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ABSTRACT
In this paper, we pursue a materialist approach to tool use to account for the material dimensions of learning mathematics with technology while decentralizing the human body from activity. We root this vision in theoretical advances on embodied mathematics learning and tool use that have been recently offered as alternatives to traditional theories of learning that draw attention only to the individual learner or the mind. Focus is on the ways that learners, tools and mathematics are entangled in provisional reconfigurations of the world and on how the tool partakes in this movement. Starting from our theoretical commitment, we discuss some episodes from a research study in which grade 9 students were introduced to function through a graphical approach. The study took advantage of graphing motion technology, which allows working with couples of position over time graphs, capturing spatio-temporal relationships. The episodes provide insight into the complex nature of learning in this context, valuing the mobile way that mathematics emerges from activity with the tool.

KEYWORDS
Tool; entanglement; animation; movement; function; graphs

Introduction

> Like philosophy, the digital is also an insatiable beast, and like philosophy, the digital is also inescapable today. Digital machines dominate the planet, in rich and poor regions alike, while so-called digital thinking–the binarisms of being and other or self and world–is often synonymous with what it means to think at all. (Galloway, 2014, p. xviii)

Drawing on Galloway, we cannot separate the digital from everyday life nowadays. Beyond binarisms, the digital reconfigures what people feel, do and think, and, in particular, what it means to learn at all, inside and outside school. Within the context of educational research, understanding mathematical practice with digital technologies continues to be a major topic of interest (e.g. Calder, Larkin, & Sinclair, 2018; Drijvers, 2015; Faggiano, Ferrara, & Montone, 2017; Hoyles & Lagrange, 2010). Theoretical approaches and generations of research have been embraced to study and improve practice with tools (see Sinclair, 2014) and they have given insights into different aspects of the use/implementation of the technology, for example: tools' affordances and influence (Arzarello, Ferrara, & Robutti, 2012; Trouche & Drijvers, 2010), teacher's role (Bartolini

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Bussi & Mariotti, 2008; Drijvers, Doorman, Boon, & Gravemeijer, 2010), learners' engagement and development (Nemirovsky, Kelton, & Rhodehamel, 2013; Roorda, Vos, Drijvers, & Goedhart, 2016), communicational and representational infrastructures (Hegedus & Moreno-Armella, 2009; Hegedus & Tall, 2016), and so forth.

In this paper, we are more interested in investigating the use of digital tools in mathematics teaching and learning by drawing specific attention to the material relations between tools and learners, that is, to how tools and human bodies are not separated but rather assemble in mathematical activity. We share with Sinclair (2014) that particular focus should be on the ways in which these relations alter how learners have to do with and do mathematics. Mathematics is also altered by its encounter with new digital technologies, progressively seeing a move toward visual and dynamic mathematical expressions (Rotman, 2008). Briefly speaking, we can have a view of mathematics as being inextricable from tools, rather than seeing it as disconnected from the contexts that give rise to mathematical activity (as Roth (2011) argues drawing on a cultural-historical activity theory perspective).

We want to expand traditional positions that see learning as occurring solely within the individual and neglect the material encounters with tools as partaking in the mathematical doing. In particular, we shift the focus of mathematical practice from the student or the tool to the *fused* entity 'student-tool' (Chorney, 2014), meaning that it is not that the student masters the tool, or that the tool simply resists or provides opportunities, instead the engagement of student and tool is a becoming of process from which mathematics emerges. We also recognize the provisional chaotic nature of perceptual investments with tools, for which tools constantly reconfigure our sensations and offer up new perceptual engagements with mathematics (de Freitas, 2016). We challenge the common view whereby knowledge is abstracted from the learning situation and we turn attention to the way that knowledge emerges out of an entanglement of tools, context, materials and tasks. Out of this intention, our main interest is on *movement*. Indeed, the entanglement, or assemblage, of student, tool, and concept defines mathematics practice. Drawing from Chorney (2014), each part of the assemblage is continually becoming as is shaped through the process. Even though one may identify different parts (the name of the tool, the student or the concept) for the sake of convenience, meaning does not emerge from individual aspects but from the assemblage becoming through a process of movement and change.

In this paper, our aim will be to investigate *how* tool is an active animate part of the movement through which learners, tool and mathematics are entangled in mathematical practice, while reconsidering the *force* of the tool in shaping activity.

Reanimating tools

1. Thinking in movement

Decentralizing the human allows for the implication of the tool, in a way which moves us to rethink the tool not as inert and static but as re-animated through movement in the situation. The notion of *animation* comes from Sheets-Johnstone (2009, 2011), who draws on Husserl's phenomenology and Merleau-Ponty's theory of perception to elucidate the primacy of movement in the life of animate beings. Sheets-Johnstone exposes animation as 'the ground floor of our being alive in all its affective, perceptual, cognitional, and imaginative guises, stages, practices, and surrounding worlds' (Sheets-Johnstone, 2009, p. 390). For her, movement is not equivalent to a mere local change in position, but it is our primary way of making sense of the world at both human and evolutionary scale (that is to say, in terms of human development and with respect to the evolution of animate forms). In particular, Sheets-Johnstone (2011) examines the experience of 'thinking in movement' and its foundational character to the creation of a kinetic bodily *logos*. Taking a first-person experience of an improvisational dance, she describes a paradigmatic example of thinking in movement. She shows how in the dance there is no gap between the global dynamic world which is perceived and the kinetic world in which she is moving. In other words, the world that is *explored* in movement cannot be separated from the world that is *created* in movement. Following Sheets-Johnstone, thinking in movement refuses a separation between thinking and its expression, that is, the fact that thoughts in one's head could exist prior to, and independently from, their corporeal expressions. In addition, saying that thinking in movement is a way of being in the world and a natural mode of being a body, she challenges representational visions of the body, which see language as one primary means by which a body mediates its way about the world. Her perspective has at least two important consequences: we might rethink what it means "to have meaning", and movement itself might be meaningful. In our understanding of it, the notion of 'thinking in movement' implies not just a temporal overlapping but the mutual constitution and implication of moving and thinking. Of particular interest in the present study is how movement and thinking are contiguous and build up each other in mathematical situations (Ferrara & Ferrari, 2018). Therefore, we aim to adhere to non-representational conceptualizations of gesture and bodily movement (e.g. de Freitas & Ferrara, 2015; de Freitas & Sinclair, 2014; Kelton & Ma, 2018; Maheux & Proulx, 2015; Nemirovsky et al., 2013; Roth & Maheux, 2015; Sedaghatjou & Campbell, 2017) with a visceral intention toward the way in which movement might be better characterized and analysed in the context of the mathematics classroom.

Borrowing from this vision, we take a dynamic perspective on mathematics thinking and learning with tools and propose to explore how tools are animate in mathematical activity. In this way, we hope to contribute to a line of discussion about dynamic approaches to mathematical thinking, which address mathematical thinking in movement in a way that learning and movement are not reduced to schemas.

2. Perceptuomotor activity and mathematical tool use

Of interest for this dynamic orientation is the non-dialectic approach to mathematical tool use offered by Nemirovsky et al. (2013). The authors define a mathematical instrument as a material and semiotic implement experienced interactively through a set of embodied practices and continuous body movements to create and transform expressive forms, such as graphs, equations and diagrams. The position of Nemirovsky and colleagues regarding the use of a mathematical instrument is clearly stated in the idea that one cannot talk about mathematical expertise divorcing it from perceptual and motor aspects of the activity with the tool. In particular, the lived experience of students, who work with mathematical instruments or incorporate the imagined presence of tools and others, can be described in terms of temporal flows of perceptuomotor activity: past and future permeate any perceptuomotor activity, in a way that it is always infused with expectations, recollections, fantasies, moods, etc.

For the authors, the achievement of a mutual combination and harmonization (interpenetration) of perceptual and motor aspects of tool use (perceptuomotor integration) is central to the user's perceiving and acting with a holistic sense of unity and flow and implicates that the activity is enacted as a whole over time. In other words, the emergence of perceptuomotor integration is a crucial quality of gaining expertise with the tool, and mathematical understanding emerges from temporal flows of the bodily engagement with the tool. We understand perceptuomotor integration as an emerging property that implies transformations in learners' lived experiences, moving towards a gradual harmonization of perceptual and motor aspects of the activity. Therefore, we see this vision as significant for a discourse about movement and change and for our commitment to an account of embodied tool use as constitutive of mathematical thinking and learning (see also Sedaghatjou & Campbell, 2017, where the perceptuomotor approach is applied to the study of a child's interactions with a multitouch technology).

3. Agency and material engagement

Pushing our theoretical orientation even further into an ontological commitment, we are broadly operating in this study from an inclusive materialist ontology of embodiment, mathematical objects and events, which recognizes and values the material and bodily dimensions of learning mathematics and mathematical experiences (de Freitas & Sinclair, 2014). De Freitas and Sinclair consider the force of inclusive materialism in how it breaks with divisions between organic and inorganic, animate and inanimate, to reanimate matter more generally and rethink it in terms of potentiality and emergent generative force (de Freitas & Sinclair, 2014). Particularly, we regard knowledge as a practice or a process, in which tools and other bodies (other than the human mind) possess a force of animating the human body, rather than as an abstract object to be transferred to students. Following Chorney (2014), we view teaching and learning as acts of engagement with heterogeneous parts emerging through the movement of things, concepts and people. For this study, we draw specific attention to the ways in which tools are always interacting with each other and with the human body.

This perspective also demands that we move away from specific questions about *agency* and how it might be at work with tools in mathematical activity (Ferrara & Ferrari, 2017). Mathematics educators pose as problematic a figure of agency that maintains focus on a relation of the kind subject-verb-object (e.g. Roth, 2016), which tends to emphasize the close borders of different entities. Ingold (2011) clarifies that it is a problem of our own making: we add agency both to human and non-human bodies (materials) as an extraneous extra ingredient that livens them up. However, the problem can be overcome focusing on the direct engagement(s) with the materials themselves, drawing attention to what happens to them 'as they circulate, mix with one another, solidify and dissolve in the formation of more or less enduring things' (p. 16). It is this direct engagement, especially with tools, in which we are most interested. This work aims to study how the tool has its own force and capacity to affect rather than being simply taken up and acted upon by the human agent. Therefore, we invoke agency according to Barad (2007), as dynamically created in/through activity, and not as a quality of one or the other entity. This invites us to focus on the tool-student relation as always provisionally individuated in the activity, while also placing value on the tool.

Stated another way and putting this all together, we operate in this study from the theoretical commitment that all mathematical activity is always material in that it is always playing out across material encounters (assemblages) of human and non-human bodies, from which knowledge emerges instead of being produced as a mental construct. Moreover, agency is spread, dispersed, plural and distributed across the learning assemblage, so that tools also have some *degree* of agency. Philosophically speaking, then, we consider mathematical activity to be always *mobile*. The learning assemblage constantly reconfigures the event. But movement also signifies for us not only a theoretical commitment to the material and embodied dimensions of all mathematical activity, but importantly, attention to and curiosity about the possibilities held by particular kinds of tasks in which tools and their interactionist energy are made more relevant to mathematics than is often the case in traditional studies of technology in mathematics education (e.g. Drijvers, 2015; Hoyles & Lagrange, 2010; for major discourse about the collaborative force of mathematical tasks see de Freitas, Ferrara, & Ferrari, 2017, 2019).

In this paper, we investigate the learners' speculative investments with a tool for the graphical study of function, and how the tool reconfigures mathematical experiences in the classroom and shapes mathematics-as-movement. Focus is on the use of a particular technology, WiiGraph, which engages pairs of students with functions through graphing motion. Using our video data, we explore ways in which aspects of function can be encountered and provide examples of how the movement implicated in the mathematical activity provides insights into the learners' understanding of functional relationships.

The study

Participants and context

The episodes examined in this paper come from a wider research with the overall aim of investigating the role of movement in mathematics and graphing motion technologies as a resource for mathematics teaching and learning. The episodes are part of a classroom-based intervention (Stylianides & Stylianides, 2013), whose main element was the teaching and learning of function with a software application called WiiGraph. The software has been developed by R. Nemirovsky and colleagues at the Center of Research in Mathematics and Science Education of San Diego State University (Nemirovsky, Bryant, & Meloney, 2012). The study took place in a public secondary school in northern Italy with average socioeconomic status. The participants consisted of 30 ninth-grade students and their mathematics teacher. The students were all between 14 and 15 years old and were engaged in mathematical investigations about graphing motion with WiiGraph. The teacher and students provided consent to participate, upon invitation by the researchers, on the basis that the study posed no conflict of interests and harm but would be beneficial to their teaching and learning experience.

In 3 months, the participating students took part in 9 days of 2-hour activities in a lab room used as a laboratory space for mathematics lessons. An interactive whiteboard (IWB), a traditional whiteboard and a desk were available for use in the room. In the usual setting, the students were arranged in a semicircle in front of the whiteboards, with a wide in-between interaction space for experiencing motion. The students worked on individual written tasks and participated in group work and collective discussions. For group work, the class was divided into small groups of three people, and sometimes pairs of groups were requested to face tasks together. The participants had never used the software before this study.

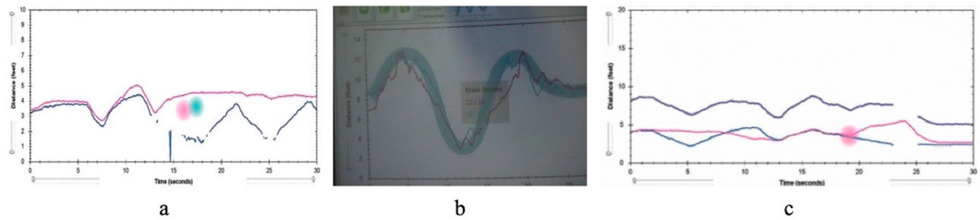

Figure 1. (a) Standard *Line* graphs, (b) *Make your own Maze!* session, (c) *Line* option for $a + b$.

WiiGraph

WiiGraph is a mathematical instrument that takes advantage of two controllers (or remote controls) to display in real time, on a single Cartesian plane, position versus time graphs associated to the movement of the controllers in front of a sensor bar. The activity with the whole system occurs through two students' body or ample arm movement, while they hold the controllers in their hands. The position is captured by distance from the sensor over time (the controllers need to be pointed toward the sensor for the software to function; in that case, two coloured circles appear on the graph area; Figure 1(a)). Bodily activity with WiiGraph therefore embeds proprioceptive and kinaesthetic experiences with the remote controls, and with the graphs and symbolic operations that the instrument provides. Many options for graph types, composite operations, targets and challenges can in fact be chosen. In the standard modality (*Line*), WiiGraph produces two coloured (pink and blue) lines, each corresponding to the movement of one controller. The two resulting graphs (Figure 1(a)) capture the functions $a(t)$ and $b(t)$, where a and b label distances from the sensor and t labels time. The creation of the graphs was shared via the IWB within the classroom.

Of interest in this paper are the *Make your own Maze!* target and the $a + b$ operation (non-standard choices in *Line*). The first option implicates the creation of a target maze to be traversed. The maze is built with a number of inflection points, thickness and tension, therefore, implies a degree of difficulty for the traversal. The maze appears as a thick light blue line and remains visible for the entire session, while two graphs originate in real time as two users move the controllers (Figure 1(b)). Each graph is also associated to a certain amount of maze traversal. The second option, $a + b$, introduces a third new graph on the screen, which emerges from the sum of a and b over time: the graph of $(a + b)(t)$ (Figure 1(c)).

Method

Having planned the target activities with the regular classroom teacher, the authors were both present during the activities to observe and interact with students while collecting video data, beyond leading the collective discussions. Simultaneously, they were also partaking in the same space and time as the study's subjects during the intervention. Intervention studies particularly put forward the idea of 'action taken to improve a situation' of Stevenson and Lindberg (2012) and directly take into account 'the practice of teaching and learning mathematics in classrooms' (Stylianides & Stylianides, 2013, p. 334). They align with the methodology of design (and teaching) experiments, which entails the design of an intervention (an action in the classroom) and the study of its impact, for example investigating opportunities of educational improvement, like when learners are engaged with new situations. In addition, these experiments draw on the idea of collaboration between teacher and researchers, which also featured the form of our intervention, together with the researchers experimenting, and reflecting on, their own teaching practice in the classroom (making the particular case of teaching experiments). This kind of research is of great interest to us because it 'can offer an "existence proof" that students can participate in creative or inventive mathematics' (Swinyard, 2011, p. 93), fostering their situated engagement with new mathematical experiences. Due to our commitment to the body, movement and dynamic, mobile mathematics, we also want to trace the phenomenology of these experiences and how it might contribute to a better understanding of mathematical knowledge as practice. Therefore, we employ a qualitative method, principally based on observation and ethnographic (e.g. Streeck & Mehus, 2005) analyses – which call for the presence of researchers and teachers as both participants and observers – for which the source of the data remains the relational moments of engagement that entail all kinds of complex social and material forces, which are to a greater or lesser extent provisionally assembled (de Freitas, Lerman, & Parks, 2017). We are particularly attentive to focus on the flow or quality in the empirical data, in order to go beyond the use of any simplistic code or story to capture it. The intervention was video recorded through the use of two mobile cameras. Data for the analyses comes from the transcriptions of video data and the group and individual write-ups. We present traces of the study in this way below, familiarizing readers with episodes from the classroom.

Classroom episodes

The episodes we offer here are concerned with different aspects related to function and engage students with different types of graphs. In the episodes, the students deal with tasks that do not require the physical use of the tool, even though they recall dynamic activity with it. Each episode is first presented and then analysed. Particular focus is on the new (change, novelty) that emerges out of the learners' ways of moving and talking (bodily activity) as they engage with mathematical concepts, with a commitment to how the tool specifically partakes of the movement in the classroom, informing mathematical thinking.

Before the first episode, the students already worked with pairs of position-time graphs in *Line*, discussing about their mutual relationships.

Episode 1: "If I had moved faster"

Part I. The first episode involves a task that asked groups of three students to challenge each other to a maze traversal, using a *Make your own Maze!* target graph. From the mathematical point of view, the task is rich in possibilities of experiencing time as that variable which cannot be neither controlled nor stopped. In order to traverse the maze, the students have to look for coordination between the flowing time and their position over time, while avoiding positions that might entail unexpected sudden moves that leave the maze.

In the activity, different target mazes were assigned to different couples of groups, which chose each one volunteer to play. We consider the case of Emanuele and Oliver, whose target maze was a thick curved line with two bumps (Figure 1(b); the figure also shows the

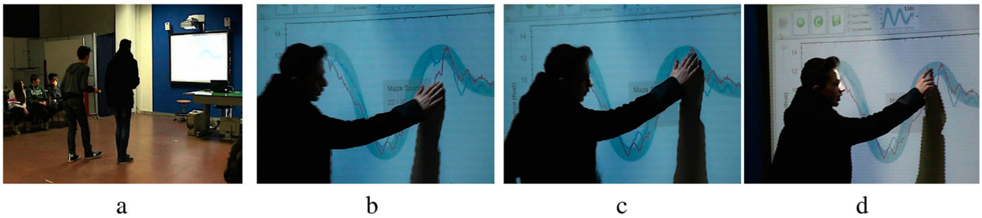

Figure 2. (a) The challenge experienced, (b,c) Oliver's gesture and posture at the IWB, (d) running the maze.

two traversals). Before moving, each student received indications from his group. Then, Emanuele and Oliver moved, quite close to each other (Figure 2(a)), affecting each other's real-time production of the line: in particular, some tension flew through the students' posture, voice and laugh. At the end Emanuele scored 28 out of 35 for traversing the maze (blue line) against 22 out of 35 (pink line), obtained by Oliver. Oliver, unsatisfied, perceived his result as mistaken, as he affirmed when the researcher asked the students their feeling about this experience:

Oliver: I made a mistake on the second [bump] (*Points towards the bump on the maze, from his chair*) when going up, because I'd to move faster, I didn't realise that

Researcher: Where, do you say? (*Invites Oliver to go to the IWB*)

Oliver: Here, I've really gone out (*Quite close in front of the IWB, head and body softly tended towards the graphs, moves his open right hand smoothly along the increasing pink line piece, which leaves the maze: 2b and 2c*), because if I'd moved faster (*His right index finger runs inside the corresponding piece of the maze: 2d. Turns to look at the researcher, smiles*) I'd have been more inside the graph (*Repeats the gesture faster*)

Analysis. The apparent discrepancy between the pink line and the target maze prompted Oliver to move and examine the configuration of lines on the screen. While the verb "go out" captures the pink line that left the maze, the subject "I", coupled with the past tense of the verb and "really", implicates an assemblage of student, line and tool. It was the graph not Oliver that did leave the maze, but Oliver produced the leaving graph, and these movements occurred simultaneously. The discrepancy speaks at once to the temporality of the experience, the position-time graphs and the feeling of being mistaken – a feeling amplified by Oliver's smooth hand movement and tension towards the graphs (Figure 2(b,c)).

Oliver and the line begin again to move together. The "if" in the explanation suddenly put forward an imaginary dimension, in which a new pink line arises from a "faster" movement and creates a new graphical configuration. The running finger actualizes a steeper line that traverses the maze instead of leaving it, while the speeding up brings forth the different speed of the new movement (Figure 2(d)). The being "more inside the graph" is also a faster, concrete steep movement captured through the body. We interpret this as a *power* of the maze to affect Oliver, causing him to move faster and to imagine the creation of the new traversal. Through movement, Oliver infuses the configuration of maze and lines with a mobile character of past and future (what *was* before and what *might be*). We see here initial insights into the temporal and spatial relationships that sustain the maze and the multiple lines that *might* traverse it, which unfold understanding of the steepness-speed relationship. The new line emerges out of an interpenetration of perceptual and motor features of the activity with the tool. It is as if a new session was played, one in which a suitable change of speed allows for a challenging traversal – challenging both with respect to the other student and to the resistance of the tool, in this case the flow of time. In fact, once one is out of the maze, returning to traverse it means adjusting the production of the line. This speaks directly to the way that the maze also establishes boundaries for the human bodies and their movements. We begin to better understand how, through the maze, the tool partakes in the coordinated movement of bodies and lines and in the reconfiguring of the activity.

Part II. After the challenge, the groups faced a written activity together. The activity asked the students to write down their sensations, distinguishing the voices of the groups, and produce a collective strategy, considering to explain it to a friend, who never used the tool. The answers given by each group were similar (Figure 3: the dotted line divides the answers). Oliver and his mates wrote about the sensation

> of changing speed in order to create a different slope for creating different kinds of "bumps". The second difficulty lies in finding the starting point so that it is possible to have a "secure" point from which the graph is generated. (Figure 3, left)

Emanuele and his mates wrote: "We had few difficulties. I didn't have many difficulties apart from the change of speed in the curves. Our major difficulty was to find the correct starting point." (Figure 3, right).

The collective strategy was expressed as follows:

> The relevant thing is to stay inside the maze and point the controller towards the sensor, and to remember that the closer you are to the sensor, the closer you are to zero on the graph. A thing to look at to remain inside the maze is that the steeper a piece is, the bigger speed has to be.

We focus on the moment immediately after, when the students used WiiGraph to validate their strategy with any maze. A new maze was displayed on the screen as the six students

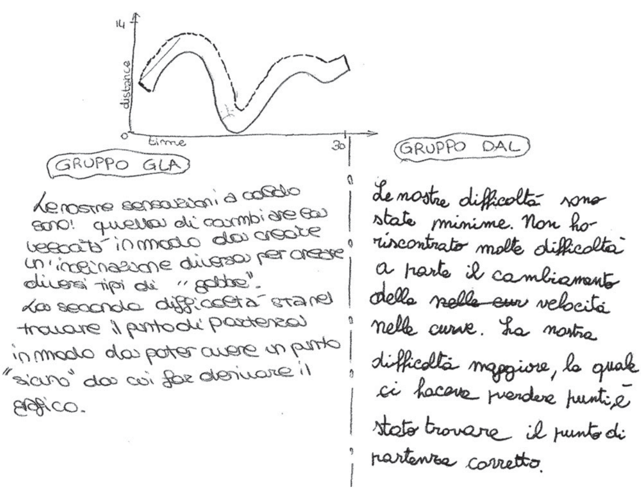

Figure 3. Emanuele's group (right) and Oliver's group's (left) answers.

a b c d

Figure 4. (a) The starting point, (b) agreement, (c,d) Emanuele and Oliver's coordinated movement.

came in front of the sensor. The four students who did not move before tried in pairs, and the awkward issue of speed appeared again. Emanuele and Oliver were thus invited to try another time to exploit the strategy while creating a better traversal. Attention is specifically drawn to this unscripted, brief experiment, in which the two students talked little and centred on joint movement in a considerable dynamic effort to coordinate bodies and movements (Figure 4).

The students first focused on positions on the floor to choose a starting point (Figure 4(a)). Then, they began to walk coordinated with and next to each other, searching for a kind of gentle agreement in movement (Figure 4(b)) and stressing that "the important thing is to create the best traversal" (Emanuele) and "we're in a collaboration" (Oliver) as the researcher asked whether they were no longer in a challenge. The two students proceeded, silent, focused and attentive to the screen, with similar pace and posture, keeping the controllers parallel and very close, their facing arms almost touching each other (Figure 4(c,d)).

Analysis. The activity shows the *force* of the maze in the case of a collaborative written task. The groups share the aspects that they considered as critical in the maze traversal, that is, the starting point and the speed of movement, assembling perceptual and motor features of their experience in the written. We see that the tool participates both off-line and on-line in the activity. The off-line way brings about the potential relations between shape and speed that are embodied in each graph, and speaks directly to the strategy, that is, to the changes that are necessary in order to produce better graphs. The on-line partaking has to do with the students' actual ways of moving, with each experiment with the technology that speaks about and to movement itself. Speed is open to mobility through the maze, thanks to new, real or imagined, traversal experiences.

The maze is a unique graph on the screen, which potentially requires a single movement to capture the best traversal. Passing from the experience to the best strategy put forward a new dimension for movement, captured by a search for agreement and coordination: a unique way of moving out of two movements, a new coordinated movement. The maze awakes the students to joint mobility: there is a whole preparation of the moving bodies to the shape of the maze; there is tension at stake in the coordinated movements, and a dynamic grasp of the relationships that sustain the maze. Therefore, the maze *is* also all the potential lines that *might* "stay inside", a "relevant" constraint in terms of the strategy. We can note that the definition of a general strategy is difficult for the students. But, as soon as the perceptual interpenetrate with the kinaesthetic, the effort of coordination appears in movement and in the bodily apprehension towards the graph, in posture, gaze and closeness. The activity is now reconfigured as a lived challenge with the maze and the continuous flow of time. It is as if the students collaborated to respond to the obligation of the task, finding ways of coordinating their bodies and movements to traverse the maze in an accurate manner. We interpret this coordinated effort as a way in which the tool affects the students and partakes in the learning assemblage.

Episode 2. "As if there were a c"

The second episode is about activity with the sum, $a+b$. While two students moved in front of the sensor, the class discovered that a third line could appear in real-time on the screen, produced by adding the values of a and b over time (back to Figure 1(c)). After this experience, the students took part in a classroom discussion, in which they argued about ways to obtain a specific line as sum. One week later, the students faced in groups a written task, which asked them to imagine describing how the sum works to a new classmate. We draw attention to the group formed by three students: Alessandro, Luisa and Massimiliano. Since Massimiliano was absent when the sum was introduced, he took on the role of the new classmate. We centre precisely on the initial moment of group work, with the three students seated all around a table, when Alessandro and Luisa explained the sum to Massimiliano:

Luisa So, there are two people, that their graph, that is, each of them performs a movement (*With the pen in her right hand mimes some bumps in the air in front, looking at Massimiliano*), which is on the graph (*Gazes and points with the pen to the graph area of WiiGraph*) and, that is, the graph (*Mimes again the bumps in the air with her right hand, the pen in the left hand*) is the sum of these movements of two people (*Looks at Alessandro, smiling*), and so

Alessandro It is as if there were, so, that is, it is as if, say, there were three people, that is, there are two people (*Turns and looks at the interaction space where the people should be*), who perform two movements (*Mimes the two people moving with his two open right hand fingers little moving back and forth in the air, gazing to the interaction space: 5a*), and it is as, that is. If they stay, one at 1 [feet] and one at 2 [feet] (*Looks back at the interaction space*), it is as if there really were a third person, who moves (*Turns towards the researcher, mimes a quick movement in front of him, with his right hand moving a little forward*

a b c

Figure 5. (a,b) Alessandro and (c) Luisa describing $a+b$.

	in front of his torso: 5b) at 3 [feet] (*Turns again towards the interaction space*). It is a sum, that is, the typical a plus b equal to c (*Looks at Massimiliano*)
Massimiliano	Ah, yeah (*Nods*)
Luisa	As if there were a c (*Looks at Massimiliano*)
Alessandro	Right
Luisa	That is, there the movement is that of c (*Repeats the previous bumping gesture in the air with her right hand: 5c*)

Analysis. The tool was not in use in the episode, and this is a peculiarity of the activity, which gave the students the specific task of imagining how to describe the sum. The students did not see graphs on any screen. Nevertheless, Alessandro and Luisa's ways of talking and moving above and around the table are unexpected in how they infuse (the experience of) the sum with a mobile character, disrupting and breaking up boundaries between lines and movements. The sum is reanimated in imagination and moves the two students to move in precise ways, merging perceptual and motor aspects of their experience.

In fact, Luisa first imagines the initial lines *as* movements of two people, who are thought of as moving "on the graph" simultaneously ("each of them"; her gesture and sight reveal particular attention to the screen although it is not in use). The sum is therefore clearly envisioned as "the sum of these movements of two people", even if Luisa seems to search for help from Alessandro with her gaze and smile. Alessandro's ways of talking and moving, prevalently above the table and insistently towards the interaction space (especially with gaze), introduce a new, dynamic element to the activity: a "third person". Alessandro imagines a new dimension of movement, with the third person "really" present and moving, like the two people already imagined, and actualizes all the movements through his bodily actions over the table. The imaginary situation ("as if" repeated many times) involves a potential group of "three people", who move together in the interaction space, coordinated according to the specific relationship for which the third mover's position depends over time on the mutual positions of the other movers. Thus, if one stays at distance 1 feet and the other stays at distance 2 feet, the third person has to be at distance 3 feet. The coordinated movement captures the obligation of the third person to the task. It is as if the other two people moved her ("If they stay"), exactly like the other two lines move the third line on the screen. In the real experience with the tool, the third graph does *not* capture any physical movement. However, adding the third person fills in the empty interaction space with new traversals of the experiential domain of the task, and infuses the graph of the sum with the qualities of a movement in front of the sensor, making it a function of position versus time, simultaneous to the other lines on the screen. It implicates a reconfiguration of the activity, with a potential set of three controllers, all imaginatively moved in the interaction space according to a well-coordinated bond.

The algebraic symbolism for the sum: "the typical *a* plus *b* equal to *c*" is implicated in all this movement and speaks directly to the bond. It emerges out of the imaginative activity, introducing the new variable *c* for the third graph/movement ("As if there were a *c*", "there the movement is that of *c*"). Therefore, the new variable is at the same time a new person/controller, her/its distance from the sensor, but also a new, flexible way in which the tool partakes in reconfiguring the activity. We interpret this as a *power* of the tool, a capacity to affect and being affected: the imaginative process takes an inventive line

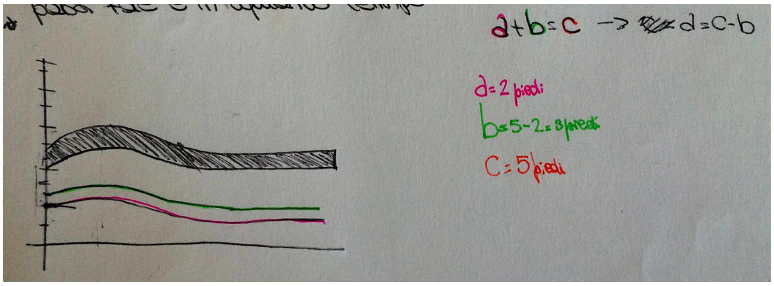

Figure 6. Written protocol for $a + b$.

of thought, moving the students in unscripted ways, which extend the use of the software beyond the real to including novelties, like the third person and the *c*, that express the sum as a mathematical function. The third person/movement is present because there is a third graph. It is not only the case that the technology allowed the students to encounter the sum of functions as a new function produced instant-by-instant by a standard numerical sum. Instead, the tool is reinvented through a reconfiguration of its peculiar functioning, which brings about a new, dynamic vision of the sum.

These aspects are also actualized in the written, when the students say: "The two people move in front of the sensor in the same way that they moved the other times but, on the graph, a third movement is represented, which is the sum of the first two". Attention is also drawn to the coordinated bond: "the first thing is that it's necessary to collaborate". This collaborative nature of movement captures the homogeneity of the sum (c) with the pink and blue lines, and the development of a symbolic understanding of the sum: beyond $a + b = c$, $a = c\text{-}b$ and $b = 5\text{-}2$ if $a = 2$ and $c = 5$ (see Figure 6).

Concluding discussion

In this paper, we have pursued a materialist vision of learning mathematics with technology, drawing on recent literature (Chorney, 2014; de Freitas & Sinclair, 2014; Nemirovsky et al., 2013; Sheets-Johnstone, 2009, 2011). This vision adopts a posthuman approach, that is, one that strives to challenge basic dichotomies between the human and the non-human and traditional assumptions to mathematical thinking and learning with tools, which direct attention only to the individual learner or the mind as the source of knowledge.

While decentralizing the human body from the activity, we want to rethink how tools are animated through activity and partake in the mathematical doing. In particular, our focus is on the material entanglements and arrangements of students and tools rather than on determinate borders between distinct bodies. We are interested in the question of how these boundaries are mobile and dynamic, and always shift reconfiguring the mathematical activity and the students' encounters with concepts. In particular, we are concerned with studying this question in regular classroom situations because of our aim to investigate tools as a resource for mathematics teaching and learning.

Substantially, therefore, we have drawn attention to the ways in which students assemble with the technology and how the tool is not inert but an active part of the learning assemblage, with potentiality and capacity to affect and be affected (see also de Freitas, Ferrara & Ferrari, 2017, 2019). In so doing, we trouble a vision of agency as a quality of one or the other entity, and rather tend to reconsider it as dynamically emerging out of the assemblage, in a word: distributed (Ferrara & Ferrari, 2017).

The episodes that we have presented offer extracts of classroom activities that involve a specific tool for graphing motion, which provides the potential for varied modalities of interaction with position-time graphs. The episodes are attempts to exemplify how the force and power of the tool ramify for the activity, shaping mathematics-as-movement, and to give evidence of the way that the tool engenders new kinds of mathematical experiences for the students.

With the first episode, we have seen how the tool is animated by means of bodily and imaginary movement in two different moments of an activity that engages the students in interactions with a target maze. The task first asks the students to challenge each other in the maze traversal. After the experience, Oliver recovers the temporality of his engagement with the tool and infuses the interpretation of the graphs with a mobile character, introducing a set of potential traversals through his gestures. The (imaginary) new pink line that brings about a new speed, a faster movement, also creates a new graphical configuration. The tool partakes in the coordinated movements of the line and the body and reconfigures the activity towards an understanding of time and the steepness-speed relationship as a central ingredient of a challenging maze traversal.

The following moment of the activity differently involves a written task, which requires the production of a collective strategy for the best maze traversal. As new traversals are imagined or produced in this context, speed is open to mobility and the maze moves the students to collaborate and join in new, coordinated efforts to create lines that remain inside. These coordinated movements are students' ways of responding to the tool, which capture its material force. They reconfigure the activity with the tool towards new bodily ways of navigating the spatio-temporal relationships in a given graph, which are central to understanding the mathematics of the experience.

The second episode is based on activity that involves the sum of two functions. We have discussed how the sum propels the students to move in particular ways, even when the tool is not in use. Movement is both bodily and imaginary and actualizes in gesture and talk a different conceptualization of the tool, which involves the third person as a new characteristic brought forth by the third line. Exactly like the third graph arises from the coordinated bond of two lines, the movement of the third person has to agree with two other movements. These imagined coordinated movements, of lines and people, are again ways of responding to the tool, but also unexpected and unscripted ways of affecting it, by reinventing it, which extend the use of the tool beyond its real functioning to including the third person, the new variable.

We have seen that, in the episodes, aspects of coordination and imagination push the mathematical activity further no matter whether the tool is in use or not. The activity is continuously reconfigured and shows how the tool is an animate part of the learning assemblage, through the peculiar ways in which the students' encounter with the mathematics is infused with coordinated movements and imaginative experiences that bring novelty or difference into the situation. Perceptual and motor ingredients of the activity with the tool constitute these movements and experiences as speculative material investments with the technology. This inextricable engagement with the tool turns our attention to the qualitative, dynamic dimension of mathematical activity, with the students experiencing new lines of flight (for example, the single movement out of two, or the third person attached to the sum), which provides insight into the complex nature of learning beyond accounts that restrict activity to cognition. The mathematics of movement emerges out of this mobility, with steepness of a graph, speed of movement, time and sum of functions being the concepts in motion together with learners and tool, in our classroom. They change along various dimensions, like the digital and the material. We see how the tool is not simply used and mastered by the students or subject to their intention. Rather, it actively partakes in the movement of things, people and concepts, through the way in which its capacity to affect and be affected ramifies for the activity. Briefly speaking, according to Chorney (2014), the student and the tool are on more equal footing.

In this paper, we moved away from discourses that attribute thinking and learning to the individual learner towards a discourse that intends to trouble the use-value of the tool in serving human will and to investigate how the tool animates and is reanimated in mathematics through the movement of becoming of the mathematical doing. The tool and the rich nature of the entanglement with it cannot be considered as a mere prosthesis in one such discourse, which wants to value and better understand the material dimensions of working with specific technologies and what they elicit for learners within the mathematics classroom.

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