Unmediated Communication in Games with Complete and Incomplete Information*

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Abstract

In this paper we study the effects of adding unmediated communication to static, finite games of complete and incomplete information. We characterize $S^U(G)$, the set of outcomes of a game G, that are induced by sequential equilibria of cheap talk extensions. A cheap talk extension of G is an extensive-form game in which players communicate before playing G. A reliable mediator is not available and players exchange private or public messages that do not affect directly their payoffs. We first show that if G is a game of complete information with five or more players and rational parameters, then $S^U(G)$ coincides with the set of correlated equilibria of G. Next, we demonstrate that if G is a game of incomplete information with at least five players, rational parameters and full support (i.e. all profiles of types have positive probability), then $S^U(G)$ is equal to the set of communication equilibria of G.

KEYWORDS: Communication, correlated equilibrium, communication equilibrium, sequential equilibrium, mechanism design, revelation principle.

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1 Introduction

Communication allows players to share information with one another and to coordinate their actions. By doing this, in many games players can attain outcomes that are not feasible otherwise. In this paper, we study the effects of adding communication to static, finite games of complete and incomplete information.

A system of communication specifies the rules according to which players communicate. Adding one to a game defines a new, extended game. In a game with communication players exchange messages before choosing their actions. The notions of correlated equilibrium (Aumann (1974)), communication equilibrium and the revelation principle (see Forges (1986) and Myerson (1982)) characterize the set of outcomes that players can achieve in games with communication. In particular, this set of outcomes coincides with the set of correlated equilibria if the game is one of complete information, and with the set of communication equilibria if the game is one of incomplete information. This is a very powerful and useful result. Although there are infinitely many ways in which players can exchange messages, it is very easy to characterize what they can achieve with communication. In fact, correlated and communication equilibria are simply defined by a number of linear inequalities. Therefore, the set of correlated or communication equilibria has a simple and tractable mathematical structure (it is a convex polyhedron). Consider, for example, a game of complete information, and suppose that we are interested in finding the highest payoff that a given player can achieve with communication. In principle, there are infinitely many games with communication to consider. However, given the characterization above, it is enough to find the best correlated equilibrium for that player. This simply involves maximizing a linear function over a convex polyhedron.

Since correlated and communication equilibria are important and commonly used solution concepts, it is worthwhile to examine carefully the conditions under which they can be applied. Forges (1986) and Myerson (1982) show that the set of correlated or commu-

nication equilibria coincides with the set of outcomes that players can achieve with communication under two critical assumptions. First, not only do players exchange messages with one another, but they can also communicate with a reliable and impartial mediator (*mediated* communication). Secondly, the *Nash equilibrium* concept is used to analyze the games with communication.

In many situations, however, the assumption that there is a reliable mediator is too strong, and players are restricted to exchange messages among themselves. In a bargaining setting, for example, the buyer and the seller usually undergo a number of rounds of direct conversations before reaching a final agreement, and no third party mediates their positions or filters their messages. Similarly, political leaders often communicate directly with their advisors. It is true that the mediator could be considered a machine that is programmed to perform a certain number of operations. Still, one has to assume that an impartial individual, different from the players, is available to program the machine. We cannot rely on the players, who would take advantage of the opportunity, and would program the machine in their best interest. When a mediator is not available, and players can only communicate with one another, we say that communication is direct or *unmediated*.

As far as the solution concept is concerned, Nash equilibrium is a legitimate candidate. However, a game with communication is an extensive-form game, and Nash equilibrium does not prevent players from using non credible threats and behaving irrationally in events that are reached with probability zero. In a game with communication, it is therefore appropriate to use a stronger solution concept, such as *sequential equilibrium* (Kreps and Wilson (1982)), which requires players to behave optimally on and off the equilibrium path.

The goal of this paper is to characterize the outcomes that players can achieve when an impartial mediator is not available and we adopt a solution concept stronger than Nash equilibrium. We further address the related question of whether it is possible to use correlated and communication equilibria when the assumptions made by Forges (1986) and Myerson (1982) are not satisfied.

Depending on the presence of a mediator and on the adopted solution concept, the existing literature on games with communication can be classified into four categories: (i) mediated communication, Nash equilibrium; (ii) mediated communication, sequential equilibrium; (iii) unmediated communication, Nash equilibrium; (iv) unmediated communication, sequential equilibrium. To illustrate our results in more details, and to describe how our findings contribute to the existing body of work, we now discuss these categories.

We have already mentioned that in case (i) the set of correlated or communication equilibria describes the outcomes that players can achieve with communication. To implement a correlated or communication equilibrium, a mediator usually performs lotteries and sends players private messages. However, Lehrer and Sorin (1997) show that correlated and communication equilibria can be implemented in communication systems in which the mediator behaves deterministically and sends only public messages (see also Lehrer (1996)).

What happens when communication is mediated and the solution concept is stronger than Nash equilibrium? For static games of complete information, the characterization of the outcomes attainable with communication in terms of correlated equilibria trivially extends to the case in which the sequential equilibrium concept is used.[1] A similar result can be derived for games of incomplete information with full support (i.e. games in which all profiles of players' types have positive probability). In this case, the set of outcomes that players can achieve with mediated communication and the sequential equilibrium concept coincides with the set of communication equilibria. This result does not hold in games without full support. In Section 3, we show that there exist communication equilibria of games without full support that cannot be implemented. We also demonstrate the failure of the revelation principle in games that do not have full support.

There is an extensive literature on games with unmediated communication. Aumann

[1] However, Dhillon and Mertens (1996) show that the outcomes achievable with communication cannot be described by correlated equilibria when the perfect equilibrium concept is used to analyze the extended games with communication. They demonstrate the failure of the revelation principle in this context, and provide a characterization of the outcomes achievable with mediated communication only in two-player games.

and Hart (1999) use the Nash equilibrium concept and characterize the set of outcomes implementable for two-player games in which only one player has private information. Amitai (1996) generalizes[2] Aumann and Hart's (1999) results to two-player games in which both sides have private information. Urbano and Vila (1997, 2001) demonstrate that if the two players have bounded rationality and can solve only problems of limited computational complexity, then it is possible to implement all correlated equilibria and all communication equilibria without mediation. Results have also been obtained for specific games with two players, like double auctions (Matthews and Postlewaite (1989)) and the "battle of the sexes" (Banks and Calvert (1992)).

For games with four or more players, Barany (1992) and Forges (1990) provide a complete characterization of the effects of unmediated communication when the solution concept is Nash equilibrium. Barany (1992) considers static games of complete information and shows that any rational correlated equilibrium (i.e. any correlated equilibrium with rational components) can be implemented. Moreover, if a game also has rational parameters, then the set of outcomes attainable with unmediated communication coincides with the set of correlated equilibria (see Forges (1990)). Games of incomplete information are considered by Forges (1990). She demonstrates that if a game has four or more players and rational parameters, then the set of communication equilibria completely characterizes the outcomes achievable with unmediated communication.[3]

To implement a correlated equilibrium, Barany (1992) constructs a scheme of unmediated communication such that an action profile is chosen according to the correlated equilibrium distribution, and each player learns only her own action. Each message is sent by two players to a third one, and public verification of the past record is possible. To prevent

[2]The first version of Aumann and Hart (1999) dates from 1993.

[3]Forges (1990) proposes a mechanism of communication in which players exchange messages at the ex-ante stage (before they learn their types) and at the interim stage (after each player learns her own type). However, Gerardi (2000) shows that in games with four or more players and rational parameters, it is possible to implement all communication equilibria even if communication takes place only at the interim stage.

unilateral deviations in the communication phase, Barany (1992) assumes that a player who receives two different messages stops the communication process and reports that cheating has occurred. Then messages are verified, and the deviator is identified and punished by the opponents who play a "minmax" strategy against her in the original game. If a receiver deviates and reports that cheating has occurred when it has not, then she is punished at her "minmax" level. Note that there are two situations in which Barany's (1992) equilibrium fails to satisfy the sequential rationality criterion (Ben-Porath (1998, 2000)). First, the strategy profile that minmaxes a deviator is not necessarily a Nash equilibrium of the original game. Second, a receiver who observes a deviation may not have an incentive to report that cheating has occurred. These problems extend to Forges (1990), since she uses the communication scheme proposed by Barany (1992) to prove her result.

Ben-Porath (2000) uses the solution concept of sequential equilibrium. He provides sufficient conditions for a communication equilibrium to be implemented with unmediated communication in games with three or more players. Specifically, Ben-Porath (2000) shows that a communication equilibrium can be implemented provided that the game admits a Bayesian-Nash equilibrium in which the payoff of every type of each player is lower than the communication equilibrium payoff. The basic idea is that if a player deviates during the communication phase, then she is punished by her opponents, who play the Bayesian-Nash equilibrium. Using this approach one cannot implement, for example, a communication equilibrium in which the payoff of a player is lower than all Bayesian-Nash equilibrium payoffs: no punishment is available to prevent the player from deviating in the communication phase.

We propose a different system of communication that avoids this problem. We provide a complete characterization of the set of outcomes that can be implemented with unmediated communication in games with five or more players and rational parameters. Our solution concept in the extended games with communication is sequential equilibrium. We show that in games of complete information an outcome can be implemented with direct

communication if and only if it is a correlated equilibrium. For games with incomplete information and full support, we demonstrate that the set of outcomes achievable with unmediated communication coincides with the set of communication equilibria.

Our results provide support for the use of correlated and communication equilibria to describe the effects of communication. If a game has at least five players, rational parameters and full support, one can use the simple structure of the set of correlated or communication equilibria even if an impartial mediator is not available and players are sequentially rational.

The rest of the paper is organized as follows. In Section 2 we analyze games with complete information and in Section 3 we study games with incomplete information. In Section 4 we illustrate an application of our results in the context of mechanism design. Section 5 contains the proof of the main theorem and Section 6 concludes.

2 Games with Complete Information

In this section we restrict attention to games of complete information, and characterize the set of outcomes that players can achieve with unmediated communication.

Let $\Gamma = \langle P_1, ..., P_I, S_1, ..., S_I, u_1, ..., u_I \rangle$ be a finite normal-form game, where $P_1, ..., P_I$ are the players, S_i is the set of actions available to player P_i, $S = \prod_{i=1}^{I} S_i$ is the set of action profiles, and $u_i : S \to \mathbb{R}$ is the payoff function of player P_i. We let $S_{-i} = \prod_{j \neq i} S_j$ denote the set of profiles of actions of players different from P_i. The set of probability distributions over S is denoted by $\Delta(S)$.

We extend Γ by allowing players to communicate before they choose their actions. A *cheap talk extension* of Γ is an extensive-form game that consists of a communication phase and an action phase. During the communication phase, players exchange "cheap" messages, i.e. messages that do not affect *directly* their payoffs. Then, in the action phase, the original game Γ is played.

To describe our results on unmediated communication, it is convenient first to recapitu-

late the more general case of mediated communication. We consider cheap talk extensions in which players exchange messages among themselves and communicate with an impartial mediator (see, for example, Myerson (1991)).

A strategy profile in a cheap talk extension of Γ induces an outcome in Γ, i.e. a probability distribution over S. Let $N(\Gamma)$ denote the set of outcomes in Γ induced by Nash equilibria of cheap talk extensions of Γ (where mediated communication is allowed), and $S(\Gamma)$ denote the set of outcomes induced by sequential equilibria. In other words, a probability distribution $r \in \Delta(S)$ is in $N(\Gamma)$ (respectively, $S(\Gamma)$) if and only if there exists a cheap talk extension of Γ and a Nash equilibrium (respectively, sequential equilibrium) of that extension that induces r.

To characterize the sets $N(\Gamma)$ and $S(\Gamma)$, we need to introduce the notion of correlated equilibrium. A probability distribution $r \in \Delta(S)$ is a correlated equilibrium of Γ if and only if:

$$\sum_{s \in S} r(s) \left(u_i(s) - u_i(s_{-i}, \delta_i(s_i))\right) \geqslant 0, \quad i = 1, ..., I, \quad \forall \delta_i : S_i \to S_i. \tag{1}$$

We let $C(\Gamma)$ denote the set of correlated equilibria of Γ. Being defined by finitely many linear inequalities, $C(\Gamma)$ is a convex polyhedron.

In the analysis below, we shall devote special attention to the class of correlated equilibria with rational components. We say that a correlated equilibrium r is *rational* if for every action profile in S, the probability $r(s)$ is a rational number.

Any correlated equilibrium is in the set $S(\Gamma)$ (and therefore in $N(\Gamma)$). Let r be a correlated equilibrium, and consider the following cheap talk extension of Γ. The mediator randomly selects an action profile in S according to r, and informs each player P_i only of the ith component of the chosen action profile. Then the players simultaneously choose their actions. Inequality (1) guarantees the existence of a sequential equilibrium in which each player follows the mediator's recommendation. Clearly, the induced outcome in Γ is the correlated equilibrium r.

The revelation principle for normal-form games states that any probability distribution

in $N(\Gamma)$ is a correlated equilibrium of Γ (Forges (1986) and Myerson (1982)). Thus, it follows that both $N(\Gamma)$ and $S(\Gamma)$ coincide with $C(\Gamma)$, the set of correlated equilibria of Γ.

We now consider unmediated communication. We assume that a reliable mediator is not available, and consider cheap talk extensions defined as follow. We start by introducing *plain* cheap talk extensions. Given a normal-form game Γ, a plain cheap talk extension of Γ consists of a finite number of steps of communication, at the end of which Γ is played. For each step, the plain cheap talk extension specifies the senders, that is, the players who are allowed to send a message. Further, for each sender, it specifies the receivers and the set of messages. Plain cheap talk extensions allow for simultaneous messages (if the sets of senders contain more than one player) and sequential messages (if the sets of senders are singleton). Further, we can have private messages, public messages, and intermediate situations in which a sender sends a message to a subset of her opponents.

We are ready to define a cheap talk extension with unmediated communication. In the first step, a number of players (possibly zero) sends *public* messages. To each vector of messages the cheap talk extension associates a plain cheap talk extension.

In a plain cheap talk extension, the identity of senders and receivers and the set of messages of each step are predetermined and do not depend on previous messages. In contrast, in a cheap talk extension, the public messages sent in the first step may affect the set of messages and the identity of senders and receivers of subsequent steps. Of course, if the number of players sending public messages in the first step is zero, then the cheap talk extension is plain. In other words, plain cheap talk extensions are a special case of cheap talk extensions with unmediated communication.

Let $S^U(\Gamma)$ denote the set of outcomes of Γ induced by a sequential equilibrium of some cheap talk extension with unmediated communication. Clearly, since mediated communication is more general than unmediated communication, $S^U(\Gamma)$ is included in $S(\Gamma)$ (and, therefore, in $N(\Gamma)$ and $C(\Gamma)$).

Our first result shows that, if there are at least five players, the set $S^U(\Gamma)$ contains all

rational correlated equilibria of Γ (i.e. all correlated equilibria with rational components). When a correlated equilibrium r belongs to $S^U(\Gamma)$, we say that r can be implemented.

Theorem 1 *Let Γ be a finite normal-form game with five or more players, and let r be a rational correlated equilibrium of Γ. Then $r \in S^U(\Gamma)$.*

A formal proof of Theorem 1 is in Section 5, where we construct $\bar{\Gamma}(r)$, a finite plain cheap talk extension of Γ with unmediated communication, and a sequential equilibrium $\Psi(r)$ of $\bar{\Gamma}(r)$ that induces the distribution r on S.

To implement a rational correlated equilibrium r, we propose a scheme of communication such that, if all players follow it, an action profile in S is chosen according to the distribution r. Moreover, when the communication phase ends, player P_i, $i = 1, ..., I$, learns *only* which action in S_i she has to play. Therefore, each player has an incentive to play the action that she learns, provided that her opponents behave likewise.

To give an intuition of our proof and explain how players learn their actions, let us consider the following simple example. There are three players, P_1, P_2, and P_3, and three action profiles, s, s', and s''. Given an action profile, say s, let $pr_3(s)$ denote the action of P_3 in s. We assume $pr_3(s) = pr_3(s')$, and $pr_3(s) \neq pr_3(s'')$. The action profiles s and s' specify the same action for P_3, while s'' specifies a different action. Suppose we want the players to select an action profile at random, according to the uniform distribution. Moreover, P_3 has to learn her action in the chosen profile, while P_1 and P_2 do not have to learn anything.[4] This can be accomplished in the following way. P_1 selects α and σ, two permutations on $\{s, s', s''\}$, at random, according to the uniform distribution, and independently of each other. P_2 selects x, an element of $\{s, s', s''\}$, at random, according to the uniform distribution. The chosen profile is $\sigma(x)$. Clearly, $\sigma(x)$ has a uniform distribution and is unknown to all players. We require P_1 to send the permutation $\alpha\sigma$ to

[4] To keep the example as simple as possible, we require that only P_3 learns her action. Obviously, to implement a correlated equilibrium, each player has to learn her action in the chosen profile (see Section 5 for details).

P_2. Since α and σ are independent of each other and uniformly distributed, P_2 does not learn anything about σ (the conditional distribution of σ, given $\alpha\sigma$, is uniform). Then P_2 sends the element $\alpha\sigma(x)$ to P_3. Finally, P_1 sends $pr_3 \circ \alpha^{-1}$ (a mapping from $\{s, s', s''\}$ to P_3's set of actions) to P_3. At this point P_3 computes $pr_3 \circ \alpha^{-1}(\alpha\sigma(x)) = pr_3(\sigma(x))$ and learns her action in the chosen profile.

This system of communication, although simple, presents a serious problem: P_1 and P_2 might have an incentive to deviate from it. If, for example, P_2 sends P_3 a message different from $\alpha\sigma(x)$, P_3 will not learn the correct action. Clearly, in a game P_2 might prefer to induce P_3 not to learn the correct action.

We construct a more complex communication scheme that solves this problem and gives all players an incentive to follow it. We require that a message is sent by three different players to a fourth one. In equilibrium, all senders report the correct message. Moreover, we show that it is sequentially rational for a receiver to follow the message sent by the *majority* of the senders. This implies that a sender does not have profitable deviations during the communication stage. Even if she deviates, the receiver will receive the correct message from the other two senders.

Our communication scheme differs from the systems of communication proposed by Barany (1992) and Ben-Porath (1998, 2000) in other important aspects. In fact, the idea of using the majority rule to prevent unilateral deviations in the conversation phase cannot be applied to their communication schemes if there are only five players. Similarly to the example above, in Barany (1992) and Ben-Porath (1998, 2000), a combination of two random variables determines the action profile in S that players play once communication is over (in our example the two variables are σ and x). Clearly, it is crucial that a player does not know both random variables, otherwise she learns her opponents' actions. However, this requirement cannot be satisfied if there are five players and every message is sent by three different players. In this case, at least one player must know both random variables. In our communication scheme, the action profile chosen depends on four random variables.

In this way, we are able to construct a communication scheme in which every message is sent by three players and no player learns her opponents' actions.

This also explains our need for at least five players. A message goes from three players to a fourth one. This already requires four players. But if there are only four players all messages are public, and all players can learn the chosen action profile. At least another player is needed to generate private messages. Thus five is the smallest number of players that we need to use our communication scheme.

Finally, in contrast to Barany (1992) and Ben-Porath (1998), our system of communication does not require public verification of past messages. In the game that we present is Section 5, players can exchange oral messages.

Theorem 1 considers only rational correlated equilibria. An obvious question is whether players can implement correlated equilibria that involve probabilities which are irrational numbers. We now show that a correlated equilibrium with irrational components can be implemented provided that it can be expressed as a convex combination of rational correlated equilibria. Let $r_1, ..., r_K$ be K rational correlated equilibria of a game, Γ, with five or more players. Consider the correlated equilibrium $r = \sum_{k=1}^{K} \rho_k r_k$, where $\rho_1, ..., \rho_K$ are positive numbers (rational or irrational) such that $\sum_{k=1}^{K} \rho_k = 1$. Theorem 1 guarantees that for every rational correlated equilibrium r_k, $k = 1, ..., K$, there exists a plain cheap talk extension $\bar{\Gamma}(r_k)$, and a sequential equilibrium $\Psi(r_k)$ of $\bar{\Gamma}(r_k)$ that induces the distribution r_k on S.

To implement r, we let players perform a "jointly controlled lottery"[5] in which every correlated equilibrium r_k is selected with probability ρ_k. Specifically, consider the following cheap talk extension $\tilde{\Gamma}(r)$. Players P_1 and P_2 simultaneously announce (to all players) two positive numbers in the unit interval. Let ω^i denote the number announced by player P_i,

[5] A jointly controlled lottery (Aumann, Maschler and Stearns (1968)) is a communication scheme that allows two or more players to randomly select an outcome. The scheme is immune against unilateral deviations in the sense that no player can, by her own decision, influence the probability distribution.

$i = 1, 2$. Let $\omega = \chi(\omega^1, \omega^2)$, where

$$\chi(\omega^1, \omega^2) := \begin{cases} \omega^1 + \omega^2 & \text{if } \omega^1 + \omega^2 \leqslant 1 \\ \omega^1 + \omega^2 - 1 & \text{if } \omega^1 + \omega^2 > 1 \end{cases}, \tag{2}$$

and define ρ_0 to be equal to zero. If $\omega \in \left(\sum_{j=0}^{k-1} \rho_j, \sum_{j=0}^{k} \rho_j\right]$, for some $k = 1, ..., K$, then the game $\bar{\Gamma}(r_k)$ is played.

Although Kreps and Wilson (1982) define sequential equilibria only for finite games, it is easy to extend their definition to $\tilde{\Gamma}(r)$. Notice that after P_1 and P_2 announce ω^1 and ω^2, respectively, a proper subgame is induced. We require that the equilibrium strategies and beliefs of $\tilde{\Gamma}(r)$, when restricted to a subgame $\bar{\Gamma}(r_k)$, constitute a sequential equilibrium of $\bar{\Gamma}(r_k)$. Moreover, player P_i, $i = 1, 2$, chooses the number ω^i to maximize her expected payoff (given her strategy in the rest of the game and her opponents' strategies).

Consider the following assessment. In the first step, both P_1 and P_2 randomly select a number in the interval $(0, 1]$, according to the uniform distribution. If the cheap talk extension $\bar{\Gamma}(r_k)$ is selected, players play the sequential equilibrium $\Psi(r_k)$. It is easy to check that this assessment constitutes a sequential equilibrium of the game $\tilde{\Gamma}(r)$. The only thing to note here is that, since ω^2 is independent of ω^1 and uniformly distributed, ω is also independent of ω^1 and uniformly distributed. This implies that P_1 is indifferent between all possible announcements in the first step (clearly, the same argument can be applied to P_2). Therefore, we have proved that the correlated equilibrium r can be implemented.

For an important class of games, we are able to provide a complete characterization of the set of outcomes that can be implemented. We say that a normal-form game Γ is rational if all its parameters are rational numbers, i.e. if for every $i = 1, ..., I$, and for every strategy profile s in S, the payoff $u_i(s)$ is a rational number. When Γ is rational, any correlated equilibrium can be expressed as a convex combination of rational correlated equilibria.[6] Thus, any correlated equilibrium is in the set $S^U(\Gamma)$. Since it is always the case that $S^U(\Gamma)$ is contained in $C(\Gamma)$, we conclude that the two sets are identical. The

[6] If Γ is rational, the vertices of $C(\Gamma)$ are rational correlated equilibria.

next corollary summarizes our results.

Corollary 1 *Let Γ be a finite normal-form game with five or more players. Let r be a convex combination of rational correlated equilibria of Γ. Then $r \in S^U(\Gamma)$. If Γ is rational, $S^U(\Gamma) = C(\Gamma)$.*

For any two-player game Γ, it is easy to show that $S^U(\Gamma)$ is the convex hull of the Nash equilibrium outcomes of Γ. Intuitively, when there are only two players, all messages are public, and therefore only lotteries over Nash equilibria can be implemented (see Aumann and Hart (1999)). Thus, in the class of games with rational parameters, a complete characterization of the set $S^U(\Gamma)$ is unavailable only when Γ has three or four players. Obviously, our communication scheme cannot be applied if there are less than five players, but this does not rule out the possibility that all correlated equilibria can be implemented with some other communication scheme. We do not have any example of a correlated equilibrium in a game with three or four players that cannot be implemented. Barany (1992) provides the example of a rational correlated equilibrium in a three-player game that cannot be implemented (in Nash equilibrium) if the message spaces are finite. However, Forges (1990) shows that the correlated equilibrium considered by Barany (1992) is induced by a Nash equilibrium in a cheap talk extension where a continuum of messages is used. For a game Γ with three or four players, a partial characterization of the set $S^U(\Gamma)$ is provided by Ben-Porath (2000), who gives sufficient conditions for a rational correlated equilibrium to be implemented.

We conclude this section with a comment on our definition of unmediated communication. The cheap talk extensions defined in this section are not the most general form of unmediated communication. One could think, for example, of situations in which a sender can choose the receivers of her message. However, Corollary 1 shows that restricting attention to our cheap talk extensions is without loss of generality in games with five or more players and rational parameters. For these games, allowing for more general forms of un-

mediated communication does not expand the set of outcomes that players can implement.

3 Games with Incomplete Information

We now consider games of incomplete information and show how unmediated communication allows players to expand the set of equilibrium outcomes.

Let $G = \langle P_1, ..., P_I, T_1, ..., T_I, S_1, ..., S_I, p, u_1, ..., u_I \rangle$ be a finite Bayesian game. Players are $P_1, ..., P_I$, and S_i denotes the set of actions of P_i. As in Section 2, S denotes the set of action profiles, S_{-i} is the set of profiles of actions of players different from P_i, and $\Delta(S)$ denotes the set of probability distributions over S. In addition, T_i is the set of types of P_i, and $T = \prod_{i=1}^{I} T_i$ is the set of type profiles. We let $T_{-i} = \prod_{j \neq i} T_j$ denote the set of profiles of types of players different from P_i. The payoffs of P_i are described by $u_i : T \times S \to \mathbb{R}$. Finally, p is a probability distribution over T.[7] We say that game G has *full support* if every profile of types occurs with positive probability, i.e. if $p(t) > 0$ for every t in T. Games in which players' types are independent constitute an obvious example of games with full support.

We proceed as in Section 2, and extend G by introducing pre-play communication. Although we are interested in unmediated communication, it is convenient to start our analysis by considering first the case of mediated communication.

A strategy profile in a cheap talk extension of G induces an outcome in G, i.e. a mapping from T to $\Delta(S)$. As in the previous section, we let $N(G)$ denote the set of outcomes in G induced by a Bayesian-Nash equilibrium of some cheap talk extension of G (where mediated communication is allowed). Similarly, $S(G)$ denotes the set of outcomes in G induced by a sequential equilibrium of some cheap talk extension of G.

The set $N(G)$ can be easily characterized in terms of communication equilibria. A

[7] For notational simplicity, we assume that beliefs in G are consistent.

function $q : T \to \Delta(S)$ is a communication equilibrium if and only if:

$$\sum_{t_{-i} \in T_{-i}} \sum_{s \in S} p(t_{-i}|t_i)\, q(s|(t_{-i}, t_i))\, u_i((t_{-i}, t_i), s) \geqslant \sum_{t_{-i} \in T_{-i}} \sum_{s \in S} p(t_{-i}|t_i)\, q(s|(t_{-i}, t'_i))\, u_i((t_{-i}, t_i), (s_{-i}, \delta_i(s_i))), \quad (3)$$
$$i = 1, ..., I, \quad \forall (t_i, t'_i) \in T_i^2 \quad \forall \delta_i : S_i \to S_i.$$

We let $CE(G)$ denote the set of communication equilibria of G. $CE(G)$ is defined by finitely many linear inequalities, and therefore is a convex polyhedron. We also say that a communication equilibrium q is rational if, for every action profile s in S and every type profile t in T, the probability $q(s|t)$ is a rational number.

To see that a communication equilibrium q belongs to the set $N(G)$, consider the following cheap talk extension $G^D(q)$, usually called the *canonical* game. First, each player P_i sends the mediator a message in T_i. The mediator, after receiving a vector of messages t, randomly selects an action profile in S according to the probability distribution $q(\cdot|t)$ and informs each player P_i only of the ith component of the chosen profile. Finally, all players simultaneously choose their actions. It follows from inequality (3) that there exists a Bayesian-Nash equilibrium of $G^D(q)$ in which every player reports her type truthfully to the mediator, and follows the mediator's recommendation. The notion of communication equilibrium is a generalization of the notion of correlated equilibrium for games with incomplete information. In a correlated equilibrium, a mediator allows players to coordinate their actions. In a communication equilibrium, the mediator has two roles: she helps players coordinate their actions, and exchange their private information. The two notions of equilibria coincide in games with complete information.

On the other hand, it follows from the revelation principle that any outcome in $N(G)$ is a communication equilibrium (see Forges (1986) and Myerson (1982)). Therefore, for any finite Bayesian game G, the sets $N(G)$ and $CE(G)$ coincide.

The characterization of the set $S(G)$ is, in general, less immediate. Clearly, for any game G, $S(G)$ is included in $CE(G)$. Further, the set $S(G)$ is non-empty (sequential equilibria exist in any extension of G) and convex (the mediator conducts a lottery among

different extensions and announces publicly the outcome of the lottery). To investigate whether any communication equilibrium of G is in $S(G)$, we first need some definitions.

Given a communication equilibrium q of G, consider the canonical game $G^D(q)$. We say that q is *regular* if there is a sequential equilibrium of $G^D(q)$ in which each player reports her type truthfully to the mediator, and obeys the mediator's recommendation *after* being honest (a player may disobey the mediator's recommendation if she did not report her type truthfully). Clearly, any regular communication equilibrium of G belongs to $S(G)$.

When G has full support, it is easy to show that any communication equilibrium is regular. When all type profiles have positive probability, a player never learns that some other player lied to the mediator. This fact and inequality (3) imply that for a player who reports her type sincerely, it is sequentially rational to obey the mediator's suggestion. Moreover, inequality (3) also guarantees that no player has an incentive to lie to the mediator. Thus, if G has full support, $S(G)$ is equal to $CE(G)$.

This result does not hold in games without full support. As we show in the next example, it is possible to construct a game G', without full support, such that the set $S(G')$ is strictly included in the set of communication equilibria $CE(G')$.

Example 1 *A Bayesian game G' with $CE(G') \nsubseteq S(G')$.*

G' is a two-player game. The set of types of players P_1 and P_2 are $T_1 = \{t_1^1, t_1^2\}$ and $T_2 = \{t_2^1, t_2^2\}$, respectively. P_1 has to choose an action from the set $S_1 = \{s^1, s^2, s^3\}$, while P_2 does not choose an action. The probability distribution over the set of type profiles is:

$$p(t_1^1, t_2^1) = p(t_1^1, t_2^2) = p(t_1^2, t_2^1) = \tfrac{1}{3}, \quad p(t_1^2, t_2^2) = 0.$$

Payoffs are described in Table 1, where, for each combination of type profile and action, we report the corresponding vector of payoffs (the first entry denotes P_1's payoff).

First, let us consider the game G' without communication. In any Bayesian-Nash equilibrium, type t_1^2 plays action s^2, and type t_1^1 chooses either s^1, or s^3, or a randomization between the two actions.

		t_2^1	t_2^2
t_1^1	s^1	$(1,0)$	$(0,0)$
	s^2	$(-1,0)$	$(-1,0)$
	s^3	$(0,2)$	$(1,1)$
t_1^2	s^1	$(-1,-1)$	$(0,0)$
	s^2	$(1,1)$	$(0,0)$
	s^3	$(0,1)$	$(0,0)$

Table 1: Payoffs of the game G'

We now provide a complete characterization of the set $S(G')$. Consider a cheap talk extension of G'. Since type t_1^2 of player P_1 knows that player P_2 has type t_2^1, sequential rationality implies that in every information set in which t_1^2 has to choose an action from S_1, she will play s^2.

Further, type t_1^1 of P_1 never chooses action s^2. In fact, independent of P_2's type, action s^2 is dominated by actions s^1 and s^3. We now show that if q belongs to $S(G')$, then $q(t_1^1, t_2^1) = q(t_1^1, t_2^2)$. Both types of P_2 prefer action s^3 to any other action when P_1 has type t_1^1. Suppose, by contradiction, that there exists a sequential equilibrium of a cheap talk extension that induces an outcome q, with $q(s^3|t_1^1, t_2^1) > q(s^3|t_1^1, t_2^2)$. Then type t_2^2 would have an incentive to deviate and mimic the behavior of type t_2^1 to increase the probability of action s^3. Similarly, suppose that a sequential equilibrium induces an outcome q such that $q(s^3|t_1^1, t_2^1) < q(s^3|t_1^1, t_2^2)$. Type t_2^1 knows that, independent of her strategy, type t_1^2 of P_1 will play action s^2. Therefore t_2^1 has an incentive to mimic the behavior of t_2^2 : if P_1 has type t_1^1, the probability of action s^3 will increase.

We conclude that $S(G')$ coincides with the set of equilibrium outcomes of G'. If the players are sequentially rational, communication cannot expand the set of equilibrium outcomes of G'. Given this, it is very easy to show that $S(G')$ is strictly included in the set of communication equilibria of G'. A communication equilibrium that does not belong to $S(G')$ is, for example, q' defined by:

$$q'(t_1^1, t_2^1) = q'(t_1^2, t_2^2) = s^1, \quad q'(t_1^1, t_2^2) = s^3, \quad q'(t_1^2, t_2^1) = s^2,$$

where we adopt the convention of writing, for example, $q'(t_1^1, t_2^1) = s^1$ to denote $q'(s^1|t_1^1, t_2^1) = 1$.

Although not all communication equilibria of G' belong to $S(G')$, the revelation principle is still valid in the above example. Any element of $S(G')$ is a regular communication equilibrium, and can be implemented with the canonical game. This result could suggest that for a Bayesian game G, $S(G)$ coincides with the set of regular communication equilibria. However, it turns out that this conjecture is incorrect. As the next example demonstrates, when a game G does not have full support, $S(G)$ may contain communication equilibria that are not regular. In other words, the revelation principle does not hold in games without full support when the solution concept is sequential equilibrium.

Example 2 *The failure of the revelation principle.*

Consider the following three-person game G''. The set of types of players P_1 and P_2 are $T_1 = \{t_1^1, t_1^2\}$, and $T_2 = \{t_2^1, t_2^2\}$, respectively. P_3 does not have private information and is the only player to choose an action, from the set $S_3 = \{s^1, s^2, s^3, s^4\}$. The probability distribution over the set of profiles of types is given by:

$$p(t_1^1, t_2^1) = p(t_1^1, t_2^2) = p(t_1^2, t_2^1) = \tfrac{1}{3}, \quad p(t_1^2, t_2^2) = 0.$$

Finally, Table 2 describes the vector of payoffs for each pair of type profile and action (the first entry refers to P_1, the second one to P_2).

The communication equilibrium of G'' that maximizes P_3's expected payoff is unique and equal to q'', where:

$$q''(t_1^1, t_2^1) = s^1, \quad q''(t_1^1, t_2^2) = q''(t_1^2, t_2^1) = s^2, \quad q''(s^3|t_1^2, t_2^2) = q''(s^4|t_1^2, t_2^2) = \tfrac{1}{2}.$$

We now show that the communication equilibrium q'' is not regular. Consider the canonical game $G''^D(q'')$. P_3 has four information sets, one for each recommendation that the mediator can send. After receiving a recommendation, P_3 computes her beliefs over

		t_2^1	t_2^2
t_1^1	s^1	$(-1,-1,1)$	$(1,0,0)$
	s^2	$(1,1,0)$	$(1,1,1)$
	s^3	$(0,0,-1)$	$(-3,0,1)$
	s^4	$(0,0,-1)$	$(1,0,-1)$
t_1^2	s^1	$(0,1,0)$	$(0,0,0)$
	s^2	$(1,1,1)$	$(0,0,0)$
	s^3	$(0,1,-1)$	$(0,0,0)$
	s^4	$(0,-3,1)$	$(0,0,0)$

Table 2: Payoffs of the game G''

the set $\{(t_1^1, t_2^1), (t_1^1, t_2^2), (t_1^2, t_2^1)\}$. In any consistent assessment of $G''^D(q'')$, P_3's beliefs after recommendation s^3 must coincide with her beliefs after s^4.[8] Notice that it is sequentially rational for P_3 to choose action s^3 only if she assigns probability one to the profile of types (t_1^1, t_2^2). On the other hand, it is optimal for P_3 to play s^4 only if she assigns probability one to (t_1^2, t_2^1). But since P_3 has the same beliefs after s^3 and s^4, it cannot be sequentially rational to obey both recommendations. We conclude that q'' cannot be implemented with the canonical game $G''^D(q'')$.

However, q'' does belong to the set $S(G'')$. Consider the following cheap talk extension $\widehat{G''}(q'')$. Players P_1 and P_2 simultaneously report their messages to the mediator. Then the mediator recommends an action to P_3. Finally, P_3 chooses an action from S_3. The set of messages of P_i, $i = 1, 2$, is $\widehat{T}_i = \{t_i^1, t_i^2, t_i^3, t_i^4\}$, where t_i^3 and t_i^4 are two arbitrary messages. In Table 3, we report the mediator's recommendation to P_3 for each vector of reports of P_1 and P_2.

The mediator adopts a deterministic behavior unless P_1 and P_2 report messages t_1^2 and t_2^2, respectively. In this case the mediator randomizes, with equal probability, between

[8] The only thing that P_3 learns after receiving recommendations s^3 or s^4 is that the mediator received message t_1^2 from P_1, and t_2^2 from P_2. The mediator performs the lottery between s^3 and s^4 after P_1 and P_2 send their messages. Therefore, to prove the consistency of an assessment of $G''^D(q'')$, we cannot choose trembles for P_1 or P_2 that depend on the outcome of the mediator's lottery. This, in turn, implies that P_3's beliefs after s^3 cannot differ from her beliefs after s^4.

	t_2^1	t_2^2	t_2^3	t_2^4
t_1^1	s^1	s^2	s^1	s^1
t_1^2	s^2	$\frac{1}{2}s^3$, $\frac{1}{2}s^4$	s^2	s^2
t_1^3	s^1	s^2	s^3	s^1
t_1^4	s^1	s^2	s^1	s^4

Table 3: Mediator's recommendation in $\widehat{G''}(q'')$

recommendations s^3 and s^4.

It is easy to show that the cheap talk extension $\widehat{G''}(q'')$ admits a sequential equilibrium in which P_1 and P_2 "reveal their types truthfully" (i.e. t_i^j sends message t_i^j, $i = 1, 2$, $j = 1, 2$), and P_3 obeys the mediator's recommendation (clearly, this equilibrium induces q''). Intuitively, by introducing the new messages t_i^3 and t_i^4, $i = 1, 2$, we allow player P_3 to have *different* beliefs after recommendations s^3 and s^4 (recommendation s^3 can be induced by reports (t_1^3, t_2^3), and s^4 by reports (t_1^4, t_2^4)). In this way, both obedience to s^3 and obedience to s^4 can be sequentially rational (see Gerardi (2001) for details).

A complete characterization of the set $S(G)$ when G is a game without full support is still an open question and beyond the scope of this paper. In Gerardi (2001), we provide the solution for a special class of games. Specifically, we characterizes $S(G)$ when G is a game in which one player is uninformed and has to choose an action, while all other players have private information but do not choose an action.

We now assume that an impartial mediator is not available and turn to unmediated communication. A cheap talk extension of G is defined as follows. First, Nature selects a type profile t according to the probability distribution p and each player learns her own type. Then, in the communication phase, players exchange "cheap" messages as described in Section 2. Finally, in the action phase, players simultaneously choose their actions.

We denote by $S^U(G)$ the set of outcomes in a Bayesian game G induced by a sequential equilibrium of some cheap talk extension with unmediated communication. Clearly, $S^U(G)$ is included in $CE(G)$.

We say that a communication equilibrium can be implemented if it belongs to $S^U(G)$. We are ready to state our first result for games of incomplete information.

Theorem 2 *Let G be a finite Bayesian game with five or more players, and let q be a rational and regular communication equilibrium of G. Then $q \in S^U(G)$.*

The proof of Theorem 2 (presented in Appendix B) consists of two steps. In the first one, we use a result due to Forges (1990). Consider a communication equilibrium (not necessarily rational) q of a finite Bayesian game G with at least *four* players. Forges (1990) constructs a cheap talk extension $G^F(q)$ in which, first, the players receive messages from a mediator, then exchange public and private messages, and, finally, choose their actions. Notice that in $G^F(q)$ the players do not send messages to the mediator, i.e. the mediator is a correlation device. Forges (1990) shows that q is the outcome induced by a Bayesian-Nash equilibrium of $G^F(q)$. We demonstrate that if q is regular, then the equilibrium in $G^F(q)$ that implements q can be made sequential. However, communication in $G^F(q)$ is still mediated, since the mediator has to send messages to the players at the beginning of $G^F(q)$. In the second part of our proof, we use the fact that the communication equilibrium is rational and that there are at least five players. Under these assumptions, we show that the players, after learning their types, can use the communication scheme presented in Section 5 to generate the mediator's messages. In other words, we construct a finite plain cheap talk extension $\bar{G}(q)$ which starts with the communication scheme described in Section 5. At the end of it, the players exchange public and private messages as in $G^F(q)$ and, finally, choose their actions. We conclude our proof by showing that $\bar{G}(q)$ admits a sequential equilibrium, $\Phi(q)$, that induces q.

Theorem 2 does not pertain to communication equilibria that are not regular or that have some irrational components. However, these equilibria can be implemented, provided that they can be expressed as a convex combination of rational and regular communication

equilibria.[9] As in Section 2, we require the players to conduct a jointly controlled lottery among the cheap talk extensions in which the rational and regular communication equilibria are implemented.

Specifically, consider a game G with at least five players. Let q be a convex combination of K regular and rational communication equilibria $q_1, ..., q_K$, with weights $\rho_1, ..., \rho_K$, respectively. It follows from Theorem 2 that for every q_k $(k = 1, ..., K)$ there exists a plain cheap talk extension $\bar{G}(q_k)$, and a sequential equilibrium $\Phi(q_k)$ of $\bar{G}(q_k)$ that induces q_k. We construct the following cheap talk extension $\tilde{G}(q)$. At the beginning of $\tilde{G}(q)$, players P_1 and P_2 simultaneously announce to all players two positive numbers in the unit interval (P_i announces ω^i, $i = 1, 2$). Let $\omega = \chi(\omega^1, \omega^2)$, where $\chi(\omega^1, \omega^2)$ is defined in equation (2), and let ρ_0 be equal to zero. If $\omega \in \left(\sum_{j=0}^{k-1} \rho_j, \sum_{j=0}^{k} \rho_j\right]$, for some $k = 1, ..., K$, then the cheap talk extension $\bar{G}(q_k)$ is played.

The cheap talk extension $\tilde{G}(q)$ is not a finite game, since the first two players can announce any number in the interval $(0, 1]$. Moreover, after P_1 and P_2 report ω^1 and ω^2, respectively, a proper subgame is not induced, since the players have private information about their types. In general, consistent beliefs have not been defined in games with infinite strategy sets. Some authors have extended the notion of sequential equilibrium only to specific classes of infinite games.[10] However, for our purposes, it is enough to consider assessments of $\tilde{G}(q)$ in which all types of P_i, $i = 1, 2$, select ω^i at random, according to the uniform distribution. We call these assessments *simple*. For a simple assessment, it is easy to define consistent beliefs. In this case, observing ω^1 and ω^2 does not provide any information about the types of P_1 and P_2. A simple assessment is a sequential equilibrium if the strategies and the beliefs, when restricted to a game $\bar{G}(q_k)$, form a sequential equilibrium of $\bar{G}(q_k)$. Moreover, for every type of P_i, $i = 1, 2$, selecting ω^i at random according to the uniform distribution, is optimal among *all* behavioral strategies.

[9] Notice that in games without full support, the set of regular communication equilibria need not be convex.

[10] Manelli (1996), for example, considers signaling games.

Consider the simple assessment where the players play the sequential equilibrium $\Phi(q_k)$ if game $\bar{G}(q_k)$ is selected, i.e. if $\omega \in \left(\sum_{j=0}^{k-1} \rho_j, \sum_{j=0}^{k} \rho_j\right]$. The analysis at the end of Section 2 shows that for every type of P_i $(i = 1, 2)$, it is optimal to choose ω^i randomly according to the uniform distribution. Therefore, we conclude that q can be implemented.

For some games, it is possible to provide a precise characterization of the set of communication equilibria that can be implemented. We say that a game has rational parameters if for every $i = 1, ..., I$, every action profile s in S and every profile of types t in T, the payoffs $u_i(t, s)$ and the probability $p(t)$ are rational numbers. If G has rational parameters, the vertices of $CE(G)$ are rational communication equilibria. Moreover the vertices of $CE(G)$ are regular communication equilibria if G has full support (remember that in this case all communication equilibria are regular). Therefore, when G has at least five players, rational parameters and full support, every communication equilibrium can be implemented. We can summarize our findings as follows.

Corollary 2 *Let G be a finite Bayesian game with five or more players. Let q be a convex combination of rational and regular communication equilibria of G. Then $q \in S^U(G)$. If G has full support and rational parameters, $S^U(G) = CE(G)$.*

It is an open question whether it is possible to implement communication equilibria that cannot be expressed as convex combinations of rational and regular communication equilibria. As Corollary 2 suggests, a complete characterization of the set of outcomes that can be implemented with unmediated communication is not available for games without full support or with irrational parameters or with less than five players. As has already been stated, for a game G of incomplete information with at least three players, a partial characterization of $S^U(G)$ is provided by Ben-Porath (2000).

4 Mechanism Design with Imperfect Commitment

This section illustrates how our results on unmediated communication can be applied to solve a mechanism design problem. We consider a decision maker (or principal) who has to select an action, the payoff of which depends on the unknown state of the world. A number of experts (or agents) have private information about the state and are affected by the principal's decision.

We model this environment as a Bayesian game, $G_0 = \langle T_1, ..., T_{I-1}, S_I, p, u_1, ..., u_I \rangle$. The principal, player I, does not have private information and her set of actions is S_I. Players $1, ..., I-1$ are agents, and T_i denotes the set of types of agent i. We let $T = \prod_{i=1}^{I-1} T_i$ denote the set of the states of the world, and p is a probability distribution over T. As usual, we use T_{-i} to denote the set of profiles of types of agents different from i. Finally, the payoffs of each player i are described by the function $u_i : T \times S_I \to \mathbb{R}$. We assume that G_0 is finite and has full support, rational parameters and at least four agents.

The decision maker faces the problem of finding a way to elicit as much information as possible from the experts. We start the analysis by considering the case where the principal can fully commit to a mechanism. In our environment, a mechanism is any function of the form $\varkappa : \Sigma_1 \times ... \times \Sigma_{I-1} \to \Delta(S_I)$, where Σ_i denotes the set of strategies available to agent i. For every profile of agents' strategies, the mechanism specifies the probability distribution according to which the principal will select an action in S_I. Direct-revelation mechanisms constitute a very important class of mechanisms. In a direct-revelation mechanism the set of strategies of each agent i coincides with T_i, the set of her types.

A mechanism defines, in an obvious way, a game among the agents. As in the previous sections, our solution concept is sequential equilibrium. For any profile of agents' strategies the principal can compute her expected payoff. The goal of the decision maker is to find the mechanism that gives her the highest sequential equilibrium payoff.[11] The revelation

[11] Actually, when full commitment to a mechanism is possible, the solution to the principal's problem does not change if the solution concept is Bayesian-Nash equilibrium.

principle greatly simplifies this problem (see Dasgupta, Hammond and Maskin (1979), Harris and Townsend (1981) and Myerson (1979), among others). According to this powerful result, the decision maker can restrict attention to truthful equilibria[12] of direct-revelation mechanisms.

Let q denote a mapping from the set of states T to $\Delta(S_I)$, the set of probability distributions over S_I. The principal's highest payoff is associated with the truthful equilibrium of the direct-revelation mechanism q^{FC}, where q^{FC} is a solution to the following problem:

$$\max_{q:T\to\Delta(S_I)} \sum_{t\in T} \sum_{s_I\in S_I} p(t)\, q(s_I|t)\, u_I(t, s_I), \tag{4}$$

subject to:

$$\begin{gathered} \sum_{t_{-i}\in T_{-i}} \sum_{s_I\in S_I} p(t_{-i}|t_i)\, q(s_I|(t_{-i}, t_i))\, u_i((t_{-i}, t_i), s_I) \geqslant \\ \sum_{t_{-i}\in T_{-i}} \sum_{s_I\in S_I} p(t_{-i}|t_i)\, q(s_I|(t_{-i}, t'_i))\, u_i((t_{-i}, t_i), s_I), \\ i = 1, ..., I-1, \quad \forall (t_i, t'_i) \in T_i^2. \end{gathered} \tag{5}$$

The decision maker recognizes that agents need incentives to reveal their information. The constraints in inequality (5), usually called informational incentive constraints, guarantee that it is rational for every agent to report her type truthfully to the principal, provided that all the other agents behave likewise.

The game defined by the mechanism q^{FC} may have other equilibria, in which the agents do not report their types truthfully to the decision maker. When we use the revelation principle, we implicitly assume that not only does the principal choose the mechanism, but she can also designate a specific equilibrium of it. For example, the decision maker can recommend (by a public announcement) all agents to play the truthful equilibrium in the direct-revelation mechanism q^{FC} (i.e. the truthful equilibrium becomes a focal point).

However, in many situations, a decision maker cannot fully commit to a mechanism. Experts often express non-binding opinions and the principal has the power to choose any action she desires. Therefore, we now model a situation of imperfect commitment.

[12] As the name suggests, in a truthful equilibrium all agents honestly report their types.

Specifically, we assume that the decision maker can commit to a particular cheap talk extension of G_0, but she cannot commit to a strategy in the extended game. In other words, once the cheap talk extension is chosen, the principal becomes a player.[13] Further, we assume that an impartial mediator is not available, and thus, we restrict attention to unmediated communication systems.

The problem of the decision maker is to find the cheap talk extension of G_0 with the highest sequential equilibrium payoff. Bester and Strausz (2001) and Wolinsky (1999) analyze a similar problem of mechanism design with imperfect commitment. Bester and Strausz (2001) consider the case where there is only one agent, and restrict attention to cheap talk extensions in which the agent sends a costless message, and the principal selects an action. They show that in order to maximize her expected utility, the decision maker chooses the game in which the set of messages coincides with the set of types of the agent. Wolinsky (1999) considers the case of multiple agents and allows the decision maker to partition the set of experts into different groups. Agents in the same group communicate among themselves and send a joint report to the principal who, in turn, chooses an action. Wolinsky (1999) shows that to elicit more information from the experts, the principal should allow partial communication among the agents, that is, the size of a group should be greater than one but smaller than the number of agents.

The results derived in the previous section allow us to find the *optimal* mechanism for the decision maker. Since G_0 has at least four agents (i.e. there are five or more players), rational parameters and full support, the set of outcomes that are induced by sequential equilibria in cheap talk extensions of G_0 coincides with the set of communication equilibria of G_0. So, we proceed in two steps. We first compute the communication equilibrium that maximizes the decision maker's expected payoff. This simply requires maximization of the

[13] To give an example of what we mean by imperfect commitment, suppose that the principal chooses a cheap talk extension that contains an information set in which she has to send either message 0 or message 1. Full commitment to a specific behavioral strategy is not possible, but when the information set is reached the principal has only two options: she can say either 0 or 1 (she cannot send any other message or be silent).

expression (4) subject to the informational incentive constraint (5) and to the following constraint:

$$\sum_{t\in T}\sum_{s_I\in S_I} p(t)\, q(s_I|t)\left(u_I(t,s_I)-u_I(t,\delta_I(s_I))\right)\geqslant 0,\quad \forall \delta_I : S_I\to S_I. \tag{6}$$

Constraint (6) is an obedience constraint, that provides the principal with the incentive to follow the mediator's recommendation. Since G_0 has rational parameters, the maximization problem above admits a solution with rational components, which we denote by q^*. Moreover, the communication equilibrium q^* is regular, since G_0 has full support.

The second step of our procedure concerns the implementation of q^*. To do this, the decision maker selects the cheap talk extension $\overline{G_0}(q^*)$ and recommends[14] the sequential equilibrium $\Phi(q^*)$ (see Appendix B).

5 Proof of Theorem 1

In this section we prove that when a normal-form game Γ has five or more players, any rational correlated equilibrium can be implemented. Given a rational correlated equilibrium r of Γ, we construct a finite plain cheap talk extension $\bar{\Gamma}(r)$, and a sequential equilibrium $\Psi(r)$ of $\bar{\Gamma}(r)$ that induces the distribution r on S.

We first illustrate how the players can generate the probability distribution r. Since r is rational, there exists a positive integer $\tilde{m}$ (greater than one) and, for every s in S, a non-negative integer $\tilde{m}_s$ such that $r(s)=\frac{\tilde{m}_s}{\tilde{m}+1}$. Define $X=\{1,...,\tilde{m},\tilde{m}+1\}$, and let $\{X_s\}_{s\in S}$ be a partition of X such that $|X_s|=\tilde{m}_s$ for every s in S. For $i=1,...,I$, let the projection $pr_i : S\to S_i$ be defined by $pr_i(s)=s_i$ if $s=(s_1,...,s_i,...,s_I)$. We extend each projection pr_i to X as follows:

$$pr_i(x)=s_i \quad \text{if } x\in X_s \quad \text{and} \quad pr_i(s)=s_i.$$

[14] The cheap talk extension $\overline{G_0}(q^*)$ admits multiple equilibria. Similarly to the case of full commitment, we assume that the decision maker can induce all agents to play the equilibrium she prefers.

It is easy to verify that if an element $y \in X$ is randomly selected according to the uniform distribution, and if every player P_i chooses the action $pr_i(y)$, then every action profile s in S is chosen with probability $r(s)$.

Let $\Lambda(X)$ denote the set of permutations on X, and let $m+1$ denote the cardinality of $\Lambda(X)$. The sets X, $\Lambda(X)$ and the projections $pr_1, ..., pr_I$ are common knowledge among the players.

It is useful to divide the game $\bar{\Gamma}(r)$ into several steps. For each step, we first illustrate the game and then present equilibrium behavioral strategies (hereafter simply called equilibrium strategies) and equilibrium beliefs. Then, we prove that the assessment $\Psi(r)$ that we propose is sequentially rational. The proof that $\Psi(r)$ is also consistent is relegated to Appendix A.

Step 0. *Random choices.*

In this step, which consists of several substeps, players "jointly select" random permutations on X and a random element of X. We will explain later how the joint selections are made. We first list the random choices that players make. P_1, P_2, P_3, P_4, P_6, ..., P_I jointly select permutation σ.[15] P_2, P_3, P_4, P_5, P_6, ..., P_I choose two permutations, τ and β_{41}. P_2, P_3, P_4, P_5 choose permutation β_{31}. P_1, P_3, P_4, P_5, P_6, ..., P_I select two permutations, φ and β_{42}. P_1, P_3, P_4, P_5 choose permutation β_{32}. P_1, P_2, P_5 select an element $x \in X$. Finally, for $i = 1, ..., 4$, players in the set $\{P_1, P_2, P_3, P_4\} \setminus \{P_i\}$ choose permutation α_i.

We let $\mathfrak{I}_i$ denote the set of random choices known to P_i at the end of Step 0. It is convenient to summarize Step 0 in Table 4, where for every player P_i, we list the elements of the set $\mathfrak{I}_i$.

There is a separate substep for each random choice. Every choice is made according to the uniform distribution over the underlying probability space, and every choice is made independently of all others.

[15] The proof we present is valid both for the cases $I = 5$ and $I > 5$. Clearly, any reference to player P_k, with $k > 5$, is relevant only if there are more than five players.

Player	Random Choices											
P_1	σ		φ	x		α_2	α_3	α_4			β_{32}	β_{42}
P_2	σ	τ		x	α_1		α_3	α_4	β_{31}	β_{41}		
P_3	σ	τ	φ		α_1	α_2		α_4	β_{31}	β_{41}	β_{32}	β_{42}
P_4	σ	τ	φ		α_1	α_2	α_3		β_{31}	β_{41}	β_{32}	β_{42}
P_5		τ	φ	x					β_{31}	β_{41}	β_{32}	β_{42}
P_6	σ	τ	φ							β_{41}		β_{42}
$\vdots$	$\vdots$	$\vdots$	$\vdots$							$\vdots$		$\vdots$
P_I	σ	τ	φ							β_{41}		β_{42}

Table 4: Random Choices of Step 0

We now describe how the players jointly select a random permutation, or an element of X. Consider, for example, the substep in which P_1, P_2, P_3, P_4, P_6, ..., P_I have to select the random permutation σ. The two players with the lowest indices (P_1 and P_2 in this case) make two announcements simultaneously. Specifically, P_i, $i = 1, 2$, announces a permutation $\sigma^i \in \Lambda(X)$ to players in the set $\{P_1, P_2, P_3, P_4, P_6, ..., P_I\} \setminus \{P_i\}$. The chosen permutation will be $\sigma = \sigma^1\sigma^2$ (notice that σ is common knowledge among P_1, P_2, P_3, P_4, P_6, ..., P_I). In equilibrium, P_i selects a permutation σ^i at random, according to the uniform distribution on $\Lambda(X)$. This implies that the random permutation σ is uniformly distributed.

A similar procedure is used to make the remaining random choices.[16] In every substep, the two players who have to make an announcement choose their messages randomly, according to the uniform distribution on the underlying probability space, independently of the messages that they have already sent or received.

Equilibrium beliefs are very simple. At the end of Step 0, a player either knows the realization of a given random variable, or believes that the random variable is uniformly distributed.

[16] To select the random element x, P_1 and P_2 simultaneously announce $x^1 \in X$ and $x^2 \in X$, respectively. The chosen element is $x = x^1 + x^2$, where $+$ is $\mod(\tilde{m} + 1)$.

Sketch of $\bar{\Gamma}(r)$.

To provide the reader with a better understanding of our construction, let us outline the rest of the game $\bar{\Gamma}(r)$ before presenting the next steps. The state $y := \sigma\tau\varphi(x)$ determines the action profile of the original game Γ that players choose in the action phase (see below). In equilibrium, y has a uniform distribution on X since σ, τ, φ and x are uniformly distributed. Notice that no player knows the realization of y at the end of Step 0. This is crucial, since a player who is informed about y knows the actions of the game Γ that her opponents play when communication is over. However, Step i, $i = 1, ..., 4$, is carefully designed to make P_i learn $\alpha_i(y)$ and nothing more (P_5 learns $\alpha_2(y)$ in Step 2, and for $k > 5$, P_k learns $\alpha_4(y)$ in Step 4). The permutations $\alpha_1, ..., \alpha_4$ prevent a player from learning the state y. In Step 5 P_i, $i = 1, ..., 4$, learns the function $pr_i\alpha_i^{-1}$ (P_5 learns $pr_5\alpha_2^{-1}$, and for $k > 5$, P_k learns $pr_k\alpha_4^{-1}$). So player P_i learns her action $pr_i(y)$. Since the action profile $(pr_1(y), ..., pr_I(y))$ is chosen according to the correlated equilibrium distribution r, P_i has exactly the same information as in the case in which she receives recommendation $pr_i(y)$ from a mediator who implements r. In the sixth and last step, every player P_i chooses action $pr_i(y)$. Notice that $pr_i(y)$ is optimal for P_i given her information and her opponents' actions.

After Step 0, the game $\bar{\Gamma}(r)$ proceeds as follows.

Step 1. *P_2, P_3 and P_4 send P_1 the permutation $\alpha_1\sigma\tau$.*

Player P_i, $i = 2, 3, 4$, sends P_1 a permutation on X. The three senders report their messages simultaneously. Let a be a permutation on X. The meaning of message a in Step 1 is "The realization of the random permutation $\alpha_1\sigma\tau$ is a". To simplify the presentation we say that in Step 1, P_2, P_3 and P_4 send P_1 the permutation $\alpha_1\sigma\tau$ (we adopt this terminology to describe the next steps).

The equilibrium strategies prescribe that each sender reports the realization of $\alpha_1\sigma\tau$ to P_1. For example, if the realizations of the random permutations α_1, σ and τ are $\widehat{\alpha_1}$, $\widehat{\sigma}$ and

$\widehat{\tau}$, respectively, in equilibrium P_2, P_3 and P_4 send P_1 the message $\widehat{\alpha_1}\widehat{\sigma}\widehat{\tau}$.[17]

Before presenting equilibrium beliefs, let us clarify our terminology. As we will see, in Steps 1-5 of the game $\bar{\Gamma}(r)$, each message is sent by three players to a fourth one. We will often consider the message sent either by the majority of the senders, if a majority exists, or by the first sender in the description of the step (P_2 in Step 1), if the senders report three different messages. For expositional reasons, we refer to this message simply as the message sent by the majority of the senders. Moreover, we occasionally refer to the message sent by the majority as the message that the receiver "learns". Finally, we let w_j denote a generic triple of messages sent in Step j, and we use z_j^i to denote the message that P_i sends in Step j.

We are now ready to describe equilibrium beliefs. At the end of Step 1, P_1 assigns probability one to the event that the realization of $\alpha_1\sigma\tau$ coincides with the message sent by the majority of the senders (we denote this message by $\overline{\alpha_1\sigma\tau}$). For any triple of messages w_1 we have:

$$\Pr\left(\alpha_1\sigma\tau = \overline{\alpha_1\sigma\tau}|w_1, \mathfrak{I}_1\right) = 1.$$

This result is trivial if the three senders follow their equilibrium strategies and report the same message. To obtain the same result off the equilibrium path, in Appendix A we construct a sequence of completely mixed strategies that satisfies two requirements. First, P_2 is less likely to deviate than P_3 and P_4. Second, the probability that any two senders deviate converges to zero faster than the probability that only the third sender deviates. These two assumptions guarantee that in the limit (i.e. as the probability of deviating converges to zero) the majority of the senders report the true message with probability one.

In Appendix A, we also show that, conditional on P_1's information, both α_1 and τ are uniformly distributed over $\Lambda(X)$. Intuitively, knowing the realization of $\alpha_1\sigma\tau$ does not make any realization of τ more likely than the others. Every realization of τ is made

[17] We adopt the following notation: o for a random variable and $\widehat{o}$ for its realization.

compatible with the realization of $\alpha_1\sigma\tau$ by one (and only one) realization of α_1. Since α_1 and τ are independent and uniformly distributed, the marginal distribution of τ, given $\alpha_1\sigma\tau$, is uniform. This, in turn, implies that the conditional distribution of the state $y = \sigma\tau\varphi(x)$ is uniform over X. In other words, P_1 does not learn anything new about y.

Our equilibrium beliefs do not say anything about β_{31} and β_{41}, two random permutations unknown to P_1. To keep notation simple, we adopt the convention that the equilibrium beliefs specified in Step 0 hold when we are silent about a random variable. Thus, conditionally on P_1's information, both β_{31} and β_{41} have a uniform distribution over $\Lambda(X)$.

Notice that P_1 knows (from Step 0) the permutation φ and the element x. On the equilibrium path, she receives the realization of the permutation $\alpha_1\sigma\tau$ (from all senders) and therefore learns $\alpha_1\sigma\tau\varphi(x) = \alpha_1(y)$.

Step 2.

In this step, P_5 and P_2 learn $\alpha_2(y)$. Since P_2 does not know the permutation φ, she receives the permutation $\alpha_2\sigma\tau\varphi$. This message should be sent by three senders. However, only two players, P_3 and P_4, know the realizations of σ, τ and φ. We divide Step 2 into two substeps: 2.1 and 2.2. In the first substep, P_5 (who knows τ and φ) receives the permutation $\alpha_2\sigma$ from P_1, P_3 and P_4. Notice that P_5 does not know α_2 and therefore does not learn anything about σ. In the second substep, P_3, P_4 and P_5 send the permutation $\alpha_2\sigma\tau\varphi$ to P_2.

Substep 2.1. *P_1, P_3 and P_4 send P_5 the permutation $\alpha_2\sigma$.*

In equilibrium, P_1, P_3 and P_4 report the realization of $\alpha_2\sigma$ to P_5. This means that P_1, P_3 and P_4 send the true message in *any* information set of Substep 2.1. As far as equilibrium beliefs are concerned, they are very similar to those derived in Step 1. Specifically, P_5 assigns probability one to the event that the majority of the senders sent the true message. We let $\overline{\alpha_2\sigma}$ denote the permutation sent by the majority. For any triple of messages w_{21} received by P_5, we have:

$$\Pr(\alpha_2\sigma = \overline{\alpha_2\sigma} | w_{21}, \mathfrak{I}_5) = 1.$$

Moreover, conditional on the information of P_5 in Substep 2.1, both α_2 and σ have a uniform distribution over $\Lambda(X)$.

P_5 knows the random permutations τ and φ, and the random element x. Since in equilibrium P_5 learns $\alpha_2\sigma$, she can compute $\alpha_2\sigma\tau\varphi(x) = \alpha_2(y)$.

Substep 2.2. *P_3, P_4 and P_5 send P_2 the permutation $\alpha_2\sigma\tau\varphi$.*

The equilibrium strategies prescribe that P_3 and P_4 report the realization of $\alpha_2\sigma\tau\varphi$ to P_2. In equilibrium, P_5 sends P_2 the permutation $\overline{\alpha_2\sigma}\tau\varphi$.

As usual, we denote the message sent by the majority of the senders to P_2 by $\overline{\alpha_2\sigma\tau\varphi}$. In Substep 2.2, P_2 knows z_1^2, the message that she sent in Step 1. For any message z_1^2 and for any triple of messages w_{22} received in Substep 2.2, P_2's beliefs are given by:[18]

$$\Pr\left(\alpha_2\sigma\tau\varphi = \overline{\alpha_2\sigma\tau\varphi}, \overline{\alpha_2\sigma} = \alpha_2\sigma | w_{22}, z_1^2, \mathfrak{I}_2\right) = 1.$$

With probability one, the realization of the random permutation $\alpha_2\sigma\tau\varphi$ is the message sent by the majority of the senders. Further, independently of the triple of messages received, P_2 assigns probability one to the event that P_5 learnt the realization of $\alpha_2\sigma$ in Substep 2.1. In Appendix A, we also show that, conditional on P_2's information, both φ and α_2 are uniformly distributed over $\Lambda(X)$. This implies that the conditional distribution of the random state y is uniform over X.

P_2 knows the realization of x, so she can compute $\overline{\alpha_2\sigma\tau\varphi}(x)$. In equilibrium, $\overline{\alpha_2\sigma\tau\varphi} = \alpha_2\sigma\tau\varphi$, i.e. P_2 learns the realization of $\alpha_2\sigma\tau\varphi$. Thus, at the end of Step 2, P_2 knows $\alpha_2\sigma\tau\varphi(x) = \alpha_2(y)$.

Step 3.

This step is designed to make P_3 learn the random element $\alpha_3(y)$. Since no player knows y, we need some preliminary steps in which the senders of the message to P_3 learn

[18] Note that from P_5's point of view, $\overline{\alpha_2\sigma}$ is a degenerate random permutation (P_5 knows the message that the majority of her senders report in Substep 2.1). In Substep 2.2, P_2 does not know the realization of $\overline{\alpha_2\sigma}$. Technically speaking, $\overline{\alpha_2\sigma}$ in Substep 2.1 is a different object from $\overline{\alpha_2\sigma}$ in Substep 2.2. However, since the meaning of $\overline{\alpha_2\sigma}$ in every step is clear and no ambiguity arises, we decide not to introduce further notation. We follow this convention in the rest of the section.

the necessary information. Specifically, in Substep 3.1, P_5 receives the permutation $\alpha_3\sigma$ from P_1, P_2 and P_4. Notice that P_5 does not know α_3 and so she does not learn anything new about σ. In Substep 3.2, P_2, P_4 and P_5 send player P_1 the permutation $\beta_{31}\alpha_3\sigma\tau$. The permutation β_{31} is unknown to P_1 and is used to prevent her from learning the realization of τ. Then, in Substep 3.3, P_2 receives the permutation $\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi$ from P_1, P_4 and P_5. The fact that P_2 does not know β_{32} implies that she does not get any information about the unknown permutation φ. Finally, in Substep 3.4, P_1, P_2 and P_5 send the element $\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi(x)$ to P_3. Notice that P_3 knows both β_{32} and β_{31}. Therefore, she can compute $\beta_{31}^{-1}\beta_{32}^{-1}(\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi(x)) = \alpha_3(y)$.

Substep 3.1. *P_1, P_2 and P_4 send P_5 the permutation $\alpha_3\sigma$.*

The equilibrium strategies prescribe that the three senders report the realization of $\alpha_3\sigma$. The message sent by the majority is denoted by $\overline{\alpha_3\sigma}$. At the end of Substep 3.1, P_5 knows the random choices in $\mathfrak{I}_5$, the message that she sent in Substep 2.2, z_{22}^5, and the triples of messages received in Substeps 2.1 and 3.1, w_{21} and w_{31}. Given this information, P_5's beliefs are given by:

$$\Pr\left(\alpha_2\sigma = \overline{\alpha_2\sigma}, \alpha_3\sigma = \overline{\alpha_3\sigma} | w_{21}, w_{31}, z_{22}^5, \mathfrak{I}_5\right) = 1.$$

According to these beliefs, the majority of the senders report the true message in Substep 3.1. Further, the additional information that P_5 obtains in Substep 3.1 does not modify her beliefs from Substep 2.1. In Appendix A, we also demonstrate that in this step, P_5 does not learn anything about the state y. The conditional distribution of σ is uniform.

Substep 3.2. *P_2, P_4 and P_5 send P_1 the permutation $\beta_{31}\alpha_3\sigma\tau$.*

In equilibrium, P_2 and P_4 send the realization of $\beta_{31}\alpha_3\sigma\tau$. P_5 sends the permutation $\beta_{31}\overline{\alpha_3\sigma}\tau$. We let $\overline{\beta_{31}\alpha_3\sigma\tau}$ denote the message sent by the majority of the senders to P_1. Equilibrium beliefs are given by:

$$\Pr\left(\beta_{31}\alpha_3\sigma\tau = \overline{\beta_{31}\alpha_3\sigma\tau}, \overline{\alpha_3\sigma} = \alpha_3\sigma, \alpha_1\sigma\tau = \overline{\alpha_1\sigma\tau} | w_{32}, w_1, z_{31}^1, z_{21}^1, \mathfrak{I}_1\right) = 1.$$

First, with probability one, the message sent by the majority is the realization of the

random permutation $\beta_{31}\alpha_3\sigma\tau$. Second, P_5 learnt the realization of $\alpha_3\sigma$ in Substep 3.1 with probability one. Moreover, the additional information that P_1 gets in Substep 3.2 does not modify her beliefs from Step 1. Finally, the proof in Appendix A shows that P_1 does not update her beliefs about the marginal distributions of τ, α_1 and β_{31} (conditional on P_1's information, these random permutations are uniformly distributed).

Substep 3.3. *P_4, P_1 and P_5 send P_2 the permutation $\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi$.*

The equilibrium strategy of P_4 is to send P_2 the realization of the random permutation $\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi$ (notice that P_4 knows this permutation). Moreover, in equilibrium, P_1 reports the realization of $\beta_{32}\overline{\beta_{31}\alpha_3\sigma\tau}\varphi$ and P_5 sends the realization of $\beta_{32}\beta_{31}\overline{\alpha_3\sigma}\tau\varphi$.

Equilibrium beliefs are similar to those described in Substep 3.2. Specifically, P_2 assigns probability one to the event that the realization of the permutation $\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi$ coincides with $\overline{\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi}$, the message sent by the majority of the senders. Moreover, with probability one, both P_5 and P_1 learnt the truth in Substeps 3.1 and 3.2. Finally, equilibrium beliefs derived in Substep 2.2 still hold. Formally, we have:

$$\begin{gathered}\Pr(\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi = \overline{\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi}, \overline{\beta_{31}\alpha_3\sigma\tau} = \beta_{31}\alpha_3\sigma\tau, \overline{\alpha_3\sigma} = \alpha_3\sigma, \\ \alpha_2\sigma\tau\varphi = \overline{\alpha_2\sigma\tau\varphi}, \overline{\alpha_2\sigma} = \alpha_2\sigma | w_{33}, w_{22}, z_{32}^2, z_{31}^2, z_1^2, \mathfrak{I}_2) = 1.\end{gathered}$$

The proof that the assessment $\Psi(r)$ is consistent also shows that P_2 does not obtain any information about the state y. The conditional distribution of the permutation φ is uniform.

Substep 3.4. *P_1, P_2 and P_5 send P_3 the element $\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi(x)$.*

P_1, P_2 and P_5 simultaneously send P_1 an element of the set X. In equilibrium, P_1 sends the message $\beta_{32}\overline{\beta_{31}\alpha_3\sigma\tau}\varphi(x)$, P_2 reports the message $\overline{\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi}(x)$ and P_5 chooses the element $\beta_{32}\beta_{31}\overline{\alpha_3\sigma}\tau\varphi(x)$.

The message sent by the majority of the senders is denoted by $\overline{\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi(x)}$. P_3's beliefs are given by:

$$\begin{gathered}\Pr(\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi(x) = \overline{\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi(x)}, \overline{\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi} = \beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi, \\ \overline{\beta_{31}\alpha_3\sigma\tau} = \beta_{31}\alpha_3\sigma\tau, \overline{\alpha_3\sigma} = \alpha_3\sigma | w_{34}, z_1^3, z_{21}^3, z_{22}^3, \mathfrak{I}_3) = 1.\end{gathered}$$

According to equilibrium beliefs, the message sent by the majority coincides with the realization of $\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi(x)$. Further, each sender knows the realization of the random permutation $\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi$. In Appendix A, we also show that, conditional on the information that P_3 has in Substep 3.4, α_3 and x are uniformly distributed on the underlying probability spaces. This, in turn, implies that the conditional distribution of the state y is uniform (in other words, P_3 learns $\alpha_3(y)$ but nothing more).

Step 4.

We provide a brief description of this step, since it is almost identical to the previous one. The purpose of Step 4 is to let P_4 (and $P_6, ...P_I$, if $I > 5$) learn the realization of the random element $\alpha_4(y)$.

In Substep 4.1, P_1, P_2 and P_3 send P_5 the permutation $\alpha_4\sigma$. In Substep 4.2, P_1 receives the permutation $\beta_{41}\alpha_4\sigma\tau$ from P_2, P_3 and P_5. Then, in Substep 4.3, P_3, P_1 and P_5 report the permutation $\beta_{42}\beta_{41}\alpha_4\sigma\tau\varphi$ to P_2. In Substep 4.4, P_1, P_2 and P_5 send P_4 the element $\beta_{42}\beta_{41}\alpha_4\sigma\tau\varphi(x)$. P_4 knows the permutations β_{41} and β_{42}, and so she learns $\alpha_4(y)$. As usual, $\overline{\beta_{42}\beta_{41}\alpha_4\sigma\tau\varphi(x)}$ denotes the message sent by the majority of the senders to P_4 in Substep 4.4.

Step 4 ends here if there are exactly five players, otherwise it continues as follows. Consider $k = 6, ..., I$. In Substep 4.$(k-1)$, P_1, P_2 and P_5 send P_k the element $\beta_{42}\beta_{41}\alpha_4\sigma\tau\varphi(x)$.[19] The message sent by the majority of the senders to P_k is denoted by $\overline{\beta_{42}\beta_{41}\alpha_4\sigma\tau\varphi(x)}^k$.

The equilibrium strategies of Step 4 are identical to the equilibrium strategies specified in Step 3. Every sender behaves sincerely, and uses the message sent by the majority of her senders when she does not know a random permutation.

According to equilibrium beliefs, the realization of the permutation that a player receives coincides with the message sent by the majority. Each receiver assigns probability one to the event that all her senders learnt the exact realization of the permutation that they report. Moreover, the information that a player obtains in Step 4 does not modify her

[19] Then P_k can use the permutations β_{41} and β_{42} to compute $\alpha_4(y)$.

former beliefs. Finally, the conditional distribution of each random variable is uniform on the underlying probability space.

Step 5.

At this point of the game, every player knows an element of the set X. However, this element alone does not reveal any information about the state y. In this step every player P_i, $i = 1, ..., I$, learns which action in the set S_i corresponds to y. We divide Step 5 into the following substeps.

Substep 5.1. *P_2, P_3 and P_4 send P_1 the function $pr_1\alpha_1^{-1}$.*

In this step, P_2, P_3 and P_4 simultaneously send three messages to P_1. The (finite) set of feasible messages is denoted by R_1 and contains any mapping from X to S_1 that can be generated by applying the function pr_1 to some permutation on X (i.e. a message is feasible if and only if it can be expressed as $pr_1\varsigma$, where ς is a permutation on X).

The equilibrium strategies prescribe that each sender sends P_1 the function $pr_1\alpha_1^{-1}$. Note that all senders know the realization of the random permutation α_1, and the projection pr_1 is common knowledge among the players.

As usual, according to equilibrium beliefs, the message sent by the majority (which we denote by $\overline{pr_1\alpha_1^{-1}}$) coincides with the realization of the random function $pr_1\alpha_1^{-1}$. Moreover, P_1 does not modify her former beliefs. Formally, let $\mathfrak{M}_1$ denote the sequence of messages sent and received by P_1 in Steps 1-5.1. We have:

$$\Pr(pr_1\alpha_1^{-1} = \overline{pr_1\alpha_1^{-1}}, \beta_{41}\alpha_4\sigma\tau = \overline{\beta_{41}\alpha_4\sigma\tau}, \overline{\alpha_4\sigma} = \alpha_4\sigma,$$
$$\beta_{31}\alpha_3\sigma\tau = \overline{\beta_{31}\alpha_3\sigma\tau}, \overline{\alpha_3\sigma} = \alpha_3\sigma, \alpha_1\sigma\tau = \overline{\alpha_1\sigma\tau}|\mathfrak{M}_1, \mathfrak{I}_1) = 1.$$

Note that not all permutations on X are compatible[20] with the message $\overline{pr_1\alpha_1^{-1}}$. Thus, at the end of this step, P_1 assigns probability zero to some realizations of α_1. This, combined with the fact that P_1 knows the realization of $\alpha_1\sigma\tau$ and σ, implies that some realizations of the random permutation τ have zero probability. Therefore, the conditional distribution

[20] A permutation ς on X is compatible with message $\overline{pr_1\alpha_1^{-1}}$ if $pr_1\varsigma^{-1} = \overline{pr_1\alpha_1^{-1}}$.

of the state y cannot be uniform over X anymore. We will come back to this point when we show that the assessment $\Psi(r)$ is sequentially rational.

The rest of Step 5 is similar to Substep 5.1. Specifically, consider $i = 2, 3, 4$. In Substep 5.i, players in the set $\{P_1, P_2, P_3, P_4\} \setminus \{P_i\}$ send P_i the function $pr_i\alpha_i^{-1}$. In Substep 5.5, P_1, P_3 and P_4 send P_5 the function $pr_5\alpha_2^{-1}$. If there are more than five players, P_1, P_2 and P_3 send P_k the function $pr_k\alpha_4^{-1}$ in Substep 5.k (where $k = 6, ..., I$). Sets of feasible messages, equilibrium strategies and equilibrium beliefs are similar to those described in Substep 5.1.

Step 6. *The game* Γ *is played.*

In Step 6 all players simultaneously choose an action and then the game $\bar{\Gamma}(r)$ ends. P_i's set of (pure) strategies in Step 6 is S_i, her set of actions in the original game.

Equilibrium strategies are formally described in Table 5. Roughly speaking, to choose an action, a player applies the projection function learned in Step 5 to the element of X computed before Step 5. Consider, for example, P_1. In Step 1, she receives the permutation $\overline{\alpha_1\sigma\tau}$ from the majority of the senders. P_1 knows (from Step 0) the realizations of φ and x and computes the element $\overline{\alpha_1\sigma\tau}\varphi(x)$. In Substep 5.1, the majority of the senders send P_1 the function $\overline{pr_1\alpha_1^{-1}}$. Her equilibrium strategy in Step 6 is to choose the action $\overline{pr_1\alpha_1^{-1}}\overline{\alpha_1\sigma\tau}\varphi(x)$. The other players adopt similar strategies (see Table 5). Notice that the equilibrium strategy of a player in Step 6 does not depend on the messages that she sends in Steps 1-5.

Sequential rationality.

We now demonstrate that the assessment $\Psi(r)$ is sequentially rational. In particular, we restrict attention to player P_1 and show that, given her beliefs, she does not have profitable deviations. Our proof can be easily applied to any other player. Given the so-called "one-shot-deviation principle", we only need to verify that deviations in a single information set are not profitable for P_1.

Player	Equilibrium strategies
P_1	$\overline{pr_1\alpha_1^{-1}}\overline{\alpha_1\sigma\tau}\varphi(x)$
P_2	$\overline{pr_2\alpha_2^{-1}}\overline{\alpha_2\sigma\tau\varphi}(x)$
P_3	$\overline{pr_3\alpha_3^{-1}\beta_{31}^{-1}\beta_{32}^{-1}\overline{\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi}(x)}$
P_4	$\overline{pr_4\alpha_4^{-1}\beta_{41}^{-1}\beta_{42}^{-1}\overline{\beta_{42}\beta_{41}\alpha_4\sigma\tau\varphi}(x)}$
P_5	$\overline{pr_5\alpha_2^{-1}}\overline{\alpha_2\sigma}\tau\varphi(x)$
P_6	$\overline{pr_6\alpha_4^{-1}\beta_{41}^{-1}\beta_{42}^{-1}\overline{\beta_{42}\beta_{41}\alpha_4\sigma\tau\varphi}(x)}^6$
$\vdots$	
P_I	$\overline{pr_I\alpha_4^{-1}\beta_{41}^{-1}\beta_{42}^{-1}\overline{\beta_{42}\beta_{41}\alpha_4\sigma\tau\varphi}(x)}^I$

Table 5: Equilibrium Strategies in Step 6

Consider an information set in Step 6. Given P_1's beliefs, we may say that she *knows*[21] the realization of $\alpha_1\sigma\tau$ and the realizations of the random variables in the set $\widetilde{\mathfrak{I}}_1$, where

$$\widetilde{\mathfrak{I}}_1 = \left\{\sigma, \varphi, x, \alpha_2, \alpha_3, \alpha_4, \beta_{32}, \beta_{42}, \beta_{31}\alpha_3\sigma\tau, \beta_{41}\alpha_4\sigma\tau, pr_1\alpha_1^{-1}\right\}.$$

Clearly, this implies that P_1 knows the realization of $pr_1(y)$, which we denote by s_1. Notice that y is independent of $\widetilde{\mathfrak{I}}_1$, since the random permutation τ is independent of $\widetilde{\mathfrak{I}}_1$.

In any information set of Step 6, P_1 assigns probability one to the event that every player P_k, $k = 2, ..., I$, chooses the action $pr_k(y)$. We now compute the probability that P_1's opponents play the action profile $(s_2, ..., s_I)$ given her information. We have:

$$\begin{array}{c}\Pr\left(pr_2(y) = s_2, ..., pr_I(y) = s_I | pr_1(y) = s_1, \widetilde{\mathfrak{I}}_1, \alpha_1\sigma\tau\right) = \\ \frac{\Pr\left(pr_1(y)=s_1, pr_2(y)=s_2, ..., pr_I(y)=s_I, \widetilde{\mathfrak{I}}_1, \alpha_1\sigma\tau\right)}{\Pr\left(pr_1(y)=s_1, \widetilde{\mathfrak{I}}_1, \alpha_1\sigma\tau\right)} = \\ \frac{\Pr\left(pr_1(y)=s_1, pr_2(y)=s_2, ..., pr_I(y)=s_I, \widetilde{\mathfrak{I}}_1\right)}{\Pr\left(pr_1(y)=s_1, \widetilde{\mathfrak{I}}_1\right)} = \\ \frac{\Pr(pr_1(y)=s_1, pr_2(y)=s_2, ..., pr_I(y)=s_I)}{\Pr(pr_1(y)=s_1)} = r(s_2, ..., s_I | s_1),\end{array} \tag{7}$$

where r is the correlated equilibrium distribution that we want to implement. The third equality follows from y being independent of $\widetilde{\mathfrak{I}}_1$, and the last equality comes from the fact that y is uniformly distributed over X. To explain the second equality, we need to

[21] In fact, P_1 assigns probability one to the event that the messages sent by the majority coincide with the realizations of the corresponding permutations.

introduce additional notation. Let $\widehat{\varphi}$, $\widehat{x}$ and $\widehat{pr_1\alpha_1^{-1}}$ denote the realizations of φ, x and $pr_1\alpha_1^{-1}$ respectively. We say that the permutation $\widehat{\alpha_1\sigma\tau}$ on X is compatible with $\widetilde{\mathfrak{I}}_1$ and s_1 if $\widehat{pr_1\alpha_1^{-1}}\widehat{\alpha_1\sigma\tau}\widehat{\varphi}(\widehat{x}) = s_1$. We let Q denote the number of realizations of $\alpha_1\sigma\tau$ that are compatible with $\widetilde{\mathfrak{I}}_1$ and s_1. Then it is easy to show that for any compatible permutation $\widehat{\alpha_1\sigma\tau}$, the following holds:

$$\Pr\left(pr_1(y) = s_1, pr_2(y) = s_2, ..., pr_I(y) = s_I, \widetilde{\mathfrak{I}}_1, \alpha_1\sigma\tau = \widehat{\alpha_1\sigma\tau}\right) = \frac{1}{Q}\Pr\left(pr_1(y) = s_1, pr_2(y) = s_2, ..., pr_I(y) = s_I, \widetilde{\mathfrak{I}}_1\right),$$

and

$$\Pr\left(pr_1(y) = s_1, \widetilde{\mathfrak{I}}_1, \alpha_1\sigma\tau = \widehat{\alpha_1\sigma\tau}\right) = \frac{1}{Q}\Pr\left(pr_1(y) = s_1, \widetilde{\mathfrak{I}}_1\right).$$

The two equalities above show that all realizations of $\alpha_1\sigma\tau$ compatible with $\widetilde{\mathfrak{I}}_1$ and s_1 are equally likely.

According to equation (7), P_1 has the same information as in the case where she receives recommendation s_1 from a reliable mediator who implements the correlated equilibrium r. Thus, the action s_1 maximizes P_1's expected payoff. In other words, the equilibrium strategy of Step 6 is optimal for player P_1.

Consider now any P_1's information set in Steps 1-5. Remember that P_1 assigns probability one to the event that all senders know the realization of the permutation that they report. Since every receiver follows the message sent by the majority of the senders, and since P_1's action in Step 6 does not depend on her messages in Steps 1-5, we conclude that a deviation is not profitable.

Finally, we examine Step 0. Consider, for example, the substep in which P_1 and P_2 send two messages (σ^1 and σ^2, respectively) to determine the permutation $\sigma = \sigma^1\sigma^2$. Since σ^2 is uniformly distributed, the permutation σ is independent of σ^1 and uniformly distributed. Any strategy is optimal in this substep for P_1, including the uniform randomization over the set $\Lambda(X)$. Clearly, a similar argument can be used to show that P_1 does not have profitable deviations in any other substep of Step 0. We conclude that the assessment $\Psi(r)$ is sequentially rational.

6 Conclusion

In this paper we characterize the outcomes of static games that sequentially rational players can implement with direct communication. We show that for a large class of games there is no difference between mediated and unmediated communication, in the sense that both forms of communication allow players to implement the same outcomes. Specifically, if a game has five or more players, rational parameters and full support, then the set of outcomes that can be implemented with unmediated communication coincides with the set of communication equilibria.

We use sequential equilibrium to analyze cheap talk extensions of a static game. However, our results provide a complete characterization of the effects of unmediated communication even for the case in which a solution concept weaker than sequential equilibrium, but stronger than Nash equilibrium (such as subgame-perfect equilibrium or perfect Bayesian equilibrium), is considered. In fact, by using a solution concept weaker than sequential equilibrium, one can always implement all the outcomes than we implement in this paper. If the solution concept is stronger than Nash equilibrium, only communication equilibria can be implemented. Thus, if a game satisfies our assumptions (five or more players, rational parameters and full support), the set of communication equilibria coincides with the set of outcomes that are induced by perfect Bayesian or subgame-perfect equilibria of cheap talk extensions with unmediated communication.

Future research is needed to obtain a complete characterization of the effects of direct communication in games that do not satisfy our assumptions. Moreover, we restrict our attention to static games. For extensive-form games, two important articles deal with mediated communication. Forges (1986) extends the notion of correlated and communication equilibria to multistage games. Myerson (1986) introduces a sequential rationality criterion in the context of multistage games with mediated communication. An interesting extension of this paper would be the analysis of unmediated communication in dynamic games.

Appendix A: Consistency of the Assessment $\Psi(r)$

In this appendix we show that the assessment $\Psi(r)$ presented in Section 5 is consistent. We construct a sequence of completely mixed strategies that converges to the equilibrium strategies. We compute beliefs along the sequence and show that, in the limit, they coincide with equilibrium beliefs. We proceed as in Section 5 and divide our analysis into different steps.

As far as Step 0 is concerned, we use the equilibrium strategies of Step 0 to construct a (constant) sequence of completely mixed strategies (in Step 0 all strategies are played with positive probability). This implies that, along the sequence, each random variable chosen in Step 0 has a uniform distribution on the underlying probability space. The assessment $\Psi(r)$ trivially satisfies consistency in Step 0.[22]

Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in the unit interval, converging to zero.

Step 1. *P_2, P_3 and P_4 send P_1 the permutation $\alpha_1\sigma\tau$.*

For any integer n, let $\varepsilon_{1,n} = \varepsilon_n^3$. We assume that in Step 1, P_2 follows her equilibrium strategy (and sends P_1 the realization of $\alpha_1\sigma\tau$) with probability $1-\varepsilon_{1,n}^3$. To simplify the notation, we drop the subscript n and say that P_2 reports the truth with probability $1-\varepsilon_1^3$. P_3 and P_4 send the true message with probability $1-\varepsilon_1^2$. All messages different from the realization of $\alpha_1\sigma\tau$ are equally likely. Suppose that the realization of the permutation $\alpha_1\sigma\tau$ is $\widehat{\alpha_1}\widehat{\sigma}\widehat{\tau}$. Along the sequences of completely mixed strategies, P_2 sends the message $\widehat{\alpha_1}\widehat{\sigma}\widehat{\tau}$ with probability $1-\varepsilon_1^3$ and any other message in $\Lambda(X)$ with probability $\frac{\varepsilon_1^3}{m}$. P_3 and P_4 send the message $\widehat{\alpha_1}\widehat{\sigma}\widehat{\tau}$ with probability $1-\varepsilon_1^2$, and any other message with probability $\frac{\varepsilon_1^2}{m}$. We refer to these completely mixed strategies by saying that P_2 trembles from her equilibrium strategy with probability ε_1^3 and P_3 and P_4 tremble from their equilibrium strategies with probability ε_1^2. Trembles are independent across information sets.

We list the messages that a player receives in the order in which the senders appear in

[22] Remember that at the end of Step 0, a player either knows the realization of a random variable or believes that the random variable is uniformly distributed.

the description of the step. For example, suppose that a, b and c are three permutations on X. In Step 1, the triple (abc) means that the messages sent by P_2, P_3 and P_4 are a, b and c, respectively. At the end of Step 1, P_1 faces one of the following five situations:

- (aaa), which means that P_2, P_3 and P_4 sent the same message, a;
- (aab), (aba) and (baa) : in these three situations two players sent the same message, a, and one player sent a different message, b;
- (abc) : the senders reported three different messages.

We let $M = \{(aaa), (aab), (aba), (baa), (abc)\}$ denote the set of triples of messages in which a is the message sent by the majority of the senders. Suppose that the realization of σ is $\widehat{\sigma}$ (note that P_1 knows $\widehat{\sigma}$). Let $\widehat{\tau}^1, ..., \widehat{\tau}^{m+1}$ be $m+1$ different permutations on X. Then for every $l = 1, ..., m+1$, there exists a permutation $\widehat{\alpha_1}^l$ in $\Lambda(X)$ such that $\widehat{\alpha_1}^l \widehat{\sigma}\widehat{\tau}^l = a$. Moreover, $l \neq l'$ implies $\widehat{\alpha_1}^l \neq \widehat{\alpha_1}^{l'}$. We let $\Pr^n\left(\alpha_1 = \widehat{\alpha_1}^l, \tau = \widehat{\tau}^l | w_1, \sigma = \widehat{\sigma}\right)$ denote the probability that $\alpha_1 = \widehat{\alpha_1}^l$ and $\tau = \widehat{\tau}^l$ given that P_1 receives the triple of messages w_1, the realization of σ is $\widehat{\sigma}$ and the trembles of P_2, P_3 and P_4 are defined by $\varepsilon_{1,n}$.

We have to show that it satisfies consistency to assign probability one to the event that the message sent by the majority of the senders to P_1 is the realization of $\alpha_1\sigma\tau$. Further, conditional on P_1's information in Step 1, both α_1 and τ are uniformly distributed over $\Lambda(X)$. To prove our claims it is enough to show that:

$$\lim_{n\to\infty} \Pr^n\left(\alpha_1 = \widehat{\alpha_1}^l, \tau = \widehat{\tau}^l | w_1, \sigma = \widehat{\sigma}\right) = \frac{1}{m+1},$$

for any triple of messages w_1 in M, and every $l = 1, ..., m+1$.[23] So, let us first assume that

[23] Note that P_1 changes her beliefs about the joint distribution of α_1 and τ. After Step 0, P_1 believes that the pair of random permutations (α_1, τ) is uniformly distributed on $\Lambda(X) \times \Lambda(X)$. On the other hand, at the end of Step 1, P_1 assigns probability $\frac{1}{m+1}$ only to the pairs of permutations $\left(\widehat{\alpha_1}^l, \widehat{\tau}^l\right)$, $l = 1, ..., m+1$. However, this does not modify P_1's beliefs about the marginal distribution of τ (which is uniform over $\Lambda(X)$).

P_1 receives the triple of messages (aaa) (this represents the situation that P_5 faces on the equilibrium path). By applying Bayes' rule, we obtain:

$$\Pr^n\left(\alpha_1=\widehat{\alpha_1}^l,\tau=\widehat{\tau}^l|\,(aaa),\sigma=\widehat{\sigma}\right)=\frac{\left(1-\varepsilon_1^3\right)\left(1-\varepsilon_1^2\right)^2}{(m+1)\left[\left(1-\varepsilon_1^3\right)\left(1-\varepsilon_1^2\right)^2+m\frac{\varepsilon_1^7}{m^3}\right]}. \tag{8}$$

Suppose now that P_1 observes (aab) or (aba). In both cases we have:

$$\begin{aligned}\Pr^n\left(\alpha_1=\widehat{\alpha_1}^l,\tau=\widehat{\tau}^l|\,(aab),\sigma=\widehat{\sigma}\right)&=\Pr^n\left(\alpha_1=\widehat{\alpha_1}^l,\tau=\widehat{\tau}^l|\,(aba),\sigma=\widehat{\sigma}\right)\\&=\frac{\left(1-\varepsilon_1^3\right)\left(1-\varepsilon_1^2\right)\frac{\varepsilon_1^2}{m}}{(m+1)\left[\left(1-\varepsilon_1^3\right)\left(1-\varepsilon_1^2\right)\frac{\varepsilon_1^2}{m}+\left(1-\varepsilon_1^2\right)\frac{\varepsilon_1^5}{m^2}+(m-1)\frac{\varepsilon_1^7}{m^3}\right]}.\end{aligned} \tag{9}$$

When P_1 observes the messages (baa) her beliefs are given by:

$$\Pr^n\left(\alpha_1=\widehat{\alpha_1}^l,\tau=\widehat{\tau}^l|\,(baa),\sigma=\widehat{\sigma}\right)=\frac{\left(1-\varepsilon_1^2\right)^2\frac{\varepsilon_1^3}{m}}{(m+1)\left[\left(1-\varepsilon_1^2\right)^2\frac{\varepsilon_1^3}{m}+\left(1-\varepsilon_1^3\right)\frac{\varepsilon_1^4}{m^2}+(m-1)\frac{\varepsilon_1^7}{m^3}\right]}. \tag{10}$$

Finally, for the case (abc), we have:

$$\Pr^n\left(\alpha_1=\widehat{\alpha_1}^l,\tau=\widehat{\tau}^l|\,(abc),\sigma=\widehat{\sigma}\right)=\frac{\left(1-\varepsilon_1^3\right)\frac{\varepsilon_1^4}{m}}{(m+1)\left[\left(1-\varepsilon_1^3\right)\frac{\varepsilon_1^4}{m}+2\left(1-\varepsilon_1^2\right)\frac{\varepsilon_1^5}{m^2}+(m-2)\frac{\varepsilon_1^7}{m^3}\right]}. \tag{11}$$

As n goes to infinity, the expressions in equations (8)-(11) converge to $\frac{1}{m+1}$.

To ease the presentation in the next steps, we introduce some notation. Consider a triple of messages w in the set M. We let $g\left(\varepsilon,\overline{m};w\right)$ denote the value of the numerator of the expression above corresponding to the triple w, when ε_1 is equal to ε and m is equal to $\overline{m}$. For example, $g\left(\varepsilon,\overline{m};(baa)\right)$ is equal to $\left(1-\varepsilon^2\right)^2\frac{\varepsilon^3}{\overline{m}}$. Further, we let $v\left(\varepsilon,\overline{m};w\right)$ denote the value of the denominator of the expression corresponding to the triple w when ε_1 is equal to ε and m is equal to $\overline{m}$. Note that for any triple of messages w in M, we have $\lim_{\varepsilon\to 0}\frac{g(\varepsilon,\overline{m};w)}{v(\varepsilon,\overline{m};w)}=\frac{1}{\overline{m}+1}$.

Step 2.

Substep 2.1. *P_1, P_3 and P_4 send P_5 the permutation $\alpha_2\sigma$.*

We assume that along the sequence of completely mixed strategies, P_1 trembles from her equilibrium strategy with probability ε_1^3 (that is, P_1 sends the realization of $\alpha_2\sigma$ with probability $1-\varepsilon_1^3$ and any other message with probability $\frac{\varepsilon_1^3}{m}$). P_3 and P_4 tremble from their equilibrium strategies with probability ε_1^2. The proof that the assessment $\Psi(r)$ satisfies consistency in Substep 2.1 is identical to the proof in Step 1, and is therefore omitted.

Substep 2.2. *P_3, P_4 and P_5 send P_2 the permutation $\alpha_2\sigma\tau\varphi$.*

Let $\tilde{\varepsilon}_{1,n}=\varepsilon_{1,n}$. The probabilities of trembling are $\tilde{\varepsilon}_1^3$ for P_3 and $\tilde{\varepsilon}_1^2$ for P_4 and P_5.

The senders' strategies in Step 2 do not depend on z_1^2, the message sent by P_2 in Step 1. Thus P_2's beliefs are not a function of z_1^2.

Suppose that the message sent by the majority of the senders to P_2 is a, and the realizations of σ and τ are $\widehat{\sigma}$ and $\widehat{\tau}$, respectively (P_2 knows these realizations). Let $\widehat{\varphi}^1,...,\widehat{\varphi}^{m+1}$ be $m+1$ different permutations on X. Then for every $l=1,...,m+1$, we can find a permutation $\widehat{\alpha_2}^l$ such that $\widehat{\alpha_2}^l\widehat{\sigma}\widehat{\tau}\widehat{\varphi}^l=a$. Moreover, $\widehat{\alpha_2}^l\neq\widehat{\alpha_2}^{l'}$ if l differs from l'. To prove that the assessment $\Psi(r)$ satisfies consistency in Substep 2.2, it is enough to show that for any triple of messages w_{22} in the set M, and for every $l=1,...,m+1$, the following equality holds:

$$\lim_{n\to\infty}\Pr^n\left(\alpha_2=\widehat{\alpha_2}^l,\varphi=\widehat{\varphi}^l,\overline{\alpha_2\sigma}=\widehat{\alpha_2}^l\widehat{\sigma}|w_{22},\sigma=\widehat{\sigma},\tau=\widehat{\tau}\right)=\frac{1}{m+1}. \tag{12}$$

The probability in equation (12) can be expressed as follows:

$$\Pr^n\left(\alpha_2=\widehat{\alpha_2}^l,\varphi=\widehat{\varphi}^l,\overline{\alpha_2\sigma}=\widehat{\alpha_2}^l\widehat{\sigma}|w_{22},\sigma=\widehat{\sigma},\tau=\widehat{\tau}\right)=$$
$$\frac{\Pr^n\left(w_{22}|\alpha_2=\widehat{\alpha_2}^l,\varphi=\widehat{\varphi}^l,\overline{\alpha_2\sigma}=\widehat{\alpha_2}^l\widehat{\sigma},\sigma=\widehat{\sigma},\tau=\widehat{\tau}\right)\Pr^n\left(\overline{\alpha_2\sigma}=\widehat{\alpha_2}^l\widehat{\sigma}|\alpha_2=\widehat{\alpha_2}^l,\sigma=\widehat{\sigma}\right)}{\sum_{h,j,k=1}^{m+1}\Pr^n\left(w_{22}|\alpha_2=\widehat{\alpha_2}^h,\varphi=\widehat{\varphi}^j,\overline{\alpha_2\sigma}=\widehat{\alpha_2}^k\widehat{\sigma},\sigma=\widehat{\sigma},\tau=\widehat{\tau}\right)\Pr^n\left(\overline{\alpha_2\sigma}=\widehat{\alpha_2}^k\widehat{\sigma}|\alpha_2=\widehat{\alpha_2}^h,\sigma=\widehat{\sigma}\right)}. \tag{13}$$

Suppose that the realization of α_2 is $\widehat{\alpha_2}^h$, $h=1,...,m+1$. Given the sequences of completely mixed strategies that we have constructed in Step 0 and Substep 2.1, the probability

that the message sent by the majority of the senders to P_5 coincides with the permutation $\widehat{\alpha_2}^h\widehat{\sigma}$ (i.e. the probability that P_5 learns the true realization of $\alpha_2\sigma$) is given by:

$$\Pr^n\left(\overline{\alpha_2\sigma}=\widehat{\alpha_2}^h\widehat{\sigma}|\alpha_2=\widehat{\alpha_2}^h,\sigma=\widehat{\sigma}\right)=1-\varepsilon_1^4\left(\frac{1}{m}+2\varepsilon_1-\varepsilon_1^3\left(1+\frac{1}{m}\right)\right):=1-\upsilon\left(\varepsilon_1\right). \quad (14)$$

The probability that P_5 learnt some other permutation $\widehat{\alpha_2}^k\widehat{\sigma}$, different from the realization $\widehat{\alpha_2}^h\widehat{\sigma}$ is:

$$\Pr^n\left(\overline{\alpha_2\sigma}=\widehat{\alpha_2}^k\widehat{\sigma}|\alpha_2=\widehat{\alpha_2}^h,\sigma=\widehat{\sigma}\right)=\frac{\varepsilon_1^4}{m}\left(\frac{1}{m}+2\varepsilon_1-\varepsilon_1^3\left(1+\frac{1}{m}\right)\right):=\frac{\upsilon\left(\varepsilon_1\right)}{m}. \quad (15)$$

Note that the probability that P_5 learnt a wrong state is an infinitesimal of the same order as $\frac{\varepsilon_1^4}{m^2}$. If P_5 did not learn the true realization of $\alpha_2\sigma$, then at least two senders deviated from their equilibrium strategies in Substep 2.1. The two most likely players to deviate are P_3 and P_4 and the probability that they send a pair of wrong messages is $\frac{\varepsilon_1^4}{m^2}$. Substituting equations (14) and (15) into equation (13) yields:

$$\Pr^n\left(\alpha_2=\widehat{\alpha_2}^l,\varphi=\widehat{\varphi}^l,\overline{\alpha_2\sigma}=\widehat{\alpha_2}^l\widehat{\sigma}|w_{22},\sigma=\widehat{\sigma},\tau=\widehat{\tau}\right)=$$
$$\frac{(1-\upsilon(\varepsilon_1))g(\tilde{\varepsilon}_1,m;w_{22})}{(1-\upsilon(\varepsilon_1))v(\tilde{\varepsilon}_1,m;w_{22})+\frac{\upsilon(\varepsilon_1)}{m}\sum\limits_{h\neq k}\Pr^n\left(w_{22}|\alpha_2=\widehat{\alpha_2}^h,\varphi=\widehat{\varphi}^j,\overline{\alpha_2\sigma}=\widehat{\alpha_2}^k\widehat{\sigma},\sigma=\widehat{\sigma},\tau=\widehat{\tau}\right)}.$$

It is easy to verify that, for any triple of messages w_{22}, the above probability converges to $\frac{1}{m+1}$ as n goes to infinity.

To give an intuition for this result, let us consider the different situations that P_2 can face in Substep 2.2. First of all, observing the triple (aaa) is consistent with the fact that all senders in Substeps 2.1 and 2.2 played their equilibrium strategies. Therefore, with probability one, P_5 and P_2 received the true realizations of $\alpha_2\sigma$ and $\alpha_2\sigma\tau\varphi$, respectively.

We now consider the case where P_2 receives the triple (aab). Although P_5's message differs from the other two messages, P_2 believes that with probability one P_5 received the true realization of $\alpha_2\sigma$. Suppose that P_3 and P_4 sent the true message (the probability of

this event is one in the limit). There are three possible reasons why P_5 sent a different message: (i) P_5 received the true realization of $\alpha_2\sigma$ in Substep 2.1, but she deviated in Substep 2.2. (ii) P_5 received a wrong message in Substep 2.1, but followed her equilibrium strategy in Substep 2.2. (iii) P_5 received a wrong message in Substep 2.1, and then deviated. Note that only one deviation (by P_5) is needed for case (i). In case (ii) we need at least two deviations in Substep 2.1. Finally at least three deviations are required for case (iii). Since we assume that the probabilities of deviations in Substeps 2.1 and 2.2 are the same, in the limit we assign probability one to case (i), which requires the fewest deviations.

When P_2 receives (aba), at least one player among P_3 and P_4 has deviated. A deviation by P_4 from her equilibrium strategy (in Substep 2.2) does not require any other deviation to justify the triple of messages (aba) (in particular, a deviation by P_4 is consistent with the fact that P_5 learnt the true realization of $\alpha_2\sigma$). On the contrary, a deviation by P_3 requires at least another deviation (either P_5 learnt the truth and deviated, or she did not learn the truth). Again, since deviations occur with arbitrarily small probabilities, we assign probability one to the event that P_4 deviated in Substep 2.2.

We now turn to the triple of messages (baa). We can justify this triple with a single deviation by P_3 in Substep 2.2. A deviation by P_4 requires at least another deviation. It is true that P_3 trembles with probability $\tilde{\varepsilon}_1^3$ in Substep 2.2, while P_3 trembles with probability $\tilde{\varepsilon}_1^2$, but any other deviation is independent and occurs at least with probability $\tilde{\varepsilon}_1^2$ or ε_1^2. Since $\tilde{\varepsilon}_1$ is equal to ε_1, we conclude that with probability one P_3 deviated in Substep 2.2 and P_5 learnt the truth in Substep 2.1.

Finally, we consider the case (abc). Note that, independently of what P_5 learns in Substep 2.1, the probability that P_2 receives three different messages converges to zero faster than or at the same rate as $\tilde{\varepsilon}_1^2$. Since P_3 and P_4 have sent different messages, clearly, at least one of them has deviated from her equilibrium strategy in Substep 2.2 (in the limit we assign probability one to the event that the deviator is P_4, since she is more likely to tremble than P_3). As far as P_5 is concerned, it could be the case that she did not learn

the truth in Substep 2.1, which implies that at least two senders deviated in that substep. However, it could also be the case that P_5 did receive the true message in Substep 2.1, but then she deviated in Substep 2.2. We assign probability one to the last event, since it requires only one deviation.

Step 3.

Substep 3.1. *P_1, P_2 and P_4 send P_5 the permutation $\alpha_3\sigma$.*

We assume that P_1 trembles from her equilibrium strategy with probability ε_1^3, and that P_2 and P_4 tremble with probability ε_1^2. With this sequence of completely mixed strategies it is easy to show that, given any triple of messages w_{31}, in the limit, P_5 assigns probability one to the event that the realization of $\alpha_3\sigma$ is the message sent by the majority (see Step 1, above). This result does not change when we take into account w_{21}, the triple of messages received by P_5 in Substep 2.1, and regarding the permutation $\alpha_2\sigma$. This follows from the fact that the random permutations α_2, α_3 and σ are independent of each other and uniformly distributed, and trembles are independent across information sets. For the same reason, in this step, P_5 does not update her beliefs from Substep 2.1. These results also imply that, conditional on P_5's information, the marginal distribution of any of the three random permutations σ, α_2 and α_3, is uniform over $\Lambda(X)$.

Substep 3.2. *P_2, P_4 and P_5 send P_1 the permutation $\beta_{31}\alpha_3\sigma\tau$.*

For any integer n, let $\varepsilon_{2,n} = \varepsilon_n^2$ (notice that $\lim_{n\to\infty} \frac{\varepsilon_{1,n}}{\varepsilon_{2,n}} = 0$). P_2 trembles from her equilibrium strategy with probability ε_2^3, and P_4 and P_5 tremble with probability ε_2^2.

Note that P_1's senders' strategies do not depend on her message in Substep 2.1. Thus, P_1's beliefs are independent of z_{21}^1. However, P_1's beliefs do depend on z_{31}^1. The message that P_5 sends in Substep 3.2 is a function of the triple of messages that she receives in Substep 3.1. This triple, in turn, depends on P_1's report, z_{31}^1.

Let $\widehat{\alpha_3}^1, ..., \widehat{\alpha_3}^{m+1}$ be $m+1$ different permutations. We let $\widehat{\alpha_3}^1$ and $\widehat{\sigma}$ denote the realizations of α_3 and σ, respectively ($\widehat{\alpha_3}^1$ and $\widehat{\sigma}$ are known to P_1). We also assume that the message sent by the majority of the senders to P_1 is the permutation a. To every

permutation[24] $\widehat{\tau}^l$, $l = 1, ..., m+1$, we can assign a different permutation $\widehat{\beta_{31}}^l$ such that $\widehat{\beta_{31}}^l \widehat{\alpha_3}^1 \widehat{\sigma} \widehat{\tau}^l = a$.

We now show that for any triple of messages w_{32} in the set M, for every $l = 1, ..., m+1$, and for every message z_{31}^1, the following equality holds:

$$\lim_{n\to\infty} \Pr^n \left(\beta_{31} = \widehat{\beta_{31}}^l, \tau = \widehat{\tau}^l, \overline{\alpha_3\sigma} = \widehat{\alpha_3}^1 \widehat{\sigma} | w_{32}, z_{31}^1, \sigma = \widehat{\sigma}, \alpha_3 = \widehat{\alpha_3}^1 \right) = \frac{1}{m+1}.$$

This result, combined with the result of Step 1, and the fact that trembles are independent across information sets, and all random choices are independent of each other and uniformly distributed, implies that the assessment $\Psi(r)$ satisfies consistency in Substep 3.2.

Along the sequence of completely mixed strategies, we have:

$$\Pr^n \left(\beta_{31} = \widehat{\beta_{31}}^l, \tau = \widehat{\tau}^l, \overline{\alpha_3\sigma} = \widehat{\alpha_3}^1 \widehat{\sigma} | w_{32}, z_{31}^1, \sigma = \widehat{\sigma}, \alpha_3 = \widehat{\alpha_3}^1 \right) = \frac{\Pr^n \left(w_{32} | \beta_{31} = \widehat{\beta_{31}}^l, \tau = \widehat{\tau}^l, \overline{\alpha_3\sigma} = \widehat{\alpha_3}^1 \widehat{\sigma}, \sigma = \widehat{\sigma}, \alpha_3 = \widehat{\alpha_3}^1 \right) \Pr^n \left(\overline{\alpha_3\sigma} = \widehat{\alpha_3}^1 \widehat{\sigma} | \sigma = \widehat{\sigma}, \alpha_3 = \widehat{\alpha_3}^1, z_{31}^1 \right)}{\sum_{h,j,k=1}^{m+1} \Pr^n \left(w_{32} | \beta_{31} = \widehat{\beta_{31}}^h, \tau = \widehat{\tau}^j, \overline{\alpha_3\sigma} = \widehat{\alpha_3}^k \widehat{\sigma}, \sigma = \widehat{\sigma}, \alpha_3 = \widehat{\alpha_3}^1 \right) \Pr^n \left(\overline{\alpha_3\sigma} = \widehat{\alpha_3}^k \widehat{\sigma} | \sigma = \widehat{\sigma}, \alpha_3 = \widehat{\alpha_3}^1, z_{31}^1 \right)}. \quad (16)$$

At this point we need to specify the action of P_1 in Substep 3.1. We first analyze the case where P_1 follows her equilibrium strategy, i.e. $z_{31}^1 = \widehat{\alpha_3}^1 \widehat{\sigma}$. In this case, the probability that P_5 learns the realization of $\alpha_3\sigma$ is given by:

$$\Pr^n \left(\overline{\alpha_3\sigma} = \widehat{\alpha_3}^1 \widehat{\sigma} | \sigma = \widehat{\sigma}, \alpha_3 = \widehat{\alpha_3}^1, z_{31}^1 = \widehat{\alpha_3}^1 \widehat{\sigma} \right) = 1 - \frac{\varepsilon_1^4}{m}. \quad (17)$$

The probability that P_5 learns some other realization $\widehat{\alpha_3}^k \widehat{\sigma}$, $k \neq 1$, is:

$$\Pr^n \left(\overline{\alpha_3\sigma} = \widehat{\alpha_3}^k \widehat{\sigma} | \sigma = \widehat{\sigma}, \alpha_3 = \widehat{\alpha_3}^1, z_{31}^1 = \widehat{\alpha_3}^1 \widehat{\sigma} \right) = \frac{\varepsilon_1^4}{m^2}. \quad (18)$$

Since P_1 is the sender with the lowest probability of trembling, and she reports the truth, P_5 can learn a wrong state $\widehat{\alpha_3}^k \widehat{\sigma}$ if and only if both P_2 and P_4 send the message $\widehat{\alpha_3}^k \widehat{\sigma}$ (the probability of this event is $\frac{\varepsilon_1^4}{m^2}$).

[24] The permutations $\widehat{\tau}^1, ..., \widehat{\tau}^{m+1}$ were introduced in Step 1 of this section.

We substitute equations (17) and (18) into equation (16) and obtain:

$$\Pr^n\left(\beta_{31}=\widehat{\beta_{31}}^{\,l},\tau=\widehat{\tau}^l,\overline{\alpha_3\sigma}=\widehat{\alpha_3}^1\widehat{\sigma}|w_{32},z_{31}^1=\widehat{\alpha_3}^1\widehat{\sigma},\sigma=\widehat{\sigma},\alpha_3=\widehat{\alpha_3}^1\right)=$$
$$\frac{\left(1-\frac{\varepsilon_1^4}{m}\right)g(\varepsilon_2,m;w_{32})}{\left(1-\frac{\varepsilon_1^4}{m}\right)v(\varepsilon_2,m;w_{32})+\left(\frac{\varepsilon_1^4}{m^2}\right)\sum\limits_{k\neq 1}\Pr^n\left(w_{32}|\beta_{31}=\widehat{\beta_{31}}^{\,h},\tau=\widehat{\tau}^j,\overline{\alpha_3\sigma}=\widehat{\alpha_3}^k\widehat{\sigma},\sigma=\widehat{\sigma},\alpha_3=\widehat{\alpha_3}^1\right)}.$$

It is easy to verify that, as n goes to infinity, the above probability converges to $\frac{1}{m+1}$. Consider the event that P_5 knows the permutation $\alpha_3\sigma$, and the message sent by the majority of the senders to P_1 coincides with the permutation $\beta_{31}\alpha_3\sigma\tau$. From the analysis of Substep 2.2 above, we know that this event has probability one if we are ignorant about P_1's action in Substep 3.1 and the trembles in Substeps 3.1 and 3.2 are equally likely. Now, since P_1 reports the true message to P_5, and deviations are less likely in Substep 3.1 than in Substep 3.2, we conclude that, *a fortiori*, P_5 is informed about the realization of $\alpha_3\sigma$, and that the message sent by the majority of the senders is the realization of $\beta_{31}\alpha_3\sigma\tau$.

We now assume that in Substep 3.1, P_1 deviates from her equilibrium strategy and sends the message $\widehat{\alpha_3}^{\widetilde{k}}\bar{\sigma}$ to P_5, for some $\widetilde{k}=2,...,m+1$. In this case, P_5 learns the truth if and only if both P_2 and P_4 report the realization of $\alpha_3\sigma$. This event occurs with probability:

$$\Pr^n\left(\overline{\alpha_3\sigma}=\widehat{\alpha_3}^1\widehat{\sigma}|\sigma=\widehat{\sigma},\alpha_3=\widehat{\alpha_3}^1,z_{31}^1=\widehat{\alpha_3}^{\widetilde{k}}\widehat{\sigma}\right)=\left(1-\varepsilon_1^2\right)^2. \tag{19}$$

The probability that P_5 learns the permutation sent by P_1 is:

$$\Pr^n\left(\overline{\alpha_3\sigma}=\widehat{\alpha_3}^{\widetilde{k}}\widehat{\sigma}|\sigma=\widehat{\sigma},\alpha_3=\widehat{\alpha_3}^1,z_{31}^1=\widehat{\alpha_3}^{\widetilde{k}}\widehat{\sigma}\right)=\varepsilon_1^2\left(2-\varepsilon_1^2\left(\frac{m^2+m-1}{m^2}\right)\right). \tag{20}$$

Clearly, P_5 is more likely to learn the permutation sent by P_1 than any other permutation $\widehat{\alpha_3}^k\widehat{\sigma}$, for $k\neq 1,\widetilde{k}$. In fact, in the last case we have:

$$\Pr^n\left(\overline{\alpha_3\sigma}=\widehat{\alpha_3}^k\widehat{\sigma}|\sigma=\widehat{\sigma},\alpha_3=\widehat{\alpha_3}^1,z_{31}^1=\widehat{\alpha_3}^{\widetilde{k}}\widehat{\sigma}\right)=\frac{\varepsilon_1^4}{m^2}. \tag{21}$$

Substituting equations (19)-(21) into equation (16) yields:

$$\Pr^n\left(\beta_{31}=\widehat{\beta_{31}}^{l},\tau=\widehat{\tau}^{l},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^{1}\widehat{\sigma}|w_{32},z_{31}^1=\widehat{\alpha_3}^{\tilde{k}}\widehat{\sigma},\sigma=\widehat{\sigma},\alpha_3=\widehat{\alpha_3}^{1}\right)=\frac{\left(1-\varepsilon_1^2\right)^2 g\left(\varepsilon_2,m;w_{32}\right)}{B_1},$$

where:

$$B_1=\left(1-\varepsilon_1^2\right)^2 v\left(\varepsilon_2,m;w_{32}\right)+\varepsilon_1^2\left(2-\varepsilon_1^2\left(\tfrac{m^2+m-1}{m^2}\right)\right)$$
$$\sum_{h,j}\Pr^n\left(w_{32}|\beta_{31}=\widehat{\beta_{31}}^{h},\tau=\widehat{\tau}^{j},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^{\tilde{k}}\widehat{\sigma},\sigma=\widehat{\sigma},\alpha_3=\widehat{\alpha_3}^{1}\right)+$$
$$\left(\tfrac{\varepsilon_1^4}{m^2}\right)\sum_{k\neq 1,\hat{k}}\Pr^n\left(w_{32}|\beta_{31}=\widehat{\beta_{31}}^{h},\tau=\widehat{\tau}^{j},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^{k}\widehat{\sigma},\sigma=\widehat{\sigma},\alpha_3=\widehat{\alpha_3}^{1}\right).$$

A sufficient condition for the above probability to converge to $\frac{1}{m+1}$ is that $\lim_{n\to\infty}\frac{\varepsilon_{1,n}}{\varepsilon_{2,n}}=0$. This condition is also necessary, at least in the cases where P_1 receives the triples (aab) and (abc). Suppose, for example, that P_2 and P_4 report the same message, a, and that P_5 sends a different message. Consider a pair of permutations, $\widehat{\beta_{31}}^{l}$ and $\widehat{\tau}^{l}$, such that $\widehat{\beta_{31}}^{l}\widehat{\alpha_3}^{1}\widehat{\sigma}\widehat{\tau}^{l}=a$. Let us assume that P_5's message is $b=\widehat{\beta_{31}}^{l}\widehat{\alpha_3}^{\tilde{k}}\widehat{\sigma}\widehat{\tau}^{l}$. The following two situations are both compatible with the triple (aab). In the first scenario, we assume that P_5 learnt the true permutation $\widehat{\alpha_3}^{1}\widehat{\sigma}$, but then deviated in Substep 3.2. The probability of this tremble is of the same order as ε_2^2. Suppose now that P_5 learnt the permutation $\widehat{\alpha_3}^{\tilde{k}}\widehat{\sigma}$ and then she followed her equilibrium strategy. The probability of this event is an infinitesimal of the same order as ε_1^2. In Substep 3.2, P_1 can assign probability zero to the last event only if we assume that $\varepsilon_{1,n}$ converges to zero faster than $\varepsilon_{2,n}$, i.e. only if the trembles are much less likely in Substep 3.1 than in Substep 3.2.[25]

Substep 3.3. *P_4, P_1 and P_5 send P_2 the permutation $\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi$.*

Let $\varepsilon_{3,n}=\varepsilon_n$ for every n, and note that $\lim_{n\to\infty}\frac{\varepsilon_{1,n}}{(\varepsilon_{3,n})^2}=0$ and $\lim_{n\to\infty}\frac{\varepsilon_{2,n}}{\varepsilon_{3,n}}=0$. We assume that along the sequence of completely mixed strategies, P_4 trembles with probability ε_3^3, and P_1 and P_5 tremble with probability ε_3^2. Clearly, P_2's beliefs are a function of z_{31}^2 and z_{32}^2, the messages that she sends in Substeps 3.1 and 3.2, respectively. However, her beliefs do not depend on z_1^2, the message sent by P_2 in Step 1.

[25] A similar problem arises when P_1 receives the triple of messages (abc).

As in the previous step, $\widehat{\alpha_3}^1$ and $\widehat{\sigma}$ denote the realizations of α_3 and σ, respectively.[26] In addition to them, P_2 knows the realizations of β_{31} and τ, which we denote by $\widehat{\beta_{31}}$ and $\widehat{\tau}$, respectively. Consider the $m+1$ permutations $\widehat{\varphi}^1, ..., \widehat{\varphi}^{m+1}$ defined in Substep 2.2 of this section. For every $l = 1, ..., m+1$ there exists a different permutation $\widehat{\beta_{32}}^l$ such that $\widehat{\beta_{32}}^l \widehat{\beta_{31}} \widehat{\alpha_3}^1 \widehat{\sigma} \widehat{\tau} \widehat{\varphi}^l = a$ (we assume that a is the message sent by the majority of the senders to P_2).

In order to demonstrate that the equilibrium beliefs of Substep 3.3 are consistent, it is enough to show that for any pair of messages z_{31}^2, z_{32}^2, for any triple of messages w_{33} in M, and for every $l = 1, ..., m+1$, the following equality is satisfied:

$$\lim_{n\to\infty} \Pr^n \left(\widehat{\beta_{32}}^l, \widehat{\varphi}^l, \overline{\beta_{31}\alpha_3\sigma\tau} = \widehat{\beta_{31}}\widehat{\alpha_3}^1\widehat{\sigma}\widehat{\tau}, \overline{\alpha_3\sigma} = \widehat{\alpha_3}^1\widehat{\sigma} | w_{33}, z_{31}^2, z_{32}^2, \widehat{\beta_{31}}, \widehat{\alpha_3}^1, \widehat{\sigma}, \widehat{\tau} \right) = \frac{1}{m+1}.$$

The probability above can be expressed as:

$$\Pr^n \left(\widehat{\beta_{32}}^l, \widehat{\varphi}^l, \overline{\beta_{31}\alpha_3\sigma\tau} = \widehat{\beta_{31}}\widehat{\alpha_3}^1\widehat{\sigma}\widehat{\tau}, \overline{\alpha_3\sigma} = \widehat{\alpha_3}^1\widehat{\sigma} | w_{33}, z_{31}^2, z_{32}^2, \widehat{\beta_{31}}, \widehat{\alpha_3}^1, \widehat{\sigma}, \widehat{\tau} \right) = \frac{e(l,l,1,1)}{\sum_{h,j,\iota,k=1}^{m+1} e(h,j,\iota,k)}, \tag{22}$$

where:

$$e(h,j,\iota,k) = \Pr^n \left(w_{33} | \widehat{\beta_{32}}^h, \widehat{\varphi}^j, \overline{\beta_{31}\alpha_3\sigma\tau} = \widehat{\beta_{31}}\widehat{\alpha_3}^\iota\widehat{\sigma}\widehat{\tau}, \overline{\alpha_3\sigma} = \widehat{\alpha_3}^k\widehat{\sigma}, \widehat{\beta_{31}}, \widehat{\alpha_3}^1, \widehat{\sigma}, \widehat{\tau} \right)$$
$$\Pr^n \left(\overline{\beta_{31}\alpha_3\sigma\tau} = \widehat{\beta_{31}}\widehat{\alpha_3}^\iota\widehat{\sigma}\widehat{\tau} | z_{32}^2, \overline{\alpha_3\sigma} = \widehat{\alpha_3}^k\widehat{\sigma}, \widehat{\beta_{31}}, \widehat{\alpha_3}^1, \widehat{\sigma}, \widehat{\tau} \right) \Pr^n \left(\overline{\alpha_3\sigma} = \widehat{\alpha_3}^k\widehat{\sigma} | z_{32}^1, \widehat{\alpha_3}^1, \widehat{\sigma} \right).$$

Depending on whether P_2 reports the true messages in Substeps 3.1 and 3.2 or not, we have to consider four different cases. We start by assuming that P_2 follows her equilibrium strategies in both steps, i.e. $z_{31}^2 = \widehat{\alpha_3}^1\widehat{\sigma}$ and $z_{32}^2 = \widehat{\beta_{31}}\widehat{\alpha_3}^1\widehat{\sigma}\widehat{\tau}$. We first need to compute the probability that P_5 learns a given permutation. We have:

[26] We also use the permutations $\widehat{\alpha_3}^1, ..., \widehat{\alpha_3}^{m+1}$, introduced in Substep 3.2 of this section.

$$\Pr^n\left(\overline{\alpha_3\sigma}=\widehat{\alpha_3}^k\widehat{\sigma}|z_{32}^1=\widehat{\alpha_3}^1\widehat{\sigma},\widehat{\alpha_3}^1,\widehat{\sigma}\right)=\left\{\begin{array}{ll}1-\varepsilon_1^5 & k=1\\ \frac{\varepsilon_1^5}{m} & k\neq 1\end{array}\right. . \tag{23}$$

Suppose that P_5 learns the truth in Substep 3.1. The probability that P_1 learns a given permutation in Substep 3.2 is:

$$\Pr^n\left(\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^\iota\widehat{\sigma\tau}|z_{32}^2=\widehat{\beta_{31}}\widehat{\alpha_3}^1\widehat{\sigma\tau},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^1\widehat{\sigma},\widehat{\beta_{31}},\widehat{\alpha_3}^1,\widehat{\sigma},\widehat{\tau}\right)=\left\{\begin{array}{ll}1-\frac{\varepsilon_1^4}{m} & \iota=1\\ \frac{\varepsilon_1^4}{m^2} & \iota\neq 1\end{array}\right. . \tag{24}$$

Substituting equations (23) and (24) into equation (22) yields:

$$\Pr^n\left(\widehat{\beta_{32}}^l,\widehat{\varphi}^l,\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^1\widehat{\sigma\tau},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^1\widehat{\sigma}|w_{33},z_{31}^2=\widehat{\alpha_3}^1\widehat{\sigma},z_{32}^2=\widehat{\beta_{31}}\widehat{\alpha_3}^1\widehat{\sigma\tau},\widehat{\beta_{31}},\widehat{\alpha_3}^1,\widehat{\sigma},\widehat{\tau}\right)=\frac{\left(1-\varepsilon_1^5\right)\left(1-\frac{\varepsilon_2^4}{m}\right)g(\varepsilon_3,m;w_{33})}{B_2},$$

where:

$$\begin{array}{c}B_2=\left(1-\varepsilon_1^5\right)\left(1-\frac{\varepsilon_2^4}{m}\right)v\left(\varepsilon_3,m;w_{33}\right)+\\ \left(1-\varepsilon_1^5\right)\frac{\varepsilon_2^4}{m^2}\sum_{\substack{h,j\\ \iota\neq 1}}\Pr^n\left(w_{33}|\widehat{\beta_{32}}^h,\widehat{\varphi}^j,\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^\iota\widehat{\sigma\tau},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^1\widehat{\sigma},\widehat{\beta_{31}},\widehat{\alpha_3}^1,\widehat{\sigma},\widehat{\tau}\right)+\\ \frac{\varepsilon_1^5}{m}\sum_{\substack{h,j,\iota\\ k\neq 1}}\Pr^n\left(w_{33}|\widehat{\beta_{32}}^h,\widehat{\varphi}^j,\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^\iota\widehat{\sigma\tau},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^k\widehat{\sigma},\widehat{\beta_{31}},\widehat{\alpha_3}^1,\widehat{\sigma},\widehat{\tau}\right)\\ \Pr^n\left(\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^\iota\widehat{\sigma\tau}|z_{32}^2=\widehat{\beta_{31}}\widehat{\alpha_3}^1\widehat{\sigma\tau},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^k\widehat{\sigma},\widehat{\beta_{31}},\widehat{\alpha_3}^1,\widehat{\sigma},\widehat{\tau}\right).\end{array}$$

Our assumptions regarding the trembles are sufficient to guarantee that the above probability converges to $\frac{1}{m+1}$ as n goes to infinity.

We now consider the case where P_2 deviates in Substep 3.1 (she reports the message $z_{31}^2=\widehat{\alpha_3}^{\widetilde{k}}\widehat{\sigma}$, for some $\widetilde{k}=2,...,m+1$) and follows her equilibrium strategy in Substep 3.2. The probability that P_5 learns a given permutation in Substep 3.1 is given by:

$$\Pr^n\left(\overline{\alpha_3\sigma}=\widehat{\alpha_3}^k\widehat{\sigma}|z_{32}^1=\widehat{\alpha_3}^{\widetilde{k}}\widehat{\sigma},\widehat{\alpha_3}^1,\widehat{\sigma}\right)=\begin{cases}(1-\varepsilon_1^3)\left(1-\frac{\varepsilon_1^2}{m}\right) & k=1\\ \frac{\varepsilon_1^2}{m}\left(1+\varepsilon_1-\frac{\varepsilon_1^3}{m}\right) & k=\widetilde{k}\\ \frac{\varepsilon_1^3}{m}\left(1-\frac{\varepsilon_1^2}{m}\right) & k\neq 1,\widetilde{k}\end{cases}. \tag{25}$$

We now substitute equations (24) and (25) into equation (22). This gives us:

$$\Pr^n\left(\widehat{\beta_{32}}^l,\widehat{\varphi}^l,\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^1\widehat{\sigma\tau},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^1\widehat{\sigma}|w_{33},z_{31}^2=\widehat{\alpha_3}^{\widetilde{k}}\widehat{\sigma},z_{32}^2=\widehat{\beta_{31}}\widehat{\alpha_3}^1\widehat{\sigma\tau},\widehat{\beta_{31}},\widehat{\alpha_3}^1,\widehat{\sigma},\widehat{\tau}\right)=\frac{\left(1-\varepsilon_1^3\right)\left(1-\frac{\varepsilon_1^2}{m}\right)\left(1-\frac{\varepsilon_2^4}{m}\right)g(\varepsilon_3,m;w_{33})}{B_3},$$

where:

$$B_3=\left(1-\varepsilon_1^3\right)\left(1-\tfrac{\varepsilon_1^2}{m}\right)\left(1-\tfrac{\varepsilon_2^4}{m}\right)v\left(\varepsilon_3,m;w_{33}\right)+$$
$$\left(1-\varepsilon_1^3\right)\left(1-\tfrac{\varepsilon_1^2}{m}\right)\tfrac{\varepsilon_2^4}{m^2}\sum_{\substack{h,j\\ \iota\neq 1}}\Pr^n\left(w_{33}|\widehat{\beta_{32}}^h,\widehat{\varphi}^j,\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^\iota\widehat{\sigma\tau},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^1\widehat{\sigma},\widehat{\beta_{31}},\widehat{\alpha_3}^1,\widehat{\sigma},\widehat{\tau}\right)+$$
$$\tfrac{\varepsilon_1^2}{m}\left(1+\varepsilon_1-\tfrac{\varepsilon_1^3}{m}\right)\sum_{h,j,\iota}\Pr^n\left(w_{33}|\widehat{\beta_{32}}^h,\widehat{\varphi}^j,\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^\iota\widehat{\sigma\tau},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^{\widetilde{k}}\widehat{\sigma},\widehat{\beta_{31}},\widehat{\alpha_3}^1,\widehat{\sigma},\widehat{\tau}\right)$$
$$\Pr^n\left(\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^\iota\widehat{\sigma\tau}|z_{32}^2=\widehat{\beta_{31}}\widehat{\alpha_3}^1\widehat{\sigma\tau},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^{\widetilde{k}}\widehat{\sigma},\widehat{\beta_{31}},\widehat{\alpha_3}^1,\widehat{\sigma},\widehat{\tau}\right)+$$
$$\tfrac{\varepsilon_1^3}{m}\left(1-\tfrac{\varepsilon_1^2}{m}\right)\sum_{\substack{h,j,\iota\\ k\neq 1,\widetilde{k}}}\Pr^n\left(w_{33}|\widehat{\beta_{32}}^h,\widehat{\varphi}^j,\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^\iota\widehat{\sigma\tau},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^k\widehat{\sigma},\widehat{\beta_{31}},\widehat{\alpha_3}^1,\widehat{\sigma},\widehat{\tau}\right)$$
$$\Pr^n\left(\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^\iota\widehat{\sigma\tau}|z_{32}^2=\widehat{\beta_{31}}\widehat{\alpha_3}^{\widetilde{\iota}}\widehat{\sigma\tau},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^k\widehat{\sigma},\widehat{\beta_{31}},\widehat{\alpha_3}^1,\widehat{\sigma},\widehat{\tau}\right).$$

Again, given our assumptions, we can conclude that the limit value of the above probability is $\frac{1}{m+1}$.

We now assume that P_2 plays her equilibrium strategy in Substep 3.1 and deviates in Substep 3.2 (we assume that she sends message $z_{32}^2=\widehat{\beta_{31}}\widehat{\alpha_3}^{\widetilde{\iota}}\widehat{\sigma\tau}$, for some $\widetilde{\iota}=2,...,m+1$). Consider the case where P_5 learns the true realization of $\alpha_3\sigma$ (the probability of this event is given by equation (23)). Then the probability that P_1 learns a given permutation in Substep 3.2 is:

$$\Pr^n\left(\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^{\iota}\widehat{\sigma\tau}|z_{32}^2=\widehat{\beta_{31}}\widehat{\alpha_3}^{\tilde{\iota}}\widehat{\sigma\tau},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^1\widehat{\sigma},\widehat{\beta_{31}},\widehat{\alpha_3}^1,\widehat{\sigma},\widehat{\tau}\right)=\begin{cases}\left(1-\varepsilon_2^2\right)^2 & \iota=1\\ \varepsilon_2^2\left(2-\varepsilon_2^2\left(\frac{m^2+m-1}{m^2}\right)\right) & \iota=\tilde{\iota}\\ \frac{\varepsilon_2^4}{m^2} & \iota\neq 1,\tilde{\iota}\end{cases}. \tag{26}$$

We substitute equations (23) and (26) into equation (22) and obtain:

$$\Pr^n\left(\widehat{\beta_{32}}^l,\widehat{\varphi}^l,\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^1\widehat{\sigma\tau},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^1\widehat{\sigma}|w_{33},z_{31}^2=\widehat{\alpha_3}^1\widehat{\sigma},z_{32}^2=\widehat{\beta_{31}}\widehat{\alpha_3}^{\tilde{\iota}}\widehat{\sigma\tau},\widehat{\beta_{31}},\widehat{\alpha_3}^1,\widehat{\sigma},\widehat{\tau}\right)=\frac{\left(1-\varepsilon_1^5\right)\left(1-\varepsilon_2^2\right)^2 g(\varepsilon_3,m;w_{33})}{B_4},$$

where:

$$\begin{aligned}B_4=&\left(1-\varepsilon_1^5\right)\left(1-\varepsilon_2^2\right)^2 v\left(\varepsilon_3,m;w_{33}\right)+\left(1-\varepsilon_1^5\right)\varepsilon_2^2\left(2-\varepsilon_2^2\left(\tfrac{m^2+m-1}{m^2}\right)\right)\\ &\sum_{h,j}\Pr^n\left(w_{33}|\widehat{\beta_{32}}^h,\widehat{\varphi}^j,\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^{\tilde{\iota}}\widehat{\sigma\tau},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^1\widehat{\sigma},\widehat{\beta_{31}},\widehat{\alpha_3}^1,\widehat{\sigma},\widehat{\tau}\right)+\\ &\left(1-\varepsilon_1^5\right)\tfrac{\varepsilon_2^4}{m^2}\sum_{\substack{h,j\\ \iota\neq 1,\tilde{\iota}}}\Pr^n\left(w_{33}|\widehat{\beta_{32}}^h,\widehat{\varphi}^j,\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^{\iota}\widehat{\sigma\tau},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^1\widehat{\sigma},\widehat{\beta_{31}},\widehat{\alpha_3}^1,\widehat{\sigma},\widehat{\tau}\right)+\\ &\tfrac{\varepsilon_1^5}{m}\sum_{\substack{h,j,\iota\\ k\neq 1}}\Pr^n\left(w_{33}|\widehat{\beta_{32}}^h,\widehat{\varphi}^j,\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^{\iota}\widehat{\sigma\tau},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^1\widehat{\sigma},\widehat{\beta_{31}},\widehat{\alpha_3}^1,\widehat{\sigma},\widehat{\tau}\right)\\ &\Pr^n\left(\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^{\iota}\widehat{\sigma\tau}|z_{32}^2=\widehat{\beta_{31}}\widehat{\alpha_3}^{\tilde{\iota}}\widehat{\sigma\tau},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^k\widehat{\sigma},\widehat{\beta_{31}},\widehat{\alpha_3}^1,\widehat{\sigma},\widehat{\tau}\right).\end{aligned}$$

It is easy to check that our conditions regarding the trembles are sufficient for the above probability to be $\frac{1}{m+1}$ in the limit. We now show that the condition $\lim_{n\to\infty}\frac{\varepsilon_{2,n}}{\varepsilon_{3,n}}=0$ is also necessary to obtain the previous result. Suppose that P_2 receives message a from P_4 and P_5 and message b from P_1. Suppose that both P_1 and P_5 learnt the true realization of the permutation they received. Then the probability that P_2 receives the triple of messages[27] (aba) is an infinitesimal of the same order as ε_3^2. Now consider the scenario where P_5 learns the realization of $\alpha_3\sigma$, but P_1 learns the wrong permutation $\widehat{\beta_{31}}\widehat{\alpha_3}^{\tilde{\iota}}\widehat{\sigma\tau}$, i.e. the message sent by P_2. The probability of this event is an infinitesimal of the same order as ε_2^2. Suppose that

[27]Remember that the first message in the triple refers to P_4, who is the least likely to tremble among the senders.

message b is equal to $\widehat{\beta_{32}}^{l}\widehat{\beta_{31}}\widehat{\alpha_3}^{\widetilde{\iota}}\widehat{\sigma}\widehat{\tau}\widehat{\varphi}^{l}$, for some $l=1,...,m+1$. In this case, P_2 receives the triple (aba) only if every sender follows her equilibrium strategy in Substep 3.3 (in other words, no deviation is needed in Substep 3.3 in order to justify the triple (aba)). In the limit, to assign probability zero to the latter event, we have to assume that $\varepsilon_{2,n}$ converges to zero faster than $\varepsilon_{3,n}$. A similar argument shows that the condition $\lim_{n\to\infty}\frac{\varepsilon_{2,n}}{\varepsilon_{3,n}}=0$ is necessary for the case of (abc), too.

Finally, we consider the case in which P_2 deviates in both substeps. We assume that P_2 sends message $z_{32}^1=\widehat{\alpha_3}^{\widetilde{k}}\widehat{\sigma}$ in Substep 3.1, and message $z_{32}^2=\widehat{\beta_{31}}\widehat{\alpha_3}^{\widetilde{\iota}}\widehat{\sigma}\widehat{\tau}$ in Substep 3.2, where $\widetilde{k}$ and $\widetilde{\iota}$ are different from one. As usual, we substitute equations (25) and (26) into equation (22). We obtain:

$$\Pr^n\left(\widehat{\beta_{32}}^{l},\widehat{\varphi}^{l},\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^{1}\widehat{\sigma}\widehat{\tau},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^{1}\widehat{\sigma}|w_{33},z_{31}^2=\widehat{\alpha_3}^{\widetilde{k}}\widehat{\sigma},z_{32}^2=\widehat{\beta_{31}}\widehat{\alpha_3}^{\widetilde{\iota}}\widehat{\sigma}\widehat{\tau},\widehat{\beta_{31}},\widehat{\alpha_3}^{1},\widehat{\sigma},\widehat{\tau}\right)=\frac{\left(1-\varepsilon_1^3\right)\left(1-\frac{\varepsilon_1^2}{m}\right)\left(1-\varepsilon_2^2\right)^2 g(\varepsilon_3,m;w_{33})}{B_5},$$

where:

$$\begin{aligned}B_5=&\left(1-\varepsilon_1^3\right)\left(1-\tfrac{\varepsilon_1^2}{m}\right)\left(1-\varepsilon_2^2\right)^2 v\left(\varepsilon_3,m;w_{33}\right)+\left(1-\varepsilon_1^3\right)\left(1-\tfrac{\varepsilon_1^2}{m}\right)\varepsilon_2^2\left(2-\varepsilon_2^2\left(\tfrac{m^2+m-1}{m^2}\right)\right)\\&\sum_{h,j}\Pr^n\left(w_{33}|\widehat{\beta_{32}}^{h},\widehat{\varphi}^{j},\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^{\widetilde{\iota}}\widehat{\sigma}\widehat{\tau},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^{1}\widehat{\sigma},\widehat{\beta_{31}},\widehat{\alpha_3}^{1},\widehat{\sigma},\widehat{\tau}\right)+\\\left(1-\varepsilon_1^3\right)\left(1-\tfrac{\varepsilon_1^2}{m}\right)\tfrac{\varepsilon_2^4}{m^2}&\sum_{\substack{h,j\\\iota\neq1,\widetilde{\iota}}}\Pr^n\left(w_{33}|\widehat{\beta_{32}}^{h},\widehat{\varphi}^{j},\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^{\iota}\widehat{\sigma}\widehat{\tau},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^{1}\widehat{\sigma},\widehat{\beta_{31}},\widehat{\alpha_3}^{1},\widehat{\sigma},\widehat{\tau}\right)+\\\tfrac{\varepsilon_1^2}{m}\left(1+\varepsilon_1-\tfrac{\varepsilon_1^3}{m}\right)&\sum_{h,j,\iota}\Pr^n\left(w_{33}|\widehat{\beta_{32}}^{h},\widehat{\varphi}^{j},\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^{\iota}\widehat{\sigma}\widehat{\tau},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^{\widetilde{k}}\widehat{\sigma},\widehat{\beta_{31}},\widehat{\alpha_3}^{1},\widehat{\sigma},\widehat{\tau}\right)\\&\Pr^n\left(\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^{\iota}\widehat{\sigma}\widehat{\tau}|z_{32}^2=\widehat{\beta_{31}}\widehat{\alpha_3}^{\widetilde{\iota}}\widehat{\sigma}\widehat{\tau},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^{\widetilde{k}}\widehat{\sigma},\widehat{\beta_{31}},\widehat{\alpha_3}^{1},\widehat{\sigma},\widehat{\tau}\right)+\\\tfrac{\varepsilon_1^3}{m}&\sum_{\substack{h,j,\iota\\k\neq1,\widetilde{k}}}\Pr^n\left(w_{33}|\widehat{\beta_{32}}^{h},\widehat{\varphi}^{j},\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^{\iota}\widehat{\sigma}\widehat{\tau},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^{k}\widehat{\sigma},\widehat{\beta_{31}},\widehat{\alpha_3}^{1},\widehat{\sigma},\widehat{\tau}\right)\\&\Pr^n\left(\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^{\iota}\widehat{\sigma}\widehat{\tau}|z_{32}^2=\widehat{\beta_{31}}\widehat{\alpha_3}^{\widetilde{\iota}}\widehat{\sigma}\widehat{\tau},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^{k}\widehat{\sigma},\widehat{\beta_{31}},\widehat{\alpha_3}^{1},\widehat{\sigma},\widehat{\tau}\right).\end{aligned}$$

Similarly to the previous cases, our assumptions regarding the trembles guarantee that in the limit, the above probability is $\frac{1}{m+1}$. Here we show that the condition $\lim_{n\to\infty}\frac{\varepsilon_{1,n}}{(\varepsilon_{3,n})^2}=$

0 is indeed necessary to obtain the previous result. Suppose that P_5 learns the permutation $\widehat{\alpha_3}^{\widetilde{k}}\widehat{\sigma}$ sent by P_2 (the probability of this event is an infinitesimal of the same order as ε_1^2). Then the probability that in Substep 3.2 P_1 learns the permutation $\widehat{\beta_{31}}\widehat{\alpha_3}^{\widetilde{\iota}}\widehat{\sigma\tau}$ (sent by P_2) converges to one. Suppose now that $\widetilde{k}$ is different from $\widetilde{\iota}$. If in Substep 3.3 every sender follows her equilibrium strategy, P_2 receives three different messages. Remember that the probability of this event converges to zero at the same speed as ε_1^2. Now consider the event in which both P_1 and P_5 receive the true permutation. Then the probability that three different messages are sent in Substep 3.3 is an infinitesimal of the same order as ε_3^4. To assign probability zero to the first event (in the limit), we need to assume that $\lim_{n\to\infty}\frac{\varepsilon_{1,n}}{(\varepsilon_{3,n})^2}=0$.

Substep 3.4. *P_1, P_2 and P_5 send P_3 the element $\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi(x)$.*

For any integer n, let $\varepsilon_{4,n}=\varepsilon_n$. We assume that P_1 trembles from her equilibrium strategy with probability ε_4^3. P_2 and P_5 tremble with probability ε_4^2.

With a slight abuse of notation, we let a denote an element of X, and M denote the set of triples of messages in which a is the message sent by the majority. P_3 knows the permutations β_{32}, β_{31}, σ, τ and φ (we denote their realizations by $\widehat{\beta_{32}}$, $\widehat{\beta_{31}}$, $\widehat{\sigma}$, $\widehat{\tau}$ and $\widehat{\varphi}$, respectively). Let $\widehat{x}^1,...,\widehat{x}^{\tilde{m}+1}$ denote $\tilde{m}+1$ different elements of X. For every element $\widehat{x}^l$ in X there are $\tilde{m}!$ different permutations[28] $\widehat{\alpha_3}^{l(\tilde{m}!)+1},...,\widehat{\alpha_3}^{(l+1)(\tilde{m}!)}$ such that $\widehat{\beta_{32}}\widehat{\beta_{31}}\widehat{\alpha_3}^{\widetilde{l}}\widehat{\sigma\tau\varphi}\left(\widehat{x}^l\right)=a$, for every $\widetilde{l}=l(\tilde{m}!)+1,...,(l+1)(\tilde{m}!)$. Further, $l\neq l'$ implies $\widehat{\alpha_3}^{j}\neq\widehat{\alpha_3}^{j'}$, for every $j=l(\tilde{m}!)+1,...,(l+1)(\tilde{m}!)$, and every $j'=l'(\tilde{m}!)+1,...,(l'+1)(\tilde{m}!)$.

P_3's beliefs do not depend on her actions in Steps 1 and 2.[29] Thus, in order to prove that the assessment $\Psi(r)$ satisfies consistency in Substep 3.4, it is sufficient to show that for every triple of messages w_{34} in M, for every $l=1,...,\tilde{m}+1$, and for every $\widetilde{l}=l(\tilde{m}!)+1,...,(l+1)(\tilde{m}!)$, the following equality holds:

[28] We relabel the indices of the permutations, if necessary.

[29] The equilibrium strategies in Step 3 do not depend on P_3's messages in Steps 1 and 2.

$$\lim_{n\to\infty}\Pr{}^n(\widehat{x}^l,\widehat{\alpha_3}^{\tilde{l}},\overline{\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi}=\widehat{\beta_{32}}\widehat{\beta_{31}}\widehat{\alpha_3}^{\tilde{l}}\widehat{\sigma\tau\varphi},\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^{\tilde{l}}\widehat{\sigma\tau},$$
$$\overline{\alpha_3\sigma}=\widehat{\alpha_3}^{\tilde{l}}\widehat{\sigma\tau}|w_{34},\widehat{\beta_{32}},\widehat{\beta_{31}},\widehat{\sigma},\widehat{\tau},\widehat{\varphi})=\tfrac{1}{m+1}.$$

Consider a step in which all the senders know the realization of the permutation that they have to send. Moreover, suppose that their trembles are defined by ε. Then the probability that the receiver learns the truth is $1-\upsilon(\varepsilon)$, and the probability that the receiver learns any other message is $\frac{\upsilon(\varepsilon)}{m}$ (see equations (14) and (15)). Thus, we have:

$$\Pr{}^n\left(\widehat{x}^l,\widehat{\alpha_3}^{\tilde{l}},\overline{\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi}=\widehat{\beta_{32}}\widehat{\beta_{31}}\widehat{\alpha_3}^{\tilde{l}}\widehat{\sigma\tau\varphi},\overline{\beta_{31}\alpha_3\sigma\tau}=\widehat{\beta_{31}}\widehat{\alpha_3}^{\tilde{l}}\widehat{\sigma\tau},\overline{\alpha_3\sigma}=\widehat{\alpha_3}^{\tilde{l}}\widehat{\sigma}|w_{34},\widehat{\beta_{32}},\widehat{\beta_{31}},\widehat{\sigma},\widehat{\tau},\widehat{\varphi}\right)=$$
$$\tfrac{(1-\upsilon(\varepsilon_1))(1-\upsilon(\varepsilon_2))(1-\upsilon(\varepsilon_3))g(\varepsilon_4,\tilde{m};w_{34})}{B_6},$$

where:

$$B_6=(1-\upsilon(\varepsilon_1))(1-\upsilon(\varepsilon_2))(1-\upsilon(\varepsilon_3))\left(\tfrac{m+1}{\tilde{m}+1}\right)\upsilon(\varepsilon_4,\tilde{m};w_{34})+(1-\upsilon(\varepsilon_1))(1-\upsilon(\varepsilon_2))\tfrac{\upsilon(\varepsilon_3)}{m}$$
$$\sum_{h=1}^{\tilde{m}+1}\sum_{\substack{j,k=1\\k\neq j}}^{m+1}\Pr{}^n\left(w_{34}|\widehat{x}^h,\widehat{\alpha_3}^j,\widehat{\beta_{32}}\widehat{\beta_{31}}\widehat{\alpha_3}^k\widehat{\sigma\tau\varphi},\widehat{\beta_{31}}\widehat{\alpha_3}^j\widehat{\sigma\tau},\widehat{\alpha_3}^j\widehat{\sigma},\widehat{\beta_{32}},\widehat{\beta_{31}},\widehat{\tau},\widehat{\varphi}\right)+$$

$$(1-\upsilon(\varepsilon_1))\tfrac{\upsilon(\varepsilon_2)}{m}\sum_{h=1}^{\tilde{m}+1}\sum_{\substack{j,k,\iota=1\\\iota\neq j}}^{m+1}\Pr{}^n\left(w_{34}|\widehat{x}^h,\widehat{\alpha_3}^j,\widehat{\beta_{32}}\widehat{\beta_{31}}\widehat{\alpha_3}^k\widehat{\sigma\tau\varphi},\widehat{\beta_{31}}\widehat{\alpha_3}^\iota\widehat{\sigma\tau},\widehat{\alpha_3}^j\widehat{\sigma},\widehat{\beta_{32}},\widehat{\beta_{31}},\widehat{\tau},\widehat{\varphi}\right)$$
$$\Pr{}^n\left(\widehat{\beta_{32}}\widehat{\beta_{31}}\widehat{\alpha_3}^k\widehat{\sigma\tau\varphi}|\widehat{\beta_{31}}\widehat{\alpha_3}^\iota\widehat{\sigma\tau},\widehat{\alpha_3}^j\widehat{\sigma},\widehat{\alpha_3}^j,\widehat{\sigma},\widehat{\beta_{32}},\widehat{\beta_{31}},\widehat{\tau},\widehat{\varphi}\right)+$$
$$\tfrac{\upsilon(\varepsilon_1)}{m}\sum_{h=1}^{\tilde{m}+1}\sum_{\substack{j,k,\iota,\eta=1\\\eta\neq j}}^{m+1}\Pr{}^n\left(w_{34}|\widehat{x}^h,\widehat{\alpha_3}^j,\widehat{\beta_{32}}\widehat{\beta_{31}}\widehat{\alpha_3}^k\widehat{\sigma\tau\varphi},\widehat{\beta_{31}}\widehat{\alpha_3}^\iota\widehat{\sigma\tau},\widehat{\alpha_3}^\eta\widehat{\sigma},\widehat{\beta_{32}},\widehat{\beta_{31}},\widehat{\tau},\widehat{\varphi}\right)$$
$$\Pr{}^n\left(\widehat{\beta_{32}}\widehat{\beta_{31}}\widehat{\alpha_3}^k\widehat{\sigma\tau\varphi}|\widehat{\beta_{31}}\widehat{\alpha_3}^\iota\widehat{\sigma\tau},\widehat{\alpha_3}^\eta\widehat{\sigma},\widehat{\alpha_3}^j,\widehat{\sigma},\widehat{\beta_{32}},\widehat{\beta_{31}},\widehat{\tau},\widehat{\varphi}\right)\Pr{}^n\left(\widehat{\beta_{31}}\widehat{\alpha_3}^\iota\widehat{\sigma\tau}|\widehat{\alpha_3}^\eta\widehat{\sigma},\widehat{\alpha_3}^j,\widehat{\sigma},\widehat{\beta_{31}},\widehat{\tau}\right).$$

It is tedious but simple to check that as n goes to infinity, the above probability converges to $\frac{1}{m+1}$.

To provide some intuition, suppose that P_3 receives the same message from two senders, and a different one from the third. Consider the two following scenarios. It is possible that all senders know the truth[30] and only one of them deviated in Substep 3.4. Alternatively,

[30] The probability of this event converges to one as n goes to infinity.

it is possible that some of the senders did not learn the truth. However, this implies that at least two players deviated in Substeps 3.1-3.3. Since deviations in Substep 3.4 are at least as likely as deviations in earlier steps, we conclude that, in the limit, the latter event has probability zero.

The case where P_3 receives three different messages can be analyzed in a similar way. If all senders know the truth, the probability of receiving three different messages is an infinitesimal of the same order as ε_4^4. If P_1 and/or P_5 did not learn the truth, then at least two players deviated in Substeps 3.1-3.2. Since $\varepsilon_{4,n}$ converges to zero faster than $\varepsilon_{1,n}$ and $\varepsilon_{2,n}$, we assign probability one to the event that both P_1 and P_5 know the truth. Given this, the probability that P_2 does not learn the truth in Substep 3.3 is an infinitesimal of the same order as ε_3^4. It is true that $\varepsilon_{3,n}$ is equal to $\varepsilon_{4,n}$, but note that P_3 cannot receive three different messages if both P_1 and P_5 follow their equilibrium strategies in Substep 3.4. At least one other deviation is needed. We conclude that all senders know the truth and two of them deviated in the last substep.

Step 4.

Consider the four sequences $\{\varepsilon_{j,n}\}_{n=1}^{\infty}$, $j = 1, ..., 4$, introduced in Step 3. In Substep 4.j, $j = 1, ..., 4$, the first sender in the description of the step trembles from her equilibrium strategy with probability $\varepsilon_{j,n}^3$. Each of the other two senders trembles with probability $\varepsilon_{j,n}^2$ (trembles are always independent across information sets). In each of the other steps (if any) P_1 trembles with probability ε_4^3, and P_2 and P_5 tremble with probability ε_4^2.

The analysis in Step 3 shows that, given the information acquired in Step 0 and Step 4, a receiver should assign probability one to the event that the majority of the senders report the true message and all of them learnt the realization of the corresponding permutation. Again, we use the fact that trembles are independent across information sets and all random choices are independent of each other and uniformly distributed to conclude that the above result does not depend on information from Steps 1-3. At the same time, the additional information in Step 4 does not modify the beliefs of earlier steps.

Step 5.

Substep 5.1. *P_2, P_3 and P_4 send P_1 the function $pr_1\alpha_1^{-1}$.*

We assume that, along the sequence of mixed strategies, P_2 sends P_1 the realization of $pr_1\alpha_1^{-1}$ with probability $(1-\varepsilon_n^3)$, and any other message with probability $\frac{\varepsilon_n^3}{|R_1|-1}$, where $|R_1|$ is the number of feasible messages in Substep 5.1. P_3 and P_4 tremble with probability ε_n^2.

With another slight abuse of notation, let us assume that a is the message sent by the majority of the senders to P_1 in Substep 5.1. Let R^c denote the number of permutations compatible with a. We denote these permutations by $\widehat{\alpha_1}^1, ..., \widehat{\alpha_1}^{R^c}$ (again, we relabel the indices if necessary). Consider a triple of messages w_{51} such that a is the message sent by the majority. Given our sequence of mixed strategies, it is easy to verify that for any $l = 1, ..., R^c$, the following equality holds:

$$\lim_{n\to\infty} \Pr^n\left(\alpha_1 = \widehat{\alpha_1}^l | w_{51}\right) = \frac{1}{R^c}.$$

This is sufficient to demonstrate that the assessment $\Psi(r)$ satisfies consistency in Substep 5.1.

Finally, notice that the equality above implies that, conditional on P_1's information in Substep 5.1, the distribution of τ assigns probability $\frac{1}{R^c}$ only to the permutations $\widehat{\tau}^1, ..., \widehat{\tau}^{R^c}$ (the same result holds for the random distributions β_{31} and β_{41}).

Since Substeps 5.2-5.I are identical to Substep 5.1, we conclude that the assessment $\Psi(r)$ is consistent.

Appendix B: Proof of Theorem 2

In this appendix we show how to implement a rational and regular communication equilibrium, q, of a Bayesian game G with at least five players. As anticipated in Section 3, we proceed in two steps. First, we show that q is the outcome induced by a sequential

equilibrium in a cheap talk extension, $G^F(q)$, in which players use a correlation device to communicate. Then, we construct a plain cheap talk extension $\bar{G}(q)$, and a sequential equilibrium of $\bar{G}(q)$ that induces q.

Remember that, given a communication equilibrium q of G, the cheap talk extension $G^D(q)$ is the following extensive-form game. In Step 1, players simultaneously send the mediator their messages (P_i sends a message in T_i). If the vector of reports is t, the mediator randomly selects an action profile in S according to the probability distribution $q(\cdot|t)$, and informs each player P_i only of the ith component of the chosen profile. Then, in Step 2, players simultaneously choose their actions (P_i chooses an action in S_i).

An information set of P_i in Step 2 is characterized by her type, t_i, her report to the mediator in Step 1, t_i', and the mediator's recommendation, s_i. Obviously, the report t_i' and the recommendation s_i can be in the same information set only if there exist a profile of opponents' types t_{-i}, and a profile of opponents' action s_{-i}, such that $q(s_i, s_{-i}|t_i', t_{-i}) > 0$. We let $L_i(t_i')$ denote the set of recommendations that P_i can receive from the mediator after sending message t_i' in Step 1. Formally:

$$L_i(t_i') = \left\{ s_i \in S_i | \sum_{s_{-i} \in S_{-i}} \sum_{t_{-i} \in T_{-i}} q(s_i, s_{-i}|t_i', t_{-i}) > 0 \right\}. \tag{27}$$

Since q is regular, $G^D(q)$ admits a sequential equilibrium $\Phi^D(q)$ that satisfies the following two requirements. First, each player reports her true type to the mediator. Second, each player obeys the mediator's recommendation after reporting her type truthfully to the mediator. We let ψ_i^D denote the equilibrium strategy of P_i in Step 2. For any information set (t_i, t_i', s_i) in Step 2, $\psi_i^D(t_i, t_i', s_i)$ specifies a probability distribution over S_i. The second requirement on $\Phi^D(q)$ implies that for any type t_i in T_i, and for any action s_i in $L_i(t_i)$, $\psi_i^D(t_i, t_i, s_i)$ assigns probability one to s_i. Clearly, $\Phi^D(q)$ induces the outcome q.

The assessment $\Phi^D(q)$ also specifies the players' beliefs. We denote P_i's beliefs in Step 2 by μ_i^D. For each information set (t_i, t_i', s_i), $\mu_i^D\left(\left(t_j, t_j', s_j\right)_{j \neq i} | t_i, t_i', s_i\right)$ is the probability

that each opponent P_j has type t_j, has reported t'_j to the mediator and has received recommendation s_j. Notice that $\mu_i^D(t_i, t'_i, s_i)$ may be a probability distribution over a *proper* subset of $T_{-i}^2 \times S_{-i}$. However, $\mu_i^D(t_i, t'_i, s_i)$ can be easily extended to $T_{-i}^2 \times S_{-i}$ by assigning the value zero to those elements of $T_{-i}^2 \times S_{-i}$ that are not in the original domain of $\mu_i^D(t_i, t'_i, s_i)$. Without introducing further notation, hereafter we refer to $\mu_i^D(t_i, t'_i, s_i)$ as a probability distribution over $T_{-i}^2 \times S_{-i}$.

Finally, let us introduce the sequence of completely mixed strategies that is used to show that the assessment $\Phi^D(q)$ is consistent. Consider the nth element of the sequence, and let t_i and t'_i be two different types of player P_i. We let $\lambda_{i,n}(t'_i|t_i)$ denote the probability that P_i sends the mediator message t'_i when her type is t_i (clearly, $\lim_{n\to\infty} \lambda_{i,n}(t'_i|t_i) = 0$). We denote by λ_n the smallest probability that all players deviate from their equilibrium strategies in Step 1, that is:

$$\lambda_n := \min_{\left(t_i, t'_i | t_i \neq t'_i\right)_{i=1}^I} \prod_{i=1}^I \lambda_{i,n}(t'_i|t_i). \tag{28}$$

We now turn to $G^F(q)$, the cheap talk extension proposed by Forges (1990).[31] First, we need to describe how the mediator chooses her messages. The mediator randomly selects a function f from T to S according to the following probability distribution:

$$\Pr(f) = \Pr\left((f(t))_{t\in T}\right) = \prod_{t\in T} q(f(t)|t).$$

Given two different type profiles, t and t', $f(t)$ is chosen independently of $f(t')$. Further, for each $i = 1, ..., I$, the mediator randomly selects a permutation γ_i on T_i according to the uniform distribution. Finally, for every $i = 1, ..., I$, the mediator also selects a permutation π_i on $T \times S_i$ at random, according to the uniform distribution. $f, \gamma_1, ..., \gamma_I, \pi_1, ..., \pi_I$ are chosen independently of each other.

Given $f, \gamma_1, ..., \gamma_I$, let $f' : T \to S$ be defined by:

[31] Since $G^F(q)$ is analyzed in great detail in Forges (1990), we proceed quickly through its description.

$$f'(t_1,...,t_I) = f\left(\gamma_1^{-1}(t_1),...,\gamma_I^{-1}(t_I)\right) \quad \forall (t_1,...,t_I) \in T,$$

and let $f_i' = T \to S_i$ be the projection of f' on S_i. Further, let Υ be the identity mapping on T and define $(\Upsilon \times f_i') : T \to T \times S_i$ by $(\Upsilon \times f_i')(t) = (t, f_i'(t))$. Then the mediator constructs the mapping $\zeta_i = \pi_i \circ (\Upsilon \times f_i')$ from T to $T \times S_i$, and the mapping $\theta_i = pr_{S_i} \circ \pi_i^{-1}$ from $T \times S_i$ to S_i (i.e. θ_i is the projection on S_i of π_i^{-1}). Clearly, $\theta_i \circ \zeta_i = f_i'$ and θ_i is independent of f_i'. Moreover, since the mapping $(\Upsilon \times f_i')$ is injective, ζ_i is also independent of f_i' (see Forges (1988, 1990)).

The extensive-form game $G^F(q)$ is as follows. The mediator reports γ_i, ζ_{i+1}, ζ_{i+2}, ζ_{i+3} (where $+$ is mod I) and θ_i to every player P_i. In Step 1, players simultaneously announce *public* messages (P_i sends a message in T_i). Step 2 of $G^F(q)$ is divided into I substeps. In Substep 2.i, player P_{i-h} (where $h = 1,2,3$ and $P_{i-h} := P_{i-h+I}$ if $i - h \leqslant 0$) sends P_i a message in $T \times S_i$. The three senders send their messages simultaneously.[32] Finally, in Step 3 players simultaneously choose their action (P_i chooses an action in S_i).

Forges (1990) shows that if q is a communication equilibrium of a Bayesian game G with at least *four* players, then $G^F(q)$ admits a Bayesian-Nash equilibrium that induces q. Additional work is required to show that if, in addition to Forges' (1990) assumptions, the communication equilibrium q is also regular, then there exist a sequential equilibrium $\Phi^F(q)$ of $G^F(q)$ that induces q.

We now describe the assessment $\Phi^F(q)$. Suppose P_i has type t_i, and has received the permutation γ_i from the mediator. In equilibrium, P_i announces the message $\gamma_i(t_i)$ in Step 1 (notice that this message is not a function of ζ_{i+1}, ζ_{i+2}, ζ_{i+3} or θ_i). We also need to construct a sequence of completely mixed strategies. Suppose, as above, that P_i has type t_i and has received the permutation γ_i. Consider a type t_i' different from $\gamma_i(t_i)$. In the nth element of the sequence of mixed strategies, the probability that P_i reports message t_i'

[32] In Forges (1990), messages of Step 2 may, but need not, be public. Here, we insist on *private* messages: in Substep 2.i only P_i observes the messages sent by P_{i-3}, P_{i-2} and P_{i-1}. We will explain later why we need private messages in Step 2.

in Step 1 is equal to $\lambda_{i,n}\left(\gamma_i^{-1}\left(t_i'\right)|t_i\right)$. That is, the probability that P_i (whose type is t_i) reports t_i' in Step 1 of $G^F(q)$ coincides with the probability that she sends the mediator the message $\gamma_i^{-1}\left(t_i'\right)$ in the canonical game $G^D(q)$.

We now consider the equilibrium strategy of P_i in Substep 2.$(i+h)$ (where $h=1,2,3$). In addition to her type t_i, and to the mediator's message $\left(\gamma_i,\zeta_{i+1},\zeta_{i+2},\zeta_{i+3},\theta_i\right)$, P_i knows the vector of reports in Step 1, say t', and the messages that she has already sent and received in Step 2. The equilibrium strategy prescribes that P_i sends message $\zeta_{i+h}(t')$ to P_{i+h}. Note that in equilibrium, in Step 2 each player receives the same message from the three senders.

As far as the sequence of completely mixed strategies is concerned, let $\{\epsilon_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in the unit interval, converging to zero and satisfying the following condition A:

$$\lim_{n\to\infty}\frac{\epsilon_n}{\lambda_n}=0, \tag{A}$$

where λ_n is defined in equation (28). Consider the nth element of the sequence of mixed strategies. Consider P_i in Substep 2.$(i+h)$ and suppose that t' is the vector of reports in Step 1. We assume that with probability $(1-\epsilon_n)$ P_i follows the equilibrium strategy and sends P_{i+h} message $\zeta_{i+h}(t')$. All other messages are equally likely. Condition (A) guarantees that the probability that a single player deviates in a substep of Step 2 converges to zero faster than the probability that all players deviate in Step 1.

Finally, we analyze Step 3. We let d_i^j denote the message sent by P_i to P_j in Substep 2.j (where $j=i+1,i+2,i+3$, and $+$ is mod I). An information set of P_i in Step 3 is characterized by her type t_i, the mediator's message $\left(\gamma_i,\zeta_{i+1},\zeta_{i+2},\zeta_{i+3},\theta_i\right)$, the vector of reports in Step 1 t', the three messages sent in Substeps 2.$(i+1)$ to 2.$(i+3)$ $\left(d_i^{i+1},d_i^{i+2},d_i^{i+3}\right)$, and the three messages received in Substep 2.i $\left(d_{i-3}^i,d_{i-2}^i,d_{i-1}^i\right)$. We say that an information set in Step 3 is of class 1 if the two following conditions are satisfied. First, at least two senders sent the same message in Substep 2.i. We let $\overline{d}^i$ denote the message sent by the

majority of the senders. Second, the message $\overline{d}^i$ is such that $\theta_i\left(\overline{d}^i\right)$ belongs to the set $L_i\left(\gamma_i^{-1}\left(t_i'\right)\right)$, where t_i' is the report of P_i in Step 1 and the set $L_i\left(\cdot\right)$ is defined in equation (27). The second requirement implies that being in an information set of class 1 is not definitive evidence that the majority of the senders deviated from their equilibrium strategies in Substep 2.i. In an information set of class 1, P_i may think that the majority of her senders played the equilibrium strategy, even though this may require a deviation by some opponents in Step 1 (we will come back to this point shortly).

We let ψ_i^F denote P_i's equilibrium strategy in Step 3. To every information set of P_i in Step 3, ψ_i^F assigns a probability distribution on S_i. For any information set of class 1, ψ_i^F satisfies:

$$\psi_i^F\left(t_i,\gamma_i,\zeta_{i+1},\zeta_{i+2},\zeta_{i+3},\theta_i,t',d_i^{i+1},d_i^{i+2},d_i^{i+3},d_{i-3}^i,d_{i-2}^i,d_{i-1}^i\right)=\psi_i^D\left(t_i,\gamma_i^{-1}\left(t_i'\right),\theta_i\left(\overline{d}^i\right)\right), \tag{29}$$

where, remember, $\overline{d}^i$ denotes the message sent by the majority in Substep 2.i, and ψ_i^D is the equilibrium strategy of player P_i in Step 2 of $G^D\left(q\right)$. That is, P_i chooses the same strategy that she plays in $G^D\left(q\right)$ after sending the mediator the message $\gamma_i^{-1}\left(t_i'\right)$, and receiving the recommendation $\theta_i\left(\overline{d}^i\right)$.

To define the equilibrium strategies in information sets that are not of class 1, we first need to analyze equilibrium beliefs. First of all notice that in *any* information set of Step 3, P_i assigns probability one to the event that all her opponents are in information sets of class 1. In fact, messages in Step 2 are private, and the senders of Substep 2.j in equilibrium send P_j the message $\zeta_j\left(t'\right)$ (where t' is the vector of reports in Step 1). It might be that P_i is a sender of P_j and P_i deviated in Substep 2.j. However, with probability one P_j received the message $\zeta_j\left(t'\right)$ from the other two senders.

Consider now any information set of class 1 of P_i. Condition (A) implies that P_i assigns probability one to the event that the majority of her senders played their equilibrium strategies in Substep 2.i. To give an intuition for this result, consider the following case.

Suppose that two senders reported the same message $\overline{d}^i$ (with $\theta_i\left(\overline{d}^i\right) \in L_i\left(\gamma_i^{-1}\left(t_i'\right)\right)$), and that the third sender sent a different message. Consider the nth element of the sequence of mixed strategies. The ratio between the probability that P_i's information set is reached and the majority of the senders deviated, and the probability that P_i's information set is reached and the majority played the equilibrium strategy is smaller than $\epsilon_n/\left(\left(1-\epsilon_n\right)^2\lambda_n\right)$. Condition (A) guarantees that in the limit (as n grows large), P_i assigns probability zero to the event that the majority of the senders deviated from their equilibrium strategies in Substep 2.i. The same conclusion can be derived, *a fortiori*, in an information set of class 1 in which all senders have reported the same message.

The above result, and the fact that the trembles in Step 1 are derived from the trembles of the equilibrium $\Phi^D(q)$ of $G^D(q)$, imply that in any information set of class 1, P_i's (consistent) beliefs satisfy the following condition:

$$\begin{aligned}\Pr((t_j,\gamma_j^{-1}\left(t_j'\right)=\bar{t}_j,\theta_j\left(\zeta_j\left(t'\right)\right)=s_j)_{j\neq i}|t_i,\gamma_i,\zeta_{i+1},\zeta_{i+2},\zeta_{i+3},\theta_i,t',d_i^{i+1},d_i^{i+2},d_i^{i+3},\\ d_{i-3}^i,d_{i-2}^i,d_{i-1}^i)=\mu_i^D\left(\left(t_j,\bar{t}_j,s_j\right)_{j\neq i}|t_i,\gamma_i^{-1}\left(t_i'\right),\theta_i\left(\overline{d}^i\right)\right),\end{aligned} \tag{30}$$

for any vector $\left(t_j,\bar{t}_j,s_j\right)_{j\neq i}$.

We now turn to analyze information sets that are not of class 1. Given the sequence of completely mixed strategies, P_i has some consistent beliefs (we will see that it does not matter what these beliefs are). However, as was said above, P_i assigns probability one to the event that each opponent P_j is in an information set of class 1. Since we have already specified equilibrium strategies in information sets of class 1, it is easy to compute equilibrium strategies in all other information sets of Step 3. P_i selects the strategy that maximizes her expected payoff, given her beliefs and her opponents' strategies in information sets of class 1.

Note that in information sets of class 1 the equilibrium strategy of opponent P_j depends only on t_j (her type), $\gamma_j^{-1}\left(t_j'\right)$ (t_j' is P_j's report in Step 1) and $\theta_j\left(\overline{d}^j\right)$. Given all the information that P_i has in Step 3, her beliefs over the realizations of $\left(t_j,\gamma_j^{-1}\left(t_j'\right),\theta_j\left(\overline{d}^j\right)\right)_{j\neq i}$

do not depend on the messages that she sent in Step 2. Therefore, equilibrium strategies in information sets that are not of class 1 may be selected so that they do not depend on the messages that a player sends in Step 2 (similarly to what happens in information sets of class 1). This concludes the description of the assessment $\Phi^F(q)$.

Checking that $\Phi^F(q)$ is indeed a sequential equilibrium is straightforward. Consider an information set of class 1 of P_i. She assigns probability one to the event that all her opponents are in information sets of class 1 (where the equilibrium strategies are defined by equation (29)). Since her beliefs satisfy equation (30), if P_i has an incentive to deviate in an information set of class 1, then, given the assessment $\Phi^D(q)$, she would have an incentive to deviate from the strategy ψ_i^D in some information set of Step 2 of $G^D(q)$, which is impossible. The strategies in information sets of Step 3 that are not in class 1 are optimal by construction. A deviation by a player in Step 2 does not have any impact on her action and on her opponents' actions in Step 3. The optimality of the strategies of Step 1 comes from the fact that, given the assessment $\Phi^D(q)$, no player has an incentive to lie to the mediator in $G^D(q)$. Finally, beliefs of $\Phi^F(q)$ are consistent, since they are derived by applying Bayes' rule to completely mixed strategies converging to the equilibrium strategies.

We now use the assumptions that q is rational and that G has five or more players, and show how unmediated communication can replace the mediator in $G^F(q)$. Let Ξ_i denote the set of messages that P_i can receive from the mediator in $G^F(q)$, and let ξ denote the probability distribution over $\Xi = \prod_{i=1}^{I} \Xi_i$ according to which the mediator selects the random vector $\left(\gamma_i, \left(\zeta_{i+h}\right)_{h=1}^{3}, \theta_i\right)_{i=1}^{I}$. If q is rational, ξ involves probabilities that are rational numbers. Therefore, there exists a finite set $\tilde{X}$, and functions $\widetilde{pr}_i : \tilde{X} \to \Xi_i$ (for $i = 1, ..., I$), such that picking at random an element $\tilde{x} \in \tilde{X}$ according to the uniform distribution, and then computing $\left(\widetilde{pr}_1(\tilde{x}), ..., \widetilde{pr}_I(\tilde{x})\right)$, generates the probability distribution ξ on Ξ. Note that by applying our communication scheme (Steps 0-5 in Section 5) to $\tilde{X}$, $\widetilde{pr}_1, ..., \widetilde{pr}_I$, players may randomly select a vector $\left(\gamma_i, \left(\zeta_{i+h}\right)_{h=1}^{3}, \theta_i\right)_{i=1}^{I}$ according to the probability distribution ξ, in such a way that every player P_i learns only $\left(\gamma_i, \left(\zeta_{i+h}\right)_{h=1}^{3}, \theta_i\right)$.

We therefore combine our communication scheme and $G^F(q)$ to obtain $\bar{G}(q)$, a plain cheap talk extension. $\bar{G}(q)$ is an extensive form game that can be divided in nine steps. Steps 0-5 of $\bar{G}(q)$ coincide with Steps 0-5 of Section 5 applied to $\tilde{X}, \widetilde{pr}_1, ..., \widetilde{pr}_I$. At the end of our communication protocol, Steps 1-3 of $G^F(q)$ are performed.

We now construct $\Phi(q)$, the sequential equilibrium of $\bar{G}(q)$ that induces q. In the first six steps of $\bar{G}(q)$ the equilibrium strategies of a player do *not* depend on her type. Specifically, every type of P_i $(i = 1, ..., I)$ adopts the equilibrium strategies of P_i in Steps 0-5 of Section 5.

To describe equilibrium strategies in the last three steps of $\bar{G}(q)$ we first need to discuss the information that a player has at the end of our communication scheme. Note that according to the terminology of Section 5, the message sent by the majority of the senders in a given step is a well-defined concept (if the senders send three different messages, the majority message is the message sent by the first player in the description of the step). P_i, $i = 1, ..., 4$, combines the random variables known in Step 0 with the majority messages in Step i and in Substep 5.i to construct a message in Ξ_i (as described in Table 5). We refer to this message, as the message that P_i learns after Step 5 of $\bar{G}(q)$. P_5 learns her message in Ξ_5 by using the majority messages received in Step 2 and in Substep 5.5 (and the random variables known in Step 0). If G has more than five players, P_k $(k = 6, ..., I)$ learns her message in Ξ_k by using the majority messages received in Step 4 and in Substep 5.k (and the random variables known in Step 0).

We are ready to define the equilibrium strategies of $\Phi(q)$ in the last three steps of $\bar{G}(q)$. Suppose that P_i learns message $\left(\bar{\gamma}_i, \left(\bar{\zeta}_{i+h}\right)_{h=1}^{3}, \bar{\theta}_i\right)$. Then P_i adopts the same behavioral strategies that P_i chooses in equilibrium $\Phi^F(q)$ of $G^F(q)$ after receiving message $\left(\bar{\gamma}_i, \left(\bar{\zeta}_{i+h}\right)_{h=1}^{3}, \bar{\theta}_i\right)$ from the mediator (note that messages of Steps 0-5 influence the equilibrium strategies of P_i in the last three steps of $\bar{G}(q)$ only through the message that P_i learns after Step 5).

To construct consistent beliefs, in Steps 0-5, for *every* type of P_i we take the sequence

of completely mixed strategies of P_i described in Appendix A.[33] The mixed strategies of Steps 0-5 are therefore described by the converging sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ (see Appendix A). In Steps 6 and 7 of $\bar{G}(q)$, the sequences of completely mixed strategies are those used above to show consistency of the assessment $\Phi^F(q)$. Moreover, we assume that ε_n converges to zero faster than the probability that *all* players deviate in Steps 6 and 7. Roughly speaking, this condition requires that deviations in Steps 0-5 are much less likely than deviations in Steps 6 and 7.

Note that if the majority of the senders follow their equilibrium strategies in Steps 0-5, then a random element $\tilde{x}$ of $\tilde{X}$ is selected according to the uniform distribution, and every player P_i learns $\widetilde{pr}_i(\tilde{x})$ after Step 5. Our assumption above on the different order of infinitesimality of the deviations guarantees that at *every* information set in Steps 6-8, a player assigns probability one to the event that every player P_j learnt $\widetilde{pr}_j(\tilde{x})$. This implies that the strategies specified by $\Phi(q)$ in Steps 6-8 satisfy the sequential rationality requirement. Sequential rationality of the assessment $\Phi(q)$ in the first five steps follows from our proof in Section 5. We conclude that $\Phi(q)$ is a sequential equilibrium of $\bar{G}(q)$ that induces q.

[33] Note that at the end of Step 5, a player's beliefs over her opponents' types are described by the prior distribution p. In other words, in the first five steps a player does not learn anything new about her opponents' types. This follows from the fact that in those steps, the equilibrium strategies and the sequences of mixed strategies are not functions of a player's type.

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