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Ernst Cassirer's transcendental account of mathematical reasoning

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ABSTRACT

Cassirer's philosophical agenda revolved around what appears to be a paradoxical goal, that is, to reconcile the Kantian explanation of the possibility of knowledge with the conceptual changes of nineteenth and early twentieth-century science. This paper offers a new discussion of one way in which this paradox manifests itself in Cassirer's philosophy of mathematics. Cassirer articulated a unitary perspective on mathematics as an investigation of structures independently of the nature of individual objects making up those structures. However, this posed the problem of how to account for the applicability of abstract mathematical concepts to empirical reality. My suggestion is that Cassirer was able to address this problem by giving a transcendental account of mathematical reasoning, according to which the very formation of mathematical concepts provides an explanation of the extensibility of mathematical knowledge. In order to spell out what this argument entails, the first part of the paper considers how Cassirer positioned himself within the Marburg neo-Kantian debate over intellectual and sensible conditions of knowledge in 1902–1910. The second part compares what Cassirer says about mathematics in 1910 with some relevant examples of how structural procedures developed in nineteenth-century mathematics.

1. Introduction

Ernst Cassirer's epistemological work is being discussed again in a variety of contexts, from Michael Friedman's dynamics of reason to different variants of ontic and structural realism.[1] However, much of this discussion focuses on Cassirer's account of scientific objectivity while calling into question the feasibility of his philosophical project. This is mainly due to the fact that Cassirer's neo-Kantian agenda revolves around what appears to be a paradoxical goal, that is, to reconcile the Kantian explanation of the possibility of knowledge with the conceptual changes of nineteenth and early twentieth-century science.

This paper offers a new discussion of one way in which this paradox manifests itself in Cassirer's philosophy of mathematics. Beginning in 1910, Cassirer articulated a unitary perspective on mathematics as an investigation of structures independently of the nature of individual objects making up those structures. However, a tension remains between Cassirer's demand for the unity of knowledge and his reliance on the structural methods of nineteenth-century mathematics. Cassirer tried to resolve this tension in his early works by pointing out that the loss of unity with regard to the subject-matter of modern mathematics – insofar as this ceases to define itself as the science of numbers and quantities – is compensated by the deeper unity of its method (Cassirer, 1907b, p. 31; 1910, p. 36). However, after the development of modern axiomatics, Cassirer realized ever more clearly that mathematics (including the most abstract parts of it) raises new problems of its own. In general, beginning in the 1920s, he acknowledged different types of objectivity at stake in the different ways to understand the world, which he called "symbolic forms." This seems to suggest that in order to account for the unity of mathematics in the latter sense, it would be inevitable to call into question the unity of knowledge in Cassirer's original account.

More recent discussions of Cassirer's philosophy of mathematics reflect the same tension. Heis (2010) suggests that a charitable way to read Cassirer today would have to offer a unitary account of mathematical objectivity. By contrast, Mormann (2008) maintains that the central thesis of Cassirer's philosophy from 1910 to his later works is that mathematical and physical knowledge are of the same kind (*sameness thesis*). Mormann offers a series of examples of how the extension of both kinds of knowledge requires the introduction of ideal elements. It follows, however, that a consistent development of the sameness thesis in the light of twentieth-century mathematics would have to acknowledge incompatible idealizations. In other words, quite contrary to Heis, Mormann's suggestion is to allow for a plurality of conceptual frameworks in the philosophy of mathematics in order to retain the main insight of the sameness thesis.

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[1] For a discussion of the receptions of Cassirer in contemporary philosophy of science, along with further references, see Heis (2014b).

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The aim of this paper is to clarify how both aspects of Cassirer's philosophy stand together by spelling out the sameness thesis in terms of a transcendental claim that can be summarized as follows: *Structural reasoning is an indispensable precondition for mathematical objectivity and for the applicability of mathematics in scientific theories*. The corresponding claim in Kant's transcendental philosophy is that: *Spatiotemporal structures are necessary conditions for the apodictic certainty of mathematics and for the applicability of mathematics to all objects of experience.*[2] A number of authors have emphasized that Cassirer departs from Kant, insofar as Kant's mediating structures for connecting mathematical to empirical concepts (i.e., the forms of intuition), in Cassirer's claim, are replaced with a structuralist view of mathematics (see Friedman, 2001; Heis, 2010; 2014a). This allows him to reaffirm the applicability of mathematics with respect to the concrete interpretations of mathematical structures in scientific theories. However, this would imply a shift from Kant's conditions for the possibility of experience to the ideas of unity and stability that regulate the organization of scientific laws. This seems to come at the price of abandoning Kant's transcendental deduction, that is, the proof that the categories of the understanding correctly apply to all objects of experience via the mediating synthesis of space and time.

I will argue that, by contrast, the very notions of structural and mathematical reasoning in Cassirer's view are derived from the reading of the Kantian theory of experience articulated by Cassirer's teachers, Cohen and Natorp, and further developed by Cassirer (1907a). I will then turn to how Cassirer connects the account of mathematical reasoning that emerges from this reading to the structural turn of nineteenth-century mathematics.

My suggestion is that a more careful consideration of the key examples for Cassirer's account can shed light on his long-term strategy to resolve the tension between his emphasis on the unity of mathematics and the sameness thesis. This tension is sharpened by the fact that the contemporary literature on Cassirer's philosophy of mathematics focuses on the embedding of a particular domain into a larger structure. Paradigmatic examples of this are the introduction of irrational numbers as limits of converging series of rational numbers and the generalization of the Euclidean plane to the projective plane. While these examples underpin a unitary perspective on specific mathematical disciplines, they seem to suggest an oversimplified account of how mathematical concepts extend beyond the original ground for their development. I will argue that there is a no less essential aspect of concept formation in Cassirer's sense, that is the transposition of structural methods from one domain to another. Three examples are particularly relevant here: (1) Dedekind's definition of natural numbers, (2) Felix Klein's use of transfer principles, (3) the construction of a numerical scale on the projective line. The epistemological implications of Cassirer's approach follow from the more articulated picture of structural mathematics that emerges from such examples, or so I will argue.

2. The philosophical roots of Cassirer's account

The philosophical roots of Cassirer's account are found in Hermann Cohen's reading of Kant and the further development of Cohen's ideas in the critical idealism of the Marburg School. This is the view that scientific theories are the only facts available for the investigation of knowledge and should not be confused with something given directly to the senses. This posed the problem of clarifying the relation of sensible and intellectual conditions of knowledge, that is, the pillar of Kant's critique of reason. Cohen developed his views in this regard in several stages of his intellectual career, which caused a controversy with Paul Natorp and other members of the school. At the center of the debate was Cohen's claim that the structure of Kant's *Critique* ought to be substantially rethought by identifying Kant's Anticipations of Perception with the Leibnizian principle of the infinitesimal method.

The following section considers how Cassirer positioned himself in the debate about Kant's relation to Leibniz. It will become clear that Cassirer started from interpretative issues that were lively discussed in the Marburg School to address the tasks of the neo-Kantian epistemology.

2.1. The debate about Kant's relation to Leibniz in the Marburg School

Cohen presented his interpretation of Kant's critique of reason as an investigation into the conceptual structure of experience in 1871. However, it was only in the 1880s that Cohen elaborated on the implications of his interpretation for the relation between transcendental philosophy and the sciences. He identified the Kantian notion of a possible experience in general with the objective and scientific meaning of experience, or what he also called "the fact of science," as opposed to subjective and psychological experience (see Cohen, 1883, pp. 4–7; 1885, pp. 66–79). The reliance of the fact of science is, in Cohen's eyes, what makes the transcendental philosophy the most consistent development of critical idealism: experience is given as the actual fact of science; the transcendental apparatus is required to investigates how this fact has become possible as *a priori* valid. While this understanding of "transcendental" (as concerning the mode of cognition which makes actual objective knowledge possible) coincides with Kant's definition,[3] Cohen introduced a dynamic element into the transcendental inquiry by focusing on the achievements of scientific knowledge. Richardson spelled out Cohen's view by saying that: "Epistemology investigates how [objective] knowledge is possible by understanding it to be not a given but an achievement that is possible only on the basis of certain necessary presuppositions. Those presuppositions are the *a priori* conditions of the possibility of experience" (Richardson, 2003, p. 60).

While Marburg neo-Kantians relied on Cohen's characterization of transcendental philosophy, Cohen's own attempt to implement such an inquiry in *The Principle of the Infinitesimal Method and Its History* was the source of much debate.[4] Cohen maintained that Leibniz discovered the calculus in an attempt to conceptually establish the reality of differentials as something instantaneous and prior to extension. According to Cohen, Leibniz's discovery offered a more precise formulation of what Kant called Anticipations of Perception according to the principle that *in all appearances the real, which is an object of sensation, has intensive magnitude* (B208). This led Cohen to substantially rethink his work on Kant, which is evident in a second, revised edition of Cohen's *Kant's Theory of Experience* (1885). Here, Cohen emphasized the centrality of the Analytic of Principles to the point of denying that there can be a direct mode of cognition besides thinking or what Kant called "intuition" in the Transcendental Aesthetic (A20/B34). Cohen now maintained that the heterogeneity of the sources of knowledge in Kant's *Critique* hampered a consequent implementation of critical idealism. Cohen's efforts in this direction culminated with his *Logic of Pure Knowledge* (1902), which outlined his project of a new explanation of the possibility of knowledge as originating in pure thinking alone.

The publication of Cohen's *Logic* caused a lively discussion with

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[2] Kant wrote: "The synthesis of spaces and time, as the essential form of all intuition, is that which also makes possible the apprehension of the appearance, and thus all outer experience, and therefore all cognition of the objects of experience; and what mathematics in its pure use demonstrates of the former, it is also necessarily valid for the latter" (A165–166/B206). All quotation from Kant's *Critique of Pure Reason* refer to the B edition. References to the corresponding passages in the A edition are given, when available. English translations are from Kant (1998).

[3] Kant wrote: "I call all cognition *transcendental* that is occupied not so much with objects but with our manner of cognition of objects insofar as this is to be possible *a priori*" (B25).

[4] I rely in the following especially on Ferrari (1988) and Giovanelli (2016). I will not engage in a critical discussion of Cohen's principle here. My focus is on the aspects of his reading of Kant, which had an influence on Cassirer.

Natorp, which also brought to the fore Natorp's disagreement with Cohen's interpretation of the history of calculus. Most of this discussion remained internal to the school. Natorp wrote a draft on Cohen's logic following Hans Vaihinger's invitation to publish a review in *Kant-Studien*. However, Natorp renounced to publish his review after Cohen's reactions to his criticisms.[5] Natorp's main point was to defend the idea of an internal articulation of the transcendental apparatus against Cohen's identification of intuitions as concepts and thought as knowledge. To Natorp, this extreme version of idealism appeared to completely break with critical idealism, that is, the version of idealism that is supposed to rely on science. The problematic aspect of this move particularly concerned the explanation of the possibility of mathematics, which in Cohen's view seemed to depend on the nonmathematical concept of intensive magnitude. Natorp had been working since the 1890s on the problem of investigating the logical foundations of mathematics, including its most recent achievements. By contrast, Cohen seemed to neglect much of the modern mathematical debate in favor of a premodern view of the foundations of calculus.

Cassirer sided with Natorp in two important respects. Firstly, Cassirer (1902) offered a symbolic interpretation of Leibniz's talk about "intensive magnitudes" .[6] Secondly, Cassirer (1907b) deemed the development of mathematics a "new fact," which critical philosophy can no longer ignore. As we will see in the following, Cassirer found a new expression of this "fact" in *Substance and Function* (1910) by identifying the fundamental type of mathematical concepts as functions. Cassirer's distance from Cohen is clearly expressed here. It is revealing that Cohen, in turn, disagreed with the central thesis of Cassirer's work.[7]

Nevertheless, Cassirer's early works provide evidence of an attempt to mediate between his teachers' views. It is revealing, for example, that Cassirer relied on his interpretation of Leibniz to defend Cohen's logic against Leonard Nelson's attack.[8] Another important indication of Cassirer's mediation is his assessment of Kant's relation to Leibniz in Cassirer (1907a). Following Natorp, Cassirer urged an investigation of the logical foundations of mathematics more in line with Leibniz: "If one judges Kant as a pure logician, if one considers only what he contributed to formal logic and to the abstract doctrine of principles of pure mathematics, then there is no doubt that he stayed far behind his great rationalist predecessors, especially Leibniz" Cassirer (1907a), p. 554). At the same time, as Richardson (2003) has pointed out, Cassirer agreed with the formulation of the task of epistemology that derived from Cohen's reading of Kant. Therefore, Cassirer also claimed that the characteristic advantage of the Kantian approach is this: "Kant's focus is on the principles of empirical knowledge. Mathematics itself is considered only insofar as it proves to be applicable to concrete actual objects" (1907a, p. 554).

Cassirer went on to point out that Kant's account of concept formation was based on the syllogistic logic of his time. Accordingly, a general concept is abstracted from all distinctive properties of a group of things as their common denominator and contains its instantiations under itself. However, this kind of abstraction did not account for the formation of mathematical concepts, such as by infinite partition. The very possibility of these kinds of procedures, which were common practice in Euclidean geometry, rests on the presupposition of a single space in which the partition takes place. Kant expressed this by saying that space contains spatial determinations *within* rather than *under* it (A25/B40). Kant famously used this distinction to characterize the pure intuitions of space and time in his Dissertation of 1770 and in the *Critique of Pure Reason*. However, Kant also called space one of the "concepts of pure understanding" (*conceptus intellectus puri*) in a Reflection written around 1769 (in Kant, 1884, n. 513). According to Cassirer, This shows that Kant's characterization of space originally agreed with Leibniz's. In the Leibnizian terminology, space is a pure product of the mind itself ("*intellectus ipse*"), which cannot be derived from the perception of extended objects because it establishes the order of coexistence. Kant's connection with Leibniz in 1769 suggests that: "The two moments, which in the *Critique of Pure Reason* are opposed to each other, are still wholly interchangeable here; the 'intuitive' does not stand in contrast with the 'intellectual' but is a particular determination of it" (Cassirer, 1907a, p. 627).

As it will be shown in the next sections, the above definition of "intuitive" as a particular determination of "intellectual" plays a central role both in Cassirer's interpretation of Kant's critical philosophy and in Cassirer's own account of mathematical reasoning.

Summing up the discussion so far, Cassirer used new materials from Kant's Reflections to argue in favor of Cohen's reading of Kant. The original designation of space and time as singular concepts shows that Kant came only gradually to distinguish these concepts from the rest of the fundamental system of pure concepts of the understanding; they were "objective principles of synthesis," before they became "concepts of intuition" and ultimately "forms of sensibility" (Cassirer, 1907a, p. 625). Cassirer agreed with Cohen that this circumstance also highlights the fact that space and time retain a fundamental role in the critique of reason as objective principles of synthesis rather than as independent faculties of the mind. At the same time, Cassirer sided with Natorp in urging a new account of mathematical reasoning up to the standards of modern mathematical logic. We can now consider how Cassirer combined these two ideas in his interpretation of the Transcendental Deduction.

2.2. *Cassirer on the transcendental deduction and the transcendental account of concept formation*

Kant's Transcendental Deduction aims to demonstrate, against empiricist psychology, that the categories of the understanding are capable of taking up the elements of cognition into the unity of consciousness. The argument proceeds as follows[9]: 1) All of our representations of objects require a faculty for ordering mental states (synthetic unity of the apperception) that is distinguished from apprehension (B132–136); 2) The action of the understanding, through which the manifold of given representations is brought under an apperception in general, is the logical function of judgments (B143); 3) The categories that are schematized for our way of cognizing (A137/B176ff) are generated from the forms of judgment in the process of synthesizing intuitions by the addition of spatial and temporal content. Kant argues, by 2) and 3), that the manifold in a given intuition is necessarily subject to the categories (B143). He completes the argument by showing that, 4) because space and time are the only forms in which appearances become

[5] Two drafts for Natorp's review are found is Holzhey (1986), pp. 5–96.

[6] Quite contrary to Cohen's thesis, it followed from Cassirer's reading that Leibniz had actually foreshadowed the modern account of "differentials" in terms of limits.

[7] Cohen wrote in a letter from August 10, 1910: "I admittedly confess that after my first reading of your book I still cannot discard as wrong what I told you in Marburg: you put the center of gravity upon the concept of relation and you believe that you have accomplished with the help of this concept the idealization of all materiality. The expression even escaped you that the concept of relation is a category; yet it is a category only insofar as it has a function, and a function unavoidably demands the infinitesimal element in which alone the root of the ideal reality can be found" (English translation from Giovanelli, 2016, p. 20).

[8] As Giovanelli (2016) showed, Nelson's review of Cohen's *Logic* contained perhaps the most dismissive of a long series of attacks against Cohen's interpretation of the foundations of calculus.

[9] The following presentation is focused on the steps of Kant's argument that are directly relevant to Cassirer's interpretation. See Pereboom (2018) for a more thorough analysis of Kant's argument, along with references to the main existing interpretations. While a discussion of this debate is beyond the purposes of this paper, I will point out that an interpretation in line with Cassirer's has been proposed by Friedman (2012).

accessible to us, the unity of sensibility is no other than that which the categories prescribe to the manifold of a given intuition. By 4), there is a necessary agreement between experience and the concepts of its objects (B167).

It is important to notice that, as Kant himself emphasizes at each step of the argument, the different types of syntheses involved are interconnected. Any determinate intuition is only possible through the consciousness of the rule-governed action of the imagination on the empirical manifold (B154). Such an action rests, in turn, on the spontaneity of the understanding, whose function is to order different representations under a common one (A68/B93). The synthetic unity of apperception is the highest principle for all use of the understanding, including the whole of logic (B133).

Cassirer further emphasized the interdependencies of the conditions of knowledge by interpreting Kant's argument as outlining a unitary but internally articulated process of "synthesis"[10]. As Cassirer (1907a, p. 698) put it: "The synthesis constitutes a unitary and continuous process. However, this process can be determined and characterized either from its starting point or from its goal. This synthesis has its origin in the understanding, but it turns to pure intuition as soon as it aims at attaining, through its mediation, to empirical reality." As a result, the apparent separation of concepts and intuitions in the Kantian argument is really a "logical correlation" (Cassirer, 1907a, p. 698).

In order to explain how such a process is possible, Cassirer outlined the transcendental apparatus starting from the lowest level of synthesis, that is, the sequence of singular representations as they present themselves in time. Even the apprehension of such a sequence presupposes what Kant called a "schema," that is, a general rule for the representation of an object that is distinguished from the representation itself. The sum of the parts of an object, for example, is presupposed by, but distinguished from, the representation of the counted objects. Kant identified number as the pure schema of the concept of magnitude, where the rule for representing this concept is "the successive addition of one (homogeneous) magnitude to another" (A142/B182). The fact that Kant called this operation "successive" seems to suggest that this happens in time. However, he went on to explain that "number is nothing other than the unity of the synthesis of the manifold of a homogeneous intuition in general, because I generate time itself in the apprehension of the intuition" (A142/B182). In other words, Kant's successive synthesis here indicates the indefinite iteration of the addition of one unit to another.

In Cassirer's view, the repetition of one single rule constitutes a synthesis of second order, which is able to differentiate a plurality of objects as positions in an ordered series. Finally, a third-order synthesis of such rules is required for the objective unity of experience:

> In the place of the mere undifferentiated copy by association there appears now a rich variety of different synthetic rules for individuation and concept formation. The manifold must be conceived of, not only as it presents itself in an arbitrary and undifferentiated sequence, but as belonging to the various relations of interdependency, if the consciousness is to form a system or an objective unity of it (Cassirer, 1907a, pp. 711–12).

Cassirer emphasized the gap between a "copy theory of concept formation by association, according to which concepts retain only a partial image of what different sense impressions have in common, and the transcendental investigation of experience as a whole system of rules of individuation. We have seen that Kant himself relied on such mathematical examples as numbers and geometrical definitions to account for the fact that the general rule for the representation of an object is distinguished from the representation itself. The decisive step for Cassirer's own argument in 1910 is a further generalization of the transcendental account of concept formation to the notion of mathematical structures. This notion is clearly implied in the following passage:

> The example of number is particularly instructive, because it suffices to take into account the further scientific development of the concept of number in pure mathematics, the progression from rational to irrational numbers, and from the reals to imaginary numbers, in order to recognize immediately that the concepts produced here are not copies of actual sense impressions, but rather the outcome of purely intellectual operations (Cassirer, 1907a, p. 714).

We can now consider a more precise formulation of Cassirer's claim: *the validity of structural procedures rests on the presupposition that mathematical objects are nothing other than products of intellectual operations*. However, this formulation also makes apparent the tension that we mentioned in the beginning between the different aspects of Cassirer's view. On the one hand, this claim amounts to a form of methodological structuralism, that is, a view mainly concerned with the implications of structural methods in modern mathematical practice (Reck, 2003). On the other hand, the neo-Kantian epistemology advocated by Cassirer is mainly concerned with the applicability of mathematics and the possibility of the mathematical science of nature. It is the applicability of mathematics in the natural sciences that, according to the neo-Kantian reconstruction, provides a proof that *a priori* concepts correctly apply to the objects experience.

Following Marburg neo-Kantians, Friedman (2012) has explored the possibility of using the discussions of natural science in Kant's *Prolegomena* and the *Metaphysical Foundations of Natural Science* to illuminate the corresponding discussions in his *Critique*. Friedman refers, more specifically, to the way in which Kant accounts for the necessary lawfulness of nature in § 36–38 of the *Prolegomena*, where he refers to Newton's deduction of the law of universal gravitation. Kant begins his discussion with the fundamental properties of circles established in Book III of Euclid's *Elements*. He goes on to point out that these properties are presupposed in the characterization of the relative motions of the satellites in the solar system with respect to their primary bodies. While these phenomena present themselves in Kepler's rules as inductive generalizations, Newton's universal gravitation allows him to derive all the possible orbits by stating a necessary connection between the concepts of force, distances and masses. According to Friedman, this example shows what is essential to Kant's conception of objective judgment, that is, "a procedure of synthetic determination" that begins by applying the transcendental unity of apperception to the pure forms of spatiotemporal intuition (Friedman, 2012, p. 322). We then move through the schemata of the individual categories to the principles of pure understanding corresponding to these categories. And at this level, we can distinguish three distinct steps. We first consider the formal conditions for the possibility of experience, which in the above example correspond to Euclid's propositions. We are next presented with the material conditions for actual perception, which correspond to Kepler's phenomena. Finally, with Newton's law, we are in a position to determine what is actual as in necessary agreement with the conditions of experience.

With regard to Cassirer's claim, the question arises whether the step from formal to material conditions of experience can take place without the mediating term of pure intuitions.[11] This step requires Cassirer to show that *the kind of reasoning at work in the definition of mathematical structures also accounts for the applicability of mathematics in scientific theories*. I take Cassirer (1907a) discussion of concept formation to elucidate a necessary premise for this argument by saying that

[10] Cassirer's terminology is derived from Kant's characterization of the combination of a manifold in general as an action of the understanding. Kant calls such a combination "synthesis" to indicate that "we can represent nothing as combined in the object without having previously combined it ourselves" (B130).

[11] Friedman's objection to Cassirer on this point is discussed in Section 3.2.

mathematical reasoning – in the above sense of structural reasoning – is capable of an a priori synthesis in its own domain. The following section considers how Cassirer continued to rely on his transcendental account of concept formation to articulate a neo-Kantian philosophy of mathematics.

3. Cassirer's neo-Kantian philosophy of mathematics

This section considers how Cassirer addressed the problem to account for mathematical reasoning from a modern standpoint. Cassirer was one of the first neo-Kantians to take the new mathematical logic (also known as "logistic") into serious consideration. As Heis (2010, 2011) has suggested, the neo-Kantian background of Cassirer's philosophy led him to address different questions from now better-known approaches, such as Frege's and Russell's. These questions include the unity of mathematics over time, despite the conceptual and ontological revolutions that it has undergone and the applicability of mathematics as a precondition of objective knowledge.

I will argue that Cassirer's philosophical background also played an important role in the development of his strategy to solve the tension mentioned at the beginning between a unitary and a more pluralistic perspective on mathematics.

3.1. The sameness thesis

Cassirer first presented his view in "Kant and Modern Mathematics" (1907). The goal of this article was to counter Louis Couturat's and Bertrand Russell's view, that the logistic proves the analyticity of mathematics, or at least rule out the view that all mathematical propositions are synthetic *a priori* in the Kantian sense. As Cassirer implied in his considerations on Kant's relation to Leibniz, Cassirer agreed with the proponents of the logistic on the problematic aspects of the Kantian view. Accordingly, Cassirer did not defend either of Kant's definitions of synthetic judgments: 1) as judgments whose subject does not include the predicate by definition (A9/B13), and 2) as judgments whose concepts subsume a pure intuition (B73). The first definition, which is derived from syllogistic logic, appears to be too restrictive to even express mathematical judgments. Regarding 2), we have seen (2.1) that Cassirer ruled out intuition as a source of mathematical certainty in favor of a Leibnizian approach. As Cassirer (1907b, pp. 31–32) made clear, he agreed with Couturat and Russell that the new logic provided a further, mathematical proof that mathematics is independent of the representations of space and time.

These points of agreement notwithstanding, Cassirer contended that there are mathematical "syntheses" in the sense elucidated in (Cassirer, 1907a), that is, as cognitive procedures that have their origin in the laws of thought, and their goal in the application to empirical reality. Cassirer wrote:

> So, a new task begins at the very point where logistic leaves off. What critical philosophy searches and must require is a *logic of objective knowledge* […]. Only once we will have understood that the same fundamental syntheses, which lie at the foundation of logic and of mathematics, rule over the scientific articulation of empirical knowledge, that only these syntheses enable us to establish a lawful order of the appearances, and therefore their objective meaning, then, and only then we will obtain a true justification of the principles (Cassirer, 1907b, pp. 44–45).

This is the argument that sets the main goal of Cassirer's philosophy of mathematics. The central claim (that the fundamental syntheses, which lie at the foundation of logic and mathematics, rule over the scientific articulation of empirical knowledge) became known in the literature as the sameness thesis.

In addition, Cassirer contended that "intuition" still plays a role in some branches of mathematics, if understood as a particular determination of intellectual procedures.[12] Here Cassirer referred, more specifically, to cases where there is a choice between incompatible logical possibilities. Examples of this include the choice among different principles, which one can assume with equal logical right in the foundation of geometry. According to Euclid's parallel postulate, there exists exactly one parallel to any given line. However, this is denied without contradiction in the non-Euclidean hypotheses.

What this suggests is that there are two aspects of mathematics that Cassirer calls synthetic. One aspect pertains to mathematical methodology: even if the logicist program was to provide a logical derivation of some parts of mathematics, intuitive considerations still play a role in other branches of mathematics. The other aspect is that mathematical syntheses play an indispensable role in the system of the principles of knowledge. We can see how the said tension arises here. In the former sense, the syntheticity of mathematics is given by a demand for unity in the face of different possible initial assumptions. In the latter sense, mathematics proves to be synthetic, insofar as the same syntheses apply to the order of appearances. A possible reading of the argument would make the syntheticity of mathematics dependent on the sameness thesis, which, in Mormann's interpretation, would allow for a more liberal attitude towards incompatible idealizations.

My suggestion is that Cassirer resolved this tension in 1910 by spelling out the meaning of "synthetic" as fulfilling the goal of uniquely determining the subject-matter of mathematical theories and what valid inferences can be made about their content. It follows that, while mathematics is not grounded in an independent faculty of intuition, mathematical reasoning has an intuitive side in itself (as a particular determination of the intellectual) and provides a synthesis of space and time as required by Kant's transcendental claim.

3.2. Cassirer's view in 1910

Cassirer reached a new perspective for the articulation of his view in *Substance and Function* (1910). The book begins by pointing out the logic at work in the evolution of mathematics and gradually identifies the elements of this logic as preconditions of experience. Finally, this logic is generalized to a logic of objective knowledge in the above sense.

This new meaning of "logic" is characterized by what Cassirer called a reversal in the traditional, Aristotelian conception of the thing/property relation. The comparison of things with regard to some property, in the Aristotelian logic, serves the purpose of abstracting from distinctive properties of individuals in order to form the concepts of classes to which they belong. Classes of individuals can be classified into genus and species according to their degree of abstraction following the principle that the less properties belong to the class, the more general the concept. However, this poses the problem of how to select such properties. This is the point at which, according to Cassirer, traditional logic presupposes the Aristotelian metaphysics: the process of abstraction reflects the idea that the ultimate commonalities of things depend on their ultimate causes or substances, and the goal of knowledge is to reproduce these causal relations as faithful as possible in thought.

Cassirer maintained that a new model of concept formation emerged for the characterization of mathematical concepts such as series and limits (Cassirer, 1910, p. III). One can avoid the epistemological problem of abstraction here by establishing a law for the specification of (possibly infinite) individual cases. Cassirer pointed out that, in this way, "the more universal concept reveals itself also as the richer in content" (p. 20). Whereas the particular determinations are included as predicates in

[12] Cassirer wrote: "Even within the field of pure mathematics 'intuition' retains a broader domain and a greater significance than then the logistic can acknowledge. Although the progress of mathematics has shown that intuition fails as an autonomous means of proof, it is nonetheless necessary to indicate the ultimate task of our logical syntheses and thereby determine their direction" (Cassirer, 1907b, p. 46).

the concept of the subject in the traditional view, the mathematical concept of function provides a general rule of individuation. Cassirer maintained that this concept provides a new standpoint for the transcendental account of concept formation and objective judgment.

According to Friedman, Cassirer's view is the most consistent implementation of the critical idealism of the Marburg School. However, this also shows how the neo-Kantian account of objective judgment fundamentally differs from Kant's. Kant's account, as outlined in 2.2, does not depend on the logical structures of judgment alone, but also on the forms of intuition. Since space and time no longer function as independent forms of pure sensibility in Marburg neo-Kantianism, the logic of objective knowledge must now proceed on the basis of purely conceptual – and thus non-spatiotemporal – *a priori* structures (Friedman, 2000, p. 28). Friedman goes on to point out that Cassirer identifies formal logic with the new theory of relations developed especially by Russell (1903). This suggests that mathematics is incorporated into formal logic and the given manifold of sensation is replaced with the methodological progression of mathematical natural science (Friedman, 2000, pp. 32–33).

We saw that a quite different picture emerged from Cassirer's attempt to position himself in the debates over Cohen's logic and on the implications of symbolic logic. One of the purposes of *Substance and Function* was to find a new expression for the view of knowledge as a unitary but internally articulated process. The concept of function offered a relevant basis, because "function" in Kant's usage indicates "the unity of the action of ordering different representations under a common one," where "concepts are grounded on the spontaneity of thinking" (B93). As we have seen in 2.2, Cassirer accounted for the further articulation of this concept as a three-step synthesis consisting of: apprehension of a manifold; ordering of a homogeneous manifold in series; and coordination of different series in an objective unity. So the question arises how Cassirer can recover these different steps starting from what appears to be a purely formal account of judgment in terms of the mathematical concept of function.

A more nuanced interpretation of Cassirer is being proposed by Heis (2014a). Heis agrees with Friedman that there is no fundamental divergence between Cassirer and philosophers in the analytic tradition, such as Russell and the logical positivists, with regard to Kant's pure intuitions. So, Heis suggests that a more promising way to locate Cassirer's contribution in the history of analytic philosophy is to clarify the different levels of meaning that "function" assumes in Cassirer's work, as: 1) a one-one or many-one relation; 2) the role or purpose of particular mathematical or scientific concepts within this or that particular scientific field; 3) a rule-governed activity of the mind. The first, logical meaning of function is derived from Russell. However, 2) is derived from the methodology at work in Dedekind's theory of numbers. Finally, Heis points out that Cassirer's account of objectivity ultimately depends on the Kantian meaning of function expressed by 3). Heis's clarification of the different meanings of function in Cassirer philosophy is very helpful for a better assessment of his view. However, this differentiation leaves out what seems, to me, to be the main point of the philosophy of the concept of function, namely, that function as a rule-governed activity of the mind still requires a schematization or additional and mediating structures for its empirical use. Accordingly, 2) has to do primarily with the transposition of structural reasoning from numerical to spatiotemporal domains.

It is apparent that Cassirer bore in mind his earlier reading of Kant in the following passage, which is worth quoting at length, because Cassirer here also summarized the main steps of his transcendental argument throughout the book:

> Every mathematical function represents a universal law, which, by virtue of the successive values which the variable can assume, contains within itself all the particular cases for which it holds. If, however, this is once recognized, a completely new field of investigation is opened for logic. In opposition to the logic of the generic concept, which, as we saw, represents the point of view and influence of the concept of substance, there now appears the *logic of the mathematical concept of function*. However, the field of application of this form of logic is not confined to mathematics alone. On the contrary, it extends over into the field of the *knowledge of nature*; for the concept of function constitutes the general schema and model according to which the modern concept of nature has been molded in its progressive historical development (Cassirer, 1910, p. 21).

This definition of function suggests a parallel with what Kant called the successive addition of one homogeneous magnitude to another (2.2). Another important indication of the fact that Cassirer had Kant's schematism in mind is the fact that Cassirer used "schema" to describe how the model of the concept of function extends to empirical laws. Not only does schematization play some role in this process, but the explanation of how such extension takes place is the very objective of the logic of knowledge. This requires Cassirer to take into account the concrete instantiations of the general schema of the concept of function, that is, the function-concepts from the exact sciences. The goal of the first part of the book is to show how paradigmatic examples of function-concepts in mathematics enable a synthesis of space and time. This is the basis for Cassirer's account of scientific objectivity in the second part of the book. Notably, there is a clear parallel between the function-concepts under consideration and the three-step synthesis outlined by him in Cassirer (1907a). In a nutshell, the first step is to abstract away from any empirical representation. The second step consists in the formulation of a serial principle, according to which all the individuals falling under the concept of the series are defined as its members. The third step is the discovery of lawful connections (coordination) between different kinds of manifolds. As Ryckman (1991) has pointed out, the notion of coordination occupies a central place in Cassirer's epistemology as a new basis for the transcendental grounding of objectivity. This also relates to the fact that the term "coordination" was widely used outside mathematics to indicate the relation between the symbolism of scientific theories and their empirical content.[13]

In what follows, I will argue that the step from serial principles to coordination in Cassirer's view finds its justification in the formation of mathematical concepts. I will turn back, in the conclusion, to how Cassirer's claim *that mathematical reasoning is capable of an priori synthesis in its own domain* relates to his transcendental argument as a whole.

4. Function-concepts from nineteenth-century mathematics

This section examines three specific function-concepts that illustrate Cassirer's notion of synthesis in its different steps.

4.1. The series of natural numbers

Cassirer's (1910) first paradigmatic example of function-concepts is Dedekind's characterization of numbers as "free creations of the human mind". Dedekind ascribed a creative power to the ability of the mind to relate things and let one thing correspond to another. The function that maps any number onto its successor instantiates such an ability by generating the whole series.

Whether Dedekind's talk of "creations" should be taken literally or not, and in the former sense, whether it should be understood as implying a psychological and subjective view of numbers, has been

[13] As pointed out by Ryckman (1991), the term "coordination" (*Zuordnung*) originally indicated a one-to-one relation in mathematical contexts and found related, but additional, applications in epistemological and methodological discussions within the mathematical sciences of nature.

much discussed.[14] However, there is evidence that Dedekind viewed his characterization of numbers as "logical" in two important senses. The first sense derives from Dedekind's methodology. He developed new structural procedures which would become standard set-theoretical techniques. An important difference to current set theory, however, is the fact that Dedekind took the concept of function to be as primitive as, rather than reducible to, sets. Dedekind did not discuss how functions can be presented. His notions of correspondence and mapping can be, nevertheless, seen as an attempt to capture the informal concept of a law-like correspondence, which was used in nineteenth-century geometry (see Sieg & Schlimm, 2017).

The second aspect of Dedekind's approach is that it has a conceptual, rather than formal character: numbers are "created" in a logical process, insofar as they are identified as a new system of objects, which is neither located in the physical world nor coinciding with any previously constructed sets. Reck has called this view "logical structuralism" in order to emphasize the philosophical implications of Dedekind's methodology, in particular the view that the structural properties of numbers determine uniquely a certain "conceptual possibility" (Reck, 2003, p. 400).

In order to see how Dedekind's approach implies such a view, let us recall briefly his main argument. Dedekind initially laid down some basic set-theoretical notions.[15] An infinite system is such that it can be mapped one-to-one onto a proper part of itself. This means that there is at least one element in the original system, which is not included in its part. The natural numbers, for example, can be mapped one-to-one onto the even numbers. Dedekind called such a system "simple," if it has a base element and there is a successor function that maps any element of the system onto a proper part of it. Again, the natural numbers are the main example here. However, Dedekind emphasized that numbers can be obtained only after admitting the existence of simply infinite systems, by considering a simply infinite system N ordered by a mapping ϕ and abstracting away from the particular nature of the elements. Dedekind spelled out his view by saying that: "With reference to this liberation of the elements from every other content (abstraction) we are justified in calling the numbers a free creation of the human mind" (Dedekind, 1888, p. 68). He went on to show that the properties which make numbers distinguishable (i.e. ordinal properties) are exactly those that are required to have arithmetical operations. In other words, Dedekind identified the structural properties of numbers as their properties *qua* numbers. Any additional properties is to be clarified as nonarithmetic. This characterization is completed by Dedekind's proof that any simply infinite system is isomorphic to the natural numbers. In current terminology, Dedekind proved the categoricity of the natural numbers.

Cassirer's notion of function offers a viable interpretation of Dedekind's view in the following sense: the ability to relate one thing to another in the above quote corresponds to the Kantian meaning of function as the rule-governed activity that is characteristic of rationality. Without referring to Kant, Dedekind himself seems to suggest such a reading in calling this "an ability without which no thinking is possible" (Dedekind, 1888, p. 32). Once this meaning of function is admitted, Cassirer emphasized that the relation established by the mind is not one of copy of one thing by another, but of "ideal correlation by which we bind otherwise totally diverse elements into a systematic unity" (Cassirer, 1910, p. 36). The relata are not supposed to have an independent existence prior to their being related but, as far as the arithmetician is concerned, they exist as serial objects generated by a serial principle.

Cassirer went on to point out that Dedekind's talk of "creation" with regard to his procedure has its counterpart in the particular meaning of "abstraction" of natural numbers from the structure of a simply infinite system. The determining power of function-concepts increases in the measure that they are abstracted in Dedekind's sense: the position of each number "is clearly determined by the others" (Cassirer, 1910, p. 38). This is the purpose of the successor function that establishes an asymmetric and transitive relation between all the members of the series. Cassirer emphasized the logical nature of Dedekind's abstraction and ruled out the psychological interpretation as follows: "Here abstraction has, in fact, the character of a liberation; it means logical concentration on the relational connection as such with rejection of all psychological circumstances, that may force themselves into the subjective stream of presentations, but which form no actual constitutive aspect of this connection" (p. 39).

As Yap pointed out, Cassirer provides the appropriate philosophical background to Dedekind's abstraction by clarifying the claim that is implicit in it as follows: "The claim that we can forget about the special character of the elements in a simply infinite system has to be based on the claim that the essential relations between elements are completely determined" (Yap, 2017, p. 15). Yap also rightly points out that, at this point, categoricity is essential for Dedekind, although Cassirer does not mention it. Yap's suggestion is that this is likely because Cassirer might have had other examples of mathematical concepts in mind, which are defined by non-categorical sets of axioms (e.g., groups).

My suggestion is that the further, geometrical examples considered by Cassirer are instrumental to his transcendental account. Cassirer began with the concept of number because, on the one hand, he saw numbers as "rooted in the substance of rational knowledge," but on the other hand, "in the thought of number all the power of knowledge seems contained, all possibility of the logical determination of the sensuous" (Cassirer, 1910, p. 27). The first claim relates to the notion of function as a precondition for thinking. Cassirer's second claim relates to the three-step synthesis that is required for schematization in our mode of cognition. The definition of numbers presupposes three things. Firstly, that one abstracts away from all psychological representations. Secondly, a serial principle is required for the determination of numbers as an ordered manifold. Finally, that the natural numbers are completely determined by the definition of arithmetical operations on the manifold of a simply infinite system: i.e. complete determination presupposes the notion of a mathematical structure and the use of what are currently known as structural or axiomatic definitions (Sieg & Schlimm, 2017). It is no less essential to Cassirer's view that the methodology of Dedekind's work on numbers can be applied in different branches of mathematics. Such a view can be spelled out by saying that - once the subject-matter of a mathematical discipline has been conceptualized in terms of structures, the mathematical investigation can proceed to examine how a variety of phenomena are structurally related. This is essential if mathematical reasoning is to enable the conceptualization of empirical objects.

4.2. *Transformation groups*

Cassirer relied on Klein's "Comparative Review of Recent Researches in Geometry" for the characterization of transformation groups (1872).[16] "Transformation" in this context indicates a one-to-one mapping of space onto itself. Informally, transformations form a group if: i) The product of any two transformations of the group also belongs to the group; ii) for every transformation of the group, there exists in the group an inverse transformation.

Klein showed that the transformation group determines what

[14] I rely in the following especially on Reck (2003) and Yap (2017), whose interpretations agree with Cassirer's in taking "creation" as a logical and objective derivation of numbers independently of all nonarithmetical properties. For the psychological interpretation, cf. Dummett (1991).

[15] I will rely in what follows on Dedekind's terminology and refer to sets as "systems."

[16] The implementation of Klein's comparative approach required a further development of group-theoretic techniques, which was mainly due to other mathematicians (notably Sophus Lie). I have argued elsewhere (Biagioli, 2018) that this partially explains the delayed reception of Klein's ideas (1872) in philosophical contexts.

geometrical figures are. Elementary geometry reflects this fact, insofar as distinct figures can be identified by superposition in a congruent way. Group theory allows for a more precise way to express this fact by defining congruence as an invariant under isometries. It is also clear from this viewpoint that what makes figures identical depends on the group. Two different figures in ordinary geometry, such as a circle and a parabola, can be identified as a conic in projective geometry. This means that a circle can be transformed into a parabola by collineations.

Having clarified that all groups are equally justifiable from a logical viewpoint, Klein showed how to compare different geometries by introducing the following principle of transfer: Suppose that a manifold A has been investigated with reference to a group B, and by any transformation A is converted into A', then B becomes B' and the B'–based treatment of A' can be derived from the B–based treatment of A (Klein, 1872, p. 72). This holds true independently of the particular choice of the elements (e.g. points, lines or planes): for each proposition that follows from the assumption of a particular element, a corresponding assumption follows from any other element, which can be taken arbitrarily. A classic example of this in projective plane geometry is the principle of duality, according to which any true statement concerning relations of points can be obtained by substituting "point" for "line," "collinear" for "concurrent," and "meet" for "join." Klein's formulation of the transfer principle was derived from Otto Hesse and was used in analytic geometry. In generalizing this principle, Klein gained insights into the structural nature of the projective consideration of figures, and arrived at the idea of a classification of geometries as relative invariants of transformation groups.[17]

I have argued elsewhere that Klein attached great epistemological importance to a form of mathematical structuralism derived from Dedekind, although it is characteristic of Klein's approach that he used structuralism to translate from analytic geometry to other mathematical practices at use in descriptive geometry (Biagioli, n.d. forthcoming). Without going into the details of how Klein developed his view, I will limit myself to drawing attention to Klein's enunciation of what the use of transfer principles implies: "As long as our geometrical investigations are based on one and the same transformation group, the geometric content remains unvaried […]. The essential thing is the transformation group" (Klein, 1872, pp. 73–74).

Ihmig (1997) has shown in detail that there is a series of analogies between Cassirer's account of concept formation and Klein's comparative approach. What I would like to add to this comparison is that the analogies depend, more specifically, on Cassirer's appreciation of how transfer principles work: two complexes of judgments, one of which deals with lines and planes and the other with the circles and spheres, are regarded as *equivalent* to each other and having the same "content of conceptual dependencies" (Cassirer, 1910, p. 93).[18] Cassirer goes on to point out that even the formal definition of transformation groups implies that the content of geometry is characterized by structural properties. His main point, however, is that this way of considering geometries enables one to draw new inference by identifying equivalent properties across different domains. Again, the shift from functions defined on a particular set to functions between completely different domains here is consistent with the fact that function in this mathematical tradition is a primitive notion. So, the crucial step for the introduction of function-concepts in geometry according to Cassirer is not the definition of groups (which, indeed, are defined by non-categorical sets of axioms), but the discovery of a principle for the classification of a plurality of geometries.[19] Cassirer emphasized the fruitfulness of this procedure even in cases where imaginary points have to be postulated. This is the case with the projective definition of distance in relation to the circle at infinity. Klein developed a model of non-Euclidean geometry by considering this circle a limiting case (in which a point at infinity is taken twice); the other cases being an imaginary second-order surface and the inner points of a real, non-degenerate surface of second order. He classified geometries into parabolic, elliptic, and hyperbolic, respectively, which correspond to different possible hypotheses about metrical relations in space. With reference to this classification, Cassirer argued that a new synthesis of space and time is achieved by the introduction of structural procedures in geometry:

> Intuition seems to grasp the content as an isolated self-contained existence; but as soon as we go on to characterize this existence in judgment, it resolves into a web of related structures which reciprocally support each other. Concept and judgment know the individual only as a member, as a point in a systematic manifold; here as in arithmetic, the manifold, as opposed to all particular structures, appears as the real logical *prius*. The determination of the individuality of the elements is not the beginning but the end of the conceptual development; it is the logical goal, which we approach by the progressive connection of universal relations. The procedure of mathematics here points to the analogous procedure of theoretical natural science, for which it contains the key and the justification (Cassirer, 1910, p. 94).

As anticipated 3.1, "intuition," as the *terminus a quo* of concept formation, retains a role in determining the task of showing that mathematical and natural concepts are of the same kind. As Cassirer also puts it, "synthesis builds the real goal of mathematical operations" (1910, p. 96). However, Cassirer now also emphasizes that the synthetic procedures under consideration are immanent to the evolution of mathematics. The extensibility of such procedures is justified by their structural nature, which, even within mathematics, allows for an expansion of knowledge from the comparison of known structures. The following section considers an elementary illustration of how such a comparison works.

4.3. Projective coordinates

Cassirer's (1910) chapter on geometry and the concept of space in *Substance and Function* offers a very rich overview of how different traditions in nineteenth-century geometry foreshadowed Klein's structuralist ideas. Among these traditions, Cassirer attached particular importance to the introduction of projective coordinates in the manner

[17] For a modern presentation of Klein's transfer principles and their implications for mathematical structuralism, see Schiemer (2020).

[18] For a detailed discussion of the notion of "transfer" in Cassirer (1910), see Schiemer (2018).

[19] The available English translation of Cassirer (1910) seems to suggest that such a principle would follow directly from the definition of group. A more literal translation would read: "The definition of 'groups' already contains a new and important logical moment, insofar as through it is brought to intellectual unity, not so much a whole of individual elements or figures, as a system of operations. […] At this point, though, with the concept of the group, a general principle of classification is gained by which the different possible kinds of geometry can be unified under a single point of view and surveyed in their symmetric connection" (Cassirer, 1910, p. 89).

of Christian von Staudt; that is, without the presupposition of metrical concepts.[20]

Von Staudt (1847, pp. 43–49) defined the fundamental invariant of projective geometry by using a construction that uniquely determines the fourth harmonic to three collinear points. In virtue of the same relation, it is possible to introduce a numerical scale on a projective line in the following way. Given two points, 0 and 1, on the line and a point at infinity, ∞, the construction of the fourth harmonic point 2 can be reiterated to generate 3, 4, and so on in order to generate a series that corresponds to the integer numbers. Negative and fractional numbers can be determined by projecting the same construction from an external point. However, there is a final step which requires a postulate analogous to Dedekind's assumption of the continuity of the line. In Dedekind's (1872) formulation: *if all points of a line fall into two classes, such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes*.[21] Dedekind went on to show that for all rational numbers there is a corresponding division into distinct classes. However, it is not the case that for any such division there is a rational number. Dedekind introduced symbols for irrational numbers in correspondence with the latter divisions. His conclusion was that continuity was thus proved to be independent of our intuition of space. In fact, we do not know whether space is continuous or discrete; what we know is that, even in the hypothesis that space was discrete, it would be possible to imagine it otherwise by filling the gaps in thought. As Dedekind put it: "This filling up the gaps would consist in a creation of new point-individuals and would have to be effected in accordance with the above principle" (Dedekind, 1872, p. 12).

The step that corresponds to Dedekind's continuity in the Staudt-Klein construction is the postulate that, in principle, the construction of the fourth harmonic can be reiterated indefinitely in order for all intermediate points to be determined. As in Dedekind's consideration about space, the postulate establishes a rule for thinking of something as continuous without implying an intuitive notion of this property, which is the same as saying that the postulate itself is compatible with different, even opposed intuitions. What the principle does, in any case, is to clarify what it would take for our intuitions to be compatible with the thought of continuity. This is even more apparent in the geometric version of the principle, insofar as there is a constructive procedure that leads from the characterization of the harmonic relation to the postulation of an indefinite harmonic progression.

Cassirer emphasized the philosophical significance of this kind of reasoning by saying that with projective coordinates there is an "inclusion of the spatial concepts in the schema of the pure serial concepts" (Cassirer, 1910, p. 87). The use of "schema" at this point shows a clear connection with Kant's schematization of concepts. Cassirer's own argument is summarized in a passage that is worth quoting at length for our following considerations:

> As in the case of number we start from an original unit from which, by a certain generating relation, the totality of the members is evolved in fixed order, so here we first postulate a plurality of points and a certain relation of position between them, and in this beginning a principle is discovered from the various applications of which issue the totality of possible spatial constructions. In this connection, projective geometry has with justice been said to be the universal "*a priori*" science of space, which is to be placed besides arithmetic in deductive rigor and purity. Space is here deduced merely in its most general form as the "possibility of coexistence." While no decision is made concerning its special axiomatic structure (in particular concerning the validity of the axiom of parallels), it can be shown that by the addition of special completing conditions, the general projective determination can be successively related to the different theories of parallels and thus carried into the special "parabolic," "elliptic" or "hyperbolic" determinations." (Cassirer, 1910, p. 88).

Cassirer appreciated the fact that points and numbers can be mapped onto each other while remaining distinct. At the same time, he pointed out that the mapping brings to the fore a more general structure than Euclidean space. In this sense, Cassirer claimed that the principle discovered (i.e. the geometric equivalent of Dedekind's continuity) has various applications in Klein's classification of geometries, and the projective model provides a "deduction" of space in its most general form as the "possibility of coexistence." The fact that Cassirer used Kantian and Leibnizian terminologies as interchangeable here is reminiscent of his 1907a reading of Kant, and so is the view that mathematical abstraction has its counterpart in an increased capability of determining something as conceptually distinct. As we saw in 2.2, this is what Cassirer (1907a) referred to as "logical correlation of concepts and intuitions." He relied on the same view in 1910 to generalize the notion of the form of space to the non-Euclidean cases.

It is worth noticing that in this passage Cassirer contrasted his view with Russell's (1897). This might seem puzzling, because Russell here also maintained that at least some fundamental properties of projective space (i.e., continuity, homogeneity, having a finite number of dimensions) are common to both Euclidean and non-Euclidean spaces. He deemed these properties *a priori*, insofar as they are necessarily presupposed in the perception of extended objects. However, Russell deemed the concept of a projective metric a mere technicality, which served the purpose of mathematical convenience but did not challenge the Euclidean view or provide an actual classification of spaces.[22]

Russell's critique of Klein concerned, more specifically, the notion of coordinates which, for Russell, serve only as convenient designations for points that the mathematician wishes to distinguish. The distinction of the elements from each other, for Russell, is a presupposition rather than a result of the designation. The problem with projective geometry is particularly that distance, as a function of projective coordinates, involves at least four elements. Russell's requirement that each element be distinguished from each other implies that any two elements should be in such a relation, regardless of their relation to all others, as in the ordinary notion of distance (Russell, 1897, p. 35). In order to address this problem, Russell, in this early work, adopted a strategy that was not infrequent in the post-Kantian debate on the foundations of geometry[23]: our background knowledge of space includes some notions, such as that of the distance between two given points, with which we are immediately acquainted and which makes possible the perception of extended

[20] In the nineteenth-century terminology, von Staudt's approach represented the most consistent implementation of a "synthetic" or descriptive approach to projective geometry without metric foundations. This did not imply avoiding analytic procedures altogether, as von Staudt himself introduced a generalized calculus of segments (called calculus of jets). He used the properties of harmonic progressions to describe how to assign projective coordinates to the particular jets that correspond to cross-ratios and defined equality and operations with jets based on the properties of involutions. The construction described in the following relies on von Staudt's ideas, but does not require the calculus of jets and is found in Klein (1893, 337–354).

[21] This property is now known as "connectedness" and requires an explicit axiomatic formulation, insofar as it implies the existence of irrational points. Klein used Dedekind's formulation to clarify this assumption, which was implicit in von Staudt's work but not available to him at the time he wrote (cf. Klein, 1873, p. 132).

[22] Russell wrote: "Since these systems are all obtained from a Euclidean plane, by a mere alteration in the definition of distance, Cayley and Klein tend to regard the whole question as one, not of the nature of space, but of the definition of distance. Since this definition, on their view, is perfectly arbitrary, the philosophical problem vanishes – Euclidean space is left in undisputed possession, and the only problem remaining is one of convention and mathematical convenience" (Russell, 1897, p. 30).

[23] Se Biagioli (2016) for a reconstruction of other positions in this debate.

objects. These are the notions that Russell identified as *a priori*. Variations of the fundamental concepts and the idea of different forms of space are possible but logically dependent on *a priori* notions.

It is well known that Russell subsequently distanced himself from this view in favor of logicism. His earlier strategy, however, reflects back on his critique of Dedekind's characterization of natural numbers in Russell (1903, p. 249). Russell's objection is that numbers should be something more than the terms of relations that constitute a progression. They should possess intrinsic qualities which make them different from anything else.

Cassirer distanced himself both from the above strategy and from Russell's logicism in defending the philosophical significance of the emerging structuralism of nineteenth-century geometry, in particular, the priority of relations over things. It followed that the correlation of concepts and intuitions had to be established, not in the supposed evidence of our acquaintance with space, but in the extension of purely intellectual and structural procedures from numerical to spatial concepts and from there on to the appearances.

5. Concluding remarks

Summing up the argument from the consideration of these function-concepts, Cassirer began with the concept of number, not only because he saw numbers as "rooted in the substance of rational knowledge," but also because "[i]n the thought of number all the power of knowledge seems contained, all possibility of the logical determination of the sensuous" (Cassirer, 1910, p. 27). He then moved from the serial principle that determines the natural numbers to geometric progressions in order to investigate the "inclusion of the spatial concepts in the schema of the pure serial concepts" (Cassirer, 1910, p. 87). As in Kant's transcendental deduction, space and time provide the only possible types of ordering of the phenomena in a lawful way. A crucial aspect of Cassirer's account, however, is the analysis of how structural procedures developed in modern mathematics by doing abstraction from spatiotemporal notions.

I have drawn attention to Cassirer's reading of Kant in (Cassirer, 1907a) to show that there is, nevertheless, an important parallel between Cassirer's interpretation of schematism as a three-step synthesis and the account of mathematical reasoning articulated in (Cassirer, 1910). The formation of mathematical concepts presupposes, firstly, that one abstracts away from empirical contents. One can then establish the serial principles that determine the mutual relations of a manifold. The complete determination of mathematical objects and the justification of mathematical deductions rests on the further presupposition that a variety of such manifolds follow under the concept of a mathematical structure. While these manifolds are not limited to spatiotemporal ones, Cassirer's focus is on group-theoretical and other procedures that allow for an extension from mathematics to the natural sciences. His aim is to show that mathematical reasoning, considered in its own right, enables such an extension as a synthetic process of determination of individuality. This kind of reasoning is required if the application of mathematics to empirical reality is to be possible. Furthermore, the above examples show that the structure of Cassirer's argument is closely connected to his reading of Kant's Transcendental Deduction as focused on a unitary but internally articulated process of cognition. Insofar as mathematical syntheses occupy the highest level of this process, they deserve to be related to - but also distinguished from - empirical knowledge. In Cassirer's interpretation, it is this differentiation of levels in concept formation that characterizes the transcendental account.

The genuinely Kantian aspect of Cassirer's view is often overlooked, arguably because the necessary applicability of mathematics (especially Euclidean geometry) appears to be called into question by its structural turn in the nineteenth century. I have argued that the originality of Cassirer's approach, in response to this concern, lies in his appreciation of the particular structuralism that emerges in the works of mathematicians such as Dedekind and Klein. The characteristic trait of this structuralism – I have argued also with reference to the more recent literature on early mathematical structuralism – is that mathematical objects obtain a unique characterization by extending structural procedures over a domain that is not fixed in advance. Cassirer deems this kind of reasoning synthetic, insofar as it determines what can be studied mathematically as the field of possible applications of structural procedures. In doing so, he emphasizes the epistemological implications of procedures that, on the contrary, appear to be mere technicalities from both Russell's early Kantian stance and his later logicism.

A weaker version of Kant's applicability requirement follows from the idea that variations of these procedures contain at least some of the structures that are actually applied in physics. Even though a discussion of this point is beyond the scope of this paper, it is worth noticing that Cassirer continued to argue for an extension of mathematical to scientific structuralism throughout his works on the foundations of twentieth-century physics.

Declaration of competing interest

None.

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References

Biagioli, F. (2016). *Space, number and geometry from Helmholtz to Cassirer*. Cham: Springer.
Biagioli, F. (2018). Articulating space in terms of transformation groups: Helmholtz and Cassirer. In S. Edgar, & L. Patton (Eds.), *Method, science, and mathematics: Neo-Kantianism and analytic philosophy* (pp. 115–131). Journal for the History of analytical philosophy, 6.3.
Biagioli, F. (n.d.). Structuralism and mathematical practice in Felix Klein's work on non-Euclidean geometry. In preparation.
Cassirer, E. (1902). *Leibniz' System in seinen wissenschaftlichen Grundlagen*. Marburg: Elwert.
Cassirer, E. (1907a). *Das Erkenntnisproblem in der Philosophie und Wissenschaft der neueren Zeit* (Vol. 2). Berlin: B. Cassirer.
Cassirer, E. (1907b). Kant und die moderne Mathematik. (Mit Bezug auf Bertrand Russells und Louis Couturats Werke über die Prinzipien der Mathematik.). *Kant-Studien, 12*, 1–49.
Cassirer, E. (1910). Substanzbegriff und Funktionsbegriff: Untersuchungen über die Grundfragen der Erkenntniskritik. Berlin: B. Cassirer. In *In Cassirer, E. Substance and function and Einstein's theory of relativity* (English edition). Chicago: Open Court (M. C. Swabey & W. C. Swabey, Trans.), 1923.
Cohen, H. (1883). *Das Princip der Infinitesimal-Methode und seine Geschichte: Ein Kapitel zur Grundlegung der Erkenntniskritik*. Berlin: Dümmler.
Cohen, H. (1885). *Kants theorie der Erfahrung* (2nd ed.). Berlin: Dümmler.
Dedekind, R. (1872). *Stetigkeit und irrationale Zahlen*. Braunschweig: Vieweg. In *In Dedekind, R. Essays on the theory of numbers* (English edition). Chicago: Open Court (W. W. Beman, Trans.), 1901.
Dedekind, R. (1888). Was sind und was sollen die zahlen?. Braunschweig: Vieweg. In R. Dedekind (Ed.), *Essays on the theory of numbers* (English edition). Chicago: Open Court.
Dummett, M. (1991). *Frege: Philosophy of mathematics*. Cambridge, MA: Harvard University Press.
Ferrari, M. (1988). *Il giovane Cassirer e la scuola di Marburgo*. Milano: Angeli.
Friedman, M. (2000). *A parting of the ways: Carnap, Cassirer, and Heidegger*. Chicago: Open Court.
Friedman, M. (2001). *Dynamics of reason: The 1999 Kant lectures at Stanford University*. Stanford, CA: CSLI Publications.
Friedman, M. (2012). The Prolegomena and natural science. In H. Lyre, & O. Schliemann (Eds.), *Kants Prolegomena: Ein kooperativer Kommentar* (pp. 299–326). Frankfurt am Main: Klostermann.

Giovanelli, M. (2016). Hermann Cohen's *Das Princip der Infinitesimal-Methode*: The history of an unsuccessful book. *Studies in History and Philosophy of Science Part A, 58*, 9–23.
Heis, J. (2010). "Critical philosophy begins at the very point where logistic leaves off": Cassirer's response to Frege and Russell. *Perspectives on Science, 18*, 383–408.
Heis, J. (2011). Ernst Cassirer's neo-Kantian philosophy of geometry. *British Journal for the History of Philosophy, 19*, 759–794.
Heis, J. (2014a). Ernst Cassirer's Substanzbegriff und Funktionsbegriff. *HOPOS: The Journal of the International Society for the History of Philosophy of Science, 4*, 241–270.
Heis, J. (2014b). Realism, functions, and the a priori: Ernst Cassirer's philosophy of science. *Studies in History and Philosophy of Science Part A, 48*, 1–19.
Holzhey, H. (1986). *Der Marburger Neukantianismus in Quellen: Zeugnisse kritischer Lektüre; Briefe der Marburger; Dokumente zur Philosophiepolitik der Schule* (Vol. 2). Basel: Schwabe.
Ihmig, K.-N. (1997). *Cassirers Invariantentheorie der Erfahrung und seine Rezeption des "Erlanger Programms."*. Hamburg: Meiner.
Kant, I. (1884). Reflexionen Kants zur Kritik der reinen Vernunft. In B. Erdmann (Ed.), *Reflexionen Kants zur kritischen Philosophie* (Vol. 2). Leipzig: Fues.
Kant, I. (1998). In P. Guyer, & A. W. Wood (Eds.), *Critique of pure reason*. Cambridge: Cambridge Univ. Press (First published 1787).
Klein, F. (1872). Vergleichende Betrachtungen über Neuere Geometrische Forschungen. Erlangen: Deichert. (2nd Ed *Mathematische Annalen, 43*, 403–415, 1893.
Klein, F. (1873). Über die sogenannte Nicht-Euklidische Geometrie. *Mathematische Annalen, 6*, 112–145.
Klein, F. (1893). In F. Schilling (Ed.), *Nicht-Euklidische Geometrie*. Leipzig: Teubner.
Mormann, T. (2008). Idealization in Cassirer's philosophy of mathematics. *Philosophia Mathematica, 16*, 151–181.
Pereboom, D. (2018). *Kant's transcendental arguments*. The Stanford Encyclopedia of Philosophy. https://plato.stanford.edu/entries/kant-transcendental/. (Accessed 17 September 2019).
Reck, E. H. (2003). Dedekind's structuralism: An interpretation and partial defense. *Synthese, 137*, 369–419.
Richardson, A. (2003). Conceiving, experiencing, and conceiving experiencing: Neo-Kantianism and the history of the concept of *experience*. *Topoi, 22*, 55–67.
Russell, B. (1897). *An essay on the foundations of geometry*. Cambridge: Univ. Press.
Russell, B. (1903). *The principles of mathematics*. Cambridge: Univ. Press.
Ryckman, T. A. (1991). Conditio sine qua non? Zuordnung in the early epistemologies of Cassirer and Schlick. *Synthese, 88*, 57–95.
Schiemer, G. (2018). Cassirer and the structural turn in modern geometry. In S. Edgar, & L. Patton (Eds.), *Method, science, and mathematics: Neo-Kantianism and analytic philosophy* (pp. 181–212). Journal for the History of Analytical Philosophy, 6.3.
Schiemer, G. (2020). *Transfer principles, Klein's Erlangen Program, and methodological structuralism* (in press).
Sieg, W., & Schlimm, D. (2017). Dedekind's abstract concepts: Models and mappings. *Philosophia Mathematica, 25*, 292–317.
Von Staudt, C. (1847). *Geometrie der Lage*. Nürnberg: Korn.
Yap, A. (2017). Dedekind and Cassirer on mathematical concept formation. *Philosophia Mathematica, 25*, 369–389.