

## Perfect fluid geometries in Rastall's cosmology

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### Abstract

The equation of state of an ultrarelativistic perfect fluid is obtained as a necessary condition for a perfect fluid space-time in Rastall's cosmology.

*Key words:* perfect fluid space-time; Rastall's cosmology; ultrarelativistic fluid

## 1 Introduction

It is well known the importance of a perfect fluid description of gravitational sources in General Relativity [3]. It has been also stressed the relevance of compatibility of perfect-fluid solutions with modified or extended theories of gravity [1].

In this note we point out some cosmological features emerging from the request that the Rastall model [11] describe a perfect fluid geometry of space-time.

We shortly recall some features of the model introduced by Rastall in 1972; we refer to this paper for the relevant issues and stress that the chosen signature for the underlying Lorentzian manifold will be  $(-, +, +, +)$ , in accordance with such a source.

Let then  $Lor\mathbf{X}$  be the open submanifold of the subbundle of symmetric tensors constituted by regular metrics with the above choice of the Lorentzian signature. An induced fibered chart on  $J^2Lor\mathbf{X}$  has local coordinates

$$(x^\alpha, g_{\alpha\beta}, g_{\alpha\beta,\gamma}, g_{\alpha\beta,\gamma\nu})$$

and we can define other local coordinates  $g^{\mu\nu}$  and so on by the relation  $g_{\alpha\mu}g^{\mu\nu} = \delta_\alpha^\nu$ .

As well known Einstein equations read

$$G_{\mu\nu} \doteq R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa_{GR}T_{\mu\nu},$$

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where  $\kappa_{GR} = \frac{8\pi G}{c^4}$ ; the Ricci tensor is obtained from a metric connection, so that  $R_{\mu\nu} = R_{\mu\nu}(j^2g)$  and the scalar curvature  $R$  has to be intended as  $R(j^2g) = g^{\alpha\beta}R_{\alpha\beta}(j^2g)$ . These equations naturally imply the conservation of energy-momentum tensor as a consequence of the Bianchi identities.

The Rastall model for gravity coupled with matter governed by an energy-momentum tensor satisfying (with  $\lambda$  a suitable non-null dimensional constant)

$$\hat{T}_{\nu;\mu}^{\mu} = \lambda R_{,\nu}$$

where the comma denotes the partial derivative, and by field equations

$$R_{\mu\nu} - \frac{1}{2}(1 - 2\kappa_r\lambda)Rg_{\mu\nu} = \kappa_r\hat{T}_{\mu\nu}, \quad (1)$$

where  $\kappa_r$  is a dimensional constant to be determined in order to give the right Poisson equation in the static weak-field limit. This model introduces an energy-momentum tensor fulfilling the requirement  $\nabla_{\mu}\hat{T}_{\nu}^{\mu} \doteq \hat{T}_{\nu;\mu}^{\mu} \neq 0$  without violating the Bianchi identities; for a 4-vector, say  $a_{\nu}$ ,  $\hat{T}_{\nu;\mu}^{\mu} = a_{\nu}$  and  $a_{\nu} \neq 0$  on curved spacetime, but  $a_{\mu} = 0$  on flat spacetime (in agreement with Special Relativity). Of course, in general,  $\kappa_r \neq \kappa_{GR}$ . Taking the trace of Eqs.(1) gives us the so-called structural or master equation  $(4\kappa_r\lambda - 1)R = \kappa_r\hat{T}$  ( $\kappa_r\lambda \neq \frac{1}{4}$ ).

By this relation the Rastall equations (1) can be recast in the form of Einstein equations. Indeed, one can immediately write  $G_{\mu\nu} = \kappa_r S_{\mu\nu}$ , where  $S_{\mu\nu} \doteq \hat{T}_{\mu\nu} - \frac{\kappa_r\lambda}{4\kappa_r\lambda-1}g_{\mu\nu}\hat{T}$ . By construction, this new energy-momentum tensor is conserved, *i.e.*  $S_{\nu;\mu}^{\mu} = 0$ . Furthermore, (if  $u_{\mu}$  is a 4-velocity with  $g^{\mu\nu}u_{\mu}u_{\nu} = -1$ ) by assuming  $\hat{T}_{\mu\nu} = -(\rho + p)u_{\mu}u_{\nu} - pg_{\mu\nu}$ , *i.e.* the source is a perfect isentropic fluid with energy density  $\rho$  and pressure  $p$ , we can explicitly work out an expression for  $S_{\mu\nu}$ . The source turns out to be still a perfect fluid, provided we redefine its energy density and pressure. [11]

## 2 Perfect fluid space-times in Rastall's cosmology

Rastall's cosmology has been recently object of attention in scientific community; some papers are concerned with the querelle whether Rastall's stress-energy tensor corresponds to an artificially isolated part of the physical conserved stress-energy or not [2, 7, 10, 15], some others are concerned with the derivation of a related variational principle (*i.e.* with the question how to get a - eventually modified - Rastall model which could be derived by a Lagrangian and physically different from Einstein gravity), see *e.g.* Refs. [12, 14, 13].

We study this model from the point of view of the compatibility of its geometry with a perfect fluid description of space-time. Indeed, as a gravity model it provides a non minimal coupling between metric and matter and we try to understand how this is related with this compatibility requirement.

Indeed it is well known that by suitably introducing a matter density  $\mu$ , the energy density of space-time  $\bar{\rho}$  can be defined as a function of  $\mu$  satisfying

$\bar{p} = \mu\bar{\rho}' - \bar{p}$  where the prime denotes derivation with respect to  $\mu$ ; (this is equivalent with asking whether we can derive the tensor  $R_{\mu\nu} \equiv \tilde{T}_{\mu\nu}$  from a variational principle, *i.e.* with looking for a Lagrangian density depending on  $\rho(\mu)$  from which  $\tilde{T}_{\mu\nu}$  can be derived variationally; such kind of space-times appear *e.g.* in the dual Lagrangian description of the Ricci tensor for barotropic perfect relativistic fluids [6], see also Refs. [4, 5, 8]).

We therefore have

$$\tilde{T}_{\alpha\beta} = \bar{\rho}(\mu)g_{\alpha\beta} - \mu\bar{\rho}'(\mu)[g_{\alpha\beta} + u_\alpha u_\beta],$$

with trace  $\tilde{T} = 4\bar{\rho} - 3\mu\bar{\rho}'$ ; moreover  $\bar{p} = \mu^2 \frac{d}{d\mu} \left( \frac{\bar{\rho}}{\mu} \right) = \mu\bar{\rho}' - \bar{\rho}$ .

## 2.1 Ultrarelativistic equation of state from a perfect fluid geometry

By requiring the perfect fluid Ricci tensor to be *also* a Rastall's geometry the following identity should hold true.

$$\begin{aligned} \tilde{T}_{\alpha\beta} &\equiv \hat{T}_{\alpha\beta} - \frac{1}{2} \frac{2\tilde{\lambda} - 1}{4\tilde{\lambda} - 1} g_{\alpha\beta} \hat{T} = \\ &= [\rho - \mu\rho' - \frac{1}{2} \frac{2\tilde{\lambda} - 1}{4\tilde{\lambda} - 1} (4\rho - 3\mu\rho')] g_{\alpha\beta} - \mu\rho' u_\alpha u_\beta \equiv \\ &\equiv (\bar{\rho} - \mu\bar{\rho}') g_{\alpha\beta} - \mu\bar{\rho}' u_\alpha u_\beta = -\bar{p} g_{\alpha\beta} - (\bar{\rho} + \bar{p}) u_\alpha u_\beta. \end{aligned} \quad (2)$$

The above identity leads to

$$\rho' = \bar{\rho}'.$$

By inserting this and deriving with respect to  $\mu$  we get

$$\frac{2\tilde{\lambda} - 1}{4\tilde{\lambda} - 1} (4\rho' - 3(\mu\rho')') = 0.$$

Let us now exclude the case  $2\tilde{\lambda} - 1 = 0$ . We see that a necessary condition for the Ricci tensor of a nontrivial Rastall's geometry of matter to be a perfect fluid is

$$4\rho' - 3(\mu\rho')' = 0.$$

*i.e.*

$$\rho' - 3p' = 0.$$

the equation of state is therefore

$$p = \frac{1}{3}\rho + \text{const.}(\mu),$$

*i.e.* if we denote  $\alpha$  the integration constant (a constant function of  $\mu$ ) (uniquely determined by initial data)

$$p - \alpha = \frac{1}{3}\rho$$

In particular it is easy to get that  $-3\alpha = \rho - 3p = \tilde{T}$ , *i.e.*  $\alpha = -\frac{1}{3}\tilde{T}$

By redefining the pressure  $\pi = p - \alpha = p + \frac{1}{3}\tilde{T}$  the equation of state is the one of an ultrarelativistic perfect fluid with pressure  $\pi$  and energy density  $\rho$ .

$$\pi = \frac{1}{3}\rho \iff p = \frac{1}{3}(\rho - \tilde{T})$$

*i.e.* with a rescaled pressure or equivalently a *energy density rescaled by the trace of the energy-momentum tensor*. This relation gives us a model with constant (with respect to the matter density  $\mu$ ) energy-tensor trace  $\tilde{T}$  and consequently constant (with respect to  $\mu$ ) scalar curvature.

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