Some Results on Joint Record Events

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Abstract

Let $X_1, X_2, \dots$ be independent and identically distributed random variables on the real line with a joint continuous distribution function F. The stochastic behavior of the sequence of subsequent records is well known. Alternatively to that, we investigate the stochastic behavior of arbitrary $X_j, X_k, j < k$, under the condition that they are records, without knowing their orders in the sequence of records. The results are completely different. In particular it turns out that the distribution of X_k, being a record, is not affected by the additional knowledge that X_j is a record as well. On the contrary, the distribution of X_j, being a record, is affected by the additional knowledge that X_k is a record as well. If F has a density, then the gain of this additional information, measured by the corresponding Kullback-Leibler distance, is j/k, independent of F. We derive the limiting joint distribution of two records, which is not a bivariate extreme value distribution. We extend this result to the case of three records. In a special case we also derive the limiting joint distribution of increments among records.

Keywords and phrases: Records, order statistics, Kullback-Leibler distance, domain of attraction, extreme value distribution

Introduction

Let $X_1, X_2, \dots$ be independent and identically distributed (iid) random variables (rvs). The rv X_m is a record if $X_m > \max(X_1, \dots, X_{m-1})$. Clearly, X_1 is a record. Records have been investigated extensively over the past decades, see, e.g. Resnick (1987, Section 4.1), Galambos (1987, Sections 6.2 and 6.3), and Arnold, Balakrishnan, and Nagaraja (1998). Consider the indicator function $I_m := \mathbb{1}(X_m \text{ is a record})$, $m \in \mathbb{N}$. It is well known that the indicator functions $I_1, I_2, \dots$ are independent with (see, e.g., Galambos (1987, Lemma 6.3.3))

$$\Pr(I_m = 1) = m^{-1}, \quad m \in \mathbb{N}. \tag{1}$$

Suppose that the common distribution function (df) F of $X_1, X_2, \dots$ is the standard exponential df $F(x) = 1 - \exp(-x)$, $x \geq 0$. It is also well known that in this case the increments of subsequent records are iid rvs with common standard exponential distribution. Precisely, put $T(1) := 1$ and, for $n \geq 2$, $T(n) := \min\{m > n-1\colon X_m \text{ is a record}\}$. Then, $X_{T(n)}$, $n \in \mathbb{N}$, is the sequence of records among $X_1, X_2, \dots$ and $T(n)$ is the arrival time of the n-th record. The increments of subsequent records are given by the sequence $Y_n := X_{T(n)} - X_{T(n-1)}$, $n \geq 2$, $\quad Y_1 := X_1$. Then, $Y_1, Y_2, \dots$ are iid rvs with common df $F(x) = 1 - \exp(-x)$, $x \geq 0$. This yields

$$X_{T(n)} = \sum_{i=1}^{n} Y_i,\ n \in \mathbb{N}, \tag{2}$$

and, thus, characterizes the distribution of the n-th record or the joint distribution of several numbered records $\left(X_{T(n_1)}, X_{T(n_2)}, \dots, X_{T(n_m)}\right), n_1 < n_2 < \dots < n_m$, etc.

In this paper, we drop the assumption that we *know* the order of a record. Therefore, we characterize the distribution $\Pr\left(X_j \leq \cdot \mid X_j \text{ is a record}\right),\ j \in \mathbb{N}$, as well as the joint distribution of two records $\Pr\left(X_j \leq \cdot, X_k \leq \cdot \mid X_j \text{ and } X_k \text{ are records}\right),\ 1 \leq j < k$. We achieve this under the assumption that the joint df F of $X_1, X_2, \dots$ is continuous. In particular, we establish the following surprising fact: Choose integers $j < k$. The distribution of X_j, being a record, is affected when we know that X_k is a record as well. The distribution of X_k, being a record, however, is not affected when we know that X_j is a record as well. The corresponding information gain is measured by the Kullback-Leibler distance between the densities. This information gain is j/k and it is independent of the underlying F. This is the content of Section 1. In Section 2, the asymptotic joint distribution of X_j and X_k, suitably standardized, under the condition that they are records, is derived. This is achieved if the underlying df F is in the domain of attraction of an extreme value df. The limit distribution is not an extreme value distribution. We also derive the limiting joint distribution of three records. Finally, for the special case of a sequence of iid rvs with a common standard negative exponential distribution, we derive the asymptotic joint distribution of increments among records.

1 Distribution of Records

Throughout this section we suppose that $X_1, X_2 \dots$ are iid rvs with a common continuous df F. The distribution of X_n, being a record, is provided by the following important result.

Lemma 1.1. *We have for* $n \in \mathbb{N}$

$$\Pr\left(X_n \leq x \mid X_n \textit{ is a record}\right) = \Pr\left(\max_{1 \leq i \leq n} X_i \leq x\right) = F^n(x),\ x \in \mathbb{R}.$$

Therefore, the distribution of X_n, being a record, coincides with that of the largest order statistic in the sample $X_1, \dots, X_n$.

Proof. Denote by $X_{1:n} \le \dots \le X_{n:n}$ the order statistics pertaining to $X_1, \dots, X_n$, and by $R(X_i) = \sum_{j=1}^{n} \mathbb{1}(X_j \le X_i)$ the rank of X_i, $1 \le i \le n$. It is well known that the vector of order statistics $(X_{1:n}, \dots, X_{n:n})$ and the vector of ranks $(R(X_1), \dots, R(X_n))$ are independent, with $\Pr(R(X_i) = k) = n^{-1}$, $1 \le i, k \le n$; see, e.g., Rényi (1962). Therefore, we obtain from equation (1), the final result

$$\begin{aligned}\Pr\left(X_n \le x \mid X_n \text{ is a record}\right) &= n \Pr\left(X_n \le x, X_n \text{ is a record}\right) \\ &= n \Pr\left(X_{n:n} \le x, R(X_n) = n\right) = \Pr\left(X_{n:n} \le x\right). \qquad \square\end{aligned}$$

The preceding result immediately yields the limiting distribution of X_n, being a record, as n tends to infinity. The necessary tools are provided by univariate extreme value theory: Suppose that there exist constants $a_n > 0, b_n \in \mathbb{R}, n \in \mathbb{N}$, such that $F^n\left(a_n x + b_n\right) \to G(x)$, $x \in \mathbb{R}$, for $n \to \infty$ and for all continuity point x of G, where G is a non-degenerate df. Then, F is said to be in the *max-domain of attraction* of G, denoted by $F \in \mathcal{D}(G)$, and G is a univariate extreme value distribution. Precisely, G is a member of a parametric family $\{G_\alpha \colon \alpha \in \mathbb{R}\}$, indexed by $\alpha \in \mathbb{R}$, with $G_\alpha(x) = \exp\left(-(1+\alpha x)^{-1/\alpha}\right), \quad 1+\alpha x > 0$, if α is different from zero, and the convention $G_0(x) = \lim_{\alpha \to 0} G_\alpha(x) = \exp\left(-\mathrm{e}^{-x}\right), \quad x \in \mathbb{R}$, (see, e.g., Resnick 1987 Ch. 1). If we put in particular $F(x) = 1 - \exp(-x)$, $x \ge 0$, then we have $F \in \mathcal{D}(G_0)$, precisely $F^n(x + \log n) \to \exp\left(-\mathrm{e}^{-x}\right)$, $x \in \mathbb{R}$, and, thus, $\Pr\left(X_n - \log n \le x \mid X_n \text{ is a record}\right) \to \exp\left(-\mathrm{e}^{-x}\right)$, $x \in \mathbb{R}$. If we know the order of the records, then the limiting distribution of the n-th record is by equation (2) and the central limit theorem, $n^{-1/2}\left(X_{T(n)} - n\right) \xrightarrow{d} \mathcal{N}(0,1)$, $n \to \infty$, where $\xrightarrow{d}$ denotes convergence in distribution as n goes to infinity.

Next we establish the joint distribution of two records. To simplify the notation, we suppose that the underlying df of the sequence of iid rvs is the standard negative exponential df $F(x) = \exp(x)$, $x \le 0$. Instead of writing $X_1, X_2, \dots$ we use with this particular underlying df the notation $\eta_1, \eta_2, \dots$. The latter distribution is a member of the set $\{G_\alpha \colon \alpha \in \mathbb{R}\}$, with $\alpha = -1$ and shifted by -1. In this particular case we have $F^n(x) = \exp(nx) = F(nx)$, $x \le 0$.

Lemma 1.2. *We have for $1 \le j < k$ and $x_1, x_2 \in \mathbb{R}$,*

$$\begin{aligned}\Pr(\eta_j \le x_1, \eta_k \le x_2 \mid \eta_j \textit{ and } \eta_k \textit{ are records}) &= \frac{k}{k-j} \Pr\left(\eta_1 \le j x_1, \eta_2 \le (k-j) x_2, (k-j)\eta_1 < j \eta_2\right) \\ &= \begin{cases} \frac{k}{k-j}\left(\mathrm{e}^{(k-j)x_2}\,\mathrm{e}^{j x_1} - \frac{j}{k}\,\mathrm{e}^{k x_1}\right), & \textit{if } x_1 < x_2 \\ \mathrm{e}^{k x_2} = \Pr(\eta_k \le x_2 \mid \eta_k \textit{ is a record}), & \textit{if } x_1 \ge x_2. \end{cases}\end{aligned}$$

Proof. Let $\eta_1^{(r)}, \eta_2^{(r)}, \dots$, $r = 1, 2$, be two independent sequences of iid copies of η. Let $\eta_{i:n}^{(r)}$ be i-th order statistics and $R_m^{(r)}(\eta_j^{(r)})$ the rank of $\eta_j^{(r)}$ in the sample $\eta_1^{(r)}, \dots, \eta_m^{(r)}$. We split the sample $\eta_1, \dots, \eta_k$ into the two independent sub-samples $(\eta_1,\dots,\eta_j) =: (\eta_1^{(1)},\dots,\eta_j^{(1)})$ and $(\eta_{j+1},\dots,\eta_k) =: (\eta_1^{(2)},\dots,\eta_{k-j}^{(2)})$. By the independence between vectors of order statistics and ranks and the fact that the distributions of $\eta_{m:m}$ and η/m coincide for $m\in\mathbb{N}$, we obtain

$$
\begin{aligned}
&\Pr(\eta_j \le x_1, \eta_k \le x_2 \mid \eta_j \text{ and } \eta_k \text{ are records})\\
&= jk \Pr\left(\eta_{j:j}^{(1)} \le x_1,\, \eta_{k-j:k-j}^{(2)} \le x_2,\, R_j^{(1)}(\eta_j^{(1)}) = j,\, R_{k-j}^{(2)}(\eta_{k-j}^{(2)}) = k-j,\, \eta_{j:j}^{(1)} < \eta_{k-j:k-j}^{(2)}\right)\\
&= jk \Pr\left(\eta_{j:j}^{(1)} \le x_1,\, \eta_{k-j:k-j}^{(2)} \le x_2,\, \eta_{j:j}^{(1)} < \eta_{k-j:k-j}^{(2)}\right) \Pr\left(R_j^{(1)}(\eta_j^{(1)}) = j\right) \Pr\left(R_{k-j}^{(2)}(\eta_{k-j}^{(2)}) = k-j\right)\\
&= \frac{k}{k-j} \Pr(\eta_1 \le jx_1,\, \eta_2 \le (k-j)x_2,\, (k-j)\eta_1 < j\eta_2).
\end{aligned}
$$

The rest of the assertion follows from elementary computations, conditioning on η_2. □

The preceding result can be extended to $X_1, X_2, \dots$ with an arbitrary continuous df F by putting $X_i := F^{-1}(\exp(\eta_i))$, $i\in\mathbb{N}$, where $F^{-1}(q) := \inf\{t\in\mathbb{R}\colon F(t)\ge q\}$, $q\in(0,1)$, is the usual generalized inverse of F. From the general equivalence $F^{-1}(q)\le t$ iff $q\le F(t)$, $q\in(0,1)$, $t\in\mathbb{R}$, we obtain for $1\le j<k$ and $y_1,y_2\in\mathbb{R}$

$$\Pr(X_j \le y_1, X_k \le y_2 \mid X_j \text{ and } X_k \text{ are records}) = \Pr(\eta_j \le \log(F(y_1)), \eta_k \le \log(F(y_2)) \mid \eta_j \text{ and } \eta_k \text{ are records}).$$

By putting $x_i := \log(F(y_i))$, $i=1,2$, the following result is an immediate consequence of Lemma 1.2.

Corollary 1.3. *We have for integers $1\le j<k$ and $y_1,y_2\in\mathbb{R}$*

$$\Pr(X_j \le y_1, X_k \le y_2 \mid X_j \textit{ and } X_k \textit{ are records}) = \begin{cases} \frac{F^j(y_1)}{k-j}\left(kF^{k-j}(y_2) - jF^{k-j}(y_1)\right), & \textit{if } F(y_1) < F(y_2)\\ F^k(y_2) = \Pr(X_k \le y_2 \mid X_k \textit{ is a record}), & \textit{if } F(y_2) \le F(y_1). \end{cases}$$

Choose integers $1\le j<k$. Next we establish the fact that the distribution of η_j, being a record, is affected, if we know that η_k is a record as well. The distribution of η_k, being a record, however, is not affected by the additional knowledge that η_j is a record as well.

Proposition 1.4. *We have for integers $1\le j<k$ and $x_1,x_2\in\mathbb{R}$,*

$$\Pr\left(\eta_k \le x_2 \mid \eta_j \textit{ and } \eta_k \textit{ are records}\right) = \mathrm{e}^{kx_2} = \Pr(\eta_k \le x_2 \mid \eta_k \textit{ is a record}), \qquad x_2 \le 0,$$

and

$$
\begin{aligned}
\Pr\left(\eta_j \le x_1 \mid \eta_j \textit{ and } \eta_k \textit{ are records}\right) &= \frac{1}{k-j}\left(k\,\mathrm{e}^{jx_1} - j\,\mathrm{e}^{kx_1}\right) \qquad x_1 \le 0,\\
&= \frac{1}{k-j}\left(k\Pr(\eta_j \le x_1 \mid \eta_j \textit{ is a record}) - j\Pr(\eta_k \le x_1 \mid \eta_k \textit{ is a record})\right).
\end{aligned}
$$

Proof. From Lemma 1.2 we have $\Pr(\eta_k \le x_2 \mid \eta_j \text{ and } \eta_k \text{ are records}) = \frac{k}{k-j}\Pr\left(\eta_2 \le (k-j)x_2, (k-j)\eta_1 < j\eta_2\right)$

by putting $x_1 = 0$, and $\Pr(\eta_j \le x_1 \mid \eta_j \text{ and } \eta_k \text{ are records}) = \frac{k}{k-j}\Pr\left(\eta_1 \le jx_1, (k-j)\eta_1 < j\eta_2\right)$ by putting $x_2 = 0$. The assertion follows by conditioning on η_2. □

Let us consider records over a sequence $X_1, X_2, \dots$ of iid rvs with an arbitrary df F that has a density, say f. Choose integers $1 \le j < k$. From Corollary 1.3 and Proposition 1.4 we obtain that the density function of the df $G_{j,k}(x) := \Pr(X_j \le x \mid X_j \text{ and } X_k \text{ are records})$ is $g_{j,k}(x) = jk(k-j)^{-1} f(x)\left(F^{j-1}(x) - F^{k-1}(x)\right)$, $x \in \mathbb{R}$, and the density function of the df $G_j(x) := \Pr(X_j \le x \mid X_j \text{ is a record})$ is $g_j(x) = jf(x)F^{j-1}(x)$, $x \in \mathbb{R}$. Suppose X_j is a record. To summarize by a single number the information, which is inherent in the additional knowledge that X_k is a record as well, we compute the Kullback-Leibler divergence between the density $g_{j,k}$ and the density g_j. In a general context, the Kullback-Leibler divergence of a density $q(\cdot)$ from a density $p(\cdot)$, is defined by

$$D_{KL}(p\|q) := \int_{-\infty}^{+\infty} p(x) \log\frac{p(x)}{q(x)}\,\mathrm{d}x.$$

It quantifies the information lost, when $q(\cdot)$ is used to approximate $p(\cdot)$. Closely related to the Kullback-Leibler divergence, the Kullback-Leibler distance of p and q is defined by $D_{KL}(p,q) := D_{KL}(p\|q) + D_{KL}(q\|p)$. Note that $D_{KL}(p\|q) \ge 0$ by Jensen's inequality.

Proposition 1.5. *The Kullback-Leibler distance between the densities $g_{j,k}$ and g_j is given by*

$$D_{KL}(g_{j,k}, g_j) = j/k, \qquad \textit{for } k > j \ge 1.$$

Proof. Firstly, we show that $D_{KL}(f_{j,k}, f_j) = j/k$ with $f_{j,k}(x) = jk(k-j)^{-1}(\mathrm{e}^{jx} - \mathrm{e}^{kx})$ and $f_j(x) = j\,\mathrm{e}^{jx}$, $x \le 0$, i.e. in the case of a sequence of negative exponential random variables.

$$\begin{aligned}
D_{KL}(f_{j,k}\|f_j) &= \int_{-\infty}^{0} \frac{jk}{k-j}\left(\mathrm{e}^{jx} - \mathrm{e}^{kx}\right) \log\left(\frac{jk}{k-j}\left(\mathrm{e}^{jx} - \mathrm{e}^{kx}\right)\frac{\mathrm{e}^{-jx}}{j}\right)\mathrm{d}x \\
&= \frac{jk}{k-j}\left(\int_{-\infty}^{0}\left(\mathrm{e}^{jx} - \mathrm{e}^{kx}\right)\log\frac{k}{k-j}\,\mathrm{d}x + \int_{-\infty}^{0}\left(\mathrm{e}^{jx} - \mathrm{e}^{kx}\right)\log\left(1 - \mathrm{e}^{(k-j)x}\right)\mathrm{d}x\right) \\
&= \log\frac{k}{k-j} + \frac{jk}{k-j}\int_{-\infty}^{0}\left(\mathrm{e}^{jx} - \mathrm{e}^{kx}\right)\log\left(1 - \mathrm{e}^{(k-j)x}\right)\mathrm{d}x.
\end{aligned}$$

The substitution $t = 1 - \mathrm{e}^{(k-j)x}$ entails

$$\int_{-\infty}^{0}\left(\mathrm{e}^{jx} - \mathrm{e}^{kx}\right)\log\left(1 - \mathrm{e}^{(k-j)x}\right)\mathrm{d}x = \frac{1}{k-j}\int_0^1\left((1-t)^{\frac{j}{k-j}-1} - (1-t)^{\frac{k}{k-j}-1}\right)\log t\,\mathrm{d}t.$$

Note that

$$\int_0^1 (1-t)^{\frac{j}{k-j}-1}\log t\,\mathrm{d}t = B\left(1, \frac{j}{k-j}\right)\left(\psi(1) - \psi\left(1 + \frac{j}{k-j}\right)\right) = \frac{k-j}{j}\left(\psi(1) - \psi\left(1 + \frac{j}{k-j}\right)\right)$$

where $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{d}t$, $a,b > 0$, $\psi(x) = \Gamma'(x)/\Gamma(x)$, $x > 0$, $\Gamma(x) = \int_0^\infty t^{x-1}\,\mathrm{e}^{-t}\,\mathrm{d}t$, $x > 0$

are the Beta, Digramma and Gamma functions, respectively. Analogously, one obtains

$$\int_0^1 (1-t)^{\frac{k}{k-j}-1} \log t \, \mathrm{d}t = \frac{k-j}{k}\left(\psi(1) - \psi\left(1 + \frac{k}{k-j}\right)\right).$$

As a consequence, we obtain

$$D_{KL}(f_{j,k} \| f_j) = \log \frac{k}{k-j} + \psi(1) + \frac{j}{k-j}\psi\left(1 + \frac{k}{k-j}\right) - \frac{k}{k-j}\psi\left(1 + \frac{j}{k-j}\right).$$

Furthermore, we have,

$$\begin{aligned} D_{KL}(f_j \| f_{j,k}) &= -\int_{-\infty}^{0} j\,\mathrm{e}^{jx} \log\left(\frac{jk}{k-j}\left(\mathrm{e}^{jx} - \mathrm{e}^{kx}\right)\frac{\mathrm{e}^{-jx}}{j}\right) \mathrm{d}x \\ &= -\log\frac{k}{k-j}\int_{-\infty}^{0} j\,\mathrm{e}^{jx}\,\mathrm{d}x - j\int_{-\infty}^{0} \mathrm{e}^{jx}\log\left(1 - \mathrm{e}^{(k-j)x}\right)\mathrm{d}x \\ &= -\log\frac{k}{k-j} - \frac{j}{k-j}\int_0^1 (1-t)^{\frac{j}{k-j}-1}\log t\,\mathrm{d}t = -\log\frac{k}{k-j} - \psi(1) + \psi\left(1 + \frac{j}{k-j}\right). \end{aligned}$$

Finally, we obtain

$$D_{KL}(f_{j,k} \| f_j) + D_{KL}(f_j \| f_{j,k}) = \frac{j}{k-j}\left(\psi\left(1 + \frac{k}{k-j}\right) - \psi\left(1 + \frac{j}{k-j}\right)\right).$$

The functional equation $\psi(1+x) = \psi(x) + 1/x$, $x > 0$, implies

$$\psi\left(1 + \frac{k}{k-j}\right) - \psi\left(1 + \frac{j}{k-j}\right) = \psi\left(1 + \frac{k}{k-j}\right) - \psi\left(\frac{k}{k-j}\right) = \frac{k-j}{k},$$

which yields the assertion. For the case of a general sequence of random variables, we have

$$D_{KL}(g_{j,k} \| g_j) = \int_{-\infty}^{+\infty} \frac{jk}{k-j} f(x)\left(F^{j-1}(x) - F^{k-1}(x)\right)\log\left(\frac{k}{k-j}\frac{F^{j-1}(x) - F^{k-1}(x)}{F^{j-1}(x)}\right)\mathrm{d}x.$$

The substitution $t = F^{-1}(\exp(x))$ entails that the above integral equals

$$\int_{-\infty}^{0} \frac{jk}{k-j}\left(\mathrm{e}^{jx} - \mathrm{e}^{kx}\right)\log\left(\frac{jk}{k-j}\left(\mathrm{e}^{jx} - \mathrm{e}^{kx}\right)\frac{\mathrm{e}^{-jx}}{j}\right)\mathrm{d}x = D_{KL}(f_{j,k} \| f_j).$$

Equally, one shows that $D_{KL}(g_j \| g_{j,k}) = D_{KL}(f_j \| f_{j,k})$. □

Clearly, $0 < D_{KL}(g_{j,k}, g_j) < 1$. The Kullback-Leibler distance between the densities $g_{j,k}$ and g_j gets small if j/k gets small. This means that the additional knowledge that X_k is a record as well, affects the distribution of X_j, being a record, less if k gets large. On the other hand, if $k = j+1$, then the information gain approaches one if j gets large. By repeating the arguments in the proof of Lemma 1.2, one derives the joint distribution of an arbitrary number of records as it is established by the next result.

Lemma 1.6. *We have for integers* $1 \le j_1 \cdots < j_d$, $d \in \mathbb{N}$, *with* $j_0 = 0$, *and* $x_1, \dots, x_d \le 0$,

$$\Pr(\eta_{j_m} \le x_m, 1 \le m \le d \mid \eta_{j_1}, \dots, \eta_{j_d} \text{ are records})$$
$$= \frac{\prod_{m=2}^d j_m}{\prod_{m=2}^d (j_m - j_{m-1})} \Pr\left(\frac{\eta_m}{j_m - j_{m-1}} \le x_m, \frac{(j_{m+1} - j_m)\eta_m}{j_m - j_{m-1}} < \eta_{m+1}, 1 \le m \le d-1, \frac{\eta_d}{j_d - j_{d-1}} \le x_d\right).$$

The case of an arbitrary sequence of iid rvs $X_1, X_2, \dots$ with common continuous df F can immediately be deduced from the preceding result via the representation $X_i = F^{-1}(\exp(\eta_i))$, $i \in \mathbb{N}$.

2 Asymptotic Joint Distribution of Records

Let $X_1, X_2, \dots$ be iid rvs with common df F, which is in the domain of attraction of an extreme value distribution G. From Lemma 1.1 we immediately obtain the following result.

Lemma 2.1. *Under the preceding conditions we obtain*

$$\Pr\left(\frac{X_n - b_n}{a_n} \le x \mid X_n \text{ is a record}\right) \xrightarrow[n\to\infty]{} G(x), \qquad x \in \mathbb{R}.$$

In what follows we investigate the joint asymptotic distribution of two records. We start with a sequence $\eta_1, \eta_2 \dots$ of iid rvs that follow the standard negative exponential df $F(x) = \exp(x)$, $x \le 0$. From Lemma 1.2 we obtain for $x_1, x_2 \in \mathbb{R}$,

$$\Pr\left(\eta_j \le \frac{x_1}{j}, \eta_k \le \frac{x_2}{k} \mid \eta_j \text{ and } \eta_k \text{ are records}\right) = \frac{k}{k-j} \Pr\left(\eta_1 \le x_1, \eta_2 \le \frac{k-j}{k} x_2, \frac{k-j}{j}\eta_1 < j\eta_2\right).$$

We let $j = j(n)$ and $k = k(n)$ both depend on $n \in \mathbb{N}$ with

$$\lim_{n\to\infty} \frac{j}{n} = \lambda_1 > 0, \quad \lim_{n\to\infty} \frac{k}{n} = \lambda_2 > \lambda_1. \tag{3}$$

The next result is a consequence of Lemma 1.2 and elementary computations.

Proposition 2.2. *Under condition* (3), *for all* $x_1, x_2 \le 0$, $\beta_j = \lambda_j/(\lambda_2 - \lambda_1)$ *and* $j = 1, 2$, *we obtain*

$$\lim_{n\to\infty} \Pr\left(\eta_j \le \frac{x_1}{n}, \eta_k \le \frac{x_2}{n} \mid \eta_j \text{ and } \eta_k \text{ are records}\right) = H_{\lambda_1,\lambda_2}(x_1, x_2),$$

where

$$H_{\lambda_1,\lambda_2}(x_1, x_2) = \begin{cases} e^{\lambda_1 x_1}(\beta_2\, e^{(\lambda_2-\lambda_1)x_2} - \beta_1\, e^{(\lambda_2-\lambda_1)x_1}), & \text{if } x_1 < x_2, \\ e^{\lambda_2 x_2}, & \text{if } x_1 \ge x_2. \end{cases} \tag{4}$$

The marginal df of H_{λ_1,λ_2} are

$$H_1(x) = H_{\lambda_1,\lambda_2}(x, 0) = \beta_2 \, \mathrm{e}^{\lambda_1 x} - \beta_1 \, \mathrm{e}^{\lambda_2 x}, \quad H_2(x) = H_{\lambda_1,\lambda_2}(0, x) = \mathrm{e}^{\lambda_2 x}, \quad x \leq 0. \tag{5}$$

Clearly, the fact that H_2 is independent of λ_1 reflects the fact that the distribution of η_k, being a record, is not affected by the additional knowledge that η_j is a record as well, as shown in the previous section. While H_2 is a univariate extreme value distribution, H_1 is not. Therefore, the bivariate df H_{λ_1,λ_2} is not a multivariate extreme value distribution. In the next result we provide the marginal means, variances and the covariance of the margins of H_{λ_1,λ_2}.

Proposition 2.3. *Let (X, Y) be a bivariate rv with df given in (4). Then, we have for all $\lambda_2 > \lambda_1 > 0$*

(i) $\mathrm{E}(X) = -\lambda_1^{-1} - \lambda_2^{-1}, \quad \mathrm{Var}(X) = \lambda_1^{-2} + \lambda_2^{-2}, \quad \mathrm{E}(Y) = -\lambda_2^{-1}, \quad \mathrm{Var}(Y) = \lambda_2^{-2}$

(ii) $\mathrm{Cov}(X, Y) = \lambda_2^{-2}, \quad \mathrm{Corr}(X, Y) = \frac{\lambda_1}{\sqrt{\lambda_1^2+\lambda_2^2}}, \quad \mathrm{E}\left[(X - Y)^2\right] = \lambda_1^{-2}.$

Proof. Assume that the probability law of the pairs of the rvs (X, Y) is given by (4). Then

$$\begin{aligned}
\mathrm{Cov}(X, Y) &= \int_{-\infty}^{0}\int_{-\infty}^{0} H_{\lambda_1,\lambda_2}(x, y) - H_{\lambda_1,\lambda_2}(x, 0)H_{\lambda_1,\lambda_2}(0, y)\, \mathrm{d}x\, \mathrm{d}y \\
&= \int_{-\infty}^{0}\left(\int_{-\infty}^{y} \frac{\lambda_2 \, \mathrm{e}^{(\lambda_2-\lambda_1)y+\lambda_1 x} - \lambda_1 \, \mathrm{e}^{\lambda_2 x}}{\lambda_2 - \lambda_1}\, \mathrm{d}x + \int_{y}^{0} \mathrm{e}^{\lambda_2 y}\, \mathrm{d}x\right) \mathrm{d}y - \int_{-\infty}^{0} \frac{\lambda_2 \, \mathrm{e}^{\lambda_1 x} - \lambda_1 \, \mathrm{e}^{\lambda_2 x}}{\lambda_2 - \lambda_1}\, \mathrm{d}x \int_{-\infty}^{0} \mathrm{e}^{\lambda_2 y}\, \mathrm{d}y \\
&= \frac{\lambda_2^2 + \lambda_1\lambda_2 - 2\lambda_1^2}{\lambda_1\lambda_2^2(\lambda_2 - \lambda_2)} - \frac{\lambda_1 + \lambda_2}{\lambda_1\lambda_2}\frac{1}{\lambda_2} = \frac{1}{\lambda_2^2}.
\end{aligned}$$

The variance of X is

$$\mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - \mathrm{E}^2(X) = 2\int_{-\infty}^{0} -x\, \frac{\lambda_2 \, \mathrm{e}^{\lambda_1 x} - \lambda_1 \, \mathrm{e}^{\lambda_2 x}}{\lambda_2 - \lambda_1}\, \mathrm{d}x - \left(-\int_{-\infty}^{0} \frac{\lambda_2 \, \mathrm{e}^{\lambda_1 x} - \lambda_1 \, \mathrm{e}^{\lambda_2 x}}{\lambda_2 - \lambda_1}\, \mathrm{d}x\right)^2 = \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2\lambda_2^2}.$$

The marginal distribution of Y is $\exp(\lambda_2 y)$, $y \leq 0$, therefore its mean and variance are $1/\lambda_2$ and $1/\lambda_2^2$, respectively. Finally, combining these results

$$\mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{1/\lambda_2^2}{\sqrt{(\lambda_1^2 + \lambda_2^2)/\lambda_1^2\lambda_2^2 \cdot 1/\lambda_2^2}} = \frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}}. \qquad \square$$

The next result extends Proposition 2.2 to a sequence of iid rvs, whose df F satisfies $F \in \mathcal{D}(G)$.

Corollary 2.4. *Let $X_1, X_2, \ldots$ be iid copies of a rv X with a continuous distribution F. Assume that $F \in \mathcal{D}(G)$ with norming constants $a_n > 0$ and $b_n \in \mathbb{R}$, $n \in \mathbb{N}$. Then, under Condition (3), we have for $y_1, y_2 \in \mathbb{R}$*

$$\lim_{n\to\infty} \Pr\left(\frac{X_j - b_n}{a_n} \leq y_1, \frac{X_k - b_n}{a_n} \leq y_2 \mid X_j \textit{ and } X_k \textit{ are records}\right) = G_{\lambda_1,\lambda_2}(y_1, y_2),$$

where

$$G_{\lambda_1,\lambda_2}(y_1,y_2) = \begin{cases} G^{\lambda_1}(y_1)\left(\beta_2 G(y_2)^{\lambda_2-\lambda_1} - \beta_1 G(y_1)^{\lambda_2-\lambda_1}\right), & \text{if } y_1 < y_2, \\ G^{\lambda_2}(y_2), & \text{if } y_1 \ge y_2. \end{cases}$$

The marginal distributions are given by $G_{\lambda_1,\lambda_2}(y_1,\infty) = \beta_2 G(y_1)^{\lambda_1} - \beta_1 G(y_1)^{\lambda_2}$, $G_{\lambda_1,\lambda_2}(\infty, y_2) = G(y_2)^{\lambda_2}$, $y_1, y_2 \in \mathbb{R}$; note that the second marginal is independent of λ_1. Note that results on the limiting distribution of joint records with *known* orders in the sequence of records have been recently derived by Barakat and Elgawad (2017).

Proof. Put $\eta_m := \log(F(X_m))$, $m \in \mathbb{N}$. Then $\eta_1, \eta_2, \dots$ are iid standard negative exponential distribution. Since $\log(\cdot)$ and $F(\cdot)$ are monotone we obtain

$$\begin{aligned} &\Pr\left(X_j \le a_n y_1 + b_n, X_k \le a_n y_2 + b_n \mid X_j \text{ and } X_k \text{ are records}\right) \\ &= \Pr\left(\eta_j \le \frac{n\log(F(a_n y_1 + b_n))}{n}, \eta_k \le \frac{n\log(F(a_n y_2 + b_n))}{n} \mid \eta_j \text{ and } \eta_k \text{ are records}\right). \end{aligned}$$

The condition $F^n(a_n x + b_n) \to G(x)$, $x \in \mathbb{R}$, as $n \to \infty$, is equivalent to $n\log(F(a_n x + b_n)) \to \log(G(x))$, $0 < G(x) \le 1$, as $n \to \infty$. Proposition 2.2 now implies

$$\begin{aligned} &\Pr\left(\eta_j \le \frac{n\log(F(a_n y_1 + b_n))}{n}, \eta_k \le \frac{n\log(F(a_n y_2 + b_n))}{n} \mid \eta_j, \eta_k \text{ are records}\right) \\ &\xrightarrow[n\to\infty]{} H_{\lambda_1,\lambda_2}(\log(G(y_1)), \log(G(y_2))). \end{aligned}$$

□

We have established the fact that the distribution of X_k, being a record, is not affected if we know in addition that X_j is a record as well. But what happens if we know, for example, that X_j, being a record, has already exceeded a fixed threshold? The answer is a straightforward consequence of our preceding results. We obtain for $y > u \in \mathbb{R}$, under the conditions of Corollary 2.4,

$$\begin{aligned} &\Pr\left(\frac{X_k - b_n}{a_n} \le y \mid \frac{X_j - b_n}{a_n} > u, X_j \text{ and } X_k \text{ are records}\right) \\ &= \frac{\Pr\left(\frac{X_k-b_n}{a_n} \le y \mid X_j \text{ and } X_k \text{ are records}\right) - \Pr\left(\frac{X_j-b_n}{a_n} \le u, \frac{X_k-b_n}{a_n} \le y \mid X_j \text{ and } X_k \text{ are records}\right)}{1 - \Pr\left(\frac{X_j-b_n}{a_n} \le u \mid X_j \text{ and } X_k \text{ are records}\right)} \\ &\xrightarrow[n\to\infty]{} \frac{G(y)^{\lambda_2} - G^{\lambda_1}(u)\left(\beta_2 G(y)^{\lambda_2-\lambda_1} - \beta_1 G(u)^{\lambda_2-\lambda_1}\right)}{\beta_2\left(1 - G(u)^{\lambda_1}\right) - \beta_1\left(1 - G(u)^{\lambda_2}\right)}. \end{aligned}$$

The results obtained so far can be extended to the case of an arbitrary number of records. However, computations become really hard. We report the case of the asymptotic joint df of three records.

Proposition 2.5. *Let $X_1, X_2, \dots$ be iid copies of a rv X with a continuous distribution F. Assume that $F \in \mathcal{D}(G)$ with norming constants $a_n > 0$ and $b_n \in \mathbb{R}$, $n \in \mathbb{N}$. Assume also that $j = j(n)$, $k = k(n)$ and*

$r = r(n)$ *all depending on* $n \in \mathbb{N}$ *with* $j < k < r$ *and*

$$\lim_{n\to\infty} \frac{j}{n} = \lambda_1 > 0, \quad \lim_{n\to\infty} \frac{k}{n} = \lambda_2 > \lambda_1, \quad \lim_{n\to\infty} \frac{r}{n} = \lambda_3 > \lambda_2.$$

Then, for all $\boldsymbol{y} \in \mathbb{R}^3$, *we have*

$$\Pr\left(\frac{X_j - b_n}{a_n} \le y_1, \frac{X_k - b_n}{a_n} \le y_2, \frac{X_r - b_n}{a_n} \le y_3 \mid X_j,\, X_k \text{ and } X_r \text{ are records}\right) \to \boldsymbol{G}_{\boldsymbol{\lambda}}(\boldsymbol{y}),$$

as $n \to \infty$, *where*

$$\boldsymbol{G}_{\boldsymbol{\lambda}}(\boldsymbol{y}) = \begin{cases} G(y_1)^{\lambda_1} G(y_2)^{\lambda_2-\lambda_1}\left(\beta_2\beta_6 G(y_3)^{\lambda_3-\lambda_1} - \beta_4\beta_5 G(y_2)^{\lambda_3-\lambda_1}\right) \\ \quad -G(y_1)^{\lambda_2}\left(\beta_1\beta_6 G(y_3)^{\lambda_3-\lambda_2} - \beta_3\beta_4 G(y_1)^{\lambda_3-\lambda_2}\right) \quad \text{if} \quad y_1 \le y_2 \le y_3 \\ G(y_2)^{\lambda_2}\left(\beta_6 G(y_3)^{\lambda_3-\lambda_2} - \beta_4 G(y_2)^{\lambda_3-\lambda_2}\right), \quad \text{if} \quad y_2 \le y_1 \le y_3 \text{ or } y_2 \le y_3 \le y_1, \\ \beta_2\beta_5 G(y_1)^{\lambda_1} G(y_3)^{\lambda_3-\lambda_1} - \beta_1\beta_6 G(y_1)^{\lambda_2} G(y_3)^{\lambda_3-\lambda_2} + \beta_3\beta_4 G(y_1)^{\lambda_3}, \quad \text{if} \quad y_1 \le y_3 \le y_2, \\ G(y_3)^{\lambda_3}, \quad \text{if} \quad y_3 \le y_2 \le y_1 \text{ or } y_3 \le y_1 \le y_2, \end{cases}$$

and where β_1, β_2 *are as in Proposition 2.2*, $\beta_3 = \lambda_1/(\lambda_3 - \lambda_1)$, $\beta_4 = \lambda_2/(\lambda_3 - \lambda_2)$, $\beta_5 = \lambda_3/(\lambda_3 - \lambda_1)$, $\beta_6 = \lambda_3/(\lambda_3 - \lambda_2)$, *and* $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$. *In addition, let* $\boldsymbol{Y} = (Y_1, Y_2, Y_3)$ *be a rv with df* $\boldsymbol{G}_{\boldsymbol{\lambda}}(\boldsymbol{y})$, *then the variance-covariance matrix of* $\boldsymbol{Y}$ *is*

$$\begin{pmatrix} \lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} & \lambda_2^{-2} + \lambda_3^{-2} & \lambda_3^{-2} \\ & \lambda_2^{-2} + \lambda_3^{-2} & \lambda_3^{-2} \\ & & \lambda_3^{-2} \end{pmatrix}.$$

Proof. Let $\eta_1, \eta_2, \ldots$ be iid rvs with a common negative exponential distribution. First of all we compute the following non-asymptotic distribution when $x_1 < x_2 < x_3$

$$\begin{aligned} &\Pr\left(\eta_j \le x_1, \eta_k \le x_2, \eta_r \le x_3 \mid \eta_j,\, \eta_k \text{ and } \eta_r \text{ are records}\right) \\ &= \frac{kr}{(k-j)(r-k)} \Pr\left(\eta_1 \le jx_1, \eta_2 \le (k-j)x_2, \eta_3 \le (r-k)x_3, (k-j)\eta_1 < j\eta_2, (r-k)\eta_2 < (k-j)\eta_3\right) \\ &= \frac{kr}{(k-j)(r-k)} \int_{-\infty}^{jx_1} \int_{\frac{k-j}{j}z_1}^{(k-j)x_2} \Pr\left(\frac{r-k}{k-j} z_2 < \eta_3 \le x_3\right) \mathrm{e}^{z_2+z_1}\, \mathrm{d}z_2\, \mathrm{d}z_1 \\ &= \frac{kr}{(k-j)(r-k)} \int_{-\infty}^{jx_1} \mathrm{e}^{z_1} \left(\mathrm{e}^{(r-k)x_3}\left(\mathrm{e}^{(k-j)x_2} - \mathrm{e}^{\frac{k-j}{j}z_1}\right) - \frac{k-j}{r-j}\left(\mathrm{e}^{(r-j)x_2} - \mathrm{e}^{\frac{r-j}{j}z_1}\right)\right) \mathrm{d}z_1 \\ &= \frac{kr}{(k-j)(r-k)} \left(\mathrm{e}^{jx_1}\left(\mathrm{e}^{(r-k)x_3}\,\mathrm{e}^{(k-j)x_2} - \frac{k-j}{r-j}\,\mathrm{e}^{(r-j)x_2}\right) - \frac{j}{k}\,\mathrm{e}^{kx_1}\,\mathrm{e}^{(r-k)x_3} + \frac{j(k-j)}{r(r-j)}\,\mathrm{e}^{rx_1}\right). \end{aligned}$$

The cases of $x_2 < x_1 < x_3$ is obtained from the expression of the above formula by substituting x_2 in x_1. Similarly the case $x_1 < x_3 < x_2$ is obtained by substituting x_3 in x_2 and lastly the case $x_3 < x_2 < x_1$ is obtained by substituting x_3 in both x_1 and x_2. Then, the asymptotic distribution is easily obtained by computing $\lim_{n\to\infty} \Pr\left(\eta_j \le x_1/n, \eta_k \le x_2/n, \eta_r \le x_3/n \mid \eta_j,\, \eta_k \text{ and } \eta_r \text{ are records}\right)$. The case of an arbitrary distribution can be deduced by following the same reasoning of Corollary 2.4 and therefore the first assertion is derived. We compute the variance-covariance matrix. Note that the bivariate and univariate marginal distribution functions of (Y_1, Y_2) are

$$F_{\lambda_1,\lambda_2}(x_1,x_2) = \begin{cases} \left(\frac{\lambda_2\lambda_3\,\mathrm{e}^{(\lambda_2-\lambda_1)x_2}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)} - \frac{\lambda_2\lambda_3\,\mathrm{e}^{(\lambda_3-\lambda_1)x_2}}{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)}\right)\mathrm{e}^{\lambda_1 x_1} - \frac{\lambda_1\lambda_3\,\mathrm{e}^{\lambda_2 x_1}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)} - \frac{\lambda_1\lambda_2\,\mathrm{e}^{\lambda_3 x_1}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}, & \text{if} \quad x_1 \le x_2, \\ \frac{\lambda_3\,\mathrm{e}^{\lambda_2 x_2} - \lambda_2\,\mathrm{e}^{\lambda_3 x_2}}{\lambda_3-\lambda_2}, \quad \text{if} \quad x_2 \le x_1, \end{cases}$$

$$F_{\lambda_1}(x_1) = \frac{\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}\,\mathrm{e}^{\lambda_1 x_1} - \frac{\lambda_1\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)}\,\mathrm{e}^{\lambda_2 x_1} - \frac{\lambda_1\lambda_2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\,\mathrm{e}^{\lambda_3 x_1},\ x_1 \le 0,$$

$$F_{\lambda_2}(x_2) = \frac{\lambda_3\,\mathrm{e}^{\lambda_2 x_2} - \lambda_2\,\mathrm{e}^{\lambda_3 x_2}}{\lambda_3-\lambda_2},\ x_2 \le 0,$$

where we have taken the transformation $x_j = \log(G(y_j))$, $j = 1, \ldots, 3$, for simplicity. Hoeffding's covariance identity implies that

$$\begin{aligned} \mathrm{Cov}(Y_1, Y_2) &= \int_{(-\infty,0]^2} F_{\lambda_1,\lambda_2}(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 - \int_{-\infty}^{0} F_{\lambda_1}(x_1)\,\mathrm{d}x_1 \int_{-\infty}^{0} F_{\lambda_2}(x_2)\,\mathrm{d}x_2 \\ &= \frac{\lambda_3}{\lambda_1(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)} - \frac{\lambda_2}{\lambda_1(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} - \frac{\lambda_1\lambda_3}{\lambda_2^2(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)} \\ &\quad + \frac{\lambda_1\lambda_2}{\lambda_3^2(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} + \frac{\lambda_3}{\lambda_2^2(\lambda_3-\lambda_2)} - \frac{\lambda_2}{\lambda_3^2(\lambda_3-\lambda_2)} - \left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}\right)\left(\frac{1}{\lambda_2}+\frac{1}{\lambda_3}\right). \end{aligned}$$

By straightforward simplifications we obtain $\mathrm{Cov}(Y_1, Y_2) = \lambda_2^{-2} + \lambda_3^{-2}$. The other covariances are computed in a similar way. The expected values and the variances are derived by simple computations. □

Under Condition (3), another application of Lemma 1.6 yields the following results.

Theorem 2.6. *Let $\eta_1, \eta_2, \ldots$ be independent and standard negative exponential distributed rvs. Assume that $j_i = j_i(n) \in \mathbb{N}$, $i = 1, 2, \ldots$, $n = 2, 3, \ldots$ are sequences of integers satisfying $\lim_{n\to\infty} j_i/n = \lambda_i > 0$, with $0 < \lambda_1 < \lambda_2 < \cdots$. Under these conditions and every $x \le 0$, $y, y_1, \ldots, y_s > 0$ and $m \in \mathbb{N}$, we have*

$$\lim_{n\to\infty} \Pr\left(\eta_{j_{i+1}} - \eta_{j_i} \le y_i/n, 1 \le i \le s \mid \eta_{j_1} \ldots \eta_{j_s} \text{ are records}\right) = \prod_{i=1}^{s}\left(1 - \mathrm{e}^{-\lambda_i y_i}\right),$$

$$\lim_{n\to\infty} \Pr\left(\eta_j \le x/n, \eta_k - \eta_j \le y/n \mid \eta_j \text{ and } \eta_k \text{ are records}\right) = Q_{\lambda_1,\lambda_2}(x,y),$$

where

$$Q_{\lambda_1,\lambda_2}(x,y) = \begin{cases} \beta_1 \left(\mathrm{e}^{(\lambda_2-\lambda_1)y} - 1\right) \mathrm{e}^{\lambda_2 x}, & \textit{if } |x| \geq y, \\ \beta_2 \,\mathrm{e}^{\lambda_1 x} - \beta_1 \,\mathrm{e}^{\lambda_2 x} - \mathrm{e}^{-\lambda_1 y}, & \textit{if } |x| < y. \end{cases} \tag{6}$$

The marginal distributions of (6) are $Q_{\lambda_1,\lambda_2}(x) = Q_{\lambda_1,\lambda_2}(x,\infty) = \beta_2 \,\mathrm{e}^{\lambda_1 x} - \beta_1 \,\mathrm{e}^{\lambda_2 x}$, $x \leq 0$, and $Q_{\lambda_1}(y) = Q_{\lambda_1,\lambda_2}(0,y) = 1 - \mathrm{e}^{-\lambda_1 y}$, $y > 0$. These results mean that the increments $Y_1, \dots, Y_n$ among records are independent but not identically distributed. Furthermore, a generic record η_j, $j = 1, 2, \dots$ and the increment between two records η_j and η_k, $k > j$ are not independent.

Proof. For all $y_1, \dots, y_s > 0$, by following the same reasoning in the proof of Lemma 1.2 we have

$$\begin{aligned} &\Pr\left(\eta_{j_2} - \eta_{j_1} > y_1, \dots, \eta_{j_s} - \eta_{j_{s-1}} > y_{s-1} \mid \eta_{j_1} \dots \eta_{j_s} \text{ are records}\right) \\ &= \frac{\prod_{m=2}^s j_m}{\prod_{m=2}^s (j_m - j_{m-1})} \Pr\left(\frac{\eta_2}{j_2 - j_1} - \frac{\eta_1}{j_1} > z_1, \dots, \frac{\eta_s}{j_s - j_{s-1}} - \frac{\eta_{s-1}}{j_{s-1}} > z_{s-1}\right) = \frac{\prod_{m=2}^s j_m}{\prod_{m=2}^s (j_m - j_{m-1})} \cdot A \end{aligned}$$

where

$$A = \int_{-\infty}^0 \int_{-\infty}^{(j_{s-1}-j_{s-2})\left(\frac{z_s}{j_s - j_{s-1}} - y_{s-1}\right)} \cdots \int_{-\infty}^{(j_2 - j_1)\left(\frac{z_3}{j_3 - j_2} - y_2\right)} \Pr\left(\eta_1 < j_1 \left(\frac{z_2}{j_2 - j_1} - y_1\right)\right) \prod_{i=2}^s \mathrm{e}^{z_i} \,\mathrm{d}z_i.$$

We show by induction that

$$\int_{-\infty}^{(j_m - j_{m-1})\left(\frac{z_{m+1}}{j_{m+1} - j_m} - y_m\right)} \cdots \int_{-\infty}^{(j_2 - j_1)\left(\frac{z_3}{j_3 - j_2} - y_2\right)} \mathrm{e}^{\frac{j_2}{j_2 - j_1} z_2} \,\mathrm{e}^{-j_1 y_1} \prod_{i=2}^m \mathrm{e}^{z_i} \,\mathrm{d}z_i = \frac{\prod_{i=2}^m (j_i - j_{i-1})}{\prod_{i=2}^m j_i} \prod_{i=1}^m \mathrm{e}^{-j_i y_i} \,\mathrm{e}^{\frac{j_m z_{m+1}}{j_{m+1} - j_m}}.$$

At the step 1 we have

$$\int_{-\infty}^{(j_2 - j_1)\left(\frac{z_3}{j_3 - j_2} - y_2\right)} \mathrm{e}^{\frac{j_2}{j_2 - j_1} z_2} \,\mathrm{e}^{-j_1 y_1} \,\mathrm{d}z_2 = \frac{j_2 - j_1}{j_2} \,\mathrm{e}^{-j_1 y_1 - j_2 y_2} \,\mathrm{e}^{\frac{j_2 z_3}{j_3 - j_2}}.$$

True for m. At the step $m+1$ we have

$$\begin{aligned} &\int_{-\infty}^{(j_{m+1} - j_m)\left(\frac{z_{m+2}}{j_{m+2} - j_{m+1}} - y_{m+1}\right)} \cdots \int_{-\infty}^{(j_2 - j_1)\left(\frac{z_3}{j_3 - j_2} - y_2\right)} \mathrm{e}^{\frac{j_2}{j_2 - j_1} z_2} \,\mathrm{e}^{-j_1 y_1} \prod_{i=2}^{m+1} \mathrm{e}^{z_i} \,\mathrm{d}z_i \\ &= \int_{-\infty}^{(j_{m+1} - j_m)\left(\frac{z_{m+2}}{j_{m+2} - j_{m+1}} - y_{m+1}\right)} \frac{\prod_{i=2}^m (j_i - j_{i-1})}{\prod_{i=2}^m j_i} \prod_{i=1}^m \mathrm{e}^{-j_i y_i} \,\mathrm{e}^{\frac{j_m z_{m+1}}{j_{m+1} - j_m}} \,\mathrm{e}^{z_{m+1}} \,\mathrm{d}z_{m+1} \\ &= \frac{\prod_{i=2}^{m+1} (j_i - j_{i-1})}{\prod_{i=2}^{m+1} j_i} \prod_{i=1}^{m+1} \mathrm{e}^{-j_i y_i} \,\mathrm{e}^{\frac{j_{m+1} z_{m+2}}{j_{m+2} - j_{m+1}}}. \end{aligned}$$

As a consequence $A = \prod_{i=2}^s (j_i - j_{i-1}) \prod_{i=2}^s j_i^{-1} \prod_{i=1}^{s-1} \mathrm{e}^{-j_i y_i}$. We obtain the first result which shows that the increments among records are independent exponentials but with different parameters $0 < \lambda_1 < \lambda_2 <$

$\cdots < \lambda_s$. By the assumptions we have that $(1-\mathrm{e}^{j_i y_i/n}) \to (1-\mathrm{e}^{\lambda_i y_i})$ as $n \to \infty$ for any $i = 1, \ldots, s$. Next, for $x \leq 0$, $y \geq 0$ we have

$$\begin{aligned} Q(x,y) &:= \Pr\left(\eta_j \leq x, \eta_k - \eta_j \leq y \mid \eta_j \text{ and } \eta_k \text{ are records}\right) \\ &= jk \Pr\left(\eta_j \leq x, \eta_k - \eta_j \leq y, \eta_j > \frac{\eta_1}{j-1}, \eta_k > \max\left(\eta_j, \frac{\eta_2}{k-j-1}\right)\right) \end{aligned}$$

When $x \leq -y$,

$$Q(x,y) = jk \int_{-\infty}^{x} \int_{z_j}^{z_j+y} \mathrm{e}^{(k-j)z_k} \mathrm{e}^{jz_j} \, \mathrm{d}z_k \, \mathrm{d}z_j = \frac{jk}{k-j}(\mathrm{e}^{(k-j)y} - 1) \int_{-\infty}^{x} \mathrm{e}^{kz_j} \, \mathrm{d}z_j = \frac{j}{k-j}(\mathrm{e}^{(k-j)y} - 1) \mathrm{e}^{kx}.$$

Therefore, $Q(x/n, y/n) \to \beta_1 \left(\mathrm{e}^{(\lambda_2-\lambda_1)y} - 1\right) \mathrm{e}^{\lambda_2 x}$ as $n \to \infty$. When $x > -y$

$$\begin{aligned} Q(x,y) &= jk \left(\int_{-\infty}^{-y} \int_{z_j}^{z_j+y} \mathrm{e}^{(k-j)z_k} \mathrm{e}^{jz_j} \, \mathrm{d}z_k \, \mathrm{d}z_j + \int_{-y}^{x} \int_{z_j}^{0} \mathrm{e}^{(k-j)z_k} \mathrm{e}^{jz_j} \, \mathrm{d}z_k \, \mathrm{d}z_j \right) \\ &= \frac{j}{k-j}(\mathrm{e}^{(k-j)y} - 1) \mathrm{e}^{-ky} + \frac{jk}{k-j} \left(\frac{\mathrm{e}^{jx} - \mathrm{e}^{-jy}}{j} - \frac{\mathrm{e}^{kx} - \mathrm{e}^{-ky}}{k} \right) = \frac{k}{k-j} \mathrm{e}^{jx} - \frac{j}{k-j} \mathrm{e}^{kx} - \mathrm{e}^{-jy}. \end{aligned}$$

Therefore $Q(x/n, y/n) \to \beta_2 \mathrm{e}^{\lambda_1 x} - \beta_1 \mathrm{e}^{\lambda_2 x} - \mathrm{e}^{-\lambda_1 y}$ as $n \to \infty$. □

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