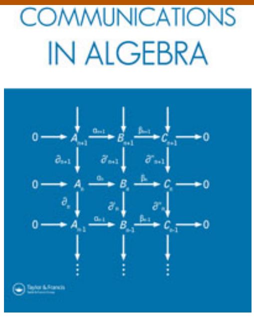

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A note on divisorial correspondences of extensions of abelian schemes by tori

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Dedicated to M. Raynaud.

ABSTRACT

Let S be a locally noetherian scheme and consider two extensions G_1 and G_2 of abelian S-schemes by S-tori. In this note we prove that the *fppf*-sheaf $\mathbf{Corr}_S(G_1, G_2)$ of divisorial correspondences between G_1 and G_2 is representable. Moreover, using divisorial correspondences, we show that line bundles on an extension G of an abelian scheme by a torus define group homomorphisms between G and $\mathbf{Pic}_{G/S}$.

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1. Introduction

In algebraic geometry, the notion of correspondences between varieties plays an important role for the study of algebraic cycles and motives. In this short note we discuss a special case: divisorial correspondences between group schemes which are extensions of an abelian scheme by a torus over a fixed locally Noetherian base S. Let G be such an extension of an abelian S-scheme by an S-torus. Denote by $p : G \to S$ its structural morphism. The **relative Picard functor** associated to G/S is the *fppf*-sheaf $\mathbf{Pic}_{G/S} = \mathrm{R}^1 p_* \mathbb{G}_m$, i.e. the *fppf*-sheaf associated to the presheaf $T/S \mapsto \mathrm{Pic}(G_T)$, where $\mathrm{Pic}(G_T)$ is the group of isomorphism classes of invertible sheaves on the T-scheme $G_T = G \times_S T$ obtained from G by the base change $T \to S$.

The S-group scheme G admits a unit section $\epsilon : S \to G$. From now on, we will assume that the structural morphism $p : G \to S$ satisfies $p_* \mathcal{O}_G = \mathcal{O}_S$ universally. With these hypotheses the *fppf*-sheaf $\mathbf{Pic}_{G/S}$ is canonically isomorphic to the étale-sheaf $\mathrm{R}^1 p_* \mathbb{G}_m$ and moreover it is canonically isomorphic to the sheaf $\mathbf{Pic}^{\epsilon}_{G/S} : T/S \mapsto \mathrm{Pic}(G_T)/\mathrm{Pic}(T)$, where $\mathrm{Pic}(G_T)/\mathrm{Pic}(T)$ is the group of isomorphism classes of invertible sheaves $\mathcal{L}$ on G_T which are rigidified along the unit section $\epsilon_T : T \to G_T$ obtained from $\epsilon : S \to G$ by the base change $T \to S$, that is it exists an isomorphism between the structural sheaf $\mathcal{O}_T$ and $\epsilon_T^* \mathcal{L}$. We call this isomorphism $\mathcal{O}_T \cong \epsilon_T^* \mathcal{L}$ a rigidification of $\mathcal{L}$ along ϵ_T.

Now consider two extensions $p_1 : G_1 \to S, p_2 : G_2 \to S$ of abelian schemes by tori. Denote by $\epsilon_i : S \to G_i$ their unit sections and suppose that p_i satisfy $p_{i_*} \mathcal{O}_{G_i} = \mathcal{O}_S$ universally for $i = 1, 2$. Consider the canonical morphism of sheaves defined by pull-backs

$$\begin{array}{cccc} \mathrm{can}: & \mathbf{Pic}_{G_1/S} \times \mathbf{Pic}_{G_2/S} & \to & \mathbf{Pic}_{G_1 \times_S G_2/S} \\ & (\mathcal{L}_1, \mathcal{L}_2) & \mapsto & pr_1^* \mathcal{L}_1 \otimes pr_2^* \mathcal{L}_2 \end{array} \quad (0.1)$$

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where $\mathrm{pr}_i : G_1 \times_S G_2 \to G_1$ are the projections to the i-th factor for $i=1, 2$. The **sheaf of divisorial correspondences between G_1 and G_2 over S**, that we denote by

$$\mathbf{Corr}_S(G_1, G_2),$$

is the *fppf*-sheaf cokernel of can (0.1). We have an exact sequence of *fppf*-sheaves

$$\mathbf{Pic}_{G_1/S} \times \mathbf{Pic}_{G_2/S} \xrightarrow{\text{can}} \mathbf{Pic}_{G_1 \times_S G_2/S} \to \mathbf{Corr}_S(G_1, G_2) \to 0.$$

Since the extensions G_i are endowed with the unit sections ϵ_i and since we have supposed $p_{i_*}\mathcal{O}_{G_i} = \mathcal{O}_S$ universally, using the rigidified version $\mathbf{Pic}^{\epsilon_i}_{G_i/S}$ of the relative Picard functor we get that for any S-scheme T the sequence

$$0 \to \mathbf{Pic}_{G_1/S}(T) \times \mathbf{Pic}_{G_2/S}(T) \xrightarrow{\text{can}} \mathbf{Pic}_{G_1 \times_S G_2/S}(T) \to \mathbf{Corr}_S(G_1, G_2)(T) \to 0, \tag{0.2}$$

is exact, that is $\mathbf{Corr}_S(G_1, G_2)(T)$ is the group of isomorphism classes of the invertible sheaves on $G_{1T} \times_T G_{2T}$ endowed with rigidifications along $\epsilon_1 \times_T G_{2T}$ and along $G_{1T} \times_T \epsilon_{2T}$ which must agree on $\epsilon_{1T} \times_T \epsilon_{2T}$.

The aim of this note is to prove that the *fppf*-sheaf $\mathbf{Corr}_S(G_1, G_2)$ of divisorial correspondences between G_1 and G_2 is representable (Theorem 1.1). Moreover, using divisorial correspondences, we show that line bundles on an extension G of an abelian scheme by a torus define group homomorphisms between G and $\mathbf{Pic}_{G/S}$ (Proposition 2.1). In [1, Thm 0.1, Thm 5.1] S. Brochard and the first author construct the morphism defined in (2.3) for 1-motives without using divisorial correspondences and they prove the Theorem of the Cube for 1-motives. In [2, Thm 5.9.] the authors prove the generalized Theorem of the Cube for 1-motives.

This paper takes the origin from an exchange of emails with M. Raynaud. We want to thank M. Brion for his comments about the hypothesis "$p_*\mathcal{O}_G = \mathcal{O}_S$ universally," we use in this paper.

2. Representability of Corr

In [5, Thm 1] Murre gives a criterion for a contravariant functor from the category of schemes over S to the category of sets to be representable by an unramified, separated S-scheme which is locally of finite type over S. Using this criterion, he proves the representability of the *fppf*-sheaf $\mathbf{Corr}_S(X_1, X_2)$ with X_1 and X_2 proper and flat S-schemes (see [5, Thm 4]). We adapt his results to extensions of abelian schemes by tori which are not proper.

Theorem 1.1. *Consider two extensions $p_1 : G_1 \to S, p_2 : G_2 \to S$ of abelian schemes by tori. Suppose that the structural morphisms p_i satisfy $p_{i_*}\mathcal{O}_{G_i} = \mathcal{O}_S$ universally for $i=1, 2$. The fppf-sheaf $\mathbf{Corr}_S(G_1, G_2)$ of divisorial correspondences between G_1 and G_2 is representable by an S-group scheme, locally of finite presentation, separated and unramified over S.*

Proof. We have to prove that the functor $\mathbf{Corr}_S(G_1, G_2)$ verifies the properties $(F_1), \ldots, (F_8)$ listed in [5, Thm 1]. Since the structural morphisms p_1, p_2 have sections, $(F_1), (F_2), (F_4)$ follow from the same properties of $\mathbf{Pic}_{G_1 \times_S G_2/S}$. Concerning property $(F3)$, by [6, Prop II 2.4 (2) (i)] the extension G_i (for $i=1, 2$) is S-pure and therefore [7, Chp 37, Lem 27.6 (2), Def 21.1] implies that there exists a universal flattening of G_i, that is the flattening functor is representable. Now using [5, Thm 2], (F_3) follows from the same property of $\mathbf{Pic}_{G_1 \times_S G_2/S}$. Property (F_5) (i.e. the fact that $\mathbf{Corr}_S(G_1, G_2)$ is formally unramified) follows by [6, Prop III 4.1]. Property (F_6) (i.e. the fact that $\mathbf{Corr}_S(G_1, G_2)$ is separated) follows by [6, Prop III 4.3]. For Properties $(F_7), (F_8)$ see Murre's proof in [5, Thm 4]. □

The assumption

$$p_*\mathcal{O}_G = \mathcal{O}_S \qquad \text{universally} \tag{1.1}$$

is not too restrictive. For example, anti-affine algebraic groups over a field k, which is not an algebraic extension of a finite field, furnish *non-trivial* examples where the condition (1.1) holds. More precisely, we have the following geometrical interpretation of the condition (1.1):

Lemma 1.2. *Let k be a field and let $\bar{k}$ its algebraic closure. Consider an extension G of an abelian variety A by a torus T defined over a field k. Let $p : G \to S = \mathrm{Spec}\ (k)$ be the structural morphism of G. Denote by $c : X^*(T)(\bar{k}) \to A^*(\bar{k})$ the $\mathrm{Gal}(\bar{k}/k)$-equivariant homomorphism which defines the extension G, where A^* is the dual abelian variety of A and $X^*(T)$ is the character group of the torus T. Then the following conditions are equivalent:*

(1) *the structural sheaf of the extension G satisfies the condition (1.1),*
(2) *the $\mathrm{Gal}(\bar{k}/k)$-equivariant homomorphism $c : X^*(T)(\bar{k}) \to A^*(\bar{k})$ is injective,*
(3) *the extension G is anti-affine, that is $\mathcal{O}_G(G) = k$.*

Proof. The equivalence between (2) and (3) is given by [3, Prop 2.1]. If (1) holds, we have that $\mathcal{O}_G(G) = p_*\mathcal{O}_G(\mathrm{Spec}\ (k)) = \mathcal{O}_S(\mathrm{Spec}\ (k)) = k$, i.e. the extension G is anti-affine. Suppose now that (3) holds. Denote by $\alpha : G \to A$ the surjective morphism of algebraic groups underlying the extension G and by $q : A \to S$ the structural morphism of A. Let $G_{\bar{k}}$ the extension obtained from G extending the scalars from k to $\bar{k}$. As observed in [3, (2.2)] $\alpha_*(\mathcal{O}_{G_{\bar{k}}}) = \oplus_{x \in X^*(T)} \mathcal{L}_x$ where $\mathcal{L}_x$ is the invertible sheaf on $A_{\bar{k}}$ algebraically equivalent to 0, which corresponds to the point $c(x)$ of $A^*_{\bar{k}}(\bar{k})$ via the isomorphism $A^* \cong \mathbf{Pic}^0_{A/S}$. Hence

$$p_*\mathcal{O}_{G_{\bar{k}}} = q_*\alpha_*\mathcal{O}_{G_{\bar{k}}} = q_* \bigoplus_{x \in X^*(T)} \mathcal{L}_x.$$

Since G is anti-affine, $\mathrm{H}^0(A_{\bar{k}}, \mathcal{L}_x) = 0$ for all $x \neq 0$. Therefore

$$p_*\mathcal{O}_{G_{\bar{k}}}(\mathrm{Spec}\ (k)) = \bigoplus_{x \in X^*(T)} \mathrm{H}^0(A_{\bar{k}}, \mathcal{L}_x) = \mathrm{H}^0(A_{\bar{k}}, \mathcal{L}_0) = \mathcal{O}_{A_{\bar{k}}}(A_{\bar{k}}) = k,$$

that is $p_*\mathcal{O}_{G_{\bar{k}}} = \mathcal{O}_S$. □

Consider an extension G of an abelian variety A by a torus T defined over a field k. As before denote by $c : X^*(T)(\bar{k}) \to A^*(\bar{k})$ the $\mathrm{Gal}(\bar{k}/k)$-equivariant homomorphism which defines this extension G. Let X'' be the biggest $\mathrm{Gal}(\bar{k}/k)$-sub-module of $X^*(T)(\bar{k})$ whose image via $c : X^*(T)(\bar{k}) \to A^*(\bar{k})$ is a torsion subgoup of $A^*(\bar{k})$. Denote by T'' the quotient torus of T whose character group is X''. Then G is an extension of the torus T'' by an extension G' of A by T/T''

$$0 \to G' \to G \to T'' \to 0.$$

Now the $\mathrm{Gal}(\bar{k}/k)$-equivariant homomorphism $X^*(T/T'')(\bar{k}) \to A^*(\bar{k})$ defining the extension G' is injective, and therefore, by the above Lemma, G' is anti-affine, that is the global functions of G' are k, or equivalently for G' the condition (1.1) holds. Since the torus T'' plays no role for the study of divisorial correspondences, we have showed that over a field we can always reduce to the case where condition (1.1) holds.

3. Linear morphisms via divisorial correspondences

Consider two extensions $p_1 : G_1 \to S, p_2 : G_2 \to S$ of abelian schemes by tori. Denote by $\epsilon_i : S \to G_i$ their unit sections and suppose that the structural morphisms p_i satisfy $p_{i_*}\mathcal{O}_{G_i} = \mathcal{O}_S$ universally for $i = 1, 2$. Let $\mathbf{Hom}_{\epsilon_1}(G_1, \mathbf{Pic}_{G_2/S})$ be the sheaf of morphisms of sheaves from G_1 to $\mathbf{Pic}_{G_2/S}$ which send the unit section ϵ_1 to the unit section of $\mathbf{Pic}_{G_2/S}$ and likewise for $\mathbf{Hom}_{\epsilon_2}(G_2, \mathbf{Pic}_{G_1/S})$. Observe that for any S-scheme T, $\mathbf{Hom}_{\epsilon_1}(G_1, \mathbf{Pic}_{G_2/S})(T)$ is just the group $\mathbf{Pic}_{G_{2T}/S}(G_{1T})$ of G_{1T}-points of $\mathbf{Pic}_{G_{2T}/S}$. Then by the short exact sequence (0.2) we have that the *fppf*-sheaf $T/S \mapsto \mathbf{Hom}_{\epsilon_{1T}}(G_{1T}, \mathbf{Pic}_{G_{2T}/S})$ is isomorphic to the *fppf*-sheaf $T/S \mapsto \mathbf{Corr}_S(G_1, G_2)(T)$. Therefore we have the isomorphisms of *fppf*-sheaves

$$\mathbf{Hom}_{\epsilon_1}(G_1, \mathbf{Pic}_{G_2/S}) \cong \mathbf{Corr}_S(G_1, G_2) \cong \mathbf{Hom}_{\epsilon_2}(G_2, \mathbf{Pic}_{G_1/S}) \tag{2.1}$$

We define by $\mathbf{Hom}_{\mathrm{Gr}}(X, Y)$ the sheaf of group homorphisms between abelian sheaves.

Proposition 2.1. *Let S be a normal scheme. Consider two extensions $p_1 : G_1 \to S, p_2 : G_2 \to S$ of abelian schemes by tori. Suppose that the structural morphisms p_i satisfy ${p_i}_* \mathcal{O}_{G_i} = \mathcal{O}_S$ universally for $i = 1, 2$. Then*

$$\mathbf{Hom}_{\mathrm{Gr}}(G_1, \mathbf{Pic}_{G_2/S}) \cong \mathbf{Corr}_S(G_1, G_2) \cong \mathbf{Hom}_{\mathrm{Gr}}(G_2, \mathbf{Pic}_{G_1/S}). \tag{2.2}$$

In particular, if G is an extension of an abelian scheme by a torus over a normal base scheme, this yields a morphism of fppf-sheaves

$$\mathbf{Pic}_{G/S} \to \mathbf{Hom}_{\mathrm{Gr}}(G, \mathbf{Pic}_{G/S}). \tag{2.3}$$

Proof. Because of (2.1), it is enough to show that if $u : G_1 \to \mathbf{Pic}_{G_2/S}$ is a morphism of sheaves which sends the unit section $\epsilon_1 : S \to G_1$ of G_1 to the unit section of $\mathbf{Pic}_{G_2/S}$, then u is in fact a group homomorphism. We will prove that the following morphism of sheaves

$$\begin{array}{rccc} v: & G_1 \times_S G_1 & \to & \mathbf{Pic}_{G_2/S} \\ & (g, g') & \mapsto & u(g+g') - u(g) - u(g') \end{array}$$

is the null morphism. The morphism v is a $G_1 \times_S G_1$-point of $\mathbf{Pic}_{G_2/S}$, that is an invertible sheaf $\mathcal{L}$ on $G_1 \times_S G_1 \times_S G_2$ that we can suppose to be rigidified along the unit section $\epsilon_2 : S \to G_2$ of G_2. Since by hypothesis $u(0) = 0$, the restriction of $\mathcal{L}$ to $G_1 \times_S S \times_S G_2$ and to $S \times_S G_1 \times_S G_2$ is trivial. Therefore $\mathcal{L}$ is rigidified along $\epsilon_1 \times_S G_1 \times_S G_2, G_1 \times_S \epsilon_1 \times_S G_2$ and $G_1 \times_S G_1 \times_S \epsilon_2$. But by [4, Chp I, §2.6], the extensions G_i over a normal base scheme satisfy the Theorem of the Cube, and so the line bundle $\mathcal{L}$ is trivial, that is v is the null morphism.

Now let G be an extension of an abelian scheme by a torus. Consider the canonical morphisms of *fppf*-sheaves

$$\mathbf{Pic}_{G/S} \to \mathbf{Pic}_{G/S} \times_S \mathbf{Pic}_{G/S} \stackrel{\text{can}}{\to} \mathbf{Pic}_{G \times_S G} \to \mathbf{Corr}_S(G, G).$$

Using (2.2) we get the expected morphism of *fppf*-sheaves. □

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