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SOME RESULTS ON CUBIC AND QUARTIC QUASI-INTERPOLATION OF OPTIMAL APPROXIMATION ORDER ON TYPE-1 TRIANGULATIONS

Abstract. In this paper we summarize the results on new families of C^1 quartic and cubic quasi-interpolating splines on type-1 triangulations approximating regularly distributed data. The splines are directly determined by setting their Bernstein-Bézier coefficients to appropriate combinations of the given data values instead of defining the approximating splines as linear combinations of compactly supported bivariate spanning functions and do not use prescribed derivatives at any point of the domain. The quasi-interpolation operators provided by the proposed schemes reproduce cubic and quadratic polynomials and yield approximation order four and three for smooth functions, respectively.

1. Introduction.

Spline quasi-interpolation is a simple procedure for constructing spline approximations of functions from their values of those of some derivatives at given points. The quasi-interpolating splines obtained in this way produce a quasi-interpolation operator defined on a space of functions $\mathcal{S}$ that is a linear and local map bounded in some relevant norm, and, moreover it reproduces some nontrivial polynomial space. Usually, $\mathcal{S}$ is the space spanned by the translates on a lattice of a nonnegative function with compact support that provide a partition of unity, and B-splines and box splines are very relevant choices [11, 12, 15, 17, 24, 29]. These functions have been used to define quasi-interpolation operators with near-minimal infinity norm (see [4, 5, 2], [1, 3, 6, 20] and [7, 8, 14, 21] for the construction in the univariate, bivariate and trivariate cases, respectively). Other functions than B-splines and box splines have been used to define quasi-interpolants. For instance, quadratic Powell-Sabin splines on nonuniform triangulations in [19] (see also [22, 23, 25]). In general, the operators are defined to reproduce a space of polynomials, but it is also possible to reproduce the whole spline space [18, 13].

A different approach has been adopted in a few papers for defining C^1 quasi-interpolating splines [26, 27, 28]. To our best knowledge, they are the unique ones in the literature dealing with this methodology until the publication of [9, 10]. In [16], the scheme presented in [28] is adapted to surfaces of varying geometric complexity, where the tiling resolution can be locally defined.

The idea of the approach in [26, 27, 28] is to set all the Bernstein-Bézier (BB-) coefficients [17] of the splines by using local portions of the data in such a way that the C^1-smoothness conditions are satisfied as well as the reproduction of the polynomials up to an appropriate total degrees. In [26], this idea is used to define C^1 quadratic quasi-interpolating splines on a type-2 triangulation of a rectangular domain reproducing quadratic polynomials. The paper [28] deals with the construction of C^1 quartic splines on a type-1 triangulation of the real plane, reproducing the space $\mathbb{P}_3$ of cubic polynomials. Finally, in [27] C^1 cubic quasi-interpolating splines on a type-6 tetrahedral partition of a rectangular, volumetric domain, are defined, and the associated operator reproduces trilinear polynomials. In all cases only point evaluations are used.

In this paper, we summarize the results in [9, 10] obtained in analyzing the construction proposed in [28]. New operators with optimal approximation orders, less computational cost and smaller uniform norms are defined.

It is foreseeable that the analysis carried out in [9, 10] can be generalized to obtain C^2 quasi-interpolating splines, although the use of a symbolic software will be even more necessary.

Here is an outline of the paper. In Section 2, we give some preliminaries on the BB-form of C^1-splines on type-1 triangulations and we introduce some useful notations used throughout the paper. In Section 3 and 4, we define families of quartic and cubic quasi-interpolating splines based on two different sets of points. We analyze the general schemes, depending on a certain number of free parameters and we present the results that some strategies provide in order to fix them. In all cases, the approximation properties of the corresponding operators are discussed.

2. Notations and preliminaries

Let Δ_h be the triangulation spanned by the vectors $e_1 := (h,h)$ and $e_2 := (h,-h)$, $h > 0$. Its vertices $v_{i,j} := ie_1 + je_2,\ i,j \in \mathbb{Z}$, define the two-dimensional lattice $\mathcal{V} := \{v_{i,j} : i,j \in \mathbb{Z}\}$. The plane is then divided into parallelograms

$$P_{i,j} := [v_{i,j}, v_{i,j+1}, v_{i+1,j+1}, v_{i+1,j}].$$

Each parallelogram $P_{i,j}$ is split into the triangles $T_{i,j} := [v_{i,j}, v_{i+1,j+1}, v_{i+1,j}]$ and $\widetilde{T}_{i,j} := [v_{i,j}, v_{i+1,j+1}, v_{i,j+1}]$ by drawing the diagonal $[v_{i,j}, v_{i+1,j+1}]$. The triangulation Δ_h can also be viewed as a collection of overlapping hexagons $H_{i,j}$ centered at $v_{i,j}$ and obtained as the union of all triangles around $v_{i,j}$.

Given a function f, the quasi-interpolants $Q_d f$ to be defined belong to the spline space

$$\mathcal{S}_d^1(\Delta_h) := \left\{ s \in C^1\left(\mathbb{R}^2\right) : s_{|T} \in \mathbb{P}_d, \text{ for all } T \in \Delta_h \right\},$$

with $d = 3,4$, where $\mathbb{P}_d$ stands for the space of polynomials of total order $d+1$. On every triangle $T := [v_1, v_2, v_3] \in \Delta_h$, $Q_d f$ can be written in terms of the Bernstein polynomials relative to T. More explicitly, $Q_d f_{|T} := \sum_{|k|=d} c_k^T B_k^T$ for some coefficients c_k^T, with $k := (k_1,k_2,k_3)$, $|k| := k_1 + k_2 + k_3$, and $B_k^T := \frac{d!}{k!} b^k$, where $b := (b_1,b_2,b_3)$ is the vector of barycentric coordinates relative to T.

For a given $\ell \geqslant 1$, ξ_ℓ^T will denote the lattice of the domain points

$$\xi_k^T := (k_1 v_1 + k_2 v_2 + k_3 v_3)/\ell, |k| = \ell.$$

The BB-coefficients c_k^T will be associated with the domain points ξ_k^T, and the surface given by $Q_d f$ on T will be contained in the convex hull of the B-net $\left\{ \left(\xi_k^T, c_k^T\right), |k| = d \right\}$.

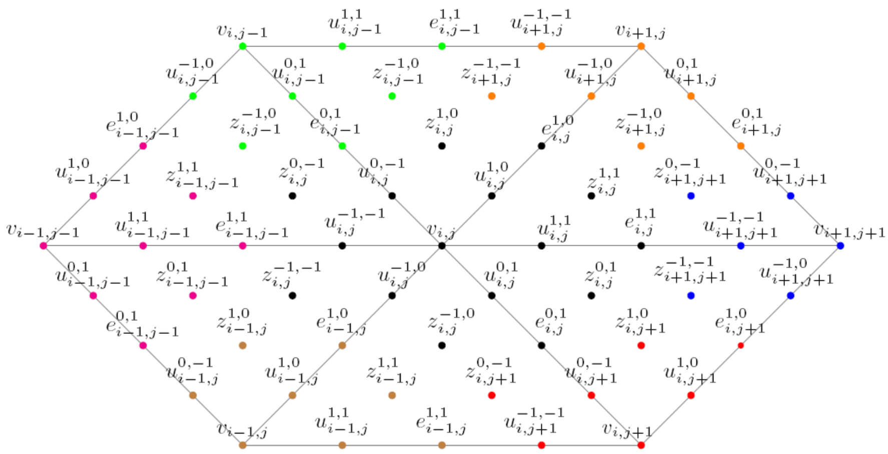

Figure 1: The points of $\mathcal{D}_4$ relative to $H_{i,j}$.

The union, without repetitions, of all domain points of each triangle in Δ_h gives rise to a set denoted by $\mathcal{D}_\ell$. For the construction of $Q_d f$ we consider the subsets $\mathcal{D}_2$ and $\mathcal{D}_3$. The resulting quasi-interpolants will be denoted $Q_{d,m}f$, $m = 2,3$.

The proposed construction is based on partitions $\left\{\mathcal{D}^\ell_{i,j}, i,j \in \mathbb{Z}\right\}$ of $\mathcal{D}_\ell$, $\ell = 2,3,4$, where $\mathcal{D}^\ell_{i,j}$ is composed by 4, 9 and 16 domain points in the triangles defining $H_{i,j}$, respectively. More precisely,

$$\begin{aligned}
\mathcal{D}^4_{i,j} &:= \Big(v_{i,j}, u^{1,1}_{i,j}, u^{1,0}_{i,j}, u^{0,-1}_{i,j}, u^{-1,-1}_{i,j}, u^{-1,0}_{i,j}, u^{0,1}_{i,j}, e_{i,j},\\
&\qquad z^{1,1}_{i,j}, e^{1,0}_{i,j}, z^{1,0}_{i,j}, z^{0,-1}_{i,j}, z^{-1,-1}_{i,j}, z^{-1,0}_{i,j}, e^{0,1}_{i,j}, z^{0,1}_{i,j}\Big),\\
\mathcal{D}^3_{i,j} &:= \Big(v_{i,j}, w^{1,1}_{i,j}, w^{1,0}_{i,j}, w^{0,-1}_{i,,j}, w^{-1,-1}_{i,j}, w^{-1,0}_{i,j}, w^{0,1}_{i,j}, t_{i,j}, \widetilde{t}_{i,j}\Big),\\
\mathcal{D}^2_{i,j} &:= \Big(v_{i,j}, e^{1,0}_{i,j}, e^{0,1}_{i,j}, e^{1,1}_{i,j}\Big),
\end{aligned}$$

where $e^{k,m}_{i,j}$ is the midpoint of $[v_{i,j}, v_{i+k,j+m}]$,

$$u^{k,m}_{i,j} := \frac{1}{4}\left(3v_{i,j} + v_{i+k,j+m}\right), \quad z^{k,m}_{i,j} := \frac{1}{4}\left(2v_{i,j} + v_{i+k,j+m} + v_{r,s}\right),$$

with $v_{r,s}$ the third vertex of $[v_{i,j}, v_{i+k,j+m}, v_{r,s}] \in \Delta_h$ counting counterclockwise, $w^{k,m}_{i,j} := \frac{1}{3}\left(2v_{i,j} + v_{i+k,j+m}\right)$, and $t_{i,j}$ and $\widetilde{t}_{i,j}$ are the barycenters of $T_{i,j}$ and $\widetilde{T}_{i,j}$, respectively. It holds $\mathcal{D}_\ell = \bigcup_{i,j} \mathcal{D}^\ell_{i,j}$, $\ell = 2,3,4$. Figs. 1, 2 and 3 show the domain points in $\mathcal{D}_\ell$, $\ell = 4,3,2$, lying in the hexagon $H_{i,j}$, respectively.

Using the notations above, the restrictions of the quasi-interpolants $Q_{d,m}f$ to the triangles $T_{i,j}$ and $\widetilde{T}_{i,j}$ can be expressed in terms of the Bernstein polynomials of degree

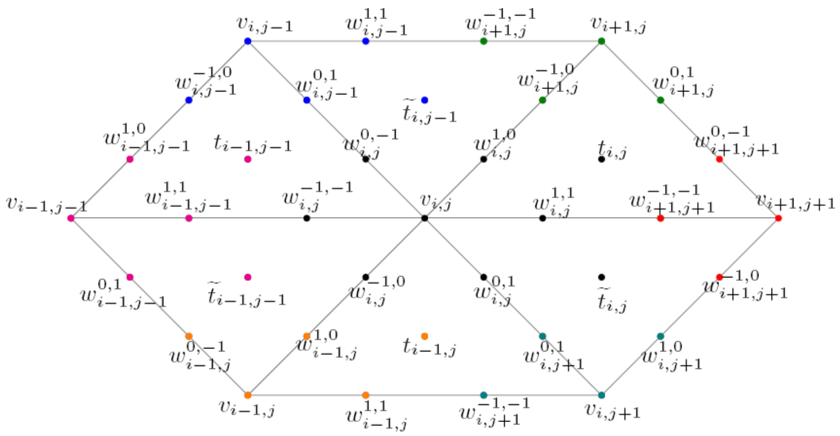

Figure 2: The points of $\mathcal{D}_3$ relative to $H_{i,j}$.

d. For $x \in T_{i,j}$ (resp. $\widetilde{T}_{i,j}$)

(2.1) $$Q_{d,m} f(x) = \sum_{p \in \mathcal{A}_{i,j}^d} c_{d,m}(p) B_P(b) \qquad \text{(resp. } Q_{d,m} f(x) = \sum_{p \in \widetilde{\mathcal{A}}_{i,j}^d} \widetilde{c}_{d,m}(p) B_P(b)\text{)},$$

where $\mathcal{A}_{i,j}^d$ (resp. $\widetilde{\mathcal{A}}_{i,j}^d$) is the subset containing the domain points $\mathcal{D}_{i,j}^d \cup \mathcal{D}_{i+1,j+1}^d \cup \mathcal{D}_{i+1,j+}^d$ (resp. $\mathcal{D}_{i,j}^d \cup \mathcal{D}_{i+1,j+1}^d \cup \mathcal{D}_{i,j+1}^d$) lying in $T_{i,j}$ (resp. $\widetilde{T}_{i,j}$), $P := (P_1, P_2, P_3)$ stands for the index k associated with p, and

$$B_P(b) = \frac{d!}{P_1! P_2! P_3!} b_1^{P_1} b_2^{P_2} b_3^{P_3}.$$

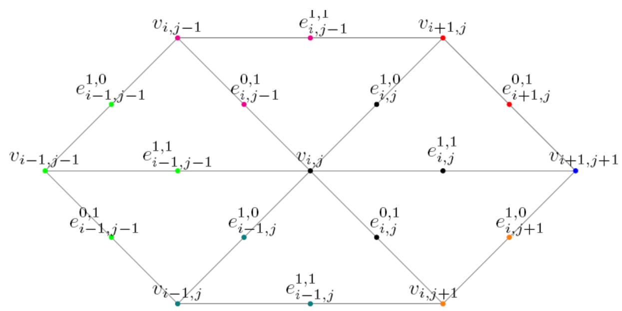

Figure 3: The points of $\mathcal{D}_2$ relative to $H_{i,j}$.

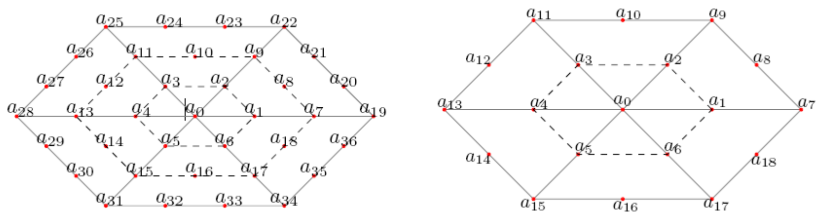

Figure 4: Notation used for enumerate $f_{i,j}(\mathcal{D}_\ell)$ and a general mask a in case $\ell=3$ (left) and $\ell=2$ (right).

3. C^1 quartic quasi-interpolating splines

In this section, we construct two different quasi-interpolating splines $Q_{4,m}f \in \mathcal{S}_4^1(\Delta_h)$, $m=2,3$, for a given function $f \in C(\mathbb{R}^2)$, by assuming to know the values $f(v)$, $v \in \mathcal{D}_m$. They will be defined by setting their BB-coefficients on the triangles $T_{i,j}$ and $\widetilde{T}_{i,j}$. Taking into account that Δ_h is a uniform triangulation, it is sufficient to define the BB-coefficients $c(p) := c_{4,m}(p)$ corresponding to the domain points $p \in \mathcal{D}_4$ denoted by the letters v, u, e and z.

Let $c(v_{i,j}) := f(v_{i,j})$. The BB-coefficients corresponding to the domain points denoted by the letters u, e and z, are expressed as linear combination of the values of f at the N_m domain points of $\mathcal{D}_m$ lying in $H_{i,j}$ with coefficients defining *masks* enumerated as in Fig. 4. The values of f involved in those expressions form the vector $f_{i,j}(\mathcal{D}_m) \in \mathbb{R}^{N_m}$, ordered as the masks, where $N_2=19$ and $N_3=37$.

So, we write $c\left(u_{i,j}^{1,1}\right) = f_{i,j}(\mathcal{D}_m)\cdot\omega$, $c\left(e_{i,j}^{1,1}\right) = f_{i,j}(\mathcal{D}_m)\cdot\alpha$, $c\left(z_{i,j}^{1,1}\right) = f_{i,j}(\mathcal{D}_m)\cdot\beta$ for masks $\omega,\alpha,\beta \in \mathbb{R}^{N_m}$, where $A\cdot B := \sum_{k=1}^{n} A_k B_k$, with n the cardinality of A and B. The BB-coefficients associated with the other u-points ($u_{i,j}^{1,0}$, $u_{i,j}^{0,-1}$, $u_{i,j}^{-1,-1}$, $u_{i,j}^{-1,0}$, and $u_{i,j}^{0,1}$) are defined in a similar way but using the *rotated versions* of the mask ω. The BB-coefficients associated with the other e, z-points are defined from the rotated versions of α and β, respectively. The following results hold (see [9], in particular for expressions providing the masks).

PROPOSITION 1. *There exist infinitely many masks α,β and ω depending on the fourteen parameters α_0, α_2, β_j, $j\in\{1,2,3,7,8,9,10,11,12,19,20,21\}$, such that the quasi-interpolation operator $Q_{4,3}$ defined by $Q_{4,3}(f) := Q_{4,3}f$ is exact on $\mathbb{P}_3$. If, in addition to the exactness on $\mathbb{P}_3$, the minimization of errors associated with quartic polynomials is required, then $\alpha_0=\frac{10}{21}$ and $\alpha_2=-\frac{27}{56}$, and the aforementioned parameters β_j remain free. If the minimization of quasi-interpolation errors for quartic and quintic polynomials are prescribed, then*

$$\beta_1=\beta_2=\beta_3=\beta_7=\beta_9=\beta_{11}=0,\ \beta_8=\frac{613}{448},\ \beta_{10}=-\frac{321}{4480},$$
$$\beta_{12}=\frac{653}{4480},\ \beta_{19}=\frac{1803}{11200},\ \beta_{20}=-\frac{5021}{14000},\ \beta_{21}=-\frac{7683}{28000}.$$

In all cases, for all triangle $T \in \Delta_h$ there exist constants $\bar{K}_{|\gamma|}$, independent on h, such that for every $f \in C^{r+1}\left(\mathbb{R}^2\right)$, $0 \leqslant r \leqslant 3$,

$$\|D^\gamma (f - Q_{4,3}f)\|_{\infty,T} \leqslant \bar{K}_{|\gamma|} h^{r+1-|\gamma|} \left\|D^{r+1} f\right\|_{\infty,\Omega_T},$$

where Ω_T stands for the union of the triangles in Δ_h having a non-empty intersection with T.

The same methodology allows to construct quasi-interpolating splines $Q_{4,2}f$ based on the values $f(v)$, $v \in \mathcal{D}_2$, and therefore using fewer points than $Q_{4,3}f$. Now, $c(p)$ will stand for the BB-coefficient $c_{4,2}(p)$. Thus, let $c(v_{i,j}) := f(v_{i,j})$, $c\left(e^{1,1}_{i,j}\right) = f_{i,j}(D_2) \cdot \alpha$, $c\left(z^{1,1}_{i,j}\right) = f_{i,j}(D_2) \cdot \beta$ and $c\left(u^{1,1}_{i,j}\right) = f_{i,j}(D_2) \cdot \omega$, with masks α, β and ω in $\mathbb{R}^{19}$, and define the BB-coefficients to the remaining u-points as well as the e, z-points by their corresponding rotated versions. We have the following result [10].

PROPOSITION 2. *There exist infinitely many masks α, β and ω depending on the three parameters β_1, β_2, β_3 such that the quasi-interpolation operator $Q_{4,2}$ defined by $Q_{4,2}(f) := Q_{4,2}f$ is exact on $\mathbb{P}_3$. The values of the mask α are $\alpha_0 = \alpha_7 = -\frac{1}{3}$, $\alpha_1 = \frac{2}{3}$, $\alpha_2 = \alpha_6 = \alpha_8 = \alpha_{18} = \frac{1}{3}$, $\alpha_9 = \alpha_{17} = -\frac{1}{6}$, $\alpha_j = 0$, $j \in \{3,4,5,10,11,12,13,14,15,16\}$, and the values of the masks β and ω satisfy the following conditions:*

- $\beta_0 = \frac{1}{3}$, $\beta_4 = \frac{1}{3} - \beta_1$, $\beta_5 = \frac{1}{3} - \beta_2$, $\beta_6 = -\beta_3$, $\beta_7 = -\frac{5}{8} + \frac{5}{8}\beta_1 + \frac{3}{8}\beta_2 - \frac{3}{8}\beta_3$, $\beta_8 = \frac{7}{6} - \beta_1 - \beta_2$, $\beta_9 = -\frac{5}{8} + \frac{3}{8}\beta_1 + \frac{5}{8}\beta_2 + \frac{3}{8}\beta_3$, $\beta_{10} = \frac{1}{2} - \beta_2 - \beta_3$, $\beta_{11} = -\frac{3}{8}\beta_1 + \frac{3}{8}\beta_2 + \frac{5}{8}\beta_3$, $\beta_{12} = -\frac{1}{2} + \beta_1 - \beta_3$, $\beta_{13} = \frac{11}{24} - \frac{5}{8}\beta_1 - \frac{3}{8}\beta_2 + \frac{3}{8}\beta_3$, $\beta_{14} = -\frac{5}{6} + \beta_1 + \beta_2$, $\beta_{15} = \frac{11}{24} - \frac{3}{8}\beta_1 - \frac{5}{8}\beta_2 - \frac{3}{8}\beta_3$, $\beta_{16} = -\frac{1}{2} + \beta_2 + \beta_3$, $\beta_{17} = \frac{3}{8}\beta_1 - \frac{3}{8}\beta_2 - \frac{5}{8}\beta_3$, $\beta_{18} = \frac{1}{2} - \beta_1 + \beta_3$,
- $\omega_0 = 1$, $\omega_1 = -\frac{2}{3} + \beta_1 + \beta_2$, $\omega_2 = -\frac{1}{3} + \beta_2 + \beta_3$, $\omega_3 = \frac{1}{3} - \beta_1 + \beta_3$, $\omega_4 = \frac{2}{3} - \beta_1 - \beta_2$, $\omega_5 = \frac{1}{3} - \beta_2 - \beta_3$, $\omega_6 = -\frac{1}{3} + \beta_1 - \beta_3$, $\omega_7 = -\frac{11}{12} + \beta_1 + \beta_2$, $\omega_8 = \frac{4}{3} - \beta_1 - 2\beta_2 - \beta_3$, $\omega_9 = -\frac{11}{24} + \beta_2 + \beta_3$, $\omega_{10} = \beta_1 - \beta_2 - 2\beta_3$, $\omega_{11} = \frac{11}{24} - \beta_1 + \beta_3$, $\omega_{12} = -\frac{4}{3} + 2\beta_1 + \beta_2 - \beta_3$, $\omega_{13} = \frac{11}{12} - \beta_1 - \beta_2$, $\omega_{14} = -\frac{4}{3} + \beta_1 + 2\beta_2 + \beta_3$, $\omega_{15} = \frac{11}{24} - \beta_2 - \beta_3$, $\omega_{16} = -\beta_1 + \beta_2 + 2\beta_3$, $\omega_{17} = -\frac{11}{24} + \beta_1 - \beta_3$, $\omega_{18} = \frac{4}{3} - 2\beta_1 - \beta_2 + \beta_3$.

  Moreover, the error estimates in Proposition 1 hold. If $\beta_1 = \beta_2$ and $\beta_3 = 0$, then the upper bound $\max\{\|\alpha\|_1, \|\beta\|_1, \|\omega\|_1\} = \max\left\{\sum_{k=0}^{18}|\alpha_k|, \sum_{k=0}^{18}|\beta_k|, \sum_{k=0}^{18}|\omega_k|\right\}$ *to the uniform norm of the operator $Q^*_{4,2}$ is minimized for all $\beta_1 \in \left[\frac{13}{36}, \frac{41}{84}\right]$, and for these values $\left\|Q^*_{4,2}\right\|_\infty = 3$.*

The quasi-interpolation scheme defined [28] is exact on $\mathbb{P}_3$, and its uniform norm is equal to 10. The new operator $Q^*_{4,2}$ is also exact on $\mathbb{P}_3$, but its norm is equal to 3. Moreover, the computational cost is almost halved taking into account that the BB-coefficients of the quasi-interpolants $Q^*_{4,2}f$ are computed from masks in $\mathbb{R}^{19}$ instead of $\mathbb{R}^{37}$.

4. C^1 cubic quasi-interpolating splines

The methodology used in the previous section will provide C^1-cubic quasi-interpolating $Q_{3,m}f$ splines from the values $f(v)$, $v \in \mathcal{D}_m$ in such a way that the associated operator $Q_{3,m}$ is exact on $\mathbb{P}_2$. Now the BB-coefficients associated with the domain points $v_{i,j}$, $w_{i,j}^{1,1}$, $w_{i,j}^{1,0}$, $w_{i,j}^{0,-1}$, $w_{i,j}^{-1,-1}$, $w_{i,j}^{-1,0}$, $w_{i,j}^{0,1}$, $t_{i,j}$, $\widetilde{t}_{i,j-1}$, $t_{i-1,j-1}$, $\widetilde{t}_{i-1,j-1}$, $t_{i-1,j}$ and $\widetilde{t}_{i,j}$ must be properly defined. However, there are some differences between the cubic and the quartic cases. First of all, $Q_{3,m}f$ will not interpolate f at the vertices, and therefore a new mask to define $c_{3,m}(v_{i,j})$ must be used. Moreover, the BB-coefficients $c_{3,m}(p)$ for $p \in \left\{w_{i,j}^{1,0}, w_{i,j}^{0,-1}, w_{i,j}^{-1,-1}, w_{i,j}^{-1,0}, w_{i,j}^{0,1}\right\}$ cannot be determined using the rotated masks of the one defining $c_{3,m}\left(w_{i,j}^{1,1}\right)$. Therefore, in the cubic case and based on $\mathcal{D}_2$, we look for masks α, β, $\widetilde{\beta}$, and γ_k, $0 \leqslant k \leqslant 5$, in $\mathbb{R}^{19}$ such that

$$c_{3,2}(v_{i,j}) = f_{i,j}(\mathcal{D}_2)\cdot\alpha,\ c_{3,2}(t_{i,j}) = f_{i,j}(\mathcal{D}_2)\cdot\beta,\ c_{3,2}\left(\widetilde{t}_{i,j}\right) = f_{i,j}(\mathcal{D}_2)\cdot\widetilde{\beta}$$

and

$$c_{3,2}\left(w_{i,j}^{1,1}\right) = f_{i,j}(\mathcal{D}_2)\cdot\gamma_0,\ c_{3,2}\left(w_{i,j}^{1,0}\right) = f_{i,j}(\mathcal{D}_2)\cdot\gamma_1,\ c_{3,2}\left(w_{i,j}^{0,-1}\right) = f_{i,j}(\mathcal{D}_2)\cdot\gamma_2,$$
$$c_{3,2}\left(w_{i,j}^{-1,-1}\right) = f_{i,j}(\mathcal{D}_2)\cdot\gamma_3,\ c_{3,2}\left(w_{i,j}^{-1,0}\right) = f_{i,j}(\mathcal{D}_2)\cdot\gamma_4,\ c_{3,2}\left(w_{i,j}^{0,1}\right) = f_{i,j}(\mathcal{D}_2)\cdot\gamma_5.$$

The following result holds [10],

PROPOSITION 3. *There exist unique masks α, β, $\widetilde{\beta}$, and γ_k, $0 \leqslant k \leqslant 5$, given in Table 4.1, such that the quasi-interpolation operator $Q_{3,2}^* : C\left(\mathbb{R}^2\right) \longrightarrow S_3^1(\Delta_h)$ defined as $Q_{3,2}^*(f) := Q_{3,2}^* f$ from the quasi-interpolating splines $Q_{3,2}^* f$ provided by those masks is exact on $\mathbb{P}_2$. Moreover,*

$$\left\|Q_{3,2}^*\right\|_\infty \leqslant \max\left\{\|\alpha\|_1, \|\beta\|_1, \|\gamma\|_1, \|\omega_k\|_1, 0 \leqslant k \leqslant 5\right\} = \frac{13}{3}.$$

For an arbitrary triangle $T \in \Delta_h$, let Ω_T be the union of the triangles in Δ having a non-empty intersection with T. Then, there exist constants $\bar{K}_{|\gamma|}$, independent on h, such that for every $f \in C^{r+1}\left(\mathbb{R}^2\right)$, $0 \leqslant r \leqslant 2$,

$$\left\|D^\gamma\left(f - Q_{3,2}^* f\right)\right\|_{\infty,T} \leqslant \bar{K}_{|\gamma|} h^{r+1-|\gamma|}\left\|D^{r+1} f\right\|_{\infty,\Omega_T},$$

for all $0 \leqslant |\gamma| \leqslant r$, $\gamma = (\gamma_1, \gamma_2)$.

Following the same logical scheme, it is possible to construct quasi-interpolation operators $Q_{3,3}$ based on masks in $\mathcal{D}_3$. There are infinitely many masks such that $Q_{3,3}$ is exact on $\mathbb{P}_3$ and provides C^1 quasi-interpolating splines $Q_{3,3}f$. They depend on the values of the components $\gamma_{0,0}$ and $\gamma_{2,0}$ of the masks γ_0 and γ_2 relative to the domain points $w_{i,j}^{1,1}$ and $w_{i,j}^{0,-1}$, respectively [10]. The standard upper bound to the infinity norm of the operator $Q_{3,3}$ given by the maximum of the 1-norms of the masks attains its maximum value (that it is equal to 5) at all points lying in the triangle with vertices $(-1,0)$, $\left(-\frac{7}{6},0\right)$ and $\left(-\frac{7}{6},1\right)$.

ℓ	18α	18β	$18\widetilde{\beta}$	$9\gamma_0$	$18\gamma_1$	$18\gamma_2$	$9\gamma_3$	$18\gamma_4$	$18\gamma_5$
0	-10	-17	-17	-9	-14	-6	-1	-6	-14
1	0	0	0	0	0	0	0	0	0
2	8	24	8	8	16	8	0	0	8
3	8	8	0	4	16	16	4	0	0
4	0	0	0	0	0	0	0	0	0
5	8	0	8	4	0	0	4	16	16
6	8	8	24	8	8	0	0	8	16
7	0	-1	-1	0	0	0	0	0	0
8	0	8	0	0	0	0	0	0	0
9	-1	-7	-1	-1	-2	-1	0	0	-1
10	0	0	0	0	0	0	0	0	0
11	-5	-4	0	-2	-9	-10	-3	-1	0
12	8	0	0	0	8	16	8	8	0
13	-8	0	0	0	-4	-12	-8	-12	-4
14	8	0	0	0	0	8	8	16	8
15	-5	0	-4	-2	0	-1	-3	-10	-9
16	0	0	0	0	0	0	0	0	0
17	-1	-1	-7	-1	-1	0	0	-1	-1
18	0	0	8	0	0	0	0	0	0

Table 4.1: The masks in Proposition 3.

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