Maximizing the Overall End-User Satisfaction of Data Broadcast in Wireless Mesh Networks[1]

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Abstract

We study the problem of broadcasting a common, possibly large, content into a wireless mesh network consisting of N end-users and of one or multiple access points that act as gateways to Internet. Each end-user is characterized by a maximum possible reception rate that depends on the distance and on the interface used to communicate with the associated access point. The end-user satisfaction is proportional to the actual rate received. The overall end-user satisfaction is the sum of the satisfaction of each end-user. Our goal is to maximize the overall end-user satisfaction under the constraint that the access points can retransmit at different rates the same common content at most K times.

We show that the problem can be solved by serving the end-users according to a suitable K segmentation, which is a K partition of the end-users that preserves a specific end-user order. When the access points and the end-users have a unique interface, the optimal segmentation can be found in $O(N(K + \log N))$ time by exploiting the convex Monge property of the satisfaction function. When both access points and end-users are equipped with multiple interfaces, the problem becomes computationally intractable, even for a single access point. Polynomial time algorithms are then devised for optimally solving some meaningful particular cases.

Keywords: Broadcast, Single-hop, Multi-Rate, Multi-Interface, Monge property, Dynamic Programming, NP-Completeness

1. Introduction

Wireless mesh networks (WMNs) have received much attention in recent years for implementing large-scale wireless networks in suburban and urban community because of their low-deploying and low-management cost, especially when built from commodity wireless cards and operating over unregulated spectrum [6, 7]. In such networks, some of the nodes serve as gateways for other nodes to access the Internet. Such nodes play both roles of a host and a router, and are typically stationary and not power-constrained. The remaining nodes, called *end-users*, are connected via wireless link to a single gateway. Packets are usually forwarded in a multi-hop fashion to and from the gateway nodes [6], while in a single-hop fashion from the gateway to the end-users. Previous works showed that employing multiple non-overlapping channels to serve the end-users is an effective approach to improve the network throughput and capacity [16].

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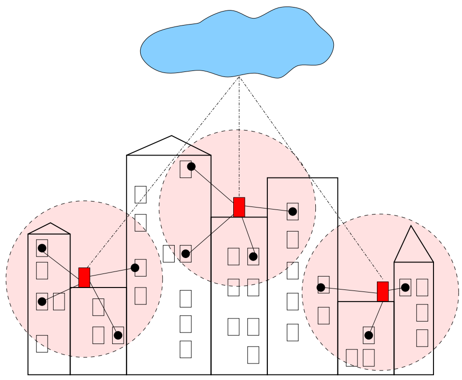

Figure 1: A large-scale urban community-based wireless mesh network.

In this work, we consider WMNs with fixed gateways, the Access Points (APs), which form a tree for communication purposes. Each AP provides one-hop network connectivity to the end-user nodes within its coverage areas, and we assume that each end-user is assigned to a single fixed AP. As an example, Figure 1 depicts a simplified view of an urban community-based wireless network, where there is a source (a provider in the cloud) which distributes a content to all APs (the red rectangles on the rooftops), which in turn forward the common content to the end-user nodes (the black dots). Each access point transmits into a disc area (the shadow circle), while each end-user refers to a fixed single AP.

As in [6, 17, 20], we assume the multi-rate capability in the PHY layer, i.e., APs may communicate with end-users at different data rates, depending on the interface, the distance, the radio signal quality, the set of modulation and coding schemes available in the system. Nowadays, many prototypes of WMNs equip each mesh node with multiple interfaces and/or radios and tune them to non-overlapping channels to augment the quality of the transmission experienced by the end-users. Also, most commodity wireless cards for AP perform adaptive modulation that changes the link transmission rate in response to the receiver distance.

Our goal in this paper is to leverage the multi-rate capability of the APs for increasing the *overall end-user satisfaction*, which is measured by the sum of the rates received by the end-users. In earlier research on mobile ad hoc networks, only control broadcasts were supported which disseminate short messages, like link status or indices, for generating or repairing routing tables. Instead in this work, we plan to disseminate a large common content (like a video) to all the end-users, which are often concentrate in small areas. We call such an operation the *data broadcast* operation. We exploit the wireless broadcast advantage by using, whenever possible, a single broadcast transmission to reach multiple end-user nodes. Note that, thanks to multiple radios/interfaces, an AP can perform parallel transmissions, each towards multiple end-users.

According to the IEEE 802.11a/b/n amendments [5, 15], the access points have a fixed maximum transmission power. If the same transmission power is used for all transmission rates, then, in general, the higher is the transmission rate, the smaller is the transmission range (although, the rate-distance variation in real life is somewhat irregular). Thus end-users closer to the access point can be reached with higher rate (see Figure 2), and experience a smaller latency. For instance, when the access point transmits up to the maximum radius (i.e. 180 meters, in Figure 2), all the end-users receive at the minimum rate, expressed typically in metric multiples of bits per second, (i.e. 6 Mbps), but if the AP transmits in the circle of minimum radius (i.e., 42 meters), then the end-user rate significantly increases by a factor of 6. So in our problem the rate received by a group of end-users served in a single transmission depends on the radius of the transmission that covers all such end-users. Specifically, for each transmission, the access point serves simultaneously all the end-users in the group at an actual rate identical to the minimum rate among all the maximum possible rates for the end-users of that group. From now on, we call the rate received by each end-user the *end-user satisfaction*.

Therefore, fixed the maximum number K of transmissions that the APs may perform to serve all the end-users, we aim to assign the end-users to the transmissions so as the overall end-user satisfaction is

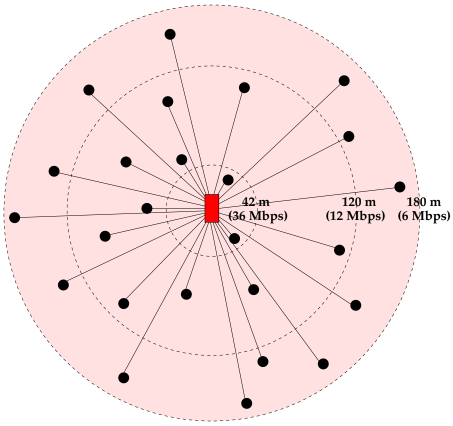

Figure 2: The disc model for access points having decreasing transmission rates for increasing transmission ranges.

maximized. In practice, the overall satisfaction can represent, for example, the profit of a provider that sells a premium data broadcast service charged on a rate basis to the end-users.

We first study the problem of maximizing the overall satisfaction assuming that there is a single interface and that the mesh consists of (*i*) a single access point or (*ii*) many access points. In both cases, there is a predefined maximum number K of transmissions reserved for the broadcast operation that must be performed by the access points.

If there is a single access point, our problem reduces to find a partition of the end-users into K groups, each served by one transmission on a different non-overlapping channel. Precisely, each end-user j has an associated maximum possible receiving rate b_j, depending on the distance of the end-user j from AP. If a group A_i of end-users is reached by means of the same transmission, all the end-users of A_i are served with the same rate that corresponds to the maximum possible rate of the furthest end-user in A_i, i.e., $\min_{j \in A_i}\{b_j\}$. Thus, the overall satisfaction to be maximized is $\sum_{i=1}^{K} |A_i| \min_{j \in A_i}\{b_j\}$.

If there are L access points, one has to decide how many transmissions out of K have to be assigned to each access point in such a way that all the end-users are served and the overall end-user satisfaction is maximized. More formally, let $p_1, p_2, \ldots, p_L$ be the access points and $A^1, A^2, \ldots, A^L$ the sets of end-users in their transmission ranges. Denoted by A_i^h the end-users served by the i-th transmission of the access point p_h, the overall satisfaction to be maximized is $\sum_{h=1}^{L} \sum_{i=1}^{k_h} |A_i^h| \min_{j \in A_i^h}\{b_j\}$, where the number k_h of transmissions assigned to p_h must be at least 1 and $\sum_{h=1}^{L} k_h = K$.

Then, we study the satisfaction problem extended to the multi-interface scenario. In fact, in real settings, the heterogeneous devices at the end-user hands are featured by several different interfaces, like WiFi, 4G, Bluetooth, etc. Hence, each end-user can receive at different rates, depending on the activated interface. In such a scenario, the access point can take advantage of the different interfaces to provide a better service for the end-users.

Multi-interface networks have been extensively studied in the last years, but using a different flavor with respect to the present paper. In fact, previous works considered which interfaces to activate to set up the network topology based on cost criteria [2, 8, 12, 14, 13]. However, little work has been done about broadcasting in a disc model where access points and end-users are equipped with multi-interfaces. Indeed, a related model has been used only in the local rate maximization problem, one of the four stages of the distributed and localized heuristics presented in [18, 19] for computing distributed 2-hop trees for broadcasting in multi-radio multi-rate multi-channel wireless mesh networks.

In our work, each end-user is associated with a subset of K possible interfaces, and with a maximum possible rate for each interface. Specifically, each end-user j holding interface i is associated with a value $b_{i,j}$ representing the maximum possible rate at which end-user j can receive transmissions from the access point when served by interface i. Without loss of generality, one can assume that each end-user is equipped with all the K interfaces because we set $b_{i,j} = 0$ to indicate that end-user j cannot receive transmissions via interface i. The access point can transmit concurrently over all possible interfaces, but each end-user can receive only by means of one interface at a time. The disc model described above still holds when a transmission is performed on a specific interface. The goal is then to decide which end-users are served by a given interface in order to maximize the satisfaction over all the end-users. We show that the problem is computationally intractable (i.e. NP-hard). Although the problem is NP-hard, polynomial time algorithms are devised for optimally solving some special cases. In particular, a reduction to a variant of a *Resource Constrained Shortest Path* (*RCSP*) problem holds when sorting the end-users by decreasing rate with respect to any interface always gives the same end-user order. Although *RCSP* is also NP-hard in general, this variant is polynomially time solvable when the number of used interfaces is polylogarithmic in the number of nodes.

The rest of this paper is structured as follows. We formally introduce two variants of the problem of maximizing the overall end-user satisfaction of data broadcast in wireless mesh networks, called the *Overall Satisfaction of K-Transmissions Multi-Rate Data Broadcast* (*K-MRB*) and *Overall Satisfaction of K-Transmissions Multi-Rate Multi-Interface Single Access Point Data Broadcast* (*K-MRIB*) problems. The former problem is defined in Section 2, where polynomial time algorithms are proposed to optimally solve it. The latter problem is formulated in Section 3, where it is shown that it is computationally intractable in general, while polynomial time algorithms are proposed for meaningful special cases. Finally, conclusions are drawn in Section 4.

2. Overall Satisfaction of K-Transmissions Multi-Rate Data Broadcast in Mesh Networks

Let $G = (V, E)$ be a mesh network, that is a connected graph, whose node set $V = P \cup U$, where P is the set of access points and U is the set of end-users, and whose edge set $E \subseteq (P \times P) \cup (P \times U)$ such that for each end-user $u \in U$ there is a single access point $p \in P$ with $(p, u) \in E$. Let $|P| = L$ and $|U| = N$. Each end-user $u \in U$ is characterized by a real number b_u which indicates the maximum possible rate that can be used to communicate towards u. Let $T = (V, E')$ be a rooted *spanning tree* of G. Note that in T, for each node $v \in V$ there is a single $p \in P$ such that $(p, v) \in E'$. A transmission of an access point p simultaneously serves a subset A_i of end-users which are children of p in T. Such a transmission is assumed to transfer data from p to A_i at a rate equal to $\min_{j \in A_i}\{b_j\}$. Given a number K of transmissions allowed for reaching all the end-users in U, the *Overall Satisfaction of K-Maximum Bandwidth Broadcast in Mesh Networks* (*K-MRB* for short) can be formally stated as follows.

K-MRB: Overall Satisfaction of K-Transmissions Multi-Rate Data Broadcast in Mesh Networks	
Input:	A spanning tree $T = (V, E')$ rooted at node r of a mesh network $G = (V, E)$, a rate function $b : U \to \mathbb{R}_0^+$, and an integer $L \le K \le N$.
Solution:	A partition $A_1, A_2, \ldots, A_K$ of U such that for every two end-users $x, y \in A_i$ there exists an access point p such that $(p, x), (p, y) \in E'$.
Goal:	Maximize $\sum_{i=1}^{K} |A_i| \min_{j \in A_i}\{b_j\}$.

In words, we aim to maximize the overall satisfaction using at most K transmissions. This implies to decide how many transmissions are performed by each access point and how the end-users associated to the access point are then partitioned among the assigned transmissions. Of course, it must hold $K \ge L$ because otherwise the end-users associated with the access points that have no transmissions will not receive the content. Indeed, by definition, each access point has at least one associated end-user.

For instance, Figure 3 depicts a mesh network composed of 4 access points (rectangles) and 9 end-users (black dots), along with a rooted spanning tree (continuous edges) and an end-user partition into $K = 5$ groups (transmissions). It is worth noting that, although different spanning trees might influence the number of hops (and hence the time) to exchange the common content among the access points, they do not influence the overall rate of the end-users. Indeed, by assumption, each end-user is connected to exactly one access point and each access point covers at least one end-user, and hence the received end-user rate is independent on the particular spanning tree.

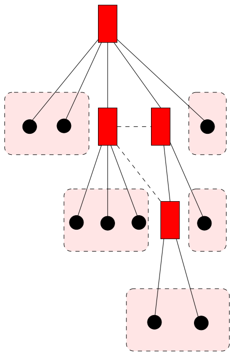

Figure 3: A partition into $K = 5$ groups of the end-users in a mesh network. The access point root is associated with 2 transmissions, while the other access points are associated with a single transmission.

In what follows, we first focus our attention to the subcase of a single access point, where $L = 1$ and $T = G$, namely both are star networks. The general case of arbitrary mesh networks will be considered later.

2.1. Star Networks

The problem on a star (i.e., on a single access point) becomes that of partitioning the children of the unique access point r (i.e. the root of the star) into K suitable subsets. For the ease of notation, let $N = |U|$ be the number of end-users, which coincides with the number of children of r in the input star. Let A_i be the group of end-users served with transmission i of access point r. The satisfaction of each end-user of group A_i is equal to $\min_i = \min_{j\in A_i}\{b_j\}$.

Lemma 1. *Let $U = \{1, 2, \ldots, N\}$ be the set of end-users indexed in such a way that $b_i \geq b_j$ whenever $i < j$. Then, there exists an optimal solution for the K-MRB problem which partitions U into K groups $A_1, \ldots, A_K$, where each group is made of consecutive end-users.*

Proof. Consider an optimal solution σ whose groups are not made of consecutive nodes. Order the partition $A_1, \ldots, A_K$ of σ in such a way that $\min_i \geq \min_j$, for $1 \leq i \leq j \leq K$. Clearly, $\min_K = \min_{1\leq j\leq N}\{b_j\} = b_N$, i.e. $N \in A_K$. Let A_t be the group not made of consecutive end-users having the largest index t. Let ℓ and p be, respectively, the smallest end-user which is missing in the group A_t and the smallest end-user in A_t out of order. Note that the end-user with rate $\min_t$ cannot be out of order, or equivalently it has index $N - \sum_{j=t+1}^{K} |A_j|$, because $A_{t+1}, \ldots, A_K$ consist of consecutive end-users and $\min_1 \geq \cdots \geq \min_K$. Thus, the missing end-user has index $\ell \leq N - \sum_{j=t+1}^{K} |A_j| - 1$. Clearly $b_p \geq b_\ell \geq b_{N-\sum_{j=t+1}^{K}|A_j|} = \min_t$. Denoted with A_{t_ℓ} the group to which ℓ belongs in σ, and recalling that ℓ is the smallest end-user out of order, $\min_{t_\ell} = b_\ell$ holds. Now, exchange end-user ℓ with end-user p. The new solution σ' achieves a transmission rate larger than or equal to that of σ because the minimum rate in group A_t remains the same, while $\min_{t_\ell}$ cannot decrease. Repeating the above process until no end-user out of order can be found, we build an optimal solution whose groups are made of consecutive end-users.

□

Hereafter, thus, it is assumed that all the end-users in U are sorted by non-increasing maximum possible rate and the optimal solutions will be sought within the class of *segmentations*, that is, the partitions that preserve the rate order, e.g. $b_1 \geq b_2 \ldots \geq b_N$.

Based on that, the satisfaction of group $\{i+1, \ldots, j\}$ is $w(i, j) = (j - i)b_j$ and the K-*MRB* problem can be solved via the following recurrence for $1 \leq k \leq K$, $k \leq n \leq N$:

$$opt(k, n) = \begin{cases} w(0, n) & \text{if } k = 1 \\ \max\limits_{k-1\leq\ell\leq n-1}\{opt(k-1, \ell) + w(\ell, n)\} & \text{if } k \geq 2 \end{cases} \tag{1}$$

where $opt(k, n)$ denotes the overall satisfaction of the optimal solution $OPT(k, n)$ for the k-MRB problem applied to the end-users $1, \dots n$. Moreover, $F[k, n]$ of $OPT(k, n)$ is the index of the last item that belongs to group G_{k-1} of $OPT(k, n)$, is:

$$F[k, n] = \arg \max_{k-1 \leq \ell \leq n-1} \{opt(k-1, \ell) + w(\ell, n)\} \tag{2}$$

Note that, in case of multiple indices that lead to the same $opt(k, n)$ value, $F[k, n]$ is set to the minimum index. Clearly, the overall satisfaction of the K-MRB problem applied to N end-users can be found in $opt(K, N)$ and the solution $OPT(K, N)$ can be built backwards from the index $F[K, N]$ in $O(N^2K)$ time.

In the rest of this section we exploit some properties of the satisfaction function w that allow to implement the Recurrence 1 in $O(NK)$ time.

A 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is convex *Monge* [2] if $a + d \geq b + c$. An $m \times n$ matrix $\mathbb{A}$ is Monge if every 2×2 submatrix is Monge. That is, for all $1 \leq i < m$ and $1 \leq j < n$,

$$A[i, j] + A[i+1, j+1] \geq A[i+1, j] + A[i, j+1].$$

A 2×2 matrix is *monotone* if the maximum of the upper row is not to the right of the maximum of the lower row. More formally, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is *monotone* if $b > a$ implies that $d > c$ and $b = a$ implies that $d \geq c$. An $m \times n$ matrix $\mathbb{A}$ is *totally monotone* if every 2×2 submatrix of $\mathbb{A}$ is monotone.

One can prove (see [1, 4, 3]) that every Monge matrix is totally monotone.

Recalling that the end-users are indexed by non-increasing maximum possible rates, it holds:

Lemma 2. *The (upper triangular) matrix $\mathbb{W}$ with $W[i, j] = (j - i)b_j$ for $0 \leq i < j \leq N$, which stores for each possible consecutive single-group its satisfaction, is a Monge matrix. Namely, for $0 \leq \ell < n < N$:*

$$w(\ell, n) + w(\ell+1, n+1) \geq w(\ell, n+1) + w(\ell+1, n).$$

If we fill the lower triangular matrix $\mathbb{W}$ with $-\infty$ values, the matrix $\mathbb{W}$ is totally monotone.

Proof. In fact, for $0 \leq \ell < n \leq N$,

$$w(\ell, n) + w(\ell+1, n+1) \geq w(\ell, n+1) + w(\ell+1, n)$$

or equivalently

$$(n - \ell)b_n + (n - \ell)b_{n+1} \geq (n + 1 - \ell)b_{n+1} + (n - \ell - 1)b_n$$

it holds if

$$b_n \geq b_{n+1}$$

□

Lemma 3. *For any fixed k, with $1 \leq k \leq K$, we prove that the matrix $\mathbb{S}^k$ that stores all the values scanned by Recurrence 1 and that is defined as:*

$$S^k_{\ell,n} = opt(k-1, \ell) + w(\ell, n), \text{ for } 1 \leq \ell \leq n \leq N, \tag{3}$$

is totally monotone.

[2] In this work, when we refer to the Monge property, we always mean the *convex* Monge property.

Input:	N end-users, with the associated maximum possible rates, and K transmissions;
Initialization:	Let $\{1, 2, \ldots, N\}$ be the end-users sorted in non-increasing order according to their maximum possible rates;
	`for` i `from` 1 `to` N `do`
	`for` k `from` 1 `to` K `do`
	`if` $k = 1$ `then` $opt_{k,i} \leftarrow w_{k,i}$ `else` $opt_{k,i} \leftarrow -\infty$;
Loop:	`for` k `from` 2 `to` K `do`
	$opt_{k,k}, \ldots, opt_{k,N} \leftarrow SMAWK(\mathbb{S}^k)$
	where $\mathbb{S}^k$ is defined by Eq. 3

Figure 4: The optimal K-*MRB*-single-access-point algorithm.

Proof. Adding the same value $opt(k-1, \ell) + opt(k-1, \ell+1)$ to both sides of the Monge condition for the single-group satisfaction matrix $\mathbb{W}$, we obtain the Monge condition for matrix $\mathbb{S}^k$:

$$\begin{aligned} &\underbrace{opt(k-1,\ell) + w(\ell,n)}_{S^k_{\ell,n}} + \underbrace{opt(k-1,\ell+1) + w(\ell+1,n+1)}_{S^k_{\ell+1,n+1}} \\ \geq\ &\underbrace{opt(k-1,\ell) + w(\ell,n+1)}_{S^k_{\ell,n+1}} + \underbrace{opt(k-1,\ell+1) + w(\ell+1,n)}_{S^k_{\ell+1,n}} \end{aligned} \quad (4)$$

□

Thus, we can compute the maximum in all rows of $\mathbb{S}^k$ in $O(N)$ time by applying an algorithm proposed by Aggarwal et al. [1], nicknamed in literature SMAWK.

Hence, for any fixed value of $k \geq 2$, to compute $opt(k, n)$ according to Recurrence 1, it is sufficient to apply the SMAWK algorithm to the n-th row of matrix $\mathbb{S}^k$, that is to the row $\mathbb{S}^k(k, n), \mathbb{S}^k(k+1, n), \ldots, \mathbb{S}^k(n, n)$. The K-*MRB* algorithm for a single access point is given in Figure 4 and it yields:

Lemma 4. *Fixed any $k \geq 2$, the K-MRB-single-access-point algorithm computes the values $opt(k, n)$ for $k \leq n \leq N$ in $O(N)$ time by invoking the SMAWK algorithm.*

Proof. To apply the SMAWK algorithm in $O(N)$ time to the totally monotone matrix $\mathbb{S}^k$ it is required that each entry $S^k_{\ell,n}$ can be computed in constant time. This is true because the values $opt(k, n)$ for $k-1 \leq n \leq N$ are calculated after the values $opt(k-1, n)$ for $k-1 \leq n \leq N$ have been computed and because each entry of the single-group satisfaction matrix $\mathbb{W}$ can be computed in constant time. □

In conclusion:

Theorem 1. *The K-MRB problem for a single access point can be solved in $O(N(K + \log N))$ time by applying $K-1$ times the SMAWK algorithm [1] to the N end-users sorted by non-increasing rates. The complexity of the K-MRB-single-access-point algorithm is optimal since it solves NK subproblems.*

2.2. General Mesh Networks

When the WMN consists of L access points, with $L > 1$, and the end-users are distributed among them, to solve the K-*MRB* problem one has to find the numbers of transmissions $k_1, k_2, \ldots, k_L$ which maximize the objective function $\sum_{h=1}^{L} \sum_{i=1}^{k_h} |A^h_i| \min_{j \in A^h_i}\{b_j\}$, where $p_1, p_2, \ldots, p_L$ are the access points and A^h_i are the end-users served by the i-th transmission of the access point p_h.

To solve the K-*MRB* problem with multiple access points, we propose the K-*MRB*-Greedy algorithm that starts assigning one transmission to each access point, namely it sets $k_1 = k_2 = \ldots = k_L = 1$, because all the end-users must be served. Then, the algorithm works in $K - L$ successive steps where, at each step, it finds where to add one more transmission in order to obtain the maximum gain in the overall satisfaction.

To study how the overall satisfaction changes when the number of transmissions increases (see Lemma 6), we first prove that the values $opt[k, n]$, for $1 \leq k \leq K$ and $1 \leq n \leq N$, form a totally monotone matrix.

Lemma 5. *Matrix $opt[k, n]$, for $1 \leq k \leq K$ and $1 \leq n \leq N$, is a Monge matrix.*

Proof. Let $k < k' \leq n' < n$. We need to prove that $opt(k,n) + opt(k',n') \geq opt(k,n') + opt(k',n)$, or equivalently $opt(k,n') - opt(k',n') \leq opt(k,n) - opt(k',n)$.

By induction on $k'-k$, $n'-n$, it is sufficient to prove for $k' = k+1$ and $n' = n+1$. Let $x = F[k, n+1]$, $y = F[k+1, n]$ be the last end-user of groups A_{k-1} and A_k of the optimal solutions $OPT(k, n+1)$ and $OPT(k+1, n)$, respectively. We proceed by induction on k. Let us consider two cases.

Case 1. $x = F[k, n+1] \geq y = F[k+1, n]$.

$$opt[k, n+1] - opt[k+1, n+1] = opt(k-1, x) + w(x, n+1) - opt(k+1, n+1)$$

substituting the optimal solution $opt(k+1, n+1)$ with a feasible solution, one has:

$$\leq opt(k-1, x) + w(x, n+1) - \underbrace{opt(k, x) - w(x, n+1)}_{-S^{k+1}_{x,n+1}} = opt(k-1, x) - opt(k, x)$$

and by inductive hypothesis,

$$\leq opt(k-1, y) - opt(k, y) = \underbrace{opt(k-1, y) + w(y, n)}_{S^k_{y,n}} - \underbrace{opt(k, y) - w(y, n)}_{-opt(k+1,n)} \leq opt(k, n) - opt(k+1, n).$$

Case 2. $x = F[k, n+1] < y = F[k+1, n]$.

$$\begin{aligned} opt[k, n+1] - opt[k+1, n+1] &= opt(k-1, x) + w(x, n+1) - opt(k+1, n+1) \\ &\leq opt(k-1, x) + w(x, n+1) - \underbrace{opt(k, y) - w(y, n+1)}_{-S^{k+1}_{y,n+1}} \end{aligned}$$

by the Monge condition on the single-group satisfaction w,

$$\begin{aligned} \leq opt(k-1, x) + w(x, n) - opt(k, y) - w(y, n) = & \underbrace{opt(k-1, x) + w(x, n)}_{S^k_{x,n}} - \underbrace{opt(k, y) - w(y, n)}_{-opt(k+1,n)} \\ & \leq opt(k, n) - opt(k+1, n). \end{aligned}$$

□

The next lemma proves that the overall satisfaction for each single access point increases as the number of allowed transmissions increases, but the gain $\delta(k, n) = opt_{k,n} - opt_{k-1,n}$ does not increase when the number of transmissions k increases.

Lemma 6. *Let* $\delta(k+2, n) = opt_{k+2,n} - opt_{k+1,n}$ *and* $\delta(k+1, n) = opt_{k+1,n} - opt_{k,n}$. *Then,* $\delta(k+2, n) \leq \delta(k+1, n)$.

Proof. Let $opt_{n,k+2} = opt_{n_3,k+1} + (n - n_3)b_n$ and $opt_{n,k+1} = optn_2, k + (n - n_2)b_n$. By the optimality of $opt(k + n1,)$, it holds $opt_{k+1,n} \geq opt_{k,n_3} + (n - n_3)b_n$. Then, $\delta(k+2, n) = opt_{k+2.n} - opt_{k+1,n} = opt_{k+1,n_3} + (n - n_3)b_n - opt_{k+1,n} \leq opt_{k+1,n_3} + (n - n_3)b_n - opt_{k,n_3} - (n - n_3)b_n = opt_{k+1,n_3} - opt_{k,n_3} = \delta(k+1, n_3)$. Since $n_3 \leq n$, by Lemma 5, $\delta(k+1, n_3) = opt(k+1, n_3) - opt(k, n_3) < opt(k+1, n) - opt(k, n) = \delta(k+1, n)$, and hence

$$\delta(k+2, n) \leq \delta(k+1, n_3) \leq \delta(k+1, n).$$

□

Input:	A mesh network with L access points and $N = \sum_{h=1}^{L} N_h$ end-users, a spanning tree T for it, and K transmissions, with $2 \leq L \leq K \leq N$

Initialization:	**for** h **from** 1 **to** L **do**
	    $k_h \leftarrow 1$;
	    compute $opt_{N_h,1}$ by means of the K-*MRB*-single-access-point algorithm;
	    compute $opt_{N_h,2}$ by means of the SMAWK algorithm;
	    $\delta(N_h, 2) \leftarrow opt_{N_h,2} - opt_{N_h,1}$;
	$K \leftarrow K - L$;

Generic step:	**while** $K > 0$ **do**
	    find j such that $\delta(N_j, k_j + 1) = \max_{1 \leq h \leq L}\{\delta(N_h, k_h + 1)\}$;
	    $k_j \leftarrow k_j + 1$;
	    $K \leftarrow K - 1$;
	    compute opt_{N_j,k_j+1} by means of the SMAWK algorithm;
	    $\delta(N_j, k_j + 1) \leftarrow opt_{N_j,k_j+1} - opt_{N_j,k_j}$;

Figure 5: The K-*MRB*-Greedy algorithm for mesh networks.

Let $\Delta_h = \delta(2, N_h), \delta(3, N_h), \ldots, \delta(K_L, N_h)$ be the list of increments achievable increasing the number of transmissions of the single access point p_h to serve all its N_h end-users. By Lemma 6, the list Δ_h is sorted in non-increasing order. Then, to find the solution the K-*MRB*-Greedy algorithm has simply to select the first $K - L$ entries of the sorted list $\Delta = \cup_{h=1}^{L} \Delta_h$. Note that the K-*MRB*-Greedy algorithm does not need to precompute the entire lists Δ_h, $1 \leq h \leq L$, but it is sufficient to compute one increment of only one list at a time. Indeed, once an increment $\delta(k_j + 1, N_j)$ is selected from list Δ_j, the $(k_j + 1)$-th transmission is assigned to access point p_j, and hence one only needs to compute the next increment $\delta(k_j + 2, N_j)$. The K-*MRB*-Greedy algorithm is given in Figure 5.

The next lemma proves that the K-*MRB*-Greedy algorithm finds the optimal solution for the K-*MRB* problem on arbitrary graphs.

Lemma 7. *The K-MRB-Greedy algorithm is optimal.*

Proof. In order to prove the claim, first observe that if in an optimal solution k_h transmissions are assigned to an access point p_h, then the partition of the N_h end-users for the single access point p_h must be optimal.

Let S be an optimal solution for the general mesh network whose satisfaction $\sum_{h=1}^{L} sol_{N_h,k_h}$ is given by the sum of the satisfactions of L independent feasible solutions for star networks. By contradiction, assume that there is an access point p_s whose N_s end-users have been suboptimally partitioned, thus leading to a satisfaction $sol_{N_s,k_s} < opt_{N_s,k_s}$, while $sol_{N_h,k_h} = opt_{N_h,k_h}$ for $h \neq s$. Then, the overall satisfaction of S is:

$$\sum_{h=1}^{L} sol_{N_h,k_h} = \sum_{1 \leq h \neq s \leq L} opt_{N_h,k_h} + sol_{N_s,k_s} \leq \sum_{h=1}^{L} opt_{N_h,k_h}$$

thus contradicting the optimality of S.

Hence, an optimal solution for the all mesh network consists of local optimal solutions for each of its star networks. Now it is shown that there is an optimal solution that follows the greedy choice, i.e., at each step it adds one transmission to the access point that gives the maximum satisfaction increment. Let us prove the existence of such a greedy solution by induction on the overall number of transmissions. The base of the induction $K = L$ is verified since the unique optimal solution has $k_1 = k_2 = \ldots = k_L = 1$. Assume that the induction is true for a generic number of transmissions $K - 1$. Suppose there is an optimal solution which adds the K-th transmission to access point p_i, while the greedy choice would add such a transmission to access point p_j. Since the K-*MRB*-Greedy algorithm selects

$$\delta(k_j + 1, N_j) = \max_{1 \leq h \leq L}\{\delta(k_h + 1, N_h)\} \geq \delta(k_i + 1, N_i)$$

the greedy choice is better than or at least as good as the optimal one, thus proving the optimality of the K-*MRB*-Greedy algorithm. □

Theorem 2. *The K-MRB problem with multiple access points can be solved by the K-MRB-Greedy algorithm in $O(N(\log N + K))$ time.*

Proof. First of all, $2L$ problems for star networks have to be computed. For each access point p_h, the above problem is solved with 1 or 2 transmissions and N_h end-users. Thus the initialization costs $O(N \log N)$ time. At each generic step, the algorithm chooses the maximum increment, which can be done in $O(\log L)$ time provided that the increments are stored into a max-heap. Then, for any j, to compute $\delta(N_j, k_j + 1)$, it is sufficient to invoke the SMAWK algorithm on the selected star network rooted at p_j. This takes $O(N_j)$ time. Since the generic step is repeated $K - L$ times, overall the K-*MRB*-Greedy algorithm takes $O(N \log N + K \log L + (K - L)N)$ time, which is $O(N(\log N + K))$ since $L \leq K \leq N$. □

3. Overall Satisfaction of *K*-Transmissions Multi-Rate Multi-Interface Data Broadcast

In this section, we extend the K-*MRB* problem to the case of multi-interface networks. Each end-user is associated with a subset of K possible interfaces, each providing a different maximum possible rate. The maximum possible rate reflects the distance of the end-user from the associated AP and the interface used to communicate with the AP.

Let G be a mesh of N end-users and L access points, and I be a set of H interfaces. Each end-user j holding interface i is associated with a value $b_{i,j}$ representing the maximum possible rate at which end-user j can receive transmissions via interface i. Note that, we set $b_{i,j} = 0$ if end-user j cannot receive transmissions through interface i. In this way, we can assume that each end-user holds all the interfaces, and we set to null the maximum possible rate of the missing interfaces. An access point can transmit over all possible interfaces to its end-users, but each end-user can receive only from one interface. As before, a transmission of an access point that simultaneously serves a subset A_i of end-users by means of interface i is assumed to transfer data at a rate equal to $\min_{j \in A_i}\{b_{i,j}\}$. Thus, $\min_{j \in A_i}\{b_{i,j}\}$ is the end-user satisfaction.

The goal is then to decide which end-users receive on a given interface in order to maximize the transmission rate over all the mesh network. As we are going to show, this new problem is quite difficult to solve even in the restricted case of star networks. Hence, we provide its formalization directly when there is a single access point. The *Overall Satisfaction of K-Maximum Bandwidth Broadcast with Multi-Interfaces and Single Access Point* (K-*MRIB* for short) can be stated as follows.

K-*MRIB* : Overall Satisfaction of K-Transmissions Multi-Rate Multi-Interface Single Access Point Data Broadcast

***Input*:**	A set U of N end-users and a set I of H interfaces. A rate function $b : U \times I \rightarrow \mathbb{R}_0^+$ and an integer $1 \leq K \leq \min\{H, N\}$.
***Solution*:**	A subset $I' \subseteq I$ of $\lvert I' \rvert = K$ interfaces and a partition $A_1, A_2, \ldots, A_K$ of U which associates each subset of end-users to one different interface in I'.
***Goal*:**	Maximize $\sum_{i \in I'} \lvert A_i \rvert \min_{j \in A_i}\{b_{i,j}\}$.

In words, we want to determine which subset of interfaces and, for each interface, which subset of end-users are reached by the same transmission in order to maximize the overall satisfaction during the broadcast.

The K-*MRIB* problem can be also visualized by considering a matrix D with H rows, one for each interface, and N columns, one for each end-user, where each entry $D[i, j] = b_{i,j}$ is the maximum possible rate for end-user i via interface j. Then, the solution selects the subset I' of K rows, and for each selected row $i \in I'$, a selection of some columns A_i in such a way that each column is associated with one and only one row.

In the multi-interface case, we introduce the generalized *multi-interface segmentation* in order to characterize optimal solutions for the K-*MRIB* problem.

Let $I' = \{r^1, r^2, \ldots, r^K\}$ be a subset of K rows and let $A_{r^1}, \ldots, A_{r^K}$ be a partition of the N columns of D, where A_{r^i} denotes the subset of columns assigned to interface r^i, with $1 \leq i \leq K$. Moreover, let the K rows r^1, r^2, $\ldots$, r^K be indexed according to the induced minimum values of the rates in the corresponding A_{r^i} columns obtaining that $r^1_{min} \geq r^2_{min} \geq \ldots \geq r^K_{min}$. In a *multi-interface segmentation*, the columns of group A_{r^i} of D served by interface r^i form a consecutive subset of columns in row r^i once the columns in $A_{r^1}, \ldots, A_{r^{i-1}}$ have been canceled from D and the remaining columns $S - (A_{r^1} \cup A_{r^2} \cdots \cup A_{r^{i-1}})$ in r^i have been sorted in non-increasing rate order. In other words, in a multi-interface segmentation, if columns x and y belong to the group of columns assigned to interface r^i and $D[r^i, x] < D[r^i, y]$, then all the columns z in row r^i not yet assigned to interfaces $r^1, \ldots, r^{i-1}$ with rate $D[r^i, x] \leq D[r^i, z] \leq D[r^i, y]$ are also associated to the same interface r^i.

Lemma 8. *The optimal solution for K-MRIB is a multi-interface segmentation.*

Proof. Given an optimal solution using K interfaces, we can order the K out of H chosen rows r^1, r^2, …, r^K according to the induced minimum values of the rates in the corresponding columns, obtaining that $r^1_{min} \geq r^2_{min} \geq \ldots \geq r^K_{min}$. Sorted row r^1 in non-increasing order and indexed the columns of D according to such an order, let us assume by contradiction that A_{r^1} consists of several distinct consecutive intervals of columns in D. All the intervals of columns not associated to r^1 before the last interval that contains $D[r^1, n_1] = r^1_{min}$ can be assigned to r^1 without affecting the solution because each column in such intervals contributes to the optimal solution with a value no greater than r^1_{min}.

After filling all the first intervals on r^1, we have $A_{r^1} = [1..n_1]$. Then, we can consider row r^2 excluding the columns of D already assigned to r^1. Sorted row r^2 in non-increasing order and indexed the columns $S - A_{r^1}$ of D according to such an order, let us assume by contradiction that A_{r^2} consists of distinct intervals of columns of D. As before, each column z not associated to r^2 on the left of column n_2 such that $D[r^2, n_2] = r^2_{min}$ can be moved in A_{r^2} without decreasing the satisfaction of the solution because they contribute at most r^2_{min} to the optimal solution.

Repeating the same until the last row r^K is considered or until all the N columns are assigned, we obtain a multi-interface segmentation that provides at least the same rate of the original optimal solution. Hence there is an optimal solution which is a multi-interface segmentation. □

Although the optimal solutions of *K-MRIB* still satisfy a kind of segmentation, the major difficulty consists in choosing the K rows along with their permutation that lead to the optimal solution. Thus, introducing multi-interfaces makes the problem much harder.

3.1. Computational Intractability

In this section we study the complexity of *K-MRIB* and we prove that the problem is computationally intractable.

Theorem 3. *K-MRIB is NP-hard.*

Proof. We prove that the underlying decisional problem, denoted by K-$MRIB_D$, is in general *NP*-complete. We need to add one bound $B \in \mathbb{R}^+_0$ such that the problem will be to ask whether there exists a partition of U composed of K subsets which induces an overall satisfaction of at least B.

The problem is in *NP*. In fact, given a partition for an instance of K-$MRIB_D$, checking whether it ensures an overall satisfaction of at least B requires linear time in the size of the instance.

The proof then proceeds by means of a polynomial reduction from the well-known *Exact Cover by* 3*-Sets* problem. Such a problem is known to be *NP*-complete [9] and it can be stated as follows:

X3C: Exact Cover by 3-Sets	
Input:	Set X with $\vert X\vert = 3q$ and a collection C of 3-element subsets of X.
Question:	Is there an exact set cover for X, i.e. a subset $C' \subseteq C$ such that $\vert C'\vert = q$ and every element of X belongs to exactly one member of C'?

Given an instance of *X3C*, we build an instance of K-$MRIB_D$ in polynomial time as follows. Let $K = q$, the set U be composed of $N = |X| = 3q$ end-users, and $|I| = H = |C|$. Hence, we define two mappings. One is between X and U, the other is between subsets C and interfaces I. For each subset $c \in C$, if an element x corresponding to end-user j belongs to c, which corresponds to an interface i, then $b_{i,j} = 1$, otherwise $b_{i,j} = 0$. It follows that each interface can be used to reach at most 3 end-users while it guarantees one unit of rate for each end-user. Finally, let $B = 3q$. We need to prove that there is a solution of *X3C* if and only if there exists a solution of the corresponding instance of K-$MRIB_D$.

(⇒): Let us suppose that *X3C* admits a solution. The covering must consist of q triples. From the K-$MRIB_D$ perspective, the q triples correspond to K subsets A_1, A_2, ..., A_K chosen to transmit data to the $3q$ end-users. Each subset corresponds to one different interface. By construction, for each $j \in U$ there is a unique subset A_i in the induced solution of K-$MRIB_D$ such that $b_{i,j} = 1$, hence A_1, A_2, ..., A_K represent a partition of U. Summing up over all the rates allowed by the corresponding interfaces, we obtain: $\sum_{i=1}^{K} |A_i| \min_{j \in A_i}\{b_{i,j}\} = \sum_{i=1}^{K} 3 = 3K = 3q = B$.

(⇐): Let us suppose that K-$MRIB_D$ admits a solution. By construction, each of the K chosen interfaces can be used to transmit one unit of rate to at most 3 end-users. Since we have $K = q$ and $B = 3q$, each interface must necessarily be used to transmit one unit of rate to 3 end-users. Hence, the partition of U provided by the assumed solution corresponds to a set of K triples in *X3C* that covers all the elements in X, that is *X3C* admits a solution. □

3.2. Polynomially solvable subcases

In this section, some special cases are considered where the K-*MRIB* problem can be efficiently solved. The following theorem can be stated.

Theorem 4. *If $K \leq 2$, then K-MRIB is polynomially solvable.*

Proof. If $K = 1$, it is easy to check which row of D admits the highest minimum among all the columns, and this clearly determines the selection of the best row. Overall, $O(NH)$ time is required.

If $K = 2$, we may consider every pair of rows. For each pair, let us sort all the columns according to the non-increasing order of the first chosen row. Once chosen the right couple of rows and the right order (either with respect to the first chosen row or to the second one), we only need to find the best index B_1 that represents the last end-user of the first group among N possibilities since the solution is a partition (see Lemma 8).

The overall complexity of the above described algorithm is $O(NH(\log N + H))$. Indeed, one can choose one of the H rows at a time, order the columns in $O(N \log N)$ time according to the just chosen row, consider all the $H - 1$ pairs of rows consisting of the just chosen row and every other row, and find in $O(N)$ time the best index B_1, for a total of $O(H(N \log N + NH))$ time. □

Consider now the particular case when there is a way of indexing the columns of D that simultaneously sorts all the rows of D. In other words, arranged the columns of D in such a way that a given row is sorted in non-increasing order, all the rows of D are also sorted in non-increasing order of their rates. When this property holds we say that the instance respects a *common* order. From now on, let K-*CMRIB* denote the K-*MRIB* problem when the common order holds. In K-*CMRIB*, we assume the columns of D indexed so that all the rows are sorted. In practice, the K-*CMRIB* problem might arise when the rate achievable by an end-user with any interface depends, in the same way, on the distance of the end-user from the access point.

When the common order holds, a multi-interface segmentation becomes a partition $A_1, \ldots, A_K$ of the columns of D where each group A_i belongs to a different row (i.e., it is assigned to a different interface) and consists of consecutive columns. In such a case a multi-interface segmentation can be denoted by the K-tuple

$$\langle (B_1; i_1), (B_2; i_2), \ldots, (B_{K-1}; i_{K-1}), (N; i_K) \rangle$$

where B_j is the index of the last column that belongs to group A_j assigned to interface i_j.

To design an optimal enumeration algorithm for K-*CMRIB*, one should consider all the possible multi-interface segmentations of the columns of matrix D and, for each subset of K rows, find the best solution using at most such rows. An improved enumeration algorithm can be achieved exploiting a reduction to a *Resource Constrained Shortest Paths* (briefly, *RCSP*) problem on directed multigraphs, which is defined as follows [11].

RCSP: Resource Constrained Shortest Paths	
Input:	A directed multigraph $G = (V, E)$ with two special vertices s and t, an integer K (which is the number of resources) and a (positive integer) resource availability vector $(R_1, R_2, ..., R_K)$. Each edge $e \in E$ has a positive weight $w(e)$ and a (positive integer) resource request vector $(r_1(e), r_2(e), ..., r_K(e))$.
Solution:	A feasible path p from s to t such that $\sum_{e \in p} r_i(e) \leq R_i$ for $1 \leq i \leq K$.
Goal:	Minimize $\sum_{e \in p} w(e)$ over all the feasible paths p.

To reduce *K-CMRIB* to *RCSP* when the set I consists of K interfaces, define a vertex i for column i of D, with $1 \leq i \leq N$, and add the vertex $s = 0$. Let vertex t be equal to N and the resource availability vector be such that $R_i = 1$, for $1 \leq i \leq K$. For every pair of vertices i, j such that $0 \leq i < j \leq N$ and every $1 \leq k \leq K$ add the edge $e = (i, j)$ with $w(e) = -(j-i)b_j$ and resource request vector $(r_1(e), r_2(e), ..., r_K(e))$ with all entries equal to zero except for $r_k = 1$. Clearly, $|V| = O(N)$ and $|E| = O(KN^2)$.

Note that *RCSP* requires positive edge weights, while the weights introduced in the reduction are negative because the *K-CMRIB* problem is a maximization problem. However, since the so constructed multigraph is acyclic, the *RCSP* problem is well defined and solvable even if the edge weights are negative.

A feasible shortest path from s to t represents an optimal solution for the *K-CMRIB* problem when $H = K$. Note that if an optimum path p has $\sum_{e \in p} r_i(e) = 0$ for some $1 \leq i \leq K$, it means that in the optimum solution interface i is not used.

Theorem 5. *If all the H rows of matrix D respect a common non-increasing order, K-CMRIB can be solved in polynomial time when*

1. $K = O(1)$*, or*
2. $H - K = O(1)$ *and* $K = O(\log^\alpha N)$*, where* $\alpha = O(1)$.

Proof. A modified version of Dijkstra's algorithm can be used to solve *RCSP*. Precisely, for each vertex v one has to save into a priority queue not only the weight of the path to reach v from s but also the set of resources used along such a path. Moreover, leaving vertex v, an edge can be used only if its resource has not already been used in the path from s to v. Since each vertex v can be reached with at most $O(2^K)$ different paths, the size of the priority queue can grow as much as $O(2^K N)$. Hence, the edges that have to be relaxed during the entire algorithm are $O(2^K KN^2)$. So, implementing the priority queue with a Fibonacci heap, *K-MRIB* requires $O(2^K KN^2 + 2^K N \log(2^K N)) = O(2^K KN^2)$ time.

In the general case, when *K-CMRIB* has $K \leq H \leq N$, *K-CMRIB* can be solved applying *RCSP* $\binom{H}{K}$ times, once for each subset of K interfaces out of the H available interfaces, for an overall time of $O(\binom{H}{K} 2^K KN^2)$. It is well known [10] that

$$\binom{H}{K} \approx \begin{cases} \frac{H^K}{K!} & \text{if } K \text{ is a constant} \\ \frac{H^{H-K}}{(H-K)!} & \text{if } H-K \text{ is a constant} \end{cases}$$

In the former case, the time complexity $O(\binom{H}{K} 2^K KN^2)$ becomes $O(H^K N^2)$, which is polynomial because $K = O(1)$ and $H \leq N$. In the latter case, if $K = O(\log^\alpha N)$ then $H = O(\log^\alpha N)$ too since $H - K$ is a constant, and thus the complexity is $O(H^{H-K} 2^K KN^2) = O((\log N)^{\alpha(H-K+1)} N^{\alpha+2})$, which is polynomial because $\alpha = O(1)$. □

As a further particular case, consider the *K-CMRIB* problem where each interface can be reused in more than one transmission. Precisely, we allow that two or more groups of the multi-interface segmentation can be associated with the same row.

Generalizing the definitions given for the single interface case, given $n \leq N$ and $k \leq K$, let $OPT_{n,k}$ denote an optimal solution for grouping end-users $1, \ldots, n$ into k groups and let $opt_{n,k}$ be its corresponding satisfaction. Let $C_{i,h;m}$ be the satisfaction of assigning consecutive end-users $i, \ldots, h$ to one group using interface m, i.e. $C_{i,h;m} = (h - i + 1)b_{m,h}$. Hence, $opt_{n,1} = \max_{m \in \{1,2,\ldots,H\}} C_{1,n;m} = n \max_{m \in \{1,2,\ldots,H\}} b_{m,n}$ for every n. For $1 < k \leq K$, the following recurrence holds:

$$opt_{n,k} = \max_{\ell \in \{1,2,\ldots,n-1\} m \in \{1,2,\ldots,H\}} \{opt_{\ell,k-1} + C_{\ell+1,n;m}\} \tag{5}$$

By the recurrence in Equation 5, a dynamic programming algorithm can be readily derived to solve this problem variant in $O(N^2HK)$ time.

4. Conclusion

We have dealt with the problem of broadcasting a common content into a wireless mesh network consisting of access points that act as gateways to Internet and of N end-users. The goal is to maximize the overall end-user satisfaction, that is defined as the sum of over all end-users of their satisfaction. The end-user satisfaction is given by the rate it receives during the data broadcast operation.

We introduced the problems of maximizing the overall satisfaction when a single interface (K-*MRB*) or multiple interfaces (K-*MRIB*) are available. The former problem, which performs the broadcast using exactly K transmissions at different rates, can be optimally solved in $O(N(K + \log N))$ time by exploiting the Monge property of the satisfaction function. The latter problem, which performs the broadcast using K transmissions, each with a different interface, is computationally intractable (i.e. NP-hard) even when restricted to star networks, that is when there is a single access point in the wireless mesh network. However, K-*MRIB* becomes polynomially solvable still for star networks in some further particular cases. Namely, when $K \leq 2$, or a *common* order in the rates holds and K is polylogarithmic in N, we provided polynomial algorithms.

As future works, it would be of interest to device good heuristics for the K-*MRIB* problem extended to multiple access points.

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