On the finiteness and periodicity of the p–adic Jacobi–Perron algorithm

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Abstract

Multidimensional continued fractions (MCFs) were introduced by Jacobi and Perron to obtain periodic representations for algebraic irrationals, analogous to the case of simple continued fractions and quadratic irrationals. Continued fractions have been studied in the field of p–adic numbers $\mathbb{Q}_p$. MCFs have also been recently introduced in $\mathbb{Q}_p$, including in particular a p–adic Jacobi–Perron algorithm. In this paper, we address two of the main features of this algorithm, namely its finiteness and periodicity. Regarding the finiteness of the p–adic Jacobi–Perron algorithm, our results are obtained by exploiting properties of some auxiliary integer sequences. It is known that a finite p–adic MCF represents $\mathbb{Q}$–linearly dependent numbers. However, we see that the converse is not always true and we prove that in this case infinitely many partial quotients of the MCF have p–adic valuations equal to -1. Finally, we show that a periodic MCF of dimension m converges to an algebraic irrational of degree less or equal than $m+1$; for the case $m = 2$, we are able to give some more detailed results.

Keywords: continued fractions, finiteness, Jacobi–Perron algorithm, multidimensional continued fractions, p–adic numbers, periodicity
2010 Mathematics Subject Classification: 11J70, 12J25, 11J61.

1 Introduction

Multidimensional continued fractions (MCFs) were introduced by Jacobi [20] to answer a question that had originally been posed by Hermite [19], namely the existence of an algorithm defined over the real numbers that eventually becomes periodic when it processes algebraic irrationalities. In other words, Hermite asked for a generalization of the classical continued fraction algorithm that produces a periodic expansion if and only if the input is a quadratic irrational. The Jacobi algorithm deals with cubic irrationals and it was generalized to higher dimensions by Perron [28]. However, the Jacobi–Perron algorithm does not solve the Hermite problem because it has not been proven that it eventually becomes periodic when processing algebraic irrationals. Nevertheless, many studies have been conducted on MCFs and their modifications; see, for example, [1], [2], [5], [13], [14], [18], [21], [24], [25], [32], [33], [34].

One–dimensional continued fractions over the p–adic numbers started to be studied in the 1970s [7], [30], [31]. From these studies, it appeared difficult to find an algorithm working on the p–adic numbers that produces continued fractions which have the same properties that hold true over the real numbers (i.e., concerning approximation, finiteness and periodicity). In particular, no algorithm was found that provides a periodic expansion for all quadratic irrationalities. Continued fractions over the p–adic numbers have also been recently studied; see, for example, [3], [4], [8], [9], [17], [22], [23], [27], [29], [35], [36].

Motivated by the research, in [26] the authors started the study of MCFs in $\mathbb{Q}_p$, and provided some results about convergence and finiteness. In particular, they gave a sufficient condition of the partial quotients of a MCF that ensures the convergence in $\mathbb{Q}_p$. Moreover, they presented an algorithm that terminates in finitely many steps when rational numbers are processed. The scope of that work was to introduce the subject and provide some general properties. However, the terminating input of this algorithm was not fully characterized and the periodicity properties were not studied. This paper represents a continuation of the previous work and it extends the investigation in these two directions. In particular, in Section 2, we fix the notation and we

show some properties that can also be of general interest for MCFs. Section 3 is devoted to the finiteness of the p-adic Jacobi–Perron algorithm. It also provides some results that improve on those presented in the previous paper [26] and which show some differences with the real case. Indeed, it is known that the real Jacobi–Perron algorithm in dimension 2 detects rational dependence (i.e. it terminates in finitely many steps if and only if it processes rational linearly dependent inputs). We show however that this is not always true in $\mathbb{Q}_p$ and we also prove that in this case infinitely many partial quotients of the MCF have p–adic valuations equal to -1. This allows us to give a condition that ensures the finiteness of the p–adic Jacobi–Perron algorithm in any dimension in terms of the p–adic valuation of its partial quotients.
In Section 4, we study the periodicity of MCFs in $\mathbb{Q}_p$. Specifically, we introduce the characteristic polynomial related to a purely periodic p–adic MCF and we see that, as in the real case, it admits a p–adic dominant root that generates a field containing the limits of the given MCF. We deduce that a periodic MCF of dimension m converges to algebraic irrationalities of a degree less than or equal to $m+1$, as in the real case. A further investigation of the characteristic polynomial allows us to characterize some cases where the degree is exactly $m+1$. We conclude our work with a conjecture that, if proven true, would give a characterization of the MCFs arising by applying the p-adic Jacobi-Perron algorithm to m-tuples consisting of $\mathbb{Q}$-linear dependent numbers.

2 Preliminaries and notation

The classical Jacobi–Perron algorithm processes an m–tuple of real numbers $\boldsymbol{\alpha}_0 = (\alpha_0^{(1)}, \dots, \alpha_0^{(m)})$ and represents them by means of an m–tuple of integer sequences $(\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(m)}) = ((a_n^{(1)})_{n\geq 0}, \dots, (a_n^{(m)})_{n\geq 0})$ (finite or infinite) determined by the following iterative equations:

$$\begin{cases} a_n^{(i)} = [\alpha_n^{(i)}], \quad i = 1, ..., m, \\ \alpha_{n+1}^{(1)} = \dfrac{1}{\alpha_n^{(m)} - a_n^{(m)}}, \\ \alpha_{n+1}^{(i)} = \dfrac{\alpha_n^{(i-1)} - a_n^{(i-1)}}{\alpha_n^{(m)} - a_n^{(m)}}, \quad i = 2, ..., m. \end{cases} \qquad n = 0, 1, 2, ... \tag{1}$$

The integer numbers $a_n^{(i)}$ and the real numbers $\alpha_n^{(i)}$, for $i = 1, \dots m$ and $n \in \mathbb{N}$, are called *partial quotients* and *complete quotients*, respectively. The sequences of the partial quotients represent the starting vector $\boldsymbol{\alpha}_0$ by means of the equations

$$\begin{cases} \alpha_n^{(i-1)} = a_n^{(i-1)} + \dfrac{\alpha_{n+1}^{(i)}}{\alpha_{n+1}^{(1)}}, \quad i = 2, ..., m \\ \alpha_n^{(m)} = a_n^{(m)} + \dfrac{1}{\alpha_{n+1}^{(1)}} \end{cases} \qquad n = 0, 1, 2, ... \tag{2}$$

which produce objects that generalize the classical continued fractions and are usually called *multidimensional continued fractions* (MCFs).

The Jacobi–Perron algorithm has been translated into the p–adic field in [26], substantially by replacing in (1) the floor function by the Browkin s-function defined below. We define the set

$$\mathcal{Y} = \mathbb{Z}\left[\frac{1}{p}\right] \cap \left(-\frac{p}{2}, \frac{p}{2}\right).$$

Definition 1. *The* Browkin s-function $s : \mathbb{Q}_p \longrightarrow \mathcal{Y}$ *is defined by*

$$s(\alpha) = \sum_{j=k}^{0} x_j p^j,$$

for every $\alpha \in \mathbb{Q}_p$ *written as* $\alpha = \sum_{j=k}^{\infty} x_j p^j$, *with* $k, x_j \in \mathbb{Z}$ *and* $x_j \in \left(-\frac{p}{2}, \frac{p}{2}\right)$.

The p–adic Jacobi–Perron algorithm processes an m–tuple of p–adic numbers $\boldsymbol{\alpha}_0 = (\alpha_0^{(1)}, \dots, \alpha_0^{(m)})$

by the following iterative rules

$$\begin{cases} a_n^{(i)} = s(\alpha_n^{(i)}) \\ \alpha_{n+1}^{(1)} = \dfrac{1}{\alpha_n^{(m)} - a_n^{(m)}} \\ \alpha_{n+1}^{(i)} = \alpha_{n+1}^{(1)} \cdot (\alpha_n^{(i-1)} - a_n^{(i-1)}) = \dfrac{\alpha_n^{(i-1)} - a_n^{(i-1)}}{\alpha_n^{(m)} - a_n^{(m)}}, \quad i = 2, ..., m \end{cases} \tag{3}$$

for $n = 0, 1, 2, \ldots$. Equations (3) define a p–adic MCF $[(a_0^{(1)}, a_1^{(1)}, \ldots), \ldots, (a_0^{(m)}, a_1^{(m)}, \ldots)]$ representing the initial m–tuple $\boldsymbol{\alpha}_0$ by the relations (2). The partial quotients satisfy the following conditions:

$$\begin{cases} |a_n^{(1)}| > 1 \\ |a_n^{(i)}| < |a_n^{(1)}|, \quad i = 2, \ldots, m \end{cases} \tag{4}$$

for any $n \geq 1$, where $|\cdot|$ denotes the p–adic norm. Moreover, for any $n \geq 1$, we have that

$$\begin{aligned} &|a_n^{(1)}| = |\alpha_n^{(1)}|, \text{ and for } i = 2, \ldots, m \\ &|a_n^{(i)}| = \begin{cases} |\alpha_n^{(i)}| \text{ if } |\alpha_n^{(i)}| \geq 1 \\ 0 \text{ if } |\alpha_n^{(i)}| < 1 \end{cases} \\ &|\alpha_n^{(i)}| < |\alpha_n^{(1)}|. \end{aligned} \tag{5}$$

Remark 1. *Conditions* (4) *ensure the convergence of a MCF in* $\mathbb{Q}_p$*, even if it is not obtained by a specific algorithm. In other words, given a sequence of partial quotients satisfying* (4)*, the corresponding MCF converges to a* m*–tuple of* p*–adic numbers, see [26].*

As with the real case, the *n–th convergents* of a multidimensional continued fraction are defined by

$$Q_n^{(i)} = \frac{A_n^{(i)}}{A_n^{(m+1)}},$$

for $i = 1, \ldots, m$ and $n \in \mathbb{N}$, where

$$A_{-j}^{(i)} = \delta_{ij}, \quad A_0^{(i)} = a_0^{(i)}, \quad A_n^{(i)} = \sum_{j=1}^{m} a_n^{(j)} A_{n-j}^{(i)} + A_{n-m-1}^{(i)} \tag{6}$$

for $i = 1, \ldots, m+1$, $j = 1, \ldots, m$ and any $n \geq 1$, where δ_{ij} is the Kronecker delta. It can be proved by induction that for every $n \geq 1$ and $i = 1, \ldots, m$, we have

$$\alpha_0^{(i)} = \frac{\alpha_n^{(1)} A_{n-1}^{(i)} + \alpha_n^{(2)} A_{n-2}^{(i)} + \ldots + \alpha_n^{(m+1)} A_{n-m-1}^{(i)}}{\alpha_n^{(1)} A_{n-1}^{(m+1)} + \alpha_n^{(2)} A_{n-2}^{(m+1)} + \ldots + \alpha_n^{(m+1)} A_{n-m-1}^{(m+1)}} \tag{7}$$

We can also use the following matrices for evaluating numerators and denominators of the convergents:

$$\mathcal{A}_n = \begin{pmatrix} a_n^{(1)} & 1 & 0 & \ldots & 0 \\ a_n^{(2)} & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_n^{(m)} & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad n \geq 0. \tag{8}$$

Indeed, if we put

$$\mathcal{B}_n = \begin{pmatrix} A_n^{(1)} & A_{n-1}^{(1)} & \ldots & A_{n-m}^{(1)} \\ A_n^{(2)} & A_{n-1}^{(2)} & \ldots & A_{n-m}^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ A_n^{(m+1)} & A_{n-1}^{(m+1)} & \ldots & A_{n-m}^{(m+1)} \end{pmatrix}$$

we have

$$\mathcal{B}_n = \mathcal{B}_{n-1}\mathcal{A}_n = \mathcal{A}_0\mathcal{A}_1 \ldots \mathcal{A}_n, \quad \det \mathcal{B}_n = (-1)^{m(n+1)}.$$

We recall some properties proved in [26].

Proposition 1. *With the notation above, we have that*

$$|A_n^{(m+1)}| = \prod_{h=1}^{n} |a_h^{(1)}|$$

for any $n \geq 1$.

Proposition 2. *Given the sequences* $(V_n^{(i)})_{n\geq -m}$, $i = 1, \ldots, m$, *defined by*

$$V_n^{(i)} = A_n^{(i)} - \alpha_0^{(i)} A_n^{(m+1)}$$

we have that

1. $\lim_{n\to+\infty} |V_n^{(i)}| = 0$
2. $V_n^{(i)} = \sum_{j=1}^{m+1} a_n^{(j)} V_{n-j}^{(i)}$
3. $\sum_{j=1}^{m+1} \alpha_n^{(j)} V_{n-j}^{(i)} = 0.$

Finally, we prove the following propositions that will be useful in the next sections.

Proposition 3. *For* $n \geq 1$, *we have that*

$$\sum_{i=1}^{m+1} \alpha_n^{(i)} A_{n-i}^{(m+1)} = \prod_{j=1}^{n} \alpha_j^{(1)}.$$

Proof. We proceed by induction on n. If $n = 1$ the left-hand side is equal to $\alpha_1^{(1)} A_0^{(m+1)} = \alpha_1^{(1)}$. For $n > 1$, we can use the inductive hypothesis and write

$$\begin{aligned}
\prod_{j=1}^{n-1} \alpha_j^{(1)} &= \sum_{i=1}^{m+1} \alpha_{n-1}^{(i)} A_{n-1-i}^{(m+1)} \\
&= A_{n-m-2}^{(m+1)} + \sum_{i=1}^{m} \alpha_{n-1}^{(i)} A_{n-1-i}^{(m+1)} \\
&= A_{n-m-2}^{(m+1)} + \sum_{i=1}^{m} \left(a_{n-1}^{(i)} + \frac{\alpha_n^{(i+1)}}{\alpha_n^{(1)}} \right) A_{n-1-i}^{(m+1)} \\
&= (\sum_{i=1}^{m} a_{n-1}^{(i)} A_{n-1-i}^{(m+1)} + A_{n-m-2}^{(m+1)}) + \sum_{i=1}^{m} \left(\frac{\alpha_n^{(i+1)}}{\alpha_n^{(1)}} \right) A_{n-1-i}^{(m+1)} \\
&= A_{n-1}^{(m+1)} + \sum_{i=1}^{m} \left(\frac{\alpha_n^{(i+1)}}{\alpha_n^{(1)}} \right) A_{n-1-i}^{(m+1)} \\
&= \frac{1}{\alpha_n^{(1)}} \left(\alpha_n^{(1)} A_{n-1}^{(m+1)} + \sum_{i=1}^{m} \alpha_n^{(i+1)} A_{n-1-i}^{(m+1)} \right) \\
&= \frac{1}{\alpha_n^{(1)}} \sum_{i=1}^{m+1} \alpha_n^{(i)} A_{n-i}^{(m+1)}.
\end{aligned}$$

proving the claim. □

Proposition 4. *For* $i = 1, \ldots, m$ *and* $n \in \mathbb{N}$ *we have that*

$$|A_n^{(i)}|_\infty < \frac{p^{n+1}}{2},$$

where $|\cdot|_\infty$ *denotes the Euclidean norm.*

Proof. We prove the claim by induction. For $n = 0$,

$$|A_0^{(i)}|_\infty = |a_0^{(i)}|_\infty < \frac{p}{2};$$

for $n \leq m$

$$A_n^{(i)} = a_n^{(1)} A_{n-1}^{(i)} + \ldots + a_n^{(n)} A_0^{(i)} + a^{(n-i)}$$

By the inductive hypothesis, and $|a_k^{(i)}|_\infty < \frac{p}{2}$ for every k,

$$|A_n^{(i)}|_\infty < \frac{p^{n+1}}{4} + \frac{p^n}{4} + \ldots + \frac{p^2}{4} + \frac{p}{2} = \frac{p^2}{4}\left(\frac{p^n - 1}{p-1}\right) + \frac{p}{2} < \frac{p^{n+1}}{2}.$$

For $n > m$, we have that

$$A_n^{(i)} = a_n^{(1)} A_{n-1}^{(i)} + \ldots + a_n^{(m)} A_{n-m}^{(i)} + A_{n-m-1}^{(i)}.$$

Again by the inductive hypothesis, and since $|a_k^{(i)}|_\infty < \frac{p}{2}$ for every k,

$$|A_n^{(i)}|_\infty < \frac{p^{n+1}}{4} + \frac{p^n}{4} + \ldots + \frac{p^{n-m+2}}{4} + \frac{p^{n-m}}{2} = \frac{p^{n-m+2}}{4}\left(\frac{p^m - 1}{p-1}\right) + \frac{p^{n-m}}{2} < \frac{p^{n+1}}{2}.$$

□

Proposition 5. *Given the MCF* $[(a_0^{(1)}, a_1^{(1)}, \ldots), \ldots, (a_0^{(m)}, a_1^{(m)}, \ldots)]$

a) every minor of $\mathcal{B}_n$ *is a polynomial in* $\mathbb{Z}[a_j^{(i)}, i = 1, \ldots, m,\ j = 0, \ldots n]$ *and each monomial has the form*

$$\lambda c_0 c_1 \ldots c_n$$

where $\lambda \in \mathbb{Z}$ *and* $c_j = 1$ *or* $c_j = a_j^{(i)}$ *for some* $i = 1, \ldots m$.

b) The summand $\lambda a_0^{(1)} \ldots a_n^{(1)}$ *does not appear in any principal minor of* $\mathcal{B}_n$ *except for the* 1×1 *minor obtained by removing all rows and columns indexed by* $2, \ldots, m+1$*; in this case* $\lambda = \pm 1$.

Proof.
a) We prove the claim by induction on n. For $n = 0$, we have $\mathcal{B}_n = \mathcal{A}_0$ and the claim immediately follows. Suppose now that the statement holds for n and consider $\mathcal{B}_{n+1}$. Let M be a square submatrix of $\mathcal{B}_{n+1}$. If $M = \mathcal{B}_{n+1}$, then $\det(M) = \pm 1$ and we are done. So we suppose some rows and columns missing in M. If M does not contain the first column, then M is a square submatrix of $\mathcal{B}_n$ and the result holds by inductive hypothesis. Therefore, we suppose that M contains the first column of $\mathcal{B}_{n+1}$, which is

$$\begin{pmatrix} A_{n+1}^{(1)} \\ A_{n+1}^{(2)} \\ \vdots \\ A_{n+1}^{(m+1)} \end{pmatrix} = \begin{pmatrix} a_{n+1}^{(1)} A_n^{(1)} + a_{n+1}^{(2)} A_{n-1}^{(1)} + \ldots + a_{n+1}^{(m)} A_{n-m+1}^{(1)} + A_{n-m}^{(1)} \\ a_{n+1}^{(1)} A_n^{(2)} + a_{n+1}^{(2)} A_{n-1}^{(2)} + \ldots + a_{n+1}^{(m)} A_{n-m+1}^{(2)} + A_{n-m}^{(2)} \\ \vdots \\ a_{n+1}^{(1)} A_n^{(m+1)} + a_{n+1}^{(2)} A_{n-1}^{(m+1)} + \ldots + a_{n+1}^{(m)} A_{n-m+1}^{(m+1)} + A_{n-m}^{(m+1)} \end{pmatrix} \tag{9}$$

The determinant of M is the sum, for $i = 1, \ldots, m+1$, of the determinants of all matrices M_i, where M_i is obtained from M by replacing the first column with a subvector of

$$a_{n+1}^{(i)} \begin{pmatrix} A_{n+1-i}^{(1)} \\ A_{n+1-i}^{(2)} \\ \vdots \\ A_{n+1-i}^{(m+1)} \end{pmatrix}$$

(to get uniform notation, we put $a_k^{(m+1)} = 1$, for every $k \in \mathbb{N}$). We then see that either two columns of M_i are proportional, so that $\det(M_i) = 0$, or $\det(M_i) = \pm a_{n+1}^{(i)} \det(M_i')$ where M_i' is a submatrix of $\mathcal{B}_n$. Thus, the claim holds by the inductive hypothesis.
b) Let M be the square submatrix obtained from $\mathcal{B}_n$ by removing all rows and columns indexed

by $I \subseteq \{1, \ldots, m+1\}$, and suppose that the summand $\lambda a_0^{(1)} \ldots a_n^{(1)}$ appears in M. Hence, by a), M must contain the first column of $\mathcal{B}_n$, so that it must contain also the first row. Moreover, since $\det(\mathcal{B}_n) = \pm 1$, at least one row and the corresponding column are missing. We argue again by induction on n. If $n = 0$, then the last row must miss, (otherwise $\det(M) \in \{1, 0\}$) so that the last column must miss too; the row indexed by m has the form $(a_0^{(m)}, 0, \ldots, 0)$ and this implies that it must miss, unless $m = 1$, so that column m is missing and so on. Thus, it follows that $I = 2, \ldots, m+1$, $\lambda = 1$. Now suppose that the result holds for $\mathcal{B}_n$. The first column of $\mathcal{B}_{n+1}$ being as in (9), we deduce by a) that $\lambda a_0^{(1)} \ldots a_n^{(1)}$ must be a summand of $\det(M_1)$, where M_1 is obtained from M by replacing the first column by a subvector of

$$a_{n+1}^{(1)} \begin{pmatrix} A_n^{(1)} \\ A_n^{(2)} \\ \vdots \\ A_n^{(m+1)} \end{pmatrix}.$$

We then see that the second column (and the second row) must miss in M (otherwise $\det(M) = 0$). Therefore $\det(M_1) = a_{n+1}^{(1)} \det(M_1')$ where M_1' is a square submatrix of B_n giving rise to a principal minor. Since $\lambda a_0^{(1)} \ldots a_n^{(1)}$ is a summand in $\det(M_1')$, by the inductive hypothesis $I = \{2, \ldots, m+1\}$ and $\lambda = 1$.

□

3 On the finiteness of the p–adic Jacobi–Perron algorithm

The following results concerning the finiteness property of the p-adic Jacobi–Perron algorithm were proved in [26].

Proposition 6. *If the p–adic Jacobi–Perron algorithm stops in a finite number of steps when processing the m–tuple $(\alpha^{(1)}, \ldots, \alpha^{(m)}) \in \mathbb{Q}_p^m$, then $1, \alpha^{(1)}, \ldots, \alpha^{(m)}$ are $\mathbb{Q}$-linearly dependent.*

Proposition 7. *For an input $(\alpha_0^{(1)}, \ldots, \alpha_0^{(m)}) \in \mathbb{Q}^m$, the p–adic Jacobi–Perron algorithm terminates in a finite number of steps.*

Thus, a full characterization of the input vectors giving rise to a finite Jacobi–Perron expansion is still missing in the p–adic case. On the other hand, the fact that the classical real Jacobi–Perron algorithm stops in a finite number of steps if and only if $1, \alpha^{(1)}, \ldots, \alpha^{(m)}$ are $\mathbb{Q}$-linearly dependent is well established for $m = 2$, whereas it is known to be false for $m \geq 3$, see [32, Theorem 44] and [11, 12]. Counterexamples in the latter case are provided by m-tuples of algebraic numbers belonging to a finite extension of $\mathbb{Q}$ of degree $< m+1$ and giving rise to a periodic MCF. This shows that the finiteness and the periodicity of the Jacobi–Perron algorithm are interrelated.

In this section we shall assume that $1, \alpha^{(1)}, \ldots, \alpha^{(m)}$ are linearly dependent over $\mathbb{Q}$. We will associate to each linear dependence relation a sequence of integers $(S_n)_{n \geq 0}$, which will be useful in the investigation of the finiteness of the p–adic Jacobi–Perron algorithm. In the case $m = 2$, we shall provide a condition that must be satisfied by the partial quotients of an infinite p-adic MCF converging to (α, β), when $1, \alpha, \beta$ are $\mathbb{Q}$-linearly dependent. We shall show in the next section that, unlike the real case, for $m = 2$ there exist some input vectors $\boldsymbol{\alpha}$ such that $1, \alpha^{(1)}, \ldots, \alpha^{(m)}$ are $\mathbb{Q}$-linearly dependent but their p-adic Jacobi–Perron expansion is periodic.

Let us consider $\boldsymbol{\alpha}_0 = (\alpha_0^{(1)}, \ldots, \alpha_0^{(m)}) \in \mathbb{Q}_p^m$ and assume that there is a linear dependence relation

$$x_1 \alpha_0^{(1)} + \ldots + x_m \alpha_0^{(m)} + x_{m+1} = 0 \tag{10}$$

with $x_1, \ldots, x_{m+1} \in \mathbb{Z}$ coprime. We associate to it the sequence

$$S_n = x_1 A_{n-1}^{(1)} + \ldots + x_m A_{n-1}^{(m)} + x_{m+1} A_{n-1}^{(m+1)} \tag{11}$$

for any $n \geq -m$, where $A_n^{(i)}$ are the numerators and denominators of the convergents of the MCF of $\boldsymbol{\alpha}_0$ defined by (6). It is straightforward to see that the following identities hold:

$$S_n\alpha_n^{(1)} + \ldots + S_{n-m+1}\alpha_n^{(m)} + S_{n-m} = 0, \text{ for any } n \ge 0; \quad (12)$$

$$S_n = a_{n-1}^{(1)}S_{n-1} + \ldots + a_{n-1}^{(m)}S_{n-m} + S_{n-m-1}, \text{ for any } n \ge 1; \quad (13)$$

$$S_n = (a_{n-1}^{(1)} - \alpha_{n-1}^{(1)})S_{n-1} + \ldots + (a_{n-1}^{(m)} - \alpha_{n-1}^{(m)})S_{n-m}, \text{ for any } n \ge 1; \quad (14)$$

$$S_n = x_1V_{n-1}^{(1)} + \ldots + x_mV_{n-1}^{(m)}, \text{ for any } n \ge -m+1. \quad (15)$$

$$\begin{pmatrix} S_n \\ S_{n-1} \\ \vdots \\ S_{n-m} \end{pmatrix} = \mathcal{B}_{n-1}^T \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m+1} \end{pmatrix} \quad (16)$$

where the superscript T denotes transposition.

Proposition 8. *Given the sequence $(S_n)_{n\ge -m}$ defined by (11), we have that $S_n \in \mathbb{Z}$, for any $n \ge -m$ and the g.c.d. of $S_n, \ldots, S_{n-m}$ is a power of p. Moreover,*

$$|S_n| < \max_{1\le i\le m}\{|S_{n-i}|\}$$

so that if the MCF for $(\alpha_0^{(1)}, \ldots, \alpha_0^{(m)})$ is infinite, then

$$\lim_{n\to+\infty} S_n = 0 \text{ in } \mathbb{Q}_p.$$

Proof. By definition $S_n \in \mathbb{Z}\left[\frac{1}{p}\right]$, for any $n \ge -m$, and $S_{-m+1}, \ldots S_0 \in \mathbb{Z}$. Then, using formula (14), and observing that $v_p(a_{n-1}^{(i)} - \alpha_{n-1}^{(i)}) > 0$, for $i = 1, \ldots, m$, where $v_p(\cdot)$ is the p-adic valuation, we get $S_n \in \mathbb{Z}$. The assertion about the g.c.d. is easily proved by induction, using formula (13).

Since $|a_n^{(i)} - \alpha_n^{(i)}| < 1$, from (14), we have that

$$|S_n| \le \max_{1\le i\le m}\{|a_n^{(i)} - \alpha_n^{(i)}||S_{n-1}|\} < \max_{1\le j\le m}\{|S_{n-i}|\}.$$

Finally, by Proposition 2 and formula (15) we see that $\lim_{n\to+\infty} S_n = 0$ in $\mathbb{Q}_p$. □

An immediate consequence of Proposition 8 is the following

Corollary 1. *For $n \ge 0$, write $n = qm + r$ with $q, r \in \mathbb{Z}$ and $0 \le r < m$; then $v_p(S_n) > q$. In particular $v_p(S_n) > \left[\frac{n}{m}\right]$ for every $n \ge 0$.*

Proposition 8 and Corollary 1 describe the behaviour of the sequence (S_n) with respect to the p-adic norm. We study now its behaviour with respect to the euclidean norm. We start by a general result.

Proposition 9. *Let $(T_n)_{n\ge -m}$ be any sequence in $\mathbb{R}$ satisfying*

$$T_n = y_n^{(1)}T_{n-1} + \ldots + y_n^{(m)}T_{n-m} + T_{n-m-1}, \quad n \ge 1$$

where $(y_n^{(1)})_{n\ge1}, \ldots, (y_n^{(m)})_{n\ge1}$ are sequences of elements in $\mathcal{Y}$; then

$$\lim_{n\to+\infty} \frac{T_n}{p^n} = 0$$

in $\mathbb{R}$.

Proof. We have

$$\begin{aligned}\left|\frac{T_n}{p^n}\right|_\infty &< \frac{1}{2}\left|\frac{T_{n-1}}{p^{n-1}}\right|_\infty + \frac{1}{2p}\left|\frac{T_{n-2}}{p^{n-2}}\right|_\infty + \ldots + \frac{1}{2p^{m-1}}\left|\frac{T_{n-m}}{p^{n-m}}\right|_\infty + \frac{1}{p^{m+1}}\left|\frac{T_{n-m-1}}{p^{n-m-1}}\right|_\infty \\ &\le K_p \max\left\{\left|\frac{T_{n-1}}{p^{n-1}}\right|_\infty, \left|\frac{T_{n-2}}{p^{n-2}}\right|_\infty, \ldots, \left|\frac{T_{n-m}}{p^{n-m}}\right|_\infty, \left|\frac{T_{n-m-1}}{p^{n-m-1}}\right|_\infty\right\},\end{aligned}$$

where $K_p = \frac{1}{p^{m+1}} + \frac{1}{2}\sum_{k=0}^{m-1} \frac{1}{p^k} < 1$. Therefore

$$\left|\frac{T_n}{p^n}\right|_\infty < K_p^{n-2} \max\left\{\left|\frac{T_m}{p^m}\right|_\infty, \left|\frac{T_{m-1}}{p^{m-1}}\right|_\infty, \dots, \left|\frac{T_1}{p}\right|_\infty, |T_0|_\infty\right\}$$

and the claim follows. □

Corollary 2. *For the sequence $(S_n)_{n\geq -m+1}$, we have*

$$\lim_{n\to+\infty} \frac{S_n}{p^n} = 0$$

in $\mathbb{R}$.

Proof. By formula (13), the sequence $(S_n)_{n\geq -m}$ satisfies the hypothesis of Proposition 9. □

Using the properties stated above, we now establish a partial converse to Proposition 6. The case $(\alpha_0^{(1)}, \dots, \alpha_0^{(m)}) \in \mathbb{Q}^m$ is managed by Proposition 7, so that we can assume $(\alpha_0^{(1)}, \dots, \alpha_0^{(m)}) \in \mathbb{Q}_p^m \setminus \mathbb{Q}^m$. Notice that in the case $m = 2$ the sequence S_n depends only on $(\alpha_0^{(1)}, \alpha_0^{(2)}) \in \mathbb{Q}_p^2$.

Proposition 10. *Assume that the sequence $\left(\frac{S_n}{p^n}\right)$ has bounded denominators. Thus there exist $k \in \mathbb{Z}$ such that $v_p(S_n) \geq n + k$, for every n. Then the Jacobi–Perron algorithm stops in finitely many steps when processing the input $(\alpha_0^{(1)}, \dots, \alpha_0^{(m)})$.*

Proof. Assume that the Jacobi–Perron algorithm does not stop. Put $z_n = p^k \frac{S_n}{p^n}$, then $z_n \in \mathbb{Z}$ and the sequence (z_n) tends to 0 in the euclidean norm, by Corollary 2. It follows that z_n (and hence S_n) is 0 for $n \gg 0$, and this is impossible by formula (12). □

The following theorem is the main result of this section. To get uniform notation, we shall put $\alpha_n^{(m+1)} = a_n^{(m+1)} = 1$ for every n.

Theorem 1. *Assume that $1, \alpha_0^{(1)}, \dots, \alpha_0^{(m)}$ are $\mathbb{Q}$-linearly dependent and*

$$v_p(a_n^{(j)}) - v_p(a_n^{(1)}) \geq j - 1 \tag{17}$$

for $j = 3, \dots, m+1$ and any n sufficiently large, then the Jacobi–Perron algorithm stops in finitely many steps when processing the input $(\alpha_0^{(1)}, \dots, \alpha_0^{(m)})$.

Notice that the condition $v_p(a_n^{(2)}) - v_p(a_n^{(1)}) \geq 1$ is always true by conditions (4).

Proof. By formula (12) we get

$$\frac{S_n}{p^n} = -\frac{S_{n-1}}{p^{n-1}}\gamma_n^{(1)} - \dots - \frac{S_{n-m}}{p^{n-m}}\gamma_n^{(m)},$$

where

$$\gamma_n^{(j)} = \frac{\alpha_n^{(j+1)}}{p^j \alpha_n^{(1)}},$$

for $j = 1, \dots, m$. By equations (4), (5) and hypotheses (17) we have $v_p(\gamma_n^{(j)}) \geq 0$ for n sufficiently large. Therefore $v_p\left(\frac{S_n}{p^n}\right) \geq \min\left\{v_p\left(\frac{S_{n-1}}{p^{n-1}}\right), \dots, v_p\left(\frac{S_{n-m}}{p^{n-m}}\right)\right\}$ for n sufficiently large, so that $v_p\left(\frac{S_n}{p^n}\right) \geq K$ for some $K \in \mathbb{Z}$. We then conclude by Proposition 10. □

In the case $m = 2$, Theorem 1 assumes the following simple form.

Corollary 3. *For $m = 2$, if $1, \alpha_0^{(1)}, \alpha_0^{(2)}$ are linearly dependent over $\mathbb{Q}$ and the p-adic Jacobi–Perron algorithm does not stop, then $v_p(a_n^{(1)}) = -1$ for infinitely many $n \in \mathbb{N}$.*

In the next section we shall present some examples where the hypotheses of Corollary 3 are satisfied.

Remark 2. *In the real case, for $m = 2$, the Jacobi–Perron algorithm detects rational dependence because the sequences $(V_n^{(1)})$ and $(V_n^{(2)})$ are bounded with respect to the euclidean norm. In fact, this implies that the set of triples (S_n, S_{n-1}, S_{n-2}) is finite and the corresponding MCF is finite or periodic. Moreover, it is possible to show that a periodic expansion can not occur and the Jacobi–Perron algorithm stops when processing two real numbers α, β such that $1, \alpha, \beta$ are $\mathbb{Q}$-linearly dependent, see [32] for details. In the p-adic case, the sequences $(V_n^{(i)})$ are bounded (because they approach zero in $\mathbb{Q}_p$, see Proposition 2); but the above argument does not apply, because the p-adic norm is non-archimedean. However, considering that $v_p(S_n) > \frac{n}{2}$ by Corollary 1, it would be interesting to focus on the sequence of integers $\left(\dfrac{S_n}{p^{n/2}}\right)$. When this sequence is bounded with respect to the euclidean norm, it is possible to argue similarly to the real case and deduce the finiteness of the p-adic Jacobi–Perron algorithm.*

4 On the characteristic polynomial of periodic multidimensional continued fractions

The classical Jacobi–Perron algorithm was introduced over the real numbers with the aim of providing periodic representations for algebraic irrationalities. However, the problem regarding the periodicity of MCFs is still open; it is not known if every algebraic irrational of degree $m + 1$ belongs to a real input vector of lenght m such that the Jacobi–Perron algorithm is eventually periodic. On the contrary, periodic MCFs have been fully studied over the real numbers. Indeed, it is known that a periodic MCF represents real numbers belonging to an algebraic number field of degree less or equal than $m + 1$, see [6] for a survey on this topic. For $m = 2$, Coleman [10] also gave a criterion for establishing when a periodic MCF converges to cubic irrationalities.
In this section, we start the study of the periodicity for p-adic MCFs. In particular, we shall see that, analogous to the real case, a periodic p-adic m-dimensional MCF represents algebraic irrationalities of degree less or equal than $m + 1$.

Let us consider a purely periodic MCF of period N:

$$(\alpha_0^{(1)}, \dots, \alpha_0^{(m)}) = \left[\left(\overline{a_0^{(1)}, \dots, a_{N-1}^{(1)}}\right), \dots, \left(\overline{a_0^{(m)}, \dots, a_{N-1}^{(m)}}\right)\right], \tag{18}$$

i.e., $a_{k+N}^{(i)} = a_k^{(i)}$ for every $k \in \mathbb{N}$ and $i = 1, \dots, m$. By (2), we also have $\alpha_{k+N}^{(i)} = \alpha_k^{(i)}$ for every $k \in \mathbb{N}$ and $i = 1, \dots, m$, so that, by (7),

$$\alpha_0^{(i)} = \frac{\alpha_0^{(1)} A_{N-1}^{(i)} + \ldots + \alpha_0^{(m)} A_{N-m}^{(i)} + A_{N-m-1}^{(i)}}{\alpha_0^{(1)} A_{n-1}^{(m+1)} + \ldots + \alpha_0^{(m)} A_{n-m}^{(m+1)} + A_{N-m-1}^{(m+1)}}. \tag{19}$$

We define the matrix

$$\mathcal{M} := \mathcal{B}_{N-1} = \prod_{j=0}^{N-1} \mathcal{A}_j = \begin{pmatrix} A_{N-1}^{(1)} & A_{N-2}^{(1)} & \dots & A_{N-m-1}^{(1)} \\ A_{N-1}^{(2)} & A_{N-2}^{(2)} & \dots & A_{N-m-1}^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ A_{N-1}^{(m+1)} & A_{N-2}^{(m+1)} & \dots & A_{N-m-1}^{(m+1)} \end{pmatrix}$$

whose characteristic polynomial $P(X)$ will be also called the characteristic polynomial of the periodic MCF (18). From equation (19), we have that

$$\mathcal{M} \begin{pmatrix} \alpha_0^{(1)} \\ \vdots \\ \alpha_0^{(m)} \\ 1 \end{pmatrix} = \left(\alpha_0^{(1)} A_{N-1}^{(m+1)} + \alpha_0^{(2)} A_{N-2}^{(m+1)} + \ldots + \alpha_0^{(m)} A_{N-m}^{(m+1)} + A_{N-m-1}^{(m+1)}\right) \begin{pmatrix} \alpha_0^{(1)} \\ \vdots \\ \alpha_0^{(m)} \\ 1 \end{pmatrix}.$$

Moreover, by Proposition 3 we know that $\sum_{i=1}^{m+1} \alpha_N^{(i)} A_{N-i}^{(m+1)} = \alpha_1^{(1)} \cdots \alpha_N^{(1)}$ and, from $\alpha_0^{(1)} = \alpha_N^{(1)}$, we have that

$$\mathcal{M} \begin{pmatrix} \alpha_0^{(1)} \\ \vdots \\ \alpha_0^{(m)} \\ 1 \end{pmatrix} = \alpha_0^{(1)} \dots \alpha_{N-1}^{(1)} \begin{pmatrix} \alpha^{(1)} \\ \vdots \\ \alpha^{(m)} \\ 1 \end{pmatrix}.$$

Therefore $\mu := \alpha_0^{(1)} \dots \alpha_{N-1}^{(1)}$ is an eigenvalue of $\mathcal{M}$. The following theorems will show that μ is the p-adic dominant eigenvalue, that is the root of $P(X)$ greatest in p-adic norm. Moreover, the algebraic properties of the limits of the periodic MCF (18) are strictly related to μ. Note that considering purely periodic MCFs is not a loss of generality, because the field generated by the n-th complete quotients of a MCF coincides with that generated by the entries of the input vector.

Theorem 2. *Given the purely periodic MCF*

$$(\alpha_0^{(1)}, \dots, \alpha_0^{(m)}) = \left[\overline{\left(a_0^{(1)}, \dots, a_{N-1}^{(1)}\right), \dots, \left(a_0^{(m)}, \dots, a_{N-1}^{(m)}\right)} \right]$$

and its characteristic polynomial $P(X)$, then $\mu = \alpha_0^{(1)} \dots \alpha_{N-1}^{(1)}$ is the greatest root in p-adic norm.

Proof. We put $a_n^{(1)} = \dfrac{\tilde{a}_n^{(1)}}{p^{k_n}}$, for any $n \geq 0$, where $k_n \geq 0$ (in fact $k_n > 0$, for $n > 0$). We define the number $k = k_0 + \ldots + k_{N-1}$ and the matrix

$$\mathcal{M}' := p^k \mathcal{M} = \mathcal{A}'_0 \dots \mathcal{A}'_{N-1}$$

where

$$\mathcal{A}'_i = p^{k_i} \mathcal{A}_i = \begin{pmatrix} \tilde{a}_i^{(1)} & p^{k_i} & 0 & \dots & 0 \\ p^{k_i} a_i^{(2)} & 0 & p^{k_i} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p^{k_i} a_i^{(m)} & 0 & 0 & \dots & p^{k_i} \\ p^{k_i} & 0 & 0 & \dots & 0 \end{pmatrix} \equiv \begin{pmatrix} \tilde{a}_i^{(1)} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \pmod{p}.$$

Therefore

$$\mathcal{M}' \equiv \begin{pmatrix} \tilde{a}_0^{(1)} \dots \tilde{a}_{N-1}^{(1)} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \pmod{p}. \tag{20}$$

Let $Q(X)$ be the characteristic polynomial of $\mathcal{M}'$, λ is an eigenvalue of $\mathcal{M}$ if and only if $p^k\lambda$ is an eigenvalue of $\mathcal{M}'$. If $\lambda_1, \dots, \lambda_{m+1}$ are the eigenvalues of $\mathcal{M}$, then

$$Q(X) = \prod_{i=1}^{m+1} (X - p^k \lambda_i) = p^{k(m+1)} \prod_{i=1}^{m+1} \left(\frac{x}{p^k} - \lambda_i \right) = p^{k(m+1)} P\left(\frac{X}{p^k} \right)$$

so that

$$P(X) = \frac{1}{p^{k(m+1)}} Q(p^k X).$$

From (20) we have that

$$Q(X) \equiv X^m (X - \tilde{a}_0^{(1)} \dots \tilde{a}_{N-1}^{(1)}) \pmod{p}.$$

Thus

$$Q(X) = X^{m+1} + \delta_m X^m + \ldots + \delta_0$$

with

$$\delta_m \equiv \tilde{a}_0^{(1)} \dots \tilde{a}_{N-1}^{(1)} \pmod{p}, \quad \delta_i \equiv 0 \pmod{p} \text{ for } i = 0, \dots, m-1, \quad \delta_0 = \pm p^{k(m+1)}.$$

It follows that

$$
\begin{aligned}
P_\mu(X) &= \frac{1}{p^{k(m+1)}} Q(p^k X) \\
&= \frac{1}{p^{k(m+1)}} (p^{k(m+1)} X^{m+1} + \delta_m p^{km} X^m + \ldots + \delta_i p^{ki} X^i + \ldots + \delta_0) \\
&= X^{(m+1)} + \frac{\delta_m}{p^k} X^m + \ldots + \frac{\delta_i}{p^{k(m+1-i)}} X^i + \ldots \pm 1 \\
&= X^{m+1} + \gamma_m X^m + \ldots + \gamma_0,
\end{aligned}
$$

where

$$
\gamma_i = \frac{\delta_i}{p^{k(m+1-i)}} \quad \text{for } i = 0, \ldots, m, \quad (\gamma_0 = \pm 1).
$$

Now we put $\mu_i = v_p(\delta_i)$ for $i = 1, \ldots, m$ and observe that

$$
\mu_m = 0, \quad \mu_i > 0 \text{ for } i = 1, \ldots, m-1, \quad \mu_0 = k(m+1).
$$

We also notice that

$$
\begin{aligned}
v_p(\gamma_m) &= v_p(a_0^{(1)} \ldots a_{N-1}^{(1)}) = \sum_{i=0}^{N-1} v_p(a_i^{(1)}) = -k \\
v_p(\gamma_i) &= v_p(\delta_i) - k(m+1-i) \\
&= \mu_i + ik - (m+1)k \quad \text{for } i = 0, \ldots, m.
\end{aligned}
$$

In order to prove that μ is the greatest root in p-adic norm, we study the Newton polygon of $P(X)$ (see [16]). The line, in the real plane, passing through the points $(i, v_p(\gamma_i))$ and $(m+1, 0)$ has equation

$$
y = \frac{v_p(\gamma_i)}{m+1-i}(-x + m + 1), \tag{21}
$$

for any $i = 1, \ldots, m-1$. We shall denote by s_i the slope of this line. From the fact that

$$
v_p(\gamma_i) = \mu_i - k(m+1-i) \text{ and } \mu_i = v_p(\delta_i) > 0,
$$

we get

$$
\frac{v_p(\gamma_i)}{m+1-i} = \frac{\mu_i}{m+1-i} - k > -k,
$$

i.e., the point in the real plane with coordinates $(m, v_p(\gamma_m)) = (m, -k)$ lies strictly under the line (21), for any $i = 1, \ldots, m-1$.

Thus, the Newton polygon associated to the polynomial $P(X)$ has slopes $(s_1, \ldots, s_m)$, which is a strictly increasing sequence, where the last slope s_m is equal to k. Hence, the claim of the theorem follows from [16, Theorem 6.4.7] and the fact that the sequence of slopes $(s_1, \ldots, s_m)$ is strictly increasing.

□

Theorem 3. *Given a purely periodic MCF*

$$
(\alpha_0^{(1)}, \ldots, \alpha_0^{(m)}) = \left[\overline{\left(a_0^{(1)}, \ldots, a_{N-1}^{(1)}\right), \ldots, \left(a_0^{(m)}, \ldots, a_{N-1}^{(m)}\right)} \right],
$$

let μ be the greatest root in p-adic norm of its characteristic polynomial. Then

a) $\mathbb{Q}(\mu) = \mathbb{Q}(\alpha_0^{(1)}, \ldots, \alpha_0^{(m)})$

b) $\mu \notin \mathbb{Q}$

Proof.
a) Certainly $\mu \in \mathbb{Q}(\alpha_0^{(1)}, \ldots, \alpha_0^{(m)})$, since $\mu = \alpha_0^{(1)} \ldots \alpha_{N-1}^{(1)}$. Conversely, by Theorem 2, the nullspace of $\mathcal{B}_{N-1} - \mu I_{m+1}$ in $\mathbb{Q}(\mu)^2$ is 1-dimensional (where I_{m+1} is the $(m+1) \times (m+1)$ identity matrix). Therefore, it is generated by a vector $\beta = (\beta_1, \ldots, \beta_{m+1})$ with entries in $\mathbb{Q}(\mu)$, which must be proportional to $(\alpha_0^{(1)}, \ldots, \alpha_0^{(m)}, 1)$. It follows that $\alpha^{(i)} = \frac{\beta_i}{\beta_{m+1}} \in \mathbb{Q}(\mu)$ for $i = 1, \ldots, m$.
b) Assume that $\mu \in \mathbb{Q}$, then $\alpha_0^{(1)}, \ldots, \alpha_0^{(m)} \in \mathbb{Q}$. But in this case the MCF corresponding to $(\alpha_0^{(1)}, \ldots, \alpha_0^{(m)})$ is finite by [26, Theorem 5], so that it cannot be periodic. □

From the previous theorems we deduce that a periodic MCF converges to a m–tuple of algebraic irrationalities of degree less or equal than $m+1$, belonging to the field generated over $\mathbb{Q}$ by the the root greatest in p-adic norm of the characteristic polynomial. When the latter is irreducible, then they generate a field of degree $m+1$.
We now investigate in more detail the roots of the characteristic polynomial. In particular we focus on $m=2$.

Lemma 1. *Let $P(X) = X^{m+1}+\gamma_m X^m+\ldots+\gamma_1 X+(-1)^{m(N+1)+1}$ be the characteristic polynomial of a purely periodic MCF $(\alpha_0^{(1)},\ldots,\alpha_0^{(m)}) = \left[\overline{\left(a_0^{(1)},\ldots,a_{N-1}^{(1)}\right),\ldots,\left(a_0^{(m)},\ldots,a_{N-1}^{(m)}\right)}\right]$. Then*

a) every γ_i is a polynomial in $\mathbb{Z}[a_j^{(i)}, i=1,\ldots,m,\ j=0,\ldots N-1]$ and each monomial has the form $\lambda c_0 c_1 \ldots c_{N-1}$ where $\lambda \in \mathbb{Z}$ and $c_j = 1$ or $c_j = a_j^{(i)}$ for some $i=1,\ldots,m$;

b) the monomial $a_0^{(1)}\cdots a_{N-1}^{(1)}$ appears only in γ_m.

Proof. Let us observe that any coefficient γ_i is the sum of the principal minors of the matrix $\mathcal{B}_{N-1}$ of order $m+1-i$, for $i=1,\ldots,m$. Hence the thesis follows from Proposition 5. ☐

Theorem 4. *Given a purely periodic MCF*

$$(\alpha_0^{(1)},\ldots,\alpha_0^{(m)}) = \left[\overline{\left(a_0^{(1)},\ldots,a_{N-1}^{(1)}\right),\ldots,\left(a_0^{(m)},\ldots,a_{N-1}^{(m)}\right)}\right],$$

every root of its characteristic polynomial has p-adic norm less than 1, except for the root greatest in p-adic norm $\mu = \alpha_0^{(1)}\cdots\alpha_{N-1}^{(1)}$.

Proof. Let $P(X) = X^{m+1}+\gamma_m X^m+\ldots+\gamma_1 X+(-1)^{m(N+1)+1}$ be the characteristic polynomial of the given MCF. By Lemma 1 and $|a_n^{(1)}|>1$, $|a_n^{(1)}|>|a_n^{(j)}|$ for $n\in\mathbb{N}$, $j=2,\ldots,m+1$, we have $|\gamma_i| \le |a_0^{(1)}\cdots a_{N-1}^{(1)}|$, for any $i=1,\ldots,m$. Moreover, this inequality becomes equality if and only if $i=m$. If $\lambda_1=\mu,\lambda_2,\ldots,\lambda_k$ are the roots of $P(X)$ with p-adic norm ≥ 1, then $\gamma_{m+1-k} \ge |\mu| = |a_0^{(1)}\cdots a_{N-1}^{(1)}|$. Recalling that γ_{m+1-k} is also the k-th elementary symmetric function of the roots, this implies $k=1$ and the thesis follows. ☐

Theorem 5. *Let z be a complex root of the characteristic polynomial $P(X)$ of a purely periodic MCF $\left[\overline{\left(a_0^{(1)},\ldots,a_{N-1}^{(1)}\right),\ldots,\left(a_0^{(m)},\ldots,a_{N-1}^{(m)}\right)}\right]$. Then*

$$|z|_\infty < p^N.$$

Proof. By Gershgorin theorem [15] there esists a row $j=1,\ldots,m+1$ in $\mathcal{M}$ such that

$$|z - A_{N-j}^{(j)}|_\infty \le \sum_{k=1,\ldots,m+1,\ k\neq j} |A_{N-k}^{(j)}|_\infty.$$

In particular

$$|z|_\infty \le \sum_{k=1}^{m+1} |A_{N-k}^{(j)}|_\infty < \frac{1}{2}\sum_{k=1}^{m+1} p^{N-k+1}$$

by Proposition 4. Moreover,

$$\frac{1}{2}\sum_{k=1}^{m+1} p^{N-k+1} = \frac{1}{2}p^{N-m}\sum_{k=0}^{m} p^k = \frac{1}{2}p^{N-m}\frac{p^{m+1}-1}{p-1} \le p^N.$$

☐

The previous theorems can be used to give some further information about the algebraic properties of the values of a periodic MCF. We firstly consider the case $N=1$.

Proposition 11. *The characteristic polynomial of a purely periodic MCF with period $N=1$ does not have any rational root. In particular when $m=2$ it is irreducible over $\mathbb{Q}$, and the limits of the MCF generate a cubic field.*

Proof. Let z be a rational root of the characteristic polynomial; by the rational root theorem it must be (up to a sign) a power of p. By Theorem 3 *b*), we know that $z \neq \mu$, and this implies that $v_p(z) \geq 1$ by Theorem 4. But $|z|_\infty < p$ by Theorem 5, a contradiction. $\square$

In general, a rational root of a MCF with period of length N must satisfy $|z|_\infty < p^N$ and $v_p(z) \geq 1$, so that for the rational root theorem it must be of the kind $\pm p^k$, with $k \leq N-1$. The next proposition gives a necessary condition for the existence of such a root, in the case $m = 2$ and $N = 2$.

Proposition 12. *Let us consider the purely periodic MCF* $\left[(\overline{a_0, a_1}), (\overline{b_0, b_1})\right]$. *Then its characteristic polynomial* $P(X)$ *is irreducible over* $\mathbb{Q}$ *unless the following condition is verified, possibly interchanging the indices 0 and 1:*

- a_0 *is of the form* $\pm\frac{1}{p} + w$ *with* $w \in \mathbb{Z}, |w|_\infty \leq \frac{p-1}{2}, w \neq 0$*; and*
- *either* $v_p(a_1 p + 1) = v_p(a_1) + 1$ *(which implies* $v_p(b_1) = v_p(a_1) + 1, v_p(b_0) = 0$*)* *or* a_1 *is of the form* $\pm\frac{1}{p} + u$ *with* $u \in \mathbb{Z}, |u| \leq \frac{p-1}{2}, u \neq 0$*;* *in the latter case one between* b_0 *and* b_1 *is zero and the other one is equal to* $-wu \pm p$*.* (22)

Proof. Write

$$P(X) = X^3 + \gamma_2 X^2 + \gamma_1 X - 1,$$

then

$$\gamma_2 = -(a_0 a_1 + b_0 + b_1), \qquad \gamma_1 = b_1 b_0 - a_0 - a_1$$

so that

$$P(X) = X(X - b_0)(X - b_1) - (a_0 X + 1)(a_1 X + 1).$$

We put $k_1 = -v_p(a_1), k_2 = -v_p(a_2), k = k_1 + k_2$. By Theorems 4 and 5 the only possible rational roots of $P(X)$ are $\pm p$. So assume

$$P(\pm p) = \pm p(\pm p - b_0)(\pm p - b_1) - (\pm a_0 p + 1)(\pm a_1 p + 1) = 0. \tag{23}$$

Notice that the valuation of the first summand is $\geq -k+3$ and that of the second summand is $\geq -k+2$. Therefore, the valuation of the second summand must be $\geq -k+3$. This implies that at least one between a_0 and a_1, say a_0, must satisfy $v(\pm a_0 p + 1) > -k_0 + 1$, that is $a_0 p \equiv \mp 1 \pmod p$. Since $a_0 \in \mathcal{Y}$ this implies $a_0 = \mp\frac{1}{p} + w$ with $w \in \mathbb{Z}, |w|_\infty \leq \frac{p-1}{2}$ and (23) becomes

$$\pm (\pm p - b_0)(\pm p - b_1) - w(\pm a_1 p + 1) = 0. \tag{24}$$

We have that $w \neq 0$, otherwise one between b_0 and b_1 should be equal to $\pm p$, which is a contradiction because $b_0, b_1 \in \mathcal{Y}$.
The right-hand side of (24) has valuation $\geq -k_1 + 1$; and $v_p(\pm p - b_0) \geq 0$, $v_p(\pm p - b_1) \geq -k_1 + 1$. If the valuation of the right side is exactly $-k_1 + 1$ then it must be $v(b_0) = 0, v(b_1) = -k_1 + 1$. On the other hand, if the valuation of the right side is $> -k_1 + 1$ then $a_1 p \equiv \mp 1 \pmod p$. As above this implies $a_0 = \mp\frac{1}{p} + u$ with $u \in \mathbb{Z}, |u|_\infty \leq \frac{p-1}{2}, u \neq 0$ and (24) becomes

$$\pm (\pm p - b_0)(\pm p - b_1) - wup = 0. \tag{25}$$

This implies that one between b_0 and b_1 is 0, the other one (say b_i) has valuation 0, and satisfies $\pm p - b_i = wu$. $\square$

The following proposition will be useful to provide numerical examples.

Proposition 13. *Let us consider the purely periodic 2-dimensional MCF* $(\alpha, \beta) = \left[(\overline{a_0, \ldots, a_{N-1}}), (\overline{b_0, \ldots, b_{N-1}})\right]$ *and suppose that its characteristic polynomial* $P(X)$ *is reducible. Let* $z = \pm p^k$ *be the (unique) rational root of* $P(X)$*, then the* 1*-dimensional eigenspace* $\mathcal{L} \subseteq \mathbb{Q}^3$ *of the transpose of* $\mathcal{B}_{N-1}$ *associated to* z *coincides with the space* $\mathcal{L}'$ *of rational vectors* (x, y, z) *such that* $x\alpha + y\beta + z = 0$*.*

Proof. Notice firstly that the space $\mathcal{L}'$ is one-dimensional, because Theorem 3 and the reducibility of $P(X)$ imply that $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}] = 2$. Therefore there is a linear dependence relation

$$x_1 \alpha + x_2 \beta + x_3 = 0$$

with coprime $x_1, x_2, x_3 \in \mathbb{Z}$, and (x_1, x_2, x_3) generates $\mathcal{L}'$. Since $\alpha_N = \alpha$, $\beta_N = \beta$, by property (12) of the sequence (S_n) defined by (11), the vector (S_N, S_{N-1}, S_{N-2}) must be a rational multiple of (x_1, x_2, x_3); then by (15) (x_0, y_0, z_0) is an eigenvector associated to a rational eigenvalue, so that it belongs to $\mathcal{L}$. $\square$

Example 1. *Condition (22) is essential. Consider the following examples.*

- *For $p=5$, the periodic MCF $(\alpha,\beta) = \left[\left(\frac{4}{5},\frac{11}{5}\right),\overline{(1,2)}\right]$ has characteristic polynomial*

$$P(X) = X^3 - \frac{119}{25}X^2 - X - 1$$

and

$$P(X) = (X-5)\left(X^2 - \frac{6}{25}X + \frac{1}{5}\right).$$

Moreover by using Proposition 13 we find the linear dependence relation between α, β and 1:

$$20\alpha + 5\beta + 4 = 0.$$

- *For $p=3$, the periodic MCF $(\alpha,\beta) = \left[\left(\frac{2}{3},\frac{5}{3}\right),\overline{(1,0)}\right]$ has characteristic polynomial*

$$P(X) = X^3 - \frac{19}{9}X^2 - \frac{7}{3}X - 1 = (X-3)\left(X^2 + \frac{8}{9}X + \frac{1}{3}\right).$$

and

$$6\alpha + 3\beta + 2 = 0.$$

- *For $p=3$, the periodic MCF $(\alpha,\beta) = \left[\left(\frac{2}{3},\frac{13}{9}\right),\overline{\left(1,\frac{1}{3}\right)}\right]$ has characteristic polynomial*

$$P(X) = X^3 - \frac{62}{27}X^2 - \frac{16}{9}X - 1 = (X-3)\left(X^2 + \frac{19}{27}X + \frac{1}{3}\right).$$

The linear dependence relation between $\alpha, \beta, 1$ is the same as in the previous case:

$$6\alpha + 3\beta + 2 = 0.$$

The above examples also provide $\mathbb{Q}$-linearly dependent numbers having a periodic (hence not finite) expansion by the p-adic Jacobi–Perron algorithm.
At the present time we were not able to find examples of m-tuples of $\mathbb{Q}$-linearly dependent p-adic numbers whose MCF is infinite and not periodic. Therefore, we state the following

Conjecture 1. *Let $\boldsymbol{\alpha} = (\alpha^{(1)},\dots,\alpha^{(m)}) \in \mathbb{Q}_p^m$ be such that $1, \alpha^{(1)},\dots,\alpha^{(m)}$ are $\mathbb{Q}$-linearly dependent. Then the p-adic Jacobi-Perron algorithm for $\boldsymbol{\alpha}$ is finite or periodic.*

Acknowledgments

We thank Matteo Semplice for stimulating conversations.

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