and domain are elements of A. □

Proof of Proposition 3.6. Let A, B be regular with $\omega \in A, B$ and $V_2 \cap A \subseteq V_2 \cap B$, i.e., for every truth value $\vartheta \subseteq 1$, it holds that $\vartheta \in A$ implies $\vartheta \in B$. Let A also be closed under the truth value operation $x, y \mapsto [\![x \in y]\!]$ and B be $\bigcup$-regular.

We will prove $A \subseteq B$ by set induction. So let $a \in A$ and for all $x \in a$, $x \in B$. We need to show $a \in B$ also.

By Lemma 3.7, there are $N, f \in A$ with $N \subseteq \omega$ and $f : N \twoheadrightarrow a$. Then for each $n \in \omega$, the truth value $\vartheta_n := [\![n \in N]\!] \in A$, so they are also elements of B. For any $n \in \omega$, it holds that

$$\forall x \in \vartheta_n \exists! y \in B. y = f(n) \tag{3.4.17}$$

Applying regularity demonstrates that for this fixed $n \in \omega$, the set $\{f(n) | \exists n \in \vartheta_n \wedge n \in \omega\} = \{f(n) | n \in N\}$, which has at most one element, is in B. As this is true for each $n \in \omega$, it is possible to collect them in a set in B:

$$\{\{f(n) | n \in N\} | n \in \omega\} \in B \tag{3.4.18}$$

So also in B is the set

$$A = \bigcup \{\{f(n) | n \in N\} | n \in \omega\} \in B \tag{3.4.19}$$

□

A possibly more elegant way to formulate the proposition would be the following:

Corollary 3.8. *(Subcountability) If $A, B \ni \omega$ are regular, A is closed under binary intersection and B is closed under union, then*

$$V_2 \cap A \subseteq V_2 \cap B \to A \subseteq B \tag{3.4.20}$$

Proof. We show that for all $x, y \in A$, the truth value $[\![x \in y]\!] \in A$. For let $x, y \in A$ and A be closed under binary intersection, then $\{y\} \cap \{x\} \in A$ and for each element in this set, there is an element of A which is equal to 0. Collecting these in a set yields precisely the truth value $[\![x \in y]\!]$. □

Thus the axiom of Subcountability can be observed once more performing an antipodal role to the axiom of the Excluded Middle: While the latter enforces a classical world where the finite stages of the von Neumann hierarchy are the same for any inaccessible and the inaccessibles differ only in height, in the inherently constructive world of Subcountability, the crucial stage is the very lowest conceivably nontrivial finite stage of the von Neumann hierarchy (i.e. V_2), and the whole inaccessible is completely determined from there.

Chapter 4

How Constructive are Large Set Axioms?

Classical set theorists can draw from a cornucopia of justifications for adopting large cardinal axioms, their reasoning ranging from pragmatic through aesthetic to onthological (e.g. [Kan03, Rat98]), and many of these can also be applied to large sets in a constructive context. However, it is easy to be tempted to leave the realm of constructively admissible reasoning by unreflectedly adding principles which are a lot less innocuous than they appear (see e.g. [AR01]), so that it is a reasonable question to ask of any new axiom whether it preserves the general constructive character of the theory.

There are several ways to approach that concern. One is to satisfy oneself that adding large sets still yields a theory that can be analyzed with generalized predicative methods of proof theory, an avenue that was successfully pursued in [Gib02, CR02]. Another is to specify metatheoretic properties that constructive set theories should have and prove them for the extensions of CZF with large set axioms. This is the route we will present here.

One hallmark for reasonable candidates for axiom systems for constructive set theories has been proposed by Aczel in [Acz07]: They should be absolute for Heyting models, reasoning that this is a fundamental set theoretic model construction under which any reasonable class of standard models should be closed. In the same vein, one can demand absoluteness for realizability models as in [Rat03b], this being the other principal model construction commonly used in constructive set theory. We will analyse absoluteness for both models in one stroke by proving absoluteness for their common generalisation as presented in [Zie07, Zie12]. Testing for this will also confirm a conjecture by Peter Aczel on $*$REA presented in [Acz07].

A second type of criteria might be the consistency with typically constructive statements such as Church's Thesis (advocated e.g. in [Bee82]), the FAN Theorem or continuity principles. Conveniently, this is already directly implied by the absoluteness for realizability models, as these statements hold in appropriate realizability models [Rat03b, Rat05a], so that this will be shown automatically by establishing good absoluteness properties.

The third group of tests of constructivity we will undertake consists of properties related to the existence property. While the existence property itself is arguably not a hallmark of constructive set theories and does in fact not even hold for CZF itself [Swa14] (although it does hold if Fullness is replaced by the more innocuous axiom of Exponentiation, see [Rat05c]), it can be considered a reasonable demand that a theory aspiring to capture the spirit of constructive thought should be able to specify which disjunct of a provable disjunction is provable (the disjunction property), provide a witness for a provable statement of the form $\exists n \in \omega.\Phi(n)$ (the numerical existence property) and ideally supply a Turing machine able to compute a function that witnesses a provable statement of the form $\exists f : \omega \to \omega.\Phi(n)$ (Church's rule).

4.1 Models of Constructive Set Theory

The two main relative model constructions available to and developed in CZF are realizability models [Rat03b] and Heyting models [Gam06][1]. They have a common general-

[1]In the inpredicative context of IZF, such models have been developed before in e.g. [Fri73] and [Gra79] respectively. However, the predicative case requires some new ideas in both cases.

ization, which in the context of CZF has been developed as models based on applicative topologies in [Zie07]. We will work with this model construction, as this makes it more economical to prove absoluteness results for both realizability and Heyting models at the same time. We will quickly review the description of models based on applicative topologies from [Zie12].

4.1.1 Applicative Topologies

Definition 4.1. *A partial order*[2] $(S, \leq)$ *together with a class* $\lhd \subseteq S \times \mathcal{P}(S)$ *is called a* ***formal topology***[3] *if the following hold:*

$$a \in p \to a \lhd p$$

$$a \leq b \lhd p \lhd q \to a \lhd q$$

$$a \lhd p \wedge a \lhd q \to a \lhd \{b | \exists c \in p, d \in q.\ b \leq c \wedge b \leq d\}$$

Here the expression $p \lhd q$ *is to be understood pointwise, i.e. as an abbreviation for* $\forall a \in p. a \lhd q$.

The formal topology is called ***set-presented*** *if there is a function* $R : S \to \mathcal{P}(\mathcal{P}(S))$ *such that*

$$a \lhd p \Leftrightarrow \exists u \in R(a)\ u \subseteq p \tag{4.1.1}$$

A set-presented formal topology depends only on set parameters, i.e. S, $\leq$ and R as $\lhd$ can be defined from R. Thus the set-presented formal topologies form a class, while the collection of all formal topologies is a hyperclass. We will only use set-presented formal topologies in these investigations.

Definition 4.2. *A set-presented formal topology* $(S, \leq, \lhd, R)$ *with an application* $\circ$: $S \times S \to_p S$*, a subset* $\nabla \subseteq S$ *and two constants* $k \in \nabla$ *and* $s \in \nabla$ *is called an* ***applicative topology*** *if the following hold for all* $a, b, c \in S$*:*

[2] i.e. $\leq$ is a binary relation on S and it always holds that $a \leq a$ and $a \leq b \leq c \to a \leq c$

[3] This is the definition used e.g. in [Gam06]. It differs from the common alternatives in the vein of [Sam87] in that it uses less structure, i.e. no binary meets or positivity predicate.

1. $(\forall x \in p, y \in q\ xy\downarrow \wedge (a \lhd p \wedge b \lhd q)) \to (ab\downarrow \wedge ab \lhd \{xy | x \in p, y \in q\})$

2. $ab\downarrow\ \wedge a, b \in \nabla \to ab \in \nabla$

3. $kab\downarrow \wedge kab \lhd \{a\}$

4. $sab\downarrow$

5. $((ac)(bc)\downarrow\ \vee sabc\downarrow) \to (sabc\downarrow\ \wedge (ac)(bc)\downarrow\ \wedge sabc \lhd \{(ac)(bc)\})$

6. $\nexists e \in \nabla\ e \lhd \emptyset$

In the following, fix an applicative topology.

We use applicative terms in the usual manner:

Definition 4.3. *1. A first order term t in the language $S \cup \{\cdot\}$ with a constant for each element in S, countably many free variables and a binary function is called an **applicative term**. For readability, we will omit brackets when the leftmost operation is meant to be carried out first, i.e. xyz is to be read as $(xy)z$. A term with no variables is called closed. If σ is a function assigning a member of S to each free variable of a term t, then $t[\sigma]$ is the result of substituting each free variable in t by its value under σ.*

2. *Define by induction the (possibly undefined) **value** t^S of a closed term t:*

$$a^S = a \textit{ for } a \in S$$

$$st = c \textit{ if } \exists a, b \in S. s^S = a \wedge t^S = b \wedge a \circ b = c$$

3. *A closed term t **denotes** (written as $t\downarrow$) if $\exists a \in S. t^S = a$.*

4. *A term t **convinces** (written as $t!$) if for all substitutions σ of elements of ∇ for the free variables, we have: $t[\sigma]\downarrow \to\ t[\sigma]^S \in \nabla$*

5. *We write $t \preceq t'$ if whenever one of the terms denotes, then both do and $t^S \lhd \{t'^S\}$.*

Like ordered pcas [HvO03], applicative topologies only have a one sided form of combinatorial completeness, which is however sufficient for our purposes:

Lemma 4.4. *For any applicative term t and variable v, there is an applicative term $\lambda v.t$ with $\lambda v.t \downarrow$ such that its free variables are exactly the free variables of t minus v (if v is free in t) and*

$$(\lambda v.t)v \trianglelefteq t \tag{4.1.2}$$

If all constants in t are elements of ∇, then $\lambda v.t!$.

If v was free in t or if $t' \downarrow$, then

$$(\lambda v.t)t' \trianglelefteq t[v := t'] \tag{4.1.3}$$

Proof. See [Zie12]. □

From there, it is possible to obtain a one sided form of the fixed point lemma. Unlike the fixed point element one obtains in partial combinatory algebras, only a fixed point term is found here:

Lemma 4.5. *Let v be a variable and for all $a \in S$, write $v := a$ for the substitution σ mapping v to a.*

Then there is an applicative term τ^{fix} with $\tau^{fix}!$ and

$$\forall a \in S.\ \tau^{fix}[v := a] \trianglelefteq a(\tau^{fix}[v := a]) \tag{4.1.4}$$

Proof. See [Zie12]. □

Remark 4.6. *There are a number of useful definable constants we will consider fixed, in particular $p, l, r, D \in S$ for pairing, left and right projection and case distinction. They fulfill:*

$$\begin{aligned} p!, l!, r!, D! \\ l(pxy) &\trianglelefteq x \\ r(pxy) &\trianglelefteq y \\ Dlxy &\trianglelefteq x \\ Drxy &\trianglelefteq y \end{aligned}$$

We will use these constants as well as $\tau^{fix}[v := a]$ as from Lemma 4.5.

4.1.2 Models of Set Theory Based on Applicative Topologies

We define a model of set theory on the following inductively defined class domain:

Definition 4.7. *Let $V(S)$ be the smallest class such that*

$$\forall a \subseteq S \times V(S).a \in V(S) \tag{4.1.5}$$

This can also be presented in stages analogous to the von Neumann stages as $V(S) = \bigcup_{\alpha \in V} V_\alpha(S)$, where

$$V_\alpha(S) = \bigcup_{\beta \in \alpha} \{X \subseteq S \times V_\beta(S)\} \tag{4.1.6}$$

We consider the following realizability relation for formulae with parameters in $V(S)$:

Definition 4.8. *Let Φ be a formula in the language $\in, \dot{\in}$ with equality. Define the formula $e \Vdash \Phi$ inductively as $e \in S$ and the appropriate clause from below:*

1. $e \Vdash \bot$ *if* $e \lhd \emptyset$
2. $e \Vdash x \dot{\in} y$ *if* $e \lhd \{e|(e, x) \in y\}$
3. $e \Vdash x \in y$ *if* $e \lhd \{f \in S|\exists z \in tc(y).lf \Vdash z \dot{\in} y \wedge rf \Vdash x = z\}$
4. $e \Vdash x = y$ *if* $\forall (f, z) \in x.lef \Vdash z \in y$ *and*

   $\forall (f, z) \in yref \Vdash z \in x$
5. $e \Vdash \Phi \wedge \Psi$ *if* $le \Vdash \Phi \wedge re \Vdash \Psi$
6. $e \Vdash \Phi \vee \Psi$ *if* $e \lhd \{f \in S|(lf \trianglelefteq l \wedge rf \Vdash \Phi) \vee (lf \trianglelefteq r \wedge rf \Vdash \Psi)\}$
7. $e \Vdash \Phi \to \Psi$ *if* $\forall f \in S.f \Vdash \Phi \to ef \Vdash \Psi$

8. $e \Vdash \exists x \in y\ \Phi(x)$ *if* $e \lhd \{f \in S | \exists (lf, a) \in y.\ rf \Vdash \Phi[a]\}$

9. $e \Vdash \forall x \in y\ \Phi(x)$ *if* $\forall (f, a) \in y.\ ef \Vdash \Phi[a]$

10. $e \Vdash \forall x \Phi(x)$ *if* $\forall a \in V(S).\ e \Vdash \Phi[a]$

11. $e \Vdash \exists x \Phi(x)$ *if* $e \lhd \{f \in S | \exists a \in V(S).\ f \Vdash \Phi[a]\}$

Use $\Vdash \Phi$ *as shorthand for* $\exists e \in \nabla . e \Vdash \Phi$ *and call such a formula* Φ ***realized****.*

Here, the atomic formulae of type $x \dot{\in} y$ are merely a technical convenience to define the other cases more easily. The clauses 8 and 9 are not actually necessary as bounded quantifiers can just be treated as defined concepts, but they are convenient to use, as has been considered to be the case in the literature about realizability [Rat03b] as well as Heyting models [Gam06].

The central result about this model is:

Theorem 4.9. *The realized formulae are a consistent*[4] *theory closed under intuitionistic implication*[5] *containing all the axioms of CZF*[6]*.*

Proof. See [Zie12]. □

As detailed in [Zie12], realizability models are a special case of the models considered here, where a pca $(A, k, s\circ)$ can be considered as an applicative topology by choosing the discrete topology on it, i.e. defining $a \leq b$ iff $a = b$, $a \lhd p$ iff $a \in p$, $R(a)$ as $\{\{a\}\}$ and ∇ as the whole of A. Similarly, Heyting models are a special case of these models, where a set-presented formal topology $S, \leq, \lhd, R)$ which w.l.o.g. contains a top element $\top$ and is closed under meets $\cap$ can be considered an applicative topology by defining $a \circ b$ as $a \cap b$, choosing s and k as equal to $\top$ and ∇ as $\{\top\}$.

[4] When spelled out as a theorem in CZF, this statement is just $\neg \Vdash \bot$.

[5] This statement even holds when spelled out in the strong formulation as a scheme of formulae of the form $(\Phi \to \Psi) \to (\vDash \Phi) \to (\Vdash \Psi)$.

[6] When spelled out as a theorem in CZF, this statement is a scheme of formulae of the form $\Vdash \Phi$ for all Φ axioms of CZF.

4.1.3 The Consistency of Tiny Large Set Axioms

The consistency of REA under realizability models has been proved by Rathjen in [Rat03b]. For Heyting models, this has been proved in [Zie07] through the model construction described above. The central lemma in this case was:

Lemma 4.10. *If A is regular and $R''S \in A$, then $A' := S \times (V(S) \cap A)$ is a set in $V(S)$ and $\Vdash A'$ is regular.*

Proof. See [Zie07]. □

Looking at the proof in [Zie07], it becomes obvious that when proving the part that A is realizedly weakly regular, only the weak regularity of A was needed, so that we also get:

Lemma 4.11. *If A is weakly regular and $R''S \in A$, then $A' := S \times (V(S) \cap A)$ is a set in $V(S)$ and $\Vdash A'$ is weakly regular.*

This can be extended to $\bigcup$-regular sets:

Lemma 4.12. *If A is $\bigcup$-regular and $R''S \in A$, then $A' := S \times (V(S) \cap A)$ is a set in $V(S)$ and $\Vdash A'$ is $\bigcup$-regular.*

Proof. By Lemma 4.10 it is already realized that A' is regular, what is left is to show that $A' \vDash$ Union. This is realized in a similar way as it is realized that V(S) itself models the Union Axiom. A possible realizer for the formula

$$\forall a \in A' \exists b \in A'. \forall c \in a \forall d \in c \; d \in b \wedge \forall d \in b \exists c \in a \; d \in c \tag{4.1.7}$$

would be

$$e := \lambda x.pk(p(\lambda yz.p(pyz)e_r)(\lambda x.p(lx)(p(rx)e_r)) \tag{4.1.8}$$

where as usual e_r is a realizer for reflexivity of equality. Indeed, for each $(f, a) \in A' \subseteq A$, there is the set

$$b := \{(pxy, d) | \exists c.(x, c) \in a \wedge (y, d) \in c\} \tag{4.1.9}$$

We want to show that $b \in A$. Consider for each element $(x, c) \in a$ with $c \in V(S)$ (and as $(f, a) \in A'$, all elements of a are of that form), consider the mapping

$$f_{x,c} : c \to a,\ f((y, d)) := (pxy, d) \tag{4.1.10}$$

As every element of c is of the form y, d, this is defined on $c \in A$ and so its image is also in A:

$$\widetilde{(x, c)} := \text{im}(f_{x,c}) \in A \tag{4.1.11}$$

Thus the map $(x, c) \mapsto \widetilde{(x, c)}$ is defined on a and has codomain A, so its image is also in A:

$$\tilde{a} := \{\widetilde{(x, c)} | (x, y) \in a\} \in A \tag{4.1.12}$$

By the union property of A, its union is an element of A as well, but this is just b:

$$b = \bigcup \tilde{a} \in A \tag{4.1.13}$$

So $(k, b) \in A'$. By construction

$$\lambda yz.p(pyz)e_r \Vdash \forall c \in a \forall d \in cd \in b \tag{4.1.14}$$

and

$$\lambda x.p(lx)(p(rx)e_r) \Vdash \forall d \in b \exists c \in ad \in c \tag{4.1.15}$$

and thus e truly realizes the Union Property. $\square$

We can also obtain a similar result for $*$REA:

Lemma 4.13. *If A is $*$-regular and $R''S \in A$, then $A' := S \times (V(S) \cap A)$ is a set in $V(S)$ and $\Vdash A'$ is $*$-regular.*

Proof. By Lemma 4.12 it is already realized that A' is $\bigcup$-regular, what is left is to show that $\Vdash A' \vDash \mathrm{RRS}_2$.

To realize this, let $e_0 \vDash D \subseteq A'$, $e_1 \vDash G \subseteq A'$ (and as per [Zie07], w.l.o.g. take $G \subseteq A$), $(e_2, a) \in G$ and $e_3 \vDash \forall x \in G \exists y \in G (x, y) \in D$.

From these, we have to construct a realizer for $\exists g \in A'.g \subseteq G \wedge a \in g \wedge \forall x \in g \exists y \in g (x, y) \in D$.

Consider the following relation R' between elements b and b' of A' which is defined to hold iff $b \subseteq b' \in A'$ and

$$\forall (f, x) \in b \exists u \in R''S \exists c \in b'. \\ e_3 f \lhd u \wedge \forall g \in u \exists (lg, y) \in c.rg \vDash (x, y) \in D \wedge (lg, y) \in G \quad (4.1.16)$$

Let $\mathcal{P}_A(X) := \{Y \in A | Y \subseteq X\}$ and note that this operation maps subsets of A to subsets of A. From the perspective of A this appears to be the powerclass operation (while viewed from the outside, it actually maps sets to sets).

We claim that $R' : \mathcal{P}_A(G) \rightrightarrows \mathcal{P}_A(G)$. Indeed, from our knowledge about e_3, we know that

$$\forall (f, x) \in G \exists u \in R''S \forall g \in u \exists (lg, y) \in G.rg \Vdash (x, y) \in D \quad (4.1.17)$$

So let $b \in A, b \subseteq G$. We have

$$\forall (f, x) \in b \exists u \in R''S \forall g \in u \exists (lg, y) \in G.rg \Vdash (x, y) \in D \quad (4.1.18)$$

Thus by (weak) regularity of A (of which the us are elements) we can collect the (lg, y)s in a set b'':

$$\forall (f, x) \in b \exists b'' \subseteq G.b'' \in A \wedge \exists u \in R''S \forall g \in u \exists (lg, y) \in b''. \\ (lg, y) \in G \wedge rg \Vdash (x, y) \in D \quad (4.1.19)$$

Using the (weak) regularity of A yet again, we can collect all neccessary b'' in a single set $B'' \in A$ – but the union $b' := b \cup \bigcup B''$ is then just as desired a set in A and a subset of G which stands in the R'-relation with b. Note that the union-regularity is used here to obtain $b' \in A$.

Using the $*$-property of A we can now find a set $g' \in A$ and subset of $\mathcal{P}_A(G)$ with $\{(e_2, a)\} \in g'$ such that $R : g' \rightrightarrows g'$. The set $g := \{(k, h)|h \in g\}$ is then as desired for the $*$-property:

It is realized by pe_2e_r that $a \in g'$. It is trivially realized that $g' \subseteq G$. Furthermore, the same realiser e_3 that realizes $D : G \rightrightarrows G$ also realizes $D : g \rightrightarrows g$, which is direct consequence of $R' : g' \rightrightarrows g'$. As all these realizers depended only in an easy way[7] on e_0, e_1, e_2 and e_3, we have shown the realizedness of "'A is $*$-regular"'.

$\square$

It does not seem surprising that the same technique also works for the $*_2$ property:

Lemma 4.14. *If A is $*_2$-regular and $R''S \in A$, then $A' := S \times (V(S) \cap A)$ is a set in $V(S)$ and $\Vdash A'$ is $*_2$ -regular.*

Proof. By Lemma 4.12 it is already realized that A' is $\bigcup$-regular, what is left is to show that $\Vdash A' \vDash$ SRRS$_2$.

To realize this, chose D, G, e_0, e_1 and e_2 as above with $e_0 \vDash D \subseteq A'$, $e_1 \vDash G \subseteq A'$, $(e_2, a) \in G$ and $e_3 \vDash \forall x, y \in G \exists z \in G(x, y, z) \in D$.

From these, we have to construct a realizer for $\exists g \in A'.g \subseteq G \wedge a \in g \wedge \forall x, y \in g \exists z \in g(x, y, z) \in D$.

Consider the following relation R' between elements b, b' and b'' of A' which is defined to hold iff $b \cup b' \subseteq b'' \in A'$ and

$$\forall (f, x) \in b \forall (f', x') \in b' \exists u \in R''S \exists c \in b''.e_3 f f' \lhd u \wedge \forall g \in u \exists (lg, y) \in c.$$
$$rg \vDash (x, x', y) \in D \wedge (lg, y) \in G \quad (4.1.20)$$

[7] namely by application of another fixed realizer

Similarly to before, we claim that $R' : \mathcal{P}_A(G) \times \mathcal{P}_A(G) \rightrightarrows \mathcal{P}_A(G)$. Indeed, from our knowledge about e_3, we know that

$$\forall (f,x) \in G \forall (f',x') \in G \exists u \in R''S \forall g \in u \exists (lg,y) \in G. \\ (lg,y) \in G \wedge rg \Vdash (x,y) \in D \quad (4.1.21)$$

So let $b, b' \in A, b, b' \subseteq G$. We have

$$\forall (f,x) \in b \forall (f',x') \in b' \exists u \in R''S \forall g \in u \exists (lg,y) \in G. \\ (lg,y) \in G \wedge rg \Vdash (x,x',y) \in D \quad (4.1.22)$$

Thus by (weak) regularity of A (of which the us are elements):

$$\forall (f,x) \in b \forall (f',x') \in b \exists b'' \subseteq G \subseteq B.b'' \in A \wedge \exists u \in R''S \forall g \in u \exists (lg,y) \in b''. \\ (lg,y) \in G \wedge rg \Vdash (x,x',y) \in D \quad (4.1.23)$$

Using the (weak) regularity of A yet again, we can collect all neccessary b'' in a single set $B'' \in A$ – but the union $b''' := b \cup \bigcup B''$ is then just as desired a set in A and a subset of G which stands in the R'-relation with b and b'. Note that the union-regularity is used here to obtain $b''' \in A$.

Using the $*_2$-property of A we can now find a set $g' \in A$ and subset of $\mathcal{P}_A(G)$ with $\{(e_2, a)\} \in g'$ such that $R : g' \times g' \rightrightarrows g'$. The set $g := \{(k,h) | h \in g\}$ is then as desired for the $*$-property:

It is realized by pe_2e_r that $a \in g'$. It is trivially realized that $g' \subseteq G$. Furthermore, the same realiser e_3 that realizes $D : G \times G \rightrightarrows G$ also realizes $D : g \times g \rightrightarrows g$, which is direct consequence of $R : g' \times g' \rightrightarrows g'$. As all these realizers depended only in an easy way[8] on e_0, e_1, e_2 and e_3, we have shown the realizedness of "'A is $*_2$-regular"'. □

[8]namely by application of another fixed realizer

This leads to the main theorem in this subsection:

Theorem 4.15. *The following statements are absolute for realizability models, Heyting models and models based on applicative topologies:*

- *wREA*
- *REA*
- $\bigcup$*REA*
- $*$*REA*
- $*_2$*REA*

Proof. Let $A \in V(S)$. By wREA, REA, $\bigcup$REA, $*$REA or $*_2$REA respectively, there is a set A' such that $A \in A'$ and $S, R''S \in A'$ and A' is weakly regular, regular, $\bigcup$-regular, $*$-regular or $*_2$-regular respectively.

Use Lemma 4.11, Lemma 4.10, Lemma 4.12, Lemma 4.13 or Lemma 4.14 respectively to see that $S \times (V(S) \cap A) \in V(S)$ is realized to have the same regularity property, and for all e we have $e \Vdash A \dot{\in} S \times (V(S) \cap A)$. $\square$

It might be noted which tiny large set axioms are absent from Theorem 4.15: fREA and sREA, those which do not deal with total relations but with functions instead. This seems to be one more indication that in the constructive context of CZF, total relations are much easier to handle than functions (and indeed, CZF's axioms themselves include Strong Collection and Fullness, not Replacement and Exponentiation).

Of course, these axioms are still realized if stronger axioms like REA or IEA respectively hold in the background theory, so they are still consistent with constructive statements like Church's Thesis or continuity principles even if fREA and sREA are not absolute for realizability models themselves.

4.1.4 The Absoluteness of Inaccessibility

Like for the tiny large set axioms, we aim to show that if a certain set containing $R''S$ enjoys a certain largeness property, then the intersection of this set with $V(S)$ is realized to enjoy the same property.

Lemma 4.16. *Let A be inaccessible and $S, R''S \in A$. Then*

$$\Vdash S \times (V(S) \cap A) \textit{ is inaccessible.} \tag{4.1.24}$$

Proof. By Lemma 4.12, A' is realized to be $\bigcup$-regular, so it is only left to show that $A' := S \times (V(S) \cap A)$ is realized to contain the natural numbers and to model Subset Collection and Binary Intersection.

The natural numbers

A variant of the natural numbers in $V(S)$ is $\bar{\omega}$, the set defined by:

$$\bar{n} := \{\underbrace{lll...lll}_{m \text{ times}} r, \bar{m})|m \in n\} \tag{4.1.25}$$

$$\bar{\omega} := \{\underbrace{lll...lll}_{n \text{ times}} r, \bar{n})|n \in \omega\} \tag{4.1.26}$$

As A is inaccessible and contains S, this is an element of A, so $\Vdash \omega \in A'$.

Subset Collection

We will prove Subset Collection rather than Fullness, which allows us to run the proof parallel to [Zie12], where it was shown that V models Subset Collection. This can be seen as the relativized version.

Let Φ be a formula with all quantifiers bounded by A'. We need a realizer for the formula

$$\forall a, b \in A' \exists c \in A' \forall u \in A'.$$
$$\forall x \in a \exists y \in b \Phi(x, y, u) \to \exists d \in c.$$
$$\forall x \in a \exists y \in d \Phi(x, y, u) \wedge \forall y \in d \exists x \in a \Phi(x, y, u) \quad (4.1.27)$$

We claim that such a realizer would be

$$j := \lambda x.\lambda y.pk(\lambda z.i) \quad (4.1.28)$$

With i being defined as:

$$i := \lambda v.pk(p(\lambda x.p(px(vx))(r(vx)))(\lambda x.p(lx)(r(rx)))) \quad (4.1.29)$$

The reason why the variables x, y and z do not play any significant role in j (all the λ-terms describe constant functions) and all the complexity of the realizer manifests in the subterm i is that the actual realizers for $a \dot{\in} A'$ do not matter — in fact, by definition of A', if $(e, a) \in A'$, then also $(f, a) \in A'$ for any $f \in S$.

Let $(x_0, a), (x_1, b) \in A'$. Define

$$\tilde{b} := \{(g, d) | \exists (g', d) \in b.l(rg) \lhd g'\} \in A' \quad (4.1.30)$$

As A models Subset Collection and Strong Collection, there is a $B \subseteq \mathcal{P}(\tilde{b})$ with $B \in A$ such that for all $(f, x) \in a$, $u \in A'$, $e \in S$ and $v \in R''S$, if

$$\forall j \in v \exists (h, y) \in \tilde{b}.j = h \wedge (lh, x) \in a \wedge (l(rh), y) \in b \wedge r(rh) \Vdash \Phi[x, y, u] \quad (4.1.31)$$

then there is a $\tilde{b}' \in A$ for which $\Psi(f, x, u, v, e, \tilde{b}')$ holds, where Ψ is the conjunction of the following two formulas:

$$\forall j \in v \exists (h, y) \in \tilde{b}'.j = h \wedge (lh, x) \in a \wedge (l(rh), y) \in b \wedge r(rh) \Vdash \Phi[x, y, u] \quad (4.1.32)$$

and

$$\forall (h, y) \in \tilde{b}' \exists j \in v.j = h \wedge (lh, x) \in a \wedge (l(rh), y) \in b \wedge r(rh) \Vdash \Phi[x, y, u] \quad (4.1.33)$$

For each $e \in S$ we use again the fact that A models Subset Collection, so that for all $e \in S$ we obtain a set $C \in A$ with $C \subseteq \mathcal{P}(B)$ such that if

$$\forall (f, x) \in a \exists \tilde{b}' \in B \exists v \in R''S.pf(ef) \lhd v \wedge \Phi(f, x, u, v, e, \tilde{b}') \quad (4.1.34)$$

Then there is a $B' \in C$ such that the following two hold:

$$\forall (f, x) \in a \exists \tilde{b}' \in B' \exists v \in R''S.pf(ef) \lhd v \wedge \Phi(f, x, u, v, e, \tilde{b}') \quad (4.1.35)$$

and

$$\forall \tilde{b}' \in B' \exists (f, x) \in a \exists v \in R''S.pf(ef) \lhd v \wedge \Phi(f, x, u, v, e, \tilde{b}') \quad (4.1.36)$$

While we obtain such a set $C \in A$ for each $e \in S$, this is not necessarily a functional relationship. Nevertheless we can use that A is regular by finding a set $C^* \in A$ such that for each $e \in S$ there is a set $C \in C^*$ with C specified as above and each $C \in C^*$ acts as a C as above for some $e \in S$.

Define

$$c := \{(k, \{(l, y) | \exists \tilde{b}' \in B'.(l, y) \in \tilde{b}'\}) | B' \in \bigcap C^*\} \in A' \quad (4.1.37)$$

We claim that this c is the desired witness[9] for the instance of Subset Collection with the formula Φ applied to a and b.

For let $(x_3, u) \in A'$ and $e \Vdash \forall x \in a \exists y \in b\ \Phi(x, y, u)$. In other words, for all $(f, x) \in a$ there is a $v \in R''S$ such that

[9]Unlike realizability models, models based on applicative topologies do not always need to decide on a single witness to realize existential statements. In this case however, this is what happens.

$$ef \lhd v \wedge \forall j \in v \exists (lh, y) \in b.j = h \wedge (lh, y) \in b \wedge rh \Vdash \Phi[x, y, u] \tag{4.1.38}$$

So there is a $v \in R''S$ with $pf(ef) \lhd v$ and

$$\forall j \in v \exists (h, y) \in \tilde{b}.j = h \wedge (lh, x) \in a \wedge (l(rh), y) \in b \wedge r(rh) \Vdash \Phi[x, y, u] \tag{4.1.39}$$

By the choice of B, we get

$$\exists v \in R''S.pf(ef) \lhd v \wedge \exists \tilde{b}' \in B.\Psi(f, x, u, v, e, \tilde{b}') \tag{4.1.40}$$

So that in total, we have

$$\forall (f, x) \in a \exists \tilde{b}' \in B \exists v \in R''S.pf(ef) \lhd v \wedge \Phi(f, x, u, v, e, \tilde{b}') \tag{4.1.41}$$

By choice of C, this implies that there is some $B' \in C$ such that

$$\forall (f, x) \in a \exists \tilde{b}' \in B' \exists v \in R''S.pf(ef) \lhd v \wedge \Psi(f, x, u, v, e, \tilde{b}') \tag{4.1.42}$$

And

$$\forall \tilde{b}' \in B' \exists (f, x) \in a \exists v \in R''S.pf(ef) \lhd v \wedge \Psi(f, x, u, v, e, \tilde{b}') \tag{4.1.43}$$

Define

$$b' := \{(l, y) | \exists \tilde{b}' \in B'.(l, y) \in \tilde{b}'\} \tag{4.1.44}$$

Then $(l(ef), b') \in c$. We want to show:

$$r(ie) \Vdash \forall x \in a \exists y \in b' \Phi(x, y, u) \wedge \forall y \in b' \exists x \in a \ \Phi[x, y, u] \tag{4.1.45}$$

To prove 4.1.45, take $(f, x) \in a$. By choice of B,

$$\exists v \in R''S\exists\tilde{b'} \in B'.pf(ef) \lhd v \wedge \Psi(f, x, u, v, e, \tilde{b}') \tag{4.1.46}$$

Thus

$$\forall j \in v\exists(h, y) \in \tilde{b}'.j = h \wedge (lh, x) \in a \wedge (l(rh), y) \in b \wedge r(rh) \Vdash \Phi[x, y, u] \tag{4.1.47}$$

This implies

$$pf(ef) \lhd \{j|\exists(j, y) \in b'.\ r(r(j)) \Vdash \Phi[x, y, u]\} \tag{4.1.48}$$

And this in turn brings us to

$$r(r(ie))f \unlhd p(pf(ef))(r(ef)) \Vdash \exists y \in b'\Phi[x, y, u] \tag{4.1.49}$$

This establishes the first conjunct of 4.1.45.

For the second part, take $(g', y) \in b'$, so there is some $\tilde{b}'$ with $(g, y) \in \tilde{b}'$ and $g' \lhd \{g''|(g'', y) \in \tilde{b}'\}$. We claim that

$$\forall g.(g, y) \in \tilde{b}' \to l(r(ie))g \Vdash \exists x \in a\Phi[x, y, u] \tag{4.1.50}$$

So let $(g, y) \in \tilde{b}'$, then by choice of B'

$$\exists(f, x) \in a, v \in R''S.pf(ef) \lhd v \wedge \forall(h, y') \in \tilde{b}'\exists j \in v.\Theta(a, b, u, j, h, x, y) \tag{4.1.51}$$

Where

$$\Theta(a,b,u,j,h,x,y) :\leftrightarrow j = h \wedge (lh,x) \in a \wedge (l(rh'),y) \in b \wedge r(rh) \Vdash \Phi[x,y,u] \tag{4.1.52}$$

If we set $h = g$ and $y = y'$, we arrive at

$$(lg,x) \in a \wedge r(rg) \Vdash \Phi[x,y,u] \tag{4.1.53}$$

And this allows us to infer

$$l(r(ie))g \trianglelefteq p(lg)(r(rg)) \Vdash \exists x \in a\Phi[x,y,u] \tag{4.1.54}$$

Thus for all g with $(g,y) \in \tilde{b}'$, we get

$$l(r(ie))g \Vdash \exists x \in a\Phi[x,y,u] \tag{4.1.55}$$

And in particular

$$l(r(ie))g' \Vdash \exists x \in a\Phi[x,y,u] \tag{4.1.56}$$

This shows the second conjunct of 4.1.45.

Binary Intersection

We make use of the following useful fact:

$$\forall a,b \in A'.\{e \in S | e \Vdash a \in b\} \in A \tag{4.1.57}$$

We want to show this by simultaneous set induction[10] over a and b. The induction only goes through if we prove at the same time the statement

[10]To be precise: Over the ordering of pairs of sets where $(x,y) < (u,v)$ if $x \in u \wedge y \in v$. This admits induction as an easy consequence of Set Induction.

$$\forall a, b \in A'.\{e \in S | e \Vdash a = b\} \in A \tag{4.1.58}$$

So let $a, b \in A'$ and let by induction hypothesis 4.1.57 and 4.1.58 hold for all (x, y) with $x \in a, y \in b$. Then the class[11] from equation 4.1.57 is equal to

$$\{e \in S | e \lhd \{f \in S | \exists y \in tc(b).lf \Vdash z\dot{\in}b \wedge rf \Vdash x = y\} \tag{4.1.59}$$

Using the set-presentedness of the formal topology, this is equal to

$$\{e \in S | \exists q \in R(e).q \subseteq \{f \in S | \exists y \in tc(b).lf \Vdash z\dot{\in}b \wedge rf \Vdash x = y\}\} \tag{4.1.60}$$

Note that $z\dot{\in}b$ can be written equivalently as a Δ_0 formula with parameters in A':

$$z\dot{\in}b \leftrightarrow \exists q' \in R(z).\forall e \in q'.(e, z) \in b \tag{4.1.61}$$

This is a Δ_0 formula with parameters in A' as $R : S \to R''S$ is an element of A by regularity.

Combining this with the induction hypothesis and A modeling Δ_0 collection yields that the inner class from 4.1.60 is in A, i.e.

$$\{f \in S | \exists y \in tc(b).lf \Vdash z\dot{\in}b \wedge rf \Vdash x = y\} \in A \tag{4.1.62}$$

And so by again using that A models Δ_0-Separation, we get that the whole class 4.1.60 is an element of A.

It remains to show that A also contains the class[12] from equation $4.1.58$, i.e.

$$\{e \in S | (\forall (f, z) \in a.lef \Vdash z \in b) \wedge (\forall (f, z) \in b.ref \Vdash z \in a)\} \tag{4.1.63}$$

[11] Actually, this class is a set by the results of [Zie12].

[12] Again, this is a set by the results of [Zie12].

But this is clear from the induction hypothesis and A modelling Δ_0-Separation.

This proves 4.1.57. We can now show the following statement which implies that A' models the Binary Intersection axiom.

$$\Vdash \forall a \in A' \forall b \in A' \exists c \in A'. \forall x \in a(x \in b \to x \in c) \wedge \forall x \in c(x \in a \wedge x \in b) \quad (4.1.64)$$

Let e_r be any realizer for reflexivity, then a realizer for the formula above is:

$$e_{\text{Sep}} := \lambda uv.pk(p(\lambda xy.p(pxy)e_r)(\lambda x.p(p(lx)e_r)(rx))) \quad (4.1.65)$$

For let $(x_0, a), (x_1, b) \in A'$. We claim that the existential we need to realize can be realized with a single witness, e.g.

$$c := \{(pef, x) | (e, x) \in a \wedge f \Vdash x \in b\} \quad (4.1.66)$$

For let $(e, x) \in a$ and $f \Vdash x \in b$. Then

$$l(r(e_{\text{Sep}}uv))ef \trianglelefteq p(pef)e_r \quad (4.1.67)$$

And indeed, as $(pef, x) \in c$, we have

$$p(pef)e_r \Vdash x \in c \quad (4.1.68)$$

Conversely, take an arbitrary element of c, it is of the form $(pef, x) \in c$ with $e, x \in a$ and $f \Vdash x \in b$.

Then

$$r(r(e_{\text{Sep}}uv))(pef) \trianglelefteq p(pee_r)f \quad (4.1.69)$$

And indeed $pee_r \Vdash x \in a$ and $f \Vdash x \in b$.

This proves that A' models the Binary Intersection axiom, which was the last part of Inaccessibility we needed to show. $\square$

4.1.5 The Consistency of Small Large Set Axioms

Inaccessibility is absolute by the results of the last subsection, but what about its generalisation α-inaccessibility? In a certain sense, absoluteness of this notion seems a moot point seeing that α does not necessarily have a direct representative in $V(S)$. However, we can still define a sensible injection from V to $V(S)$ and this will actually preserve the notion of α-inaccessibility.

Definition 4.17. *Define recursively a function $a \mapsto a_k : V \hookrightarrow V(S)$ by*

$$a_k := \{(k, x_k) | x \in a\} \tag{4.1.70}$$

Define a left inverse $a \mapsto a^S : V(S) \twoheadrightarrow V$ recursively by

$$a^S := \{x^S | \exists e \in S.(e, x) \in a\} \tag{4.1.71}$$

Then we get the following:

Lemma 4.18. *Let $\alpha \in V(S)$. Let A be α^S-inaccessible and $S, R''S \in A$. Define $A' := S \times (V(S) \cap A)$, then*

$$\Vdash A' \text{ is } \alpha\text{-inaccessible.} \tag{4.1.72}$$

Proof. We will prove that there is a uniform realizer $e_{\text{inacc}} \in \nabla$ for the desired statement which does not depend on α by induction over α^S. Let $e_{\text{i}} \in \nabla$ be the realizer whose existence the proof of Lemma 4.16 showed, i.e. whenever B is inaccessible and $S, R''S \in B$, then e_{i} realizes that $S \times (V(S) \cap B)$ is inaccessible. Let by the fixed point theorem $e_{\text{inacc}} \in \nabla$ be such that

$$le_{\text{inacc}} \lhd e_{\text{i}} \;\wedge\; \forall ef.re_{\text{inacc}}ef \lhd pke_{\text{inacc}} \tag{4.1.73}$$

We will show by induction on α^S that whenever A is α^S-inaccessible and $S, R''S \in A$, then e_{inacc} is a realizer for $S \times (V(S) \cap A)$ being α-inaccessible.

So let by induction hypothesis for all $(e, \beta) \in \alpha$, and for all B which are β^S-inaccessible

$$e_{\text{inacc}} \Vdash S \times (B \cap V(S)) \text{ is } \beta\text{-inaccessible.} \tag{4.1.74}$$

Let A be α^S-inaccessible and $S, R''S \in A$ and $A' := S \times (V(S) \cap A)$.

To show the desideratum, we need to show two things: Firstly, le_{inacc} needs to realize the inaccessibility of A' (which it does by Lemma 4.16 as A is α^S-inaccessible and thus also inaccessible), and secondly we need

$$re_{\text{inacc}} \Vdash \forall x \in A' \forall \beta \in \alpha \exists B' \in A'.x \in B' \wedge B' \text{ is } \beta\text{-inaccessible.} \tag{4.1.75}$$

So let $(x_0, x) \in A'$ and $(x_1, \beta) \in \alpha$. Then there is a $B \in A$ which is β^S-inaccessible and contains x, S and $R''S \in A$ as $\beta^S \in \alpha^S$. So by induction hypothesis, e_{inacc} realizes that $B' := B \cap V(S) \in A$ is β-inaccessible. As $(k, B') \in A'$, we are done. □

Corollary 4.19. *Let $\alpha \in V(S)$. If A is rk(α)-inaccessible and $S, R''S \in A$, then $A' := S \times (V(S) \cap A)$ fulfills*

$$\Vdash A' \textit{ is } \alpha\textit{-inaccessible.} \tag{4.1.76}$$

The result above can also be extended to the Mahlo-hierarchy:

Lemma 4.20. *Let $\alpha \in V(S)$. If A is α^S-Mahlo and $S, R''S \in A$, then $A' := S \times (V(S) \cap A)$ fulfills*

$$\Vdash A' \textit{ is } \alpha\textit{-Mahlo.} \tag{4.1.77}$$

Proof. Under the specified assumptions, it is realized that $A' := S \times (V(S) \cap A)$ is inaccessible by Lemma 4.16.

Like in the proof of Lemma 4.18, the α-Mahlo property needs an induction over α^S to show the existence of a uniform realizer which does not depend on A' or α.

So let the statement above hold for all B which are x^S-Mahlo for any $(x_0, x) \in \alpha$. Let

$$e \Vdash \forall a \in A' \exists b \in A'.(a, b) \in R \tag{4.1.78}$$

This means that for all $(x_1, a) \in A'$

$$\exists q \in R(ex_1).q \subseteq \{f | \exists (lf, b) \in A'.rf \Vdash (a, b) \in R\} \tag{4.1.79}$$

For any such q with $q \subseteq \{f | \exists (lf, b) \in A'.rf \Vdash (a, b) \in R\}$, we can collect the necessary b into a single set by A' modelling Strong Collection (note that $q \subseteq$ ... is a statement of the form $\forall f \in q\exists$...). Also note that $(lf, b) \in A' \leftrightarrow b \in A \cap V(S)$. So for all $(x_1, a) \in A'$

$$\exists B \in A.\exists q \in R(ex_1).q \subseteq \{f | \exists b \in B \cap V(S).rf \Vdash (a, b) \in R\} \tag{4.1.80}$$

This does not quite amount to a total relation from A to A, but only to a total relation from A' to A, as such a $b \in A$ only exists for each $(x_1, a) \in A'$. So we cannot apply the Mahlo property directly — however, we can apply it indirectly, as there is a surjective function from A to A', e.g. the function $H : A \twoheadrightarrow A'$ defined by

$$H(a) := a \cap S \times V(S) \tag{4.1.81}$$

So there is a total relation $T : A \rightrightarrows A$ defined by

$$(a', B) \in T :\leftrightarrow$$
$$\exists a, x_1.H(a') = (x_1, a) \wedge \exists q \in R(ex_1).q \subseteq \{f | \exists b \in B \cap V(S).rf \Vdash (a, b) \in R\} \tag{4.1.82}$$

For any $(x_2, x) \in \alpha$, this relation T is reflected in a x^S-Mahlo set $B \in A$ with $S, R''S \in B$[13] (as A is α^S-Mahlo). Then by realizers obtainable from the induction hypothesis and e, it is realized that $B' := B \cap V(S)$ is x-Mahlo and

$$\forall a \in B' \exists b \in B'.(a, b) \in R \tag{4.1.84}$$

Which is a direct consequence of $T : B \rightrightarrows B$.

The realizer itself can again be constructed from the fixed point property Lemma 4.5, which is not especially complicated, but tedious (and depends on the exact definition of Mahloness). $\square$

This has an obvious Corollary:

Corollary 4.21. *If A is Mahlo and $S, R''S \in A$, then*

$$\Vdash S \times (V(S) \cap A) \textit{ is Mahlo.} \tag{4.1.85}$$

Proof. Note that being Mahlo is equivalent to being 1-Mahlo. A version of 1 in $V(S)$ is $\{(k, \emptyset)\}$ and $(k, \emptyset)^S = 1$. So applying Lemma 4.20 to $\alpha := \{(k, \emptyset)\}$ yields the desideratum. $\square$

[13] It is an easy lemma that Mahlo sets reflect the totality of any relation not just anywhere, but also in sets with specified members, i.e. for all $m \in \alpha^S$:

$$z \in A \wedge U : A \rightrightarrows A \rightarrow \exists V.z \in V \wedge U : V \rightrightarrows V \wedge V \ m - \text{Mahlo} \tag{4.1.83}$$

This leads to the main theorem in this subsection:

Theorem 4.22. *The following statements are absolute for realizability models, Heyting models and models based on applicative topologies:*

1. *There is an inaccessible set.*
2. *IEA*
3. *There is a Mahlo set.*
4. *MEA*
5. *Axiom M*
6. *For all α, there is an α-inaccessible set.*
7. *For all α, A, there is an α-inaccessible set I with $A \in I$.*
8. *For all α, there is an α-Mahlo set.*
9. *For all α, A, there is an α-Mahlo set I with $aA \in I$.*
10. *For all A, there is an I-inaccessible set I with $A \in I$.*
11. *For all A, there is an I-Mahlo set I with $A \in I$.*

Proof.

1. This is the same proof as for the next point except we only consider the special case $A = 0$.
2. Let $A \in V(S)$. By IEA, there is an inaccessible set A' such that $A \in A'$ and $S, R''S \in A'$.

   Use Lemma 4.16 to see that $S \times (V(S) \cap A')$ is realized to be inaccessible, and for all e we have $e \Vdash A \dot{\in} A' \cap V(S)$.
3. Analogous to the above (use Corollary 4.21 instead of Lemma 4.16).
4. Analogous to the above (use Corollary 4.21 instead of Lemma 4.16).

5. Let $e \Vdash \forall x \exists y \Phi(x, y)$. Then for all $a \in V(S)$ there is a set $q \in R(e)$ such that

$$q \subseteq \{f | \exists b. f \Vdash \Phi(a, b)\} \tag{4.1.86}$$

By Collection, these b can be found in some set B, so that we have

$$\forall a \in V(S) \exists x. x = (B, R''S) \wedge \exists q \in R(e). q \subseteq \{f | \exists b. f \Vdash \Phi(a, b)\} \tag{4.1.87}$$

Using the function $H : V \twoheadrightarrow V(S)$ as above, this yields a total relation on V which is then reflected in some inaccessible set A. Then $A' = S \times (A \cap V(S))$ is inaccessible by Lemma 4.16 and $\Vdash \forall x \in A' \exists y \in A'. \Phi(x, y)$.

6. This is the same proof as for the next point except we only consider the special case $A = 0$.

7. Let $\alpha, A \in V(S)$. By the statement holding in the background, there is an α^S-inaccessible A' such that $A \in A'$ and $S, R''S \in A$'.

   Use Lemma 4.18 to see that $S \times (V(S) \cap A')$ is realized to be α-inaccessible, and for all e we have $e \Vdash A \dot{\in} A' \cap V(S)$.

8. This is the same proof as for the next point except we only consider the special case $A = 0$.

9. Analogous to the point two above (use Lemma 4.20 instead of Lemma 4.18).

10. Let $A \in V(S)$. By the statement holding in the background, there is an A'-inaccessible A' such that $A \in A'$ and $S, R''S \in A'$.

    We need to show that the following is (uniformly) realized:

$$\Vdash \forall a, \alpha \in S \times (V(S) \cap A') \exists b \in A' \cap V(S). a \in b \wedge b \text{ is } \alpha\text{-inaccessible.} \tag{4.1.88}$$

    So let $(x_0, a), (x_1, \alpha) \in A'$, then $\alpha^S \in A'$ as A' is inaccessible and contains S and thus closed under $x \mapsto x^S$. As A' is A'-inaccessible, there is a $b \in A'$

with $S, R''S \in b$ and b is α^S inaccessible. So $S \times (b \cap V(S))$ is realized to be α-inaccessible by Lemma 4.18 (with a fixed realizer), and for all e we have $e \Vdash a\dot{\in}S \times (b \cap V(S))$.

11. Analogous to the above (use Lemma 4.20 instead of Lemma 4.18) and argue as in the proof of Lemma 4.20.

□

4.1.6 The Consistency of Large Large Set Axioms

This subsection is meant to investigate the absoluteness of axioms dealing with elementary embeddings under models based on applicative topologies (and thus realizability and Heyting models). However, as these axioms are cast in an enriched language, this requires an extension of the realizability definition for the new symbols of that language j (a unary function symbol) and M (a unary relation symbol).

Throughout this subsection, work in the language containing j and M and assume the axiom $j : V \overset{\equiv}{\hookrightarrow} M$. Also assume that j is constant on all elements of S and R and $j(S) = S$, $j(R) = R$. As all standard applicative topologies (in particular the Kleene algebra or the usual Heyting algebras) are a member of every inaccessible, this is a reasonable assumption when investigating measurable or Reinhardt sets.

Definition 4.23. *Let Φ be a formula in the language $\in, \dot{\in}, j, M$ with equality. Define the fomula $e \Vdash \Phi$ inductively as $e \in S$ and the appropriate clause from below:*

1. $e \Vdash \bot$ *if* $e \lhd \emptyset$
2. $e \Vdash j^n(x)\dot{\in}j^m(y)$ *if* $e \lhd \{e|(e, j^n(x)) \in j^m(y)\}$
3. $e \Vdash j^n(x) \in j^m(y)$ *if* $e \lhd \{f \in S|\exists z \in tc(j^m(y))lf \Vdash z\dot{\in}j^m(y) \wedge rf \Vdash j^n(x) = z\}$
4. $e \Vdash j^n(x) = j^m(y)$ *if* $\forall (f, z) \in j^n(x).lef \Vdash z \in j^m(y)$ *and*
   $\forall (f, z) \in j^m(y).ref \Vdash z \in j^n(x)$

5. $e \Vdash M(j^n(x))$ *if* $e \lhd \{f \in S | \exists z \in M \cap V(S). f \Vdash j^n(x) = z\}$

6. $e \Vdash \Phi \wedge \Psi$ *if* $le \Vdash \Phi \wedge re \Vdash \Psi$

7. $e \Vdash \Phi \vee \Psi$ *if* $e \lhd \{f \in S | (lf \trianglelefteq l \wedge rf \Vdash \Phi) \vee (lf \trianglelefteq r \wedge rf \Vdash \Psi)\}$

8. $e \Vdash \Phi \rightarrow \Psi$ *if* $\forall f \in S. f \Vdash \Phi \rightarrow ef \Vdash \Psi$

9. $e \Vdash \exists x \in y\ \Phi(x)$ *if* $e \lhd \{f \in S | \exists (lf, a) \in y.\ rf \Vdash \Phi[a]\}$

10. $e \Vdash \forall x \in y\ \Phi(x)$ *if* $\forall (f, a) \in y.\ ef \Vdash \Phi[a]$

11. $e \Vdash \exists x \in M\ \Phi(x)$ *if* $e \lhd \{f \in S | \exists (lf, a) \in S \times M \cap V(S).\ rf \Vdash \Phi[a]\}$

12. $e \Vdash \forall x \in M\ \Phi(x)$ *if* $\forall (f, a) \in S \times M \cap V(S).\ ef \Vdash \Phi[a]$

13. $e \Vdash \forall x \Phi(x)$ *if* $\forall a \in V(S).\ e \Vdash \Phi[a]$

14. $e \Vdash \exists x \Phi(x)$ *if* $e \lhd \{f \in S | \exists a \in V(S).\ f \Vdash \Phi[a]\}$

As usual, use $\Vdash \Phi$ *as shorthand for* $\exists e \in \nabla . e \Vdash \Phi$ *and call such a formula* Φ *realized.*

For formulae not containing M and j, this is the usual realizability as presented before. Note that treating the quantifiers $\forall x \in M$ and $\exists x \in M$ as quantifiers in their own right in this way conforms to intuitionistic logic with the exact same proof as for bounded quantification as presented in [Zie07] and analogous to [Rat03b].

Note that a priori it is not clear whether this definition even realizes j to be a function (i.e. whether $\Vdash \forall a, b. a = b \rightarrow j(a) = j(b)$, and if $j : V \overset{\equiv}{\hookrightarrow} M$ had not be assumed, this would indeed not be provable.

Remark 4.24. *(Abuse of Notation) This entails an abuse of notation as it is not clear whether e.g. the* j *in* $e \Vdash j(a) = b$ *is to be read as "*e *realizes the formula* $j(v_1) = v_2$ *where* v_1 *has been substituted by the parameter* $a \in V(S)$ *and* v_2 *has been substituted by the parameter* $b \in V(S)$*", or rather as "*e *realizes the formula* $v_1 = v_2$ *where* v_1 *has been substituted by the parameter* $j(a) \in V(S)$ *and* v_2 *has been substituted by the parameter* $b \in V(S)$*". This could be avoided by choosing a different symbol in each* *case, however we will refrain from this as the definition above interprets both formulae equivalently.*

First, we need to establish that the definition integrates well into our set theory:

Lemma 4.25. *For any set a:*

$$a \in V(S) \leftrightarrow j(a) \in V(S) \tag{4.1.89}$$

And

$$a \in M \cap V(S) \leftrightarrow a \in M \wedge (a \in V(S))^M \tag{4.1.90}$$

Proof. As $V(S)$ was introduced in a recursively Δ_0 definition (and the one parameter, S, is unchanged under j), the first part of the statement is a direct consequence from Lemma 2.19 and the second part follows directly by induction over a. □

Then the elementarity of j extends to $\Vdash$ for bounded formulae:

Lemma 4.26. *For any bounded formula $\Phi(\overrightarrow{x})$ with all free variables displayed in the language $\in$ with equality (but without j and M):*

$$\forall \overrightarrow{a} \in V(S), e \in S. e \Vdash \Phi(\overrightarrow{a}) \leftrightarrow e \Vdash \Phi(\overrightarrow{j(a)}) \leftrightarrow (e \Vdash \Phi(\overrightarrow{j(a)}))^M \tag{4.1.91}$$

Proof. Note that the second equivalence follows directly from the definition, as there is no unbounded quantification in the formula $e \Vdash \Phi(j(\overrightarrow{a}))$ and all parameters are in M (in particular, S and R). The first equivalence is proved by induction over the complexity of Φ.

- For the atomic cases $\perp$, $\cdot\in$, $\in$ and $=$, note that these were again introduced in a recursively Δ_0 definition (and the parameters, S and R, are unchanged under j and elements of M), so the statement is again a direct consequence from Lemma 2.19. Note that the atomic case concerning M is precluded by the condition that Φ not contain M or j.

- $e \Vdash \Phi(\overrightarrow{a}) \wedge \Psi(\overrightarrow{a})$ iff $le \Vdash \Phi \wedge re \Vdash \Psi$ which by induction hypothesis is equivalent to $le \Vdash \Phi(\overrightarrow{j(a)}) \wedge re \Vdash \Psi(\overrightarrow{j(a)})$ which is equivalent to $e \Vdash (\Phi \wedge \Psi)(\overrightarrow{j(a)})$.

- $e \Vdash \Phi(\overrightarrow{a}) \vee \Psi(\overrightarrow{a})$ if $e \lhd \{f \in S | (lf \trianglelefteq l \wedge rf \Vdash \Phi(\overrightarrow{a})) \vee (lf \trianglelefteq r \wedge rf \Vdash \Psi(\overrightarrow{a}))\}$. By induction hypothesis, this is equivalent to

$$e \lhd \{f \in S | (lf \trianglelefteq l \wedge rf \Vdash \Phi(\overrightarrow{j(a)}) \vee lf \trianglelefteq r \wedge rf \Vdash \Psi(\overrightarrow{j(a)})\} \tag{4.1.92}$$

And thus to $e \Vdash (\Phi \vee \Psi)(\overrightarrow{j(a)})$.

- $e \Vdash \Phi(\overrightarrow{a}) \rightarrow \Psi(\overrightarrow{a})$ iff $\forall f \in S.f \Vdash \Phi(\overrightarrow{a}) \rightarrow ef \Vdash \Psi(\overrightarrow{a})$. By induction hypothesis, this is equivalent to

$$\forall f \in S.f \Vdash \Phi(\overrightarrow{j(a)}) \rightarrow ef \Vdash \Psi(\overrightarrow{j(a)}) \tag{4.1.93}$$

And thus to $e \Vdash (\Phi \rightarrow \Psi)(\overrightarrow{j(a)})$.

- $e \Vdash \exists x \in y\ \Phi(x, \overrightarrow{a})$ if $e \lhd \{f \in S | \exists (lf, b) \in y.\ rf \Vdash \Phi(\overrightarrow{x}, b)\}$. By elementarity and induction hypothesis[14], this is equivalent to

$$e \lhd \{f \in S | \exists (lf, b) \in j(y).\ rf \Vdash \Phi(\overrightarrow{x}, j(b))\} \tag{4.1.94}$$

And thus to $e \Vdash (\exists x \in y \Phi(x))(\overrightarrow{j(a)})$.

- $e \Vdash \forall x \in y\ \Phi(\overrightarrow{a}, x)$ if $\forall (f, b) \in y.\ ef \Vdash \Phi(\overrightarrow{a}, b)$. By elementarity and induction hypothesis, this is equivalent to

$$\forall (f, b) \in j(y).\ ef \Vdash \Phi(\overrightarrow{j(a)}, b) \tag{4.1.95}$$

And thus to $e \Vdash (\forall x \in y \Phi(x))(\overrightarrow{j(a)})$.

[14]Note that this proof can be read as an induction over Φ that formulae of the form $e \Vdash \Phi(\overrightarrow{a})$ can be included in the elementarity scheme — as these formulae are equivalent to formulae in the language $\in, =$, for this it suffices to prove that $(e \Vdash \Phi(\overrightarrow{a})) \leftrightarrow (j(e) \Vdash \Phi(\overrightarrow{j(a)}))$ and the more complex instances of the elementarity scheme follow.

□

This has one direct consequence we can not do without:

Proposition 4.27. *The realizer $skk = \lambda x.x$ realizes that j is functional, i.e.*

$$skk \Vdash \forall x, y.x = y \to j(x) = j(y) \tag{4.1.96}$$

The realizer e_r which realizes reflexivity also realizes that j maps V to M, i.e.

$$e_r \Vdash \forall x.M(j(x)) \tag{4.1.97}$$

Also, M is realized to be transitive, i.e.

$$\Vdash \forall x.M(x) \to \forall y \in x.M(y) \tag{4.1.98}$$

Proof. If $e \Vdash x = y$ then by Lemma 4.26 also $e \Vdash j(x) = j(y)$, so skk realizes j to be functional.

Let $x \in V(S)$. For the second statement, we want to show that there is a $y \in M$ with $e_r \Vdash j(x) = y$, but as $j(x) \in M$, defining $y := j(x)$ is as required.

Let x be such that $e_1 \Vdash M(x)$, i.e. there is some $q \in R(e_1)$ with

$$q \subseteq \{f \in S | \exists z \in M \cap V(S).f \Vdash x = z\} \tag{4.1.99}$$

Let $(e_2, y) \in x$. Then as M is transitive, also

$$q \subseteq \{f \in S | \exists z \in M \cap V(S).lfe_2 \vdash y = z\} \tag{4.1.100}$$

For the set on the right hand side of 4.1.99 is just a subset of the set on the right hand side of 4.1.100.

Thus $\lambda xy.lxy$ is a realizer for the transitivity of M.

□

Then j is realized to be an elementary embedding:

Lemma 4.28.

$$\Vdash j : V \overset{\equiv}{\hookrightarrow} M \tag{4.1.101}$$

In particular for any formula $\Phi(\overrightarrow{x})$ with all free variables displayed and not containing M or j,

$$\Vdash \forall \overrightarrow{a}.\Phi(\overrightarrow{a}) \leftrightarrow \Phi^M(\overrightarrow{j(a)}) \tag{4.1.102}$$

Proof. Building on Proposition 4.27, all that is left to prove is that the elementarity scheme is realized. This is done by induction on Φ. The atomic cases follow directly from Lemma 4.26. The junctor cases and bounded quantification work exactly the same as in the proof of Lemma 4.26. We still need to present unbounded universal and existential quantification:

Let $\Phi(\overrightarrow{x})$ be $\forall y \Psi(y, \overrightarrow{x})$. Then $e \in S$ realizes this if

$$\forall y.e \Vdash \Psi(y, \overrightarrow{x}) \tag{4.1.103}$$

By elementarity and induction hypothesis, this is equivalent to

$$\forall y \in M.e \Vdash \Psi(y, \overrightarrow{j(x)}) \tag{4.1.104}$$

And this is just $\Phi^M(\overrightarrow{j(a)})$.

Alternatively, let $\Phi(\overrightarrow{x})$ be $\exists y \Psi(y, \overrightarrow{x})$. Then $e \in S$ realizes this if

$$e \lhd \{f | \exists y.f \Vdash \Psi(y, \overrightarrow{x}\} \tag{4.1.105}$$

By elementarity and induction hypothesis, this is equivalent to

$$e \lhd \{f | \exists y \in M.f \Vdash \Psi(y, \overrightarrow{j(x)}\} \tag{4.1.106}$$

And this is just $\Phi^M(\overrightarrow{j(a)})$. $\square$

Remark 4.29. *Not only is j any elementary embedding, but symbols for j and M can also appear in the axiom schemes of set theory and they still hold. The reason for this is that j and M can appear in the axiom schemes in the background universe, so the exact same proof that Strong Collection and Set Induction (and Subset Collection if one should chose not to work with Fullness instead) are absolute for models based on applicative topologies goes through when the schemes are allowed to contain j and M.*

To imbue $j : V \overset{\equiv}{\hookrightarrow} M$ with any strength at all, we need statements about a critical point of the embedding. These are mirrored in the realizability model:

Lemma 4.30. *Let K be an inaccessible set, $K \in j(K)$ and $\forall x \in K.j(x) = x$. Let $K' := S \times (K \cap V(S))$. Then*

$$\Vdash K' \textit{ inaccessible } \wedge K' \in j(K') \wedge \forall x \in K'.j(x) = x \tag{4.1.107}$$

Proof. K' is realized to be innaccessible by Lemma 4.16. As $j(K)$ is inaccessible and $K \in j(K)$, also $K' \in j(K)$ as $V(S)$ has a recursive Δ_0 definition and intersections of such classes with a set are again sets, a fact which is reflected in inaccessible sets like $j(K)$.

As $K' \in j(K)$ and also $K' \in V(S)$, we have $(k, K') \in j(K') = S \times (j(K) \cap V(S))$ and so for e_{r} a realizer for reflexivity,

$$pke_{\mathrm{r}} \Vdash K' \in j(K') \tag{4.1.108}$$

Also if $(e, x) \in K'$, then by K being regular, $(e, x) \in K$ and thus $j(x) = x$. So in this case,

$$e_{\mathrm{r}} \Vdash j(x) = x \tag{4.1.109}$$

So the universal statement is realized by $\lambda x.e_{\mathrm{r}}$. □

The route to Reinhardt sets is given by

Lemma 4.31. *Let $V = M$. Then $\Vdash V = M$.*

Proof. If $V = M$ then for all $x \in V(S)$, any realizer e_{r} for reflexivity also realizes the equality of x to an element of M, i.e. itself, and thus $M(x)$. So in this case $e_{\mathrm{r}} \Vdash \forall x.M(x)$. □

Thus we arrive at the conclusion:

Theorem 4.32. *The following statements are absolute for realizability models, Heyting models and models based on applicative topologies:*

- $j : V \overset{\equiv}{\hookrightarrow} M$.
- *There is a measurable set.*
- *There is a Reinhardt set.*

Proof. Lemmata 4.28, 4.30 and 4.31. □

4.2 The Existence of Witnesses

A number of pleasing metamathematical properties of constructive set theories can be proved using a modified realizability structure due to Rathjen [Rat05b], who employed it to prove these properties for CZF and CZF + REA. The properties we will consider are:

Definition 4.33.

- *A theory T is said to have the **disjunction property** if whenever $T \vdash \Phi \vee \Psi$, then either $T \vdash \Phi$ or $T \vdash \Psi$.*
- *A theory T is said to have the **numerical existence property** if whenever $T \vdash \exists n \in \omega\Phi(n)$, then there is a natural number n such that $T \vdash \Phi(\bar{n})$[15].*
- *A theory T is said to be **closed under Church's rule** if whenever $T \vdash \forall n \in \omega\exists m \in \omega\Phi(n, m)$ then there is some natural number e such that T proves that the Turing machine with number $\bar{e}$ maps all numbers n to a number m which fulfills $\Phi(n, m)$.*
- *A theory T is said to be **closed under the extended Church's rule** if for any formula $\Psi(x)$ such that $T \vdash \forall n.\Psi(n) \leftrightarrow \neg\neg\Psi(n)$, whenever $T \vdash \forall n \in \omega.\Psi(n) \to \exists m \in \omega\Phi(n, m)$ then there is some natural number e such that T proves that the*

[15] In CZF, numerals are not an explicit part of the language, but they can be defined via $\bar{0} := 0$ and $\overline{n+1} := \bar{n} \cup \{\bar{n}\}$

Turing machine with number $\bar{e}$ maps all numbers n which fulfill Ψ to a number m which fulfills $\Phi(n, m)$.

- *A theory T is said to be* ***closed under the variant of Church's rule*** *if whenever it proves the existence of a function $f : \omega \to \omega$ with a certain property (definable without parameters), there is a natural number e such that T proves that the function computed by the Turing machine with number $\bar{e}$ enjoys said property.*

- *A theory T is said to fulfill* ***Unzerlegbarkeit***[16] *if whenever*

$$T \vdash \forall x.\Phi(x) \vee \Psi(x) \tag{4.2.110}$$

  then either $T \vdash \forall x.\Phi(x)$ or $T \vdash \forall x.\Psi(x)$.

- *A theory T is said to fulfill* ***Uniformity*** *if whenever*

$$T \vdash \forall x \exists n \in \omega.\Phi(n, x) \tag{4.2.111}$$

  then $T \vdash \exists n \in \omega \forall x.\Phi(n, x)$.

The first four items are ordered by strength and all are properties that might reasonably be expected from a constructive theory with the collection of all sets as objects of discourse, especially the disjunction property and the numerical existence property.

4.2.1 Realizability with Truth

The following definition will help introduce the variant of usual realizability called realizability with truth. We will present it in a slightly different, but equivalent way to [Rat05b].

Definition 4.34. *Let $b \in V(S)$. Then a pair (l, a) is called a* ***labeling*** *of b with label l if there is a bijection from b to a such that an element $(e, y) \in b$ is mapped to a pair (e, x) with x being a labeling of y with label in l.*

[16] This is a slightly different definition than in [Rat05b], yet still goes through in all of the proofs.

If x is a labeling, then the first component of x (the label) is refered to as x^0 and the second component as x^*.

The class of all labelings is called $V'(S)$. In other words, $V'(S)$ is the smallest class such that

$$\forall b \subseteq S \times V'(S) \forall a.(\forall (n,c) \in b \forall (a',b') \in c.a' \in a) \to (a,b) \in V'(S) \tag{4.2.112}$$

Proof. If (l_0, a_0) is a labeling of b, then we prove by induction over b^S that (l_0, a_0) belongs to every class Γ fulfilling

$$\forall b \subseteq S \times \Gamma \forall a.(\forall (n,c) \in b \forall (a',b') \in c.a' \in a) \to (a,b) \in \Gamma \tag{4.2.113}$$

Unraveling the definition of (l_0, a_0) being a labeling of b, it follows that each element of a_0 is of the form (e,x) with x being a labeling of some y with $(e,y) \in b$ and $x^0 \in y$. As $y^S \in b^S$, the induction hypothesis implies that $x \in \Gamma$.

So a_0 is a subset of $\omega \times \Gamma$ and for any $(n,c) \in a_0$ and $(a',b') \in c$, it follows that $a' \in l_0$ as (l_0, a_0) is a labeling. Thus $(l_0, a_0) \in \Gamma$.

On the other hand, let x be in the smallest class Γ such that

$$\forall b \subseteq S \times \Gamma \forall a.(\forall (n,c) \in b \forall (a',b') \in c.a' \in a) \to (a,b) \in \Gamma \tag{4.2.114}$$

We prove by induction over the definition of Γ that there is a $b \in V(S)$ such that x is a label of b. So let by induction hypothesis x be of the form (a,b) with $b \subseteq \omega \times \Gamma$ and $(\forall (n,c) \in b \forall (a',b') \in c.a' \in a)$ such that for all (n,c) in b, the set c is a labeling of some set in $V(S)$. By Lemma 4.35 part 1 (which does not build on the characterization that is being proved here), there is a function $f : b \to V(S)$ such that for all $(n,c) \in b$, the second component c is a labeling of $f(n,c)$. According to the premise above, this labeling must have a label in a, which means that (a,b) is a labeling of the set

$$\{(n, f(n,c)) | (n,c) \in b\} \in V(S) \tag{4.2.115}$$

□

Lemma 4.35. *1. Every labeling (l, a) has a unique set $s((l, a))$ of which it is the labeling, namely*

$$s((l, a)) = \{(e, s(x))|(e, x) \in a\} \quad (4.2.116)$$

2. *Every $b \in V(S)$ has a canonical labeling $l(b)$ with label b^S, namely*

$$l(b) := (b^S, \{(e, l(y))|(e, y) \in b\}) \quad (4.2.117)$$

3. *With $l : V(S) \to V'(S)$ and $s : V'(S) \to V(S)$ as above, s is a left inverse of l, i.e.*

$$s \circ l = id_{V(S)} \quad (4.2.118)$$

Proof. 1. This is proved by induction over the label l. If (l, a) is a labeling of some $b \in V(S)$, then by the definition of labelings,

$$b = \{(e, y)|\exists(e, x) \in a.x \text{ is a labeling of } y\} \quad (4.2.119)$$

Thus by induction hypothesis,

$$b = \{(e, y)|\exists(e, x) \in a.y = s(x)\} \quad (4.2.120)$$

And this means that $b = s(l, a)$.

2. This follows immediately by induction over b^S, as by induction hypothesis and definition, all elements of $l(b)^*$ are of the form $(e, l(y))$ with $(e, y) \in b$ and $l(y)^0 = y^S \in b^S$. Note that the 1:1 correspondence of (e, y)s and $(e, l(y))$s requires part 1 of this lemma.

3. Using rank induction over b, direct calculation shows that

$$s(l(b)) = s(b^S, \{(e, l(y))|(e, y) \in b\}) =$$
$$\{(e, s(x))|(e, x) \in \{(e, l(y))|(e, y) \in b\}\} = b \quad (4.2.121)$$

The last equality uses that for all relevant y, $s(l(y)) = y$ by induction hypothesis.

□

Realizability with truth defines a realizability relation for formulae with parameters from $V'(S)$. For such a formula Φ, define Φ^0 as the corresponding statement about the labels, i.e. let

1. $\bot^0 :\leftrightarrow \bot$
2. $(a \in b)^0 :\leftrightarrow a^0 \in b^0$
3. $(a = b)^0 :\leftrightarrow a^0 = b^0$
4. $(\Phi j \Psi)^0 :\leftrightarrow \Phi^0 j \Psi^0$ for any binary connective j.
5. $(Qx.\Phi)^0 :\leftrightarrow Qx.\Phi^0$ for any quantifier Q.

The following definition is the adaption of the corresponding definition from [Rat05b] for applicative topologies:

Definition 4.36. *Let Φ be a formula in the language $\in$ with equality. Define the formula $e \Vdash_{rt} \Phi$ inductively as $e \in S$ and the appropriate clause from below:*

1. $e \Vdash_{rt} \bot$ *if* $\bot$
2. $e \Vdash_{rt} x \dot{\in} y$ *if* $e \lhd \{e | (e, x) \in y^*\}$
3. $e \Vdash_{rt} x \in y$ *if* $(x \in y)^0$ *and* $e \lhd \{f \in S | \exists z \in tc(y^*) lf \Vdash_{rt} z \dot{\in} y \wedge rf \Vdash_{rt} x = z\}$
4. $e \Vdash_{rt} x = y$ *if* $(x = y)^0$ *and* $\forall (f, z) \in x^*. lef \Vdash_{rt} z \in y$ *and* $\forall (f, z) \in y^* ref \Vdash_{rt} z \in x$
5. $e \Vdash_{rt} \Phi \wedge \Psi$ *if* $le \Vdash_{rt} \Phi \wedge re \Vdash_{rt} \Psi$
6. $e \Vdash_{rt} \Phi \vee \Psi$ *if* $e \lhd \{f \in S | (lf \trianglelefteq l \wedge rf \Vdash_{rt} \Phi) \vee (lf \trianglelefteq r \wedge rf \Vdash_{rt} \Psi)\}$
7. $e \Vdash_{rt} \Phi \rightarrow \Psi$ *if* $(\Phi \rightarrow \Psi)^0$ *and* $\forall f \in S. f \Vdash_{rt} \Phi \rightarrow ef \Vdash_{rt} \Psi$
8. $e \Vdash_{rt} \exists x \in y\ \Phi(x)$ *if* $e \lhd \{f \in S | \exists (lf, a) \in y^*.\ rf \Vdash_{rt} \Phi[a]\}$
9. $e \Vdash_{rt} \forall x \in y\ \Phi(x)$ *if* $(\forall x \in a.\Phi(x))^0$ *and* $\forall (f, a) \in y^*.\ ef \Vdash_{rt} \Phi[a]$

10. $e \Vdash_{rt} \forall x \Phi(x)$ *if* $\forall a \in V'(S).\ e \Vdash_{rt} \Phi[a]$

11. $e \Vdash_{rt} \exists x \Phi(x)$ *if* $e \lhd \{f \in S | \exists a \in V'(S).\ f \Vdash_{rt} \Phi[a]\}$

While only explicitely demanded for the clauses for $\in, =, \rightarrow$ and $\forall x \in a.\Phi$, it follows that the theory of realized formulae is a subtheory of the theory of true formulae, or in other words:

Lemma 4.37. *Let* $e \in \nabla$. *Then:*

$$(e \Vdash_{rt} \Phi) \rightarrow \Phi^0 \tag{4.2.122}$$

Proof. [Rat05b] □

While realizability with truth might also be fruitfully employed with other applicative topologies, for the metamathematical properties considered here it suffices to consider only the case $S = Kl$ where Kl is the Kleene pca of natural numbers with application $e \circ f = g$ if the Turing machine with number e applied to the input $\bar{f}$ terminates with output $\bar{g}$ for a suitable representation $n \mapsto \bar{n}$ of natural numbers on Turing tape. The topology can be chosen to be the discrete topology with $e \lhd q :\leftrightarrow e \in q$ and $\nabla = \omega$. In such a case, realizability with the applicative topology becomes just the usual realizability with a pca ([Zie12]) while the above definition for realizability with truth becomes just what is described in [Rat05b].

In the rest of this chapter, we only consider $S := Kl$.

Then the main method from [Rat05b] can be described as:

Fact 4.38. *If* T *is an extension of CZF and*

$$T \vdash \Vdash_{rt} \Phi \textit{ for each axiom } \Phi \textit{ of } T \tag{4.2.123}$$

Then T *has all the properties from Definition 4.33.*

Proof. [Rat05b]. □

Using this, all that is left to do is to demonstrate absoluteness for the different concepts of largeness, which will then automatically establish the good metamathematical properties of the large set axioms.

4.2.2 Absoluteness Proofs

Lemma 4.39. *If A is regular and $\omega \in A$, then*

$$A' := (A, \{k\} \times (V'(Kl) \cap A)) \tag{4.2.124}$$

is a set in $V'(Kl)$ and $\Vdash_{rt} A'$ is regular.

Proof. This is proved in [Rat05b]. □

Note that in analogy to the previous subsections, we could also have defined A' as $(A, \omega \times (V'(Kl) \cap A))$, however, those two sets are realizedly equal.

Looking at the proof in [Rat05b], it becomes obvious that when proving the part that A is realizedly weakly regular, only the weak regularity of A was needed, so that we also get:

Lemma 4.40. *If A is weakly regular and $\omega \in A$, then*

$$A' := (A, \{k\} \times (V'(Kl) \cap A)) \tag{4.2.125}$$

is a set in $V'(Kl)$ and $\Vdash_{rt} A'$ is weakly regular.

This can be extended to $\bigcup$-regular sets:

Lemma 4.41. *If A is $\bigcup$-regular and $\omega \in A$, then*

$$A' := (A, \{k\} \times (V'(Kl) \cap A)) \tag{4.2.126}$$

is a set in $V'(Kl)$ and $\Vdash_{rt} A'$ is $\bigcup$-regular.

Proof. By Lemma 4.39 it is already realized that A' is regular, what is left is to show that $A' \vDash$ Union. This is realized in a similar way as it is realized that V'(Kl) itself models the Union Axiom. A possible realizer for the formula

$$\forall a \in A' \exists b \in A'.\forall c \in a \forall d \in cd \in a \wedge \forall d \in a \exists c \in ad \in c \tag{4.2.127}$$

would be

$$e := \lambda x.pk(p(\lambda yz.p(pyz)e_r)(\lambda x.p(lx)(p(rx)e_r))) \tag{4.2.128}$$

where as usual e_r is a realizer for reflexivity of equality. Indeed, the statement about the labels, i.e. (Union Axiom)^0, holds and for each $(k,a) \in A'^* \subseteq A$, there is the set

$$b := (\bigcup a^0, \{(pxy,d) | \exists c \in V'(Kl).(x,c) \in a^* \wedge (y,d) \in c^*\}) \in V'(Kl) \tag{4.2.129}$$

with $(k,b) \in A'^*$: $a^* \in A$ by transitivity and for each $(x,c) \in a^*$, we have $c* \in A$ and for each $(y,d) \in c*$ it holds that $(pxy,d) \in A$. So by Regularity $\{(pxy,d)|(y,d) \in c^*\} \in A$ and by Union-Regularity $b^* \in A$. Thus $b \in A$. Note that $b \in V'(Kl)$ follows from the fact that for $(pxy,d) \in b^*$, we have $d^0 \in c^0 \in a^0$ for a c with $(x,c) \in a^* \wedge (y,d) \in c^*$, so $d^0 \in a^0$.

We need to show

$$\lambda yz.p(pyz)e_r \Vdash_{\text{rt}} \forall c \in a \forall d \in c.d \in b \tag{4.2.130}$$

and

$$\lambda x.p(lx)(p(rx)e_r \Vdash_{\text{rt}} \forall d \in b \exists c \in a.d \in c \tag{4.2.131}$$

Together, they imply that e truly realizes the Union Property.

To prove statement 4.2.130, first note that the statement is true about the labels: $\forall c \in a^0 \forall d \in c.d \in b^0$, because b^0 was chosen as $\bigcup a^0$. Let $(f,c) \in a^*$. Note that the inner statement is true about the labels: $\forall d \in c^0.d \in b^0$ because $c^0 \in a^0$ and thus $b^0 = \bigcup a^0$. Let $(g,d) \in c^*$. We need to show

$$p(pfg)e_r \Vdash_{\text{rt}} d \in b \tag{4.2.132}$$

But this is the case since $(pfg, d) \in b^*$ and $d^0 \in b^0 = \bigcup a^0$.

To prove statement 4.2.131, first note that the statement is true about the labels: $\forall d \in b^0 \exists c \in a^0 d \in c$, because b^0 was chosen as $\bigcup a^0$. Let $(f, d) \in b^*$. Then by definition of b, there is a $c \in V'(Kl)$ such that $(x, c) \in a^*$ and there is a $(y, d) \in c^*$ such that $f = pxy$. In particular, $(lf, c) \in a^*$ and $(rf, d) \in c^*$. As also $d^0 \in c^0$, we get

$$p(lf)(p(rx)e_r) \Vdash \exists c \in a.d \in c \tag{4.2.133}$$

□

We can also obtain a similar result for $*$REA:

Lemma 4.42. *If A is $*$-regular and $\emptyset \in A$, then*

$$A' := (A, \{k\} \times (V'(Kl) \cap A)) \tag{4.2.134}$$

is a set in $V'(Kl)$ and $\Vdash_{rt} A'$ is $*$-regular.

Proof. First note that all $*$-regular sets containing $\emptyset$ also contain ω as an element. An induction over n shows that $\forall n \in \omega . n \in A$, for let this statement be true for all $m < n$ and $n > 0$, then consider the following relation defined on $n \subseteq a$:

$$R : n \rightrightarrows n, R(m_1, m_2) :\leftrightarrow m_2 = m_1 + 1 \vee m_2 = m_1 = n - 1 \tag{4.2.135}$$

This relaton is reflected in some element of A containing 0, but this can only be the set n. Thus $\omega \subseteq A$, and to see that also $\omega \in A$, it suffices to consider the relation

$$R : \omega \rightrightarrows \omega,\ R(m_1, m_2) :\leftrightarrow m_2 = m_1 + 1 \tag{4.2.136}$$

If this relation is reflected in a set containing 0, this set can only be ω, which is thus in A.

This means that we can apply Lemma 4.41, which means it is already realized that A' is $\bigcup$-regular. What is left is to show that $\Vdash_{\text{rt}} A' \vDash \text{RRS}_2$.

To realize this, first note that the statement $(A' \vDash \mathrm{RRS}_2)^0$ is true about the labels, as well as its pertinent substatements[17], as $A'^0 = A$ is $*$-regular.

So let $e_0 \Vdash_{\mathrm{rt}} D \subseteq A'$, $e_1 \Vdash_{\mathrm{rt}} G \subseteq A'$ (and again w.l.o.g. take $D, G \subseteq A$ with $D, G \in V'(Kl)$), $(e_2, a) \in D^*$ and $e_3 \Vdash_{\mathrm{rt}} \forall x \in D \exists y \in D.(x, y) \in G$.

From these, we have to construct a realizer for $\exists d \in A'.d \subseteq D \wedge a \in d \wedge \forall x \in d \exists y \in d(x, y) \in G$.

Consider the following relation R between elements b and b' of $A \cap V'(Kl)$ which is defined to hold iff $b \subseteq b' \subseteq D$ and

$$e_3 \Vdash_{\mathrm{rt}} \forall x \in b \exists y \in b'.(x, y) \in G \tag{4.2.138}$$

Let

$$D' := \{c \in A' | \lambda x.pxe_r \Vdash_{\mathrm{rt}} c \subseteq D\} \tag{4.2.139}$$

Then $R : D' \rightrightarrows D'$ by choice of e_3. Using the $*$-property of A we can now find a set $d' \in A$ with $R : d' \rightrightarrows d'$ and $\{a\} \in d'$). Define

$$d := (\{c^0 | c \in A'\}, \{(k, c) | c \in A'\}) \tag{4.2.140}$$

This set is realized to be in A' and it is realized that its union is as desired for the $*$-property, as $R : d' \rightrightarrows d'$ implies that it is realized that

[17] The statement $(A' \vDash RRS_2)^0$ is of the form

$$\forall G \forall D \forall d \in D.\Phi_1 \to \Phi_2 \to ... \to \Psi \tag{4.2.137}$$

The whole statement $(\forall G...)^0$ is true, as well as for all $G \in V'(Kl)$ the statement $(\forall D \forall ...)^0[G := G^0]$ as well as for all $G \in V'(Kl), D \in V'(Kl)$ the statement $(\Phi_1 \to ...)^0[G, D := G^0, D^0]$, as well as for all $G \in V'(Kl), D \in V'(Kl)$ the statement $\Phi_1^0[G, D := G^0, D^0] \to (\Phi_2 \to ...)^0[G, D := G^0, D^0]$ and so on, up until for all $G \in V'(Kl), D \in V'(Kl)$ the statement $\Phi_1^0[G, D := G^0, D^0] \to \Phi_2^0[G, D := G^0, D^0] \to ... \to \Psi^0[G, D := G^0, D^0]$.

$$\forall b \in d \exists b' \in d. \forall x \in b \exists x' \in b'.(x, x') \in G \tag{4.2.141}$$

As A' is realized to model the Union axiom, this finishes the proof of the lemma. $\square$

Again, the same technique works for the $*_2$ property:

Lemma 4.43. *If A is $*_2$-regular and $\emptyset \in A$, then*

$$A' := (A, \{k\} \times (V'(Kl) \cap A)) \tag{4.2.142}$$

is a set in $V(Kl)$ and $\Vdash_{rt}$ A' is $*_2$ -regular.

Proof. As in the proof of Lemma 4.42, it is realized that A' is $\bigcup$-regular. What is left is to show that $\Vdash_{\text{rt}} A' \vDash \text{SRRS}_2$.

To realize this, first note that the statement $(A' \vDash \text{SRRS}_2)^0$ is true about the labels, as well as its pertinent substatements, as $A'^0 = A$ is $*_2$-regular.

So let $e_0 \Vdash_{\text{rt}} D \subseteq A'$, $e_1 \Vdash_{\text{rt}} G \subseteq A'$ (and again w.l.o.g. take $D, G \subseteq A$ with $D, G \in V'(Kl)$), $(e_2, a) \in D^*$ and $e_3 \Vdash_{\text{rt}} \forall x, x' \in D \exists y \in D.(x, x', y) \in G$.

From these, we have to construct a realizer for $\exists d \in A'.d \subseteq D \wedge a \in d \wedge \forall x, x' \in d \exists y \in d(x, x', y) \in G$.

Consider the following relation R between elements b, b' and b'' of $A \cap V'(Kl)$ which is defined to hold iff $b \cup b' \subseteq b'' \subseteq D$ and

$$e_3 \Vdash_{\text{rt}} \forall x \in b \forall x' \in b' \exists y \in b''.(x, x', y) \in G \tag{4.2.143}$$

Let

$$D' := \{c \in A' | \lambda x.pxe_r \Vdash_{\text{rt}} c \subseteq D\} \tag{4.2.144}$$

Then $R : D' \times D' \rightrightarrows D'$ by choice of e_3. Using the $*$-property of A we can now find a set $d' \in A$ with $R : d' \times d' \rightrightarrows d'$ and $\{a\} \in d'$). Define

$$d := (\{c^0|c \in A'\}, \{(k, c)|c \in A'\}) \tag{4.2.145}$$

This set is realized to be in A' and it is realized that its union is as desired for the $*$-property, as $R : d' \times d' \rightrightarrows d'$ implies that it is realized that

$$\forall b, b' \in d \exists b'' \in d. \forall x \in b, x' \in b' \exists x'' \in b''.(x, x', x'') \in G \tag{4.2.146}$$

As A' is realized to model the Union axiom, this finishes the proof of the lemma. $\square$

Moving up the ladder of large sets, we arrive at

Lemma 4.44. *Let A be inaccessible and define*

$$A' := (A, \{k\} \times (V'(Kl) \cap A)) \tag{4.2.147}$$

Then

$$\Vdash_{rt} A' \textit{ is inaccessible} \tag{4.2.148}$$

Proof. By Lemma 4.41, A' is realized to be $\bigcup$-regular, so it is only left to show that A is realized to contain the natural numbers and to model Subset Collection and Binary Intersection. This can be proved the same way as [Rat05b] proved that the existence of natural numbers, Subset Collection and Δ_0-Collection were realized in $V'(S)$, just with the predicate "is a set" replaced by "is an element of A". It is easy to check that all witnesses used in [Rat05b] are indeed in A. $\square$

To deal with α-inaccessible and α-Mahlo sets, we need to employ the map $x \mapsto x^0$ instead of $x \mapsto x^S$ here.

Lemma 4.45. *Let $\alpha \in V'(Kl)$. If A is α^0-inaccessible, then*

$$A' := (A, \{k\} \times (V'(Kl) \cap A)) \tag{4.2.149}$$

is a set in $V(Kl)$ and

$$\Vdash_{rt} A' \textit{ is } \alpha\textit{-inaccessible.} \tag{4.2.150}$$

Proof. This proof mainly mimics the proof of Lemma 4.18. Let $e_{\rm i}$ be the realizer whose existence the proof of Lemma 4.44 showed, i.e. whenever B is inaccessible, then $e_{\rm i}$ realizes that $(B, \{k\} \times (V'(Kl) \cap B))$ is inaccessible.

Let by the fixed point theorem $e_{\rm inacc} \in Kl$ be such that

$$le_{\rm inacc} = e_{\rm i} \ \wedge \ \forall ef.re_{\rm inacc}ef = pke_{\rm inacc} \tag{4.2.151}$$

We will show by induction on α^0 that whenever A is α^0-inaccessible, then $e_{\rm inacc}$ is a realizer for $(A, \{k\} \times (V'(Kl) \cap A)$ being α-inaccessible.

So let by induction hypothesis for all $(e, \beta) \in \alpha^*$ (which implies $x^0 \in \alpha^0$), and for all B which are x^0-inaccessible

$$e_{\rm inacc} \Vdash_{\rm rt} (B, \{k\} \times (V'(Kl) \cap B)) \text{ is } \beta\text{-inaccessible.} \tag{4.2.152}$$

Let A be α^0-inaccessible and $(A, \{k\} \times (V'(Kl) \cap A))$. Note that the statement we want to realize and its pertinent substatements are true about the labels as $A'^0 = A$ is α^0 inaccessible.

To complete the proof, we need to show two things: Firstly, $le_{\rm inacc}$ needs to realize the inaccessibility of A' (which it does by Lemma 4.44 as A is α^0-inaccessible and thus also inaccessible), and secondly we need

$$re_{\rm inacc} \Vdash_{\rm rt} \forall x \in A' \forall \beta \in \alpha \exists B' \in A'.x \in B' \wedge B' \text{ is } \beta\text{-inaccessible.} \tag{4.2.153}$$

So let $(x_0, x) \in A'^*$ and $(x_1, \beta) \in \alpha^*$. Then there is a $B \in A$ which is β^0-inaccessible and contains x as $\beta^0 \in \alpha^0$. So by induction hypothesis, $e_{\rm inacc}$ realizes that the set

$$B' := (B, \{k\} \times (V'(Kl) \cap B)) \in A \tag{4.2.154}$$

is β-inaccessible. It is an element of A as the definition of $V'(Kl)$ is a recursive Δ_0 definition and as such is absolute for transitive models of CZF, CZF implies that the intersection of $V'(Kl)$ with any set is again a set and A is a transitive model of CZF.

As thus $(k, B') \in A'^*$, we are done. □

The result above can also be extended to the Mahlo-hierarchy:

Lemma 4.46. *Let $\alpha \in V(S)$. If A is α^0-Mahlo, and if*

$$A' := (A, \{k\} \times (V'(Kl) \cap A)) \tag{4.2.155}$$

Then

$$\Vdash_{rt} A' \textit{ is } \alpha\textit{-Mahlo.} \tag{4.2.156}$$

Proof. Under the specified assumptions, it is realized that A' is inaccessible by Lemma 4.44.

Like in the proof of Lemma 4.45, the α-Mahlo property needs an induction over α^0 to show the existence of a uniform realizer which does not depend on A' or α.

So let the statement above hold for all B which are β^0-Mahlo for any $(x_0, \beta) \in \alpha^*$ (then $\beta^0 \in \alpha^0$). Note that the statement we need to realize (i.e. the statement that A' is α-Mahlo) and its pertinent substatements are true about the labels by $A'^0 = A$ being α^0-Mahlo. These substatements are:

- The statement

$$\forall R \forall \beta \in \alpha^0 . R : A'^0 \rightrightarrows A'^0 \to \exists B \in A'^0 . R : B \rightrightarrows B \wedge B \text{ is } \beta \text{ Mahlo} \tag{4.2.157}$$

- For all $R \in V'(Kl)$ the statement

$$\forall \beta \in \alpha^0 . R^0 : A'^0 \rightrightarrows A'^0 \to \exists B \in A'^0 . R^0 : B \rightrightarrows B \wedge B \text{ is } \beta \text{ Mahlo} \tag{4.2.158}$$

- For all $R \in V'(Kl)$, $(n, \beta) \in \alpha^*$ the statement

$$R^0 : A'^0 \rightrightarrows A'^0 \to \exists B \in A'^0 . R^0 : B \rightrightarrows B \wedge B \text{ is } \beta^0 \text{ Mahlo} \tag{4.2.159}$$

- For all $R \in V'(Kl)$, $(n, \beta) \in \alpha^*$ such that $R^0 : A'^0 \rightrightarrows A'^0$ the statement

$$\exists B \in A'^0 . R^0 : B \rightrightarrows B \wedge B \text{ is } \beta^0 \text{ Mahlo} \tag{4.2.160}$$

For the realizability proof proper, let $R \in V'(Kl)$, $(x_2, \beta) \in \alpha^*$ and let

$$e \Vdash_{\text{rt}} \forall a \in A' \exists b \in A'.(a,b) \in R \tag{4.2.161}$$

Then in particular for each $a \in A = A'^0$ there is a $b \in A$ such that $(a,b) \in R^0$.[18] Also, for each $(k,x) \in A^*$ there is a $(l(ek), y) \in A*$ with $r(ek) \Vdash_{\text{rt}} (a,b) \in R$. This can be made into a total relation $A \rightrightarrows A$ by using the function $H : A \twoheadrightarrow A^*$, where H is defined recursively as

$$\begin{aligned} &H(a) := \\ &\quad (\{x | \exists y.a = (x,y)\}, \bigcup_y \{(n,z) \in y \cap (\omega \times V'(Kl)) | \exists x.a = (x,y) \wedge z^0 \in x\}) \end{aligned} \tag{4.2.163}$$

Let $B \in A$ be a β^0-Mahlo set reflecting the totality of both relations[19], i.e. $R^0 : B \rightrightarrows B$ and for each $(k,x) \in B \cap V'(Kl)$ there is a $(l(ek), y) \in B \cap V'(Kl)$ with $r(ek) \Vdash_{\text{rt}} (a,b) \in R$.

By induction hypothesis it is realized that

$$B' := (B, \{k\} \times B \cap V'(Kl)) \tag{4.2.164}$$

is β-Mahlo and B' is an element of A, so $(k, B') \in A'^*$. By its totality properties, it is also obviously the case that

$$e \Vdash_{\text{rt}} \forall a \in B' \exists b \in B'.(a,b) \in R \tag{4.2.165}$$

[18]This follows when unraveling the abbreviation $(a,b) \in R$ to the more exact

$$\exists x \in R \forall y.y \in x \leftrightarrow (\forall z(z \in y \leftrightarrow z = x) \vee \forall z(z \in y \leftrightarrow (z = x \vee z = y))) \tag{4.2.162}$$

This is preserved under the shift $\phi \mapsto \phi^0$.

[19]Obviously, two relations can be reflected at the same time — let $R_1, R_2 : A \rightrightarrows A$, then the relation $\{(x,y) | \exists z_1, z_2.y = (z_1, z_2) \wedge (x, z_1) \in R_1 \wedge (x, z_2) \in R_2\}$ is total on A and any inaccessible reflecting its totality also reflects the totality of R_1 and R_2.

□

This has a direct Corollary:

Corollary 4.47. *If A is Mahlo, and if*

$$A' := (A, \{k\} \times (V'(Kl) \cap A)) \tag{4.2.166}$$

Then

$$\Vdash_{rt} A' \text{ is Mahlo.} \tag{4.2.167}$$

Proof. Note that being Mahlo is equivalent to being 1-Mahlo. A version of 1 in $V'(Kl)$ is $(1, \{(k, \emptyset)\})$. So applying Lemma 4.46 to $\alpha := (1, \{(k, \emptyset)\})$ yields the desideratum. □

4.2.3 Realizing Elementary Embeddings

To deal with axioms concerning elementary embeddings, we amend the realizability definition analogously to before in Definition by extra clauses concerning j and M.

Definition 4.48. *Let Φ be a formula in the language $\in, \dot{in}, j, M$ with equality. Define the fomula $e \Vdash \Phi$ inductively as $e \in \omega$ and the appropriate clause from below:*

1. $e \Vdash_{rt} \bot$ *if* $\bot$
2. $e \Vdash_{rt} j^n(x)\dot{\in}j^m(y)$ *if* $(e, j^n(x)) \in j^m(y)\}$
3. $e \Vdash_{rt} j^n(x) \in j^m(y)$ *if* $\exists(le, z) \in j^m(y).re \Vdash_{rt} j^n(x) = z\}$
4. $e \Vdash_{rt} j^n(x) = j^m(y)$ *if* $\forall(f, z) \in j^n(x).lef \Vdash_{rt} z \in j^m(y)$ *and*

   $\forall(f, z) \in j^m(y) ref \Vdash_{rt} z \in j^n(x)$

   $e \Vdash_{rt} M(j^n(x))$ *if* $\exists y \in M.e \Vdash_{rt} j^n = y$

... *The other clauses are as in Definition 4.36.*

As usual, use $\Vdash_{rt} \Phi$ as shorthand for $\exists e.e \Vdash_{rt} \Phi$ and call such a formula Φ realized.

Then realizability with truth is absolute by the same reasons as standard realizability:

Lemma 4.49. *For any set a:*

$$a \in V'(Kl) \leftrightarrow j(a) \in V'(Kl) \tag{4.2.168}$$

And

$$a \in M \cap V'(Kl) \leftrightarrow a \in M \wedge (a \in V'(Kl))^M \tag{4.2.169}$$

Proof. As for Lemma 4.25 □

Then the elementarity of j extends to $\Vdash_{\mathrm{rt}}$ for bounded formulae:

Lemma 4.50. *For any bounded formula $\Phi(\overrightarrow{x})$ with all free variables displayed in the language $\in$ with equality (but without j and M):*

$$\forall \overrightarrow{a} \in V'(Kl), e \in S.e \Vdash_{rt} \Phi(\overrightarrow{a}) \leftrightarrow e \Vdash_{rt} \Phi(\overrightarrow{j(a)}) \leftrightarrow (e \Vdash_{rt} \Phi(\overrightarrow{j(a)}))^M \tag{4.2.170}$$

Proof. This goes through exactly as the proof of Lemma 4.26, noting that the truth of the statements about the labels is not affected by j. □

Just like for normal realizability, this implies directly:

Proposition 4.51. *The realizer $skk = \lambda x.x$ realizes that j is functional, i.e.*

$$skk \Vdash_{rt} \forall x, y.x = y \rightarrow j(x) = j(y) \tag{4.2.171}$$

The realizer e_r which realizes reflexivity also realizes that j maps V to M, i.e.

$$e_r \Vdash_{rt} \forall x.M(j(x)) \tag{4.2.172}$$

Also, M is realized to be transitive, i.e.

$$\Vdash_{rt} \forall x.M(x) \rightarrow \forall y \in x.M(y) \tag{4.2.173}$$

And with this it is possible to prove that j is realized to be an elementary embedding:

Lemma 4.52.

$$\Vdash_{rt} j : V \overset{\equiv}{\hookrightarrow} M \tag{4.2.174}$$

In particular for any formula $\Phi(\overrightarrow{x})$ with all free variables displayed and not containing M or j,

$$\Vdash_{rt} \forall \overrightarrow{a}.\Phi(\overrightarrow{a}) \leftrightarrow \Phi^M(\overrightarrow{j(a)}) \tag{4.2.175}$$

Proof. Just as for 4.28, noting that the truth of the statements about the labels is not affected by j. $\square$

As for usual realizability, we obtain:

Lemma 4.53. *1. Let K be an inaccessible set, $K \in j(K)$ and $\forall x \in K.j(x) = x$. Let*

$$K' := (K, \{k\} \times (V'(Kl) \cap K)) \tag{4.2.176}$$

Then

$$\Vdash_{rt} K' \textit{ regular } \wedge K' \in j(K') \wedge \forall x \in K'.j(x) = x \tag{4.2.177}$$

2. Let $V = M$. Then $\Vdash V = M$.

Proof. K' is realized to be regular by Lemma 4.44. As $j(K)$ is inaccessible and $K \in j(K)$, also $K' \in j(K)$ as $V'(Kl)$ has a recursive Δ_0 definition and intersections of such classes with a set are again sets, a fact which is reflected in inaccessible sets like $j(K)$.

As $K' \in j(K)$ and also $K' \in V'(Kl)$, we have $(k, K') \in j(K'^*) = \{k\} \times (V'(Kl) \cap K)$ and so for e_{r} a realizer for reflexivity,

$$pke_{\mathrm{r}} \Vdash_{\mathrm{rt}} K' \in j(K') \tag{4.2.178}$$

Also if $(e, x) \in K'$, then by K being regular, $(e, x) \in K$ and thus $j(x) = x$. So in this case,

$$e_{\mathrm{r}} \Vdash_{\mathrm{rt}} j(x) = x \tag{4.2.179}$$

So the universal statement is realized by $\lambda x.e_{\mathrm{r}}$.

For the second part, if $V = M$ then for all $x \in V'(Kl)$, any realizer e_{r} for reflexivity also realizes the equality of x to an element of M, i.e. itself, and thus $M(x)$. So in this case $e_{\mathrm{r}} \Vdash_{\mathrm{rt}} \forall x.M(x)$. □

4.2.4 Theories with Good Properties

Theorem 4.54. *The following theories enjoy all the properties from Definition 4.33, in particular the disjunction property and the numerical existence property:*

1. *CZF*
2. *CZF + wREA*
3. *CZF + REA*
4. *CZF + $\bigcup$REA*
5. *CZF + $*$REA*
6. *CZF + $*_2$REA*
7. *CZF + There is an inaccessible set.*
8. *CZF + IEA*
9. *CZF + There is a Mahlo set.*
10. *CZF + MEA*
11. *CZF + Axiom M*
12. *CZF + For all α, there is an α-inaccessible set.*
13. *CZF + For all α, a, there is an α-inaccessible set containing a.*
14. *CZF + For all α, there is an α-Mahlo set.*

15. *CZF + For all α, a, there is an α-Mahlo set containing a.*

16. *CZF + For all a, there is a I-inaccessible set I containing a.*

17. *CZF + For all a, there is a I-Mahlo set I containing a.*

18. *CZF + There is a measurable set.*

19. *CZF + There is a Reinhardt set.*

Proof. As all these statements are absolute for realizability with truth (the proof uses Lemmata 4.40, 4.39, 4.41, 4.42, 4.43, 4.44, 4.45, 4.46, 4.52, 4.53 and Corollary 4.21 in the same way as the proof of theorems 4.22, 4.15 and 4.32), the methods from [Rat05b] as recapped in Fact 4.38 yield the desired result. □

This theorem extends the results of [Rat05b], which focussed primarily on the theories CZF and CZF + REA and also notes that the same methods can also be applied to deal with theories containing axioms of largeness like CZF + IEA and CZF + MEA. The theorem also mirrors results of [FŠ84], who proved similar desirable properties for several extensions of IZF with large cardinals.

Chapter 5

How are Large Sets arranged in the Universe?

In this chapter, we will introduce a new tool and apply it to obtain results about the arrangement of large sets within the universe and in relation to each other. This tool will in some aspects take on the role the cumulative hierarchy plays in classical set theory.

Classical set theory benefits greatly from John von Neumann's cumulative hierarchy which he originally used to prove the relative consistency of the axiom of Foundation ([vN29]), but which has since become almost omnipresent in set-theoretic proofs and definitions. By iterating the powerset operation, one obtains a hierarchy of sets indexed by the ordinals and spanning the whole universe and thus providing the set-theorist with a Rosetta-stone-like link from the class V of all sets to the class $\mathrm{O_n}$ of ordinals, not only unlocking more applications for what is already a powerful tool (reasoning with ordinals), but also providing a sort of map to understand the structure of the set-theoretic universe (Figure 5.1).

However, in the constructive case, this close connection breaks down at least partially, as the absence of the Powerset axiom leads to CZF not proving that the von Neumann hierarchy acts as a hierarchy of sets anymore, thereby removing the most direct connection between $\mathrm{O_n}$ and V. Coupled with the fact that ordinals seem to be more difficult to

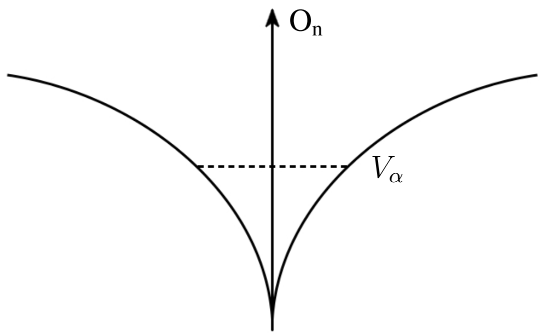

Figure 5.1: An Informal Map of the Universe (reproduced from [Zie14])

tame in the context of a constructive set theory, this may be one of the reasons why they have not been studied as extensively as in classical set theory and are generally regarded as less useful.

In this chapter however, we will propose an alternative hierarchy of sets that covers the whole set-theoretic universe, which will at least to some extent unlock the power of reasoning with ordinals for CZF. While the von Neumann hierarchy is based on iterating applications of the Powerset axiom, its analogue in CZF is less well suited for this: Fullness is not an explicit set axiom, i.e. it does not postulate the existence of a concrete definable set, but rather the existence of some set with certain properties, which many different sets might fulfill of which it is difficult to single one out ([Acz09]) and actually impossible in the general case by the results of [Swa14]. As such, it can not iterated quite so simply.

Instead, we will modify the powerclass operation by adding only some subsets by imposing a bound on the complexity of the subset relation as measured by the truth values $(\llbracket x \in a \rrbracket)_{x \in b}$ for $a \subseteq b$, which will slow down the increases of the hierarchy in each step to a constructively controllable speed. Without such a bound one obtains the original von Neumann hierarchy (which does not provably consist of sets in CZF) while with a constant bound one could never hope to cover more than a small fraction of V. Thus we will choose a bound that increases with the stage of the hierarchy and depends on what has already been covered, thus continuing to adhere to von Neumann's basic idea of constructing the universe from below.

This method has several applications, including determining how large sets are arranged in the universe.

Most of the results in this chapter have also been published by the author in [Zie14].

5.1 Ordinals and the Cumulative Hierarchy

In the context of CZF, it is very hard to obtain actual well-orderings in the classical sense. Indeed, if even a two element set could be ordered so that it enjoys the minimal element property, the excluded middle for all bounded formulae would immediately follow (consider for $a < b$ the inhabited class $\{b\} \cup \{a|\Phi\}$, which is a set if Φ is Δ_0 and whose minimal element determines whether Φ is true or false). Still, if one considers the essence of ordinals to be rooted in induction rather than minimal elements, this can be captured quite well by CZF.

Ordinals can then be generated inductively very simply: They should be transitively ordered and for each set A of ordinals, there should be a least ordinal larger than all the elements of A, which we will call next(A).

Definition 5.1. *The **ordinals** are the members of the smallest class O_n such that for all sets $A \subseteq O_n$, it follows that*

$$\mathit{next}(A) \in O_n \tag{5.1.1}$$

where

$$\mathit{next}(A) := A \cup \bigcup A \tag{5.1.2}$$

Note that the definition of next(A) as $A \cup \bigcup A$ is the minimal set which still guarantees transitivity. It contains exactly the elements we can be sure of that they must be smaller than the least element greater than anything in A: The elements of A itself, and their elements because of transitivity. Had we forgone the transitivity and just set next(A) as

A instead, we would arrive at a definition for the well founded sets (which, according to Set Induction, are all the sets).

We usually refer to ordinals with lower case Greek letters from the beginning of the alphabet. We also use these to refer to sets which might not be ordinals, but for which their rank, i.e. their place in the von Neumann hierarchy, is the most important feature. In the context of ordinals, the relations $\in$ and $\subseteq$ will also be written as $<$ and $\leq$.

Our definition of $\mathrm{O_n}$ is equivalent to the usual way ordinals are introduced in CZF (see e.g. [AR01]):

Lemma 5.2. *A set α is an ordinal if and only if α is a transitive set all of whose elements are also transitive.*

Proof. Let α be a transitive set all of whose elements are also transitive. We prove by set induction on α that $\alpha \in \mathrm{O_n}$. By induction hypothesis, $\alpha \subseteq \mathrm{O_n}$, so

$$\mathrm{O_n} \ni \mathrm{next}(\alpha) = \alpha \cup \bigcup \alpha = \alpha \tag{5.1.3}$$

As being transitive implies that $\bigcup \alpha \subseteq \alpha$.

For the other direction, we prove by set induction on α the statement:

$$\forall \alpha \in \mathrm{O_n} \forall a \in \alpha \forall b \in a. b \in \alpha \wedge \forall c \in b. c \in a \tag{5.1.4}$$

So let $\alpha \in \mathrm{O_n}$. As $\mathrm{O_n}$ is chosen as the minimal class closed under $x \mapsto \mathrm{next}(x)$, there is some $A \subseteq \mathrm{O_n}$ with $\alpha = \mathrm{next}(A)$. By the definition of next, the elements of A are also elements of α, so the induction hypothesis applies to them: A is a set all of whose elements are transitive and so are their elements. Let $b \in a \in \alpha$. Then by $\alpha = \mathrm{next}(A)$, either $a \in A$ or $a \in \bigcup A$. As 5.1.4 holds for all elements of A, in the first case $b \in a \in A$ implies $b \in A$ and in the second case $b \in a \in a' \in A$ implies $b \in a' \in A$. In either case, $b \in \bigcup A$ and thus $b \in \alpha$. Thus α is transitive.

To show that the elements of α are also transitive, let $c \in b \in a \in \alpha$. Again, either $a \in A$ in which case $a \in \mathrm{O_n}$ is transitive and $c \in a$, or $a \in \bigcup A$, in which case $a \in a' \in A$ for some $a' \in A \subseteq \mathrm{O_n}$. By definition of next, also $a' \in \alpha$, so the induction hypothesis implies that a' and all of its elements are transitive — including a. $\square$

Definition 5.3. *For a formula Φ, the class $0 \leq [\![\Phi]\!] \leq 1$ is defined as $\{0|\phi\}$ and if Φ is a Δ_0 formula it is the unique ordinal between 0 and 1 which is 1 exactly iff Φ holds. It is also called the **truth value** of Φ.*

The class $\mathcal{P}(1)$ is known as the class of truth values[1] *and each of its elements is a truth value of a formula*[2].

Definition 5.4. *The stages of the **von Neumann hierarchy** and the **rank function** are defined recursively as follows:*

1. *For any set α, set*
$$V_\alpha = \bigcup_{\beta \in \alpha} \{x | x \subseteq V_\beta\} \tag{5.1.5}$$
2. *Set*
$$V = \bigcup_{\beta \in O_n} V_\beta \tag{5.1.6}$$
3. *For any set a, set*
$$rk(a) = next(rk''a) = \bigcup_{x \in a} rk(x) + 1 \tag{5.1.7}$$

Note that the definition of V in part 2 of Definition 5.4 does not clash with our previous definition of $V := \{x|\top\}$ by parts 1 and 4 of the following remark.

The following are basic facts about the cumulative hierarchy:

Remark 5.5. 1. *Every set enters the hierarchy at some point, i.e.*
$$\forall x \exists \alpha . x \in V_\alpha \tag{5.1.8}$$
2. *For $\alpha > 1$, CZF does not prove that V_α is a set.*
3. *The hierarchy is cumulative in the sense that for $\alpha \leq \beta$ we have $V_\alpha \subseteq V_\beta$ and also in the sense that we have $\mathcal{P}(V_\alpha) \subseteq \mathcal{P}(V_\beta)$.*
4. *The hierarchy factors over rank, i.e. for all a we have $V_a = V_{rk(a)}$*

[1] although strictly speaking it only contains the truth values of bounded formulae

[2] as $\forall a \in \mathcal{P}(\Omega).\ a = [\![0 \in a]\!]$

Proof. 1. By set induction on a we prove

$$\forall a.a \subseteq V_a \tag{5.1.9}$$

Let this be true for all $b \in a$. For any $b \in a$ by definition of V_a, the statements $b \subseteq V_b$ and $b \in a$ imply that $b \in V_a$. Thus also $a \subseteq V_a$.

2. If $1 \in \alpha$, and $a \in \mathcal{P}(1)$, then $a \subseteq V_1 = 1$ and thus $a \in V_\alpha$. So $\mathcal{P}(1) = \{\vartheta \in V_\alpha | \vartheta \subseteq 1\}$ and if V_α were a set, $\mathcal{P}(1)$ would be a set as well, which is not provable in CZF ([AR01]).

3. Let $\alpha \leq \beta$, then for all $a \in V_\alpha$, there is some $\gamma \in \alpha$ with $a \subseteq V_\gamma$. Then $\gamma \in \beta$ and thus $a \in V_\beta$. For all $x \subseteq V_\alpha$ it follows that $x \subseteq V_\beta$ by transitivity of $\subseteq$.

4. This is proved by set induction over a. The statement $b \in V_a$ is equivalent to $\exists x \in a.b \subseteq V_b$ and by induction hypothesis, this is equivalent to $\exists x \in a.b \subseteq V_{\text{rk}(x)}$. As $\text{rk}(a) = \bigcup_{x \in a} \text{rk}(x) + 1$, this is implies to $\exists x \in \text{rk}(a).b \subseteq V_{\text{rk}(x)}$. It is actually equivalent, for let $x \in \text{rk}(a)$ and $b \subseteq V_{\text{rk}(x)}$, then either $x = \text{rk}(x')$ for some $x' \in a$, in which case $b \subseteq V_{\text{rk}(x')}, x' \in a$ or $x \in \text{rk}(x')$ for some $x' \in a$, in which case $b \subseteq V_{\text{rk}(x')}, x' \in a$ by 3. In any case

$$b \in V_a \leftrightarrow \exists x \in \text{rk}(a).b \subseteq V_{\text{rk}(x)} \tag{5.1.10}$$

The left part is equivalent to $\exists x \in \text{rk}(a).b \subseteq V_x$ by induction hypothesis, and this is just the definition of $b \in V_{\text{rk}(x)}$.

□

5.2 The Modified Hierarchy

As is the case with the classical von Neumann hierarchy, the modified version will also consist of stages that are reached by iterating a step operator.

Definition 5.6. *1. The **modified powerclass** operator $X \mapsto \mathcal{MP}(X)$ is defined on classes $X \subseteq V$ by*

$$\mathcal{MP}(X) := \{x \subseteq X | \forall y \in X. [\![y \in x]\!] \in X \cup \{0,1\}\} \tag{5.2.11}$$

*We will also call this the **step operator** of the modified hierarchy.*

2. We define a collection of classes $\hat{V}_\alpha$ indexed by sets $\alpha \in V$ by

$$\hat{V}_\alpha := \bigcup_{\beta \in \alpha} \mathcal{MP}(\hat{V}_b) \tag{5.2.12}$$

*We will use the parlance "a **enters the hierarchy** at α" for $a \in \mathcal{MP}(\hat{V}_\alpha)$, whether α is the smallest such set or not.*

3. Set

$$\hat{V} := \bigcup_{\beta \in O_n} \hat{V}_a \tag{5.2.13}$$

On the natural numbers, the modified von Neumann hierarchy exhibits the behaviour which is well known for the usual von Neumann hierarchy in a classical context, although in a constructive context the finite stages of the usual von Neumann hierarchy itself are decidedly less orderly — in fact, the finite ranks starting from V_2 contain so many elements that they can not even be proved to be sets.

Example 5.7. *(Behaviour at finite stages) For finite $n \in \omega$, $\hat{V}_n$ has $\underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{n-1}$ elements (where by convention $\underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{-1} := 0$), namely exactly the hereditarily finite sets of rank $< n$.*

Another important example is the very beginning of the modified hierarchy:

Example 5.8. *(Behaviour for $0 \leq \alpha \leq 1$) The map $\alpha \mapsto \hat{V}_\alpha$ when restricted to the class of all truth values $\mathcal{P}(1)$ is just the identity. In particular, for $[\![\Phi]\!] \in \alpha$, it follows that $[\![\Phi]\!] \in \hat{V}_\alpha$.*

Proof. A direct calculation yields $\hat{V}_0 = 0, \hat{V}_1 = \{0\}$. Let $\alpha \subseteq 1$. Then $\hat{V}_\alpha \subseteq \{0\}$ and this is inhabited iff α is. $\square$

The following are basic facts about the modified hierarchy:

Remark 5.9. *1. The step operator $\mathcal{MP}(X)$ maps sets to classes containing them, i.e.*

$$\forall a.a \in \mathcal{MP}(a) \tag{5.2.14}$$

In particular, if $\hat{V}_\alpha$ is a set, then $\hat{V}_\alpha \in \hat{V}_\beta$ for $\alpha \in \beta$.

2. *Each ordinal stage $\hat{V}_\alpha$ for $\alpha \in O_n$ of the hierarchy is transitive. In fact, for any transitive α, the class $\hat{V}_\alpha$ is transitive as well.*

3. *The hierarchy is cumulative in the sense that for $\alpha \leq \beta$ we have $\hat{V}_\alpha \subseteq \hat{V}_\beta$.*

4. *The hierarchy factors over rank, i.e. for all a we have $\hat{V}_a = \hat{V}_{rk(a)}$*

Proof. 1. $X \subseteq X$ and for each $x \in X$, the truth value $[\![x \in X]\!] = 1 \in X \cup \{0, 1\}$.

2. This is proved by set induction over α. Let $a \in b \in \hat{V}_\alpha$. Then for some $\beta \in \alpha$, $b \in \mathcal{MP}(\hat{V}_\beta)$. In particular $b \subseteq \hat{V}_\beta$ and thus $a \in \hat{V}_\beta$, so for some $\gamma \in \beta$ we have $a \in \mathcal{MP}(\hat{V}_\gamma)$. As α is transitive it follows that $\gamma \in \alpha$ and so $a \in \hat{V}_\alpha$. If α is an ordinal, transitivity is guaranteed by Lemma 5.2.

3. Let $\alpha \leq \beta$, then for all $a \in \hat{V}_\alpha$, there is some $\gamma \in \alpha$ with $a \in \mathcal{MP}(\hat{V}_\gamma)$. Then $\gamma \in \beta$ and thus $a \in \hat{V}_\beta$.

4. This is proved by set induction over a. Then $b \in V_a$ is equivalent to $\exists x \in a.b \subseteq V_b$ and by induction hypothesis, this is equivalent to $\exists x \in a.b \subseteq V_{\text{rk}(x)}$. As $\text{rk}(a) = \bigcup_{x \in a} \text{rk}(x) + 1$, this is implies to $\exists x \in \text{rk}(a).b \subseteq V_{\text{rk}(x)}$. It is actually equivalent, for let $x \in \text{rk}(a)$ and $b \subseteq V_{\text{rk}(x)}$, then either $x = \text{rk}(x')$ for some $x' \in a$, in which case $b \subseteq V_{\text{rk}(x')}, x' \in a$ or $x \in \text{rk}(x')$ for some $x' \in a$, in which case $b \subseteq V_{\text{rk}(x')}, x' \in a$ by 3. In any case

$$b \in V_a \leftrightarrow \exists x \in \text{rk}(a).b \subseteq V_{\text{rk}(x)} \tag{5.2.15}$$

The left part is equivalent to $\exists x \in \text{rk}(a).b \subseteq V_x$ by induction hypothesis, and this is just the definition of $b \in V_{\text{rk}(x)}$.

□

Comparing Remarks 5.5 and 5.9, it might be noted that the cumulativity property of the von Neumann hierarchy is stronger: If a set enters the hierarchy at one point, it does so again and again at every later point. This is not the case for the modified hierarchy, a complication which arises from the non-monotonicity of the modified powerclass operator:

Proposition 5.10. *The statement that for all a, α, if a enters the modified hierarchy at stage α, it also enters it at every stage $\beta \ni \alpha$, implies the weak exluded middle for Δ_0-formulae, i.e. the scheme $\neg\Phi \vee \neg\neg\Phi$ for all bounded formulae Φ.*

This is also true if α and β are restricted to ordinals.

Proof. Let Φ be a Δ_0 formula. Direct application of the definition of the modified hierarchy yields

$$\begin{array}{lcl} V_{\llbracket\Phi\rrbracket} & = & \llbracket\Phi\rrbracket \\ \mathcal{MP}(V_{\llbracket\Phi\rrbracket}) & = & \{\emptyset, \llbracket\Phi\rrbracket\} \\ V_{\llbracket\Phi\rrbracket+1} & = & \mathcal{MP}(V_{\llbracket\Phi\rrbracket}) \cup V_{\llbracket\Phi\rrbracket} = \{\emptyset, \llbracket\Phi\rrbracket\} \\ V_{(\llbracket\Phi\rrbracket+1)\cup 2} & = & \{0, \llbracket\Phi\rrbracket, 1\} \end{array}$$

So $\llbracket\Phi\rrbracket \in \mathcal{MP}(V_{\llbracket\Phi\rrbracket})$ and $\llbracket\Phi\rrbracket \in (\llbracket\Phi\rrbracket + 1) \cup 2$, but if we assume

$$\llbracket\Phi\rrbracket \in \mathcal{MP}(V_{(\llbracket\Phi\rrbracket+1)\cup 2}), \tag{5.2.16}$$

then this would mean that for each element of $V_{(\llbracket\Phi\rrbracket+1)\cup 2}$, the truth value of the statement that $\llbracket\Phi\rrbracket$ lies in that element is 0, 1 or in $V_{(\llbracket\Phi\rrbracket+1)\cup 2}$, which contains the truth values 0, 1 and $\llbracket\Phi\rrbracket$.

In particular, for $1 \in V_{(\llbracket\Phi\rrbracket+1)\cup 2}$, we have

$$\llbracket\llbracket\Phi\rrbracket \in 1\rrbracket \in \{0, 1, \llbracket\Phi\rrbracket\} \tag{5.2.17}$$

$\llbracket\llbracket\Phi\rrbracket \in 1\rrbracket$ is just $\llbracket\neg\Phi\rrbracket$, so we get

$$[\![\neg\Phi]\!] = 0 \vee [\![\neg\Phi]\!] = 1 \vee [\![\neg\Phi]\!] = [\![\Phi]\!] \tag{5.2.18}$$

So either Φ is not not true, or it is not true, as the third case can not occur. $\square$

This means that while $\alpha \mapsto \hat{V}_\alpha$ is monotone, $\alpha \mapsto \mathcal{MP}(\hat{V}_\alpha)$ can not proved to be monotone.

5.3 The Modified Hierarchy as a Hierarchy of Sets

The modified hierarchy exhibits the desired central properties which the von Neumann hierarchy enjoys in the classical case:

Theorem 5.11. *1. The stages of the hierarchy are sets, i.e.*

$$\forall\alpha\exists x.\hat{V}_\alpha = x \tag{5.3.19}$$

2. Every set enters the hierarchy at some ordinal point, i.e.

$$\forall x\exists\alpha \in O_n.x \in \hat{V}_\alpha \tag{5.3.20}$$

Proof. 1. We will show that the step operator $a \mapsto \mathcal{MP}(a)$ maps sets to sets, which implies that the whole hierarchy consists of sets by transfinite recursion over the index. So let a be a set and consider the following function $f : {}^{a}(a \cup 2) \to \mathcal{P}(a)$ defined by

$$f(g) := \{x \in a | g(x) = 1\} \tag{5.3.21}$$

By exponentiation and replacement, the range of this function is a set, and the range is a superset of $\mathcal{MP}(a)$, as every $X \in \mathcal{MP}(a)$ has the preimage

$$g : a \to a \cup 2, x \mapsto \{0 | x \in X\} \tag{5.3.22}$$

Thus $\mathcal{MP}(a)$ can by Δ_0-Separation be seen to be the set of those X elements of the range of f which fulfill the Δ_0-formula

$$\forall x \in a.[\![x \in X]\!] \in a \cup 2 \tag{5.3.23}$$

2. We proceed by $\in$-induction over x. By the induction hypothesis, take x to fulfill

$$\forall y \in x \exists \alpha . y \in \hat{V}_\alpha \tag{5.3.24}$$

By Strong Collection, we obtain α_0 with

$$x \subseteq \hat{V}_{\alpha_0} \tag{5.3.25}$$

While $x \in \mathcal{P}(\hat{V}_{\alpha_0})$, we would need $x \in \mathcal{MP}(\hat{V}_{\alpha_0})$ to be done.

So define an increasing sequence $(\alpha_i)_{1 \leq i \in \omega}$ with $\alpha_i \subseteq \alpha_{i+1}$ inductively by

$$\alpha_i = \alpha_{i-1} \cup \{[\![y \in x]\!] | y \in \hat{V}_{\alpha_{i-1}}\} \tag{5.3.26}$$

Finally set $\alpha := \bigcup_{i \in \omega} \alpha_i$. Now $x \in \mathcal{MP}(\hat{V}_\alpha)$, because it is a subset ($x \subseteq \hat{V}_{\alpha_0} \subseteq \hat{V}_\alpha$) and for any $y \in \hat{V}_\alpha$, we have $y \in \hat{V}_{\alpha_i}$ for some $i > 0$ and then $[\![y \in x]\!] \in \alpha_{i+1} \subseteq \alpha$, so by the Example 5.8, $[\![x \in a]\!] \in \hat{V}_\alpha$.

□

In fact, the second part of the proof achieves more than just proving the statement of the theorem: It shows that every set enters unboundedly often[3], and how to pinpoint the place where it enters.

Definition 5.12. *Let $\alpha \in O_n$ and $i \in \omega$, then define by recursion on α and i functions $\hat{rk}^i_\alpha : V \to O_n$ and $\hat{rk}_\alpha : V \to O_n$ by*

$$\hat{rk}^0_\alpha(a) := \alpha \cup \bigcup_{x \in a} \hat{rk}_\alpha(x) + 1 \tag{5.3.27}$$

$$\hat{rk}^{i+1}_\alpha(a) := \hat{rk}^i_\alpha(a) \cup \{[\![x \in a]\!] | x \in \hat{V}_{\hat{rk}^i_\alpha}(a)\} \tag{5.3.28}$$

$$\hat{rk}_\alpha(a) := \bigcup_{i \in \omega} \hat{rk}^i_\alpha(a) \tag{5.3.29}$$

[3] In fact, Example 7.6 further on will be that the class of ordinals where it enters is a club.

Finally define the ***modified rank*** *of a set a as*

$$\hat{rk}(a) := \hat{rk}_{\emptyset}(a) \tag{5.3.30}$$

These functions return locations where sets enter the hierarchy:

Lemma 5.13.

$$\forall a, \alpha . a \in \mathcal{MP}(\hat{V}_{\hat{rk}_{\alpha}(a)}) \tag{5.3.31}$$

Proof. The definitions are crafted to make explicit the proof of Theorem 5.11. □

So in analogy to the classical hierarchy we get:

Corollary 5.14.

For $\hat{rk}(a) \in \alpha$, we have $a \in \hat{V}_{\alpha}$.

Classically, the converse direction holds as well for the usual von Neumann hierarchy, i.e. if a is in some von Neumann stage, this stage is at least $\mathrm{rk}(a)$. This is not the case in CZF. In fact, not only does the rank function fail to pinpoint a least point where a set enters the hierarchy, such a least point fails to exist in the first place:

Proposition 5.15.

1. *The statement that for all sets a there is a least ordinal α such that $a \in V_{\alpha}$ implies the principle of excluded middle for Δ_0-formulae.*
2. *The statement that for all sets a there is a least ordinal α such that $a \in \hat{V}_{\alpha}$ implies the principle of excluded middle for Δ_0-formulae.*
3. *The statement that for all sets a there is a least ordinal α such that $a \in \mathcal{P}(V_{\alpha})$ implies the principle of excluded middle for Δ_0-formulae.*
4. *The statement that for all sets a there is a least ordinal α such that $a \in \mathcal{MP}(\hat{V}_{\alpha})$ implies the principle of excluded middle for Δ_0-formulae.*

Proof. Let Φ be a Δ_0-formula.

1. $0 \in V_1$ and $0 \in V_{[\![\Phi]\!]+1}$, so if there were a least α such that $0 \in V_\alpha$, this would have to fulfill $\alpha \subseteq 1 \cap ([\![\Phi]\!] + 1) = \{0|\Phi \vee \neg\Phi\} = [\![\Phi \vee \neg\Phi]\!]$. As V_α needs to be inhabited, α needs to be as well, so $[\![\Phi \vee \neg\Phi]\!] = 1$.

2. $0 \in \hat{V}_1$ and $0 \in \hat{V}_{[\![\Phi]\!]+1}$, so if there were a least α such that $0 \in \hat{V}_\alpha$, this would have to fulfill $\alpha \subseteq 1 \cap ([\![\Phi]\!] + 1) = \{0|\Phi \vee \neg\Phi\} = [\![\Phi \vee \neg\Phi]\!]$. As $\hat{V}_\alpha$ needs to be inhabited, α needs to be as well, so $[\![\Phi \vee \neg\Phi]\!] = 1$.

3. $\{[\![\Phi]\!]\} \in \mathcal{P}(V_2)$ and $\{[\![\Phi]\!]\} \in \mathcal{P}(V_{[\![\Phi]\!]+1})$, so if there were a least α such that $\{[\![\Phi]\!]\} \in \mathcal{P}(V_\alpha)$, this would have to fulfill $\alpha \subseteq 2 \cap [\![\Phi]\!] + 1 = \{0|\Phi \vee \neg\Phi\} \cup \{1|\Phi\}$. By monotonicity this implies

$$\{[\![\Phi]\!]\} \in \mathcal{P}(V_{\{0|\Phi\vee\neg\Phi\}\cup\{1|\Phi\}}) \tag{5.3.32}$$

   And thus

$$[\![\Phi]\!] \in V_{\{0|\Phi\vee\neg\Phi\}\cup\{1|\Phi\}} = \{0|\Phi \vee \neg\Phi\} \cup \{\vartheta \in \mathcal{P}(1)|\Phi\} \tag{5.3.33}$$

   So either $[\![\Phi]\!] = 0$ and $\Phi \vee \neg\Phi$, or Φ holds. In either case, $\Phi \vee \neg\Phi$.

4. $1 \in \mathcal{MP}(V_1)$ and $1 \in \mathcal{MP}(V_{([\![\Phi]\!]+1)\cup([\![\neg\Phi]\!]+1)}) = \mathcal{MP}(\{0, [\![\Phi]\!], [\![\neg\Phi]\!]\})$, so if there were a least α such that $1 \in \mathcal{MP}(V_\alpha)$, this would have to fulfill

$$\alpha \subseteq 1 \cap (([\![\Phi]\!] + 1) \cup ([\![\neg\Phi]\!] + 1)) = [\![\Phi \vee \neg\Phi]\!] \tag{5.3.34}$$

   As α needs to be inhabited for V_α to be inhabited (and after all V_α needs to contain $0 \in V_\alpha$ or 1 could not be a subset), $[\![\Phi \vee \neg\Phi]\!]$ needs to be 1, so $\Phi \vee \neg\Phi$.

□

5.4 Interaction with Large Sets

ZFC and CZF work very differently when it comes to the interaction of the usual von Neumann hierarchy with inaccessibles.

Proposition 5.16. *1. ZFC proves that a set A is inaccessible iff $A = V_\kappa$ for some regular strong limit cardinal*[4] *κ.*

2. CZF does not prove that any set A which is inaccessible is equal to some V_κ. In fact, this statement would imply the Powerset axiom and thus[5] *be far stronger than the existence of a mere inaccessible.*

Proof. 1. It is a well known fact that the transitive models of second order set theory ZF_2 are exactly the V_κ with κ an inaccessible cardinal (e.g. [Sha91]). Of course, ZFC proves that every transitive model of CZF_2 also models the excluded middle and is consequently also a model of the Foundation and Powerset axioms as well as full Separation and thus ZF_2.

2. Assume that for some inaccessible A it were true that $A = V_\kappa$. This means that $\{x \in A | x \subseteq 1\}$ would be a set and equal to $\mathcal{P}(1) \subseteq V_\kappa$. This would then imply the Powerset axiom (see [AR01]).

□

The interaction of the modified hierarchy with inaccessibles is however much more productive, mainly because all its relevant functions are reflected in inaccessible sets. For the usual von Neumann hierarchy, this is only the case with the rank function, not for the step operator $x \mapsto \mathcal{P}(x)$ or the stage enumerator $\alpha \mapsto V_\alpha$.

Lemma 5.17. *Let I be inaccessible. Then the following functions are reflected in it:*

1. $a \mapsto \hat{rk}(a) : I \to I$
2. $a, \alpha \mapsto \hat{rk}_\alpha(a) : I \times (I \cap O_n) \to I$
3. $a \mapsto \mathcal{MP}(a) : I \to I$
4. $a \mapsto \hat{V}_a : I \to I$

[4] i.e. what is usually known as an inaccessible cardinal.

[5] See e.g. [CR02] and [AR01].

Proof. 3. Let $a \in I$, then

$$\mathcal{MP}(a) = \{x \subseteq a | \forall y \in a.[\![y \in x]\!] \in X \cup \{0, 1\}\} \tag{5.4.35}$$

The set of functions ${}^{a}(a \cup 2)$ is included in every set that is full in $(a, a \cup 2)$ as a Δ_0-definable subset. As I models Δ_0-Separation and is closed under fullness, this implies ${}^{a}(a \cup 2) \in I$. As I models second order replacement, the following set is also in I:

$$S := \{\{y \in a | f(y) = 1\} | f \in {}^{a}(a \cup 2)\} \in I \tag{5.4.36}$$

For every $x \in \mathcal{MP}(a)$ the function $y \in a \mapsto [\![y \in x]\!]$ serves as f with $x = \{y \in a | f(y) = 1\}$, so $\mathcal{MP}(a) \subseteq S$. As the formula $x \in \mathcal{MP}(a)$ is a Δ_0 formula with all parameters in I, the set

$$\{x \in S | x \in \mathcal{MP}(a)\} = \mathcal{MP}(a) \in I \tag{5.4.37}$$

4. This is a direct set induction over a: Assume as induction hypothesis

$$\forall x \in a.\hat{V}_x \in I \tag{5.4.38}$$

Then by the previous part of the lemma and Union-Replacement,

$$V_a = \bigcup_{x \in a} \mathcal{MP}(V_x) \in I \tag{5.4.39}$$

2. Let I be $\bigcup$-regular and $\alpha \in I$. We prove by induction on a with side induction on i the statement:

$$\forall a \in I \forall i \in \omega.\hat{\mathrm{rk}}^{i}_{\alpha}(a) \in I \tag{5.4.40}$$

For $i = 0$, recall that

$$\hat{\mathrm{rk}}^{0}_{\alpha}(a) = \alpha \cup \bigcup_{x \in a}((\bigcup_{i \in \omega} \hat{rk}^{i}_{\alpha}(x)) + 1) \tag{5.4.41}$$

By the main induction hypothesis, for all $x \in a$ we have $\hat{rk}^i_\alpha(x) \in I$. So by Union-Replacement, this is again an element of $I \ni \hat{rk}^0_\alpha(a)$.

For the step case, recall that

$$\hat{\mathrm{rk}}^{i+1}_\alpha(a) = \hat{\mathrm{rk}}^i_\alpha(a) \cup \{[\![x \in a]\!] | x \in \hat{V}_{\hat{rk}^i_\alpha(a)}\} \tag{5.4.42}$$

By the induction hypothesis and point 4 of this lemma, this is again an element of $I \models CZF$.

1. This is a special case of point 2 for $\alpha := \emptyset$.

□

The consequence is that a multitude of stages coincide at the same point:

Theorem 5.18. *Let I be inaccessible, then the following sets are equal:*

$$\begin{aligned} I &= \hat{V}_I \\ &= \hat{V}_{O_n \cap I} \\ &= \hat{V}_{rk(I)} \\ &= \hat{V}_{\hat{rk}(I)} \end{aligned}$$

Proof. For the numbering of cases, refer to I, $\hat{V}_I$, $\hat{V}_{\mathrm{O_n} \cap I}$, $\hat{V}_{\mathrm{rk}(I)}$ and $\hat{V}_{\hat{\mathrm{rk}}(I)}$ as $1., 2., 3., 4.$ and $5.$ respectively.

1.⊆2. Let $a \in I$. By Lemma 5.17 this implies $\hat{\mathrm{rk}}(a) \in I$ and so by Lemma 5.13,

$$a \in \mathcal{MP}(\hat{V}_{\hat{rk}(a)}) \subseteq \bigcup_{x \in I} \mathcal{MP}(\hat{V}_x) = \hat{V}_I \tag{5.4.43}$$

1.⊇2. Let $x \in \hat{V}_I$. Recall that

$$\hat{V}_I = \bigcup_{a \in I} \mathcal{MP}(\hat{V}_a) \tag{5.4.44}$$

So for some $a \in I$, we have $x \in \mathcal{MP}((\hat{V}_a))$. As I is closed under the functions $a \mapsto \hat{V}_a$ and $x \mapsto \mathcal{MP}(x)$ by Lemma 5.17, we have $\forall a \in I . \mathcal{MP}(\hat{V}_a) \in I$.

2.=3. By cumulativity, $\hat{V}_I \supseteq \hat{V}_{I\cap \mathrm{O_n}}$. For the other direction, let $a \in I = \hat{V}_I$, then by 5.4.40, this implies $\hat{\mathrm{rk}}(a) \in I \cap \mathrm{O_n}$ and so by Lemma 5.13 and 5.4.40,

$$a \in \mathcal{MP}(\hat{V}_{\hat{rk(a)}}) \subseteq \bigcup_{x\in I} \mathcal{MP}(\hat{V}_x) = \hat{V}_I \tag{5.4.45}$$

3.=4. A quick set induction over α shows that

$$\alpha \in \mathrm{O_n} \to \mathrm{rk}(\alpha) = \alpha \tag{5.4.46}$$

For let $\alpha \in \mathrm{O_n}$, then $\mathrm{rk}(\alpha) = \bigcup_{\beta\in\alpha} \mathrm{rk}(\beta) + 1$ and by induction hypothesis this is $\bigcup_{\beta\in\alpha} \beta + 1$ and by transitivity of α, this is α. Thus $\mathrm{rk}''I \supseteq \mathrm{O_n} \cap I$. Conversely, $\mathrm{rk}''I \subseteq \mathrm{O_n} \cap I$ as $\mathrm{rk} : I \to I$ and $\mathrm{rk} : V \to \mathrm{O_n}$. So

$$\mathrm{rk}(I) = \bigcup_{a\in I} \mathrm{rk}(a) + 1 = rk''I \cup \bigcup_{a\in I} \mathrm{rk}(a) = rk''I = I \cap \mathrm{O_n} \tag{5.4.47}$$

The third equation holds by transitivity: If $a \in I$ and $\alpha \in \mathrm{rk}(a)$, then for some $b \in a \in I$, either $\alpha \in \mathrm{rk}(b) \in I$ or $\alpha = \mathrm{rk}(b) \in I$. In either case, $\alpha \in I$ and as α is an ordinal, $\alpha = \mathrm{rk}(\alpha) \in \mathrm{rk}''I$.

Thus cumulativity implies the desired equality.

1.=5. By Lemma 5.17, we have $\forall a \in I.\hat{\mathrm{rk}}(a) \in I$, so $\hat{\mathrm{rk}}(I) \subseteq I$ and thus by the previous theorem

$$\hat{V}_{\hat{\mathrm{rk}}(I)} \subseteq I \tag{5.4.48}$$

Conversely, let $a \in I$. By the definition of $\hat{\mathrm{rk}}$ this implies $\hat{\mathrm{rk}}(a) \in \hat{\mathrm{rk}}(I)$ and thus by Lemma 5.13

$$a \in \mathcal{MP}(V_{\hat{\mathrm{rk}}(a)}) \in \hat{V}_{\hat{\mathrm{rk}}(I)} \tag{5.4.49}$$

□

Armed with Theorem 5.18, it is possible to uncover a lot of new facts about the structure and configuration of large sets within the universe. In particular, it is possible to classify inaccessibles purely by the ordinals they contain, once more strengthening the connection between large sets and certain ordinals even for the constructive case.

Corollary 5.19. *Let I_1, I_2 be inaccessibles.*

1. *If $rk(I_1) \subseteq rk(I_2)$ or $O_n \cap I_1 \subseteq I_2$, then $I_1 \subseteq I_2$.*
2. *If $rk(I_1) = rk(I_2)$ or $O_n \cap I_1 = O_n \cap I_2$, then $I_1 = I_2$.*
3. *If $rk(I_1) \in rk(I_2)$ or $O_n \cap I_1 \in I_2$, then $I_1 \in I_2$.*

Proof. All of this follows directly from Theorem 5.18 with Lemma 5.17. □

In particular, the inaccessibles do not lie scattered arbitrarily throughout the universe, but are ordinally ordered:

Corollary 5.20. *Let I be the class of inaccessible sets.*

Then there is a subclass $\Gamma \subseteq O_n$ of ordinals and a bijective class function $\Delta : \Gamma \hookrightarrow\!\!\!\!\rightarrow I$ such that Δ is an isomorphism with respect to the structure generated by $\in, \subseteq$ and $=$.

5.5 Further Applications of the Methods Developed in this Chapter

While this chapter set out to answer the question of how large sets were arranged in the universe, it should be noted that the modified von Neumann hierarchy can also be fruitfully employed to research other topics. One of these are different characterizations of Mahlo sets which will be developed later in Setion 7.2. Another is to investigate clubs in a constructive context which will take place in Section 7.1. A third is found in Section 9.1 where we use the modified von Neumann hierarchy to extract more closure properties from critical points of elementary embeddings.

Not directly related to large cardinals is the following application, which uses the von Neumann hierarchy to break down the complexity of V to the complexity of $\mathrm{O_n}$ plus Δ_0 formulae, which is not only of philosophical interest, but may also be expressed in the form of a quantifier elimination theorem.

Theorem 5.21. *There is a definitional extension of CZF with the property that for every formula Φ there is a formula Φ' provably equivalent to Φ such that Φ' contains only quantifiers of the forms*

$$\forall x \in y, \exists x \in y, \forall \alpha \in O_n, \exists \alpha \in O_n \tag{5.5.50}$$

I.e. all quantifiers can be bounded by sets or the class of ordinals.

Proof. The extension of CZF is obtained by adding a unary function symbol $\hat{V}.$ and axioms defining it. As it is provable in CZF that the $\hat{V}_\alpha$s are unique sets, by [AR01] this extension is conservative.

Then Φ' can be defined by structural recursion as follows:

$$\begin{aligned}
\Phi' &= \Phi \text{ if } \Phi \text{ is atomic.} \\
(\Phi j \Psi)' &= \Phi' j \Psi' \text{ for any binary connective } j \\
(\forall x \Phi(x))' &= \forall \alpha \in \mathrm{O_n} \forall x \in \hat{V}_\alpha . \phi' \\
(\exists x \Phi(x))' &= \exists \alpha \in \mathrm{O_n} \exists x \in \hat{V}_\alpha . \phi'
\end{aligned}$$

□

While always pleasing to eliminate, unbounded quantification plays an especially recalcitrant role in CZF as it may not appear in the Δ_0-Separation scheme. In this vein, the theorem above allows us to conclude:

Corollary 5.22. *Full Separation is equivalent to the scheme that all class functions $\Gamma : O_n \to O_n$ on ordinals have an infimum $\beta \in O_n$ and bounded suprema $\gamma_\delta \in O_n$ for each $\delta \in O_n$, where*

$$\beta = \bigcap \{f(\beta) | \beta \in O_n\} \tag{5.5.51}$$

$$\gamma_\delta = \bigcup \{f(\beta) \in \delta | \beta \in O_n\} \tag{5.5.52}$$

Proof. Obviously Full Separation implies the existence of suprema and infima. Conversely, working in the definitional extension from above, we prove by structural recursion over Φ that

$$\forall a.\{x \in a|\Phi\} \text{ is a set.} \tag{5.5.53}$$

By the theorem above, we can assume w.l.o.g. that Φ does not contain completely unbounded quantification but only quantification of one of the forms

$$\forall x \in y, \exists x \in y, \forall \alpha \in \mathrm{O_n}, \exists \alpha \in \mathrm{O_n} \tag{5.5.54}$$

The atomic cases are immediate by Δ_0-Separation. For the cases $\wedge$, $\vee$ and $\rightarrow$, note that union and intersection of sets are sets as is $\{x \in a|x \in b \rightarrow x \in c\}$ by Δ_0-Separation.

For $\Phi = \forall y \in b.\Psi$ (potentially b may be a while y of course must be new) note that for all $y \in b$, the class $\{x \in a|\Phi(y)\}$ is a set, and so there is a function $f : b \rightarrow V$ realizing this relation. Then

$$\{x \in a|\forall y \in b.\Psi\} = \{x \in a|\forall y \in b \exists z \in f, b' \in tc(z).z = (y, b') \wedge a \in b'\} \tag{5.5.55}$$

And thus it is a set as well.

For $\Phi = \exists y \in b.\Psi$ (potentially b may be a while y of course must be new) note that for all $y \in b$, the class $\{x \in a|\Phi(y)\}$ is a set, and so there is a function $f : b \rightarrow V$ realizing this relation. Then

$$\{x \in a|\exists y \in b.\Psi\} = \{x \in a|\exists y \in b \exists z \in f, b' \in tc(z).z = (y, b') \wedge a \in b'\} \tag{5.5.56}$$

And thus it is a set as well.

For $\Phi = \forall \alpha \in \mathrm{O_n}.\Psi$ consider the class function $\Gamma : \mathrm{O_n} \rightarrow \mathrm{O_n}$ with $\Gamma(\alpha) := [\![\Psi(\alpha)]\!]$ and let a be a set. For each element of $x \in a$, the class $\{x|\forall\alpha\Psi(\alpha)\}$ is a set by the premise and Replacement, so by Union-Replacement, $\{x \in a|\forall\alpha\Psi(\alpha)\}$ is a set.

For $\Phi = \exists \alpha \in \mathrm{O_n}.\Psi$ consider the class function $\Gamma : \mathrm{O_n} \rightarrow \mathrm{O_n}$ with $\Gamma(\alpha) := [\![\Psi(\alpha)]\!]$ and let a be a set. For each element of $x \in a$, the class $\{x|\forall\alpha\Psi(\alpha)\}$ is a set by the premise (set $\delta := 1$) and Replacement, so by Union-Replacement, $\{x \in a|\forall\alpha\Psi(\alpha)\}$ is a set. $\square$

Chapter 6

How many Large Sets are there?

In the classical case, the relationship between the number of postulated inaccessibles and the strength of the theory is a simple one: The more inaccessibles, the higher the consistency strength. This can be expressed precisely as follows:

Remark 6.1. *Let $\Phi_1(x)$, $\Phi_2(x)$ be formulae describing a (possibly class-sized) cardinality, i.e.*

$$ZFC \vdash \forall x.\Phi_1(x) \rightarrow (x \in \mathit{O_n} \wedge \nexists f : x \twoheadrightarrow \{y|\Phi_1(y)\} \wedge \forall y.y \in x \rightarrow \Phi_1(y)) \quad (6.0.1)$$

$$ZFC \vdash \forall x.\Phi_2(x) \rightarrow (x \in \mathit{O_n} \wedge \nexists f : x \twoheadrightarrow \{y|\Phi_2(y)\} \wedge \forall y.y \in x \rightarrow \Phi_2(y)) \quad (6.0.2)$$

Let the first be provably larger than the second, i.e.

$$ZFC \vdash \exists y.y = \{x|\Phi_2(x)\} \wedge \Phi_1(y) \quad (6.0.3)$$

Let the second cardinality not be inflatable by von Neumann universes, i.e.

$$ZFC \vdash (V_\kappa \models ZFC \wedge x \in V_\kappa \wedge V_\kappa \models \Phi_2(x)) \rightarrow \Phi_2(x) \quad (6.0.4)$$

Consider the two theories T_1 and T_2, which consist of ZFC plus the claim that there is a class of inaccessible cardinals[1] *in 1:1 correspondence with the sets fulfilling $\Phi_1(x)$ respectively $\Phi_2(x)$.*

Then T_1 implies $Con(T_2)$.

[1] i.e. regular strong limit cardinals

Proof. Without loss of generality, we can assume

$$\{x|\Phi_1(x)\} = \{x|\Phi_2(x)\}^+ \tag{6.0.5}$$

for otherwise, define

$$\Phi_1'(x) :\leftrightarrow x \in \{x|\Phi_2(x)\}^+ \tag{6.0.6}$$

Then this also describes a cardinality provably larger than the extension of Φ_2 and the statement that there are extension of Φ_1 many inaccessible cardinals implies the statement that there are extension of Φ_1' many inaccessible cardinals.

Working in T_1, let I be a class of inaccessibles in 1:1 correspondence with $\{x|\Phi_1(x)\}$. For all ordinals α with $\Phi_1(\alpha)$, let $f(\alpha)$ be recursively defined as the least inaccessible in I bigger than all of the $f(\beta)$ for $\beta < \alpha$. For all these α there is such an ordinal by T_1. Define the ordinal class κ as

$$\kappa := f(\{\alpha|\Phi_2(\alpha)\} + 1) \tag{6.0.7}$$

As Φ_1 describes a cardinality bigger than Φ_2, it is still the case that $\Phi_1(\{\alpha|\Phi_2(\alpha)\} + 1)$, so that κ is well-defined and is an inaccessible cardinal. So V_κ is a model of ZFC, and by the condition on Φ_2, we have

$$\{x \in V_\kappa | V_\kappa \vDash \Phi_2(x)\} \subseteq \{x|\Phi_2(x)\} \tag{6.0.8}$$

So as there are extension of Φ_2 many inaccessibles in V_κ (and being inaccessible is absolute), V_κ is also a model of there being Φ_2 many inaccessibles. So T_2 has a model. □

Cardinalities fulfilling these conditions include all numerals, ω, ω_1, 2^ω, "class many" and most other usual cardinalities.

Despite being somewhat peculiar, the absoluteness condition in the remark above is necessary, as illustrated by the following example, where κ is inflatable by von Neumann universes:

Example 6.2. *It is possible to define a cardinality κ such that*

$$ZFC + \text{“There are at least } \kappa \text{ distinct inaccessibles”}$$
$$\equiv ZFC + \text{“There are at least } \kappa^+ \text{ distinct inaccessibles”}$$

Proof. Let

$$\kappa = \{x \in 0 | \nexists \mu.\mu \text{ inaccessible}\} \tag{6.0.9}$$

Then

$$\kappa = \begin{cases} 0 & \text{if there exists at least one inaccessible} \\ 1 & \text{if there exists no inaccessible} \end{cases} \tag{6.0.10}$$

and the two theories ZFC + “There are at least κ distinct inaccessibles” and ZFC + “There are at least κ^+ distinct inaccessibles” are the same, and both are equivalent to the existence of at least one inaccessible. □

Barring such pathological examples however (e.g. by restricting on absolute cardinalities), the classical case can be said to be quite simple: The consistency strength strictly increases with the number of inaccessibles postulated.

The constructive case however is vastly different. Here it turns out that the sheer number of inaccessibles alone is irrelevant to the consistency strength. The remainder of this chapter will be concerned with proving the following theorem:

Theorem 6.3. *When added to CZF, the following statements all lead to theories of equal consistency strength:*

1. *There is an inaccessible.*
2. *There are two different inaccessibles.*
3. *There are ω different inaccessibles.*
4. *There are class many inaccessibles in the weak sense that while there is at least one inaccessible, the inaccessibles do not form a set.*

5. *There are class many inaccessibles in the sense that while there is at least one inaccessible, there is no set that contains all inaccessibles.*

6. *There are class many inaccessibles in the strong sense that for each set there is an inaccessible which is not a member of that set.*

The following sections will be concerned with constructing models establishing these equiconsistencies. The step from 1 to 2 will be a modification of the formulae as classes construction found in [Rat06] and will yield two inaccessibles I_1, I_2 with $I_1 \subsetneq I_2$. The steps from 2 to 6 hinge on adding certain axioms (in particular Subcountability) to this configuration and then reasoning in set theory to obtain the desired inaccessibles.

6.1 A Realizability Model with Inaccessible pca

In this section, let I be an inaccessible. We will construct a realizability model similar to Rathjen's formulae-as-classes construction [Rat06], although the pca in question will be a set pca based on I instead of a class pca based on V. Nevertheless, most ideas concerning the pca and many naming conventions come directly from [Rat06]. This construction can be seen as (one version of) the initial object in the category of all pcas on I where certain basic set-theoretic functions are represented.

As in [Rat06], define a pca structure $(s, k, \circ)$ on I which represents functions for the principal combinators k and s, pairing p, l and r, successor s_N, predecessor p_N and decision by cases d_N on natural numbers, π and σ for set-theoretic dependent products and sums, $\overline{fa}$ for function application and $\overline{ab}$ for function abstraction. We add a means to represent images of functions.

Definition 6.4. *Choose constants $s, k, \pi, \sigma, \overline{im}, \overline{fa}, \overline{ab} \in \omega$. Then define $\circ : I \times I \to_p I$ as the smallest set such that for all $a, b, c, d \in I$:*

1. $k \circ a = (k, a)$

2. $(k, a) \circ b = a$

3. $s \circ a = (s, a)$

4. $(s, a) \circ b = (s, a, b)$

5. $(s, a, b) \circ c = (a \circ c) \circ (b \circ c)$ *in case* $(a \circ c) \circ (b \circ c)$ *is defined.*

6. $\pi \circ a = (\pi, a)$

7. $(\pi, a) \circ b = \{f : a \to I | \forall x \in a.f(x) \in b(x)\}$ *in case* $b : a \to I$.

8. $\sigma \circ a = (\sigma, a)$

9. $(\sigma, a) \circ b = \{(x, z) | z \in b(x)\}$ *in case* $b : a \to I$.

10. $\overline{fa} \circ a = (\overline{fa}, a)$

11. $(\overline{fa}, a) \circ b = c$ *in case* $\forall i \in I.(b, i) \in a \leftrightarrow i = c$.

12. $\overline{ab} \circ a = (\overline{ab}, a)$

13. $(\overline{ab}, a) \circ b = \{(x, y) | x \in b \wedge y \in I \wedge a \circ x = y\}$ *in case* $\forall x \in b \exists y \in I.a \circ b = y$.

14. $\overline{im} \circ a = (\overline{im}, a)$

15. $(\overline{im}, a) \circ b = a''b$ *in case* $\forall x \in b \exists! y \in I.(x, y) \in a$.

For $a \circ b$ *we will also write* ab.

Proof. This definition contains two claims: First that there is a smallest set $\circ$ and second that it is a partial function $\circ : I \times I \to_p I$. Both draw on the Inductive Definition Theorem from [AR01]: All stages of the inductive definition of $\circ$ are elements of I, so their union is a rk(I)-indexed union of elements of I and thus a set, and all stages are partial functions by induction, and thus so is their union. $\square$

Note that $a \circ b$ can only have a value if a is a tuple with a natural number as first element. By the clause regarding k and s, $(I, \circ, k, s)$ is a obviously a pca ([TVD88]) or an applicative topology when setting $\nabla := I$, $x \leq y :\leftrightarrow x = y$ and $x \lhd p :\leftrightarrow x \in p$, which amounts to the discrete topology over I. So it makes sense to talk about the realizability model $V(I)$ over this pca. By the usual method presented in [Rat03b] for regular sets and developed in chapter 4 for inaccessibles, the set

$$\bar{I} := I \times (I \cap V(I)) \tag{6.1.11}$$

can not be proved to be realizedly inaccessible directly, as the condition from Lemma 4.16 was that the whole pca is an element of I, while here it is only a subset. This does not however change things:

Theorem 6.5.

$$V(I) \Vdash \bar{I} \textit{ is inaccessible.} \tag{6.1.12}$$

Proof. All parts go through exactly as before, with two slight exceptions: Realizing that $\bar{I}$ is regular and that it models Subset Collection.

For the regularity, let $(x_0, a) \in \bar{I}$, $e \in I$ and $R \in V(I)$ such that

$$e \Vdash \forall x \in a \exists y \in I.(x, y) \in R \tag{6.1.13}$$

Then for all $(f, x) \in a$ there is a pair $(f, y) \in I$ such that $r(ef) \Vdash (x, y) \in R$. As $a \in I$, we can use regularity to collect these pairs into a set $b \in I$ with $b \subseteq I \times V(I)$ and

$$\forall (f, x) \in a \exists (f, y) \in b.r(ef) \Vdash (x, y) \in R \tag{6.1.14}$$

As well as

$$\forall (f, y) \in b \exists (f, x) \in a.r(ef) \Vdash (x, y) \in R \tag{6.1.15}$$

We claim that

$$p(\lambda x.px(r(ex)))(\lambda x.px(r(ex))) \Vdash \begin{cases} \forall x \in a \exists y \in b\ (x, y) \in R \\ \wedge\ \forall y \in b \exists x \in a\ (x, y) \in R \end{cases} \tag{6.1.16}$$

For let $(f, x) \in a$, then there is

$$(l(pf(r(ef))), y) = (f, y) \in b \tag{6.1.17}$$

with

$$r(pf(r(ef))) = r(ef) \Vdash (x, y) \in R \tag{6.1.18}$$

Conversely, let $(f, y) \in b$, then there is $(l(pf(r(ef))), x) = (f, x) \in a$ which fulfills

$$r(pf(r(ef))) = r(ef) \Vdash (x, y) \in R \tag{6.1.19}$$

As $(k, b) \in \bar{I}$, regularity is realized.

For Subset Collection, choose rather to realize that for each $a, b \in \bar{I}$ there is a $c \in \bar{I}$ which is full in mv(a, b), which is an equivalent condition. So let $(x_0, a), (x_1, b) \in \bar{I}$. Let $c' \in I$ be full in mv(a, b) and set

$$c := \{(k, \bar{r}) | r \in c'\} \in V(I) \tag{6.1.20}$$

where for $r \in c'$ we define

$$\bar{r} := \{(k, \overline{\text{pair}}(x, y)) | (x, y) \in R\} \in V(I) \tag{6.1.21}$$

with help of a function $\overline{\text{pair}} : I \times I \to I$ which internalizes pairing, i.e. for some $e_p \in I$

$$\forall x, y \in I. e_p \Vdash (x, y) = \overline{\text{pair}}(x, y)) \tag{6.1.22}$$

Then $(k, c) \in \bar{I}$ and it is uniformly realized that c is full in mv(a, b). For let $R \in V(I)$ with

$$e \Vdash \forall x \in a \exists y \in b.(x, y) \in R \tag{6.1.23}$$

Then for all $(f, x) \in a$ there is a pair $(f, y) \in I$ such that $r(ef) \Vdash (x, y) \in R$. By c' being full in mv(a, b), there is an $r \in c'$ such that

$$\forall x \in a \exists y \in b.(x, y) \in r \wedge (x, y) \in R \tag{6.1.24}$$

So it is (uniformly) realized $\Vdash \bar{r} \subseteq R$ and $\Vdash \bar{r} \in c$. Thus $c \in V(I) \cap I$ is realized to be full in a, b, which is what we needed to prove. $\square$

6.2 A second Inaccessible

This model contains also a second inaccessible based on I which we intend to define now:

Definition 6.6. *Define inductively a subset $\tilde{I}_0 \subseteq \bar{I}$:*

If $A \in \bar{I}$ such that for all $(a, b) \in A$, it holds that $b \in \tilde{I}_0$ and for suitable e, c, we have $a = pce$ and

$$b = \{z | \exists x \in c.e \circ x = z\} \tag{6.2.25}$$

Then also $A \in \tilde{I}_0$.

Define $\tilde{I} \in V(I), \tilde{I} \subseteq \bar{I}$ as follows:

If $a \in I, b \in \tilde{I}_0$ such that it holds that for suitable e, c, we have $a = pce$ and

$$b = \{z | \exists x \in c.e \circ x = z\} \tag{6.2.26}$$

Then $(a, b) \in \tilde{I}$.

This also describes an inaccessible set. To prove it, we need to use a member of the pca calculating an element of $\tilde{I}$ by its address in $\tilde{I}$:

Lemma 6.7. *There is a realizer $e_{\tilde{I}}$ such that for all $(a, b) \in \tilde{I}$, we have*

$$e_{\tilde{I}} \circ a = b \tag{6.2.27}$$

Proof. This can easily be pieced together from the basic functions represented in the pca. One possibility for $e_{\tilde{I}}$ is:

$$e_{\tilde{I}} := \lambda x.\overline{im}(\overline{ab}(rx)(lx))(lx) \tag{6.2.28}$$

□

We will use the constant $e_{\tilde{I}}$ for the realizer constructed in this proof for the rest of this chapter.

Theorem 6.8.

$$V(I) \Vdash \tilde{I} \textit{ is inaccessible.} \tag{6.2.29}$$

Proof. We show that $\tilde{I}$ is transitive, regular and models several set theoretic axioms:

1. Transitivity: If e_r is a realizer for reflexivity, then

$$\lambda xy.pye_r \Vdash \forall a \in \tilde{I} \forall x \in a.x \in \tilde{I} \tag{6.2.30}$$

   by construction of $\tilde{I}$.

2. Regularity: Let $R \in V(I)$, $(pa_0f, a) \in \tilde{I}$ an arbitrary element of $\tilde{I}$ and

$$e \Vdash \forall x \in a \exists y \in \tilde{I}.(x, y) \in R \tag{6.2.31}$$

   Let $b \in \tilde{I}_0$ be defined as

$$b := e_{\tilde{I}} \circ (pa_0(\lambda x.pxe_{\tilde{I}}(l(e(fx))))) \tag{6.2.32}$$

   And note that

$$(pa_0(\lambda x.pxe_{\tilde{I}}((e(fx))_0)), b) \in \tilde{I} \tag{6.2.33}$$

   Then we have

$$\lambda x.ex \Vdash \forall x \in a \exists y \in b.(x, y) \in R \tag{6.2.34}$$

   For let $(g, x) \in a$, then there is $(l(eg), y) \in I$ with $r(eg) \Vdash (x, y) \in R$. By definition of b, also $(l(eg), y) \in b$ and thus $ex \Vdash \exists y \in b.(x, y) \in R$. Also we have

$$\lambda x.ex \Vdash \forall x \in a \exists y \in b.(x, y) \in R \tag{6.2.35}$$

   For let $(g, y) \in b$, then by definition of b, there for some $g, x \in a$, we have $(l(eg), y) \in I$ and $r(rg) \Vdash (x, y) \in R$. Thus $ex \Vdash \exists y \in b.(x, y) \in R$.

   This proof demonstrates a useful motif which we will have occasion to reapply often: When it is more convenient to construct the realizer why some set should be in $\tilde{I}$ than writing down that set itself (in this case b), Lemma 6.7 can be used to obtain a complete element of $\tilde{I}_0$ just from the realizer for why it should be in $\tilde{I}$.

3. Pairing: Let $(pa_0f, a), (pb_0f, b) \in \tilde{I}$, then a version of the pair of a and b is represented by the set

$$\{(0, a), (1, b)\} \tag{6.2.36}$$

and with $h : 2 \twoheadrightarrow \{(0, a), (1, b)\}, h(0) := (0, a), h(1) := (1, b)$

$$(p2(\lambda x.\bar{f}ahx))), \{(0, a), (1, b)\}) \in \tilde{I} \tag{6.2.37}$$

Furthermore, their relationship is computable inside the pca.

4. Union: Let $(pa_0f, a) \in \tilde{I}$, then a version of the union of a is represented by the set

$$\{(pxy, c) | \exists b.(x, b) \in a \wedge (y, c) \in b\} \tag{6.2.38}$$

and this is a surjective image of

$$d := \bar{\sigma}(a_0(p_0(fa))) \tag{6.2.39}$$

courtesy of the function represented by

$$g := \lambda x.\bar{(}px_0x_1, \overline{pr_1}\overline{fa}(fx_0)x_1\bar{)} \tag{6.2.40}$$

where $\bar{(},\bar{)}$ is an computation for internal ordered pairs and $\overline{pr_1}$ for the internal projection on the first component possible by the constructions for pairing above. So $(d, g) \in \tilde{I}$ being computable in the pca witnesses Union.

5. Natural numbers: Starting with the empty set which is represented by $\emptyset \in \tilde{I}$, there is a canonical representation of every natural number n in $\tilde{I}$ by recursively repeating the constructions given above for pairing and union. Let $f : \omega \to \tilde{I}_0$ the function enumerating this. Then the following set represents a version of the natural numbers.

$$(p\omega(\lambda x.\bar{f}afx), e_{\tilde{I}}p\omega(\lambda x.\bar{f}afx)) \in \tilde{I} \tag{6.2.41}$$

6. Δ_0-Separation: Let $\Phi(x, \overrightarrow{a})$ be a Δ_0 formula with parameters in $\tilde{I}_0$. By the results of [Rat06] (in particular Lemma 3.7), the set of realizers for $\exists x \in b\Phi(x, \overrightarrow{a})$ is computable in the pca, so the set $(\{e | e \Vdash \exists x \in b\Phi(x, \overrightarrow{a}\}, \lambda x.lb(lx))$, which serves as the Separation set, is computably in $\tilde{I}$.

7. Exponentiation: Let $(pa_0f, a), (pb_0f, b) \in \tilde{I}$. Then any set $f \in \tilde{I}_0$ which is realized to be a function from a to b by some realizer e must send elements $(g, x) \in a$ to elements $(e'g, y) \in b$ with e' easily computable from e (and depending on the exact logical form in which we express $f : a \to b$), i.e. be a member of $\pi a_0 g$ where g is the constant function $\overline{ab}a_0(\lambda x.b_0)$. Thus $\pi a_0 g$ serves as a computable set to enumerate (a superset of the set of) all functions realized to be from a to b (and which can be computed from an element of $\pi a_0 g$ in a straightforward but tedious way).

8. Fullness: While this could also be proved directly, it conveniently follows from $\tilde{I}$ modelling Exponentiation and the Presentation Axiom (see Proposition 6.12 below) as those two statements together implying Fullness (see [AR01]).

□

This concludes the proof of theorem 6.3, $1 \to 2$, considering that

Remark 6.9.

$$\Vdash \bar{I} \neq \tilde{I} \tag{6.2.42}$$

In fact, we have the more useful

Lemma 6.10.

$$\Vdash \tilde{I} \subsetneq \bar{I} \tag{6.2.43}$$

Proof. As $\tilde{I} \subseteq \bar{I}$ holds in the background universe, the subset relation is also realized. If there were a realizer for the other direction, it would imply the existence of an e such that for all $(f, a) \in \bar{I}$ there is an $(ef, b) \in \tilde{I}$ with $\Vdash a = b$. In particular, for each $a \in \tilde{I}$, $(0, a) \in \bar{I}$ and so there is an $(e0, b) \in \tilde{I}$ with $\Vdash a = b$. This implies that every $a \in \tilde{I}$ is a surjective image of $e0$. This would apply in particular to a in the canonical copy of I contained in $\tilde{I}$ and is thus a direct contradiction to I being regular. □

Summarizing this part of the proof of Theorem 6.3, this amounts to a relativized model construction such that

Corollary 6.11.

$$\Vdash \textit{There are two different inaccessibles such that one is a proper subset of the other.} \tag{6.2.44}$$

$\tilde{I}$ and choice principles

While not strictly necessary for our goal to prove the consistency of larger numbers of inaccessible cardinals[2], the following propostition might be of general interest.

Recall that the Presentation Axiom PAx states that every set is the surjective image of a base, i.e. a set such that every total relations whose domain is this set extend a function defined on the same domain.

Proposition 6.12.

$$\Vdash (\tilde{I} \vDash PAx) \tag{6.2.45}$$

Proof. Let $(pa_0 f, a) \in \tilde{I}$. Recall that skk is an index for the identity function. We have

$$b := (pa_0(skk), a_0) \in \tilde{I} \tag{6.2.46}$$

And it is realized that $\tilde{I}$ models that a is a surjective image of b by virtue of the (unique) element of $\tilde{I}$ with first component

$$pa_0(\lambda x.\bar{(}x, fx\bar{)}) \tag{6.2.47}$$

Where $\bar{(},\bar{)}$ is an computation for ordered pairs possible by the portions on pairing in the proof above.

It is also realized that $\tilde{I}$ models its element b to be a base, as from

$$e \Vdash \forall x \in b \exists y \in \tilde{I}.(x, y) \in R \tag{6.2.48}$$

[2]While the proof of $\tilde{I}$ modelling Fullness as presented above does refer to $\tilde{I}$ modelling the Presentation Axiom, of course Fullness could also have been proved directly.

it follows that a function containing pairs of the form $(x, e_{\tilde{I}}(ex)_1)$ is included in R and this is computable in the pca.

Note that while b is realized to be modeled a base in $\tilde{I}$, it is not necessarily realized to be an actual base respective to the whole of $V(I)$. This bears some resemblance to realizability models proving $AC_{\omega,\omega}$, but not necessarily AC_ω itself. □

A further interesting feature of $\tilde{I}$ is the following:

Proposition 6.13.

$$\Vdash \tilde{I} \textit{ is } * \textit{-regular.} \tag{6.2.49}$$

Proof. Let e, C, D be such that

$$e \Vdash \forall x \in \tilde{I}.x \in C \to \exists y \in \tilde{I}.x \in C \wedge (x, y) \in D \tag{6.2.50}$$

and let $(pa_0 f, a) \in \tilde{I}$ such that $f \Vdash a \in C$. By virtue of the pca admitting recursion, find an index g such that $f \circ 0 = pa_0 f$ and for a representing natural numbers,

$$g \circ s_N a = (e(ga))_0 \tag{6.2.51}$$

Then the set

$$p(\overline{im} g\omega)(\lambda x.x), \overline{im} g\omega) \in \tilde{I} \tag{6.2.52}$$

is computable in the pca and as required by the $*$-property, as it is computably realized that a is an element and the set is closed under the total relation C. □

This section can be distilled to the following corollary.

Corollary 6.14. *$\tilde{I}$ models the Presentation Axiom and the axiom RDC of relativized dependent choices.*

Proof. Arguing internally, the $*$-property implies that the Relation Reflection Scheme RRS is modelled. PAx implies DC and DC plus RRS implies RDC. □

6.3 A third Inaccessible

It is interesting to note that $I \times \tilde{I}_0$ is also realized to be inaccessible. This is a uniformized set that relates to $\tilde{I}$ much like $\omega_0 := \{(k, \bar{n}) | n \in \omega\}$ relates to $\bar{\omega} = \{(n, \bar{n}) | n \in \omega\}$ (where $\bar{n}$ and $\bar{\omega}$ are internal representations of the ordinals n and ω respectively). In fact,

Proposition 6.15. *There is an $e \in I$ such that both*

$$e \Vdash \tilde{I} \textit{ is inaccessible} \tag{6.3.53}$$

And

$$e \Vdash I \times \tilde{I}_0 \textit{ is inaccessible} \tag{6.3.54}$$

Proof. This is just the realizer constructed in the proof of Theorem 6.8. Note that for $I \times \tilde{I}$, the realizers for $a \dot{\in} I \times \tilde{I}$ need not be so complicated, but complicated realizers work as well, and as l and r are always defined, so are the choices in the proof of Theorem 6.8. $\square$

Remark 6.16. *Of course, we have again that*

$$V(I) \Vdash I \subsetneq I \times \tilde{I}_0 \tag{6.3.55}$$

6.4 Another ω Inaccessibles

Consider the Cartesian product of ω many pcas such as from the proof of theorem 6.3, $1 \to 2$.

Definition 6.17. *On the set ${}^{\omega}I$ define a partial application $\circ_\omega : {}^{\omega}I \to_p {}^{\omega}I$ such that*

$$f \circ_\omega g \downarrow :\leftrightarrow \forall n \in \omega . f(n) \circ g(n) \downarrow \tag{6.4.56}$$

And if $f \circ_\omega g \downarrow$, then define

$$(f \circ_\omega g)(n) := f(n) \circ g(n) \tag{6.4.57}$$

Remark 6.18. *1. We will also sometimes write fg for $f \circ_\omega g$, which of course overloads the term somewhat, as it could be interpreted as $\circ : I \to_p I$ or $\circ_\omega : {}^\omega I \to_p {}^\omega I$ (note that ${}^\omega I \subseteq I$). We will take care not to cause confusion as to which of the two application functions we are talking about in any given context. It helps that the application from last section can only denote a value when the first element is a finite tupel with a natural number in first place, while this one can only denote a value if the first element is a function from ω to finite tupels such that the first component is always a natural number.*

2. As the cartesian products of pcas are again pcas, $({}^\omega I, \circ_\omega)$ describes a pca, or an applicative topology when setting $\nabla = {}^\omega I$, $a \le b :\leftrightarrow a = b$ and $a \lhd p :\leftrightarrow a \in p$.

We are interested in the model V(${}^\omega I$). Inside that model we will construct a set that behaves like $\tilde{I}$ in some components and like $I \times \tilde{I}_0$ in others.

Definition 6.19. *Define inductively a set $I'_0 \in V({}^\omega I)$:*

If $A \in I \cap \mathcal{P}({}^\omega I \times I)$ such that for all $(a, b) \in A$, it holds that $b \in I'_0$ and for all $n \in \omega$, there are $e, c \in I$, such that we have $a(n) = pce$ and

$$b = \{z | \exists x \in c.e \circ x = z\} \tag{6.4.58}$$

Then also $A \in I'_0$.

For $N \subseteq \omega$, define $I^N \in V({}^\omega I), I^N \subseteq I$ as follows:

If $a \in {}^\omega I, b \in I'_0$ such that it holds that for all $n \in N$, there are $e, c \in I$, such that we have $a(n) = pce$ and

$$b = \{z | \exists x \in c.e \circ x = z\} \tag{6.4.59}$$

Then $(a, b) \in I^N$.

Theorem 6.20. *Let $N \subseteq \omega$. Then it is uniformly realized that*

$$V({}^\omega I) \Vdash I^N \text{ is inaccessible} \tag{6.4.60}$$

Proof. This is completely analogous to the proof of Theorem 6.8. The only crucial observation is that the same realizer that realizes the inaccessibility of $\tilde{I}$ also realizes the inaccessibility of the components of I^N (by Proposition 6.15), so that there need not be any case distinction whether a component n is in N or not. □

It is now easy to find ω different inaccessibles in this model. One candidate would be the sequence $I^{\{0\}}, I^{\{1\}}, I^{\{2\}}, ...$:

Lemma 6.21. *Define $f \in V({}^{\omega}I)$ as*

$$f := \{\bar{(}\bar{n}, I^{\{n\}}\bar{)}|n \in \omega\} \tag{6.4.61}$$

Then

$$V({}^{\omega}I) \Vdash f : \omega \hookrightarrow \{I|I \text{ inaccessible}\} \tag{6.4.62}$$

Proof. By its definition, f is realized to be a function with domain $\bar{\omega}$. Its images are inaccessible by the above theorem. The injectivity follows because there can only be a realizer for $I^{\{n\}} = I^{\{m\}}$ if the numbers n and m are actually equal (just as there was no realizer for $\bar{I} = \tilde{I}$ and using that if two natural numbers cannot be different, then they are equal) and then the realizer for reflexivity realizes $n = m$. □

This also proves Theorem 6.3, $1 \to 3$.

6.5 A Proper Class of Inaccessibles

Recall the Subcountability axiom stating that

$$\forall a \exists f, b. b \subseteq \omega \wedge f : b \twoheadrightarrow a \tag{6.5.63}$$

As shown by Rathjen in [Rat02], adding the Subcountability scheme to CZF or CZF + REA does not increase the consistency strength of the theory. This is done by providing an interpretation of set theory into an appropriate variant of Martin-L"of type theory into

Kripke-Platek set theory, and that interpretation validates Subcountability. For the corresponding slightly stronger type theories and KP set theories ([GRT05], [Gib02]), this works exactly the same for CZF plus the existence of an inaccessible, so that Subcountability can also be added to the theory from theorem 6.3, part 1.

Thus for the rest of this section, work in the theory CZF plus the existence of one inaccessible plus the axiom of Subcountability.

There is no reason to suppose that Subcountability is absolute for realizability models, but a very easy consequence (e.g. proved in [Rat02]) is the negation of the Powerset axiom, and this is absolute for realizability models:

Lemma 6.22. *The negation of the Powerset axiom is absolute for realizability models. In other words, for any pca A, we have*

$$\nexists x.x = \mathcal{P}(1) \to V(A) \Vdash \nexists x.x = \mathcal{P}(1) \tag{6.5.64}$$

Proof. Suppose there is no powerset of 1 in the background universe, but there is an $a \in V(A)$ and an e such that

$$e \Vdash \forall x.x \subset 1 \to x \in a \tag{6.5.65}$$

Define a set $P \subseteq \mathcal{P}(1)$ by

$$P := \{\{0|x \text{ inhabited}\}|(f,x) \in a\} \tag{6.5.66}$$

then for each $\vartheta \in \mathcal{P}(1)$, we have

$$\lambda x.p00 \Vdash \{(0,\emptyset)|\vartheta \text{ inhabited}\} \subseteq 1 \tag{6.5.67}$$

So we have

$$e(\lambda x.p00) \Vdash \{(0,\emptyset)|\vartheta \text{ inhabited}\} \in a \tag{6.5.68}$$

Thus for some $(f,x) \in a$,

$$(e(\lambda x.p00)0)_1 \Vdash \{(0,\emptyset)|\vartheta \text{ inhabited}\} = x \tag{6.5.69}$$

It follows directly from the definition of realizability for equality that if two sets are realizedly equal, one is inhabited if and only if the other is. So

$$x \text{ inhabited} \leftrightarrow \vartheta \text{ inhabited} \tag{6.5.70}$$

This means that

$$P \ni \{0|x \text{ inhabited}\} = \{0|\vartheta \text{ inhabited}\} = \vartheta \tag{6.5.71}$$

Thus $P = \mathcal{P}(1)$, which is a contradiction. $\square$

Working in the realizability model $V(I)$ from before, the relevant consequence of lemma 6.22 is that there can be no powerset of 1. If there were a set A containing all inaccessibles, then in particular there would be a set

$$B := \{J \in A|\tilde{I} \subseteq J \subseteq \bar{I}\} \tag{6.5.72}$$

For each $\vartheta \subseteq 1$, there is thus another inaccessible

$$\tilde{I} \cup \{i|i \in \bar{I} \wedge \vartheta \text{ inhabited}\} \in B \tag{6.5.73}$$

Thus, the following equality holds

$$\{\{0|x = \bar{I}\}|x \in B\} = \mathcal{P}(1) \tag{6.5.74}$$

In particular $\mathcal{P}(1)$ would be a set. This is a contradiction to the negation of the Powerset axiom and thus concludes the proof of Theorem 6.3, parts $1 \to 4$ and $1 \to 5$.

6.6 Proper Classes and Inexhaustible Classes

One of the strongest sensible ways to express the classical concept of properly "class many" constructively is to require that for every set of objects, there needs to exist a new one not contained in that set. This demands more than just that the class not be a set, but that it is possible to find new members of that class which are different from any number of previously given. We will call this concept inexhaustible and a class with inexhaustibly many elements an inexhaustible class.

Definition 6.23. *Call a class Γ an **inexhaustible** class if $\forall A \subseteq \Gamma \exists a \in \Gamma . a \notin A$. Say that there are **inexhaustibly many** sets with a certain property if their class is inexhaustible.*

Example 6.24.

1. *There are inexhaustibly many sets, as for any set $a \in V$, there is a set distinct from each member of a (namely a itself, as implied by Set Induction). In fact, even without the axiom of Set Induction, the following argument from [AR01] can be read as a proof that there are inexhaustibly many sets: Associate to each set a of sets its Russel set $\{x \in a | x \notin x\} \notin a$. By the same token, Set Induction is not needed to show that the Russel class $\{x | x \notin x\}$ is an inexhaustible class.*

2. *There are inexhaustibly many ordinals, as for any set of ordinals $A \subseteq O_n$, the ordinal next$(A) = A \cup \bigcup A$ is different from each ordinal in A.*

3. *The axiom of Subcountability not only implies that there are weakly class many subsets of 1 (in the sense that their class does not form a set), but also that there are inexhaustibly many subsets of ω.*

   Proof. This is just a way of looking closely at Cantor's diagonal argument: Let $A \subseteq \mathcal{P}(\omega)$, $a \subseteq \omega$ and $f : a \twoheadrightarrow A$, then a new set would be

$$\{n \in \omega | n \in a \wedge n \notin f(n)\} \in \mathcal{P}(\omega) - A \qquad (6.6.75)$$

   $\square$

The last example indicates that while the concept of weakly class many can not distinguish between the powerclasses of different inhabited sets (as was e.g. proved in [AR01], the powerclass of an inhabited set is a proper class if and only if any other powerclass of an inhabited set is), this concept of inexhaustibly many can: While under Subcountability there are inexhaustibly many subsets of ω, there is no reason to suppose there to be inexhaustibly many subsets of 1.

Open Question 6.25. *Is it consistent with CZF that there are two inhabited sets A, B such that A has inexhaustibly many subsets, while B has not?*

For even a slight weakening of the concept of inexhaustibly many subsets, this fails:

Proposition 6.26. *If one inhabited set A has the property that for every set $B \in \mathcal{P}(A)$ there is a $B' \in \mathcal{P}(A)$ with $B \subsetneq B'$, then every inhabited set has this property.*

Proof. Assume A has the property from the proposition. We only need to show that $\mathcal{P}(1)$ has this property, then it directly follows for all inhabited sets. To see that, let $B \in \mathcal{P}(1)$ be a set. Consider

$$C := \{X \in \mathcal{P}(A) | \forall a \in A. [\![a \in C]\!] \in B\} \tag{6.6.76}$$

By Exponentiation, this is a set. Thus by assumption, there is a D with $C \subsetneq D \subseteq \mathcal{P}(A)$. Let

$$E := \{[\![a \in d]\!] | a \in A, d \in D\} \tag{6.6.77}$$

Then $B \subsetneq E \subseteq \mathcal{P}(1)$, as $B = E$ would imply $C = D$. $\square$

6.7 An Inexhaustible Class of Inaccessibles

While Subcountability is not absolute for $V(I)$ or $V({}^{\omega}I)$, one of its consequences is:

Proposition 6.27. *The statement that ω has inexhaustibly many subsets is absolute for realizability models. In other words, for any pca S, if ω has inexhaustibly many subsets, then*

$$V(S) \Vdash \omega \textit{ has inexhaustibly many subsets.} \tag{6.7.78}$$

Proof. Let $A \in V(S)$ and $e \Vdash A \subseteq \mathcal{P}(\omega)$, i.e. for all $(f, a) \in A$ and $(g, x) \in a$ there is a natural number n such that $l(efg) = \underline{n}$ and $r(efg) \Vdash x = \bar{n}$ for canonical representations $n \mapsto \underline{n}$ of natural numbers in S and $n \mapsto \bar{n}$ of natural numbers in $V(S)$ (compare [Rat03b]). Define

$$A' := \{\{n \in \omega | \exists (g, x) \in a. \Vdash x = \bar{n}\} | (f, a) \in A\} \subseteq \mathcal{P}(\omega) \tag{6.7.79}$$

Then this is a collection of subsets of ω, so by Example 6.24, there is an $N' \subseteq \omega$ with $N' \notin A'$. Define

$$N := \{(\underline{n}, \bar{n} | n \in N')\} \subseteq \bar{\omega} = \{(\underline{n}, \bar{n}) | n \in \omega\} \tag{6.7.80}$$

Then $\lambda x.pxe_r$ realizes that this is a subset of the natural numbers, since it is a subset of $\bar{\omega}$, a canonical representation of the natural numbers in $V(S)$. And any realizer realizes that it is not an element of A, for assume otherwise, i.e. $\Vdash N \in A$. Then there is $(f, a) \in A$ with an $\Vdash a = N$. We want to show that

$$N' = \{n \in \omega | \exists (g, x) \in a. \Vdash x = \bar{n}\} \in A' \tag{6.7.81}$$

From $\Vdash a \subseteq N$ we conclude that for all $(g, x) \in a$ there is an $n \in N'$ with some $l(hg) = \underline{n}$ and $r(hg) \Vdash x = \bar{n}$. If also $\Vdash x = \bar{n'}$, then $\Vdash \bar{n} = \bar{n'}$ and so $n = n'$. Thus any element in $\{n \in \omega | \exists (g, x) \in a. \Vdash x = \bar{n}\}$ is also in N'.

On the other hand, let $n \in N'$. From $\Vdash N \subseteq a$ we conclude that there is an $(g, x) \in A$ with some $l(h\underline{n}) = g$ and $r(h\underline{n}) \Vdash x = \bar{n}$. Thus $n \in \{n \in \omega | \exists (g, x) \in a. \Vdash x = \bar{n}\}$.

This however is a contradiction to $N' \notin A'$. So no realizer can realize $N \in A$ and thus any realizer realizes $N \notin A$. $\square$

This means that assuming Subcountability in the background, the realizability model $V({}^{\omega}I)$ realizes that there are inexhaustibly many subsets of ω. We use this to find inexhaustibly many inaccessibles, basically relying that we have $\mathcal{P}(\omega)$ many different inaccessibles in $V({}^{\omega}I)$.

Lemma 6.28. *(Subcountability)*

$$V({}^{\omega}I) \Vdash \textit{There are inexhaustibly many inaccessibles.} \tag{6.7.82}$$

Proof. Let $A \in V(\,^{\omega}I)$ and $\Vdash \forall x \in A.x$ inaccessible. We need to find an $I \in V(\,^{\omega}I)$ such that some fixed realizer realizes $\Vdash I$ is inaccessible and $\Vdash I \notin A$.

While the class $\{B \subseteq \omega | \Vdash I^B \in A\}$ can not easily seen to be a set, the following class is a set by Δ_0-Separation (note that the class of realizers for a bounded formula is a set by [Rat03b]), Union and Replacement:

$$N := \{\{n \in \omega | \Vdash I^{\{n\}} \subseteq a\} | (f, a) \in A\} \subseteq \mathcal{P}(\omega) \tag{6.7.83}$$

By Subcountability, there is a $N' \subseteq \omega$ with $N' \notin N$. Then consider the set $I^{N'}$ which is realized to be an inaccessible. If there were a realizer $\Vdash I^{N'} \in A$, then for some $(f, a) \in A$, it is realizer $\Vdash I^{N'} = a$. But then for all $n \in N'$, it is realized that $\Vdash I^{\{n\}} \subseteq a$ by transitivity of equality and conversely it can only be realized that $\Vdash I^{\{n\}} \subseteq I^{N'}$ if actually $n \in N$. This is a contradiction to $N' \notin N$. □

This finalizes the proof of the theorem.

Remark 6.29. *(Stacking Inaccessibles) The results in this chapter produced inaccessibles $I_1 \subsetneq I_2$ and could be used to produce longer chains $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \ldots$ as well by iteration. However, in the absence of a principle like the trichotomy of ordinals, this does not produce a stack $I_1 \in I_2 \in \ldots$ of ordinals as in the classical case. This is no accident: It is obvious that claiming the existence of such a stack would increase the proof theoretic strength for purely Gödelian reasons alone, as I_2 would then be a model for CZF plus the existence of an inaccessible and its existence would prove the consistency of CZF plus an inaccessible and similarly I_3 would be a model of CZF plus a stack of inaccessibles of length 2 and so on.*

In other words, in CZF the consistency strength of inaccessibles does not stem from sheer number but from their arrangement in the universe and towards each other.

Chapter 7

A Closer Look into Mahlo Sets

This chapter aims at investigating several facets of Mahloness.

In Section 7.1 we will explore the constructive theory of clubs, a concept closely connected to Mahlo sets. This turns out to be not only a natural object of study in connection with large sets but also an area of application for axioms developed to substitute choice, namely RRS and $*$-regularity axioms. It will also lead to some new characterisations for Axiom M (and thus for Mahlo sets, as these are the inaccessibles modeling Axiom M).

In Section 7.2 we will combine our results on clubs with the modified von Neumann hierarchy developed in Chapter 5 to reconcile the classical and constructive definitions of Mahloness. It will turn out that even though the classical definition at first glance looks sterile without the excluded middle, it is actually equivalent to the one used by constructivists instead.

Finally in Section 7.3 we will use Mahloness to prove new and useful induction principles and find out exactly how much induction is contained in Axiom M (and thereby in Mahlo sets, as these are the inaccessibles modeling Axiom M). For this we will use induction principles related to Induction Recursion in type theory and a nondeterministic variation of this.

7.1 A Constructive Rendering of Clubs

7.1.1 Preliminaries and Definitions

The constructive version of Mahloness uses total relations, which are a very useful tool in the absence of the excluded middle. Classically however, clubs are used, which allows the even more powerful tool of ordinals to enter the arena. Ordinals being more unwieldy in a constructive setting seems to be the reason why clubs have been discarded as a medium to cast the concept of Mahloness in CZF. However, with care this can be remedied, although some attention needs to be paid to choose a rendering of the concept of club that retains its potency in a constructive context.

Definition 7.1. *Let $C \subseteq O_n$ be a class of ordinals. An ordinal α is called a* ***limit point*** *of C, written $\alpha \in Lim(C)$ if it is inhabited and*

$$\forall \beta \in \alpha \exists \gamma \in \alpha . \beta \in \gamma \in C \tag{7.1.1}$$

The class C is called ***closed*** *if it contains all its limit points, i.e., if $Lim(C) \subseteq C$.*

The class C is called ***unbounded*** *if*

$$\forall \beta \exists \gamma . \beta \in \gamma \in C \tag{7.1.2}$$

The class C is called a ***club*** *if it is closed and unbounded. It is called a* ***club in*** *A if A is a transitive class which models*

$$A \vDash C \cap A \text{ is a club.} \tag{7.1.3}$$

Usually, the concept of C being a club in A is only used if A is an ordinal or an inaccessible set. In any case, C is a club in A iff it is a club in $O_n \cap A$ and in particular for A being inaccessible, the clubs in A are exactly the clubs in $rk(A)$. Generally, the following is a nice characterisation for which classes are clubs in some fixed ordinal.

Lemma 7.2. *Let $C \subseteq O_n$ be a class of ordinals and $\alpha \in O_n$. Then C is a club in α iff the following conditions are both met:*

1. *Lim*$(C) \cap \alpha \subseteq C$
2. $\alpha \in$ *Lim*(C)

Proof. This is just examining what $\alpha \vDash C \cap \alpha$ closed and $\alpha \vDash C \cap \alpha$ unbounded mean, respectively. The second part is direct, for the first one, the literal unwinding of the definition would be

$$\text{Lim}(C \cap \alpha) \cap \alpha \subseteq C \cap \alpha \tag{7.1.4}$$

So assume Equation 7.1.4. Let $\beta \in \text{Lim}(C) \cap \alpha$, so $\beta \in \alpha$ and for all $\gamma \in \beta$ there is a $\delta \in \beta$ with $\gamma \in \delta \in C$. By transitivity, also for all $\gamma \in \beta$ there is $\delta \in \beta$ with $\gamma \in \delta \in C \cap \alpha$, so $\beta \in \text{Lim}(C \cap \alpha)$ and also $\beta \in \alpha$, so 7.1.4 implies $\beta \in C \cap \alpha$, in particular $\beta \in C$.

Conversely, assume $\text{Lim}(C) \cap \alpha \subseteq C$, we want to show 7.1.4. So let $\beta \in \text{Lim}(C \cap \alpha) \cap \alpha$. For any $\gamma \in \beta$, there is $\delta \in \beta$ with $\gamma \in \delta \in C \cap \alpha$, so in particular $\beta \in \text{Lim}(C)$ and thus $\beta \in C$. So $\beta \in C \cap \alpha$. $\square$

Limit points can also be characterised differently:

Lemma 7.3. *Let $C \subseteq O_n$, $\alpha \in O_n$. Then the following are equivalent:*

1. $\alpha \in$ *Lim*(C)
2. α *is inhabited and* $\alpha = \bigcup(\alpha \cap C)$

Proof. Let $\alpha \in \text{Lim}(C)$ (in particular α is inhabited) and $\beta \in \alpha$. Then for some $\gamma \in \alpha \cap C$, we have $\beta \in \gamma$. So $\alpha \subseteq \bigcup(\alpha \cap C)$. On the other hand, if $\beta \in \bigcup(\alpha \cap C)$, then by transitivity $\beta \in \alpha$, so $\alpha = \bigcup(\alpha \cap C)$. This proves $1 \to 2$.

Let $\alpha = \bigcup(\alpha \cap C)$ be inhabited. Then for each $\gamma \in \alpha$, there is a $\gamma' \in \alpha \cap C$ with $\gamma \in \gamma'$. Thus $2 \to 1$. $\square$

The limit points of $\mathrm{O_n}$ are also called limit ordinals. By Lemma 7.3, these are just the inhabited fixed points of the $\bigcup$-function, i.e. those inhabited α which fulfill $\alpha = \bigcup \alpha$.

In a classical context, the limit ordinals are often defined as the nonzero ordinals closed under successors, invoking the powerful classical duality that each ordinal is either closed under successors (limit ordinals and 0) or a successor itself (successor ordinals). In a constructive context however, it can happen that a limit ordinal not only is not closed under successors, but actually does not contain any successors at all:

Remark 7.4. *The principle that each limit ordinal contains at least one successor is equivalent to the principle of the excluded middle for Δ_0-formulae.*

In particular, the principle that each limit ordinal contains at least one successor is equivalent to the principle that each limit ordinal is actually closed under successors.

Proof. The only relevant direction is that the principle of all limit ordinals containing a successor implies the principle of the excluded middle for Δ_0-formulae. So assume that all limit ordinals contain a successor.

Let Φ be a Δ_0-statement. Define a class function $s_\Phi : \mathrm{O_n} \to \mathrm{O_n}$ by setting

$$s_\Phi(\alpha) := \alpha \cup \{\alpha | \Phi\} \cup \{\alpha \cup \{\alpha | \Phi\}\} \tag{7.1.5}$$

This defines an ordinal, because all its elements are either elements of α (and thus subsets), or equal to α itself and Φ is true (and thus the element is a subset) or equal to $\alpha \cup \{\alpha|\Phi\}$ and this is a subset: All its elements are either of the form $\beta \in \alpha$ and are thus also in $s_\Phi(\alpha)$ or are equal to α and Φ is true. But if Φ is true, then $\alpha \in s_\Phi(\alpha)$.

The class function $s_\perp$ is just the ordinary successor function $\alpha \mapsto \alpha + 1$ while $s_\top$ is the double successor function $\alpha \mapsto \alpha + 2$.

Consider

$$\omega_\Phi := \bigcup_{n\in\omega} (s_\Phi)^n(0) \tag{7.1.6}$$

We want to show that ω_Φ is a limit ordinal. In order to do this, let $\alpha \in \omega_\Phi$, so $\alpha \in (s_\Phi)^n(0)$ for some n. Then

$$\alpha \in (s_\Phi)^n(0) \cup \{(s_\Phi)^n(0)|\Phi\} \in (s_\Phi)^{n+1}(0) \subseteq \omega_\Phi \tag{7.1.7}$$

Thus there is another element of ω_Φ which contains α.

Now we want to show $\Phi \vee \neg\Phi$. In order to do this, prove the following by induction over n:

$$\forall\alpha.\alpha + 1 \in (s_\Phi)^n(0) \to \Phi \vee \neg\Phi \tag{7.1.8}$$

Let $\alpha + 1 \in (s_\Phi)^n(0)$, then $n > 0$. So there are three cases: Either $\alpha + 1 \in (s_\Phi)^{n-1}(0)$, in which case we are done by the induction hypothesis. Or $\alpha + 1 = (s_\Phi)^{n-1}(0)$ and Φ, in which case $\Phi \vee \neg\Phi$ holds. The third possibility is

$$\alpha + 1 = (s_\Phi)^{n-1}(0) \cup \{(s_\Phi)^{n-1}(0)|\Phi\} \tag{7.1.9}$$

In this case, α is an element of the right hand side, so either $\alpha \in (s_\Phi)^{n-1}(0)$, which implies $\neg\Phi$ as otherwise there would be an element larger than α in $\alpha + 1$ (namely $(s_\Phi)^{n-1}(0)$), or otherwise $\alpha = (s_\Phi)^{n-1}(0)$ and Φ holds. In any case, $\Phi \vee \neg\Phi$ holds. This proves statement 7.1.8.

By assumption, there is some successor ordinal $\alpha + 1 \in \omega_\Phi$ and the definition of ω_Φ implies that it must be an element of some $(s_\Phi)^n(0)$, so by 7.1.8, we are done.

□

There is a plentiful multitude of clubs in the set theoretic universe:

Example 7.5. *1. O_n is a club. For all $\alpha \in O_n$, the set α is a club in α.*

2. *The limit ordinals form a club.*

3. *The clubs in ω are exactly the infinite sets of ordinals.*

4. *For any $\alpha \in O_n$ the principal club C_α is a club, where*

$$C_\alpha = \{\beta \in O_n | \alpha \in \beta\} \tag{7.1.10}$$

Proof. For the unboundedness, let $\beta \in \mathrm{O_n}$. Classically, the ordinal $(\beta \cup \alpha) + 1$ contains both β and α and is thus an element of C_α. This however lies on the trichotomy of ordinals and does not hold in CZF. However, the ordinal $(\beta + 1) \cup (\alpha + 1)$ contains both β and is in C_α.

If $\beta \in \mathrm{Lim}(C_\alpha)$, then there is some $\beta_0 \in \beta \cap C_\alpha$. Thus $\alpha \in \beta_0 \in \beta$, which implies $\beta \in C_\alpha$ and so C_α is closed. □

5. *For a set $S \subseteq O_n$, the filter F_S generated by S is a club, where*

$$F_S = \{\beta \in O_n | S \subseteq \beta\} \tag{7.1.11}$$

Proof. For the unboundedness, let $\beta \in \mathrm{O_n}$. Consider the ordinal

$$\gamma := (\bigcup_{\delta \in S} \delta + 1) \cup (\beta + 1) \tag{7.1.12}$$

Then $\beta \in \gamma$ and for all $\delta \in S$, we have $\delta \in \gamma$ and thus $\gamma \in F_S$. □

Another example illustrates that the chosen definition of club connects well to the modified von Neumann hierarchy:

Example 7.6. *For any set a, let class E_a be the class of ordinals where a enters the modified von Neumann hierarchy, i.e.*

$$E_a := \{\alpha | a \in \mathcal{MP}(\hat{V}_\alpha)\} \tag{7.1.13}$$

1. *E_a is a club.*

2. *For any inaccessible I, the set E_a is a club in I (or equivalently in rk(I)).*

Proof. 1. E_a is unbounded as for any $\alpha \in \mathrm{O_n}$, Lemma 5.13 implies that

$$a \in \mathcal{MP}(\hat{V}_{\hat{rk}_{\alpha+1}(a)}) \tag{7.1.14}$$

So $\hat{rk}_{\alpha+1}(a) \in E_a$. But Definition 5.12 directly implies that $\alpha + 1 \subseteq \hat{rk}_{\alpha+1}(a)$, so for arbitrary $\alpha \in \mathrm{O_n}$ there is a member of E_a containing α.

E_a is closed as for any $\alpha \in \mathrm{Lim}(E_a)$ and $X \in \hat{V}_\alpha$, there is a $\beta \in \alpha$ such that $X \in \mathcal{MP}(\hat{V}_\beta)$. Then as α is a limit point there is a $\gamma \in \alpha \cap E_a$ with $\beta \in \gamma$ so $a \in \mathcal{MP}(\hat{V}_\gamma)$. This can only be the case if $[\![a \in X]\!] \in \hat{V}_\gamma \subseteq \hat{V}_\alpha$ and $a \subseteq \hat{V}_\gamma \subseteq \hat{V}_\alpha$. Thus $a \in \mathcal{MP}(\hat{V}_\alpha)$ or in other words $\alpha \in E_a$.

2. We show that both conditions from Lemma 7.2 are met. Obviously $\mathrm{Lim}(C) \cap \mathrm{rk}(I) \subseteq \mathrm{Lim}(C) \subseteq C$ by the first part of the proof. So for the second condition, let $\beta \in \mathrm{rk}(I)$. Then by Lemma 5.17 also $\beta \in \hat{\mathrm{rk}}_{\beta+1}(a) \in I$ and by Lemma 5.13 $\hat{\mathrm{rk}}_{\beta+1}(a) \in E_a$, so $I \in \mathrm{Lim}(E_a)$.

□

7.1.2 Limits of Clubs

The results of this section usually either require full AC or at least require that ordinals admit choice, i.e. that in the theory ZF, any total relation $R \subseteq a \times \alpha$ has a function $f : a \to \alpha$ as a subset. This is a consequence of the minimal element principle which does not hold in CZF.

Instead, following e.g. [Acz08], we will use Aczel's principles of RRS and $*$-REA to eliminate the use of choice.

Proposition 7.7. *1. Let $C \subseteq O_n$ be unbounded (e.g. a club) and assume RRS. Then Lim(C) is a club.*

2. Let A be inaccessible, $*$-regular and $C \subseteq O_n$ be unbounded in A (e.g. a club in A). Then Lim(C) is a club in A.

Proof. 1. Let C be a club. Then $\mathrm{Lim}(C)$ is closed, for let

$$\alpha \in \mathrm{Lim}(\mathrm{Lim}(C)) \tag{7.1.15}$$

be a limit point of $\mathrm{Lim}(C)$, i.e. let α be inhabited and

$$\forall \beta \in \alpha \exists \gamma \in \alpha . \beta \in \gamma \in \mathrm{Lim}(C) \tag{7.1.16}$$

We want to show that $\alpha \in \mathrm{Lim}(C)$. So let $\beta \in \alpha$. Then there is a $\gamma \in \alpha \cap \mathrm{Lim}(C)$ with $\beta \in \gamma$. So by definition of $\mathrm{Lim}(C)$, there is a $\delta \in \gamma \cap C$ with $\beta \in \delta$. As by transitivity $\delta \in \alpha \cap C$, this proves that α is a limit point of C.

The main part is to show that $\mathrm{Lim}(C)$ is unbounded. For this, fix $\alpha_0 \in \mathrm{O_n}$. Define the class relation $R : \mathrm{O_n} \rightrightarrows \mathrm{O_n}$ by

$$(\alpha, \beta) \in R :\leftrightarrow \alpha \in \beta \wedge \beta \in C \tag{7.1.17}$$

As C is unbounded, this is total. So let by RRS $D \subset \mathrm{O_n}$ be a set with $\alpha_0 \in D$ and $R : D \rightrightarrows D$. Set

$$\gamma := \bigcup D \tag{7.1.18}$$

Then as R is total on D, there is an element $\beta \in D$ with $(\alpha, \beta) \in R$, so $\alpha \in \beta \in D$ and thus $\alpha \in \gamma = \bigcup D$.

We claim that $\gamma \in \mathrm{Lim}(C)$. To see this take an arbitrary $\delta \in \gamma$. Then there exists some $\delta' \in D$ with $\delta \in \delta'$. So by $R : D \rightrightarrows D$, there is a $\delta'' \in D$ with $\delta'' \in C$ and $\delta' \in \delta''$. Again by $R : D \rightrightarrows D$, there is a $\delta''' \in D$ with $\delta'' \in \delta'''$. So

$$\delta \in \delta' \in \delta'' \in \delta''' \;\wedge\; \delta, \delta', \delta'', \delta''' \in D \;\wedge \delta'' \in C \tag{7.1.19}$$

Thus

$$\delta \in \delta'' \in \gamma \;\wedge\; \delta'' \in C \tag{7.1.20}$$

And this is what was to show.

2. Analogous. All these calculations can be made inside a set modeling the theory used for them, i.e. CZF plus RRS.

□

The second part of this proposition can be improved in two different ways by tweaking the condition on α: First it will be shown that α does not need to fulfill the $*$-condition if it is Mahlo (a proof theoretically stronger concept, but one that does not directly imply $*$-regularity), then it will be shown that A does not need to fulfill inaccessibility but only regularity as long as it models RRS.

The proof will make use of the following lemma.

Lemma 7.8. *If A is $\bigcup$-regular and $2 \in A$, then rk : $A \to A$ and*

$$rk(A) = \bigcup_{a \in A} rk(a) \tag{7.1.21}$$

Proof. Use induction over a to show the statement

$$a \in A \to \mathrm{rk}(a) + 1 \in A \tag{7.1.22}$$

Let $a \in A$ and for all $b \in a$ we have $\mathrm{rk}(b) + 1 \in A$. Then by regularity there is $c \in A$ with $\forall b \in a.\mathrm{rk}(b) + 1 \in c$. So $\bigcup c = \mathrm{rk}(a) \in A$. As $2 \in A$, the regular set A is closed under pairing, so $\bigcup c + 1$ is also in A.

This shows that A is closed under rk.

To prove the claimed equality, let $\alpha \in \mathrm{rk}(I)$, i.e., let $\alpha \in \mathrm{rk}(a) + 1$ for some $a \in A$. Then $\mathrm{rk}(a) + 1 \in A$ and thus

$$\alpha \in \mathrm{rk}(a) + 1 = \mathrm{rk}(\mathrm{rk}(a) + 1) \subseteq \bigcup_{a \in A} \mathrm{rk}(a) \tag{7.1.23}$$

Conversely, let $\alpha \in \bigcup_{a \in A} \mathrm{rk}(a)$, i.e., let $\alpha \in \mathrm{rk}(a)$ for some $a \in A$, then also $\alpha \in \mathrm{rk}(a) + 1$ and thus $\alpha \in \mathrm{rk}(I)$. $\square$

Proposition 7.9. *Let M be Mahlo and $C \subseteq O_n$ be unbounded in M. Then Lim(C) is a club in M.*

Proof. Mahlo cardinals reflect all total relations, but as it was defined in Equation 7.1.17 of the proof of Proposition 7.7, R was not total on M but only on $\mathrm{rk}(M)$. However, there is an easy trick to extend it to a total relation by use of the rk-function:

To show that $\mathrm{Lim}(C)$ is unbounded in M, fix $\alpha \in \mathrm{O_n} \cap M$. Define a relation $R : M \rightrightarrows M$ by

$$(a,(b_0,b_1)) \in R :\leftrightarrow \mathrm{rk}(a) \in b_0 \wedge b_0 \in C \wedge b_1 = \alpha \tag{7.1.24}$$

As C is unbounded and $\mathrm{rk} : M \to M$, this is total. So let $D \in M$ be inaccessible with $R : D \rightrightarrows D$. Set

$$\gamma := \mathrm{rk}(D) \tag{7.1.25}$$

Then as R is total on D, there is an element of the form $(b,\alpha) \in D$ with $(0,(b,\alpha)) \in R$, so $\alpha \in D$ and

$$\alpha = \mathrm{rk}(\alpha) \in \mathrm{rk}(D) = \gamma \tag{7.1.26}$$

We claim that $\gamma \in \mathrm{Lim}(C)$. To see this take an arbitrary $\delta \in \gamma$. Then by Lemma 7.8 there exists some $d' \in D$ with $\delta \in \mathrm{rk}(d') =: \delta' \in D$ as D is closed under rk. So by $R : D \rightrightarrows D$, there is a $d'' \in D$ with $\mathrm{rk}(d'') =: \delta'' \in C$ and $\delta' \in \delta''$. Again by $R : D \rightrightarrows D$, there is a $d''' \in D$ with $\mathrm{rk}(d''') =: \delta''' \in D$ with $\delta'' \in \delta'''$. So

$$\delta \in \delta' \in \delta'' \in \delta''' \;\wedge\; \delta, \delta', \delta'', \delta''' \in D \;\wedge \delta'' \in C \tag{7.1.27}$$

Thus

$$\delta \in \delta'' \in \gamma \;\wedge\; \delta'' \in C \tag{7.1.28}$$

And this is what was to show.

Closedness follows just the same as before. □

Alternatively the limits of a club also form a club with a proof theoretically weaker assumption:

Proposition 7.10. *Let A be $*$-regular and $C \subseteq O_n$ be unbounded in A. Then Lim(C) is a club in A.*

Proof. Just as Proposition 7.7, noting that the constructions can all be carried out in a $*$-regular set. We need the $*$-property once to construct $D \in A$ and the $\bigcup$-property once to obtain $\gamma := \bigcup D$. □

7.1.3 Intersections of Clubs

It is a well known fact that in classical set theory, if $\beta \neq 0$ and $(C_\alpha)_{\alpha<\beta}$ are clubs, then so is $\bigcap_{\alpha\in\beta} C_\alpha$ (e.g. [Kan03]).[1] The closedness is easy to prove, and the unboundedness follows in a straightforward way as well: Define recursively a monotone function $f : \beta \cdot \omega \to \mathrm{O_n}$ with a fixed starting value $f(0)$ being fixed and $f(\beta \cdot n + \alpha) \in C_\alpha$. This can easily be done as all the C_α are unbounded, so there is an ordinal in C_α which is bigger than the range of the function defined so far. Then the supremum of the function is an element of the intersection $\bigcap_{\alpha\in\beta} C_\alpha$ that is bigger than $f(0)$.

This however made use of a choice principle when defining the function f, namely $AC(\mathrm{O_n}, \mathrm{O_n})$. From the perspective of ZF, this choice principle is perfectly valid as it follows from the classical fact that inhabited classes of ordinals have a least element. This is not the case constructively however, so a constructive theory of clubs that wants to admit intersections must either include this choice principle or find a way to circumvert its use.

It turns out that working with RRS (and RRS-regularity for clubs in sets) can replace this use of choice, just as for the situation with limits of clubs. This is somewhat unusual, as commonly this principle is used to replace (relative) dependent choice [Acz08], not ordinal choice.

Theorem 7.11. *(RRS) Let I be a set and for each $i \in I$, let $C_i \subseteq O_n$ be a club. Then*

$$C := \bigcap_{i\in I} C_i \tag{7.1.29}$$

is a club.

Proof. Set

$$C := \bigcap_{i\in I} C_i \tag{7.1.30}$$

[1] While this is the form the theorem is usually stated, it can be proved for the index set an arbitrary nonzero set instead of an ordinal in the same way, without needing additional choice principles.

To see closedness, let $\alpha \in \mathrm{Lim}(C)$. Then α is inhabited and for all $\beta \in \alpha$, there is a $\gamma \in \alpha$ with $\beta \in \gamma \in C$. As for any $i \in I$ we have $C \subseteq C_i$, it also holds that $\gamma \in C_i$ for each $i \in I$. Thus for each $i \in I$, it follows that $\alpha \in \mathrm{Lim}(C_i)$. As the C_i are closed, it follows that

$$\alpha \in \bigcap_{i \in I} \mathrm{Lim}(C_i) \subseteq \bigcap_{i \in I} C_i = C \tag{7.1.31}$$

To see unboundedness, let $\alpha_0 \in \mathrm{O_n}$. Define the class Γ as

$$\Gamma = \bigcup_{n \in \omega} \{r \subseteq (n \times I) \times \mathrm{O_n} | r : n \times I \rightrightarrows \mathrm{O_n}\} \tag{7.1.32}$$

Define a relation $R : \Gamma \rightrightarrows \Gamma$ by setting $(r, r') \in R$ iff for suitable n and n' the domains of r and r' are $n \times I$ and $n' \times I$ respectively, and the following all hold

1. $n' = n + 1$
2. $\forall m < n \forall i \in I \forall \alpha \in \mathrm{O_n}.(m, i, \alpha) \in r \rightarrow (m, i, \alpha) \in r'$
3. $\forall i \in I \forall \alpha \in \mathrm{O_n}.(n, i, \alpha) \in r' \rightarrow \alpha \in C_i$
4. $\forall m < n \forall i, j \in I \forall \alpha, \beta \in \mathrm{O_n}.(n, i, \alpha) \in r' \wedge (m, j, \beta) \in r \rightarrow \beta \in \alpha$

We claim that R is total on Γ. For let $r \in \Gamma$, i.e. for some $n \in \omega$ let

$$r : n \times I \rightrightarrows \mathrm{O_n} \tag{7.1.33}$$

Now set

$$\alpha_{sup} := \bigcup_{\exists i \in I.(m,i,\alpha) \in r} (\alpha + 1) \in \mathrm{O_n} \tag{7.1.34}$$

Note that

$$\forall i \in I, m < n, \alpha \in \mathrm{O_n}.(m, i, \alpha) \in r \to \alpha \in \alpha_{sup} \tag{7.1.35}$$

Now by unboundedness of the C_i, it is true that

$$\forall i \in I \exists y \exists \alpha . y = (n, i, \alpha) \wedge \alpha \in C_i \wedge \alpha_{sup} \in \alpha \tag{7.1.36}$$

By Strong Collection, collect enough such y in a set, i.e. find a set a such that

$$\forall i \in I \exists y \in a \exists \alpha . y = (n, i, \alpha) \wedge \alpha \in C_i \wedge \alpha_{sup} \in \alpha \tag{7.1.37}$$

and

$$\forall y \in a \exists i \in I \exists \alpha . y = (n, i, \alpha) \wedge \alpha \in C_i \wedge \alpha_{sup} \in \alpha \tag{7.1.38}$$

Then $r \cup y \in \Gamma$ as $r : n \times I \rightrightarrows \mathrm{O_n}$ and $y : \{n + 1\} \times I \rightrightarrows \mathrm{O_n}$. We claim that $r \cup y$ is a witness for totality, in other words:

$$(r, r \cup y) \in R \tag{7.1.39}$$

The first condition is clear as r is defined on one input. The second is immediate as $r' \cap (n \times I \times \mathrm{O_n}) = r$. The third follows from the second conjunct in statement 7.1.38 and the fourth from the third conjunct in 7.1.38, noting statement 7.1.35 about α_{sup}.

So by RRS, there is a set D with $R : D \rightrightarrows D$ and $\{((0, i), \alpha_0) | i \in I\} \in D$. Let D' be the thrid projection of D, i.e.

$$D' := \{\alpha \in \mathrm{O_n} | \exists r \in D \exists n \in \omega \exists i \in I.((n, i), \alpha) \in r\} \tag{7.1.40}$$

As D' is a set of ordinals, its union is again an ordinal.

$$\beta := \bigcup D' \in \mathrm{O_n} \tag{7.1.41}$$

We claim that for all $i \in I$, the ordinal β is a limit point of C_i. Obviously it is inhabited as $\alpha_0 \in \beta$.

Let $\gamma \in \beta$. By definition of β and D', this implies

$$\exists r \in D, n \in \omega, i \in I.((n,i),\gamma) \in r \tag{7.1.42}$$

Then for such r, n, i, find by totality of R some $r' \in D$ with $(r, r') \in R$. Then by definition of R,

$$\exists \delta \in \mathrm{O_n}.((n+1,i)\delta) \in r' \wedge \gamma \in \delta \wedge \delta \in C_i \tag{7.1.43}$$

As such a δ is then also an element of β, this yields

$$\exists \delta \in \beta \cap C_i.\gamma \in \delta \tag{7.1.44}$$

This proves β to be a limit point of C_i for all $i \in I$, so in particular $\beta \in C_i$ for all i and thus $\beta \in C$. So $\alpha_0 \in \beta \in C$ and β witness unboundedness of C.

This finishes the proof. □

7.1.4 An Ordinalless Approach

Just as a class $C \subseteq \mathrm{O_n}$ of ordinals is called a club if it is unbounded and closed in the class of ordinals, for a class $C \subseteq V$ of arbitrary sets similar properties can be considered.

Definition 7.12. *Let $C \subseteq V$ be a class. An transitive set A is called a* ***set limit point*** *of C, written $A \in$ SLim(C) if it is inhabited and*

$$\forall a \in A \exists b \in A.a \in b \wedge b \in C \tag{7.1.45}$$

The class C is called ***set closed*** *if it contains all its set limit points, i.e., if SLim$(C) \subseteq C$.*

The class C is called ***set unbounded*** *if*

$$\forall a \exists b. a \in b \wedge b \in C \tag{7.1.46}$$

The class C is called a ***set club*** *if it is set closed and set unbounded. It is called a* ***set club in*** *A if A is a transitive class which models the statement that $C \cap \alpha$ is a set club.*

Note that the set limit points of a class Γ are just the inhabited sets of $I(\Gamma)$ from Definition 2.13.

Remark 7.13. *This definition generalizes Definition 7.1 in the sense that for $C \subseteq$ On,*

1. *SLim(C) = Lim(C)*
2. *C is a club iff C is a set club in On.*
3. *C is a club in $\alpha \in$ On iff C is a set club in α.*

Proof.

1. Let $A \in \mathrm{SLim}(C)$. Then every element $a \in A$ is transitive, as $a \in b$ for some $b \in C$ and all $b \in C$ are ordinals. As A itself is transitive, it is an ordinal and thus $A \in \mathrm{Lim}(C)$. The converse is immediate from the definition.
2. When the quantifiers in the definition of set club are restricted to ordinals, the result is exactly the definition of a club.
3. Ditto.

□

Remark 7.14. *Extension axioms postulate that certain classes are set unbounded, e.g.*

1. *REA is equivalent to the statement that the class of regular sets is set unbounded.*
2. *IEA is equivalent to the statement that the class of inaccessible sets is set unbounded.*

3. *MEA is equivalent to the statement that the class of Mahlo sets is set unbounded.*

The main properties of clubs established in the previous section carry over to set clubs, in particular the following:

Theorem 7.15. *(RRS) Let I be a set and for each $i \in I$, let $C_i \subseteq O_n$ be a set club.*

Then $\bigcap_{i \in I} C_i$ is also a set club.

Proof. This is analogous to the proof of theorem 7.11. □

This has direct implications for the theory of large cardinals. Axiom M as mentioned before can be read as the statement that every set club has an inaccessible member, making use of the following way to translate between total relations and set clubs.

Proposition 7.16. *(RRS) The following two statements about a class R are equivalent:*

1. $R : V \rightrightarrows V$

2. *The following class is a set club*

$$\{x | R : x \rightrightarrows x\} \tag{7.1.47}$$

Proof. $1 \rightarrow 2$ Let $R : V \rightrightarrows V$ and set

$$C := \{x | R : x \rightrightarrows x\} \tag{7.1.48}$$

Then C is set closed, for let $A \in \text{SLim}(C)$ be transitive and $x \in A$. Then there is an $a \in A$ with $R : a \rightrightarrows a$ and $x \in a$. So $\exists y \in a.(x, y) \in R$. As A is transitive, $y \in A$ and thus $A \in C$.

To see that C is set unbounded, consider the relation $R' : V \rightrightarrows V$ with $(a, b) \in R'$ if $\forall x \in a \exists y \in b.(x, y) \in R \wedge a \in y$. This is total by Collection and thus RRS implies $\forall x \exists a \ni x.R : a \rightrightarrows a$, and $R : a \rightrightarrows a$ implies $a \in \text{SLim}(C)$.

$2 \to 1$ Take an arbitrary set a, then as $\{x|R : x \mapsto x\}$ is a set club, there is a b with $a \in b$ and $R : b \rightrightarrows b$. Thus there is a $c \in b$ with $(a, c) \in R$.

□

Using the now established facts about clubs, it can be seen that axiom M is equivalent to what on the face of it seems to be weaker, namely only requiring that every club has a regular member.

On the other hand, it can also be strengthened by showing that axiom M also implies that every club has a member that is not only inaccessible, but also for any α one that is α-inaccessible.

Theorem 7.17. *(RRS) The following schemes are equivalent:*

1. *Every set club has a regular member.*
2. *Every total class relation $R : V \rightrightarrows V$ is reflected in a regular set.*
3. *Every set club has a $\bigcup$-regular member.*
4. *Every total class relation $R : V \rightrightarrows V$ is reflected in a $\bigcup$-regular set.*
5. *Every set club has an inaccessible member.*
6. *Axiom M*
7. *For any $\alpha \in O_n$, every set club has an α-inaccessible member.*
8. *For any $\alpha \in O_n$, total class relation $R : V \rightrightarrows V$ is reflected in an α-inaccessible set.*

Proof. We only need to prove the odd statements to be equivalent, since every odd statement is equivalent to its successor by Proposition 7.16. The odd statements are ordered in (weakly) increasing logical strength, so only the implications from lower to higher numbered statements need to be proved.

$1 \to 3$ Assume 1 and let C be a set club. Define

$$C_{\bigcup} := \{x | \forall a \in x. \bigcup a \in x\} \tag{7.1.49}$$

This is a set club: $C_{\bigcup}$ is unbounded, as for every set a, we can define

$$a' := \{\underbrace{\bigcup ... \bigcup}_{n \text{ times}} x | n \in \omega, x \in a \cup \{a\}\} \tag{7.1.50}$$

Then $a \in a'$ and $a' \in C_{\bigcup}$.

The class $C_{\bigcup}$ is also set closed, for let $x \in \text{SLim}(C_{\bigcup})$ and $a \in x$. Then for some $y \in C_{\bigcup} \cap x$, $a \in y$. So $\bigcup a \in y$ and as x is transitive, also $\bigcup a \in x$. Thus $x \in C_{\bigcup}$.

So by Theorem 7.15, $C \cap C_{\bigcup}$ is a club and thus has a regular member. But as this regular set is also in $C_{\bigcup}$, it is actually $\bigcup$-regular.

$3 \to 5$ Assume 3 and let C be a set club. Define some other classes:

$$C_\omega := \{x | \omega \in x\} \tag{7.1.51}$$

$$C_\cap := \{x | \forall (a, b) \in x. a \cap b \in x\} \tag{7.1.52}$$

$$C_{\text{full}} := \{x | \forall (a, b) \in x. \exists c \in x. c \text{ full in } \text{mv}(a, b)\} \tag{7.1.53}$$

These are all set clubs:

(a) C_ω is the principal set club generated by ω (just as principal clubs are indeed clubs, so principal set clubs are set clubs by the same argument).

(b) $C_\cap$ is unbounded, as for every set a, we can define

$$a' := \{x_1 \cap ... \cap x_n | n \in \omega, x_1, ..., x_n \in a \cup \{a\}\} \tag{7.1.54}$$

Then $a \in a'$ and $a' \in C_\cap$.

The class $C_\cap$ is also set closed, for let $x \in \text{Lim}(C_\cap)$ and $(a, b) \in x$. Then for some $y \in C_2 \cap x$, $(a, b) \in y$. So $a \cap b \in y$ and as x is transitive, also $a \cap b \in x$. Thus $x \in C_\cap$

(c) C_{full} is unbounded, for consider the total relation $R : V \rightrightarrows V$ defined by

$$(x, y) \in R :\leftrightarrow \forall (a, b) \in x \exists c \in y.c \text{ full in mv}(a, b) \tag{7.1.55}$$

This is total by Fullness and Collection, so by RRS there are unbounded fixed points c with $R : c \rightrightarrows c$, and these are all in C_{full}.

The class C_{full} is also set closed, for let $x \in \text{SLim}(C_{\text{full}})$ and $(a, b) \in x$. Then for some $y \in C_{\text{full}} \cap x$, $(a, b) \in y$. So there is a set full in mv(a, b) in y and as x is transitive, this is also in x, so $x \in C_{\text{full}}$.

So by Theorem 7.15, $C \cap C_\omega \cap C_\cap \cap C_{\text{full}}$ is a club and thus has a $\bigcup$-regular member. But as this $\bigcup$-regular set is also in C_ω, it contains ω. As it is in $C_\cap$ and is a regular set containing $2 \in \omega$, for all a, b in the set, (a, b) is also in the set, so it models binary intersection. Similarly, as it is in C_{full}, it is also closed under Fullness and consequently, it is inaccessible.

$5 \to 7$ Consider the classes

$$C_\alpha := \{x | \forall a \in x \exists b \in x.a \in b \wedge b \text{ is } \alpha\text{-inaccessible}\} \tag{7.1.56}$$

The classes C_α are all closed for the same reasons as $C_{\bigcup}$, $C_\cap$ and C_{full} and we prove that they are also unbounded (and thus set clubs) by induction over α.

So let $\alpha \in \text{O}_\text{n}$ and C_β be a set club for all $\beta \in \alpha$ by induction hypothesis. Let a be a set. By Theorem 7.15, $\bigcap_{\beta \in \alpha} C_\alpha$ is a set club as well and so is its intersection with $\{x | a \in x\}$, the principal set club generated by a. So this intersection has an inaccessible element, which contains a and is closed under β-inaccessibles for all $\beta \in \alpha$ — thus it is α-inaccessible. By abstraction, every set is included in an α-inaccessible and thus by RRS, C_α is a club.

□

7.2 Classical and Constructive Mahloness

Aim of this section is to apply the modified von Neumann hierarchy from Chapter 5 to Mahlo sets and to obtain a new characterization of Mahloness. This will work towards bridging the gap between the usual classical definition for Mahloness (e.g. [Kan03]) and the definition established in a constructive context (e.g. [AR01]). Many of the results in this section have also been published by the author in [Zie14].

7.2.1 Characterising Mahlo Sets using Dependent Choice

Using total relations and reflections, the constructive rendering of the concept of Mahlo relies on concepts which retains their potency even in the absence of the law of the excluded middle. Classical mathematicians however profit greatly from the powerful tool of working with clubs and ordinals to arrive at a definition equal or very similar to the following:

Definition 7.18. *An inaccessible set M is called **classically Mahlo** if for each set $C \subseteq M \cap \mathbf{O_n}$ that is a club in M there is an inaccessible set whose rank is in C.*

*An inaccessible set M is called **classically α-Mahlo** if for each set $C \subseteq M \cap \mathbf{O_n}$ that is a club in M and each $\beta \in \alpha$ there is a classically β-Mahlo set whose rank is in C.*

Note that the classically β-Mahlo set whose rank is in C is automatically an element of M by the results of Chapter 5: As an inaccessible, this set (call it I) is equal to the stage of its modified rank, i.e. $I = \hat{V}_{\mathrm{rk}\hat{(}I)}$. But $\mathrm{rk}(I) \in C \subseteq M$ and as an inaccessible M is closed under the function $x \mapsto \hat{V}_x$.

In the absence of full Separation, it makes sense to question the restriction that C needs to be a set. Admitting classes into the definition would mean that the statement "M is classically 1-Mahlo" would be an infinite scheme of formulae, and the statement of "M is classically 2-Mahlo" would on the face of it require existential quantification over a concept only expressible in an infinite scheme of formulae — unless this scheme could be recognized to be equivalent to a single formula, which will be proved in Theorem 7.20. Thus only the proof of Theorem 7.20 actually shows that the following definition can be carried out in CZF.

Definition 7.19. *(DC) An inaccessible set M is called* ***very classically Mahlo*** *if for each class $\Gamma \subseteq O_n$ that is a club in M there is an inaccessible set whose rank is in Γ.*

An inaccessible set M is called ***very classically α-Mahlo*** *if for each class $\Gamma \subseteq O_n$ that is a club in M and each $\beta \in \alpha$ there is a classically β-Mahlo set whose rank is in Γ.*

The following characterisation reconciles the constructive and classical approach:

Theorem 7.20. *(DC) Let M and α be sets. The following are equivalent:*

1. *M is α-Mahlo.*
2. *M is classically α-Mahlo.*
3. *M is very classically α-Mahlo.*

Proof. We prove the equivalence $1 \leftrightarrow 2 \leftrightarrow 3$ by induction on α. So let β-Mahlo, classically β-Mahlo and very classically β-Mahlo be equivalent for all $\beta \in \alpha$.

$1 \to 3$: Let M be α-Mahlo, the class $C \subseteq \mathrm{O_n}$ be a club in M and $\beta \in \alpha$. We need to show that there is a very classically β-Mahlo set whose rank is in C.

Consider the relation $R \subseteq M \times M$ defined by

$$R := \{(a, b) \in M \times M | \mathrm{rk}(a) \in \mathrm{rk}(b) \in C\} \tag{7.2.57}$$

This is a total relation, as for each a, there is a $b \in \mathrm{O_n}$ with $\mathrm{rk}(a) \in b \in C$ because of the unboundedness of C. Then $(a, b) \in R$. The class R is not necessarily a set, but by Strong Collection, there is a subrelation $r : M \rightrightarrows M$ with $r \subseteq R$ such that r is a set.

By M being α-Mahlo, let $I \in M$ be β-Mahlo such that r is total on I. By induction hypothesis, I is also very classically β-Mahlo. It remains to show that

$\mathrm{rk}(I) \in C$ for which it suffices to show that $\mathrm{rk}(I) \in \mathrm{Lim}(C)$ by Lemma 7.2 as C is a club in M and $\mathrm{rk}(I) \in \mathrm{rk}(M)$.

So let $\gamma \in \mathrm{rk}(I)$, then $\gamma \in I$ and as r and thus R is total on I, there is some element $b \in I$ with $(\gamma, b) \in R$, which in particular implies $\gamma \in \mathrm{rk}(b)$ and $b \in C$. As $C \subseteq \mathrm{O_n}$, this implies $\mathrm{rk}(b) = b$ and thus $\gamma \in b$.

$3 \to 2$: This is just specializing to the case that $\Gamma \subseteq \alpha$ is a set.

$2 \to 1$: Let M be classically α-Mahlo. Let $\beta \in \alpha$ and $R \subseteq M \times M$ such that

$$\forall a \in M \exists b \in M.(a, b) \in R \tag{7.2.58}$$

Consider the following subset of the ordinals of M:

$$C := \{\alpha \in M \cap \mathrm{O_n} | \forall a \in \hat{V}_\alpha \exists b \in \hat{V}_\alpha.(a, b) \in R\} \tag{7.2.59}$$

We claim that C is a club in M.

To see that it is closed, let $x \in M$. Then $C \cap x \in M$ as C is defined by a Δ_0-formula (this would fail if C had been defined analogously using the von Neumann hierarchy instead of the modified hierarchy). We claim that

$$\forall a \in \hat{V}_{\bigcup(C\cap x)} \exists b \in \hat{V}_{\bigcup(C\cap x)}.(a, b) \in R \tag{7.2.60}$$

For let $a \in \hat{V}_{\bigcup(C\cap x)}$, in other words let $\alpha' \in C \cap x$, let $\beta' \in \alpha'$ and $a \in \mathcal{MP}(\hat{V}_\beta)$. As $a \in \hat{V}'_\alpha$ and $\alpha' \in C$, there is a $b \in \hat{V}'_\alpha$ with $(a, b) \in R$. This b is then element of some $\mathcal{MP}(\hat{V}_\gamma)$ for some $\gamma \in \alpha'$ and thus

$$b \in \hat{V}_{\bigcup(C\cap x)} \tag{7.2.61}$$

This demonstrates closedness, as for any $x \in \mathrm{Lim}(C)$, also $x = \bigcup C \cap x$ and then by the argument above $x \in C$.

To see that it is unbounded, we take an arbitrary $\gamma \in M$. In order to apply DC, define a relation $S \subseteq M \times M$ by

$$(a, b) \in S :\leftrightarrow \forall x \in \hat{V}_{\mathrm{rk}(a)} \exists y \in \hat{V}_{\mathrm{rk}(b)}.(x, y) \in R \tag{7.2.62}$$

We claim that this relation is total on M, i.e.

$$\forall a \in M \exists b \in M.(a, b) \in S \tag{7.2.63}$$

For let $a \in M$, then as R is total on M and $\hat{V}_{\mathrm{rk}(a)} \in M$ (another point in the proof where it is crucial to use the modified hierarchy instead of the von Neumann hierarchy, as $V_{\mathrm{rk}(a)} \notin M$), by regularity there exists a set $B \in M$ such that

$$\forall x \in \hat{V}_{\mathrm{rk}(a)} \exists b \in B.(x, b) \in R \tag{7.2.64}$$

Now

$$\beta' := \bigcup_{b \in B} \hat{\mathrm{rk}}(b) + 1 \tag{7.2.65}$$

is an element of M (for M is closed under $\hat{rk}$) and $(a, \beta') \in S$.

As $S \subseteq M \times M$ is total, by DC there is some $A \in M$ with $\gamma + 1 \in A$ and

$$\forall x \in A \exists y \in A.(x, y) \in S \tag{7.2.66}$$

Then by definition of S, the relation R is total on the sets belonging to universes with rank in rk$(\bigcup A)$. So rk$(\bigcup A) \in C$ by the definition of C and $\gamma \in \bigcup A$, which demonstrates unboundedness.

As C is therefore a club and M is classically α-Mahlo, C must have a classically β-Mahlo ordinal γ as element. But then $\hat{V}_\gamma \in M$ is a classically β-Mahlo set and by induction hypothesis also a constructively β-Mahlo one. By definition of C then $\hat{V}_\gamma$ is a β-Mahlo set such that R is total on it, which concludes the proof.

□

7.2.2 A Variation without Choice

Instead of relying on DC to ensure that total relations are reflected inside the Mahlo set, this can also be incorporated in the definition. Thus define the two related concepts:

Definition 7.21. *An inaccessible set M is called **classically** $+$**-Mahlo** if for each set $C \subseteq M \cap O_n$ that is a club in M there is an inaccessible set whose rank is in C and for every $R : M \rightrightarrows M$ and $a \in M$ there is an $a \in A \in M$ with $R : A \rightrightarrows A$.*

*An inaccessible set M is called **classically** α**-**$+$**-Mahlo** if for each $\beta \in \alpha$ the following holds:*

For every set $C \subseteq M \cap O_n$ that is a club in M and there is a classically β-Mahlo set whose rank is in C and for every $R : M \rightrightarrows M$ and $a \in M$ there is an $a \in A \in M$ with $R : A \rightrightarrows A$.

Lemma 7.22. *Let M be a set. If it is classically α-$+$-Mahlo, then it is classically α-Mahlo. If it is classically α-Mahlo and DC holds, then it is classically α-$+$-Mahlo.*

Proof. This is direct by induction over α. Note that DC implies that all regular sets are $*$-regular ([Acz08]) and thus also have the $+$-property. $\square$

A choice free version of Theorem 7.20 is the following theorem, which relates Mahlo sets to classically $+$-Mahlo sets.

Theorem 7.23. *Let M and α be sets. The following are equivalent:*

1. *M is α-Mahlo.*
2. *M is classically α-$+$-Mahlo.*

Proof. Both proofs proceed analogously to the proof of Theorem 7.20 and by induction on α.

$1 \to 2$ This is just like $1 \to 3$ from the proof of 7.20. Note that DC was not used in that part of the proof and that the $+$-property, i.e. that all total relations are reflected in some set, is implied directly by the Mahlo property that all relations are reflected in an inaccessible set (and both only need to be fulfilled for inhabited α).

$2 \to 1$ This is just like $2 \to 1$ from the proof of 7.20. Note that DC was only used to conclude that if $S : M \rightrightarrows M$ and $\gamma + 1 \in M$, then there is some $A \in M$ with $S : A \rightrightarrows A$ and $\gamma + 1 \in A$, and this is also implied by the $+$-property (and is only needed for inhabited α).

□

7.2.3 A Club Based Characterisation for Axiom M

Just as Mahlo sets could be reconciled with their classical characterisation via clubs, in the same vein a connection between axiom M and clubs can be formed which can be seen as the extension of Theorem 7.17 to the classically established characterisation with ordinals.

Theorem 7.24. *(RRS) The following are equivalent:*

1. *Axiom M, i.e. the scheme that every for every class $R : V \rightrightarrows V$ there is an $M \in V$ such that $R : M \rightrightarrows M$.*
2. *The scheme that every club $C \subseteq O_n$ has a member which is the rank of an inaccessible.*

Proof. Let Axiom M hold and C be a club. Consider the class relation $R \subseteq V \times V$ defined by

$$(x, y) \in R :\leftrightarrow \mathrm{rk}(x) \in \mathrm{rk}(y) \in C \tag{7.2.67}$$

Unboundedness of C implies that $R : V \rightrightarrows V$, so by Axiom M there is an inaccessible I with $R : I \rightrightarrows I$. For $\alpha \in \mathrm{rk}(I) \cap \mathrm{O_n}$ also $\alpha \in I$ and thus $\exists \beta \in I. \alpha \in \mathrm{rk}(\beta) \in C$. As also $\mathrm{rk}(\beta) \in \mathrm{rk}(I)$, this means that $\mathrm{rk}(I)$ is a limit point of $C \supseteq \mathrm{Lim}(C)$ and thus $\mathrm{rk}(I) \in C$.

Conversely, let every club $C \subseteq \mathrm{O_n}$ have a member which is the rank of an inaccessible and let the class R fulfill $R : V \rightrightarrows V$. Consider the class $C \subseteq \mathrm{O_n}$ defined by

$$\alpha \in C :\leftrightarrow \alpha \in \mathrm{O_n} \wedge R : \hat{V}_\alpha \rightrightarrows \hat{V}_\alpha \tag{7.2.68}$$

This is closed since $\alpha \in \mathrm{Lim}(C)$ implies that for all $b \in \hat{V}_\alpha$ there is $\beta \in \alpha$ with $b \in \mathcal{MP}(\hat{V}_\alpha)$. As α is a limit point, there is a $\gamma \in \alpha \cap C$ with $\beta \in \gamma$, so also $b \in \hat{V}_\gamma$. Thus by $\gamma \in C$ there is a $c \in \hat{V}_\gamma$ with $(b,c) \in R$ and as $\gamma \subseteq \alpha$ by transitivity, also $c \in \hat{V}_\alpha$. Thus $\alpha \in C$.

C is also unbounded. For this, consider the relation S defined by

$$(\alpha, \beta) \in S :\leftrightarrow R : \hat{V}_\alpha \rightrightarrows \hat{V}_\beta \tag{7.2.69}$$

Then $S : \mathrm{O_n} \rightrightarrows \mathrm{O_n}$ since for $\alpha \in \mathrm{O_n}$, the relaton R is total on $R : \hat{V}_\alpha \rightrightarrows V$ and $\hat{V}_\alpha$ is a set, so by Strong Collection there is a set b such that $R : \hat{V}_\alpha \rightrightarrows b$. Then $\beta := \bigcup_{x \in b}(\hat{rk}(x)+1)$ fulfills $(\alpha,\beta) \in S$. So for any $\alpha \in \mathrm{O_n}$, by RRS there is a set $A \subseteq \mathrm{O_n}$ such that $\alpha+1 \in A$ and $S : A \rightrightarrows A$. Set $\beta := \bigcup A$. We claim that $\alpha \in \beta \in C$. To see this, let $a \in \hat{V}_\beta$, so for some $\gamma \in A$ we have $a \in \hat{V}_\gamma$. By choice of A there is some $\gamma' \in A$ and $b \in \hat{V}_{\gamma'}$ such that $(a,b) \in R$. But then by monotonicity also $b \in \hat{V}_\beta$ and β is as desired.

As C is closed and unbounded, C is a club and must have an element $C \ni \alpha = \mathrm{rk}(I)$ for some inaccessible set I. Then as $\hat{V}_{\mathrm{rk}(I)} = I$, the definition of C implies

$$R : I \rightrightarrows I \tag{7.2.70}$$

□

7.3 How much Induction is contained in Mahloness?

One of the most direct applications for large sets is that they allow more complex inductions to take place in the universe. The classical example is REA which implies that bounded inductive definitions produce sets.

Definition 7.25. *Let $\Gamma : V \to V$ be a class function. This is called an inductive definition. A class Δ is called a* ***fixed point*** *of Γ if $\Gamma : \mathcal{P}(\Delta) \to \mathcal{P}(\Delta)$.*

An inductive definition is called ***local*** *if for all sets $A \in V$, the class $\Gamma''\mathcal{P}(A)$ is a set.*

*The **support** supp*(Γ) *of an inductive definition is defined as the class*

$$\mathit{supp}(\Gamma) := \{a | \exists b \in \Gamma(a)\} \tag{7.3.71}$$

*An inductive definition is called **bounded** if there is some set* b *such that*

$$\forall a \in \mathit{supp}(\Gamma) \exists b_0 \in b \exists f : b_0 \twoheadrightarrow a \tag{7.3.72}$$

Remark 7.26. *[AR01] uses a slightly more general concept of inductive definitions. The one introduced here implicitely demands that the class* $\Gamma(a)$ *of consequences of a premise* a *needs to be a set, a provision [AR01] do not work into their definition but use as a condition in their set existence theorem. The results presented here need it as premise as well and as it seems implausible to arrive at the existence of any nontrivial set fixed points from an inductive definition where the class of consequences of a premise does not form a set (at least in all relevant cases), it seems more straightforward in our case to choose a formulation of inductive definitions which requires this — especially since the formulation chosen above can easily be altered to accommodate nondeterminism, which will be of use later.*

We have already used many inductive definitions in this thesis. The central facts about them as proved in [AR01] are:

Fact 7.27.

1. *Let* Γ *be an inductive definition. Then there is a smallest class* Δ *which is a fixed point of* Γ.
2. *(wREA) Let* Γ *be a bounded inductive definition. Then there is a smallest set* A *which is a fixed point of* Γ.

The second statement can also be cast as follows:

Lemma 7.28. *(wREA) Let* $\Gamma : V \to V$ *be a class and* $b \in V$ *be a set. Then there is a smallest set* A *such that:*

$$\forall a \subseteq A.(\exists b_0 \in b \exists f : b_0 \twoheadrightarrow a) \to \Gamma(a) \subseteq A \tag{7.3.73}$$

In this lemma, the boundedness condition could be skipped as only a bounded part of Γ was used for the induction.

A bound on the complexity of the premises used in an inductive definition is very important as otherwise even seemingly innocuous definitions can lead to very large fixed points:

Example 7.29. *The only class which is a fixed point of the inductive definition given by $\Gamma(a) := \{a\}$ is V, and in particular there is no set which is a fixed point.*

However, the bound on the complexity does not need to be constant, as will be explored below.

7.3.1 Unleashing M

If there are stronger large set assumptions than REA present, we can obtain stronger induction principles. In particular, assuming Axiom M, the bound b of the induction can increase inductively as the induction proceeds. This can be seen as a set theoretic analogon of the type theoretic method of induction recursion where the complexity of conditions is also increased inductively (see e.g. [Dyb00]). Consider the following scheme:

Set-Theoretic Induction Recursion. For any class $\Gamma : V \to V$:

For all sets B there is a smallest set $A \supseteq B$ such that for all $a \subseteq A$ with

$$\exists b_0 \in A, f : b_0 \twoheadrightarrow a \tag{7.3.74}$$

it holds that

$$\Gamma(a) \subseteq A \tag{7.3.75}$$

The way this induction principle is formulated, the empty set is always a trivial fixed point. Adding the set B is needed to get the induction started. Classical inductive definitions as in the previous subsection have no need of being jump-started in this way, as there the constant bound b can contain $\emptyset$ as an element and the definition Γ some sets of the form $(\emptyset, x)$ which infer x without any assumptions. This is however just a technicality:

Lemma 7.30. *Set-Theoretic Induction Recursion is equivalent to the scheme that for any class $\Gamma : V \to V$:*

There is a smallest set A such that for all $a \subseteq A$ with

$$\exists b_0 \in A \cup \{0\}, f : b_0 \twoheadrightarrow a \tag{7.3.76}$$

it holds that

$$\Gamma(a) \subseteq A \tag{7.3.77}$$

Proof. To see that the statement in this lemma is implied by Set-Theoretic Induction Recursion, set $B := \Gamma(0)$. To see that Set-Theoretic Induction Recursion is implied by the statement of this lemma, apply it to

$$\Gamma'(a) := \Gamma(a) \cup B \tag{7.3.78}$$

$\square$

Theorem 7.31. *Axiom M implies Set-Theoretic Induction Recursion.*

Proof. Let $\Gamma : V \to V$ be an inductive definition and B a set. By Axiom M there is an inaccessible set I closed under Γ with $B \in I$ (e.g. the inaccessible set closed under $a \mapsto (\Gamma(a), B)$).

Define inductively a sequence of sets $(\Gamma^a)_{a \in I} \in I$ by

$$\Gamma^a = B \cup \bigcup_{b \in a} \{\bigcup \Gamma(x) | x \subseteq \Gamma^b \wedge \exists b_0 \in \Gamma^b, f : b_0 \twoheadrightarrow x\} \tag{7.3.79}$$

Note that this is monotone, i.e. if $a \subseteq b$ then $\Gamma^a \subseteq \Gamma^b$.

These are all elements of I by induction over $a \in I$: Assume that for all $b \in a$ the class Γ^b is in I, then $\bigcup b_0 \in \Gamma^b$ is in I and so is $\bigcup_{b_0 \in \Gamma^b} {}^{b_0}(\Gamma^b)$. Thus

$$\Gamma^a = \bigcup_{b \in a} \{\bigcup \mathrm{im}(f) | f \in \bigcup_{b_0 \in \Gamma^b} {}^{b_0}(\Gamma^b)\} \in I \tag{7.3.80}$$

Note that this also implies that $\Gamma^I := \bigcup_{b \in I} \{\bigcup \mathrm{im}(f) | f \in \bigcup_{b_0 \in \Gamma^b} {}^{b_0}(\Gamma^b)\}$ is a subset of I. We claim that it is closed by the induction principle under consideration.

To check this, let $x \subseteq \Gamma^I$ and $b_0 \in \Gamma^I$ with $f : b_0 \twoheadrightarrow x$. By $f : b_0 \twoheadrightarrow x$, the set x is not only a subset of I but actually an element $x \in I$. Each of its elements $y \in x$ is in some Γ^j for a $j \in I$ because $y \in \Gamma^I$ implies $y \in B$ (in which case $y \in \Gamma^0$) or $y \in \{\bigcup \Gamma(x) | x \subseteq \Gamma^b \wedge \exists b_0 \in \Gamma^b, f : b_0 \twoheadrightarrow x\}$ for some $i \in I$ in which case $y \in \Gamma^{i+1}$. Similarly, there is some $b \in I$ with $b_0 \in \Gamma^b$.

So collect enough of these j in a set $J \in I$, which fulfills $x \subseteq \bigcup_{j \in J} \Gamma^j$. Then $x \subseteq \Gamma^{\{J\} \cup b}$ and $b_0 \in \Gamma^{\{J\} \cup b}$. So $\Gamma(x) \subseteq \Gamma^{\{J\} \cup b} \subseteq \Gamma^I$.

Γ^I is also the smallest such set, for assume that $A \supseteq B$ and for all $a \subseteq A$, $b_0 \in A$ and $f : b_0 \twoheadrightarrow a$, if $(a, x) \in \Gamma$ then $x \in A$. Then a direct set induction over a shows that

$$\forall a. \Gamma^a \subseteq A \tag{7.3.81}$$

□

Axiom M implies set theoretic induction recursion, and the other direction holds to some extent as well. However, inductive definitions are better suited to deal with functional relationships which correspond to explicit set existence axioms than with total relations.

Definition 7.32. *Let CZF$_{explicit}$ be the theory CZF with Exponentiation instead of Fullness and Replacement instead of Strong Collection or alternatively Myhill's CST without DC.*

*Call a set I **explicitly inaccessible** if it is a transitive model for the theory $(CZF_{explicit})_2$.*

*Let the scheme "**The universe is explicitly Mahlo**" be: For all class functions $F : V \to V$ there is an explicitly inaccessible set I with $F : I \to I$.*

Remark 7.33. *(ZF)*

1. *I is explicitly inaccessible iff it is inaccessible.*

2. *The universe is explicitey Mahlo iff Axiom M holds.*

In the absence of the excluded middle however, being explicitly inaccessible is weaker than being inaccessible.

Theorem 7.34. *Set-Theoretic Induction Recursion implies that the universe is explicitly Mahlo.*

Proof. Let Set-Theoretic Induction Recursion hold and $F : V \to V$ be a class function. Define an inductive definition Γ by:

$$\begin{aligned}
\Gamma(a) = \quad & \{F(x) | a = \{x\}\} && (7.3.82)\\
\cup \quad & \{y \in x | a = \{x\}\} && (7.3.83)\\
\cup \quad & \{ {}^{x}y | a = \{x, y\}\} && (7.3.84)\\
\cup \quad & \{x \cap y | a = \{x, y\}\} && (7.3.85)\\
\cup \quad & \{\bigcup(x) | a = \{a\}\} && (7.3.86)\\
\cup \quad & \{a\} && (7.3.87)
\end{aligned}$$

Then let $A \supseteq \omega + 1$ be a fixed point for the Induction Recursion defined by Γ, i.e. let for all $b_0 \in A$ and $f : b_0 \twoheadrightarrow a \subseteq A$ hold $\Gamma(a) \subseteq A$. We claim that A is an explicitly inaccessible fixed point of F.

A models Replacement, for let $a \in A$ and $f : a \to A$, then $f : a \twoheadrightarrow f''a \subseteq A$ and thus by A being a fixed point and $f''a \in \Gamma(f''a)$, part 7.3.87 of the definition of Γ implies $f''a \in A$.

A is transitive, for let $a \in A$, then as $1 \in \omega + 1 \subseteq A$, $\{a\} \subseteq a$ is the surjective image of an element of A, so by A being a fixed point, part 7.3.83 of the definition of Γ implies $a \subseteq A$.

A models Infinity because $\omega \in A$. As A is functionally regular, this also implies A is closed under ordered and unordered pairs.

A models Exponentiation, Binary Intersection and Union by parts 7.3.84, 7.3.85 and 7.3.86, always noting that the conditions needed are all surjective images of elements of A (namely of 1 and 2).

A models Δ_0-Collection because by [AR01] this is implied by Binary Intersection on the basis of Extensionality, Pairing, Union and Replacement.

Finally $F : A \to A$ as for any $a \in A$, $\{a\} \subseteq A$ is a surjective image of an element of A (namely of 1) and thus by A being a fixed point, part 7.3.82 of the definition of Γ implies $\{F(a)\} \subseteq A$.

□

7.3.2 Nondeterministic Inductive Definitions equivalent to M

If the full Axiom M is supposed to be retrieved, nondeterministic inductive definitions need to be studied, i.e. total relations $\Gamma : V \rightrightarrows V$ instead of functions $\Gamma : V \to V$. A fixed point of such an inductive definition is then a set A with $\Gamma : \mathcal{P}(A) \rightrightarrows \mathcal{P}(A)$, i.e. for all $a \subseteq A$ there is some b with $(a, b) \in \Gamma$ and $b \subseteq A$. As they are nondeterministic, we obviously can not expect a smallest fixed point to exist; even the trivial definition $\Gamma := \{(a, b) | b = \{0\} \vee b = \{1\}\}$ has the fixed points $\{0\}$ and $\{1\}$, but not $\{0\} \cap \{1\} = \emptyset$. Adapting the deterministic parts of the induction principle from last subsection to nondeterministic definitions leads to:

Nondeterministic Set-Theoretic Induction Recursion. For any class $\Gamma : V \rightrightarrows V$:

For all sets B there is a set $A \supseteq B$ such that whenever $b_0 \in A$ and $R : b_0 \rightrightarrows A$, there is a subrelation $R' \subseteq R$ with $R' : b_0 \rightrightarrows A$ and its image $a := \{y | \exists x \in b_0.(x, y) \in R'\} \subseteq A$ fulfills

$$\exists b \subseteq A.(a, b) \in \Gamma \tag{7.3.88}$$

Then the concluding theorem is:

Theorem 7.35. *Axiom M is equivalent to Nondeterministic Set-Theoretic Induction Recursion.*

Proof. 1. Assume Axiom M. Let $\Gamma : V \rightrightarrows V$ be a nondeterministic inductive definition and B be a set. By Axiom M there is an inaccessible set I closed under Γ with $B \in I$ (e.g. the inaccessible set closed under $\{(a,(b,B))|(a,b) \in \Gamma\}$).

Define a relation $R \subseteq I \times I$ by induction over the first component such that for all $a, b \in I$:

$$\begin{aligned}(a,b) \in R :\leftrightarrow b \in B \vee \exists c \in a \exists b_0, x, r \in I.\\ r : b_0 \leftleftarrows\!\rightrightarrows x \ \wedge (c, b_0) \in R \wedge \forall y \in x\ (c,y) \in R \wedge \exists d.\\ b \in d \wedge (x,d) \in \Gamma\end{aligned} \tag{7.3.89}$$

Write Γ^i for $\{b \in I | \exists a \in i.(a,b) \in R\}$. We claim that $\Gamma^I := \{b \in I | \exists a \in I.(a,b) \in R\} \supseteq B$ is a fixed point of the nondeterministic set-theoretic induction recursion.

To check this, let $b_0 \in \Gamma^I$ and $r : b_0 \rightrightarrows \Gamma^I$. By regularity of $I \supseteq \Gamma^I$, there are some $r', x \in I$ with $r' : b_0 \leftleftarrows\!\rightrightarrows x$ and $r' \subseteq r$. There is an $i \in I$ such that $b_0 \in \Gamma^i$. Thus by definition of R, the set $x \in \Gamma^{i+1}$ and thus $x \in \Gamma^i$ as desired.

2. Let Nondeterministic Set-Theoretic Induction Recursion hold and $R_0 : V \rightrightarrows V$ be a class. Define a nondeterministic inductive definition Γ by:

$$(a,b) \in \Gamma :\leftrightarrow \quad \exists z \in b.R_0 : a \leftleftarrows\!\rightrightarrows z \tag{7.3.90}$$

$$\wedge \quad \forall x.a = \{x\} \to x \subseteq b \tag{7.3.91}$$

$$\wedge \quad \forall xy.a = \{x,y\} \to \exists c \in b.c \text{ full in } \mathrm{mv}(x,y) \tag{7.3.92}$$

$$\wedge \quad \forall xy.a = \{x,y\} \to x \cap y \in b \tag{7.3.93}$$

$$\wedge \quad \forall x.a = \{x\} \to \bigcup x \in b \tag{7.3.94}$$

$$\wedge \quad \{a\} \in b \tag{7.3.95}$$

This does indeed define a total relation $\Gamma : V \rightrightarrows V$, for let a be a set. By Strong Collection, there is a set b_0 with $R_0 : a \leftleftarrows\rightrightarrows b_0$. By Replacement, Pairing, Union and Δ_0-Collection, there are also the sets

$$\begin{aligned} b_1 &:= \{x | x \in a \wedge a = \{x\}\} \\ b_3 &:= \{x \cap y | x, y \in a \wedge a = \{x, y\}\} \\ b_4 &:= \{\bigcup x | x \in a \wedge a = \{x\}\} \\ b_5 &:= \{\{a\}\} \end{aligned}$$

Also, for each element of $\{(x, y) | x, y \in a \wedge a = \{x, y\}\}$, there is a set c which is full in mv(x, y), so these sets can be collected into a set b_2. Set $b := \{b_0\} \cup b_1 \cup b_2 \cup b_3 \cup b_4 \cup b_5$, then $(a, b) \in \Gamma$. Indeed, the conjucts 7.3.90, 7.3.91, 7.3.92, 7.3.93, 7.3.94 and 7.3.95 are fulfilled because $b \supseteq \{b_0\}, b \supseteq b_1, b \supseteq b_2, b \supseteq b_3, b \supseteq b_4$ and $b \supseteq b_5$ respectively. So indeed $\Gamma : V \rightrightarrows V$.

Thus let $A \supseteq \omega + 1$ be a fixed point for the nondeterministic Induction Recursion defined by Γ, i.e. for all $b_0 \in A$ and $R : b_0 \rightrightarrows A$ let there be a $r \subseteq R$ such that $\exists b \subseteq A.(\{y | \exists x.(x, y) \in r\}, b) \in \Gamma$. We claim that A is a inaccessible and $R : A \rightrightarrows A$.

A models Strong Collection, for let $a \in A$ and $R : a \rightrightarrows A$, then there is an $r \subseteq R$ and a $b \subseteq A$ with $(\{y | \exists x.(x, y) \in r\}, b) \in \Gamma$. So by part 7.3.95 of the definition of Γ also $\{y | \exists x.(x, y) \in r\} \in A$ and this is an element witnessing Strong Collection for a and R.

A is transitive, for let $a \in A$, then as $1 \in \omega + 1 \subseteq A$, $\{a\} \subseteq a$ is the surjective image of an element of A. Note that functions are minimal total relations, so if $f : 1 \twoheadrightarrow \{a\}$ any $r \subseteq f$ with $r : 1 \rightrightarrows A$ fulfills $r = f$, which means that if some subset of A is the surjective image of an element of A, then there is always some $b \subseteq A$ with $(a, b) \in \Gamma$. In this case, part 7.3.91 of the definition of Γ then implies $a \subseteq A$.

A models Infinity because $\omega \in A$. As A is functionally regular, this also implies A is closed under ordered and unordered pairs.

A models Fullness, Binary Intersection and Union by parts 7.3.92, 7.3.93 and 7.3.94, always noting that the conditions needed are all surjective images of elements of A (namely of 1 and 2).

A models Δ_0-Collection because by [AR01] this is implied by Binary Intersection on the basis of Extensionality, Pairing, Union and Replacement.

Finally $R_0 : A \rightrightarrows A$ as for any $a \in A$, $\{a\} \subseteq A$ is a surjective image of an element of A (namely of 1) and thus by A being a fixed point, part 7.3.90 of the definition of Γ implies $\{F(a)\} \subseteq A$.

$\square$

Chapter 8

Expressing Weak Compactness

8.1 Different Renderings of the same Classical Concept

Classical set theory enjoys a whole host of equivalences for the cardinal concept of weak compactness, most of which break down in the constructive case. Not all are equally useful without the excluded middle and strong choice principles and some may even lose all proof theoretic strength completely, e.g. what is classically known as the partition property $\kappa \to (\kappa)^2_2$ (e.g. [Dra74]):

Definition 8.1. *Let A be a set. The class of* ***two element subsets*** *of elements of A is defined as*

$$[A]^2 := \{\{a, b\} | a, b \in A \wedge a \neq b\} \tag{8.1.1}$$

A function f is said to be ***homogeneous*** *on an inhabited set $A \subseteq dom(f)$ if $f''A$ is a singleton or equivalently if*

$$\forall x, y \in A. f(x) = f(y) \tag{8.1.2}$$

Example 8.2. *The following theories are equiconsistent:*

1. *CZF plus the existence of an inaccessible*

2. *CZF plus the existence of an inaccessible I with the property that for every $f : [I]^2 \to 2$ there is a set $A \subseteq I$ such that f is homogeneous on $[A]^2$ and A is set unbounded in I*

2'. *CZF plus the existence of an inaccessible I with the property that for every $f : [I]^2 \to 2$ there is a set $A \subseteq I$ such that f is homogeneous on $[A]^2$ and A is bijective to I*

3. *CZF plus the existence of an inaccessible I with the property that every $f : [I]^2 \to 2$ is constant*

Proof. Let I be an inaccessible and consider the realizability model $V(Kl)$. By Lemma 4.16 the following set is realized to be inaccessible in $V(Kl)$:

$$\bar{I} := \omega \times (I \cap V(Kl)) \tag{8.1.3}$$

Let $f \in V(Kl)$ and $\Vdash f : [I]^2 \to 2$, then in particular there is some realizer e with

$$e \Vdash \forall x, y \in \bar{I}. x \neq y \to ((x, y), 0) \in f \vee ((x, y), 1) \in f \tag{8.1.4}$$

Note that $(g, x) \in \bar{I}$ is equivalent to $(k, x) \in \bar{I}$ for any g and x. So for all $x, y \in I \cap V(Kl)$, if $g \Vdash x \neq y$, then $ekkg \Vdash (x, 0) \in f \vee (x, 1) \in f$. However, if $g \Vdash x \neq y$, then also $k \Vdash x \neq y$, so for all realizedly different x and $y \in I \cap V(Kl)$, it holds that $ekkk \Vdash ((x, y), 0) \in f \vee ((x, y), 1) \in f$. So either $lekkk = 0$ and $rekkk \Vdash ((x, y), 0) \in f$ or $lekkk = 1$ and $rekkk \Vdash ((x, y), 1) \in f$.

In the first case, it holds that for all $(g_1, x), (g_2, y) \in \bar{I}$ with $g_3 \Vdash x \neq y$ we have $eg_1g_2g_3 \Vdash ((x, y), 0) \in f$ and in the second case for all $(g_1, x), (g_2, y) \in \bar{I}$ with $g_3 \Vdash x \neq y$ we have $eg_1g_2g_3 \Vdash ((x, y), 1) \in f$. As $\bar{I}$ has two realizedly different members (e.g. $(k, 0)$ and $(k, \{(k, 0)\})$), one of those cases actually holds. $\square$

The situation would not be significantly altered using total relations instead of functions. In fact, the same proof idea yields:

Example 8.3. *The following theories are equiconsistent:*

1. *CZF plus the existence of an inaccessible*

2. *CZF plus the existence of an inaccessible I with the property that for all $A, B \subseteq I$ if $A \cup B = [I]^2$ then either A or B are set unbounded in I*

2'. *CZF plus the existence of an inaccessible I with the property that for all $A, B \subseteq I$ if $A \cup B = [I]^2$ then either A or B are bijective to I*

3. *CZF plus the existence of an inaccessible I with the property that for all $A, B \subseteq I$ if $A \cup B = [I]^2$ then either $A = I$ or $B = I$*

In both examples, the second equivalent describes what in ZFC amounts to a weakly compact cardinal[1] while the last equivalent amounts to a straightforward inconsistency in the presence of the excluded middle.

A fruitful way to express weak compactness in a constructive setting is inspired by the π^1_1 indescribability and developed in [Rat98] and [Gib02] as 2-strongness. This formulation is classically equivalent to weak compactness and its logical strength is wedged satisfyingly between measurability and Mahloness:

Proposition 8.4. *Let K be 2-strong. Then the set of I-Mahlo sets $I \in K$ is stationary, i.e., whenever $R : K \rightrightarrows K$ there is an I-Mahlo set $I \in K$ with $R : I \rightrightarrows I$.*

Proof. First show by induction on $a \in K$ that K is a-Mahlo for any $a \in K$. So let $a \in K$ and for all $c \in b \in a$ and $R : K \rightrightarrows K$ let there be a c-Mahlo set I such that $R : I \rightrightarrows I$. To see that K is a-Mahlo, let $R : K \rightrightarrows K$ and $b \in a$. Consider the set

[1] As these properties depend only on the cardinality of the set in question, it makes no difference whether they are demanded for the weakly compact cardinal κ or its von Neumann stage V_κ.

$S \subseteq K^3$ defined by

$$\begin{aligned}(u,v,x) \in S \quad \leftrightarrow \quad & (u,v,x) \in K^3 \wedge \\ & v : u \cap b \rightrightarrows K \wedge \\ & \forall (c,i) \in v \cap (b \times K).x : i \rightrightarrows i \wedge \\ & \forall (c,i) \in v \cap (b \times K).i \text{ is } c\text{-Mahlo} \wedge \\ & \exists (b,u') \in v.(u,u') \in R\end{aligned}$$

We claim that

$$\forall R' : K \rightrightarrows K \forall u \in K \exists v \in K \exists x \in K.x \subseteq R' \wedge (u,v,x) \in S \tag{8.1.5}$$

For let $R' : K \rightrightarrows K$ and $u \in K$. For each $c \in u \cap b \in K$ it holds by induction hypothesis that there is an $i \in K$ such that i is c-Mahlo and $R : i \rightrightarrows i$. Using regularity, collect enough of these pairs (c,i) in a set $v_0 \in K$ such that $\forall c \in u \cap b \exists (c,i) \in v_0$ such that i is c-Mahlo and $R : i \rightrightarrows i$. Let u' be some element such that $(u,u') \in R$, then set

$$v := v_0 \cup \{(b,u')\} \tag{8.1.6}$$

By $R' : K \rightrightarrows K$, for each $(c,i) \in v \cap (b \times K) \in K$ and $d \in i$ there is a $(d,e) \in R$. Using regularity two times to collect these pairs into one set, one arrives at the existence of an $x \in K$ such that $x \subseteq R$ and $\forall (c,i) \in v.x : i \rightrightarrows i$. Then by construction

$$(u,v,x) \in S \tag{8.1.7}$$

For this, also note that (b,u') can never be of the form (c,i) with $(c,i) \in v \cap (b \times K)$. This concludes the proof of Claim 8.1.5

So by 2-strongness of K, there is some inaccessible set I with

$$\forall R' : I \rightrightarrows I \forall u \in I \exists v \in I \exists x \in I.x \subseteq R' \wedge (u,v,x) \in S \tag{8.1.8}$$

We claim that this I is b-Mahlo. For let $c \in b$ and $R' : I \rightrightarrows I$, then $c \in I$ by transitivity. Let $u := \{c\} \in I$, then by Equation 8.1.8 there are $v, x \in I$ with $(u, v, x) \in S$. In particular, there is some J with $(c, J) \in v$ (and thus $J \in I$), $R' : J \rightrightarrows J$ and J is c-Mahlo. Thus I is b-Mahlo. Thus K is a-Mahlo. This concludes the proof for

$$\forall a \in K. K \text{ is } a \text{ -Mahlo} \tag{8.1.9}$$

Now let $R : K \rightrightarrows K$. Consider the set $S \subseteq K^3$ defined by

$$\begin{aligned} S = \{&(u, v, x) \in K^3 | \\ &\exists v_1, v_2 . v = (v_1, v_2) \wedge v_1 \text{ is } u\text{-Mahlo} \wedge x : v_1 \rightrightarrows v_1 \wedge (u, v_2) \in R\} \end{aligned} \tag{8.1.10}$$

Then K being K-Mahlo implies that for every $u \in K$, every relation $R' : K \rightrightarrows K$ is reflected in some u-Mahlo $v_1 \in K$ with $R : v_1 \rightrightarrows v_1$, and so by regularity there is an $x \in K$ with $x \subseteq R$ and $x : v_1 \rightrightarrows v_1$. In other words,

$$\forall R : K \rightrightarrows K \forall u \in K \exists v \in K \exists x \in K . x \subseteq R \wedge (u, v, x) \in S \tag{8.1.11}$$

By 2-strongness this is reflected down into some inaccessible $I \in K$ which is then also closed under R and for every $u \in I$, the set I is u-Mahlo. $\square$

8.2 A Simpler Characterisation of 2-Strong Sets

Recall that an inaccessible set K is called 2-strong, if for all S (wlog with $S \subseteq K^3$)

$$\forall R : K \rightrightarrows K . \forall u \in K \exists v \in K \exists x \in K . x \subseteq R \wedge (u, v, x) \in S \tag{8.2.12}$$

implies the existence of an inaccessible $I \in K$ such that

$$\forall R : I \rightrightarrows I . \forall u \in I \exists v \in I \exists x \in I . x \subseteq R \wedge (u, v, x) \in S \tag{8.2.13}$$

The statement that is reflected downwards from K to I contains four quantifiers (and even more if $R : K \rightrightarrows K$ were to be spelled out), which can make checking if a set is 2-strong somewhat tedious. Consider the following less complex properties obtained by simply omitting quantifiers from the definition of 2-strong:

Definition 8.5. *An inaccessible set K is said to have the property (+), if for all S' (wlog with $S' \subseteq K^2$)*

$$\forall R : K \rightrightarrows K.\exists v \in K \exists x \in K.x \subseteq R \wedge (v, x) \in S' \tag{8.2.14}$$

implies the existence of an inaccessible $I \in K$ such that

$$\forall R : I \rightrightarrows I.\exists v \in I \exists x \in I.x \subseteq R \wedge (v, x) \in S' \tag{8.2.15}$$

And one step further

Definition 8.6. *An inaccessible set K is said to have the property (++), if for all S'' (wlog with $S'' \subseteq K$)*

$$\forall R : K \rightrightarrows K.\exists x \in K.x \subseteq R \wedge x \in S'' \tag{8.2.16}$$

implies the existence of an inaccessible $I \in K$ such that

$$\forall R : I \rightrightarrows I.\exists x \in I.x \subseteq R \wedge x \in S'' \tag{8.2.17}$$

Remark 8.7. *If a set is 2-strong, then it has property (+) and if a set has property (+) then it also has property (++).*

This is seen readily as (+) is just 2-strong with the S restricted to those S of the form $K \times S'$ and (++) is just (+) with the S' restricted to those S' of the form $K \times S''$ (noting that the quantifiers do not run empty since as an inaccessible set, K is always inhabited).

Thus formally, these two simpler properties constitute weakenings of 2-strongness. They are however stronger than they appear. To prove this, we first need to state a simple fact about coding and decoding pairs, namely that the left inverses of pairing (i.e. the projection functions) can be extended to functions with domain V, not only $V \times V$.

Lemma 8.8. *There are definable functions* $p_1, p_2 : V \to V$ *such that*

$$\forall a, b.\ p_1((a, b)) = a \wedge p_2((a, b)) = b \tag{8.2.18}$$

and which are reflected in all inaccessible sets, i.e., whenever I *is inaccessible, then* $p_1, p_2 : I \to I$.

Proof. Set

$$p_1(x) := \{y \in tc(x) | \exists a, b \in tc(x). x = (a, b) \wedge y \in a\} \tag{8.2.19}$$

$$p_2(x) := \{y \in tc(x) | \exists a, b \in tc(x). x = (a, b) \wedge y \in b\} \tag{8.2.20}$$

This works as inaccessible sets are closed under transitive closures and both a and b as well as their elements are in $tc(a, b)$. Furthermore, if x has the form (a, b), then a and b are uniquely determined. □

We will use these p_i in the proof of the following.

Proposition 8.9. *Let* K *have the property (+). Then* K *is 2-strong.*

Proof. Let K fulfill (+) and let $S \subseteq K^3$ such that

$$\forall R : K \rightrightarrows K. \forall u \in K \exists v \in K \exists x \in K. x \subseteq R \wedge (u, v, x) \in S \tag{8.2.21}$$

We need to show that this statement is reflected at some inaccessible element of K.

The idea of the proof is the following: We want to code both u and R into a new total relation R' in a way that the quantification still works. This can be done by having R' consist of the pairs $(a, (b, u))$ for $(a, b) \in R$, then for every total R we find a total R' and vice versa. We only need one element of R' to read off u. However, we have to read it off not from R' itself, but from $x \subseteq R'$ and there is no a priori guarantee that x is inhabited. So we force it to be inhabited, say by demanding $\emptyset \in \mathrm{dom}(x)$. This loses a small bit of information, namely whether or not $\emptyset \in \mathrm{dom}(x)$ in the first place. As this information might have been important, this bit should be stored into v' which will be a pair, with the old v as first component and the information about $\emptyset \in \mathrm{dom}(x)$ in the second one, storing it as a truth value and also storing which a was used in the addition $(\emptyset, (a, u))$.

To decode x from x' and v', define the shorthand

$$f(x', v') :=$$
$$\{(a, p_1(b))|(a, b) \in x' \wedge (a = \emptyset \wedge p_1(b) = p_1(p_1(p_2(v'))) \to \emptyset \in p_2(p_2(v')))\} \quad (8.2.22)$$

Note that inaccessible sets are closed under the functions p_1, p_2 and f.

Now define

$$S' := \{(v', x')|(\emptyset, p_1(p_2(v'))) \in x' \wedge (p_2(p_1(p_2(v'))), p_1(v'), f(x', v')) \in S\} \quad (8.2.23)$$

We claim that

$$\forall R' : K \rightrightarrows K.\exists v' \in K \exists x' \in K.x \subseteq R \wedge (v', x') \in S' \quad (8.2.24)$$

To see this, take any $R' : K \rightrightarrows K$. Let d be some element of K with $(\emptyset, d) \in R'$ and let $u := p_2(d)$. Define a total relation $R : K \rightrightarrows K$ by setting

$$R := \{(a, p_1(b))|(a, b) \in R'\} \quad (8.2.25)$$

Then by the assumption, there are an $x \subseteq R$ and a v such that $(u, v, x) \in S$.

Note that for all $(a, c) \in x$ there is a pair $(a, b) \in R'$ such that $c = p_1(b)$. By Strong Collection, there is a set z of pairs such that

$$\forall (a, c) \in x \exists (a, b) \in z.c = p_1(b) \wedge \forall (a, b) \in z.(a, p_1(b)) \in x \quad (8.2.26)$$

Then define

$$x' := z \cup (\emptyset, d) \tag{8.2.27}$$

and

$$v' := (v, (d, \{\emptyset | (\emptyset, p_1(d)) \in x\})) \tag{8.2.28}$$

Then $x' \subseteq R'$ by construction. We claim that $(v', x') \in S'$. To show this, we need to establish

$$(\emptyset, p_1(p_2(v'))) \in x' \wedge (p_2(p_1(p_2(v'))), p_1(v'), f(x', v')) \in S \tag{8.2.29}$$

The first conjunct is true as $p_1(p_2(v')) = d$ and $(\emptyset, d)$ was added to x' in its definition. For the second, note that

$$
\begin{aligned}
u &= p_2(p_1(p_2(v'))) && (8.2.30)\\
v &= p_1(v') && (8.2.31)\\
x &= (x \cup (\emptyset, p_1(d))) \cap \{(\emptyset, p_1(d)) | (\emptyset, p_1(d)) \in x\} && (8.2.32)\\
&= (x \cup (\emptyset, p_1(d))) \cap \{(\emptyset, p_1(d)) | \emptyset \in p_2(p_2(v'))\} && (8.2.33)\\
&= \{(a, p_1(b)) | (a, b) \in x' \wedge && (8.2.34)\\
&\quad (a = \emptyset \wedge p_1(b) = p_1(p_1(p_2(v'))) \to \emptyset \in p_2(p_2(v')))\} && (8.2.35)\\
&= f(x', v') && (8.2.36)
\end{aligned}
$$

As $(u, v, x) \in S$ by choice of v and x, this shows Equation 8.2.29 and it follows that $(v', x') \in S'$. So by abstaction,

$$\forall R' : K \rightrightarrows K. \exists v' \in K \exists x' \in K. x' \subseteq R \wedge (v', x') \in S' \tag{8.2.37}$$

Thus we can use the property (+) and find some inaccessible $I \in K$ such that

$$\forall R' : I \rightrightarrows I.\exists v' \in I \exists x' \in I.x' \subseteq R \wedge (v', x') \in S' \tag{8.2.38}$$

We claim that this I is as desired to show that K is 2-strong, i.e. we need to show that

$$\forall R : I \rightrightarrows I.\forall u \in I \exists v \in I \exists x \in I.x \subseteq R \wedge (v, x) \in S \tag{8.2.39}$$

So let $R : I \rightrightarrows I$. Consider

$$R' := \{(a, (b, u)) | (a, b) \in R\} \tag{8.2.40}$$

Then there are $v' \in I$ and $x' \in I$ with

$$x' \subseteq R' \wedge (v', x') \in S' \tag{8.2.41}$$

Set

$$v := p_1(v') \tag{8.2.42}$$

$$x := f(x', v') \tag{8.2.43}$$

By definition of S', we know that

$$(\emptyset, p_1(p_2(v'))) \in x' \wedge (p_2(p_1(p_2(v'))), p_1(v'), f(x', v')) \in S \tag{8.2.44}$$

From $(\emptyset, p_1(p_2(v'))) \in x'$ we conclude that $(\emptyset, p_1(p_2(v'))) \in R'$ and by definition of R' this means that $p_1(p_2(v'))$ must be of the form (b, u) for some $(\emptyset, b) \in R$. Thus $p_2(p_1(p_2(v'))) = u$. Also $p_1(v') = v$ and $f(x', v') = x$ by definition. So the equation

$$(p_2(p_1(p_2(v'))), p_1(v'), f(x', v')) \in S \tag{8.2.45}$$

simplifies to $(u, v, x) \in S$ which was to show.

$\square$

It is also possible to get rid of the existential quantifier for v. In fact, this is quite a bit less messy.

Proposition 8.10. *Let K have the property (++). Then it has the property (+).*

Proof. Let K fulfill (++) and let $S \subseteq K^2$ such that

$$\forall R : K \rightrightarrows K.\exists v \in K\exists x \in K.x \subseteq R \wedge (v, x) \in S \tag{8.2.46}$$

We need to show that this statement is reflected at some inaccessible element of K.

The idea of the proof is again to code the variables we want to get rid of into those we keep in a way that will not mess with the quantification. So this time we need to code v into x. We do that by enlarging x by an element that determines v and yet cannot be confused with the previous elements of x - for example a pair of the type $((x, v), b)$ with such a b as to make this pair an element of R.

So define a set $S' \subseteq K$ by

$$S' = \{x' | \exists v, x, b.x' = x \cup \{((x, v), b)\} \wedge (v, x) \in S\} \tag{8.2.47}$$

Then this fulfills

$$\forall R : K \rightrightarrows K.\exists x' \in K.x' \subseteq R \wedge x' \in S' \tag{8.2.48}$$

Because for any such R, choose v, x as above and some b with $((x, v), b) \in R$ and then let

$$x' := x \cup \{((x, v), b)\} \tag{8.2.49}$$

Then using the property (++), this gets reflected downwards, i.e. there is an inaccessible $I \in K$ with

$$\forall R : I \rightrightarrows I.\exists x' \in I.x' \subseteq R \wedge x' \in S' \tag{8.2.50}$$

We want to show that this I is as desired, i.e. that

$$\forall R : I \rightrightarrows I.\exists v \in I \exists x \in I.x \subseteq R \wedge (v, x) \in S \tag{8.2.51}$$

To this end, take an arbitrary $R : I \rightrightarrows I$. Then there is an $x' \in I$, $x' \subseteq R$ with $x' \in S'$. Then let

$$x := \{(a, b) \in x' | \exists z, v, c \in I.(a, b) \in z \wedge ((z, v), c) \in x'\} \tag{8.2.52}$$

As $x' \in S'$, it is of the form $x_0 \cup \{((x_0, v), b)\}$ and this ensures that x is just this unique x_0. Furthermore, let v be just this unique v_0, i.e. the set such that $x' = x \cup \{((x, v), b)\}$ for some b.

Then $x \subseteq x' \subseteq R$ and

$$(v, x) \in S \tag{8.2.53}$$

Which is as needed. □

These results can be summed up in a neatly simple characterisation of 2-strong:

Theorem 8.11. *An inaccessible set K is 2-strong if and only if for all S (wlog $S \subseteq K$), whenever*

$$\forall R : K \to K \exists x \subseteq R.x \in S \cap K \tag{8.2.54}$$

then there is an inaccessible $I \in K$ with

$$\forall R : I \to I \exists x \subseteq R.x \in S \cap I \tag{8.2.55}$$

Chapter 9

Elementary Embeddings

In this chapter we will analyse several aspects regarding elementary embeddings, i.e. the axioms associated to measurable or Reinhardt sets.

We will consider several natural weakenings of the axiom "There is a measurable set": In Section 9.1 we consider demanding less closure from the critical point (an idea already brought forth by Friedman and Ščedrov) and see that the corresponding axioms still hold a surprising amount of power which can be unlocked by assuming small large cardinal axioms. In section 9.2 we analyse Δ_0 elementary embeddings where the elementarity scheme is restricted to bounded formulae, which still proves to be quite useful. Finally in Section 9.4 we will see that casting the elementarity scheme as an implication rather than an equivalence makes quite the difference and construct a model for the constructively weaker variant with only CZF as background theory. It should be noted that all three variations are equivalent to measurable cardinals when using classical logic (for the second case under one reasonable extra assumption).

One theme that will surface repeatedly during these investigations is that of cofinality, i.e. the property that the image of the elemental embedding be unbounded in M, which will be introduced in Section 9.3, which will also show that in the case of Reinhardt embeddings, very strong cofinality properties hold. For ordinals, cofinality always holds in classical set theory as $j''\mathrm{O_n}$ is unbounded in $\mathrm{O_n}$. We will show that this not necessarily the case in CZF by constructing models refuting strong cofinality in Sections 9.4 and 9.5. The first will contain an interesting (and to the author's knowledge new) partial combinatory algebra which directly leads to a weak elementary embedding in the model, while the second will work with a full elementary embedding and develop new tools to decide whether a set is or is not realized to be in a transitive model of set theory (in this case M).

Finally in Section 9.6, we will find that the caesura between small and large large cardinals in ZFC marked by their compatibility with $V = L$ is echoed by an inherently constructive principle in CZF.

During this chapter, we work in the extended language with the unary function symbol j and the unary relation symbol M and extend the schemes of CZF to formulae containing these two symbols.

9.1 How Inaccessible must the Critical Point of a Measurable be?

In [FŠ84], Friedman and Ščedrov considered two different formulations of measurable cardinals with critical points and Gibbons proved his results for the latter, more powerful version in [Gib02]. Both have in common the axiom scheme which in this thesis has been named $j : V \overset{\equiv}{\hookrightarrow} M$, i.e. an elementary embedding into a transitive class and in addition, they demand either the existence of a transitive or an inaccessible critical point, i.e. an z that fulfills one of the following:

$$\Phi_1(z) := z \in j(z) \wedge \forall x \in z\; x = j(x) \wedge z \text{ is transitive} \tag{9.1.1}$$

$$\Phi_2(z) := z \in j(z) \wedge \forall x \in z\; x = j(x) \wedge z \text{ is inaccessible} \tag{9.1.2}$$

From our perspective on large sets in CZF, it is also natural to consider the following variants, some of which will be defined later in this section:

$$\Phi_{1'}(z) := z \in j(z) \wedge \forall x \in z\ x = j(x) \wedge z \text{ is an ordinal} \tag{9.1.3}$$

$$\Phi_{1.5}(z) := z \in j(z) \wedge \forall x \in z\ x = j(x) \wedge z \text{ is union-functionally regular} \tag{9.1.4}$$

$$\Phi_{1.5'}(z) := z \in j(z) \wedge \forall x \in z\ x = j(x) \wedge z \text{ is proto-inaccessible} \tag{9.1.5}$$

$$\Phi_{2'}(z) := z \in j(z) \wedge \forall x \in z\ x = j(x) \wedge z \text{ is explicitly inaccessible} \tag{9.1.6}$$

The connections between these statements can be summarized in the following figure:

$$\begin{tikzcd}
 & \Phi_1 \arrow[r] & \Phi_{1'} \arrow[dll] & \\
\Phi_{1.5} \arrow[rrr] & & & \Phi_{1.5'} \arrow[dl] \\
 & \Phi_2 \arrow[r] & \Phi_{2'} &
\end{tikzcd}$$

Figure 9.1: The implicational relationships between the different degrees of inaccessibility of critical points. An arrow between two statements signifies that the existence of a set fulfilling one implies the existence of a set fulfilling the other (sometimes on the basis of certain small large cardinal axioms). Some trivial implications were removed for legibility.

9.1.1 Improving on Transitivity

Theorem 9.1. *($j : V \overset{\equiv}{\hookrightarrow} M$)*

$$\exists z \Phi_1(z) \rightarrow \exists \alpha \Phi_{1'}(\alpha) \tag{9.1.7}$$

Proof. Assume $\Phi_1(z)$ and set $\alpha := \text{rk}(z)$. Then to show $\Phi_{1'}(\alpha)$, we need to prove that j restricted to α is the identity and $j(\alpha) \ni \alpha$.

For the first demonstrandum, show by set induction over a that

$$\forall a.\forall b \in a\ j(b) = b \to \forall \beta \in \text{rk}(a).j(\beta) = \beta \tag{9.1.8}$$

For let $\beta \in \text{rk}(a)$, so either $\beta = \text{rk}(b)$ for some $b \in a$ or $\beta \in \text{rk}(b)$ for some $b \in a$. In the first case $j(\beta) = j(\text{rk}(b)) = \text{rk}(j(b)) = \text{rk}(b) = \beta$ where the crucial step used the commutativity of the functions j and rk by Example 2.20. In the second case $j(\beta) = \beta$ by the induction hypothesis.

For the second demonstrandum, note that $z \in j(z)$ implies $\text{rk}(z) \in \text{rk}(j(z))$, so

$$\alpha = \text{rk}(z) \in \text{rk}(j(z)) = j(\text{rk}(z)) = j(\alpha) \tag{9.1.9}$$

Here we again used the commutativity of rk and j.

This completes the proof. □

Theorem 9.1 indicates that when showing that $\Phi(z)$ implies stronger properties about z, we might without loss of generality also start from an ordinal, which is why in the following we will use variable names usually associated with ordinals, even though in fact the critical points can be arbitrary transitive sets.

Consider the following weak regularity properties, which unlike full regularity can also be expected to apply to large ordinals:

Definition 9.2. *A transitive set α is called **weakly functionally regular** if whenever $\beta \in \alpha$ and for some function $f : \beta \to \alpha$ there is some $\gamma \in \alpha$ with $f : \beta \to \gamma$.*

*A transitive set α is called **union-functionally regular** if for all $\beta \in \alpha$ and $f : \beta \to \alpha$, we have $\bigcup_{x\in\beta} f(x) \in \alpha$.*

Remark 9.3. *Obviously an ordinal $\alpha \geq 2$ can never actually be regular, or even functionally regular, as $\{1\} \subseteq \alpha$ is the unique set in functional correspondence with $\{0\} \in \alpha$ via the function $f := \{(0,1)\}$, but is not an ordinal and thus not in α.*

Still, they can be weakly functionally regular or union-functionally regular. In fact, in the presence of the excluded middle, these two properties coincide with what is usually called "regular" in a classical context, i.e. being its own cofinality.

For critical points, the first of these properties comes for free with transitivity:

Proposition 9.4. *($j : V \overset{\equiv}{\hookrightarrow} M$) If there is an α with $\Phi_1(\alpha)$, then it is weakly functionally regular.*

Proof. Let $\beta \in \alpha$ and $f : \beta \to \alpha$. Then for all $x, y \in \alpha$:

$$(x, y) \in f \to (j(x), j(y)) = (x, y) \in j(f) \tag{9.1.10}$$

and f is a function with domain β, so $j(f)$ is a function with the same domain $j(\beta) = \beta$. So in fact $f = j(f)$. So we have

$$M \vDash \exists \gamma \in j(\alpha). \forall x \in \beta \exists y \in \gamma. (x, y) \in j(f) \tag{9.1.11}$$

Namely take γ to be $\alpha \in j(\alpha) \in M$, as by transitivity $\alpha \in M$. So in our background model, there is such a $\gamma \in \alpha$. □

In fact, we can even prove:

Proposition 9.5. *($j : V \overset{\equiv}{\hookrightarrow} M$) If there is an α with $\Phi_1(\alpha)$, then it is weakly regular.*

Proof. Let $\beta \in \alpha$ and $R : \beta \rightrightarrows \alpha$. Then for all $x, y \in \alpha$:

$$(x, y) \in R \to (j(x), j(y)) = (x, y) \in j(R) \tag{9.1.12}$$

so $R \subseteq j(R)$. Thus as

$$\forall x \in \beta \exists y \in \alpha.(x, y) \in R \tag{9.1.13}$$

we also have

$$\forall x \in \beta \exists y \in \alpha.(x, y) \in j(R) \tag{9.1.14}$$

Thus $\beta = j(\beta)$ implies that setting $\gamma := \alpha$ supplies a witness for

$$M \vDash \exists \gamma \in j(\alpha).\forall x \in j(\beta) \exists y \in \gamma.(x, y) \in j(R) \tag{9.1.15}$$

So by elementarity there is also such an element $\gamma \in \alpha$ with $R : \beta \rightrightarrows \gamma$. □

9.1.2 IEA for Critical Points with more Traction

To obtain union-functional regularity, a new proof idea is needed, and it involves the existence of large sets. We want to prove that if there is an α with $\Phi_1(\alpha)$, then there is an α' with $\Phi_1(\alpha')$ such that α' is union-functionally regular.

For this, define a function which imbues an ordinal α with union-functionally regular properties. The following definition will usually be considered for $\alpha \in \mathrm{O_n}$, I an inaccessible set and $i \in I + 1$. However, for technical reasons we impose no such restriction in the definition.

Definition 9.6. *Let α, i and I be sets. Define the expression α_i^I recursively as*

$$\alpha_i^I = \alpha \cup \bigcup_{k \in i} \{ \bigcup_{x \in \beta} f(x) | \beta \in \alpha_k^I, f : \beta \to \alpha_k^I \} \cup \bigcup_{\beta \in \alpha} \beta_I^I \tag{9.1.16}$$

Note that this is Δ_0-definable, so by Lemma 2.19 always $\alpha_i^I = b \leftrightarrow (j(\alpha))_{j(i)}^{j(I)} = j(b)$ or written more directly:

$$j(\alpha_i^I) = j(\alpha)_{j(i)}^{j(I)} \tag{9.1.17}$$

Lemma 9.7. *Let I be inaccessible. Then:*

1. *For all $a, b, c \in I$, also $a_b^c \in I$.*

2. *For all $a \in I$ it holds that $a_I^I \subseteq I$.*

3. *For all a it holds that $a_I^I = \bigcup_{i \in I} a_i^I$*

Proof. These are straightforward inductions over the definition of a_b^c since inaccessibles are closed under the set theoretic functions used in the definition. □

If the indices extend an inaccessible, α_I^J does not really depend on them:

Proposition 9.8. *Let I, J be α-inaccessible sets, $A \supseteq I$ and $B \supseteq J$ and $\alpha \in I \cap J$. Then the following hold:*

1. $\forall i.\alpha_i^A = \alpha_i^B$

2. $\forall \beta \in \alpha.\beta_A^A \in B$

3. $\forall i.\alpha_i^A \subseteq \alpha_J^J \subseteq \alpha_B^B$

4. $\alpha_A^A = \alpha_B^B$

Proof. The proof of this conjunction proceeds by induction on α with a side induction on i for the the first and third claims separately. All four conjuncts are proved simultaneously and uniformly in I, J, A, B (i.e. the formula used for induction contains the universal quantification over I, J, A and B rather than treat them as parameters).

1. Let $\beta \in \alpha_i^A$. Then one of the following three cases holds by Definition 9.6:

   (a) $\beta \in \alpha$. In this case also $\beta \in \alpha_B^B$ by Definition 9.6.

(b) For some $k \in i$, $\gamma \in \alpha_k^A$ and $f : \gamma \to \alpha_k^A$, $\beta = f''\gamma$. By induction hypothesis on i also $\gamma \in \alpha_k^B$ and $f : \gamma \to \alpha_k^B$, so $\beta \in \alpha_i^B$.

(c) For some $\gamma \in \alpha$, $\beta \in \gamma_A^A$. By the fourth part of the induction hypothesis for α, we have $\gamma_A^A = \gamma_B^B$, so $\beta \in \alpha_i^B$.

So $\alpha_i^A \subseteq \alpha_i^B$, and analogously $\alpha_i^A \supseteq \alpha_i^B$.

2. Let $\beta \in \alpha$. By transitivity of J, it holds that $\beta \in J$. By J being α-inaccessible and $\beta \in \alpha$, there is some $K \in J$ which is β-inaccessible and $\beta \in K$. Then by the fourth part of the induction hypothesis, $\beta_A^A = \beta_K^K$. And as β and K are both in J, also $\beta_K^K \in J \subseteq B$ by the first part of Lemma 9.7.

3. The second inequality $\alpha_J^J \subseteq \alpha_B^B$ follows directly from monotonicity of the definition and $J \subseteq B$. For the first one, let $\beta \in \alpha_i^A$. Then one of the following three cases holds by Definition 9.6:

   (a) $\beta \in \alpha$. In this case also $\beta \in \alpha_J^J$ by Definition 9.6.

   (b) For some $k \in i$, $\gamma \in \alpha_k^A$ and $f : \gamma \to \alpha_k^A$, $\beta = f''\gamma$. By induction hypothesis on i also $\gamma \in \alpha_J^J$ and $f : \gamma \to \alpha_J^J$, so by the second and third part of Lemma 9.7, $\gamma \in J$ and for every $\delta \in \gamma$, there is a $j \in J$ such that $f(\delta) \in \alpha_j^J$. Collecting these j in a set $j_0 \in J$ and choosing $j_2 \in J$ such that $j_0 \in j_2$ and $\gamma \in \alpha_{j_1}^J$ for some $j_1 \in j_2$, one sees that $f''\gamma \in \alpha_{j_2}^J \subseteq \alpha_J^J$.

   (c) For some $\gamma \in \alpha$, $\beta \in \gamma_A^A$. By the fourth part of the induction hypothesis for α (note that the induction hypothesis applies to any superset of J, in particular to J itself), we have $\gamma_A^A = \gamma_J^J$, so $\beta \in \alpha_J^J$.

   In each case, $\beta \in \alpha_J^J$.

4. The previous part implies that

$$\alpha_A^A \subseteq \bigcup_i \alpha_i^A \subseteq \alpha_B^B \tag{9.1.18}$$

   Symmetrically, also $\alpha_B^B \subseteq \alpha_A^A$ and thus they are equal.

□

The embedding j does not necessarily always map inaccessibles to inaccessibles, although it does so if for example $j : V \overset{\equiv}{\hookrightarrow} V$. The reason for this is that ($M \models j(I)$ regular) does not imply that $j(I)$ contains total subrelations for $R : a \rightrightarrows j(I)$ with $a \in j(I)$ if $R \notin M$.

Definition 9.9. *Call an α-inaccessible I **j-α-inaccessible** if $j(I)$ is α-inaccessible as well.*

*Let j**IEA** be the statement that for every α, each set is contained in a j-α-inaccessible.*

Note that $j : V \overset{\equiv}{\hookrightarrow} M + j$IEA is a conservative extension of CZF+"For every α every set is contained in an α-inaccessible" and in case $j : V \overset{\equiv}{\hookrightarrow} V$, IEA is a consequence of Axiom M (in fact, $j : V \overset{\equiv}{\hookrightarrow} V$ + Axiom M proves that there are stationary many models of CZF$_2$ with $j : V \overset{\equiv}{\hookrightarrow} M + j$IEA).

Corollary 9.10. *Let I be j-$j(\alpha)$-inaccessible. Then $j(\alpha^I_i) = j(\alpha)^I_{j(i)}$ and $j(\alpha^I_I) = j(\alpha)^I_I$*

Proof. The first equation is implied directly by Proposition 9.8 and Equation 9.1.17. □

We want to show that if α is a critical point, α^I_i is as well, at least under certain favourable circumstances. This consists of two parts: Proving that j does not move anything below α^I_i, and proving that it moves α^I_i above itself. The first part is the following lemma:

Lemma 9.11.

$$\forall i \forall \alpha \in \boldsymbol{O}_n.(\forall \beta \in \alpha . j(\beta) = \beta) \rightarrow (\forall \beta \in \alpha^I_i . j(\beta) = \beta) \tag{9.1.19}$$

Proof. The proof proceeds by induction on α with side induction on i. Let $\beta \in \alpha^I_i$. Then one of the following three cases holds by definition of α^I_i:

1. $\beta \in \alpha$. In this case $j(\beta) = \beta$ by the antecedent.

2. For some $k \in i$, $\gamma \in \alpha^I_k$ and $f : \gamma \to \alpha^I_k$, $\beta = f''\gamma$. The induction hypothesis on i implies $j(\gamma) = \gamma$. Thus $j(f) : \gamma \to j(\alpha^I_k)$. If $(\delta, \delta') \in f$, then by elementarity $(j(\delta), j(\delta')) \in j(f)$, but by induction hypothesis $j(\delta) = \delta$ and $j(\delta') = \delta'$. So $f \subseteq j(f)$ and as they have the same domain, they are equal. In particular $\beta = f''\gamma = j(f)''j(\gamma) = j(\beta)$.

3. For some $\gamma \in \alpha$, $\beta \in \gamma^I_I$. By induction hypothesis on α, $j(\beta) = \beta$.

□

So if α is a critical point, then nothing below α^I_i gets moved. But α^I_i itself does:

Lemma 9.12. *Let I be j-$j(\alpha)$-inaccessible with $j(\alpha) \in I$. Then*

$$\alpha \in j(\alpha) \to \alpha^I_I \in j(\alpha^I_I) \tag{9.1.20}$$

Proof. $\alpha \in j(\alpha)$, so $\alpha^I_I \in j(\alpha)^I_I$ by definition. But by Proposition 9.8, $j(\alpha)^I_I = j(\alpha)^{j(I)}_{j(I)} = j(\alpha^I_I)$. □

Now we have accumulated enough facts to prove the main theorem:

Theorem 9.13. *($j : V \overset{\equiv}{\hookrightarrow} M$ + jIEA)*

$$\exists x \Phi_1(x) \to \exists x \Phi_{1.5}(x) \tag{9.1.21}$$

Proof. Assume $\exists x.\Phi_1(x)$. By Theorem 9.1 let α be an ordinal with $\Phi_1(\alpha)$ and by jIEA let I be j-$j(\alpha)$-inaccessible with $\alpha \in I$. By Lemma 9.11, j acts as the identity on α^I_I, while by Lemma 9.12, $\alpha^I_I \in j(\alpha^I_I)$. Thus $\Phi_{1.5}(\alpha^I_I)$. □

Corollary 9.14. *($j : V \overset{\equiv}{\hookrightarrow} V$ + Axiom M)*

$$\exists x \Phi_1(x) \to \exists x \Phi_{1.5}(x) \tag{9.1.22}$$

9.1.3 The Modified von Neumann Hierarchy for more Closure

In Definition 9.6 we gave a construction for steps to obtain a set α_I^I which extends α and is closed under $\beta \mapsto \beta_I^I$ and union-functionality. The proof goes through exactly the same way if instead of just closing under union-functionality, we also add another Δ_0-definable functional closure property, as long as this definition can be carried out inside an inaccessible (i.e. $(\alpha, a, b) \mapsto \alpha_a^b$ is reflected in every inaccessible — note that the α-inaccessibility was only needed for the part that α_i^J contains all the β_J^J with $\beta \in \alpha$). For example, we could have used the following property:

Definition 9.15. *Call a union-functionally regular set α with $\alpha \in \omega$ **proto-inaccessible**, if the following hold for all $\beta \in \alpha$:*

1. $\forall a, b, c \in \hat{V}_\beta . [\![c \in a \cap b]\!], [\![c \in \{a, b\}]\!], [\![c \in \bigcup a]\!] \in \alpha$
2. $\forall a, b, c \in \hat{V}_\beta . [\![c \in {}^{a}b]\!] \in \alpha \wedge \forall f. f \in {}^{a}b \to [\![c \in f]\!] \in \alpha$
3. $\forall a, b \in \hat{V}_\beta \forall f : {}^{a}b \to \alpha. \bigcup_{x \in {}^{a}b} f(x) \in \alpha$
4. $\forall a \in \hat{V}_\beta \forall f : a \to \alpha \bigcup_{x \in a} f(x) \in \alpha$
5. $\beta + 1 \in \alpha$

Then the same proofs go through (always noting that inaccessible sets are closed under the relevant functions of the modified von Neumann hierarchy), arriving at:

Theorem 9.16. *($j : V \overset{\equiv}{\hookrightarrow} M$ + jIEA)*

$$\exists x \Phi_1(x) \to \exists x \Phi_{1.5'}(x) \tag{9.1.23}$$

Up until now all properties considered for critical points (i.e. Φ_1, $\Phi_{1'}$, $\Phi_{1.5}$ and $\Phi_{1.5'}$) were consistent with being an ordinal. However, no ordinal can ever be inaccessible or explicitly inaccessible (i.e. fulfill Φ_2 or $\Phi_{2'}$). To return from the realm of ordinals to the realm large sets, we use the modified von Neumann hierarchy.

Lemma 9.17. *Let α be proto-inaccessible. Then $\hat{V}_\alpha$ is explicitly inaccessible.*

Proof. First note that $\omega \in \hat{V}_\alpha$ as $\omega \in \hat{V}_{\omega+1}$: The set $\hat{V}_\omega$ is the set of herediterily finite sets, and it is discrete by a direct induction over $\hat{\mathrm{rk}}$. The natural numbers $\omega \subseteq \hat{V}_\omega$ form a discrete subset and thus ω is contained in $\mathcal{MP}(\hat{V}_\omega)$.

Closure under pairing, union and binary intersection are completely analogous to each other, so we will only present the first. We will use that α is closed under binary unions by union-functionally regularity, for indeed if $\beta, \gamma \in \alpha$, then the union of the function range of $f : 2 \to \alpha$ with $0 \mapsto \beta, 1 \mapsto \gamma$ is in α as $2 \in \hat{V}_3$ and $3 \in \omega \in \alpha$. This union is just $\beta \cup \gamma$.

Now let $a, b \in \hat{V}_\alpha$, so by α being closed under successors, there is some $\beta \in \alpha$ with $a, b \in \hat{V}_\beta$. Then by the first part of Definition 9.15, there is a function f fulfilling

$$f_0 : \hat{V}_\beta \to \alpha,\ c \mapsto [\![c \in \{a, b\}]\!] + 1 \tag{9.1.24}$$

As $\hat{V}_\beta \in \hat{V}_{\beta+1}$ and $\beta + 1 \in \alpha$, by Part 4 of Defintion 9.15 and α being closed under binary unions, the following set is in α as well.

$$\beta_0 := \beta \cup \bigcup_{x \in \hat{V}_\beta} f_0(x) \in \alpha \tag{9.1.25}$$

Similarly, we can define recursively more functions $(f_i)_{i\in\omega}$ and elements $(\beta_i)_{i\in\omega} \in \alpha$ by setting for all $i \in \omega$

$$f_{i+1} : \hat{V}_{\beta_i} \to \alpha,\ f_{i+1}(c) := [\![c \in \{a, b\}]\!] + 1 \tag{9.1.26}$$

$$\beta_{i+1} := \beta_i \cup \bigcup_{x \in \hat{V}_{\beta_i}} f_i(x) \in \alpha \tag{9.1.27}$$

As usual, $\{a, b\} \in \hat{V}_{(\bigcup_{i\in\omega} \beta_i)+1}$ and $(\bigcup_{i\in\omega} \beta_i) + 1 \in \alpha$ as $\omega \in \alpha$.

The remaining desiderata, i.e. the modeling of Replacement and Exponentiation, share the same basic proof idea with a few subtleties added. For Replacement, let $a \in \hat{V}_\alpha$, i.e.

$a \in \hat{V}_\beta$ for some $\beta \in \alpha$ and let $f : a \to \hat{V}_\alpha$. While for all sets of the form $f(x)$ there is some $\gamma \in \alpha$ with $f(x) \in \hat{V}_\gamma$, this alone does not help us: To apply Part 4 of Definition 9.15, we need a functional relationship, i.e. a canonical γ and not just any γ. We get this from the following claim:

$$\forall a \in \hat{V}_\alpha . \hat{\mathrm{rk}}(a) \in \alpha \tag{9.1.28}$$

We show $\forall \beta \in \alpha \forall a \in \mathcal{MP}(\hat{V}_\beta).\hat{\mathrm{rk}}(a) \in \alpha$ by induction over β. So let all $\gamma \in \beta$ have the desired property and let $a \in \mathcal{MP}(\hat{V}_\beta)$. In particular, $a \in \hat{V}_{\beta+1}$ and $\beta + 1 \in \alpha$. As by induction hypothesis $\hat{\mathrm{rk}} : a \to \alpha$, the set $\beta_0 \in \alpha$ where

$$\beta_0 := \bigcup_{x \in a} \hat{\mathrm{rk}}(x) + 1 \tag{9.1.29}$$

Then $a \subseteq \hat{V}_{\beta_0}$ and for all $b \in \hat{V}_{\beta_0}$ the truth value $[\![b \in a]\!]$ is an element of α as

$$[\![b \in a]\!] = [\![b \in \{a, a\}]\!] \in \alpha \tag{9.1.30}$$

So this defines a function from $\hat{V}_{\beta_0} \in \hat{V}_{\beta_0+1}$ to α, thus the following set is also in α by Part 4 of Definition 9.15

$$\beta_1 := \beta_0 \cup \bigcup_{x \in \hat{V}_{\beta_0}} [\![x \in a]\!] + 1 \in \alpha \tag{9.1.31}$$

In fact, we can repeat the same argument to define recursively for $i \in \omega$:

$$\beta_{i+1} := \beta_i \cup \bigcup_{x \in \hat{V}_{\beta_i}} [\![x \in a]\!] + 1 \in \alpha \tag{9.1.32}$$

Since $\omega \in \alpha$, the union of all these $(\beta_i)_{i \in \omega}$ is an element of α — but this is just $\hat{\mathrm{rk}}(a)$. This concludes the demonstration of Claim 9.1.28.

Returning to the proof of Replacement, let $a \in \hat{V}_\beta$ for some $\beta \in \alpha$ and let $f : a \to \hat{V}_\alpha$, then by Equation 9.1.28

$$\hat{\mathrm{rk}} \circ f : a \to \alpha \tag{9.1.33}$$

And so by Part 4 of Definition 9.15 there is a $\gamma_0 \in \alpha$ such that $f''a \subseteq \hat{V}_{\gamma_0}$. We use the usual definition for $i \in \omega$:

$$\gamma_{i+1} := \gamma_i \bigcup_{x \in \hat{V}_{\gamma_i}} [\![x \in f''a]\!] + 1 \tag{9.1.34}$$

Then recursively using the second part of Definition 9.15, this is a sequence of ordinals in α and their union serves as a witness for $\exists \gamma \in \alpha . f''a \in \hat{V}_\gamma$.

For Exponentiation, first note that by Replacement all functions from a to b are in V_α as long as a and b are. By the third part of the definition their ranks can be collected into an ordinal in α and the second part allows the usual recursive construction of a $\gamma \in \alpha$ with ${}^{a}b \in \hat{V}_\gamma$. □

If $j : V \overset{\equiv}{\hookrightarrow} M$ and α fulfills $\Phi_{1.5'}(\alpha)$, i.e. α is a proto-inaccessible critical point of j, then as $j(\hat{V}_\alpha) = \hat{V}_{j(\alpha)}$ and $\alpha \in j(\alpha)$ we have

$$\hat{V}_\alpha \in j(\hat{V}_\alpha) \tag{9.1.35}$$

Furthermore, recursion over β yields that for all $x \in \hat{V}_\beta$ for $\beta \in \alpha$, we have $j(x) = x$ as x can be retrieved from a function whose domain and codomain are not moved by j by induction hypothesis. Putting this together with Lemma 9.17, we conclude:

Theorem 9.18. *($j : V \overset{\equiv}{\hookrightarrow} M$ + jIEA)*

$$\exists x \Phi_1(x) \to \exists x \Phi_{2'}(x) \tag{9.1.36}$$

Corollary 9.19. *($j : V \overset{\equiv}{\hookrightarrow} V$ + Axiom M)*

$$\exists x \Phi_1(x) \to \exists x \Phi_{2'}(x) \tag{9.1.37}$$

This concludes the proof of the claims made in Figure 9.1.

The step from explicit inaccessibility to full inaccessibility requires something more and can at least on the face of it not be taken without a non-deterministic induction, as there does not appear to be a direct way to specify the sets which need to be added due to Fullness. The results from Section 7.3 however suggest that this might possibly be carried out in a theory like $j : V \overset{\equiv}{\hookrightarrow} V$ + Axiom M.

9.2 The Strength of Restricted Elementary Embeddings

Instead of considering different requirements on the critical point, we could also consider restrictions to the elementarity scheme, for example a restriction to Δ_0 formulae. This makes sense as preliminary research suggests that models for these might be constructed more easily than models with a full elementary embedding, and as it turns out they still carry a large part of the strength associated with measurable sets.

In this section, let K be an inaccessible set and $j : V \to V$ a Δ_0-elementary embedding with critical point K, i.e.

$$K \in j(K) \ \wedge \ \forall x \in K. j(x) = x \tag{9.2.38}$$

and for all bounded formulae $\Phi(\overrightarrow{a})$ with all free variables displayed and without j or K included in the language of Φ (although sets like K or $j(K)$ might be parameters of course), we have

$$\Phi(\overrightarrow{a}) \leftrightarrow \Phi(\overrightarrow{j(a)}) \tag{9.2.39}$$

We also assume that the axiom schemes of set theory still hold when the formulae in them contain the symbols j or K.

Note that without extra assumptions like j being Σ_1 or j being cofinal, this is equivalent to a Δ_0 elementary embedding $j : V \to M$ into some transitive M. On the other hand, if j is cofinal into some M, then classical logic would imply j to be a full elementary embedding $V \to M$ and even constructively it would be an embedding for a significantly larger class of formulae than Δ_0:

Remark 9.20. *If j is cofinal in the sense*[1] *that $\forall x \in M \exists x_0. x \in j(x_0)$ then j is a Σ_1-elementary*[2] *embedding. Elementarity also holds for all universally quantified Σ_1-statements.*

[1]This concept will later be called "set cofinal"

[2]Here, the Σ_1-formulae are the smallest class of formulae containing all bounded formulae and being closed under bounded quantification, $\vee$, $\wedge$ and unbounded existential quantification.

Proof. Take a Δ_0 formula $\Phi(x, \overrightarrow{a})$.

If $\exists x.\Phi(x, \overrightarrow{a})$ then by elementarity also $\exists x \in M.\Phi(x, \overrightarrow{j(a)})$, namely just apply j to the witness. If on the other hand $\exists x \in M.\Phi(x, \overrightarrow{j(a)})$ then for an appropriate x_0, we have $\exists x \in j(x_0).\Phi(x, \overrightarrow{j(a)})$ and thus $\exists x \in x_0.\Phi(x, \overrightarrow{a})$, and in particular $\exists x.\Phi(x, \overrightarrow{a})$.

Thus j is a Σ_1-elementary embedding, because by virtue of Strong Collection, all Σ_1 formulae can be written in the form $\exists x.\Phi(x, \overrightarrow{a})$.

Now let $\Phi(x, \overrightarrow{a})$ be a Σ_1 formula. If $\forall x.\Phi(x, \overrightarrow{a})$, then take an arbitrary $y \in M$ and a y_0 with $y \in j(y_0)$. We have $\forall x \in y_0.\Phi(x, \overrightarrow{a})$ which is a Σ_1 formula, so by elementarity also $\forall x \in j(y_0).\Phi(x, \overrightarrow{j(a)})$, but that implies that in particular $\Phi(x, \overrightarrow{j(y)})$. As y was an arbitrary element of M we can conclude $\forall x \in M.\Phi(x, j(\overrightarrow{a}))$. If on the other hand $\forall x \in M.\Phi(x, \overrightarrow{j(a)})$ then for all x also $\Phi(j(x), \overrightarrow{j(a)})$ and thus by elementarity $\forall x.\Phi(x, \overrightarrow{a})$. $\square$

Classically, the same argument can be repeated throughout the whole Lévy hierarchy, using the fact that in formulas of the form $\forall x \in b\exists y$ the existential can be pushed to the left via Collection (which also works in CZF) and that in formulas of the form $\exists x \in b\forall y$, the universal can also be pushed to the left (which requires classical logic on top of Collection).

9.2.1 Inaccessibility and Mahloness

We now want to prove that K fulfills many large set properties, starting with:

Proposition 9.21. *K is $\alpha-$inaccessible for all $\alpha \in K$.*

Being $\alpha-$inaccessible is not a Δ_0 concept. In fact, even being regular is not a Δ_0 concept as it needs a quantifier over a class of relations. Restricting this quantifier to a set would yield a Δ_0-concept and if the set is large enough, and that suffices.

Definition 9.22. *A transitive set A is called C-**bound regular**, if for all $R \in C$ and $a \in A$ such that $R : a \rightrightarrows A$ there is a $b \in A$ with $R : a \leftleftarrows\!\rightrightarrows b$.*

Lemma 9.23. *If A is regular, then A is C-bound regular.*

If A is C-bound regular and C is full in mv(a, A) for all $a \in A$ then A is actually regular.

Proof. If A is regular, then it is also C-bound regular because the definition of C-bound regular differs only by a restriction on a universal quantification.

On the other hand, let A be C-bound regular and C be full in mv(B, A) for all $B \in A$.

If $R : a \rightrightarrows A$ with $a \in A$, then there is some $R' \subseteq R$ with $R' \in C$, so by C-regularity there is a set $b \in A$ such that

$$\forall x \in B \exists y \in b.(x, y) \in R' \tag{9.2.40}$$

$$\forall y \in b \exists x \in a.(x, y) \in R' \tag{9.2.41}$$

Then by $R' \subseteq R$, also

$$\forall x \in a \exists y \in b.(x, y) \in R \tag{9.2.42}$$

$$\forall y \in b \exists x \in a.(x, y) \in R \tag{9.2.43}$$

Which means that b is as required by regularity. □

A similar idea works to approximate inaccessibility by a Δ_0 concept:

Definition 9.24. *A regular set A is called **C-bound inaccessible** if it is C-bound regular, $\omega \in A$, A is closed under union and binary intersection and models the Fullness axiom.*

*An C-bound inaccessible set A is called **C-bound α-inaccessible** if for all $\beta < \alpha$ and all $a \in A$, there is a $I \in A$ such that $a \in I$ and I is C-bound β-inaccessible.*

Unlike the requirement that A should be closed under fullness, i.e. $\forall a, b \in A \exists c \in A$ with c full in mv(a, b), the formally weaker requirement that A models Fullness is Δ_0, so that the whole definition is Δ_0 with C as parameter. As is the definition of C-bound α-inaccessibility with C and α as parameter.

Lemma 9.25. *If A is α-inaccessible, then A is C-bound α-inaccessible.*

If A is C-bound α-inaccessible and C is full in mv(a, A) for all $a \in A$, then A is actually α-inaccessible.

Proof. Again the first part is immediate. For the second statement, proceed by induction over α. Let A be C-bound α-inaccessible.

First it needs to be shown that A is actually inaccessible, not only C-bound inaccessible. This follows from Lemma 9.23, as it implies that A is regular, and as a regular set containing ω, modelling fullness and closed under union, binary intersection and full sets, it is inaccessible.

Now let $\beta < \alpha$ and $a \in A$. There is a C-bound β-inaccessible set containing a in A and by induction hypothesis, this set is actually β-inaccessible, which is what we need. □

Now we are able to prove Proposition 9.21.

Proof of Proposition 9.21. By Collection, find a set C such that C is full in mv(a, K) for all $a \in K$. We want to show that K is C-bound α-inaccessible for all $\alpha \in K$, then by the Lemma 9.25 we are done. We proceed by induction over α. So let $\beta < \alpha$, $a \in K$.

By induction hypothesis, we know that

$$\exists x \in j(K).j(a) \in x \wedge x \text{ is } j(C)-\text{bound } j(\beta) - \text{inaccessible}) \tag{9.2.44}$$

For K itself is a witness for this, keeping in mind that $j(a) = a \in K$ and that K is truly $j(\beta) = \beta$-inaccessible by induction hypothesis, so by Lemma 9.25, it is also $j(C)$-bound $j(\beta)$-inaccessible. Then by Δ_0-elementarity and owing to $\beta = j(\beta)$, it follows that

$$\exists x \in K.a \in x \wedge x \text{ is } C-\text{bound } \beta - \text{inaccessible}) \tag{9.2.45}$$

But as for a witness x of this statement, C is full in mv(y, K) for all $y \in K$, so in particular also for all $y \in x$ by transitivity of K. Thus we can conclude by Lemma 9.25 that this x is actually β-inaccessible. □

To tackle Mahloness, we want to bound that concept as well.

Definition 9.26. *A C-bound inaccessible set A is called C**-bound** α**-Mahlo**, if for all $\beta \in \alpha$, whenever $R : A \rightrightarrows A$ for some $R \in C$, there is a C-bound β-Mahlo set $B \in A$ with $R : B \rightrightarrows B$.*

Lemma 9.27. *If A is α-Mahlo, then A is C-bound α-Mahlo.*

If A is C-bound α-Mahlo and C is full in mv(A, A) and in mv(a, A) for all $a \in A$, then A is actually α-Mahlo.

Proof. Again, the first statement is trivial and the second is proved by induction over α. So let A be C-bound α-Mahlo and C be full in mv(A, A).

First note that A is actually inaccessible as it is C-bound inaccessible and C is sufficiently large.

To show the Mahlo-reflection, let $R : A \rightrightarrows A$ and $\beta \in \alpha$. Then $R \subseteq R'$ for some $R' \in C$ and $R' : A \rightrightarrows A$. Now by C-bound α-Mahloness, there is some C-bound β-Mahlo set $B \in A$ with $R' : B \rightrightarrows B$, so also $R : B \rightrightarrows B$ and by the induction hypothesis, B is actually β-Mahlo and thus as desired. $\square$

Proposition 9.28. *K is $\alpha-$Mahlo for all $\alpha \in K$.*

Proof. Let C be full in mv(a, K) for all $a \in K \cup \{K\}$. We want to show that K is C-bound α-Mahlo for all $\alpha \in K$, then by the previous lemma we are done. By the previous results K is C-bound inaccessible, but we still need to prove Mahlo-reflection. We proceed by induction over α. So let $\beta < \alpha$, $R : K \in K$ with $R \in C$.

We claim that K itself is a witness for

$$\exists a \in j(K).j(R) : a \rightrightarrows a \wedge a \text{ is } j(C)-\text{bounded } j(\beta)-\text{Mahlo} \tag{9.2.46}$$

This is because by induction hypothesis, K is $j(C)$-bounded $\beta-$Mahlo because it is truly $j(\beta) = \beta-$Mahlo and $j(R) : K \rightrightarrows K$. This follows from $R : K \rightrightarrows K$ and $j(R) \supseteq$

$R \cap K$ the latter of which holds for $(x, y) \in R \cap K$ implies $(x, y) = (j(x), j(y)) = j(x, y) \in j(R)$.

Reflecting this statement back to K and remembering that $\beta = j(\beta)$ yields

$$\exists a \in K.R : a \rightrightarrows a \wedge atextisC-\text{bounded } \beta-\text{Mahlo} \tag{9.2.47}$$

Thus K is C-bound α-Mahlo and thus actually α-Mahlo. $\square$

9.2.2 Beyond Mahlo

Next, we want to analyse 2-strongness. Recall that the main requirement of 2-strongness is of the form

$$\forall S.\Phi^K(x) \to \exists I \in K.\Phi^I(x) \wedge I \text{ inaccessible} \tag{9.2.48}$$

We want to handle this property one S at a time:

Definition 9.29. *1. An inaccessible set A is called S-**reflecting** if*

$$\forall R : A \rightrightarrows A \forall u \in A \exists x \in A, v \in A.x \subseteq R \wedge (v, u, x) \in S \tag{9.2.49}$$

implies

$$\forall R : B \rightrightarrows B \forall u \in B \exists x \in B, v \in B.x \subseteq R \wedge (v, u, x) \in S \tag{9.2.50}$$

*2. A C-bound inaccessible set A is called C-**bound** S-**reflecting** if*

$$\forall R \in C.R : A \rightrightarrows A \to \forall u \in A \exists x \in A, v \in A.x \subseteq R \wedge (v, u, x) \in S \tag{9.2.51}$$

implies that there is some C-bound inaccessible $B \in A$ with

$$\forall R \in C.R : B \rightrightarrows B \to \forall u \in B \exists x \in B, v \in B.x \subseteq R \wedge (v, u, x) \in S \tag{9.2.52}$$

Obviously if a set is S-reflecting for all its subsets S, then it is 2-strong, and vice versa (note that in particular it is inaccessible since there is always at least one subset, namely $\emptyset$).

The corresponding version of the usual lemma holds with slightly stronger requirements on one side, as now imposing a C-bound amounts to more than just decreasing the scope of the quantifiers of a Π_1 statement.

Lemma 9.30. *If A is S-reflecting and C is full in mv(A, A), then A is C-bound S-reflecting.*

If A is C-bound S-reflecting and C is full in mv(B, B') for all $B, B' \in A$, then A is actually S-reflecting.

Proof. For the first part, assume S-reflection and the antecedent of C-bound S-reflecting. Then the antecedent of S reflecting itself holds, as each $R : A \rightrightarrows A$ can be decreased to a $R' \subseteq R$ with $R' \in C, R' : A \rightrightarrows A$ and the same x and v that work for R' also work for $R \supseteq R' \supseteq x$. So the consequent of S-reflection holds for some inaccessible $B \in A$, and thus also the consequent of C-bound S-reflection, and A is C-bound S-reflecting.

For the second part, assume C-bound S-reflection and the antecendent of S-reflection. Then obviously the antecedent of C-bound S-reflecting holds as well, thus so does the consequent of C-bound S-reflection for some C-bound inaccessible $B \in A$. This B also witnesses the consequent of S-reflection, for let $R : B \rightrightarrows B$, then there is some $R' \subseteq R$ with $R' \in C$ and the x, v for that R' also work for $R \supseteq R' \supseteq x$. Also B is truly inaccessible as C is full in mv(B, B) and in all mv(b, B) for $b \in B$ by condition. □

Proposition 9.31. *K is 2-strong.*

Proof. We will show that K is S-reflecting for all $S \subseteq K^3$. By the lemma before, we only need to show that it is C-bound S-reflecting for some fixed C that is full in mv(K, K) and mv(B, B') for all $B, B' \in K$, easily obtained by Collection.

So let

$$\forall R \in C.R : K \rightrightarrows K \to \forall u \in K \exists x \in K, v \in K.x \subseteq R \wedge (v, u, x) \in S \tag{9.2.53}$$

Note that $S \subseteq j(S)$ as $x \in S \to x = j(x) \in j(S)$.

So K is a witness for $\exists B \in j(K)$ such that B is C-bound inaccessible and

$$\forall R \in j(C).R : B \rightrightarrows B \to \forall u \in B \exists x \in B, v \in B.x \subseteq R \wedge (v, u, x) \in j(S) \quad (9.2.54)$$

For any such $R : K \rightrightarrows K$ can be trimmed down to an $R \in C$ with $R : K \rightrightarrows K, R \subseteq R'$ and the x that works for R' also works for $R \supset R' \supset x$.

So by Δ_0-elementarity, there is a $B \in K$ which is C-bound inaccessible and

$$\forall R \in C.R : B \rightrightarrows B \to \forall u \in B \exists x \in B, v \in B.x \subseteq R \wedge (v, u, x) \in S \quad (9.2.55)$$

Which yields C-bound S-reflection. □

For the previous proposition, it sufficed to handle each S separately, but to tackle larger properties like the existence of unboundedly many 2-strong sets below the critical point, we must deal with all of them simultaneously.

Definition 9.32. *For C-bound inaccessible A, let A be C-**bound 2-strong** if for all $S \in C$, whenever*

$$\forall R : A \rightrightarrows A.R \in C \to \forall u \in A \exists x \in A, v \in A.x \subseteq R \wedge (v, u, x) \in S \quad (9.2.56)$$

this property is reflected in some C-bound inaccessible $B \in A$, i.e. $\exists B$ inaccessible and

$$\forall R : B \rightrightarrows B.R \in C \to \forall u \in B \exists x \in B, v \in B.x \subseteq R \wedge (v, u, x) \in S \quad (9.2.57)$$

The demands on the bound C are now stronger than before:

Lemma 9.33. *If A is 2-strong and C is full in mv(A, A), then A is C-bound 2-strong.*

If A is C-bound 2-strong and C is full in mv(A, A) and in mv$(D \times A, A \times A)$ for some $D \subseteq C \cap$ mv(A, A) full in mv(A, A) and closed under projections, and C is full in mv(a, A) for all $a \in A$, then A is actually 2-strong.

Proof. The first statement follows from the previous lemma, as being 2-strong means being S-reflecting for all S, that a forteriori means S-reflecting for all $S \in C$ and that implies being C-bound S-reflecting for all $S \in C$, which is the same as being C-bound 2-strong.

For the second statement, let A be C-bound 2-strong and C be as specified, including full in mv(A, A) and in mv$(D \times A, A \times A)$ for some $D \subseteq C \cap \text{mv}(A, A)$ full in mv(A, A). Take $S \subseteq A^3$, we need to show that A is S-reflecting. So assume that

$$\forall R : A \rightrightarrows A.\forall u \in A \exists x \in A, v \in A.x \subseteq R \wedge (v, u, x) \in S \tag{9.2.58}$$

This can be used to define a total relation

$$Q : (D) \times A \rightrightarrows A \times A \tag{9.2.59}$$

by mapping an R and u to an x and v such that $(v, u, x) \in S$. Trim this to a $Q' \subseteq Q$ with $Q' \in C$ having the same property. So by C-bound 2-strongness applied to the projection of Q' on the last three components, we get a C-bound inaccessible (and thus inaccessible) $B \in A$ with the desired property. □

Proposition 9.34. *K contains set unboundedly many 2-strong sets below it, i.e. all elements of K lie in another element of K that is 2-strong.*

Proof. Let $a \in K$ and C be as in the last lemma, we need to find a C-bound 2-strong A such that $a \in A \in K$. But K itself is a witness for

$$\exists A \in j(K).j(a) \in A \wedge A \text{ is } j(C)-\text{bound } 2-\text{strong} \tag{9.2.60}$$

by the previous lemma as $j(C)$ is full in mv(A, A) and A is actually 2-strong and $j(a) = a \in K$. Then by elementarity,

$$\exists A \in K.a \in A \wedge A \text{ is } C-\text{bound } 2-\text{strong} \tag{9.2.61}$$

This A is then actually 2-strong by the previous lemma. □

The methods presented here then extend to K containing stationary many 2-strong sets below it, K being (2-strong)$^+$, where (2-strong)$^+$ is the same as 2-strong but with the occurrance of inaccessibility in its definition replaced by 2-strongness itself again, and many more similar concepts.

9.3 Cofinality of Elementary Embeddings

In the context of classical ZF set theory, elementary embeddings $j : V \to M$ into a transitive model M possess an important property: Cofinality. If the codomain of the embedding is V itself, as in the context of a Reinhardt set, a particularly strong version of cofinality is fulfilled.

Definition 9.35. *Let M be a transitive model of set theory.*

1. *A mapping $j : V \to M$ is called **cofinal** if for every ordinal $\alpha \in O_n \cap M$, there exists $\beta \in V$ with $\alpha \in j(\beta)$.*
2. *A mapping $j : V \to M$ is called **strongly cofinal** if for every ordinal $\alpha \in O_n$, there exists $\beta \in V$ with $\alpha \in j(\beta)$.*
3. *A mapping $j : V \to M$ is called **set cofinal** if for every set $a \in M$, there exists $b \in V$ with $a \in j(b)$.*
4. *A mapping $j : V \to M$ is called **strongly set cofinal** if for every set $a \in V$, there exists $b \in V$ with $a \in j(b)$.*

The transitivity of M lets us characterize directly how much stronger the concepts of strong cofinality actually are:

Remark 9.36. *Let $j : V \to M$ be a mapping of V into a transitive model of set theory. Then:*

1. *The mapping j is strongly cofinal if and only if and M contains all ordinals, i.e. $M \cap O_n = O_n$, and j is cofinal.*

2. *The mapping j is strongly set cofinal if and only if and M contains all sets, i.e. $M = V$, and j is set cofinal.*

Now we state the main theorem for this section. The classical side of it is of course already known:

Theorem 9.37. *1. In the theory ZFC, all elementary embeddings $j : V \to M$ are cofinal, strongly cofinal and set cofinal, but can only be strongly set cofinal if j is the trivial identity map. In particular, the elementary embedding associated to a measurable cardinal can never be strongly set cofinal.*

2. In the theory CZF, all elementary embeddings $j : V \to V$ are cofinal, set cofinal, strongly cofinal and strongly set cofinal.

The constructive part will be proved in the following subsections. The classical part relies on the following lemma which can be found and is proved e.g. in [Kun80]:

Lemma 9.38. *(ZF) If $j : V \to M$ is an elementary embedding, then for all $\alpha \in O_n$ it follows that $\alpha \subseteq j(\alpha)$.*

Proof. By induction over α, using that the induction hypothesis implies $\forall \beta \in \alpha . \beta \in j(\beta) \in j(\alpha) \vee \beta = j(\beta) \in j(\alpha)$. We used the excluded middle to conclude $\beta \in j(\beta) \vee \beta = j(\beta)$ from $\beta \subseteq j(\beta)$. ☐

This directly yields strong cofinality as then $\forall \alpha . \alpha \in j(\alpha + 1)$. For set cofinality note that for any $a \in M$ there is some α with $\mathrm{rk}(a) = \alpha$. As $rk(a) = b$ is a Δ_1 formula, it is absolute for M, so also $M \vDash \mathrm{rk}(a) = \alpha$. Using Lemma 9.38, we conclude that

$$M \vDash \mathrm{rk}(a) \in j(\alpha + 1) \tag{9.3.62}$$

Thus $a \in V^M_{j(\alpha+1)} = j(V_{\alpha+1})$ which is as required for set cofinality. Strong set cofinality of a nontrivial embedding $j : V \overset{\equiv}{\hookrightarrow} M$ however would imply the existence of a Reinhardt set which is incompatible with ZFC.

9.3.1 The Constructive Case

In the classical case, cofinality relied on $\alpha \in j(\alpha+1)$, but the proof of that relies on the trichotomy of ordinals, which is a principle to be avoided in CZF as it implies the principle of the excluded middle for bounded formulae [AR01]. This problem can not be repaired directly, as the statement $\alpha \in j(\alpha+1)$ itself implies the principle of the excluded middle for closed bounded formulae, provided that it is not completely vacuous, i.e. provided that the elementary embedding $j : V \to M$ is not trivial in the sense that it has a critical point.

Proposition 9.39. *Let M be transitive and $j : V \to M$ be an elementary embedding with critical point K, i.e. $K \in j(K)$ and $\forall a \in K.a = j(K)$.*

Then the statement $\forall \alpha \in O_n.\alpha \in j(\alpha+1)$ implies the principle of the excluded middle for bounded formulae, i.e. for each Δ_0 formula Φ, it implies $\Phi \vee \neg\Phi$.

Proof. Let $\Phi(\overrightarrow{x})$ be Δ_0 with all free variables displayed. Consider the ordinal

$$\alpha := \{\beta \in K | \beta \in \mathrm{O_n} \wedge \Phi(\overrightarrow{x})\} \tag{9.3.63}$$

Then by elementarity

$$j(\alpha) = \{\beta \in j(K) | \beta \in \mathrm{O_n} \wedge \Phi(\overrightarrow{j(x)})\} \tag{9.3.64}$$

So assuming $\alpha \in j(\alpha+1)$, we either have $\alpha \in j(\alpha)$ or $\alpha = j(\alpha)$. In the first case $j(\alpha)$ must be inhabited, which implies $\Phi(\overrightarrow{j(x)})$ which by elementarity implies $\Phi(\overrightarrow{x})$. In the second case, $\Phi(\overrightarrow{j(x)})$ can not be true, as otherwise $K \in j(\alpha)$ which would imply $K \in \alpha$ and thus the contradiction $K \in K$. So in the second case we have $\neg\Phi(\overrightarrow{j(x)})$ and so by elementarity $\neg\Phi(\overrightarrow{x})$.

In total we get $\Phi(\overrightarrow{x}) \vee \neg\Phi(\overrightarrow{x})$. $\square$

This seems to spell early doom for the endeavour of establishing strong cofinality in a constructive setting, but it turns out that this can still be achieved by a more complicated road.

9.3.2 Approximating Subset Relations

For this, we note that while the powerclass of a set can not expected to be a set again, we can approximate it with a set by not collecting all subsets into one class but only those where the question of elementhood has a sufficiently simple truth value.

Definition 9.40. *Let $\Omega \subseteq \mathcal{P}(1)$ be a class of truth values, Y be any class. Then define the class $\mathcal{P}_\Omega(Y)$ as*

$$\mathcal{P}_\Omega(Y) := \{X \subseteq Y | \forall y \in Y. [\![y \in X]\!] \in \Omega\} \tag{9.3.65}$$

Write $X \subseteq_\Omega Y$ for $X \in \mathcal{P}_\Omega(Y)$.

This concept approximates power classes in the following sense:

Remark 9.41. *1. The operation $\Omega \mapsto \mathcal{P}_\Omega(Y)$ is monotone in Ω. For $\Omega = \emptyset$ and inhabited Y, it returns $\emptyset$ and for $\Omega = \mathcal{P}(1)$ it returns $\mathcal{P}(Y)$.*

2. If Ω and Y are sets, then $\mathcal{P}_\Omega(Y)$ is a set.

Proof. The first part is immediate from the definition. The second follows from Exponentiation. To see this, consider the operation $I_1 : {}^Y\Omega \to \mathcal{P}(Y)$ defined by

$$I_1(f) = \{y \in Y | f(y) = 1\} \tag{9.3.66}$$

Then the range of this operation is included in $\mathcal{P}_\Omega(Y)$ and every set $X \subseteq_\Omega Y$ has the preimage $y \mapsto [\![y \in X]\!]$. So

$$\mathcal{P}_\Omega(Y) = \{I_1(f) | f \in {}^Y\Omega\} \tag{9.3.67}$$

□

We need to adapt this concept to a generalisation of the subset relation.

Definition 9.42. *Define the relation $a \subseteq\subseteq b$ recursively by*

$$a \subseteq\subseteq b :\leftrightarrow \forall x \in a \exists y \in b. x \subseteq\subseteq y \tag{9.3.68}$$

This is a reflexive and transitive binary relation.

Remark 9.43. *The principle of the restricted excluded middle is equivalent to the statement*

$$\forall \alpha, \beta \in \boldsymbol{O}_n . \alpha \subseteq\subseteq \beta \leftrightarrow \alpha \subseteq \beta \tag{9.3.69}$$

Proof. First note that by induction $\alpha = \beta$ always implies $\alpha \subseteq\subseteq \beta$. Thus $\alpha \subseteq \beta$ also always implies $\alpha \subseteq\subseteq \beta$. Also note that the restricted excluded middle implies trichotomy on the ordinals, i.e. that for all ordinals α, β we either have $\alpha \in \beta$, $\alpha = \beta$ or $\alpha \ni \beta$.

For the first direction, assume the restricted middle for bounded formulae and show that $\alpha \subseteq\subseteq \beta$ implies $\alpha \subseteq \beta$ by induction over β. Using trichotomy, either $\alpha \in \beta$ (in which case also $\alpha \subseteq \beta$) or $\alpha = \beta$ (in which case also $\alpha \subseteq \beta$) or $\alpha \ni \beta$. The latter case can not happen however, as then $\beta \in \alpha \subseteq\subseteq \beta$ would imply $\exists \gamma \in \beta . \beta \subseteq\subseteq \gamma$. By the induction hypothesis, $\beta \subseteq \gamma$, so in particular $\gamma \in \gamma$ which is a contradiction.

For the converse direction, let Φ be any Δ_0 formula and assume that $\alpha \subseteq\subseteq \beta$ always implies $\alpha \subseteq \beta$, so in particular this should be the case for

$$\alpha := \{\{0|\Phi\}\} \cup \{0|\Phi\} \tag{9.3.70}$$

and

$$\beta := 2 = \{\{0\}, 0\} \tag{9.3.71}$$

Indeed $\alpha \subseteq\subseteq \beta$, as it has at most two elements, and the first one is a subset of $\{0\} \in \beta$ and the second one a subset of $0 \in \beta$. However, if $\alpha \subseteq \beta$, then its one sure element $\{0|\Phi\}$ must be in β, so must either be equal to $\{0\}$, in which case Φ is true, or equal to 0, in which case Φ is false. In total, we obtain $\Phi \vee \neg\Phi$. $\square$

Collecting all sets which are in the $\subseteq\subseteq$ relation to a given class provides a generalisation of the powerset operation

Definition 9.44. *For a class Y, define the class $\mathcal{P}^{\subseteq\subseteq}(Y)$ to be*

$$\mathcal{P}^{\subseteq\subseteq}(Y) := \{X | X \subseteq\subseteq Y\} \tag{9.3.72}$$

Like the powerclass operation, we also intend to approximate this operation using truth values. For technical reasons (namely preserving a certain monotonicity property), we use not only a single set of truth values but a set of truth values for every von Neumann stage we might need (i.e. the elements of rk($\{Y\}$)).

Definition 9.45. *Let a, b be sets and $f : rk(b) + 1 \to \mathcal{P}(\mathcal{P}(1))$. Then define the relation $a \subseteq\subseteq_f b$ by recursion on b as the conjunction of the formulae*

$$\forall x \in a \exists y \in b. x \subseteq\subseteq_{f \restriction (rk(y)+1)} y \tag{9.3.73}$$

and

$$\forall y \in b \forall x \subseteq\subseteq_{f \restriction (rk(y)+1)} y. [\![x \in a]\!] \in f(rk(b)) \tag{9.3.74}$$

Define $\mathcal{P}_f^{\subseteq\subseteq}(b)$ as the class of all a with $a \subseteq\subseteq_f b$.

Note that as with $\mathcal{P}_\Omega$, this approximation is from below, i.e. by an obvious induction we have

$$\forall a, b, f. a \subseteq\subseteq_f b \to a \subseteq\subseteq b \tag{9.3.75}$$

The central facts about this construction are the following:

1. It approximates $\mathcal{P}^{\subseteq\subseteq}(b)$ with sets.
2. Every $\subseteq\subseteq$ relation is approximable by it.

Written more formally, we arrive at:

Proposition 9.46. *1. Let b be a set and $f : rk(b) + 1 \to \mathcal{P}(\mathcal{P}(1))$. Then $\mathcal{P}_f^{\subseteq\subseteq}(b)$ is a set.*

2. Let $a \subseteq\subseteq b$ be sets. Then there is an f such that $a \subseteq\subseteq_f b$.

Proof. 1. This is slightly more cumbersome than the proof of the corresponding statement about the approximation of the usual powerclass. The proof proceeds by set induction over b.

So assume by induction hypothesis that $\mathcal{P}^{\subseteq\subseteq}_{f\restriction(\mathrm{rk}(y)+1)}(y)$ are sets for all $y \in b$. Then so is

$$A := \bigcup_{y \in Y} \mathcal{P}^{\subseteq\subseteq}_{f\restriction(\mathrm{rk}(y)+1)}(y) \tag{9.3.76}$$

Define an operation

$$I_1 : \ {}^{A}f(\mathrm{rk}(b) \to \mathcal{P}^{\subseteq\subseteq}_{f}(b) \tag{9.3.77}$$

by setting

$$I_1(g) := \{x \in A | g(x) = 1\} \tag{9.3.78}$$

To see that this is well defined, let $g \in {}^{A}\Omega$. Then each element of $a := I_1(g)$ is in A, and thus

$$\forall x \in a \exists y \in b . x \subseteq\subseteq_{f\restriction(\mathrm{rk}(y)+1)} y \tag{9.3.79}$$

For any $y \in b$ and $x \subseteq\subseteq_{f\restriction(\mathrm{rk}(y)+1)} y$, we have $x \in A$, so $g(x)$ is defined here and $x \in X$ iff $g(x) = 1$. Thus $[\![x \in a]\!] = g(x) \in f(\mathrm{rk}(b))$. In other words

$$\forall y \in b \forall x \subseteq\subseteq_{f\restriction(\mathrm{rk}(y)+1)} y . [\![x \in a]\!] \in f(\mathrm{rk}(b)) \tag{9.3.80}$$

Together, 9.3.79 and 9.3.80 show that the operation $g \mapsto I_1(g)$ is well defined with codomain $\mathcal{P}^{\subseteq\subseteq}_{f}(b)$. We need to show that it is also surjective.

So let $a \subseteq\subseteq_f b$. Define a function $g : A \to f(\mathrm{rk}(b))$ by setting

$$g(x) := [\![x \in a]\!] \tag{9.3.81}$$

This is well defined because the second part of the definition of $a \subseteq\subseteq_f b$ implies that for all $x \in A$, the truth value $[\![x \in a]\!]$ is in $f(\mathrm{rk}(b))$. Then

$$I_1(g) = a \tag{9.3.82}$$

So

$$\mathcal{P}_f^{\subseteq\subseteq}(b) = \{I_1(g) | g \in {}^A f(\text{rk}(b))\} \tag{9.3.83}$$

Which is a set since A and $f(\text{rk}(b))$ are sets.

2. We first show that every relationship $a \subseteq\subseteq b$ has a skeleton set R, where we recursively call R a skeleton for $a \subseteq\subseteq b$ if $R : \text{tc}(a) \rightrightarrows \text{tc}(b)$ and for each $x \in a$ there is a $y \in b$ such that $(x, y) \in R$ and R is a skeleton for $x \subseteq\subseteq y$. In particular, if R is a skeleton for $a \subseteq\subseteq b$, then $a \subseteq\subseteq b$ holds. Write $a \subseteq\subseteq^R b$ if R is a skeleton for $a \subseteq\subseteq b$.

   Direct induction over a shows that only the part of R that is actually part of $\text{tc}(a) \times \text{tc}(b)$ matters, i.e.

$$\forall a, b, R. a \subseteq\subseteq^R b \to a \subseteq\subseteq^{R \cap (\text{tc}(a) \times \text{tc}(b))} b \tag{9.3.84}$$

   Similarly, enlarging R by forming the union with other skeletons preserves the relationship, as another direct induction over a yields:

$$\forall i \in I \; a_i \subseteq\subseteq^{R_i} b_i \wedge a \subseteq\subseteq^R b \to a \subseteq\subseteq^{R \cup \bigcup_{i \in I} R_i} b \tag{9.3.85}$$

   Such an R can always be found by recursively collecting together all the pairs $x \in \text{tc}(a), y \in \text{tc}(b)$ which were needed to establish $a \subseteq\subseteq_\Omega b$. I.e. we claim:

$$\forall a, b. a \subseteq\subseteq b \to \exists R. a \subseteq\subseteq^R b \tag{9.3.86}$$

   We show this by induction over a. Let $a \subseteq\subseteq b$, then by induction hypothesis, for all $x \in a$ there is a $y \in b$ and an R such that $x \subseteq\subseteq^R y$. By Equation 9.3.84 we can w.l.o.g. choose this $R \subseteq \text{tc}(x) \times \text{tc}(y)$.

   By Strong Collection, find a set which contains all of these R and let R_0 be the union of that set. Then by Equation 9.3.85, this R_0 is as desired.

Now let $a \subseteq\subseteq b$. By Equation 9.3.86, it has a skeleton R. By recursion over $\text{rk}(y) + 1$, construct a function $f_{\text{rk}(y)} : \text{rk}(y) + 1 \to \mathcal{P}(\mathcal{P}(1))$ such that for every pair $(x, y') \in R$ with $\text{rk}(y') \in \text{rk}(y) + 1$ we have $x \subseteq\subseteq_{f_{\text{rk}(y)}} y$ — namely extend the function so far by $(\text{rk}(y), \Omega)$ where Ω contains enough truth values such that for every such x

$$\forall y' \in y \forall x' \subseteq\subseteq_{f_{\text{rk}(y')}} y'.[\![x' \in x]\!] \in \Omega \tag{9.3.87}$$

This is possible since there are only set many such x' by part one and the skeleton R being a set.

□

The approximation of $a \subseteq\subseteq b$ by functions has definite merits by the results Proposition 9.46, however it also fails one essential requirement to be used directly in the proof of Theorem 9.37: It does not interact well with elementary embeddings. Its collapse however does:

Definition 9.47. *Let a and b be sets and $\Xi \in \mathcal{P}(\mathcal{P}(\mathcal{P}(1)))$ a set of sets of truth values. Define*

$$a \subseteq\subseteq_{\Xi}^{\exists} b :\leftrightarrow \exists f.a \subseteq\subseteq_f b \wedge f''(rk(b) + 1) = \Xi \tag{9.3.88}$$

Define

$$\mathcal{P}_{\Xi}^{\subseteq\subseteq^{\exists}} := \{x | x \subseteq\subseteq_{\Xi}^{\exists} b\} \tag{9.3.89}$$

Then this has the same good properties:

Proposition 9.48. *1. Let b be a set and $\Xi \in \mathcal{P}(\mathcal{P}(\mathcal{P}(1)))$. Then $\mathcal{P}_{\Xi}^{\subseteq\subseteq^{\exists}}(b)$ is a set.*

2. Let $a \subseteq\subseteq b$ be sets. Then there is a Ξ such that $a \subseteq\subseteq_{\Xi}^{\exists} b$.

Proof. 1. Let b be a set and $\Xi \in \mathcal{P}(\mathcal{P}(\mathcal{P}(1)))$. Then

$$\mathcal{P}_{\Xi}^{\subseteq\subseteq^{\exists}}(b) = \bigcup_{f \in {}^{\text{rk}(b)+1}\Xi} \mathcal{P}_f^{\subseteq\subseteq} \tag{9.3.90}$$

And by the first part of Proposition 9.46, this is a union of sets.

2. Let $a \subseteq\subseteq b$ be sets. By the second part of Proposition 9.46, there is an f such that $a \subseteq\subseteq_f b$. Then $\Xi := f''(\mathrm{rk}(b) + 1)$ is as desired.

$\square$

9.3.3 Truth Values and Elementary Embeddings

In the following subsections we want to connect the previously defined concepts with elementary embeddings. The results presented here only depend on the embedding being Δ_0-elementary, i.e. we demand $V \vDash \Phi(\overrightarrow{x}) \leftrightarrow M \vDash \Phi(\overrightarrow{j(x)})$ for bounded formulae Φ only.

So let $j : V \to M$ be a Δ_0-elementary embedding with transitive subcritical point K, i.e. K is transitive and $\forall a \in K.a = j(a)$ (if in addition $K \in j(K)$ and K is a set, then K would be a critical point in the usual sense). If K is a set, we cannot expect every truth value to be in K. But nevertheless, they are still preserved by j:

Remark 9.49. *For all truth values $\eta \subseteq 1$, $j(\eta) = \eta$.*

This follows from the fact that $1 \in K$ and the following slightly more general assertion:

Lemma 9.50. *Let $a \in K$, $b \subseteq a$. Then $b = j(b)$.*

Proof. Let $x \in b$. Then $x \in a \in K$ and thus by transitivity $x \in K$ and $x = j(x)$. As by elementarity $j(x) \in j(b)$, it follows that $x \in j(b)$.

Conversely, let $x \in j(b)$. By elementarity $j(b) \subseteq j(a) = a$, so $x \in a \in K$, so by transitivity $x \in K$ and $x = j(x)$. So $j(x) \in j(b)$, and thus by elementarity $x \in b$. $\square$

As an aside, this implies that $\bigcup_{a \in K} \mathcal{P}(a) \subseteq M$.

We descend one level further:

Remark 9.51. *For all sets $\Omega \subseteq \mathcal{P}(1)$ of truth values, $\Omega = j(\Omega)$.*

Again we will state it in a slightly more general way:

Lemma 9.52. *Let $a \in K$, and b be a set of subsets of a. Then $b = j(b)$.*

Proof. Let $x \in b$, so $x \subseteq a \in K$. Then by Lemma 9.50 $j(x) = x$. So $j(x) \in j(b)$ implies $x \in j(b)$.

Conversely, let $x \in j(b)$, so $x \subseteq j(a) = a \in K$. Then by Lemma 9.50 $j(x) = x$. So $x \in j(b)$ implies $j(x) \in j(b)$ and thus $x \in b$. □

And once more:

Remark 9.53. *For all sets $\Xi \subseteq \mathcal{P}(\mathcal{P}(1))$ of sets of truth values, $\Xi = j(\Xi)$*

Again we will state it in a slightly more general way:

Lemma 9.54. *Let $a \in K$, and b a set of sets of subsets of a. Then $b = j(b)$.*

Proof. Let $x \in b$, so $x \subseteq \mathcal{P}(a)$ for $a \in K$. Then by Lemma 9.52 $j(x) = x$. So $j(x) \in j(b)$ implies $x \in j(b)$.

Conversely, let $x \in j(b)$, so x is a set of subsets of $j(a) = a \in K$. Then by Lemma 9.52 $j(x) = x$. So $x \in j(b)$ implies $j(x) \in j(b)$ and thus $x \in b$. □

Note that remarks 9.49, 9.51 and 9.53 do not depend on the existence of a critical point K, as 0 and 1 are never moved by j anyways, because they are simply Δ_0-definable. The corresponding lemmata still work if the condition $a \in K$ is replaced by $j(a) = a \wedge \forall x \in \mathrm{tc}(a).j(x) = x$.

Also, the results of course generalise to arbitrary finite iterations of the powerclass operation, and in particular to the elements of V_ω.

While not directly relevant to the following results, it is maybe of general interest that using set induction, Lemma 9.50 can easily be extended to apply not only to the $\subseteq$ relation, but to the $\subseteq\subseteq$ relation.

Proposition 9.55. *Let $a \in K$, $b \subseteq\subseteq a$. Then $b = j(b)$.*

Proof. The proof proceeds by set induction over a. So let by the induction hypothesis $a \in K$ be such that for all $x \in a$, $y \subseteq\subseteq x$ implies $y = j(y)$ and take $b \subseteq\subseteq a$.

To show $b \subseteq j(b)$, let $y \in b$. Then there is some $x \in a$ with $y \subseteq\subseteq x$. By induction hypothesis, $y = j(y)$. By elementarity, $y \in b$ implies $j(y) \in j(b)$ and thus $y \in j(b)$.

To show that conversely, $j(b) \subseteq b$, take some $y \in j(b)$. The set b fulfills

$$\forall y \in b \exists x \in a. y \subseteq\subseteq x \tag{9.3.91}$$

As $a \in K$ and thus $a = j(a)$, elementarity implies

$$\forall y \in j(b) \exists x \in a. y \subseteq\subseteq x \tag{9.3.92}$$

So let $x \in a$ be such that $y \subseteq\subseteq x$. By induction hypothesis, $y = j(y)$ and so $j(y) \in j(b)$. By elementarity, this implies $y \in b$. □

This proof made use of the elementarity scheme extended by formulae containing a symbol for $\subseteq\subseteq$, a consequence of Lemma 2.19, since $\subseteq\subseteq$ has a recursive Δ_0 definition.

In the presence of the Powerset Axiom, this means that critical points can be saturated under $\subseteq\subseteq$:

Corollary 9.56. *If K is a critical point of j, then so is $\bigcup_{a \in K} \mathcal{P}^{\subseteq\subseteq}(a)$, provided it is a set.*

Proof. Let K be a critical point of j, and let

$$K' := \bigcup_{a \in K} \mathcal{P}^{\subseteq\subseteq}(a) \tag{9.3.93}$$

Let K' be a set. From Proposition 9.55, we know that for all $a \in K$, if $b \in \mathcal{P}^{\subseteq\subseteq}(a)$, then $b = j(b)$. This implies that j restricted to K' is the identity.

We also need to prove that $K' \in j(K')$. For that, note that

$$x \in K' \leftrightarrow \exists a \in K. x \subseteq\subseteq a \tag{9.3.94}$$

So by elementarity, to conclude $K' \in j(K')$, it suffices to conclude $K' \in M$ and $\exists a \in j(K).K' \subseteq\subsetneq a$. But by a direct unfolding of the definition $K' \subseteq\subsetneq K$, and so simply setting $a := K$ works.

It remains to show that $K' \in M$. To show this, since $K \in j(K) \in M$, it suffices to show that for each $a \in K$, the set $\mathcal{P}^{\subseteq\subsetneq}(a)$ is in M. This is indeed a set, since it can be obtained by Separation from K', which is a set by condition.

It is an element of M, because

$$M \ni j(\mathcal{P}^{\subseteq\subsetneq}(a)) = \mathcal{P}^{\subseteq\subsetneq}(a) \tag{9.3.95}$$

To see this, let $b \in j(\mathcal{P}^{\subseteq\subsetneq}(a))$, which considering that $a = j(a)$ is equivalent to $M \Vdash b \subseteq\subsetneq a$. A short induction shows that if the $\subseteq\subsetneq$ relation holds in M, then it also holds in V. Thus $b \subseteq\subsetneq a$ and so $b \in \mathcal{P}^{\subseteq\subsetneq}(a)$.

Conversely, if $b \in \mathcal{P}^{\subseteq\subsetneq}(a)$, then $b = j(b)$ and as $b \subseteq\subsetneq a$ implies $j(b) \subseteq\subsetneq j(a) = a$, it follows that $b \in j(\mathcal{P}^{\subseteq\subsetneq}(a))$ □

While ZF does prove that $K' := \bigcup_{a \in K} \mathcal{P}^{\subseteq\subsetneq}(a)$ is a set, the corollary is of course trivial in that environment, as classically critical points are always closed under the $\subseteq\subsetneq$ relation and so $K = K'$. However, it is a meaningful statement e.g. in the context of IZF.

9.3.4 Subset Relations and Elementary Embeddings

While classical logic provides the useful $\alpha \subseteq j(\alpha)$ for elementary embeddings, the constructive case can make use of the following analogon:

Proposition 9.57. *Let $j : V \to M$ be an elementary embedding. Then*

$$\forall \alpha \in O_n.\alpha \subseteq\subsetneq j(\alpha) \tag{9.3.96}$$

Proof. This is an easy induction over α.

Let by induction hypothesis $\forall \beta \in \alpha . \beta \subseteq\subseteq j(\beta)$. As for all $\beta \in \alpha$, $j(\beta) \in j(\alpha)$ by elementarity, this means that in the following formula, we can choose γ to be $j(\beta)$:

$$\forall \beta \in \alpha \exists \gamma \in j(\alpha) . \beta \subseteq\subseteq \gamma \tag{9.3.97}$$

This is just the definition of $\alpha \subseteq\subseteq j(\alpha)$. □

In fact, we have not even used that α is an ordinal and not an arbitrary set, and we only used that j has the property $x \in y \to j(x) \in j(y)$. So from the same proof we actually obtain:

Corollary 9.58. *Let $j : V \to M$ be a Δ_0-elementary embedding. Then*

$$\forall x . x \subseteq\subseteq j(x) \tag{9.3.98}$$

That the statement also applies to arbitrary sets x and not only to ordinals α is not overly surprising in light of the following characterisation of $\subseteq\subseteq$ for arbitrary sets, which implies that the $\subseteq\subseteq$ relation on arbitrary sets can be reduced to the ordinal case:

Proposition 9.59. *For all sets $a, b \in V$,*

$$a \subseteq\subseteq b \leftrightarrow a \subseteq V_b \tag{9.3.99}$$

Proof. Show this by induction over b. So take a set b such that for all $y \in b$ and all x, we have $x \subseteq\subseteq y$ if and only if $x \subseteq V_y$. Let a be a set. Then $a \subseteq\subseteq b$ is equivalent to

$$\forall x \in a \exists y \in b . x \subseteq\subseteq y \tag{9.3.100}$$

And by induction hypothesis this is equivalent to

$$\forall x \in a \exists y \in b . x \subseteq V_y \tag{9.3.101}$$

This is equivalent to the statement that each element of a is also in $\bigcup_{y \in b} \mathcal{P}(V_y)$. But this is exactly V_b. □

Corollary 9.60. *For all sets $a, b \in V$,*

$$a \subseteq\subseteq b \leftrightarrow \mathit{rk}(a) \subseteq\subseteq \mathit{rk}(b) \tag{9.3.102}$$

Thus in the classical case, the $\subseteq\subseteq$ relation can be characterised very economically:

Corollary 9.61. *The principle of the restricted excluded middle is equivalent to the statement*

$$\forall a, b. a \subseteq\subseteq b \leftrightarrow \mathit{rk}(a) \leq \mathit{rk}(b) \tag{9.3.103}$$

Having gathered all the necessary ingredients, we now return to the proof of Theorem 9.37.

Proof of Theorem 9.37. Let in the following $j : V \to V$ be an elementary embedding. We want to show that $\forall a. a \in \bigcup_{x \in V} j(x)$. So let a be an arbitrary set.

According to Proposition 9.57, $\forall a. a \subseteq\subseteq j(a)$. Then by Proposition 9.48 there is a set of sets of truth values Ξ witnessing this, i.e. a set fufilling

$$a \subseteq\subseteq^{\exists}_{\Xi} j(a) \tag{9.3.104}$$

Note that the concept of $\mathcal{P}^{\subseteq\subseteq^{\exists}}_{\cdot}$ could be defined by a formula without parameters, for example let

$$\Theta(x, y, z) :\leftrightarrow x = \mathcal{P}^{\subseteq\subseteq^{\exists}}_{y}(z) \tag{9.3.105}$$

Now

$$x = \mathcal{P}^{\subseteq\subseteq^{\exists}}_{\Xi}(a) \tag{9.3.106}$$

is just $\Theta(x, \Xi, a)$ and this implies $\Theta(j(x), j(y), j(z))$ by elementarity and thus

$$j(x) = \mathcal{P}^{\subseteq\subseteq^{\exists}}_{j(\Xi)}(j(a)) \tag{9.3.107}$$

Thus

$$j(\mathcal{P}^{\subseteq\subseteq^{\exists}}_{\Xi}(a)) = \mathcal{P}^{\subseteq\subseteq^{\exists}}_{j(\Xi)}(j(a)) \tag{9.3.108}$$

But as $\Xi = j(\Xi)$ by Remark 9.53, this means that

$$a \in \mathcal{P}^{\subseteq\subseteq^{\exists}}_{\Xi}(j(a)) = j(\mathcal{P}^{\subseteq\subseteq^{\exists}}_{\Xi}(a)) \tag{9.3.109}$$

is an element of a set in the range of j as desired. $\square$

Remark 9.62. *These results do not yet exhaust the question of cofinality of elementary embeddings as they do not cover elementary embeddings $j : V \to M$ where M may be different from V. While it is clear that general embeddings $j : V \to M$ cannot be proved to be strongly set cofinal (at least provided that ZFC is consistent with the existence of measurable cardinals), their strong cofinality and set cofinality still remains open and we will address it in the following sections.*

On the other hand, $\forall a.a \subseteq\subseteq j(a)$ can itself be seen as a constructive version of cofinality and this has been shown to hold for all elementary $j : V \to M$.

9.4 A Model with a Map $j : V \to M$

While in the theory ZFC all embeddings $j : V \to M$ are always strongly cofinal, we want to construct a model of CZF with a weakly elementary embedding that is not strongly cofinal. To strengthen this result, our model will actually include a weakly measurable cardinal, i.e. the axioms schemes will also hold for formulae including a symbol for j.

This construction can be carried out with only CZF as background theory. Thus it sharpens the fact mentioned in the last section that the consistence of ZFC + a measurable cardinal implies that it cannot be proved that all elementary embeddings are strongly set cofinal: In fact, this section shows that only the consistence of CZF needs to be assumed to get the result for all weakly elementary embeddings. Here a weakly elementary embedding, in contrast to a fully elementary embedding, only needs to fulfill $\Phi(\overrightarrow{x}) \to \Phi^M(\overrightarrow{j(x)})$ instead of $\Phi(\overrightarrow{x}) \leftrightarrow \Phi^M(\overrightarrow{j(x)})$. Classically of course, these two concepts are equivalent, as the first one in particular includes $\neg\Phi(\overrightarrow{x}) \to \neg\Phi^M(\overrightarrow{j(x)})$.

Our $j : V \to M$ will have a critical point K in the sense that $\forall x \in K.j(x) = x$ but $K \neq j(K)$. This implies that while in classical set theory, weakly elementary embeddings with critical points that are admissible in the axiom schemes (i.e. measurable cardinals) increase the consistency strength dramatically, in CZF they are actually already relatively consistent.

As an aside, this is also an example of a transitive proper class model of set theory that does not contain all the ordinals - classically this can only happen for transitive models which are sets.

9.4.1 The pca

The construction is a realizability model with a rather peculiar pca defined on the following set:

Definition 9.63. *Let* $^{\omega}\omega_{ec}$ *be the set of* ***eventually constant functions*** $f : \omega \to \omega$*, i.e.*

$$^{\omega}\omega_{ec} := \{f : \omega \to \omega | \exists n_0 \in \omega \forall n > n_0 . f(n) = f(n_0)\} \tag{9.4.110}$$

For $f \in {}^{\omega}\omega_{ec}$*, write* $f(\infty)$ *for its* ***eventual value****, i.e.*

$$f(\infty) := \mathit{lim\ inf}_{n \to \infty} f(n) = \{i | \exists n_0 \in \omega \forall n_1 > n_0 . i < f(n_1)\} \tag{9.4.111}$$

Application is based on the idea of pointwise Turing application with the twist that the programme has access to all previous inputs as well as the eventual limit, if it chooses to access that data. However, the amount of data to be processed needs to be bounded. Introduce the following notations for Turing computation with oracles:

Definition 9.64. *Let* $e \in \omega$*,* $f : \omega \to \omega$*. Then write* $\{e\}^f$ *for the result of the application of the* $e - th$ *Turing machine with an oracle for* f*, should this computation terminate. Also, let* $\{e\}^f \downarrow$ *be the statement that the computation terminates and* $\{e\}^f \uparrow$ *the statement that it does not terminate. If* $\{e\}^f \downarrow$ *and* $n \in \omega$*, write*

$$\{e\}^f \downarrow_n \tag{9.4.112}$$

if the run of the computation accesses at most the first n *entries of the oracle (i.e.* $f(0)$ *up to* $f(n-1)$*).*

Note that in contrast to how the symbol $\downarrow_n$ is sometimes defined alternatively, the run of $\{e\}^f\downarrow_n$ is allowed to take much longer than just n steps, as long as no more than n entries of the oracle tape are read off.

Remark 9.65. *1. Obviously*

$$\{e\}^f \downarrow \leftrightarrow \exists n \in \omega.\{e\}^f \downarrow_n \tag{9.4.113}$$

as the computation needs to be done in finite time and thus only has the capacity to process a finite amount of information.

2. Obviously

$$\{e\}^f \downarrow_n \wedge f \restriction n = g \restriction n \rightarrow \{e\}^f = \{e\}^g \tag{9.4.114}$$

as the computation did not access any information where f and g might have differed.

Now define the partial application on ${}^{\omega}\omega_{\text{ec}}$.

Definition 9.66. *Let $e, f \in {}^{\omega}\omega_{ec}$. Let $f_n : \omega \to \omega$ be defined by*

$$f_n(m) = \begin{cases} \langle f(n-m), e(n-m)\rangle & \text{if } n \geq m \\ \langle f(\infty), e(\infty)\rangle & \text{if } n < m \end{cases} \tag{9.4.115}$$

Where $\omega \ni n, m \mapsto \langle n, m\rangle \in \omega$ is some suitable computable implementation of the pairing function.

If there is an $n_0 \in \omega$ such that

$$\forall n.\{e(n)\}^{f_n} \downarrow_{n_0} \tag{9.4.116}$$

then write $e \circ f \downarrow$ and define $e \circ f : \omega \to \omega$ as

$$(e \circ f)(n) := \{e(n)\}^{f_n} \tag{9.4.117}$$

Otherwise, leave it undefined and write $e \circ f \uparrow$.

The symbol $\circ$ *may be omitted from* $f \circ g$ *when it is clear from the context that the term was written with this application in mind.*

The function f_n that is made available as oracle is just the countdown of the elements $\langle f(n), e(n)\rangle, \langle f(n-1), e(n-1), \rangle$... until it reaches $\langle f(0), e(0)\rangle$. After this point it is constantly equal to $\langle f(\infty), e(\infty)\rangle$.

The restriction on $f \circ g$ that the number of entries used from the oracle needs to be bounded might seem strange at first. The reason for imposing it will become clear later. Basically it prevents certain constructions from being computable in the pca, for example there is no f with the property

$$\forall n.(f \circ g)(n) := g(0) \tag{9.4.118}$$

While it would be easy to write an f such that when run, the Turing machine $\{f(n)\}^{g_m}$ reads off the $n-th$ entry in the oracle, this would not yield $f \circ g \downarrow$, as there is no bound on how far the oracle is tapped - indeed $f(n)$ would need to descend n steps on the oracle tape.

On the other hand, the bound is not required to be uniform in g. Thus it is perfectly possible to describe an f with the property

$$(f \circ g)(n) := \begin{cases} g(n - g(n)) & \text{if } n \geq g(n) \\ g(\infty) & \text{if } n < g(n) \end{cases} \tag{9.4.119}$$

Just let $f(n)^{g_m}$ read off the first entry in the oracle, call its first component k, and then descend k steps further down the oracle tape. If $g \in {}^{\omega}\omega_{\text{ec}}$, then there are only finitely many different such k, so they have some maximum k_{max} and thus $f \circ g \downarrow$ by virtue of the bound k_{max} - a bound which however is not at all uniform in g.

Lemma 9.67. *Let $e, f \in {}^{\omega}\omega_{ec}$ and $e \circ f \downarrow$. Then*

$$e \circ f \in {}^{\omega}\omega_{ec} \tag{9.4.120}$$

Proof. Let $e \circ f \downarrow$, so for some $n_0 \in \omega$ it holds that

$$\forall n.\{e(n)\}^{f_n} \downarrow_{n_0} \tag{9.4.121}$$

As $e, f \in {}^{\omega}\omega_{\mathrm{ec}}$, there are furthermore n_1, n_2 with

$$\forall n > n_1.e(n) = e(n_1), \forall n > n_2.f(n) = f(n_2) \tag{9.4.122}$$

Thus with $n_3 := n_0 + \max(n_1, n_2)$, it holds that

$$\forall n > n_3. \wedge f_n \restriction n_0 = f_{n_3} \restriction n_0 \wedge e(n) = e(n_3) \wedge \{e(n)\}^{f_n} \downarrow_{n_0} \tag{9.4.123}$$

Thus for all $n > n_3$,

$$\{e(n)\}^{f_n} = \{e(n_3)\}^{f_n} = \{e(n_3)\}^{f_{n_3}} \tag{9.4.124}$$

where the second equality holds by Remark 9.65. In particular, the function $e \circ f$ is eventually constant. □

Proposition 9.68. *The set ${}^{\omega}\omega_{ec}$ together with the partial application operation*

$$\circ : {}^{\omega}\omega_{ec} \times {}^{\omega}\omega_{ec} \to_p {}^{\omega}\omega_{ec} \tag{9.4.125}$$

is a pca. The combinators k and s can be chosen to be constant functions.

Proof. The combinator k is very easy: It should be the constant function that always returns an index for a Turing machine that reads off the first entry of the oracle and then returns an entry for a Turing machine that always puts out the first component of that entry (and does not read off anything from the oracle tape).

The combinator s is also relatively straightforward, as long as a few basic caveats are followed. If s is the constant function with value c, i.e. $s : \omega \to \omega, n \mapsto c$, then c needs to be an index for a Turing machine which behaves as follows:

It reads off the first entry on the oracle tape (say, $\langle a_0, b_0 \rangle$) and puts out an index for a Turing machine which does the following:

It reads off the first entry on the oracle tape (say, $\langle a_1, b_1 \rangle$) and puts out an index for a Turing machine T from which a_0 and a_1 can be read off (how this is done depends on how exactly you define Turing machines of course, but it is always possible) and which does the following:

T reads off the first entry on the oracle tape (say, $\langle a_2, b_2 \rangle$) and then simulates lazily the run of the entries of the first oracle tape $\circ$ the entries of the third one, and this $\circ$ (the entries of the second oracle tape $\circ$ the entries of the third one) to obtain the entry of $xz(yz)$ at the topmost position. That is, T will start simulating $\{a_2\}^{\cdots}$ and when this Turing machine intends to read off an entry on its oracle tape at position n, T will look at its own oracle tape at position n, and read the required $a_{0,n}$ off the first component and $a_{3,n}$ off the second component of the oracle entry to return $\langle a_{0,n}, a_{3,n} \rangle$. If it does not terminate, then $sxyz$ does not denote (and does not need to). Otherwise T arrives at some interim result d.

Now T will want to start simulating the run of $\{d\}^{\cdots}$ and when this Turing machine intends to read off an entry on its oracle tape at position n, it again needs to be calculated on the spot. It will be an entry of the form $\langle x_n, y_n \rangle$. For x_n, the same calculation just described needs to be redone just shifted by n further down the tapes and for y_n the analogon for the simulation of (yz) needs to be calculated.

This is a very inefficient process and will have to repeat the same calculations several times at different points in its run, but if indeed $(x \circ z) \circ (y \circ z) \downarrow$, then $s \circ x \circ y \circ z \downarrow$ and returns the same result. To see that s only needs to descend a bounded number of steps on the oracle tapes, note that there are upper bounds for the runs of $x \circ z$, $y \circ z$ and $(x \circ z) \circ (y \circ z)$ and the combined sum of these bounds works for the run of $(s \circ x \circ y) \circ z$. $\square$

Note that pairing and projection functions can be defined pointwise, as can the implementation of the natural numbers in the pca and case distinction.

9.4.2 The Realizability Model

Now that $({}^{\omega}\omega_{\text{ec}}, \circ)$ is established as a pca, we can build the standard realizability model over it as e.g. described in [Rat03b]. This can be done with just CZF as background theory and yields a model $V({}^{\omega}\omega_{\text{ec}}) \Vdash$ CZF whose elements are recursively defined as all sets of pairs (e, a) with $e \in {}^{\omega}\omega_{\text{ec}}$ and $a \in V({}^{\omega}\omega_{\text{ec}})$.

As we however want to model a weakly measurable cardinal, we need to extend the language of the model to include a symbol j for which we want

$$V({}^{\omega}\omega_{\text{ec}}) \Vdash j : V \to V \tag{9.4.126}$$

Let j be this symbol and by slight abuse of notation, also define j as a function on the pca and the underlying class of the model.

Definition 9.69. *1. Define $j : {}^{\omega}\omega_{ec} \to {}^{\omega}\omega_{ec}$ by*

$$j(e)(n) := \begin{cases} e(\infty) & \textit{if } n = 0 \\ e(n-1) & \textit{if } n > 0 \end{cases} \tag{9.4.127}$$

2. Define $j : V({}^{\omega}\omega_{ec}) \to V({}^{\omega}\omega_{ec})$ recursively by

$$j(a) := \{(j(e), j(x)) | (e, x) \in a\} \tag{9.4.128}$$

Remark 9.70. *1. It is clear that $j : {}^{\omega}\omega_{ec} \to {}^{\omega}\omega_{ec}$ is computable in this pca by a constant function, which by abuse of notation we will also call j, so that*

$$\forall e \in {}^{\omega}\omega_{ec}. j \circ e = j(e) \tag{9.4.129}$$

Indeed this function can be chosen to be the constant function putting out an index for a Turing machine which takes the first component of the second entry (i.e. entry number 1) of its oracle tape and returns that.

2. *It follows directly from the definitions that for* $e, f \in {}^{\omega}\omega_{ec}$,

$$j(e \circ f) \simeq j(e) \circ j(f) \tag{9.4.130}$$

Also j *behaves as the identity on constant functions, so* $j(k) = k$ *and* $j(s) = s$.

That means that j *is a representable pca-monomorphism.*

To keep track, there are five meanings of the j now: The symbol in the language we want to interpret, its interpretation, a function on $V({}^{\omega}\omega_{\mathrm{ec}})$, a function on ${}^{\omega}\omega_{\mathrm{ec}}$ which was used to define the function on $V({}^{\omega}\omega_{\mathrm{ec}})$ and an element of ${}^{\omega}\omega_{\mathrm{ec}}$ which yields the function on ${}^{\omega}\omega_{\mathrm{ec}}$ when plugged into the first component of $x, y \mapsto x \circ y$.

Definition 9.71 (Extending Realizability to function symbols)**.** *Let* $\mathcal{L}$ *be a signature containing* $\in$ *and function symbols and let* A *be a pca. For each of those function symbols* f *with arity* n, *let* $f^{V(A)} : V(A)^n \to V(A)$ *be a function.*

Define the value of each term with parameters in $V(A)$ *recursively by setting* $a^{V(A)} = a$ *for* $a \in V(A)$ *and*

$$(f(t_1, ..., t_n))^{V(A)} := f^{V(A)}(t_1^{V(A)}, ..., t_n^{V(A)}) \tag{9.4.131}$$

Extend the realizability relation $e \Vdash \Phi(\overrightarrow{a})$ *by the following clause: If* $\Phi(x_1, ..., x_n)$ *does not contain any function symbols, then let*

$$e \Vdash \Phi(t_1, ..., t_n) :\leftrightarrow e \Vdash \Phi(t_1^{V(A)}, ..., t_n^{V(A)}), \tag{9.4.132}$$

where $e \Vdash \Phi(t_1^{V(A)}, ..., t_n^{V(A)})$ *is already defined by the standard realizability in* $V(A)$.

In the following, we will interpret $\mathcal{L} = \{\in, j\}$ *in* $V({}^{\omega}\omega_{ec})$ *with* $j^V = j$.

A priori it is not at all clear that this way the interpretation of the function symbols will indeed be functions. Considering that the raw structure of $V(A)$ is not identical to its extensional structure, it is very possible that $f(x) = y \wedge x = z \to f(z) = y$ fails to be realized. On the other hand, at least $f(x) = y \wedge y = z \to f(x) = z$ is always realized as transitivity of equality was in a certain sense built into the realizability model.

Lemma 9.72. *It is realized that $j : V \to V$ is indeed a (class) function.*

Proof. We only need to find a realizer for

$$\forall a, b. a = b \to j(a) = j(b) \tag{9.4.133}$$

However, when trying to use the recursion theorem to find a realizer for this formula, the induction does not go through. What we need to construct first is an element e of the applicative structure that fulfills

$$\forall a, b \in V({}^{\omega}\omega_{\text{ec}}) \forall f \in {}^{\omega}\omega_{\text{ec}}. f \Vdash a = b \to e(j(f)) \Vdash j(a) = j(b) \tag{9.4.134}$$

Such a realizer can then be found making use of the recursion theorem. This theorem assures us that we can find an e fulfilling

$$e = \lambda x. p(\lambda y. p((x_0 y)_0, e((x_0 y)_1)), \lambda y. p((x_1 y)_0, e((x_1 y)_1))) \tag{9.4.135}$$

Now $9.4.134$ is proved by induction over a and b.

Let $f \Vdash a = b$. We need to demonstrate that $e(j(f)) \Vdash j(a) = j(b)$. To that end, take an element of $j(a)$, it is of the form

$$(j(g), j(c)) \in j(a) \tag{9.4.136}$$

The goal is to show that there is some element of $j(b)$ whose first component is equal to $((e(j(f)))_0 j(g))_0$ and whose second component is realizedly equal to $j(c)$ by the realizer $((e(j(f)))_0 j(g))_1$.

Note that by the definition of j, $(g, c) \in a$. So by the assumption on f, there is a $d \in V({}^{\omega}\omega_{\text{ec}})$ with

$$(((f_0)g)_0, d) \in b \wedge ((f_0)g)_1 \Vdash c = d \tag{9.4.137}$$

So by the induction hypothesis,

$$e(j((f_0 g)_1)) = e((j(f)_0 j(g))_1) \Vdash j(c) = j(d) \tag{9.4.138}$$

And by the definition of $j(b)$, the fact that $(((f_0)g)_0, d) \in b$ implies

$$(j((f_0g)_0), j(d)) \in b \tag{9.4.139}$$

So it only remains to show the two equations

$$((e(j(f)))_0 j(g))_0 = j((f_0 g)_0) \tag{9.4.140}$$

and

$$(e(j(f))_0 j(g))_1 = e(j((f_0 g)_1)) \tag{9.4.141}$$

These however are just equation 9.4.135 plugged in, noting that j is a monomorphism.

The other direction is of course proved analogously.

Now a realizer for formula 9.4.133 is just $\lambda x.e(j(x))$, noting that j is representable in ${}^{\omega}\omega_{\mathrm{ec}}$ as per remark 9.70. □

Thus j is realized to be an (extensional) class function. It is not just any class function however, but one with the important property that it may appear freely in all axiom schemes of CZF. The following list omits Subset Collection only because with the help of the other schemes it can be derived from the equivalent Fullness axiom which is not a scheme at all, so the following lemma does imply that j may also appear in Subset Collection.

Lemma 9.73. *Let Strong Collection$_j$, Set Induction$_j$ and Δ_0-Separation$_j$ be the axiom schemes that for each Φ in the language $L(\in, j)$ where a and b do not occur free:*

$$\forall x \in a \exists y \Phi \to \exists b. \forall x \in a \exists y \in b \Phi \wedge \forall y \in b \exists x \in a \Phi \tag{9.4.142}$$

$$(\forall b \in a \Phi(b) \to \Phi(a)) \to \forall a \Phi(a) \tag{9.4.143}$$

$$\Phi \textit{ bounded } \to \forall a \exists b. b = \{x \in a | \Phi\} \tag{9.4.144}$$

Then these axiom schemes are realized in $V({}^{\omega}\omega_{ec})$.

Proof. Set Induction$_j$ is trivial, as the usual proof that normal Set Induction is realized also goes through for Set Induction$_j$.

Strong Collection$_j$ is also trivial, as the usual proof that normal Strong Collection is realized goes through for Strong Collection$_j$, noting that j can be defined in the background universe and thus the background universe models Strong Collection$_j$.

Δ_0-Separation$_j$ also follows from j being definable in the background universe: Just as for normal Δ_0-Separation, the critical step is to show that the class of realizers for a bounded formula is actually a set, where now there is a new atomic case to consider. But since j is definable in the background universe and thus the background universe models Δ_0-Separation$_j$, the classes of realizers for $j^n(x) \in y$, $j^n(x) = y$ and $j^n(x) \ni y$ respectively do form sets for each n because $j^n(x) = x'$ for some $x' \in V({}^{\omega}\omega_{\text{ec}})$ and as already shown before by Rathjen in [Rat03b], the classes of realizers for $x' \in y$, $x' = y$ and $x' \ni y$ form sets. From the fact that the classes of realizers for bounded formulae form sets, proceed just as in [Rat03b]. $\square$

This lemma was so easy to prove because the definition of what it means to add function symbols to the language being realized was done in such a way that it would be clear that they can be used in the axiom schemes but left all the work to be done about the question whether the function symbols would actually be interpreted as (extensional) functions.

Consider the following sub-pca of ${}^{\omega}\omega_{\text{ec}}$:

$$ {}^{\omega}\omega_{\text{ec},\,0} := \{e \in {}^{\omega}\omega_{\text{ec}}.e(0) = e(\infty)\} \tag{9.4.145}$$

This is the range of j and it is easy to see that j is actually a pca isomorphism onto it:

$$ j : {}^{\omega}\omega_{\text{ec}} \xrightarrow{\cong} {}^{\omega}\omega_{\text{ec},\,0} \tag{9.4.146}$$

Let in the following $M \subseteq V({}^{\omega}\omega_{\text{ec}})$ be defined as $M = V({}^{\omega}\omega_{\text{ec},\,0})$, i.e.

$$a \in M :\leftrightarrow \forall x \in a \exists y \in M, e \in {}^{\omega}\omega_{\mathrm{ec},0}.x = (e, y) \tag{9.4.147}$$

Then for $a \in V({}^{\omega}\omega_{\mathrm{ec}})$, we have $j(a) \in M$, although of course $e \Vdash j(a) = x$ does not necessarily imply $x \in M$ (yet this implication holds if $e \in {}^{\omega}\omega_{\mathrm{ec},0}$). In a slight abuse of notation, define within $V({}^{\omega}\omega_{\mathrm{ec}})$ the class M as $\{x | \exists y.x = j(y)\}$. Then as each element of M has a preimage under j, it follows that

$$(e \Vdash x \in M) \leftrightarrow (\exists y \in M.e \Vdash x = y) \tag{9.4.148}$$

We will make use of this because it allows the interpretation of quantifiers $e \Vdash \forall x \in M.\Phi$ and $e \Vdash \exists x \in M.\Phi$ directly as $\forall x \in M.e \Vdash \Phi$ respectively $\exists x \in M.e \Vdash \Phi$ and this can thus be proved to conform to the usual definition of these quantifiers.

Now it is possible (and relatively straightforward, even though it is somewhat protracted) to show that j is weakly elementary.

Theorem 9.74. *It is realized that $j : V \to M$ is a weak elementary embedding into a transitive model of set theory, i.e. there is a realizer for*

$$\forall x \forall y \in x.x \in M \to y \in M \tag{9.4.149}$$

and for all formulae $\Phi(\overrightarrow{x})$ with all free variables displayed, there is a realizer for

$$\forall \overrightarrow{x}.\Phi(\overrightarrow{x}) \to \Phi^M(\overrightarrow{j(x)}) \tag{9.4.150}$$

Proof. In order to show transitivity, there needs to be a realizer e with the property:

$$\begin{aligned} &\forall x, y \in V({}^{\omega}\omega_{\mathrm{ec}}) \forall x' \in M \forall f, g \in {}^{\omega}\omega_{\mathrm{ec},0}. \\ &\qquad f \Vdash y \in x \wedge g \Vdash x = x' \to \exists y' \in M.efg \Vdash y = y' \end{aligned} \tag{9.4.151}$$

If e_t is a realizer for transitivity and e_s for symmetry of equality, then a realizer e that fulfills 9.4.151 would be

$$e := \lambda x, y.e_t x_1 (y_0 x_0)_1 \tag{9.4.152}$$

To show this, let $x, y \in V({}^{\omega}\omega_{\mathrm{ec}})$, $x' \in M$ and $f, g \in {}^{\omega}\omega_{\mathrm{ec}}$ with $f \Vdash y \in x$ and $g \Vdash x = x'$.

There must exist some $\overline{y}$ with

$$(f_0, \overline{y}) \in x \wedge f_1 \Vdash y = \overline{y} \tag{9.4.153}$$

Thus there exists some $\overline{y}'$ with

$$((g_0 f_0)_0, \overline{y}') \in x' \wedge ((g_0 f_0)_1 \Vdash \overline{y} = \overline{y}' \tag{9.4.154}$$

Thus by transitivity

$$e_t f_1((g_0 f_0)_1 \Vdash y = \overline{y}' \tag{9.4.155}$$

As $x' \in M$, so is $\overline{y}' \in M$ and thus

$$efg = e_t f_1((g_0 f_0)_1 \tag{9.4.156}$$

is as desired.

Now for the elementary embedding: The asserted realizer is found and proved to work by recursion on the structure of Φ. In fact, for the induction to go through the actual statement to be proved is a two pronged statement, namely that for all $\Phi(\overrightarrow{x})$ with all free variables displayed, there are e_Φ and e'_Φ such that for all $a_0, ..., a_n \in V({}^{\omega}\omega_{\mathrm{ec}})$

$$(e \Vdash \Phi(\overrightarrow{a})) \rightarrow e_\Phi j(e) \Vdash \Phi^M(\overrightarrow{j(a)}) \tag{9.4.157}$$

and

$$(e \Vdash \Phi^M(\overrightarrow{j(a)})) \rightarrow \exists e'. e'_\Phi e = j(e') \wedge e' \Vdash \Phi(\overrightarrow{a}) \tag{9.4.158}$$

While this does also imply that $\Vdash \Phi^M(\overrightarrow{j(a)}) \Rightarrow \Vdash \Phi(\overrightarrow{a})$, this implication is not represented in the pca and thus not realized. However, as j is represented in the pca, it immediately implies the existence of a realizer for $\Phi(\overrightarrow{a}) \rightarrow \Phi^M(\overrightarrow{j(a)})$, which is what was claimed in the theorem.

To avoid the use of metavariables, note that by the generic implementation of the realizability relation for $\forall x$, the order of variables in $\overrightarrow{x}$ plays no role in the realizer, so we need not distinguish for instance the cases of Φ being $x_1 = x_2$ and Φ being $x_2 = x_1$.

1. Φ is $\perp$: Then Φ^M is $\perp$ as well, and e_Φ and e'_Φ can be chosen to be arbitrary elements. Then 9.4.157 and 9.4.158 are vacuously true.

2. Φ is $x_1 = x_2$: First note that by substitution and the way that j was implemented, this also covers the case $j^n(x_1) = j^m(x_2)$. The existence of a e_Φ for this formula was already proved in lemma 9.72. The other claim that for a realizer $e \Vdash j(x_1) = j(x_2)$ basically only the values $e(1), e(2), ...$ need to matter might intuitively be clear as its inputs and outputs are all of the form $j(g)$. However, there is still something to do for e'_Φ: Even in the case that $(ef)(0) = (ef)(\infty)$ and thus it does not contain anything interesting, the run of $(ef)(1)$ is allowed to access the oracle that returns $e(0)$, so changing e to a function in the range of j might change the output. However, this problem can be circumvented with care.

   First note that in usual implementations of Turing machines (and we want to use one of these), there is the possibility to include in the index of a Turing machine what in programming is known as comments, i.e. data that can be read off of the index of the Turing machine but does not affect the run of the program. In technical terms, there are Turing machines a (add comment) and r (read off comment) such that:

$$\forall a, c, d.\{r\}(\{\{a\}e\}c) = c \wedge \{(\{\{a\}e\}c)\}d \simeq \{e\}d \tag{9.4.159}$$

   So $\{\{a\}e\}c$ contains the information of the comment c and this can be read off, but it is still the index of a Turing machine that acts exactly like $\{e\}$.

   Now $e'_\Phi(e)$ transforms e pointwise, where $e'_\Phi(1) = e'_\Phi(2) = ... = e'_\Phi(\infty)$ and only $e'_\Phi(0)$ differs in that $e'_\Phi e(0) := e'_\Phi e(\infty)$. This latter condition can be met by shifting the oracle tape by one, i.e. by setting

$$\{e'_\Phi(0)\}^g = \{e'_\Phi(\infty)\}^{n \mapsto g(n+1)} \tag{9.4.160}$$

   Now to describe $e'_\Phi(\infty)$. This is an index for a Turing machine that takes an input e (think of the input as a realizer $e \Vdash j(x_1) = j(x_2)$) and returns a pair, both components of which will be constructed analogously, so we will only describe the first one. This first one takes an input f (think of it as the first component of some $(f, j(a)) \in j(x_1)$) and returns a pair with the second input of the oracle tape as comment (if the definition of pairs are functions that return the first or second component depending on input, then pairs can be commented as well). The second component just recursively applies $e'_\Phi(\infty)$, but the first one simulates the run of e on f. However, when this run wants to descend on the oracle tape to read off a value of e at a strictly lower point (e.g. $e(0)$ if we are currently calculating the value at 1), this query is redirected to the comments of the oracle tape one above, where this information has been saved even though it may have been deleted at the point where the query has been originally directed to.

Thus $e'_\Phi e$ is always in the range of j and side induction over x_1, x_2 directly yields that if $e \Vdash j(x_1) = j(x_2)$, then the preimage of $e'_\Phi e$ under j realizes $x_1 = x_2$.

3. Φ is $x_1 \in x_2$: First note that by substitution and the way that j was implemented, this also covers the case $j^n(x_1) \in j^m(x_2)$.

   Let $e \Vdash x_1 \in x_2$, i.e.

$$\exists x_3.(e_0, x_3) \in x_2 \wedge e_1 \Vdash x_1 = x_3 \tag{9.4.161}$$

   Then by definition of j, it follows that

$$(j(e_0), j(x_3)) \in j(x_2) \tag{9.4.162}$$

   and by the induction hypothesis

$$e_{x_1=x_3} j(e_1) \Vdash j(x_1) = j(x_3) \tag{9.4.163}$$

   And so setting

$$e_\Phi := \lambda x.p(x_0, e_{x_1=x_3} x_1) \tag{9.4.164}$$

   returns a realizer as desired.

   On the other hand, let $e \Vdash j(x_1) \in j(x_2)$, i.e.

$$\exists x_3.(e_0, x_3) \in j(x_2) \wedge e_1 \Vdash j(x_1) = x_3 \tag{9.4.165}$$

Then by the definition of j, it follows that $e_0 = j(g)$ for some realizer g and $x_3 = j(x_3')$ for some $x_3 \in V({}^{\omega}\omega_{\text{ec}})$. Thus

$$\exists x_3.(e_0, j(x_3')) \in j(x_2) \wedge e_1 \Vdash j(x_1) = j(x_3') \tag{9.4.166}$$

and by the induction hypothesis,

$$j^{-1}(e'_{x_1=x_3} e_1) \Vdash x_1 = x_3' \tag{9.4.167}$$

And so setting

$$e'_\Phi := \lambda x.p(x_0, e'_{x_1=x_3} x_1) \tag{9.4.168}$$

returns a realizer as desired (note that $e'_{x_1=x_3}$ does not depend on x_1 or, more importantly, x_3).

4. Φ is $\Psi(\overrightarrow{x}) \to \Theta(\overrightarrow{x})$: Let $e \Vdash \Psi(\overrightarrow{x}) \to \Theta(\overrightarrow{x})$. Let $f \Vdash \Psi^M(\overrightarrow{j(x)})$. Then by the induction hypothesis on Ψ,

$$j^{-1}(e'_\Psi f) \Vdash \Psi(\overrightarrow{x}) \tag{9.4.169}$$

Thus

$$ej^{-1}(e'_\Psi f) \Vdash \Theta(\overrightarrow{x}) \tag{9.4.170}$$

By the induction hypothesis on Θ,

$$e_\Theta j(ej^{-1}(e'_\Psi f)) \Vdash \Theta^M(j(\overrightarrow{x})) \tag{9.4.171}$$

This implies

$$e_\Theta(j(e)(e'_\Psi f)) \Vdash \Theta^M(j(\overrightarrow{x})) \tag{9.4.172}$$

And so setting

$$e_\Phi = \lambda x.\lambda y.e_\Theta x(e'_\Psi y)) \tag{9.4.173}$$

returns a realizer as desired.

On the other hand, let $e \Vdash \Psi^M(j(\overrightarrow{x})) \to \Theta^M(j(\overrightarrow{x}))$. Let $f \Vdash \Psi(\overrightarrow{x})$. Then by the induction hypothesis on Ψ,

$$e_\Psi j(f) \Vdash \Psi^M(j(\overrightarrow{x})) \tag{9.4.174}$$

Thus

$$ee_\Psi j(f) \Vdash \Theta^M(j(\overrightarrow{x})) \tag{9.4.175}$$

By the induction hypothesis on Θ,

$$j^{-1}(e'_\Theta(ee_\Psi j(f))) \Vdash \Theta(\overrightarrow{x}) \tag{9.4.176}$$

And so setting

$$e''_\Phi := \lambda x.\lambda y.e'_\Theta(xe_\Psi j(y)) \tag{9.4.177}$$

returns a realizer almost as desired - but while $e''_\Phi fg$ is in the range of j, $e''_\Phi f$ is not necessarily in that range. But the same trick as for the case of Φ being $x_1 = x_2$ works: Set

$$e''_\Phi := \lambda x.c(\lambda y.e'_\Theta(xe_\Psi j(y))) \tag{9.4.178}$$

where $c(n)$ is the index for a Turing machine that reads some g and returns a Turing machine with the second entry on the oracle tape as a comment that simulates g while redirecting the oracle query for the first component of oracle entry number $n > 0$ to the comment of the first component of oracle query number $n - 1$, while $c(0)$ is just so that $cg(0) = cg(\infty)$. This is now a realizer as desired.

5. Φ is $\Psi \wedge \Theta$: Let $e \Vdash \Psi(\overrightarrow{x}) \wedge \Theta(\overrightarrow{x})$. Then by the induction hypothesis

$$e_\Psi j(e_0) \Vdash \Psi^M(j(\overrightarrow{x})) \text{ and } e_\Theta j(e_1) \Vdash \Theta^M(j(\overrightarrow{x})) \tag{9.4.179}$$

In other words

$$p(e_\Psi j(e_0), e_\Theta j(e_1)) \Vdash (\Psi^M(j(\overrightarrow{x})) \wedge \Theta(j(\overrightarrow{x})))^M \tag{9.4.180}$$

And by j being a monomorphism, setting

$$e_\Phi = \lambda x.p(e_\Psi x_0, e_\Theta x_1) \tag{9.4.181}$$

returns a realizer as desired.

On the other hand, let $e \Vdash \Psi^M(j(\overrightarrow{x})) \wedge \Theta^M(j(\overrightarrow{x}))$. Then by the induction hypothesis

$$j^{-1}(e'_\Psi e_0) \Vdash \Psi(\overrightarrow{x}) \text{ and } j^{-1}(e'_\Theta e_1) \Vdash \Theta(\overrightarrow{x}) \tag{9.4.182}$$

In other words

$$p(j^{-1}(e'_\Psi e_0), j^{-1}(e'_\Theta e_1)) \Vdash \Psi(\overrightarrow{x}) \wedge \Theta(\overrightarrow{x}) \tag{9.4.183}$$

And by j being a monomorphism (and the preimages being defined), setting

$$e_\Phi = \lambda x.p(e'_\Psi x_0, e'_\Theta x_1) \tag{9.4.184}$$

returns a realizer as desired.

6. Φ is $\Psi \vee \Theta$: Let $e \Vdash \Psi(\overrightarrow{x}) \vee \Theta(\overrightarrow{x})$. Then either $e_0 = 0$ and $e_1 \Vdash \Psi(\overrightarrow{x})$ or $e_0 = 1$ and $e_1 \Vdash \Theta(\overrightarrow{x})$. Using the induction hypothesis on each disjunct yields

$$(e_0 = \underline{0} \wedge e_\Psi(j(e_1)) \Vdash \Psi^M(j(\overrightarrow{x})) \vee (e_0 = \underline{1} \wedge e_\Theta(j(e_1)) \Vdash \Theta^M(j(\overrightarrow{x})) \tag{9.4.185}$$

Thus by j being a monomorphism, setting

$$e_\Phi = \lambda x.\ p(x_0, dx_0(e_\Psi x_1)(e_\Theta x_1)) \tag{9.4.186}$$

returns a realizer as desired, where d denotes some member of the pca that implements case distinctions, i.e.

$$\forall x, y. d\underline{0}xy = x \wedge d\underline{1}xy = y \tag{9.4.187}$$

On the other hand, let $e \Vdash \Psi^M(j(\overrightarrow{x})) \vee \Theta^M(j(\overrightarrow{x}))$. Then either $e_0 = \underline{0}$ and $e_1 \Vdash \Psi^M(j(\overrightarrow{x}))$ or $e_0 = \underline{1}$ and $e_1 \Vdash \Theta^M(j(\overrightarrow{x}))$. Using the induction hypothesis on each disjunct yields

$$(e_0 = \underline{0} \wedge j^{-1}(e'_\Psi e_1) \Vdash \Psi(j(\overrightarrow{x})) \vee (e_0 = \underline{1} \wedge j^{-1}(e'_\Theta e_1) \Vdash \Theta(j(\overrightarrow{x})) \tag{9.4.188}$$

Thus by j being a monomorphism, setting

$$e_\Phi = \lambda x.p(x_0, dx_0(e'_\Psi x_1)(e'_\Theta x_1)) \tag{9.4.189}$$

returns a realizer as desired, where again d denotes some member of the pca that implements case distinctions.

7. Φ is $\forall x_1 \in x_2 \Psi(x_1, \overrightarrow{x})$ or $\exists x_1 \in x_2 \Psi(x_1, \overrightarrow{x})$: These cases can be skipped since the bounded quantifiers are definable from the other logical connectives.

8. Φ is $\forall x_1 \Psi(x_1, \overrightarrow{x})$: By realizing unbounded quantifiers in a generic way, this directly reduces to the induction hypothesis since

$$e \Vdash \forall x_1 \Psi(x_1, \overrightarrow{x}) \leftrightarrow \forall x_1 . e \Vdash \Psi(x_1, \overrightarrow{x}) \tag{9.4.190}$$

and

$$e \Vdash (\forall x_1 \Psi(x_1, \overrightarrow{j(x)}))^M \leftrightarrow \forall x_1 \in M . e \Vdash \Psi^M(j(x_1), \overrightarrow{j(x)}) \tag{9.4.191}$$

This uses that all elements of M are realizedly of the form $j(x_1)$ as established above and the consequently justified convention to interpret $\forall x \in M$ generically.

9. Φ is $\exists x_1 \Psi(x_1, \overrightarrow{x})$: This is just the same as the universal quantification directly above, as in the same vein

$$e \Vdash \exists x_1 \Psi(x_1, \overrightarrow{x}) \leftrightarrow \exists x_1 . e \Vdash \Psi(x_1, \overrightarrow{x}) \tag{9.4.192}$$

and

$$e \Vdash (\exists x_1 \Psi^M(x_1, \overrightarrow{j(x)}))^M \leftrightarrow \exists x_1 \in M . e \Vdash \Psi^M(j(x_1), \overrightarrow{j(x)}) \tag{9.4.193}$$

$\square$

9.4.3 Critical Points and Cofinality in the Model

This section will show that the $j : V \to M$ realized in the last section is neither strongly set-cofinal nor even strongly cofinal (unlike all weakly elementary embeddings in classical set theory). One point where this fails will be a critical point itself in the above sense, i.e. a K such that $j \restriction K = id_K$ but $K \neq j(K)$ (or equivalently $K \subsetneq j(K)$).

In the following, use the convention of $\overline{n} \in V({}^{\omega}\omega_{\text{ec}})$ being the implementation of the natural number n in the model such that

$$\overline{n} = \{(x \mapsto m, \overline{m}) | m < n\} \tag{9.4.194}$$

Also, ω can be implemented as

$$\overline{\omega} = \{(x \mapsto m, \overline{m}) | m \in \omega\} \tag{9.4.195}$$

Then the statement $\overline{\omega} = \omega$ is realized. Note that this is not the usual implementation of natural numbers and is specific to this pca, but it enables a quicker presentation of the following arguments. This alternative implementation works as the mapping of the constant functions of the form $x \mapsto m$ to the usual implementation of natural numbers m (as certain terms in s and k) is representable in the pca, as is its inverse.

Definition 9.75. *For $n \in \omega$, let $f_n \in {}^{\omega}\omega_{ec}$ be the function defined by*

$$f_n(m) = \begin{cases} n & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases} \tag{9.4.196}$$

Define $K \in V({}^{\omega}\omega_{ec})$ as

$$K := \{(f_n, \overline{n}) | n \in \omega\} \tag{9.4.197}$$

Looking from the outside, K contains exactly the natural numbers, but with unusual realizers. From the inside, the situation is slightly more complicated.

Lemma 9.76. *The following are realized:*

1. *K is transitive, i.e.*

$$\forall x \in K \forall y \in x . y \in K \tag{9.4.198}$$

2. *K is a set of transitive sets, i.e.*

$$\forall x \in K \forall y \in x \forall z \in y . z \in x \tag{9.4.199}$$

3. *K contains all natural numbers, i.e. $\omega \subseteq K$*

4. *K consists not only of natural numbers, i.e. $\neg K \subseteq \omega$*

5. *The elements of K consist only of natural numbers, i.e. $\bigcup K = \omega$*

6. *The elements of K are fixed points of j, i.e. $\forall x \in K.j(x) = x$*

7. *K itself is no fixed point of j, i.e. $K \subsetneq j(K)$*

8. *K is not in M*

Proof. 1. Let $(e, x) \in K$, $(f, y) \in x$. Then by the definition of K, the set x must be equal to $\overline{n_1}$ for some $n_1 \in \omega$. Thus by the definition of $\overline{n_1}$,

$$\exists n_2 \in n_1.f = (m \mapsto n_2)_{m\in\omega} \wedge y = \overline{n_2} \tag{9.4.200}$$

Let e_r being the value of some constant function e_r' realizing the reflexivity of equality. Then the above implies

$$p(f_{n_2}, e_r') \Vdash y \in K \tag{9.4.201}$$

Thus the following realizer works as desired:

$$\omega \ni n \mapsto \begin{cases} \lambda e\lambda f.p(f, e_r) & \text{if } n = 0 \\ \lambda e\lambda f.p(0, e_r) & \text{if } n \neq 0 \end{cases} \tag{9.4.202}$$

Here $\lambda x.t$ and p are used as an abbreviation for the index of a Turing machine, not a member if the pca ${}^{\omega}\omega_{\text{ec}}$. It should be understood that the input $\lambda x.t$ uses for x is the second component of the first entry (entry number 0) of its oracle tape.

2. Let $(e, x) \in K$, $(f, y) \in x$, $(g, z) \in y$. Then by the definition of K, the set x must be equal to $\overline{n_1}$ for some $n_1 \in \omega$. Thus by the definition of $\overline{n_1}$, the set y must be equal to $\overline{n_2}$ for some $n_2 \in n_1$ and by the definition of $\overline{n_2}$ it follows that

$$\exists n_3 \in n_2.g = (m \mapsto n_3)_{m\in\omega} \wedge z = \overline{n_3} \tag{9.4.203}$$

This means that for some e_r being the constant value of some constant function e_r' realizing the reflexivity of equality,

$$p(f_{n_3}, e_r') \Vdash y \in K \tag{9.4.204}$$

Thus the following realizer works as desired:

$$\omega \ni n \mapsto \begin{cases} \lambda e \lambda f \lambda g.p(g, e_r) & \text{if } n = 0 \\ \lambda e \lambda f \lambda g.p(0, e_r) & \text{if } n \neq 0 \end{cases} \tag{9.4.205}$$

3. To realize the equivalent $\forall x \in \overline{\omega}.x \in K$, let $(e, x) \in \overline{\omega}$. Then by the definition of $\overline{\omega}$,

$$\exists n \in \omega.g = (m \mapsto n)_{m\in\omega} \wedge x = \overline{n} \tag{9.4.206}$$

This means that for some e_r being the constant value of some constant function e_r' realizing the reflexivity of equality,

$$p(f_n, e_r') \Vdash x \in K \tag{9.4.207}$$

Thus the following realizer works as desired:

$$\omega \ni n \mapsto \begin{cases} \lambda e.p(e, e_r) & \text{if } n = 0 \\ \lambda e.p(0, e_r) & \text{if } n \neq 0 \end{cases} \tag{9.4.208}$$

4. Assume for contradiction $e \Vdash \forall x \in K.x \in \overline{\omega}$. Then for all elements of K, i.e. for all $(f_n, \overline{n}) \in K$

$$\exists((ef_n)_0, \overline{(ef_n)_0}) \in \overline{\omega}.(ef_n)_1 \Vdash \overline{n} = \overline{(ef_n)_0} \tag{9.4.209}$$

But for all natural numbers $\overline{n_1}$ and $\overline{n_2}$, their equality can only be realized if they actually stem from equal numbers $n_1 = n_2$. So

$$n = (ef_n)_0 \tag{9.4.210}$$

Equality between functions is extensional in set theory and repeating this argument for $n + 1$, this means

$$\forall m.(ef_n)_0(m) = n \neq n + 1 = (ef_{n+1})_0(m) \tag{9.4.211}$$

But as f_n and f_{n+1} only differ in their value at point 0, this means that the calculation of the mth component of ef_n uses at least m entries of the oracle tape, which is a contradiction to the boundedness imposed on oracle consumption.

5. As $\omega \subseteq K$, it follows that $\omega = \bigcup \omega \subseteq \bigcup K$, so the only thing that remains to be demonstrated is a realizer for

$$\forall x \in K \forall y \in x.y \in \omega \tag{9.4.212}$$

Let $(e, x) \in K$ and $(f, y) \in x$. Then by the definition of K, the set x must be equal to $\overline{n_1}$ for some $n_1 \in \omega$. Thus by the definition of $\overline{n_1}$ it follows that

$$\exists n_2 \in \omega.f = (m \mapsto n_2)_{m \in \omega} \wedge y = \overline{n_2} \tag{9.4.213}$$

This means that for e'_r being some function realizing the reflexivity of equality,

$$p(f_{n_2}, e'_r) \Vdash y \in K \tag{9.4.214}$$

Thus the following realizer works as desired:

$$\lambda x \lambda y.p(y, e'_r) \tag{9.4.215}$$

6. Let $(e, x) \in K$. Then by the definition of K, the set x must be equal to $\overline{n}$ for some $n \in \omega$. A direct induction over n shows that all the sets $\overline{n}$ have the property that $j(\overline{n}) = n$, so for e_r' being some function realizing the reflexivity of equality,

$$\lambda x.e_r \tag{9.4.216}$$

is a realizer that works as desired.

7. As the elements of K are fixed points of j, it follows directly that $K \subseteq j(K)$. For the negation of the converse, suppose for contradiction that there exists a realizer e such that

$$\forall (f, x) \in j(K) \exists ((ef)_0, y) \in K.(ef)_1 \Vdash x = y \tag{9.4.217}$$

By the definitions of K and j, these (f, x) must all be of the form $(j(f_n), j(\overline{n}))$ and $j(\overline{n}) = n$, with such an element existing for every $n \in \omega$. Furthermore, $((ef)_0, y) \in K$ must be of the form $(f_m, \overline{m})$. As $\overline{n}$ is realizedly equal to $\overline{m}$, they must be equal themselves, i.e. $n = m$. In particular,

$$f_n = (ej(f_n))_0 \tag{9.4.218}$$

This means that there is an element in the pca ${}^{\omega}\omega_{\text{ec}}$ which calculates f_n from $j(f_n)$ for all n. But the value of f_n at 0, which is n, i.e. always different, must be calculated using only the value of $j(f_n)$ at zero, which is always 0, and the value of $j(f_n)$ at ∞, which is always 0 as well. This is a contradiction.

8. Recall the result that if K were realized to be in M, there would be an element $j(A)$ of M realizedly equal to K. Then because $j(A)$ needs to be a superset of K and j acts as the identity on K, in particular there is a realizer e such that

$$\exists x_0, x_1 \in M.(ef_0, x_0) \in j(A) \wedge (ef_1, x_1) \in j(A) \wedge \Vdash x_0 = 0 \wedge \ Vdash x_1 = 1 \tag{9.4.219}$$

As on the other hand $j(A)$ is also a subset of K, there is a realizer f such that

$$fef_0 = f_0 \wedge fef_1 = f_1 \tag{9.4.220}$$

But as $(ef_i) \in {}^{\omega}\omega_{\text{ec},0}$ for $i = 0, 1$ (i.e. as functions their value at place 0 is equal to their value at place ∞), the different value of the functions fef_0 and fef_1 at place 0 implies that the values of ef_1 and ef_0 need to have differed at place ∞. But that cannot be as it would be in direct contradiction with the boundedness of the consumption of oracle tape just as in point 4.

□

Theorem 9.77. *1. The weakly elementary embedding $j : V \to M$ has a critical point K in the sense that $j \restriction K = id_K$ but $K \neq j(K)$.*

2. The map $j : V \to M$ is neither strongly set cofinal nor strongly cofinal.

Proof. The critical point were points 6 and 7 of lemma 9.76. A set witnessing that j is not strongly set cofinal is K, as shown in 8 of lemma 9.76, noting that M is realized to be transitive. This is also a witness to j not being strongly cofinal as K is an ordinal as per points 1 and 2 of lemma 9.76. □

Corollary 9.78. *1. The proof theoretic strength of CZF plus a weakly measurable cardinal is equal to that of CZF.*

2. If CZF is consistent, then it does not prove that all weakly elementary embeddings are cofinal.

Both statements become false if CZF is replaced by ZFC.

9.4.4 The Limits of the Methods used in this Chapter

The model construction detailed above relies on a pca monomorphism $j : A \to A$ on some pca A with certain properties which is then lifted to a homomorphism of realizability models $j : V(A) \to V(A)$ by defining recursively

$$j(A) := \{(j(e), j(a)) | (e, a) \in A\} \tag{9.4.221}$$

This manifests itself in the model $V(A)$ as a weakly elementary embedding. A fully elementary embedding remains tantalizingly close: While it is true that

$$\Vdash \Phi(\overrightarrow{x}) \leftrightarrow \Vdash \Phi^M(\overrightarrow{j(x)}), \tag{9.4.222}$$

only the direction from left to right has been proved to be represented in the pca. This is not an incomplete result but due to the general limitations of this method:

Theorem 9.79. *If in a situation as described above there is a realizer for*

$$\forall \overrightarrow{x}.\Phi^M(\overrightarrow{j(x)}) \to \Phi(\overrightarrow{x}) \tag{9.4.223}$$

then there is a left inverse for j representable in the pca and thus

$$\Vdash \forall a.j(a) = a \tag{9.4.224}$$

Proof. Assume there was such a realizer at least in the special case consisting of the following formula:

$$e \Vdash \forall x, y.j(x) = j(y) \to x = y \tag{9.4.225}$$

Then define

$$a := \{(k, \emptyset)\} \in V(A) \tag{9.4.226}$$

For all $f \in A$, define

$$b_f := \{(f, \emptyset)\} \in V(A) \tag{9.4.227}$$

And for e_r some realizer for $\emptyset = \emptyset$, let

$$g_f := p(\lambda x.p(j(f), e_r), \lambda x.p(k, e_r)) \in A \tag{9.4.228}$$

Note that $j(a) = a$ and $j(b_f) = \{(j(f), \emptyset)\}$, so

$$g_f \Vdash j(a) = j(b_f) \tag{9.4.229}$$

Thus

$$eg_f \Vdash a = b_f \tag{9.4.230}$$

So for each $(k, \emptyset) \in a$, there is a $(((eg_f)_0 k)_0, \emptyset) \in b_f$. But as there is only one element in b_f, this means

$$((eg_f)_0 k)_0 = f \tag{9.4.231}$$

And thus by abstraction

$$\forall f.(\lambda x.((ep(\lambda y.p(j(x), e_r), \lambda y.p(k, e_r)))_0 k)_0) j(f) = f \tag{9.4.232}$$

So the left inverse of j is representable in the pca via the element

$$e' := \lambda x.((ep(\lambda y.p(j(x), e_r), \lambda y.p(k, e_r)))_0 k)_0 \tag{9.4.233}$$

From that it is very easy to find a realizer for $\forall a.a = j(a)$. For example the following realizer defined with the recursion theorem in A works as desired:

$$e'' := p(\lambda x.p(jx, e''), \lambda x.p(e'x, e_s e'')) \tag{9.4.234}$$

Where e_s is a realizer for $\forall x, y.x = y \to y = x$.

Then this realizes j being the identity by induction: For $(f, x) \in a$ there is $(j(f), j(x)) \in j(a)$ with $((e''_0)f)_0 = j(f)$ and by induction hypothesis

$$((e_0'')f)_1 = e'' \Vdash x = j(x) \tag{9.4.235}$$

Conversely for $(f, x) \in j(a)$, $f = j(f')$ there are f', x' with $x = j(x')$ and $(f', x') \in a$. Then $((e_1'')f)_0 = e'f = f'$ and by the induction hypothesis

$$((e_1'')f)_1 = e_s e'' \Vdash x = x' \text{ where x=j(x')} \tag{9.4.236}$$

$\square$

Thus if this method is to produce a nontrivial $j : V \to M$, it can at most be proved to be weakly elementary, never fully elementary.

9.5 Stronger Assumptions yield a Fully Elementary Embedding refuting Cofinality

Last section demonstrated that assuming the consistency of CZF, the embedding associated to the large set axiom of weak measurability cannot be proved to be cofinal. If we want to extend this result to fully elementary embeddings, it stands to reason that a stronger assumption is needed as then the consistency strength increases. In the result for this section, this will be the existence of a measurable set (which holds for instance in models of ZFC with a measurable cardinal).

Let in the following $j : V \to M$ be an elementary embedding. Consider the realizability model $V(Kl)$ as described in [Rat03b] augmented by realizability for the symbols for j and $\in M$ as presented in Definition 4.23. Recall that the interpretation of j is chosen in such a way that for $x \in V(Kl)$ the formula $j^n(x) = y$ is realized if and only if for $x' := j^n(x)$ the formula $x' = y$ is realized, and analogously for $\in$. A realizer for $x \in M$ is any realizer which realizes $x = y$ for some $y \in M$.

In the following we will rely on Kl being unaffected by j. Not only are its underlying set and its elements fixed points of j, but the structure of Kl is fixed as well:

Remark 9.80. *For $e, f \in Kl$*

$$\exists g \in \omega(e,f,g) \in \circ \leftrightarrow V \vDash e \circ f \downarrow \leftrightarrow M \vDash e \circ f \downarrow \leftrightarrow \exists g \in \omega(e,f,g) \in j(\circ) \quad (9.5.237)$$

and for g also in Kl

$$(e,f,g) \in \circ \leftrightarrow V \vDash e \circ f = g \leftrightarrow M \vDash e \circ f = g \leftrightarrow (e,f,g) \in j(\circ) \quad (9.5.238)$$

Proof. The relation $\circ \subseteq \omega^3$ can be and defined by a bounded formula whose quantifiers are bounded by fixed points of j (i.e. ω and its elements) and its domain consists of fixed points as well (i.e. triples of natural numbers). $\square$

9.5.1 What is and is not realized to be in M

As $M \cap V(Kl) = (V(Kl))^M$, we will use the convention $M(Kl)$ to refer to this class. By Definition 4.23, anything that is realizably equal to an element of $M(Kl)$ is realized to be in M. Thus if $e \Vdash x \in M$, the set $x \in V(Kl)$ does not actually have to be an element of $M(Kl)$. For example if $a \notin M$, then the set $a^k := \{(k, x^k)|x \in a\}$ cannot be in M either, as M is closed under the construction $x \mapsto x^k$. However, if $a \subsetneq b \in M$, then b^k is in $M(Kl)$, and b^k is realizably equal to

$$b^k \cup \{(skk, x^k)|x \in a\} \quad (9.5.239)$$

This set however can not be in $M(Kl)$, even if it is realized to be in M.

Thus to show that something is not realized to be in M, it does not suffice to show that it is not a member of $M(Kl)$. Except in special circumstances:

Definition 9.81. *Define the class of **slim** sets $a \in V(Kl)$ recursively by letting a be slim iff both of the following hold*

1. *For all $(e, x) \in a$, the set $x \in V(Kl)$ is slim.*
2. *For all $(e, x) \in a$ and $(e', x') \in a$ and $\Vdash x = x'$, it follows that $e = e'$ and $x = x'$.*

Slim sets have some convenient properties:

Definition 9.82. *If $e \in Kl$ and $a \in V(Kl)$, define recursively*

$$a_e := \{((e_1 f)_0, x_{(e_1 f)_1}) | (f, x) \in a\} \in V(Kl) \tag{9.5.240}$$

Proposition 9.83. *Let $a \in V(Kl)$ be slim, $e \in \omega$ and $b \in V(Kl)$ be such that $e \Vdash a = b$. Then*

$$a = b_e \tag{9.5.241}$$

Proof. The proof proceeds by induction over a and b.

Take an arbitrary element of b_e, it has the form $((e_1 f)_0, x_{(e_1 f)_1})$ where $(f, x) \in b$. Thus by the condition on e,

$$\exists((e_1 f)_0, y) \in b.(e_1 f)_1 \Vdash x = y \tag{9.5.242}$$

By the induction hypothesis, y must be equal to $x_{(e_1 f)_1}$. Thus $((e_1 f)_0, x_{(e_1 f)_1}) \in a$ and by abstraction $b_e \subseteq a$.

Now take an arbitrary element $(f, x) \in a$. Then

$$\exists((e_0 f)_0, y) \in b.(e_0 f)_1 \Vdash x = y \tag{9.5.243}$$

Going back to a yields

$$\exists((e_1(e_0 f)_0)_0, z) \in a.(e_1(e_0 f)_0)_1 \Vdash y = z \tag{9.5.244}$$

By transitivity, $\Vdash x = z$ and so as a is slim, $x = z$ and $(e_1(e_0 f)_0)_0 = f$. But the definition of b_e implies that 9.5.243 also yields

$$((e_1(e_0 f)_0)_0, y_{(e_1(e_0 f)_0)_1}) \in b_e \tag{9.5.245}$$

But the first component of this is equal to f as established above and the second component equal to x by the induction hypothesis, which is applicable since $(f, x) \in a$ implies that x is also slim. Thus $a \subseteq b_e$. $\square$

Corollary 9.84. *Let a, b and b' be slim sets. Then*

$$e \Vdash a = b \wedge e \Vdash a = b' \to b = b' \tag{9.5.246}$$

In particular for e_r a realizer for reflexivity,

$$b = b' \leftrightarrow e_r \Vdash b = b' \tag{9.5.247}$$

Proof. By proposition 9.83, $e \Vdash a = b$ and $e \Vdash a = b'$ imply that $b = a_e = b'$. Setting $a := b$ implies the second statement.

$\square$

Slimness is basically a bounded property:

Proposition 9.85. *If N is a transitive model of set theory, then for all $a \in N(Kl)$, a is slim iff $N \vDash a$ is slim.*

Proof. This is direct by induction over a, noting that the induction hypothesis can always be applied because $(e, x) \in a \in N$ implies $x \in N$. $\square$

For slim sets, it is easy to decide whether they are realized to be in M.

Theorem 9.86. *Let $a \in V(Kl)$ be slim and e_r be a realizer for reflexivity of equality. Then*

$$a \in M(Kl) \leftrightarrow e_r \Vdash a \in M \leftrightarrow \Vdash a \in M \tag{9.5.248}$$

Proof. The directions from left to right are trivial, so it suffices to show the implication

$$(\Vdash a \in M) \to (a \in M(Kl)) \tag{9.5.249}$$

So let for some $b \in M(Kl)$

$$e \Vdash a = b \tag{9.5.250}$$

As a is slim, proposition 9.83 implies

$$a = b_e \tag{9.5.251}$$

But a quick induction over x shows that M is closed under the operation $(x, f) \mapsto x_f$ (as it was defined by bounded recursion) and since b and e are elements of M, this then means that a must be an element of M (and thus of $M(Kl)$) as well.

□

Now we turn to the question of refuting cofinality in this model. This relies on constructive logic giving a lot of leeway to the world of ordinals and there being a great wealth of ordinals in $V(Kl)$. In particular there are ordinals in $V(Kl)$ corresponding to every element of ${}^{\alpha}\omega$ for α some ordinal from the background universe:

Definition 9.87. *Define for ordinals α and functions $f : \alpha \to \omega$ recursively over α the symbol α^f as*

$$\alpha^f := \{(f(\beta), \beta^{f\restriction\beta}) | \beta \in \alpha\} \in V(Kl) \tag{9.5.252}$$

This fulfills a strong injectivity property:

Lemma 9.88. *Let $\alpha, \beta \in O_n$ and $f : \alpha \to \omega$, $g : \beta \to \omega$. Then*

$$(\Vdash \alpha^f = \beta^g) \to (\alpha = \beta) \tag{9.5.253}$$

Proof. This is done by induction.

If $e \Vdash \alpha^f = \beta^g$, then in particular for every element $(e, \gamma^{f'}) \in \alpha^f$, there is a $(h, \delta^{g'}) \in \beta^g$ such that $\gamma^{f'}$ is realizedly equal to $\delta^{g'}$ and vice versa.

By the induction hypothesis, this means that $\gamma = \delta$, and thus for every element of α, there is an equal element of β and vice versa. In other words, $\alpha = \beta$. □

Proposition 9.89. *Let $\alpha \in O_n$ and $f : \alpha \to \omega$. Then α^f is slim and realized to be an ordinal.*

Proof. This is shown by induction over α.

For the first part of the definition of slimness, note that by the induction hypothesis, all $\beta^{f\upharpoonright\beta}$ are slim for $\beta \in \alpha$.

For the second part, take two arbitrary elements of α^f, they are of the form

$$(f(\beta), \beta^{f\upharpoonright\beta}), (f(\beta'), \beta'^{f\upharpoonright\beta}) \in \alpha^f \tag{9.5.254}$$

where $\beta, \beta' \in \alpha$. Suppose there is a realizer

$$e \Vdash \beta^{f\upharpoonright\beta} = \beta'^{f\upharpoonright\beta'} \tag{9.5.255}$$

Then by lemma 9.88, $\beta = \beta'$. Thus both elements are identical, which is as desired by the second part of the definition of slimness.

That α^f is an ordinal can be expressed as

$$\forall x \in \alpha^f \forall y \in x \forall z \in y. y \in \alpha^f \wedge z \in x \tag{9.5.256}$$

This is realized by a fixed point of the following equation where e_r is a realized for reflexivity of identity:

$$e = \lambda x \lambda y \lambda z. p(p(y, e_r), p(z, e_r)) \tag{9.5.257}$$

To see this, let $(g, x) \in \alpha^f, (h, y) \in x, (i, z) \in y$. Then the goal is to show that

$$(eghi)_0 = p(h, e_r) \Vdash y \in \alpha^f \tag{9.5.258}$$

and

$$(eghi)_1 = p(i, e_r) \Vdash z \in x \tag{9.5.259}$$

For 9.5.258, note that $h = f(\gamma)$ for some γ with $y = \gamma^{f\restriction\gamma}$ by the definition of $x \mapsto x^f$ and choice of x and y. Thus $p(h, e_r) \Vdash y \in \alpha^f$, again by the definition of $x \mapsto x^f$.

For 9.5.259, note that $i = f(\gamma)$ for some γ with $z = \gamma^{f\restriction\gamma}$ by the definition of $x \mapsto x^f$ and choice of x, y and z. Also, $x = \delta^{f\restriction\delta}$ for some $\delta \ni \gamma$. Thus $p(i, e_r) \Vdash z \in x$, again by the definition of $x \mapsto x^f$. □

Putting it all together leads to

Proposition 9.90. *If there is any $\alpha \in O_n$ and $f : \alpha \to \omega$ with $f \notin M$, then*

$$\Vdash \exists\beta \in O_n.\beta \notin M \tag{9.5.260}$$

In particular, j is not strongly cofinal.

Proof. Let $f : \alpha \to V$ be such a function $f \notin M$. Then also $\alpha^f \notin M$ as f can be retrieved from α^f by absolute (by Δ_0-recursion definable) set theoretic operations. Then by Theorem 9.86, it cannot be realized that α^f is in M. But as it is an ordinal, it is realized that there are ordinals outside of M.

If j were realized to be strongly cofinal, then there would be a $\beta \in V(Kl)$ with $\alpha^f \in j(\beta)$ being realized. By transitivity of M, it would also be realized that $\alpha^f \in M$, which contradicts the first part of the proof. □

Indeed, while classically all elementary embeddings are strongly cofinal so that all ordinals lie in M, the condition of proposition 9.90 that there is a numerical function on an ordinal that is outside of M is still easy to fulfill.

Proposition 9.91. *The following theories are equiconsistent:*

1. *ZFC plus a measurable[3] embedding $j : V \to M$*
2. *ZFC plus a measurable embedding $j : V \to M$ such that there is an ordinal α and function $f : \alpha \to 2$ for which $f \notin M$*

[3] i.e. a nontrivial elementary embedding j so that the axiom schemes of ZFC also hold in the language $L(\in, j)$

Proof. As the second part implies the first one outright, only the direction from 1 to 2 needs to be shown. Argue in the specified theory.

By the usual arguments (see for example [Kun80]), it is possible to construct a relative model to this theory with a measurable embedding $j : V \to M$ with a critical point κ such that there is an ultrafilter $U \subseteq \mathcal{P}(\kappa)$ and j is the embedding associated to this ultrafilter, i.e. M is the Mostowski collapse of the ultrapower of V over U. By [Kun80] Proposition 5.7, $U \notin M$ but $\mathcal{P}(\kappa) = \mathcal{P}(\kappa)^M \in M$.

By choice in M, let $f : \alpha \hookrightarrow\!\!\!\!\rightarrow \mathcal{P}(\kappa)$ be a well ordering of $\mathcal{P}(\kappa)$ with $f \in M$. Then define the function $g : \alpha \to 2$ by

$$g(\beta) := \begin{cases} 1 \text{ iff } f(\beta) \in U \\ 0 \text{ iff } f(\beta) \notin U \end{cases} \tag{9.5.261}$$

This is a function from an ordinal to 2 and can not be an element of M lest

$$U = \{f(\beta) | \beta \in \alpha \wedge g(\beta) = 1\} \tag{9.5.262}$$

also be in M. □

Thus we obtain a new result about the refutation of cofinality of measurable embeddings:

Corollary 9.92. *Provided measurable cardinals are consistent with classical set theory, CZF does not prove that measurable embeddings are strongly cofinal.*

9.6 Hitting the Ceiling

The main theorem in this section is the following:

Theorem 9.93. *(There is a measurable set + Subcountability)*

$$1 = 0 \tag{9.6.263}$$

This is a direct consequence of the following more positive proposition:

Proposition 9.94. *($j : V \overset{\equiv}{\hookrightarrow} M$ + Subcountability)*

$$\forall a. j(a) = a \tag{9.6.264}$$

For this we first need an easy lemma:

Lemma 9.95. *($j : V \overset{\equiv}{\hookrightarrow} M$)*

$$\forall n \in \omega. j(n) = n \wedge j(\omega) = \omega \tag{9.6.265}$$

Proof. We prove $j(n) = n$ by induction over n. For the induction beginning, note that $\nexists x. x \in 0$ so by elementarity $\nexists x \in M. x \in j(0)$. As any element of $j(0)$ would be in M by transitivity of M, this implies $j(0) = 0$.

For the induction step, assume $j(n) = n$. Then $x \in n + 1 \leftrightarrow x \in n \vee x = n$, so by elementarity

$$\forall x \in M. x \in j(n+1) \leftrightarrow x \in j(n) = n \vee x = j(n) = n \tag{9.6.266}$$

As any element of $j(n+1)$ would be in M, this implies $x \in j(n+1) \leftrightarrow x \in n \vee x = n$ for any x, so $j(n+1) = n + 1$.

To prove $j(\omega) = \omega$, note that $n \in \omega \to n = j(n) \in j(\omega)$, so $j(\omega) \supseteq \omega$. The natural numbers fulfill

$$\forall n \in \omega. n = 0 \vee \exists m \in \omega. n = m + 1 \tag{9.6.267}$$

So the same is true for $j(\omega)$ (note that $n = m + 1$ is absolute as it is Δ_0):

$$\forall n \in j(\omega). n = 0 \vee \exists m \in j(\omega). n = m + 1 \tag{9.6.268}$$

So prove by set induction that $j(\omega) \subseteq \omega$: Let $x \in j(\omega)$, then either it is 0, in which case it is an element of ω, or it is $m + 1$ for some element of $j(\omega)$ which is a natural number by induction hypothesis. In either case, $x \in \omega$. $\square$

Proof of Proposition 9.94. We proceed by set induction over a. So take an a such that

$$\forall x \in a . j(x) = x \tag{9.6.269}$$

By Subcountability, let $A \subseteq \omega$ and $f \subseteq A \times a$ with

$$f : A \twoheadrightarrow a \tag{9.6.270}$$

Then also

$$j(f) : j(A) \twoheadrightarrow j(a) \tag{9.6.271}$$

And $\mathrm{dom}(j(f)) = j(\mathrm{dom}(f)) = j(A)$.

For all $n \in \omega$, $j(n) = n$ by Lemma 9.95, so if $(n, x) \in f$ then also

$$(n, x) = (j(n), j(x)) \in j(f) \tag{9.6.272}$$

Thus

$$j(f) \upharpoonright A = f \tag{9.6.273}$$

A natural number n fulfills $n \in A$ iff $n = j(n) \in j(A)$ by elementarity, and all elements of $j(A)$ are natural numbers since $j(A) \subseteq j(\omega) = \omega$, so

$$A = j(A) \tag{9.6.274}$$

Consequently

$$j(f) = j(f) \upharpoonright j(A) = j(f) \upharpoonright A = f \tag{9.6.275}$$

And thus

$$a = f''A = j(f)''j(A) = j(a) \tag{9.6.276}$$

□

Note that the exact same proof also yields the following result, whose main motivation lies in the fact that the hereditarily subcountable sets often constitute a (class) model of CZF itself[4]:

Corollary 9.96. *($j : V \overset{\equiv}{\hookrightarrow} M$ or $j : V \overset{\equiv}{\hookrightarrow}_{\Delta_0} M$) All hereditarily subcountable sets are unmoved by j.*

This has the interesting consequence of refuting a considerable weakening[5] of Subcountability, which has recently been found to be validated in certain category theoretic models of set theory, namely the axiom $V = V_{\omega_1}$ (see [SS12]).

Definition 9.97.

$$\begin{aligned} \omega_1 \ := \ & \{rk(a) | a \textit{ is hereditarily subcountable}\}) && (9.6.277) \\ & = rk(\{a | a \textit{ is hereditarily subcountable}\}) && (9.6.278) \\ & = \{\alpha \in O_n | \alpha \textit{ is hereditarily subcountable}\} && (9.6.279) \\ & && (9.6.280) \end{aligned}$$

Note that neither CZF nor CZF + REA prove ω_1 to be a set. In contrast, [SS12] use a different definition of ω_1 which can be shown to be a set in CZF + REA. As we want the strongest possible refutation result, we aim at defining ω_1 in a way which makes $V = V_{\omega_1}$ as weak as possible, and thus define ω_1 to be as large as possible. It can easily be seen that $\omega_1 \supseteq \omega_1^\sharp \supseteq \omega_1^\flat$ where $\omega_1^\sharp$ and $\omega_1^\flat$ are the versions of ω_1 from [SS12]. So if $V = V_{\omega_1}$ does not hold in a certain model, then neither do $V = V_{\omega_1^\sharp}$ and $V = V_{\omega_1^\flat}$.

Combining Propositions 9.55 and 9.59 with the previous corollary yields the following result directly:

Corollary 9.98. *($j : V \overset{\equiv}{\hookrightarrow} M$ or $j : V \overset{\equiv}{\hookrightarrow}_{\Delta_0} M$) V_{ω_1} is a subcritical point of j, i.e. all its elements fulfill $j(x) = x$.*

[4]e.g. in the standard Kleene realizability model if all subsets of ω are bases in the background theory

[5]It is an obvious weakening when employing our Definition 9.97 of ω_1. With the alternative definitions mentioned below, it seems that it neither implies, nor is implied by, Subcountability. The inconsistency results hold for these as well.

In particular, if $V = V_{\omega_1}$, then there can not exist a measurable set.

Theorem 9.93 has both practical and philosophical implications. On the practical side, incompability with Subcountability makes it much harder to analyze the proof-theoretic strength of axioms about elementary embeddings, as the now well-worn track of interpreting constructive set theory with large cardinals via an appropriate type theory into a Kripke-Platek style set theory (e.g. presented in [GR94] and employed in [Rat02] and [Gib02]) turns into a cul-de-sac, seeing that at least without further modification, such a model would also validate Subcountability and thus not the existence of elementary embeddings.

On a more fundamental note, depending on one's preferred style of constructivism, this result can cast serious doubt on the admissibility of elementary embeddings to constructive reasoning. Actual Subcountability or at least compatibility with Subcountability can be seen as essential to some brands of constructivism (e.g. schools of thought inspired by Russian Constructivism or by Classical Finitism), so that asserting the existence of any nontrivial elementary embedding could in the light of Theorem 9.93 be regarded as taboo for the conscientious constructivist.

On the other hand, this situation might merely be interpreted as an indication that the specifically constructive axiom of Subcountability could play a role for the constructive theory of large sets similar to that which the axiom $V = L$ plays in classical set theory: An arguably sensible assumption which limits the universe to contain only the easily graspable sets (either constructible or subcountable ones) and which constitutes a watershed for axioms about largeness, being consistent with the small ones but denying the large ones which demand or imply the existence of nontrivial elementary embeddings in the universe.

Whether this watershed merits being crossed might be a debatable question, and the answer might well also depend on what exactly we find on the other side.

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