The unwarped, resolved, deformed conifold: fivebranes and the baryonic branch of the Klebanov-Strassler theory

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Abstract

We study a gravity solution corresponding to fivebranes wrapped on the S^2 of the resolved conifold. By changing a parameter the solution continuously interpolates between the deformed conifold with flux and the resolved conifold with branes. Therefore, it displays a geometric transition, purely in the supergravity context. The solution is a simple example of torsional geometry and may be thought of as a non-Kähler analog of the conifold. By U-duality transformations we can add D3 brane charge and recover the solution in the form originally derived by Butti et al. This describes the baryonic branch of the Klebanov-Strassler theory. Far along the baryonic branch the field theory gives rise to a fuzzy two-sphere. This corresponds to the D5 branes wrapping the two-sphere of the resolved conifold in the gravity solution.

1 Introduction

The conifold [1] is a very simple non compact Calabi-Yau geometry which has given us important lessons about the behavior of string theory. In particular, understanding the transition between the resolved and the deformed conifold was very important [2]. The conifold geometry was also important for constructing the Klebanov-Strassler geometry [3], which is dual to a four dimensional $\mathcal{N} = 1$ supersymmetric field theory displaying confinement. This field theory arises when one wraps fivebranes and antifivebranes on the conifold and one takes the near brane limit[1]. This theory can spontaneously break a $U(1)_B$ baryonic symmetry. The geometries corresponding to arbitrary values of the corresponding VEVs for the baryonic operators were constructed by Butti et al. in [4] and further studied in [5, 6] and several other papers.

Here we further analyze the solution in [4]. We point out that the solution in [4] is related to a simpler solution which corresponds to fivebranes wrapping the S^2 of the resolved conifold and no extra D3 brane charge. The solution we discuss is not new, it is a limit of [4], and was also discussed in [7].

Another supergravity solution with $\mathcal{N} = 1$ supersymmetry is the Chamseddine-Volkov/Maldacena-Nuñez solution [8, 9, 10]. This is the near brane region for fivebranes wrapped on the two-sphere of the resolved conifold. Indeed, we will see that both the CV-MN geometry and the KS solution arise as limits of this more general solution.

This solution with fivebranes wrapping the S^2 of the resolved conifold displays very clearly the geometric transition described in [11]. In fact, the solution depends on a non-trivial parameter which can roughly be viewed as the size of the two-sphere that the brane is wrapping. When this size is very large, the solution looks like the resolved conifold with branes and when it is very small the solution becomes the deformed conifold with three form flux on S^3. The geometry is always smooth and has the topology and also the complex structure of the deformed conifold. However, the metric is not Ricci-flat.

In fact, if we consider only NS-5 branes, this solution is a very simple example of a non-Kähler, or torsional, manifold [12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. It is a solution with three form flux H_3 which preserves $\mathcal{N} = 1$ supersymmetry. Thus, the solution we consider could be viewed as the non-Kähler analog of the conifold. Namely, it is a simple and very symmetric geometry which could be describing smaller regions of a bigger six dimensional geometry. As was discussed in [11], the addition of flux or branes smoothly connects the deformed and the resolved conifolds. Here we are stressing, that this transition can be seen rather vividly purely in the gravity context. This is in contrast to the similar transition involving D6 branes on the S^3 of the deformed conifold [22] which requires some non-geometric insight.

We then argue that starting from this solution we can construct a solution which also has D3 brane charge. This is done via a U-duality symmetry of the classical gravity equations. This configuration corresponds to a cascading theory, as in [3], but where

[1] This is sometimes called the "near horizon" limit. However, since the solutions in this paper have no horizon, it is more appropriate to call this a "near brane" limit.

the cascade stops and merges into an ordinary conifold with constant four dimensional warp factor. A further scaling limit of this solution gives the usual Klebanov-Strassler solution [3] where the cascade goes on forever. The resulting geometry represents the baryonic branch of the quiver field theory [4, 5].

This perspective on the solution is useful for understanding how the geometry behaves for large values of the VEV of the baryonic operators. For large values of the baryonic operators the solution looks like fivebranes wrapping the S^2 of the resolved conifold. The fivebranes have some dissolved D3 branes. This leads to a non-commutative theory on the fivebrane worldvolume.

In fact, one can see this S^2 emerging already in a weakly coupled version of the quiver field theory. We show that the baryonic VEVs written in the field theory analysis of [5] can be viewed as a *fuzzy two-sphere*. The spectrum of field theory excitations match the Kaluza-Klein modes on the fuzzy two-sphere. The sphere becomes less and less fuzzy as we increase the VEVs of the baryonic operators. Thus, for large baryon VEVs we recover the picture of a fivebrane wrapping the S^2 of a resolved conifold in a way that matches rather precisely the gravity description.

The emergence of the fivebrane theory from the KS theory may be also understood from a dielectric effect, in a fashion rather analogous to [23]. It was demonstrated in [24] that the fivebrane theory on the fuzzy two-sphere emerges from the mass-deformed $\mathcal{N} = 1^*$ theory. Our derivation here from the baryonic branch of the KS theory displays some novel features. In particular, in our construction the fuzzy two-sphere emerges from bifundamentals in a quiver matrix theory. Recently a very closely related construction was discussed in [25, 26] for vacua of the mass-deformed ABJM theory [27].

We expect that a similar picture would hold for other field theories coming from D3 branes on singularities plus fractional branes. Far along the baryonic branch we expect to see resolved geometries with fractional branes.

2 Fivebranes on the two-sphere of the resolved conifold

In this section we discuss the gravity solution that is associated to fivebranes wrapping the S^2 of the resolved conifold. In other words, it is the solution that takes into account the backreaction of the branes on the geometry. It is a smooth solution with flux. The final topology of the solution is that of the deformed conifold, in the sense that there is an S^3 which is not shrinking, see figure 1. This solution is a limit of the one found in [4], and was also written in [7]. We will discuss some of its properties in some detail. We will also see that this solution can be used to construct the full solution in [4].

In order to write the solution we will use the NS language. Namely, we consider NS-5 branes wrapping the S^2 of the resolved conifold. The solution corresponding to D5 branes can be easily found by performing an S-duality. The NS solution can also be interpreted as a solution of type IIA supergravity or even the heterotic string theory (or type I supergravity)[2].

The solution is a simple example of a non-Kähler (or torsional) geometry involving H_3 flux, originally studied by Strominger [12] and Hull [13]. See also [14, 15, 18] for more recent discussions of these geometries. The solution contains the NS three form, the metric and the dilaton. These are non-trivial only in six of the ten dimensions. Other non-compact non-Kähler geometries were discussed in [19].

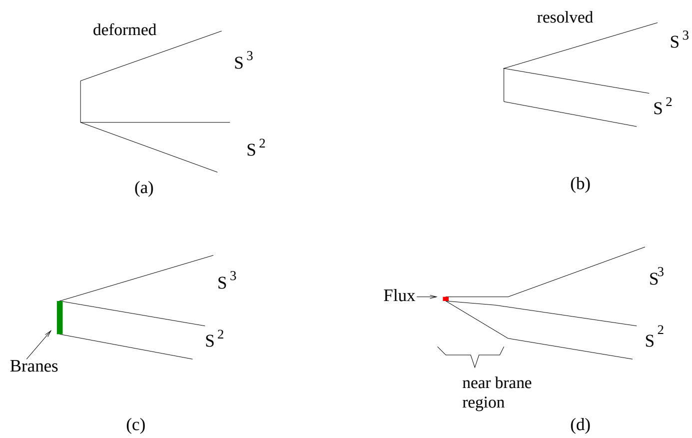

Figure 1: (a) A picture of the deformed conifold. The S^2 shrinks but the S^3 does not. (b) The resolved conifold, the S^2 does not shrink but the S^3 shrinks. In (c) we add five branes wrapping the S^2 of the resolved conifold of picture (b). (d) Backreacted geometry. The branes are replaced by geometry and fluxes. The end result is a geometry topologically similar to that of the deformed conifold with flux on the S^3. The near brane region is the CV-MN solution [8, 9, 10].

The solution is [4, 7]

$$
\begin{aligned}
ds^2_{str} &= dx^2_{3+1} + \frac{\alpha' M}{4} ds^2_6 && (2.1)\\
ds^2_6 &= c'(dt^2 + (\epsilon_3 + A_3)^2) + \frac{c}{\tanh t}(\epsilon_1^2 + \epsilon_2^2 + e_1^2 + e_2^2) + 2\frac{c}{\sinh t}(\epsilon_1 e_1 + \epsilon_2 e_2)\\
&\quad + \left(\frac{t}{\tanh t} - 1\right)(\epsilon_1^2 + \epsilon_2^2 - e_1^2 - e_2^2) && (2.2)\\
e^{2\phi} &= e^{2\phi_0}\frac{f^{1/2}c'}{\sinh^2 t}\ , \qquad H_3 = \frac{\alpha' M}{4} w_3\ , && (2.3)\\
w_3 &= (\epsilon_3 + A_3)\wedge\left[(\epsilon_1\wedge\epsilon_2 + e_1\wedge e_2) + \frac{t}{\sinh t}(\epsilon_1\wedge e_2 + e_1\wedge\epsilon_2)\right]\\
&\quad + \frac{(t\coth t - 1)}{\sinh t} dt\wedge(\epsilon_1\wedge e_1 + \epsilon_2\wedge e_2) && (2.4)
\end{aligned}
$$

[2] For the heterotic application we can embed the spin connection into the gauge group in order to cancel the $Tr[R\wedge R] - Tr[F\wedge F]$ term. Otherwise, it should also be taken into account and it would lead to a modification of the solution. This modification is small in the limit of a large number of branes.

where

$$\begin{aligned}
e_1 &= d\theta_1 \ , \qquad e_2 = -\sin\theta_1 d\phi_1 \ , \qquad\qquad A_3 = \cos\theta_1 d\phi_1 \ , \\
\epsilon_1 + i\epsilon_2 &= e^{-i\psi}(d\theta_2 + i\sin\theta_2 d\phi_2) \ , \qquad \epsilon_3 = d\psi + \cos\theta_2 d\phi_2 \ .
\end{aligned} \tag{2.5}$$

The $SU(2)$ left-invariant one-forms ϵ_i obey $d\epsilon_1 = -\epsilon_2 \wedge \epsilon_3$ and cyclic permutations. The functions $c(t)$ and $f(t)$ appearing in (2.1) obey the equations

$$f' = 4\sinh^2 t\, c \tag{2.6}$$

$$c' = \frac{1}{f}[c^2\sinh^2 t - (t\cosh t - \sinh t)^2] \tag{2.7}$$

where the primes denote derivatives with respect to t. The range of t is between zero and infinity. We will be interested in solutions to these equations with the following boundary conditions for small and large t

$$c = \gamma^2 t + \cdots \ , \qquad f = t^4\gamma^2 + \cdots \ , \qquad \text{for} \quad t \to 0 \tag{2.8}$$

$$c = \frac{1}{6}e^{\frac{2(t-t_\infty)}{3}} + \cdots \ , \qquad f = \frac{1}{16}e^{2t_\infty}e^{\frac{8(t-t_\infty)}{3}} + \cdots \ , \qquad \text{for} \quad t \to \infty \tag{2.9}$$

$$U \equiv 12 e^{\frac{2t_\infty}{3}} \tag{2.10}$$

where the dots indicate higher order terms. γ^2 and t_∞ are parameters describing a family of solutions. We have also related the parameter t_∞ to the parameter U introduced previously in the literature [5]. In general, we can only solve the equations numerically. The solutions have the property that both c and f are monotonically increasing and thus c' is positive. We plot some representative solutions in figure 2. In appendix A we comment further on the structure of these equations. The solutions

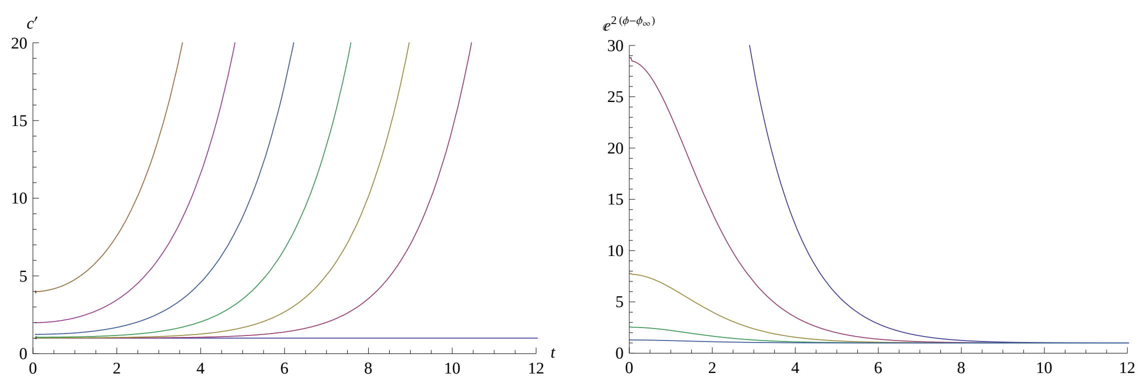

Figure 2: Plots of the solutions for some values of γ^2. On the left hand side: plots of c' for $\gamma^2 = 1.01, 1.02, 1.06, 1.25, 2, 4$. The bottom constant one is the CV-MN value, $\gamma^2 = 1$. On the right hand side: plot of $e^{2(\phi-\phi_\infty)}$ for $\gamma^2 = 1.01, 1.02, 1.06, 1.25, 2$. The CV-MN profile is not plotted since the dilaton does not asymptote to a constant.

interpolate between the conifold at $t \to \infty$ and the $t = 0$ region where there is an

S^3 which does not shrink and has flux M for the H_3 field, see figure 1(d). There is a one parameter family of solutions since the differential equations relate γ^2 to t_∞. The parameter γ^2 should be larger than one, and t_∞ can have any real value. The dilaton is a maximum at $t = 0$ and it decreases when we go large values of t, achieving a constant value ϕ_∞ asymptotically as $t \to \infty$.

This way of writing the metric and the equations makes manifest the $\mathbb{Z}_2$ symmetry corresponding to making a flop transition of the conifold (before we wrap the branes) and then wrapping an antibrane on the flopped two cycle. Explicitly, this symmetry is $c, f, M \to -c, -f, -M$. This is a symmetry of the equations, but not of the solutions.

The solution has an $SU(2) \times SU(2)$ global symmetry.

In addition, the full configuration has a second parameter which corresponds to an overall shift of the dilaton, denoted by ϕ_0. From the gravity point of view this is a rather trivial parameter. Finally, there is also an overall size parameter M. This appears also in H_3 where it gives the flux of three form on the S^3. It is thus quantized and the integer M can be viewed as number of fivebranes that we are wrapping.

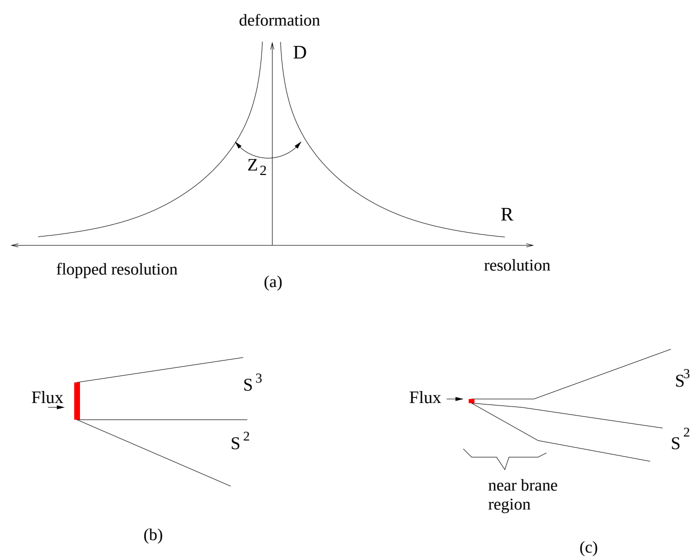

Figure 3: (a) The moduli space of the conifold with no flux or branes has two branches, denoted here by the vertical and horizontal axes. One is a deformation and the other is the resolution, which has two sides differing by flop transition. When we add flux we have a one parameter family that interpolates continuously between a deformed conifold with flux in region D and a resolved conifold with branes in region R. A $\mathbb{Z}_2$ symmetry relates this to another branch that joins the deformed conifold with the flopped resolved conifold. (b) The solution looks like the deformed conifold with flux in region D of (a). (c) In region R of (a) the solution looks like the resolved conifold with some branes, where the branes have been replaced by their near brane geometry. In all cases the topology (but not the geometry) is that of the deformed conifold.

Let us discuss in more detail the dependence of the solution on the nontrivial parameter γ^2 or t_∞. At the level of gravity solutions we can view this parameter as the size of the S^3 at the origin. Namely, the S^3 at the origin has radius squared equal to $r_{S^3}^2 = \alpha' M \gamma^2$. However, from the quantum gravity perspective, it is more convenient to define the parameter at large distances, where the geometry is more rigid, by stating how the metric deviates from the conifold metric. Naively one would think that the parameter should simply be the size of the S^2 of the resolved conifold that the brane is wrapping. More explicitly, the metric of the resolved conifold is [28]

$$
\begin{aligned}
ds^2_{RC} &= \frac{1}{\kappa(\rho)} d\rho^2 + \frac{\rho^2}{6}(e_1^2 + e_2^2) + \left(\alpha^2 + \frac{\rho^2}{6}\right)(\epsilon_1^2 + \epsilon_2^2) + \frac{\rho^2}{9}\kappa(\rho)(\epsilon_3 + A_3)^2 \\
&\qquad \text{where} \quad \kappa(\rho) = \frac{9\alpha^2 + \rho^2}{6\alpha^2 + \rho^2} \, . \qquad (2.11)
\end{aligned}
$$

This is the metric before adding any branes or fluxes. As $\rho \to 0$ the metric approaches a two-sphere S^2, of radius α, and four normal directions that are fibered over the two-sphere. Looking at the metric far away, at large ρ, we can view the resolution parameter α^2 as the difference between the coefficients of ϵ_1^2 and e_1^2, for example. This is a non-normalizable mode and we can view it as a parameter that we can control from infinity. This parameter eventually sets the size of the S^2 at $\rho = 0$, but is, in principle, defined at large ρ.

One would naively expect that the full geometry (2.1) asymptotes to the large ρ form of (2.11). On the other hand, returning to (2.1) and inserting (2.9), we see that the large t asymptotic form has an effective α^2 which is linear in t for large t, coming from the term of the form $(\frac{t}{\tanh t} - 1)$ in (2.1). Thus, we see that we cannot fix this parameter at infinity, it "runs" with the distance. This is analogous to the brane bending that appears in T-dual constructions, as we will later review. Nevertheless we can still define a parameter by selecting a trajectory via

$$
\frac{\alpha_{eff}^2}{\alpha'} = \frac{M}{2}(t-1) = \frac{M}{2}\left[3 \log \frac{\rho}{\sqrt{\alpha' M}} - 1 + t_\infty\right] \, . \qquad (2.12)
$$

Thus, we see that once we express α_{eff}^2 in terms of the physical size of the two- (or three-) sphere at infinity then t_∞ appears as an additive constant. Thus we view t_∞ as the parameter that we can control from infinity. In other words, we imagine sitting at a finite but large value of ρ and reading off the value of t_∞ via (2.12). Alternatively we could imagine that we are cutting off the geometry at a large value of ρ and embedding it into a compact space. When we Kaluza-Klein reduce to four dimensions, t_∞ will appear as a parameter for the effective field theory describing the small ρ region of the geometry. In addition, we can view $\log \rho/\sqrt{\alpha' M}$ as the bare parameter at some scale and the left hand side as the scale dependent coupling given by an RG running. At this stage this is not the running of the coupling in any decoupled, local, four dimensional field theory. It is a running in the four dimensional effective field theory that results from Kaluza-Klein reduction.

In the regime that t_∞ is very large and positive one can show that the solution has a region where it looks very close to the resolved conifold with some branes wrapping the S^2. As we get close to these branes the solution takes into account back reaction and the geometry in this "near brane " region is the Chamseddine-Volkov/Maldacena-Nuñez solution [8, 9, 10]. In this case γ^2 is very close to one. More explicitly, in the region $t \ll t_\infty$ the solution looks like the CV-MN solution. In the region $t \sim t_\infty$ the solution looks like the resolved conifold, with some branes wrapping the sphere. For larger values of t it looks like the resolved conifold with the "running" α^2_{eff}. The metric is very close to the metric of the resolved conifold for a large range of distances when $t_\infty \gg 1$. We discuss this in more detail in appendix A.

On the other hand, when t_∞ is large but negative, the solution looks like the deformed conifold with a very large S^3, see figure 3(b). The size of the S^3 is determined by the IR parameter γ^2, which is becoming very large as $t_\infty \to -\infty$. For this solution the total change in the dilaton is not very large, it is of order $1/\gamma^2$, see appendix A. In fact, for very large γ^2 it is possible to find an approximate solution of (2.7),(2.6) by ignoring the second term in (2.7), see appendix A. This gives the standard solution for the deformed conifold, which in our notation is

$$\begin{aligned} ds^2_{DC} = \frac{M\alpha'}{4}\gamma^2 \Big[& \frac{\sinh^2 t}{K(t)^2}(dt^2 + (\epsilon_3 + A_3)^2) + \frac{K(t)}{\tanh t}(\epsilon_1^2 + \epsilon_2^2 + e_1^2 + e_2^2) \\ & + 2\frac{K(t)}{\sinh t}(\epsilon_1 e_1 + \epsilon_2 e_2) \Big] \\ \text{where} \qquad K(t) = & \frac{3^{1/3}}{4^{1/3}}(\sinh 2t - 2t)^{1/3} \ . \end{aligned} \tag{2.13}$$

Thus, this solution displays very explicitly the geometric transition discussed in [11, 29]. It is a simple example where the transition happens within the supergravity description. In fact, for large positive t_∞ we can view the solutions as branes wrapping the S^2 and for large negative t_∞ we have a deformed conifold with flux, see figure 3. When the flux is zero, the deformed and resolved conifold represent distinct branches. With non-zero flux these branches are smoothly connected, see figure 3 (a). It is interesting that one can see this transition purely in supergravity. Of course, we knew from [3] that this transition happened. However, here we see it directly in a simpler setting with only one kind of branes and fluxes. This system should be contrasted to the closely related problem of D6 branes wrapping the S^3 of the deformed conifold [22, 30]. In that case, the transition could not be seen purely in supergravity.

By comparing the mixed term in the large t expansion of the metric (2.2) to the same term in the deformed conifold metric (2.13) we can define an "effective deformation" parameter

$$\varepsilon^2_{eff} = \frac{1}{3^{1/3}} e^{-2t_\infty/3} \ . \tag{2.14}$$

We see that the effective resolution and deformation parameters are not independent. In particular, for very large positive t_∞ we get a large amount of resolution, and a small deformation. We depicted this schematically in Figure 3(a). When t_∞ is very large negative, we get a large deformation parameter. However, we cannot use the definition in (2.12) to measure the effective resolution. In fact, at infinity, the effective resolution always becomes big, since it runs. Another way to say this, is that in this geometry the *complex* and (would-be) *Kähler* structures are not independent.

Note that taking t_∞ very negative does not correspond to wrapping anti branes on the flopped version of the resolved conifold. If we had no flux and we had made α^2 negative in (2.11) this would have been the case. However, in our case, even if we take t_∞ very negative we see that α^2_{eff} is still growing towards a positive value at infinity. The solution where we wrap anti-branes on the flopped version is a separate configuration related by the $\mathbb{Z}_2$ operation described above. In some sense, it is connected to the unflopped version by going to $t_\infty \to -\infty$ (or $\gamma^2 \to \infty$) and back on the other branch of figure (3) (a).

When we consider strings propagating on this geometry we have (0,2) worldsheet supersymmetry. It would be nice to see if there is a gauged linear sigma model which describes this background, in the same way that there is one for the conifold with no flux [31].

An interesting object that exists in this geometry is a domain wall that comes from wrapping an NS five-brane on the S^3. The three other directions are an $\mathbb{R}^{1,2}$ subspace of $\mathbb{R}^{1,3}$. Its tension is given by

$$T_2 = \frac{1}{(2\pi)^5 \alpha'^3 e^{2\phi(0)}} V_{S^3} = \frac{M^{3/2}}{(2\pi)^3 \alpha'^{3/2} e^{2\phi_\infty}} \frac{1}{18} e^{-t_\infty} \tag{2.15}$$

where we used that $e^{2\phi(0)} = 9\gamma^3 e^{t_\infty} e^{2\phi_\infty}$ [3]. Note that even though we are evaluating the tension of a brane located at $t = 0$, the final answer can be expressed in terms of the quantities defined at infinity. This is related to the fact that this tension is BPS and that we can compute the superpotential for this configuration in terms of the parameters at infinity. As we will see later, this is intimately related to the fact that the domain walls tension in the Klebanov-Strassler solution [3] are independent of the VEV of the baryons [5, 32].

The geometry underlying the solution can be neatly characterized in terms of a two-form J and a complex three-form Ω defining the $SU(3)$-structure, which are constructed from spinor bilinears[4]. By construction these satisfy the algebraic constraints $\Omega \wedge \bar{\Omega} = -\frac{4i}{3} J^3$ and $J \wedge \Omega = 0$. The conditions imposed by supersymmetry on these forms were derived in [12, 13], and can be written concisely as calibration conditions [15]

$$\begin{aligned} d(e^{-2\phi}\Omega) &= 0 && (2.16)\\ e^{2\phi} d(e^{-2\phi} J) &= - *_6 H_3 && (2.17)\\ d(e^{-2\phi} J \wedge J) &= 0\,. && (2.18) \end{aligned}$$

In fact, imposing these equations on a suitable ansatz [33] is one way to derive the BPS equations (2.6), (2.7). The condition (2.16) implies that the manifold is complex. In

[3] Do not confuse ϕ_0, which is just an additive constant, with $\phi(0)$ which is the full value of ϕ at $t = 0$.

[4] Explicitly, we have $\Omega_{ijk} = \epsilon^T \Gamma_{ijk} \epsilon$ and $J_{ij} = \epsilon^\dagger \Gamma_{ij} \epsilon$, where ϵ is the internal spinor solving the supersymmetry equations in string frame.

particular we can define a rescaled three-form $\Omega_{\rm hol} = e^{-2\phi}\Omega$ which is then a holomorphic (3,0)-form. The fact that the two-form J is not closed implies that the manifold is not Kähler. For the solution we are considering we have the following explicit expressions

$$\begin{aligned} J = \frac{\alpha' M}{4}\Big[& (\coth t(t\coth t - 1) - c)\, e_1 \wedge e_2 + (\coth t(t\coth t - 1) + c)\, \epsilon_1 \wedge \epsilon_2 \\ & + \frac{1}{\sinh t}\,(t\coth t - 1)\,(\epsilon_1 \wedge e_2 + e_1 \wedge \epsilon_2) - c' dt \wedge (\epsilon_3 + A_3)\Big] \end{aligned} \tag{2.19}$$

$$\begin{aligned} \Omega_{\rm hol} = \frac{\mathrm{e}^{-2\phi_0}(\alpha' M)^{3/2}}{8}\Big[& \sinh t(e_1 \wedge \epsilon_1 + e_2 \wedge \epsilon_2) - i\cosh t(\epsilon_1 \wedge e_2 + e_1 \wedge \epsilon_2) \\ & - i(e_1 \wedge e_2 + \epsilon_1 \wedge \epsilon_2)\Big] \wedge \big(dt + i(\epsilon_3 + A_3)\big) \end{aligned} \tag{2.20}$$

The holomorphic $(3,0)$-form $\Omega_{\rm hol}$ is identical to that of the deformed conifold [34] (see eq. (2.79) of this reference), implying that the solution here has the same complex structure as the latter. This agrees with the arguments in [11] which said that the addition of RR fluxes would not change the topological string, which depends on the complex structure, since the solution we are discussing is S-dual to a solution with only RR fluxes.

2.1 The superpotential

We can also discuss the superpotential for this solution. This is a generalization of the Gukov-Vafa-Witten (GVW) superpotential [35], and can be extracted, for example, from the general expression in [36, 37], see also [38]. The superpotential is

$$W \;=\; \int_{M_6} \Omega_{\rm hol} \wedge (H_3 + i dJ)\ . \tag{2.21}$$

We see that extremising this superpotential will relate the complex structure $\Omega_{\rm hol}$ to the would-be Kähler structure J, complexified by the B-field. We already remarked that this is indeed the case for our solution. On the other hand, recall that extremizing the ordinary GVW superpotential fixes the complex structure of the Calabi-Yau, in terms of the integer fluxes [39].

Of course, computing the superpotential does not require knowing the full solution. In fact, it is possible to compute it "off shell" by introducing a resolution parameter S and then extremize it to find the on shell value, as explained in [11]. Here we will simply see how one recovers the final on shell value from the classical solution. This is a simple check of the formulas for superpotentials in the literature and an example that shows how the different quantities entering in the definition of the superpotential look like in an explicit example. The uninterest reader can jump to the next subsection.

The superpotential (2.21) can be evaluated explicitly by a computation as in [29]. Namely, we can use the formula

$$\int_{M_6} \Omega_{\rm hol} \wedge (H_3 + i dJ) \;=\; \int_\Gamma \Omega_{\rm hol} \cdot \int_{S^3} (H_3 + i dJ) - \int_{S^3} \Omega_{\rm hol} \cdot \int_\Gamma (H_3 + i dJ) \tag{2.22}$$

where S^3 is the compact three-cycle and $\Gamma \simeq \mathbb{R}^3$ is the dual non-compact three-cycle. Notice that the three-forms being integrated are indeed closed. We now evaluate the terms in (2.22). First, let us define two representative three-cycles as

$$\begin{aligned} S^3 &= \{t = 0, \theta_1 = \text{constant}, \phi_1 = \text{constant}\} \\ \Gamma &= \{\theta_1 = \theta_2, \phi_1 = -\phi_2, \psi = \psi(t)\} \quad \text{with} \quad \psi(0) = 0 \ , \quad \psi(\infty) = \psi_\infty \end{aligned} \tag{2.23}$$

where ψ_∞ is a constant reference ψ. This is an additional parameter of the solution, which is related to the θ angle of the associated Yang-Mills theory via $\theta \sim M\psi_\infty$. The reason it is a parameter is because the integrals will depend on ψ_∞. We then have that

$$\Omega|_{S^3} = e^{2\phi}\Omega_{\text{hol}}|_{S^3} \ = \ e^{2(\phi(0)-\phi_0)}\frac{(M\alpha')^{3/2}}{8}\epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3 = dv_3 \tag{2.24}$$

where dv_3 is the volume element of the cycle in string frame. This shows that the three-sphere S^3 is calibrated, i.e. it is a supersymmetric cycle. The other fluxes are

$$\int_{S^3}(H_3 + idJ) = -4\pi^2\alpha' M \ , \quad \int_\Gamma (H_3 + idJ) = \int_{S_c^2}(B + iJ) \equiv b + ij \tag{2.25}$$

where S_c^2 is a two-sphere at some cut-off distance t_c. The periods of the holomorphic three form are

$$\int_{S^3} \Omega_{\text{hol}} \ = \ \frac{(\alpha' M)^{3/2}}{8}\frac{16\pi^2}{9e^{2\phi_\infty}}e^{-t_\infty - i\psi_\infty} \ , \tag{2.26}$$

$$\int_\Gamma \Omega_{\text{hol}} \ = \ \frac{(\alpha' M)^{3/2}}{8}\frac{4\pi i}{9e^{2\phi_\infty}}e^{-t_\infty - i\psi_\infty}\left[2(t_c + i\psi_\infty) + e^{-t_c - i\psi_\infty} - e^{t_c + i\psi_\infty}\right] \tag{2.27}$$

where we have multiplied the expression for Ω_{hol} in (2.20) by $e^{-i\psi_\infty}$ to make sure that it depends holomorphically on the parameter $t_\infty + i\psi_\infty$. We are free to define the phase of the three form. The period on the non-compact cycle contains a divergent term, however after changing variables as

$$e^{t - t_\infty} = \frac{\rho^3}{(\alpha' M)^{3/2}} \tag{2.28}$$

we see this term does not depend on the parameter t_∞, thus it can be dropped, as in [29]. As we discussed around (2.12) we can define a bare scale parameter as $\hat{\Lambda} = \rho_c/\sqrt{\alpha' M}$, and then, to compare directly with with [29], let us also define $S = e^{-t_\infty - i\psi_\infty}$ and a "complex coupling" $2\pi\alpha'\tilde{\alpha} = -(j + ib)$. Then we have

$$W = (\alpha' M)^{3/2}\frac{4\pi^3 i\alpha'}{9e^{2\phi_\infty}}\left[-MS\log\frac{\hat{\Lambda}^3}{S} - S\tilde{\alpha}\right] \ . \tag{2.29}$$

The prefactor can be absorbed in the definition of S. This has the expected form of the Veneziano-Yankielowicz superpotential, if we regard $\tilde{\alpha}$ as the running coupling as in [29]. Indeed, in the field theory limit we discuss below the real part of $\tilde{\alpha}$ may be interpreted as the 4d gauge coupling of the $\mathcal{N}=1$ SYM theory, and S is identified with the glueball superfield [11]. Here we can evaluate $\tilde{\alpha}$ in the solution at hand, getting

$$\tilde{\alpha} = M(t_c - 1 + i\psi_\infty) = M(t_\infty + \log \hat{\Lambda}^3 - 1 + i\psi_\infty) \tag{2.30}$$

where we have included a contribution from a flat B-field at infinity. Now, inserting this into (2.29), the logarithmically divergent terms correctly cancel and we get

$$W = (\alpha' M)^{3/2} \frac{4\pi^3 i}{9 e^{2\phi_\infty}} M\alpha' e^{-t_\infty - i\psi_\infty} \ . \tag{2.31}$$

This agrees[5] with the domain wall tension (2.15), noting that for large M, $T_2 \sim |\Delta W| \sim W/M$.

The solution depends on the parameter $\Phi = M(t_\infty + i\psi_\infty)$ which has the identification $\Phi \sim \Phi + 2\pi i$. The fact that it is not invariant under the naive shift symmetry of ψ is related to the $U(1)_R$ breaking as discussed in more detail in [10]. The pattern of breaking of this $U(1)_R$ is the simplest way to derive (2.31) [40].

Now let us discuss a particular limit of this configuration which is supposed to lead to a decoupled $\mathcal{N}=1$ pure Yang-Mills theory. When t_∞ is very large and positive, it can be interpreted as the size of the S^2 that the NS five branes are wrapping, see appendix A. In that case the full solution looks as in figure 3(c). Notice that if we reduce an NS fivebrane on an S^2 of radius α, which is very large, then the four dimensional gauge coupling[6] is given by

$$\frac{8\pi^2}{g_4^2} = 2\frac{\alpha^2}{\alpha'} = M t_\infty \tag{2.32}$$

where we have evaluated the radius of the two-sphere at the value that it has in the region where the solution looks similar to that of the resolved conifold. This is possible only if t_∞ is very large. This is the value of the coupling at the Kaluza-Klein scale set by the radius of the sphere. Note that t_∞ gives the 't Hooft coupling and it parameterizes the gravity solution, as expected. This coupling has to be very small, thus requiring that we take $t_\infty \to \infty$. In addition, we would like to decouple the fundamental strings. A fundamental string stretched along one of the non-compact four dimensional directions is a BPS state in this geometry. It can sit at any value of the radial coordinate. We want these strings to be much heavier than the branes discussed in (2.15). This can be achieved if we keep ϕ_∞ fixed as we take $t_\infty \to \infty$. At the origin we find that $\phi(0) \to \infty$ in this case. Thus, as expected, we should S-dualize to the D5 brane picture in order to analyze the limit. The limit we are taking is such that the S-dual coupling is becoming extremely small and that the same time the size of the S^2 is also becoming small in the new string units, but with (2.32) still large. This decouples the D1 branes which are S-dual to the BPS fundamental strings mentioned

[5]We have to divide by α'^4 in order to restore the correct units in the superpotential.

[6] In this limit, the above superpotential is the one arising from gluino condensation $W \propto \Lambda^3$. In particular, one gets the correct coefficient for the beta function, see also [41].

above. Of course, in this regime the gravity solution fails and would probably have to use non-critical strings to describe the large M limit of the theory. Nevertheless, the superpotential computed in terms of t_∞ continues to be valid since it is independent of ϕ_∞. So we can first take ϕ_∞ small enough so that $\phi(0)$ is not so large and we can trust the gravity description. Then we take ϕ_∞ to a larger value so that the field theory decouples. Now this discussion seems to be explicitly contradicted by the fact that the tension (2.15) depends on ϕ_∞. However, this dependence is simply a choice of units. In fact, it could also be viewed as arising from the Kähler potential in a situation where we compactify the theory and go down to four dimensions. Since those terms in the Kähler potential are determined in the bulk region of the six dimensional space, they are not corrected by the physics in the tip.

Note that the system of M branes on S^2 naively has an overall $U(1)$ gauge symmetry. This mode becomes non-normalizable in the solution. If we were to start from our configuration with 5 branes on S^2 then we can add the flux of this $U(1)$ gauge field along the four dimensional space by performing a U-duality. In this case we see that the asymptotic form of the metric changes in a non-normalizable way. Namely, a $U(1)$ flux on the branes induces lower brane charges which contribute to the logarithmic running of the resolution parameter.

2.2 Relation to brane constructions

The conifold is T-dual to two orthogonal NS branes. In other words, we have an NS brane along 012345 and an NS' brane along 012367. Strictly speaking we should introduce a compact direction along which to do the T-duality, see [42] for further discussion. The compact direction can be the direction 8. Thus we can have these branes separated along the direction 9. We have a cylinder formed by directions 8 and 9. This configuration has one more parameter, relative to the conifold, which is the radius in the 8 direction. In the limit that the radius, R_8, in the 8 direction goes to zero we expect to recover the conifold after a T-duality[7]. Thus the brane picture contains yet one more parameter. We have the string coupling, the radius of the 8th direction and the separation between the branes. We then consider a sort of near brane limit of the fivebranes where we take $r \to 0$ and $g_{IIA} \to 0$ with g_{IIA}/r fixed. At the same time we take R_8 to zero so that $g_{IIA}\frac{l_s}{R_8} = g_{IIB}$ is kept fixed. Here g_{IIB} is the value of the IIB coupling.

When we add D4 branes stretching between the NS branes we find that the NS branes bend and there is a logarithmic running of the separation between the branes, see figure 4. See also [43].

Another possibility is the M-theory construction in [44, 45]. That is obtained in the limit that $R_8 \to \infty$ and $g_{IIA} \to \infty$. Thus, adding the size of the extra circle R_8 as a parameter allows us to interpolate between the various pictures that have been proposed for describing $\mathcal{N} = 1$ SYM.

[7] It would be nice to derive the geometry that is T-dual to the system of fivebranes for a finite value of R_8.

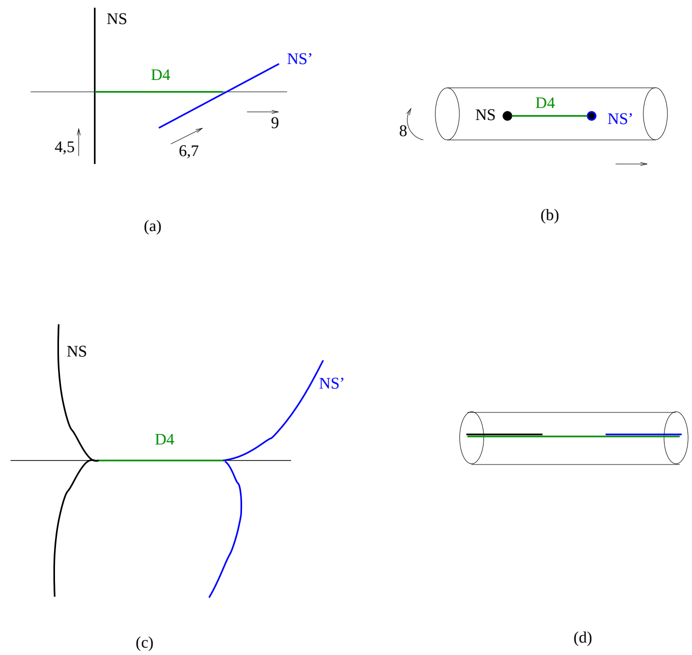

Figure 4: (a) A D4 brane stretched between two orthogonal NS fivebranes. In (b) we compactify a direction orthogonal to all the branes in (a). In the limit that the size of the 8th circle goes to zero we expect to recover the conifold. In (c) and (d) we have schematically represented the effects of brane bending. The transverse position of branes varies logarithmically. This has the same origin as the dependence of the parameter α_{eff}^2 on the radial position.

In particular, the superpotential of the theory, as a holomorphic function of the non-trivial parameter in all these pictures is expected to be the same because we do not expect any dependence of the superpotential on the parameters that we are varying. This is due to the fact that the partners of the parameters that we are varying are axions and we do not have any finite action BPS instantons which could contribute to the superpotential.

Let us discuss the various moduli in the type IIB picture, from the point of view of the geometric side. This would be a geometry similar to the conifold, except that it asymptotes to $S^1 \times \mathbb{R}^5$ at infinity, since the size of the 8th direction is finite. We are considering the background in which we wrap D5 branes, S-dual to the original solution. We can consider how various fields are paired under the supersymmetry preserved by the D5 brane wrapped on the compact two cycle. The string coupling, g_s, is paired by supersymmetry to the RR axion, a^{RR}, dual to C_2^{RR} in four dimensions, i.e. $*_4 da^{RR} = F_3^{RR}$. The corresponding instanton is a D5 fivebrane wrapped over all the six internal dimensions. The radius of the 8 direction R_8 is paired to a RR C_2 field along the internal directions. The corresponding instantons are non-compact euclidean D1 branes extended along the internal directions and wrapped along the eighth dimension at infinity. The ten dimensional RR axion is paired with the four dimensional NS axion a^{NS} which is dual to H_3^{NS} with all four dimensional indices. The corresponding instantons are NS5 branes along the internal dimensions. Finally, we have B^{NS} on the compact two cycle which is paired with C_4 on a non-compact internal four-cycle. The corresponding euclidean D3 brane instantons also have infinite action. Thus, holomorphy, plus the absence of finite action instantons, imply that the superpotential is only a function of $t_\infty + i\psi_\infty$. Thus, we can vary the other variables from the values which decouple the four dimensional theory to other values where the M theory brane picture is a good approximation.

In the case in figure 4 (b) we have a BPS string corresponding to a D2 brane wrapping the 8th direction, see figure 6(a). This T-dualizes to a D1 brane on the original picture, which is a state that we have to decouple to get to the $\mathcal{N} = 1$ pure Yang-Mills theory.

3 Solutions with D3 branes from a duality transformation

In this section we recover the solution in [4] by applying a simple chain of dualities to the solution discussed in section 2. This introduces various fluxes. In fact, the procedure that we discuss is quite general, and it can be applied to any solution with only dilaton and NS three-form turned on. In principle, the starting solution may also be non-supersymmetric. However, if it preserves supersymmetry and is therefore of the type discussed in [12], then the duality maps it to a supersymmetric solution of type IIB with non-trivial NS and RR fluxes, where the internal six-dimensional geometry is of $SU(3)$-structure type[8]. The internal geometry is not (conformally) Calabi-Yau, and the three-form fluxes are not imaginary self-dual. However, for the solution that we discuss here, we will see that the latter may be recovered by taking a certain limit. In particular, in this limit we recover the Klebanov-Strassler warped deformed conifold geometry [3].

Let us now describe the dualities. First of all, we perform an S-duality on the initial solution, which then represents D5 branes wrapped on the S^2 of the resolved conifold. The solution has non-trivial dilaton and a RR three-form flux. We then compactify on a torus three spatial world-volume coordinates of the D5 branes and perform T-dualities along these directions, obtaining a type IIA configuration of D2 branes wrapped on the S^2. This is then uplifted to M-theory, where we do a boost[9]

$$t \to \quad \cosh\beta\, t - \sinh\beta\, x_{11}\ , \qquad x_{11} \to -\sinh\beta\, t + \cosh\beta\, x_{11}\ , \tag{3.1}$$

obtaining a configuration with M2, and Kaluza-Klein momentum charges. Finally, we reduce back to type IIA and repeat the three T-dualities on the torus. The resulting

[8] The duality may be easily adapted to other supersymmetric geometries of the type $\mathbb{R}^{1,d} \times M_{9-d}$ with dilaton and NS three-form [15], producing corresponding supersymmetric solutions with non-trivial RR fluxes, both in type IIA and type IIB.

[9] The coordinate t here is the time coordinate, and should not be confused with the variable t elsewhere in the paper.

type IIB solution has D5-brane plus D3-brane charges[10]. The steps involved in the transformation are summarized by the following diagram

$$D5 \ \to\ D2 \ \to\ M2 \ \to\ M2, p_{KK} \ \to D2, D0 \ \to\ D5, D3 \tag{3.2}$$

The final result is the following solution

$$\begin{aligned}
\hat{\phi} = \phi_{here} &= -\phi_{\rm previous} = -\phi && (3.3)\\
d\tilde{s}^2_{str} &= \frac{1}{h^{1/2}} dx^2_{3+1} + \frac{e^{\hat{\phi}_\infty}\tilde{M}\alpha'}{4}\frac{h^{1/2}}{\cosh\beta} e^{-2(\phi-\phi_\infty)} ds^2_6 && (3.4)\\
h &= 1 + \cosh^2\beta (e^{2(\phi-\phi_\infty)} - 1) && (3.5)\\
F_3 &= \frac{\alpha'\tilde{M}}{4} w_3 \qquad H_3 \ = \ -\tanh\beta \frac{e^{\hat{\phi}_\infty}\tilde{M}\alpha'}{4} e^{-2(\phi-\phi_\infty)} *_6 w_3 && (3.6)\\
F_5 &= -\tanh\beta e^{-\hat{\phi}_\infty}(1 + *_{10}){\rm vol}_4 \wedge dh^{-1} && (3.7)
\end{aligned}$$

The six dimensional metric ds^2_6 and the three form w_3 are the same as in (2.1). Notice that the dilaton here is minus the dilaton in (2.1). We denote by $\hat{\phi}$ the dilaton for this solution in this frame and we continue to denote by ϕ the expression for the dilaton in (2.1). Note that all the terms involving $\phi - \phi_\infty$ do not depend on the constant ϕ_0 in (2.1). So we should think of $\hat{\phi}_\infty$ as a new parameter determining the asymptotic value of the coupling. Here F_3 denotes the RR three-form and H_3 the transformed NS three-form. The parameter $\tilde{M}$ is the quantized RR flux through a three-sphere at infinity, representing the number of D5 branes that we are wrapping. This is related to the parameter M, giving the number of NS fivebranes in the original solution as

$$\tilde{M} \ = \ \frac{1}{4\pi^2\alpha'}\int_{S^3_\infty} F_3 \ = \ e^{\hat{\phi}_\infty} M \cosh\beta \in \mathbb{N} \ . \tag{3.8}$$

Thus in particular the original gravity parameter M is not quantized in the transformed solution[11].

We see that the changes of the solution with respect to the one in section 2 are rather simple. However, crucially non-trivial RR fluxes are generated. Note that the three-form fluxes satisfy the relation

$$\cosh\beta H_3 + \sinh\beta *_6 e^{-2(\phi-\phi_\infty)} e^{\hat{\phi}_\infty} F_3 \ = \ 0 \tag{3.9}$$

which is a generalization of the imaginary-self-dual condition for supersymmetric fluxes on a Calabi-Yau geometry [39]. The boost parameter β can be thought of as an interpolating parameter [47]. When this goes to zero, the solution reduces back to the initial one. On the other hand, as we discuss below, in a certain limit of infinite

[10] Of course, the D3 Page charge is still zero. See [46] for further discussion of various definitions of charge in this background.

[11] This is a usual feature of supergravity duality transformations which are a symmetry of the gravity equations but not a symmetry of the full string theory.

boost, one can recover the warped Calabi-Yau solution, with imaginary self dual fluxes. Notice that the NS H_3 is manifestly closed. This follows from the fact that in the initial NS5 solution, this obeys the calibration condition (2.17). Then we can read off the B field in terms of the two-form J, namely

$$B = \sinh\beta e^{-2\phi} J \; . \tag{3.10}$$

The warp factor $h^{-1/2}$ is an increasing function of t which goes to one at infinity. From the small t expansions of the functions given in section 2, we see that the warp factor $h^{-1/2}$ becomes constant at $t = 0$, thus the IR geometry is essentially the same for all values of the parameter β. In particular, there is a finite-size S^3. D5-branes wrapping this S^3 corresponds to a domain wall in the four-dimensional world-volume. It can be shown that this is a BPS object, because the S^3 is calibrated by the generalized special Lagrangian calibration, namely the three-form Ω discussed in section 2. The tension of this domain wall is in fact equal to (2.15). However, after expressing it in terms of the physical $\tilde{M}$, we have

$$T_{DW} = \frac{\tilde{M}^{3/2} e^{\hat{\phi}_\infty/2}}{(2\pi)^3 \alpha'^{3/2}} \frac{1}{18\left(e^{\frac{2}{3}t_\infty}\cosh\beta\right)^{3/2}} \; . \tag{3.11}$$

Notice that this depends on the parameters β and t_∞, but only through a particular combination. We will further comment on this dependence below.

The solution (3.3) is contained in [4]. To obtain the baryonic branch solution they have set an integration constant to a particular value[12]. Here we have restored it to its more general value. In this more general solution, the warp factor goes to constant at infinity. Thus we are coupling the field theory to the string theory modes of the ordinary conifold. In particular, we are also gauging the baryonic $U(1)_B$. Then, a combination of the parameters t_∞ and β may be thought of as the value of the FI parameter for this $U(1)$, while another combination corresponds to the domain wall tension (3.11). The baryonic branch interpretation can be recovered in a limit in which we send β to infinity. We will discuss momentarily the relationship of the domain wall tension above and that computed in [5, 34, 32].

Again, we can discuss the generalized GVW superpotential for this solution. For the type of geometry we have the general expressions in [37, 36] reduce to the simple form[13]

$$W = \int_{M_6} \Omega_{\rm hol} \wedge (F_3 + i e^{-\hat{\phi}} \cos w H_3 + i d(\sin w J)) \; , \tag{3.12}$$

where we expressed it in terms of the same forms $\Omega_{\rm hol}$ and J of the previous section. The function w is the same appearing in [4]. This may be thought of as part of the data defining the $SU(3)$-structure, in addition to J and Ω. In particular, it is a degree

[12] We have that the constant η in [4] is $\eta = -\tanh\beta$.

[13] The geometry we have is of the simpler $SU(3)$ structure type, as opposed to the more general $SU(3) \times SU(3)$ structure type. Here we are setting the type IIB RR axion to zero.

of freedom in the spinor ansatz that solves the type IIB supersymmetry equations[14]. We have that $\cos w = -\tanh\beta e^{\hat\phi-\hat\phi_\infty}$, thus it is suggestive to re-express the boost parameter in terms of an angular parameter, defining $\sin\delta = -\tanh\beta$, so that

$$\cos w = e^{\hat\phi-\hat\phi_\infty}\sin\delta\ , \qquad \sin w = e^{\hat\phi-\hat\phi_\infty}h^{1/2}\cos\delta \tag{3.13}$$

and the superpotential becomes

$$W \;=\; \int_{M_6}\Omega_{\rm hol}\wedge\Big[F_3 + ie^{-\hat\phi_\infty}\Big(\sin\delta H_3 + \cos\delta d(e^{\hat\phi}h^{1/2}J)\Big)\Big]\ . \tag{3.14}$$

We see that this form interpolates between the original GVW superpotential when $\cos\delta = 0$ and the S-dual version of the one discussed in the previous section, when $\sin\delta = 0$. Although the limit $\sin\delta\to 0$ is straightforward, the infinite boost limit $\cos\delta\to 0$ should be done more carefully, but it does reproduce the correct GVW expression in the KS solution. We could also write the superpotential (3.14) using the formula (2.22) as before. Then the discussion is essentially unchanged, provided we replace $b+ij\to c_2+i(e^{-\hat\phi_\infty}\sin\delta b+\cos\delta j)$ and write the periods of the holomorphic three-form in terms of $\tilde M$. In the end the superpotential depends only on the parameter

$$L = \frac{U}{\cos\delta} = 12e^{2/3t_\infty}\cosh\beta \tag{3.15}$$

and we recover the domain wall tension (3.11). Notice that the effective "coupling constant" is then

$$\frac{8\pi^2}{g_4} = \frac{1}{2\pi\alpha'}(\cos\delta\ j + \sin\delta\ \frac{b}{g_s})\ . \tag{3.16}$$

This interpolates between the definition we discussed in section 2 for $\delta = 0$, and the definition of the coupling g_-^2 used in [48, 3] for $\delta = \pi/2$.

In fact it is rather natural to change the angle δ keeping L fixed. In the limit that $\cos\delta\to 0$ we obtain a finite limit which is simply the Klebanov-Strassler solution but with a $+1$ in the warp factor, so that the warp factor asymptotes to one at infinity. The reason we get a smooth solution in this limit is due to the fact that in this limit the function $e^{2\phi-2\phi_\infty}-1\sim U^2\times$(finite), as shown in appendix A.

3.1 Recovering the Klebanov-Strassler asymptotics

The solutions we discussed so far are such that the warp factor goes to one at infinity, which is a reasonable thing to consider.

However, one can consider a near brane limit where the warp factor grows without bound as we go to large t. This gives the solution [4] with Klebanov-Strassler [3] (or

[14]In the formalism of generalized geometry adopted in [37], e^{iw} is the zero-form part of the pure spinor Ψ_1.

Klebanov-Tseytlin [49]) asymptotics. This can be obtained from the metric above by taking $\beta \to \infty$. To obtain a finite limit we also rescale the worldvolume coordinates

$$x \to e^{\frac{\hat{\phi}_\infty}{2}} \sqrt{\tilde{M}\alpha'}\sqrt{U}\sqrt{\cosh\beta}\Lambda_0 x \tag{3.17}$$

where we have included additional factors. The factor of U will make sure that the asymptotic form of the metric is independent of U. The factor Λ_0 simply introduces a scale, which is the scale of the last step of the cascade. The other factors are just for convenience.

This then gives the solution

$$\begin{aligned}
\hat{\phi} &= -\phi && (3.18)\\
d\tilde{s}^2_{str} &= e^{\hat{\phi}_\infty}\tilde{M}\alpha'\left[\frac{1}{\hat{h}^{1/2}}U\Lambda_0^2 dx^2_{3+1} + \frac{1}{4}\hat{h}^{1/2}e^{-2(\phi-\phi_\infty)}ds_6^2\right] && (3.19)\\
\hat{h} &= e^{2(\phi-\phi_\infty)} - 1 && (3.20)\\
F_3 &= \frac{\alpha'\tilde{M}}{4}w_3 \qquad H_3 = -\frac{\alpha'\tilde{M}e^{\hat{\phi}_\infty}}{4}e^{-2(\phi-\phi_\infty)} *_6 w_3 && (3.21)\\
F_5 &= -\Lambda_0^4(\tilde{M}\alpha')^2U^2e^{\hat{\phi}_\infty}(1+*_{10})\mathrm{vol}_4\wedge d\hat{h}^{-1} && (3.22)
\end{aligned}$$

where ϕ, w_3 and ds_6^2 are the same as in (2.1). This way to write the metric shows clearly the dependence on $\tilde{M}$ and $\hat{\phi}_\infty$. They appear as simple overall factors. This implies that any gravity computation gives an answer which scales like $\tilde{M}^4e^{2\hat{\phi}_\infty} = \tilde{M}^2(e^{\hat{\phi}_\infty}\tilde{M})^2$, since this is the overall factor in the action. The fact that the metric has a U independent asymptotics comes from the fact that $\hat{h} \propto U^2te^{\frac{-4t}{3}}$, and $c \propto U^{-1}e^{\frac{2t}{3}}$ for large t, see (A.11) in appendix A. This implies that the dependence on U cancels for large t. Of course the full solution depends on U. In fact, this rescaling introduces a factor of U in the term corresponding to the running deformation parameter

$$\left(\frac{t}{\tanh t} - 1\right) \to U\left(\frac{t}{\tanh t} - 1\right) \ . \tag{3.23}$$

This deformation, which was not normalizable when the warp factor was constant asymptotically, is now normalizable. In fact U is parametrizing the VEV of the scalar operator that is an $\mathcal{N}=1$ partner of the baryonic current [5]. The rescaling (3.17) also has the effect of making the domain wall tension independent of U

$$T_{DW} \propto \tilde{M}^3e^{2\hat{\phi}_\infty}\Lambda_0^3 \sim \Lambda^3 \ . \tag{3.24}$$

This result was obtained numerically in [5] and then analytically in [32, 34]. It is simply the statement that the superpotential is constant along the baryonic branch. Here Λ^3 is the usual holomorphic Λ which is introduced for this theory [5].

The metric is closely related to the one for the simpler case with only fivebrane charge. In particular, the fact that the large U region is related to branes wrapped on the resolved conifold continues to be valid, but with some modifications. The boosting that we have done induces a large, but finite amount of D3 brane charge on the fivebranes which is proportional to t_∞ as we will show below. (Of course, the fluxes lead to a diverging D3 brane charge at infinity.) Thus, the theory on the fivebrane becomes non-commutative [50]. Nevertheless, for large values of U, this description in terms of D5 branes wrapped a resolved conifold becomes better and better.

In the next section we show that this picture also emerges from the field theory analysis in [5], the baryonic VEVs give rise to a fuzzy sphere which is building up the fivebrane wrapping the S^2. We will discuss there a more detailed comparison with the gravity description. The geometry in the large U region is divided into two parts, one region looks like the resolved conifold, namely the solution of Pando-Zayas and Tseytlin [28]. Near the origin of the resolved conifold, one has a region that looks like the near horizon geometry of fivebranes. More details are given in appendix A.

3.2 Parameters of the solutions

It is interesting to discuss a bit more explicitly the parameters of the various solutions. For the solutions of section 2, before we perform the boost, we had one discrete parameter M labeling the number of branes and two continuous parameters ϕ_∞ and t_∞. The solution depended non-trivially only on t_∞.

For the boosted solutions we now continue to have a discrete parameter $\tilde{M}$ which is the net number of fractional branes. We have a simple parameter ϕ_∞ and a non-trivial parameter t_∞ plus a parameter β. These are the parameters in the case that the warp factors asymptotes to a constant. All of these parameters are non-normalizable. Notice however that the superpotential and domain wall tension depend only on the combination L, which is paired by supersymmetry with the phase coming from the RR C_2.

If we further take the limit that leads to Klebanov-Strassler asymptotics, we loose the parameter β. Furthermore, in that case, the parameter U is normalizable and is interpreted as the baryonic branch VEV. It is interesting to understand how the parameter t_∞ becomes non-normalizable once we change the asymptotics of the solution. The KS solution also has a $U(1)_B$ global symmetry which is spontaneously broken. When we change the asymptotics from Klebanov-Strassler to a constant warp factor at infinity, we are gauging this $U(1)$ symmetry and adding an FI term for this $U(1)$. Thus, when we set the D-term to zero we relate U to the FI term for this $U(1)$ symmetry. We still have a parameter that we can change, which is the FI term, which in turn changes U, but it is now a non-normalizable parameter.

Notice also that in the solution with KS asymptotics, the superpotential and the tension of the domain wall do not depend on U. Indeed, with these asymptotics the susy partner of U is the zero-mode of the RR potential C_4 [52], which manifestly does not appear in the superpotential.

3.3 Brane picture

We can also understand the boosting procedure in the type IIA brane picture discussed in subsection 2.2. Recall that the solution with only D5 brane charge corresponds to the following IIA picture: an NS fivebrane along directions 012345 and one NS' fivebrane along 012367, with M D4-branes stretching on a segment along the direction 9. The remaining direction, 8, is a compact direction. We can now imagine moving the NS' brane along the 8th direction, keeping fixed the NS brane. Before taking into account backreaction, we have the picture in figure (5)(b). Here we naively have two parameters, the separation in the 9th direction and the separation in the 8th direction. However, once we take into account the effects of brane bending, the parameters really become a choice of RG trajectory, which can be viewed as the parameter L and an angle δ. We can think of L as a kind of renormalized distance along the direction where the branes are separated. In this picture we see that $U = L\cos\delta$ is simply the projection of the parameter L on to the 9th direction, after we identify L as in (3.15). Once we have a non-zero angle, the D4 branes wrap the 8th direction an infinite amount of times, due to the brane bending. This translates into the ever growing D3 charge we have in the IIB geometry. Note that, even in the case that the warp factor goes to a constant, we have this ever growing D3 brane charge.

Supersymmetry continues to pair the renormalized parameter L with the flux of the C_2^{RR} field on the S^2 in the IIB picture.

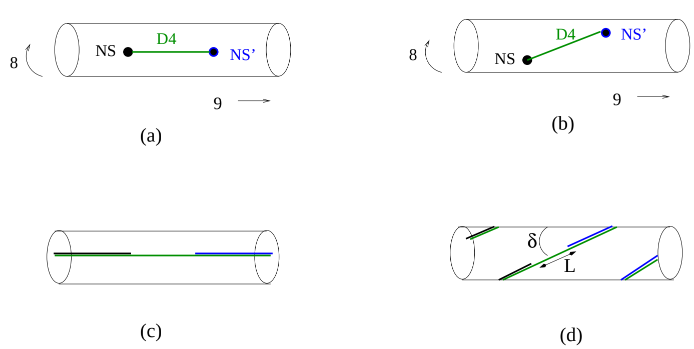

Figure 5: (a) IIA configuration with a D4 brane stretching between two orthogonal NS fivebranes. (b) We separate the fivebranes in the compact 8th dimension. In (c) we add the effects of brane bending, the NS fivebranes bend in the non-compact 9th direction. (d) When we include the effects of brane bending the branes now stretch along a slanted direction parametrized by an angle δ.

The solution with KS asymptotics, on the other hand, corresponds to rotating by 90 degrees, keeping the distance U fixed. Notice that the superpotential depends only on the distance between the NS branes, and not on the amount of rotation. The configuration of two NS branes displaced along the compact direction is known to corresponds to the KS solution [3, 42]. The boosted solution corresponds to a general rotated configuration of NS-NS'-D4. In fact, the existence of such solution was anticipated in [51].

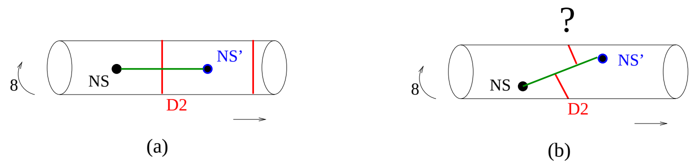

Figure 6: (a) BPS state which corresponds to a D2 brane wrapping the circle. (b) This brane picture would mislead us to expect also a BPS state for a D2 wrapping the circle. However, there is no such BPS state in the full theory.

Finally, let us comment on the fate of the BPS four dimensional string that we had for zero angle. Naively, one would expect that for a finite angle one should continue to have this BPS state. Indeed, the brane picture suggests a very natural candidate as shown in figure (6)(b). This would correspond to a string closely related to the D1 brane in these geometries (3.3). However, the analysis in [52] showed that the D1 brane is not BPS, and no other BPS strings were found[15]. Naively, from the field theory point of view, one would expect to find such BPS strings in the case that we gauge $U(1)_B$, which is what is happening when we have an asymptotically constant warp factor. In fact, the classical theory contains such strings [55]. However, due to the quantum deformation of the moduli space these strings cease to be BPS in the quantum theory, see section 4.2 of [55]. There strings are D1 branes in (3.3), which are not BPS if $\delta \neq 0$.

4 The baryonic branch and the fuzzy sphere

We have seen that the large U asymptotic form of the Butti et al solution [4], with Klebanov-Strassler boundary conditions, can be represented accurately in terms of M fivebranes wrapping the resolved conifold with a large amount of dissolved D3 brane charge. This gives a good picture for the asymptotic form of the solution far along the baryonic branch. In this section we drop the tilde in $\tilde{M}$ and denote it simply by M.

In this section we will see that a field theory analysis of the baryonic branch also leads to this picture. The field theory analysis of the various branches of the KS theory was done in detail by Dymarksy, Klebanov and Seiberg in [5]. In particular, these authors found the vacuum expectation values of the fields along the baryonic branch. In this section we argue that these VEVs represent a fuzzy two-sphere. This two-sphere is building up the D5 brane. In fact, a closely related discussion was also developed in [26] for vacua in the ABJM theory [27]. This is not a coincidence since the ABJM theory is closely related to the Klebanov-Witten [48] field theory.

We start by writing explicitly the classical solutions [5] for the baryonic branch in the weakly coupled version of the quiver field theory. We consider the quiver field theory with gauge group $SU(Mk) \times SU(M(k+1))$. For the time being k is fixed and

[15] Note, however, that in the case that $\tilde{M} = 0$, so that we study the Klebanov-Witten theory [48] in its baryonic branch [53], then such BPS strings do exist [54].

we will discuss later the effects of the cascade. See [56] for further discussion on the weak coupling version of the cascading theory. We have bifundamental fields A_i and $B_a^\dagger$ and their complex conjugates, which are anti-bifundamentals. The classical baryonic branch has two regions one with $B_a = 0$ and one with $A_i = 0$. Here we concentrate on the first and set $B_a = 0$ and $A_i = C\,\Phi_i \otimes \mathbf{1}_M$ where C is an arbitrary complex constant and Φ_i are the following two $k \times (k+1)$ matrices

$$\Phi_1 = \begin{pmatrix} \sqrt{k} & 0 & 0 & \cdot & 0 & 0 \\ 0 & \sqrt{k-1} & 0 & \cdot & 0 & 0 \\ 0 & 0 & \sqrt{k-2} & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & \cdot & 1 & 0 \end{pmatrix} \qquad \Phi_2 = \begin{pmatrix} 0 & 1 & 0 & \cdot & 0 & 0 \\ 0 & 0 & \sqrt{2} & \cdot & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & 0 & \cdot & \sqrt{k} \end{pmatrix} \tag{4.1}$$

We can view this as a solution in the $SU(k) \times SU(k+1)$ quiver theory and we recover the solution of the $SU(kM) \times SU((k+1)M)$ theory by multiplying each entry by the $M \times M$ identity matrix, $\mathbf{1}_M$. Thus we see that we can set $M = 1$ for the time being and we will restore the M dependence at the end.

Setting $B_a = 0$ the D-term equations of the theory are the following

$$\begin{aligned} A_1 A_1^\dagger + A_2 A_2^\dagger &= (k+1)|C|^2 \mathbf{1}_k \ , \\ A_1^\dagger A_1 + A_2^\dagger A_2 &= k|C|^2 \mathbf{1}_{k+1} \ . \end{aligned} \tag{4.2}$$

The constant C is a modulus of the solution since the D term constraints set to zero only the $SU(N)$ part of (4.2). The fact that the constants in the two lines of (4.2) are related follows from taking the trace on both sides. We have defined C in such a way that the expectation value of the scalar operator $\mathcal{U}$, the $\mathcal{N} = 1$ partner of the baryonic current, is[16]

$$\mathcal{U} = \frac{1}{Mk(k+1)} \mathrm{Tr}[A_i^\dagger A_i - B_i^\dagger B_i] \sim |C|^2 \ . \tag{4.3}$$

If we *were* to gauge the baryon current, then C would be the FI term we would need to have the above VEVs. However, we are *not* gauging the baryon number current. The factors of k were introduced in (4.3) because we want to normalize the baryon number current so that the bifundamental A_i has charge $\frac{1}{Mk(k+1)}$ after k steps in the cascade [57]. More explicily, the baryon operators have the schematic form $\mathcal{A} \sim (A_i)^{k(k+1)M}$, with the precise index contractions given in [58]. We want these to have baryon charge one. However, since we do not know the Kähler potential, we do not know if (4.3) will remain as the proper expression for the operator as we make the coupling stronger. In general $\mathcal{U}$ is defined as the partner of the baryon current. For the moment we will just do the computation in a weakly coupled theory for fixed k.

[16] Our normalization of $\mathcal{U}$ differs from the one in [5] by a factor M. In our normalization the baryon operator, $\mathcal{A}$ has charge one under the baryonic current.

We now define the following $k \times k$ matrices

$$\begin{aligned} L_1 &= \frac{1}{2}(\Phi_1\Phi_2^\dagger + \Phi_2\Phi_1^\dagger) \\ L_2 &= \frac{i}{2}(\Phi_1\Phi_2^\dagger - \Phi_2\Phi_1^\dagger) \\ L_3 &= \frac{1}{2}(\Phi_1\Phi_1^\dagger - \Phi_2\Phi_2^\dagger) \\ &\text{and} \quad \Phi_1\Phi_1^\dagger + \Phi_2\Phi_2^\dagger = (k+1)\mathbf{1}_k \end{aligned} \tag{4.4}$$

We find that the hermitian matrices L_i obey the $SU(2)$ commutations relations. In addition we find that the Casimir operator is

$$L_1L_1^\dagger + L_2L_2^\dagger + L_3L_3^\dagger = \frac{1}{4}(k^2-1)\mathbf{1}_k \tag{4.5}$$

Thus we have the spin $j = \frac{k-1}{2}$, or k dimensional, irreducible representation of $SU(2)$. We can do the same for the matrices multiplied in the other order. We define

$$\begin{aligned} R_1 &= \frac{1}{2}(\Phi_1^\dagger\Phi_2 + \Phi_2^\dagger\Phi_1) \\ R_2 &= \frac{i}{2}(\Phi_2^\dagger\Phi_1 - \Phi_1^\dagger\Phi_2) \\ R_3 &= \frac{1}{2}(\Phi_1^\dagger\Phi_1 - \Phi_2^\dagger\Phi_2) \\ &\text{and} \quad \Phi_1^\dagger\Phi_1 + \Phi_2^\dagger\Phi_2 = k\mathbf{1}_{k+1} \end{aligned} \tag{4.6}$$

In this case the Casimir is

$$R_1R_1^\dagger + R_2R_2^\dagger + R_3R_3^\dagger = \frac{1}{4}k(k+2)\mathbf{1}_{k+1} \tag{4.7}$$

thus it is a spin $j = \frac{k}{2}$, or $(k+1)$-dimensional, irreducible representation of $SU(2)$.

The commutation relations of these matrices with the Φ_i show that the Φ_i transform in the fundamental representation of the sum of the two $SU(2)$ groups. This is important for arguing that the $SU(2)$ global symmetry that acts on the i index of A_i is unbroken, once we combine it with appropriate gauge transformations. The appropriate gauge transformations are generated by $SU(2)$ matrices living in each of the gauge groups. These matrices are the matrices L_i and R_i discussed above. This unbroken $SU(2)$ symmetry is important for classifying fluctuations around the background.

In fact, the matrices (4.1) really define a fuzzy supersphere, [59, 60] or [61, 62] for reviews. In addition to the "even" generators L_i and R_i we also have "odd" generators given by the Φ_i. We did not find the supersphere perspective useful for what we do here, but it is a curious fact.

4.1 The spectrum of quadratic fluctuations

In order to demonstrate the emergence of a two-sphere we want to show that the quadratic fluctuations around this vacuum behave as Kaluza-Klein modes on a two-sphere.

We start considering the four dimensional gauge fields. Most of them are higgsed along the baryonic branch. We want to show that the spectrum of massive gauge fields agrees with what one expects from a Kaluza-Klein compactification of a six dimensional gauge theory on a two-sphere. In other words, when we set a non-zero VEV for the fields A_i we are higgsing the $SU(k) \times SU(k+1)$ gauge fields. We denote the four dimensional gauge fields as a^L_μ and a^R_μ. They are in the adjoint of $SU(k)$ and $SU(k+1)$ respectively. The Higgs fields mix a^L and a^R. In order to avoid unnecessary notational clutter, we sometimes drop the four dimensional index μ. Alternatively we concentrate just on one particular component of the four dimensional gauge field. The masses for the gauge bosons come from expanding the kinetic term for the Higgs fields A_i

$$\sum_i \mathrm{Tr}[D_\mu A_i^\dagger D^\mu A_i] \ , \qquad D_\mu A_i = \partial_\mu A_i + i(a^L_\mu A_i - A_i a^R_\mu) \ . \tag{4.8}$$

We get the structure $\mathrm{Tr}[(a^L\Phi_i - \Phi_i a^R)^\dagger (a^L\Phi_i - \Phi_i a^R)]$. Expanding this out we get

$$\mathrm{Tr}\left[(k+1)(a^L)^2 + k(a^R)^2 - 2\Phi_1^\dagger a^L \Phi_1 a^R - 2\Phi_2^\dagger a^L \Phi_2 a^R\right] \ . \tag{4.9}$$

We will now expand a^L and a^R in fuzzy spherical harmonics,

$$a^L \sim \sum_{l=0}^{k-1} c_l (L)^l \qquad a^R \sim \sum_{l=0}^{k} c_l (R)^l \tag{4.10}$$

where the $(L)^l = L_{(i_1} \cdots L_{i_l)}$−traces. These are simply products of the matrices introduced in (4.4). They transform in the spin l representation of the unbroken $SU(2)$. The coefficients c_l have the corresponding indices, which are the same as the indices of ordinary spherical harmonics. $SU(2)$ symmetry allows us to decouple different values of l, but since we have a^L and a^R we end up with a two by two matrix. In order to compute this matrix, we need to define an operator S via

$$S(M) \ = \ \Phi_1 M \Phi_1^\dagger + \Phi_2 M \Phi_2^\dagger \ . \tag{4.11}$$

We think of S as an operator which sends $(k+1)\times(k+1)$ matrices into $k \times k$ matrices and it commutes with the unbroken $SU(2)$. It is easy to check that

$$S(R_+^l) \quad = \quad (k+1+l) L_+^l \tag{4.12}$$

where $R_+ = \Phi_1^\dagger \Phi_2$ and $L_+ = \Phi_2 \Phi_1^\dagger$ are the raising generators (4.4), (4.6). Since the action of S respects $SU(2)$, it means that it acts in this way on any of the elements of

R^l transforming according to the lth spherical harmonic. We can similarly define an operator $\tilde{S}$ as

$$\tilde{S}(M) \;=\; \Phi_1^\dagger M \Phi_1 + \Phi_2^\dagger M \Phi_2 \ . \tag{4.13}$$

For this we have

$$\tilde{S}(L_+^l) \;=\; (k-l) R_+^l \ . \tag{4.14}$$

Using (4.14), (4.12) and (4.9) we get the following two by two matrix for each value of l

$$\lambda \begin{pmatrix} a_l^L \\ a_l^R \end{pmatrix} = \begin{pmatrix} k+1 & -(k+1+l) \\ -(k-l) & k \end{pmatrix} \begin{pmatrix} a_l^L \\ a_l^R \end{pmatrix} \tag{4.15}$$

with eigenvalues

$$\lambda_{l,\pm} \;=\; k + \frac{1}{2} \pm \sqrt{\left(k+\frac{1}{2}\right)^2 - l(l+1)} \ . \tag{4.16}$$

This formula is valid for $l = 0, 1, \cdots k-1$. For $l = k$ we have that $a^L = 0$ and the eigenvalue is simply k, which is the same as λ_-. (4.16) gives a zero eigenvalue for $l = 0$. This gives the unbroken gauge symmetry. In the case that we have an $SU(k) \times SU(k+1)$ group there is no unbroken gauge symmetry and this mode is not present. On the other hand, if we have $SU(kM) \times SU((k+1)M)$ then this mode gives the unbroken $SU(M)$ gauge symmetry. This corresponds to the $SU(M)$ gauge symmetry on M fivebranes. Let us expand the eigenvalues for large k and fixed l. We get

$$\begin{aligned} \lambda_{l,-} &= \frac{l(l+1)}{2k+1} + \dots \ , && l \ll k \ , \\ \lambda_{l,+} &= 2k+1 - \frac{l(l+1)}{2k+1} + \dots \ , && l \ll k \ . \end{aligned} \tag{4.17}$$

The lower values, $\lambda_{l,-}$, agree with our interpretation in terms of Kaluza-Klein modes on the two-sphere. The other ones, $\lambda_{l,+}$ are large, as long as k is large, and we will not offer any interpretation for them. We view them as a UV effect associated to the precise fashion in which this fuzzy sphere is approximating the ordinary sphere. It would be nice to see if they have a simple physical interpretation.

So far we have shown that the four dimensional components of the gauge field correctly reproduce what we would expect from Kaluza-Klein reducing a six dimensional theory on a two-sphere. One could do a similar analysis expanding the fields A_i around their vacuum expectation values in terms of the fuzzy sphere spherical harmonics. These get a mass from the D-terms potential. However, it is not necessary to do this explicitly, since $\mathcal{N} = 1$ supersymmetry implies that there should be a scalar partner

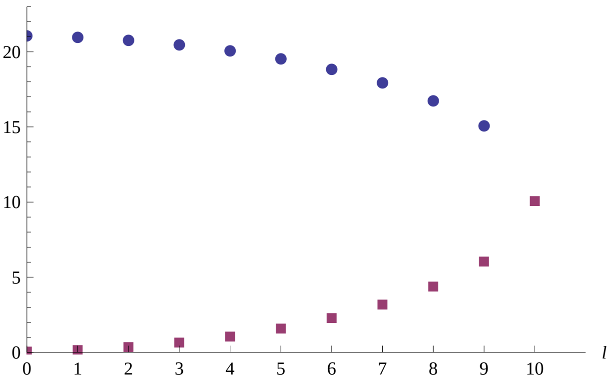

Figure 7: The eigenvalues λ_+ (upper) and λ_- (lower) as functions of l, for $k = 10$.

with the same masses as the ones from (4.17). In fact, this is enough to account for all the modes that come from A_i. The counting is the following. We have $2 \times 2 \times k(k+1)$ real components for the two complex fields A_i. On the other hand, the number of massive vector fields is $k^2 - 1 + (k+1)^2 - 1 = 2k(k+1) - 1$. Thus, we see that all components of A_i are involved in the $\mathcal{N} = 1$ Higgsing of the gauge bosons except for a single complex field, which is simply the field C.

In addition we can consider the Kaluza-Klein modes of the B_a fields. Here we can compute their masses directly from the superpotential

$$W = h\mathrm{Tr}[\epsilon^{ab}\epsilon^{ij}A_i B_a A_j B_b] \ . \tag{4.18}$$

Again we now expand B_a into spinor spherical harmonics. The reason we get spinors is that we get an odd number of fields Φ_i or $\Phi_i^\dagger$. So we have $B_a = \sum_{l=0}^{k-1} b_a^l X^l$ were $X^l \sim \Phi^i,\ \Phi^{(i_1\dagger}\Phi^{i_2}\Phi^{i_3)\dagger}, \cdots$, for $l = 0, 1, \cdots$.

The eigenvalues of the superpotential are then $\pm(l+1)$ and the spin under the unbroken $SU(2)$ is $j = l + \frac{1}{2}$. This agrees with the spectrum of the Dirac operator on the fuzzy sphere [60]. The structure of the superpotential is then

$$W = h\epsilon_{ab} \sum_{l=0}^{k-1} \langle b_a^l b_b^l \rangle \ , \tag{4.19}$$

where the angle brackets denote the antisymmetric $SU(2)$ invariant inner product of two SU(2) representations with half integer spin. The mass eigenvalues come from the F-term potential $\sum_{a=1}^{2} |\partial W/\, \partial B_a|^2$, thus in the end we have that $\lambda_{l,B} = |C|^4 h^2 (l+1)^2$.

4.2 Fuzzy sphere parameters

We found three types of fields, each organized into $SU(2)$ representations. Two towers of vectors from the gauge fields a^L, a^R, this tower also contains the $\mathcal{N} = 1$ scalar partners of the massive gauge fields coming from part of the fluctuations of A_i. Finally

fields	$SU(2)$ spin	# superfields	eigenvalues
a^L_μ , a^R_μ + scalar(δA_i)	$j = l$	1	$\lambda_{l,-}, \lambda_{l,+}$
B_a	$j = l + \frac{1}{2}$	2	$\lambda_{l,B}$

we have one tower from the fields B_a. The spectrum is summarized in the following table

For $l \ll k$ the eigenvalues are

$$\lambda_{l,-} \sim \frac{g^2|C|^2}{2k+1} l(l+1) \ , \quad \lambda_{l,+} \sim g^2|C|^2(2k+1) \ , \quad \lambda_{l,B} = |C|^4 h^2 (l+1)^2 \ . \quad (4.20)$$

The dependence on the gauge coupling comes from the fact that we normalize the YM term without a coupling constant by rescaling the gauge fields by g. We have set the two g's equal when we derived the spectrum. Of course, the couplings would run in opposite directions, and we would get a slightly more complicated expression. The correct formula for $\lambda_{l,-}$, for small values of l, is as in (4.20) but with $\frac{1}{g^2} \to \frac{1}{g_+^2} \equiv \frac{1}{g_L^2} + \frac{1}{g_R^2}$.

Neglecting the highly massive states with $\lambda_{l,+} \sim k$, we see that the spectrum is very simlar to the spectrum of the fivebrane theory compactified on an S^2, as computed in [24] (see pages 21-22 in [24]). The only difference is that in [24] the modes with half integer spin had masses which were set by the same overall scale as the modes with integer spin. Here the ratio of their masses involves $h^2|C|^2$ which is an arbitrary parameter. From the gravity dual that we discussed in the previous sections we would have expected these modes to have the same mass, as in [24]. We should not be surprised by this mismatch, the field theory computation we did here was for a weakly coupled theory. At strong coupling we expect that the coefficient of the superpotential should be determined by the other parameters. Further discussion can be found in [56].

From the expression of the masses of the four dimensional gauge fields we can read off the radius of the fuzzy sphere as well as the non-commutativity parameter

$$\frac{1}{|C|^2 R^2_{\rm Fuzzy}} = \frac{1}{\langle \mathcal{U} \rangle R^2_{\rm Fuzzy}} \propto \frac{g_+^2}{k} \ , \qquad \theta_{\rm Fuzzy} \propto \frac{1}{k} \qquad (4.21)$$

were $|C|^2 R^2_{\rm Fuzzy}$ is the radius of the sphere in units of $|C|^{-2}$, or the VEV of the operator $\mathcal{U}$. This is setting the scale of the overall mass of the gauge bosons and it is the natural scale to use.

So far, we are not finding any relation between $|C|$ and k. And indeed there is no relation in the weakly coupled field theory. However, once we include the effects of the cascade we expect k and $|C|$ to be related. For a given $|C|$, or a given VEV of the gauge invariant operator $\mathcal{U}$ (4.3), we should find the value of k corresponding to the appropriate region of the cascade. As we increase $|C|$ we see that k should increase. We know that the running of the difference between the couplings goes like $8\pi^2(\frac{1}{g_L^2} - \frac{1}{g_R^2}) \sim 3M\ell + {\rm constant}$, where $\ell = \log({\rm scale})$ is the RG time [3]. The amount of "time" or $\Delta\ell$ for each step in the cascade can be calculated by setting $g_L = \infty$ and then see how much we should run until $g_R \to \infty$. This gives $\Delta\ell_{1-\text{step}} = 8\pi^2/(3Mg_+^2)$. The natural scale is here set by the value of the VEVs which is in turn given by $|C|$. Thus the amount of RG time from the IR scale Λ to the scale $|C|$ is given by $\Delta\ell \sim \log\frac{|C|}{\Lambda}$. Then the number of steps in the cascade goes as

$$k \sim \frac{\Delta\ell}{\Delta\ell_{1-\text{step}}} \sim \frac{3Mg_+^2}{8\pi^2}\log\frac{|C|}{\Lambda} = \frac{3Mg_+^2}{16\pi^2}\log\frac{\langle\mathcal{U}\rangle}{\Lambda^2}\ , \qquad \frac{1}{g_+^2} = \frac{1}{g_L^2} + \frac{1}{g_R^2} \tag{4.22}$$

where Λ is the scale of the last step of the cascade in the IR. k is telling us how many steps away we are from the last step of the cascade. Of course, k is an integer while the right hand side is a continuous variable. Here we are considering the large k limit where the distinction is not important.

For large values of $|C|$, it is natural to measure the size of the S^2 in units of the VEV of the operator $\mathcal{U}$ which has dimension two. This simply gives from (4.21)

$$\langle\mathcal{U}\rangle R_{\text{Fuzzy}}^2 \propto \frac{k}{g_+^2} \sim \frac{3M}{16\pi^2}\log\frac{\langle\mathcal{U}\rangle}{\Lambda_0^2} + \cdots \tag{4.23}$$

where Λ_0 is the scale at the last step of the cascade, normalized with the factor of M natural from the 't Hooft counting point of view[17].

4.3 Comparison with the gravity picture

We can now compare to the quantities that we had in the gravity analysis. First we recall that the VEV of the field $\mathcal{U}$ is proportional to [5]

$$\langle\mathcal{U}\rangle \propto MU\Lambda_0^2 \propto Me^{\frac{2t_\infty}{3}}\Lambda_0^2\ . \tag{4.24}$$

We have seen that the metric in this solution is basically the resolved conifold with M fivebranes wrapping it, plus a large amount of D3 brane flux. The amount of D3 brane flux that we have on the fivebranes can be determined by computing the value of the B^{NS} field on the two-cycle near the tip of the resolved conifold. The value of the B^{NS} field only varies logarithmically, so it does not matter precisely where we evaluate it, as long as it is around $t \sim t_\infty$, which is the region where the metric looks like that of the resolved conifold. In fact we have

$$\frac{1}{(2\pi)^2\alpha'}\int_{S^2} B^{NS}\Big|_{t=t_\infty} \propto \frac{g_s}{2\pi}Mt_\infty = k\ . \tag{4.25}$$

We have identified this with the number of steps from the bottom of the cascade in the gravity approximation, since it gives us how many D3 branes we have dissolved on the D5 branes: $N_3 = kM$.

[17] In this normalization the gaugino bilinear expectation value is $\langle\text{Tr}[\psi^2]\rangle \propto M(g_+^2M)\Lambda_0^3$, see (3.24). We have ignored a factor of $M^2g_+^{2/3}$ inside the log in (4.23).

We would also like to have some way of estimating the size of the S^2 on which we put the fivebranes. We see that the radius of the S^2 of the conifold is proportional to t_∞ before we do the boosting procedure (2.12). The boosting procedure introduces the warp factor $\hat{h}$ which multiplies the spacetime direction and a similar factor multiplying the spatial directions (3.19). The radius of the S^2 is then given by

$$r_{S^2}^2 \propto \hat{h}^{1/2} g_s M t_\infty \ . \tag{4.26}$$

Note that in this region the dilaton is constant and $e^{2(\phi-\phi_\infty)} \sim 1$. We find that the fivebrane has a large amount of B field and we should use the appropriate expression for the open string metric on the fivebranes. We use the formulas for the open string metric on branes when we have a large B field in eqn (2.5) in [50]

$$G^{ij}_{\rm open} \sim \frac{r^2_{\rm closed}}{B^2} \ , \qquad \theta^{ij} \sim \frac{1}{B} \ , \qquad \text{for} \qquad r^2_{\rm closed} \ll B \tag{4.27}$$

where r is the radius of the closed string metric, which appears in (4.26). We face the problem that $\hat{h}$ diverges where the branes are sitting, but of course, this is already taking into account the backreaction of the branes. We should really evaluate $\hat{h}$ at some distance from the point where the fivebranes are sitting. It turns out that the final answer (4.23) does not depend on $\hat{h}$. However, we see that as we are in the region that the metric is accurately given by the resolved conifold, but away from the origin of the resolved conifold (say at $\rho/\alpha \gg 1/t_\infty$ but $\rho \sim \alpha$ in (2.11)), we get that $\hat{h}$ is becoming very small as $U \to \infty$. In fact, we have that $r^2_{\rm closed} \propto \sqrt{t_\infty}$ vs. $B \propto t_\infty$. See around eq. (A.37). For large t_∞, this justifies the use of (4.27).

From (4.25) and (4.27) we see that the non-commutativity parameter is indeed as in the fuzzy sphere construction (4.21). Similarly, we can compare the masses of the Kaluza-Klein modes on the fivebrane. These masses are proportional to

$$m^2_{KK} = [\hat{h}^{-1/2} U M g_s \Lambda_0^2] \frac{l(l+1)}{r^2_{\rm open}} = l(l+1) \hat{h}^{-1/2} U M \Lambda_0^2 \frac{r^2_{\rm closed}}{B^2} \sim l(l+1) \frac{U}{t_\infty} \Lambda_0^2 \tag{4.28}$$

where the factor of $[\hat{h}^{-1/2} U M g_s \Lambda_0^2]$ comes from the warping of the four dimensional space in (3.19) and we have used that $r^2_{\rm closed}$ is $r^2_{S^2}$ computed in (4.26). l labels the angular momentum on S^2. We see that once we express these modes in units of the expectation value of $\mathcal{U}$ from (4.24) we get

$$\frac{m^2_{KK}}{\langle \mathcal{U} \rangle} \propto \frac{m^2_{KK}}{M U \Lambda_0^2} \propto l(l+1) \frac{1}{M t_\infty} \propto l(l+1) \frac{g_s}{k} \ , \qquad g_s \propto g_+^2 \tag{4.29}$$

Thus we see that we get agreement also for the k dependence of the radius of the fuzzy sphere in (4.21).

Notice that the dilaton $\hat{\phi}$ is very close to a constant up to the region that is very close to the branes. Once we analyze in the near brane region we see that the dilaton $\hat{\phi} = -\phi$ starts decreasing rapidly, see (A.17) in appendix A. This implies, due to (3.21), that

$H_{NS} \to 0$ (for large t_∞) rapidly and the solution becomes very similar to the straight S-dual of (2.1). This is related to the fact that the effects of non-commutativity on the fivebrane become less important as we go to the IR.

Notice that the emergence of the fuzzy sphere relied on the VEVs for A_1 and A_2 given by (4.1), which are a solution of the D-term equations. On the other hand, we did not rely on the details of the superpotential. By setting $B_a = 0$ we ensured that $\partial W = 0$. In particular, we know that the Klebanov-Strassler theory could arise from an $\mathcal{N} = 2$ $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ plus a mass deformation [48]. Since the mass deformation only enters in the superpotential, we can easily check that the VEVs for A_i in (4.1) continue to be good vacua. Thus, when these VEVs are much larger than the mass we expect that the configuration should have a description in terms of a $SU(M(k+1)) \times SU(Mk)$ $\mathcal{N} = 2$ quiver theory, see [58] for further discussion on this theory. In this case, we expect that the proper description of the vacuum should be in terms of fivebranes that are wrapping the S^2 of an Eguchi-Hanson space. Solutions corresponding to such configurations were presented in [64]. However, we did not check the details.

5 Discussion

In this paper we have analyzed various solutions describing closely related configurations. The solutions are not new, and they are contained in [4]. Nevertheless, we think that the points we have made are not generally appreciated.

First, we have discussed the most basic solution from which all others follows. This is the solution for a number of fivebranes wrapping the S^2 of the resolved conifold. Alternatively we can just as well say that it is the solution describing a deformed conifold with flux. In both cases, the geometry is not that of the resolved or deformed conifold. The solution interpolates between the deformed conifold with flux and the resolved conifold with branes and it is a simple realization of the geometric transition described in [11]. With NS three form field strength the four dimensional string metric is unwarped, which justifies the first word in our title.

The solution we discussed is also one of the few explicit examples of torsional geometries, in the sense of [12, 13]. In particular, the geometry is complex, but not Kähler. Thus, this geometry can be viewed as a non-Kähler version of the conifold, where the metric is not Ricci-flat[18]. The solution discussed here may be describing a region of a bigger compact manifold. One difference with the conifold is that no cycle goes to zero size, and the geometry is always smooth. It is a natural arena for studying aspects of non-Kähler geometry.

Starting from this solution one can add D3 brane charge by a certain U-duality transformation. This gives a useful perspective on the solution of [4], representing the gravity dual of the baryonic branch of the Klebanov-Strassler theory. In fact, this

[18]The CV-MN solution is also a non-compact, non-Kähler geometry. But it asymptotes to a linear dilaton background.

could have been another avenue for deriving that solution. The BPS equations become simpler with only non-trivial H_3 and dilaton. It would be nice to see if other explicit interesting solutions can be constructed in this way, starting from solutions of Type I supergravity.

We have also discussed the interpretation of the U-duality transformation in a T-dual brane picture. In this context the duality corresponds to a simple *rotation* of the NS branes. Various features of the supergravity solutions may be then understood heuristically from this picture.

One basic lesson of this analysis is that going far along a baryonic branch in confining theories with fractional branes is related to resolving the singularity and wrapping some branes on the resulting two-cycles. This picture could be particularly useful for cases where one cannot find the explicit solutions. One interesting case would be the theory studied in [65] which is supposed to display a runaway behavior [66] pushing it far along the baryonic branch. Thus, it might be possible to find the gravity picture of the runaway behavior. One would start with a suitably resolved space[19], add branes, and presumably find that there is residual force pushing the branes away, as opposed to the case in this paper where we have an exact modulus.

The emergence of a two-sphere when we go along the baryonic branch is another observation that we have made. We have seen that the scaling of the size of the sphere matches quite well between the field theory and the gravity description. The fact that the fuzzy sphere arises does not depend too much on the details of the theory. It only relied on the existence of a quiver description with two different ranks. It would be nice to explore this phenomenon in more generality by considering a general class of theories. One closely related example is the picture for vacua of the massive ABJM theory discussed in [26].

The analysis in this paper is probably also useful for studying in more detail the inflationary model proposed in [68] which involves wrapping fivebranes on the S^2 of the conifold.

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[19]Several explicit Ricci-flat Kähler metrics on (partially) resolved Calabi-Yau singularities were presented in [67].

A More details on the solution

In this a appendix we consider the equations (2.6)(2.7), which we reproduce here[20]

$$f' = 4\sinh^2 t\, c \tag{A.1}$$

$$c' = \frac{1}{f}[c^2 \sinh^2 t - (t\cosh t - \sinh t)^2] \ . \tag{A.2}$$

We will collect a few facts about these equations. We can define a new variable τ via $d\tau = \sinh^2 t dt$. Then the equations become

$$\partial_\tau f = 4c \tag{A.3}$$

$$\partial_\tau c = \frac{1}{f}[c^2 - k(\tau)] \ , \qquad k(\tau) = \left(\frac{t}{\tanh t} - 1\right)^2 \tag{A.4}$$

$$\tau = \frac{1}{2}(\cosh t \sinh t - t) \tag{A.5}$$

The equations (A.3), (A.4) can be written as a second order equation for f,

$$f\partial_\tau^2 f = \frac{(\partial_\tau f)^2}{4} - 4k(\tau) \tag{A.6}$$

which is the equation of motion for the action

$$S = \int d\tau \left[\frac{1}{16}\frac{(\partial_\tau f)^2}{\sqrt{f}} + \frac{k(\tau)}{\sqrt{f}}\right] \ . \tag{A.7}$$

We could also introduce a new variable $x = f^{3/4}$. Then the lagrangian has the form $\dot{x}^2 + k(\tau)x^{-2/3}$. This is a negative potential. The particle starts at $x = 0$ at $\tau = 0$ and the rolls off down the potential as $\tau \to \infty$.

The Hamiltonian associated to the lagrangian in (A.7) is not conserved, and it is given by

$$H = \frac{1}{16}\frac{(\partial_\tau f)^2}{\sqrt{f}} - \frac{k(\tau)}{\sqrt{f}} = e^{2(\phi-\phi_0)} \tag{A.8}$$

where we noted that the Hamiltonian is equal to the dilaton in (2.3). Using (2.8), (2.9) we see that this has the following values at $t = 0$ and $t = \infty$

$$H(0) = \gamma^3 \ , \qquad H(\infty) = \frac{1}{9}e^{-t_\infty} = \frac{8}{\sqrt{3}}U^{-3/2} \ . \tag{A.9}$$

The derivative of the Hamiltonian on a solution is given by the explicit time dependence of the Lagrangian

$$\partial_\tau H = -\partial_\tau L = -\frac{(\partial_\tau k)}{\sqrt{f}} \ . \tag{A.10}$$

[20]The function c used here is related to a in [4] by $c = -\frac{a(\sinh t - t\cosh t)}{(1+a\cosh t)}$.

This is negative since $k(\tau)$ is an increasing function of τ (A.4). Thus the dilaton is a maximum at $t = 0$ and it then decreases monotonically as $\tau \to \infty$. In fact, for large times the change in the energy goes to zero due to (2.9). In fact, we can compute the first subleading term for large t which has the form

$$\hat{h}(t) = \frac{H(t)}{H(\infty)} - 1 = 2 \times 3^3\, t\, e^{-\frac{4}{3}t}\, e^{\frac{4}{3}t_\infty} = U^2 \frac{3t}{8} e^{-4/3t} \ . \tag{A.11}$$

This function appears in the expression of the boosted solution and also in the solution with Klebanov-Strassler asymptotics (3.19). This overall factor of U^2 cancels out in (3.19).

A.1 The solution for small U, or $t_\infty \ll 0$

When U is small and t_∞ is very negative, then we have that $\gamma \gg 1$. In this case the particle described by (A.7) moves very quickly to large values of f where the derivative of the energy becomes very small. Thus, in this regime the energy is conserved to first approximation and the dilaton is constant. In the limit that γ is very large we can find an approximate solution to these equations by neglecting $k(\tau)$ in (A.4). This approximate solution has the form

$$c^3 = \gamma^6\, 3\, \tau \ , \qquad f = \gamma^{-6} c^4 \ , \qquad \gamma \gg 1 \tag{A.12}$$

with τ in (A.5). This solution, inserted in the ansatz, gives the deformed conifold metric. More precisely, in a scaling limit where $\gamma^2 \to \infty$, and up to an overall scale γ^2 in the metric, we get precisely the deformed conifold, (2.13). For large and finite γ, we get a metric which is very close to the deformed conifold, but in addition we have a non-vanishing three form NS flux on the S^3 of the deformed conifold. The relation between γ and U in this regime is

$$\frac{U}{12} = e^{\frac{2t_\infty}{3}} \sim \frac{1}{3^{4/3}\gamma^2} \tag{A.13}$$

which can be obtained comparing the large t behavior of (A.12) and (2.9).

We can also find the subleading correction to the dilaton by using the energy non-conservation equation (A.10)

$$\hat{h} = e^{2(\phi - \phi_\infty)} - 1 = \frac{1}{\gamma^3} \int_t^\infty \frac{\partial_t k}{\sqrt{f}} = \frac{1}{\gamma^4} 2^{4/3} 3^{-2/3} \int_t^\infty dt' \frac{(t' \coth t' - 1)}{\sinh^2 t'} (\sinh 2t' - 2t')^{1/3} \tag{A.14}$$

This expression is necessary to recover the Klebanov-Strassler limit of the solution (3.19). Of course, the large t limit of (A.14) agrees with the general expression (A.11), after using (A.13).

A.2 The solution for large U, or $t_\infty \gg 0$

We now want to study the solution in the regime $\gamma \sim 1$. For $\gamma = 1$ we have the CV-MN [8, 9, 10] solution

$$c = t \ , \qquad f = t^2 \sinh^2 t - (t \cosh t - \sinh t)^2 = t(\sinh(2t) - t) - \sinh^2 t \ . \qquad \text{(A.15)}$$

This solution does not go over the conifold at infinity. However, as soon as $1 < \gamma$, the asymptotic form of the solution at large t changes and it becomes that of the conifold. The solution stays very close to (A.15) up to a large value of t and then it starts deviating from it. The large t form of (A.15) is

$$c = t \ , \qquad f = \frac{t}{2} e^{2t} \ , \qquad e^{2\phi - 2\phi_0} = 4\sqrt{\frac{t}{2}} e^{-t} \ . \qquad \text{(A.16)}$$

Let us now solve the equation in the region where it starts deviating from (A.15). We will call this the "fivebrane" region. We can write $c = t + \mu(t)$ and we assume that $\mu(t) \ll t$, but we do not assume that μ' is small. The transition happens around a value of t we will call t_5, we will define it better below. So we want $\mu \ll t_5$. In equation (A.3) we set $\mu = 0$ so that f remains the same. In equation (A.4) we expand to first order in μ

$$\mu' = \frac{2t \sinh^2 t}{f} \mu = \mu \quad \to \quad \mu = e^{t - t_5} \qquad \text{(A.17)}$$

where t_5 is an integration constant. When this is inserted in the expression for the metric and the dilaton we obtain

$$c' = 1 + e^{t - t_5} \ , \qquad e^{2\phi - 2\phi_0} = 4\sqrt{\frac{t}{2}} e^{-t} (1 + e^{t - t_5}) \ . \qquad \text{(A.18)}$$

We see that the transition in the behavior of c' occurs at $t \sim t_5$ and it is very rapid compared to the variation of t. Thus in the expression for the dilaton we can approximate the value of t in the prefactor as a constant equal to t_5. Therefore we can say that the dilaton changes from the large t behavior in (A.16) to basically a constant. It is possible to see that the full metric becomes that of an ordinary fivebrane if we identify $e^{t - t_5} = \frac{r^2}{M\alpha'}$. The directions along the S^2 are simply a constant.

We get the following approximate metric

$$\begin{aligned} \frac{M\alpha'}{4} ds_6^2 &\approx (M\alpha' + r^2) \left\{ \frac{1}{4} dt^2 + \frac{1}{4} [(\epsilon_3 + A_3)^2 + e_1^2 + e_2^2] \right\} + \frac{M\alpha' t_5}{2} (\epsilon_1^2 + \epsilon_2^2) \\ &\approx (1 + \frac{\alpha' M}{r^2})(dr^2 + r^2 d\tilde{\Omega}_3^2) + \frac{M\alpha' t_5}{2} d\Omega_2^2 \qquad \text{(A.19)} \\ e^{2\phi - 2\phi_0} &\approx 4\sqrt{\frac{t_5}{2}} e^{t_5} (1 + \frac{\alpha' M}{r^2}) \qquad \text{(A.20)} \end{aligned}$$

$$|t - t_5| \ll \log t_5 \ , \qquad e^{t - t_5} = \frac{r^2}{M\alpha'} \ll t_5 \ . \qquad \text{(A.21)}$$

We see that $\tilde{\Omega}_3$ is a three sphere (fibered over the S^2). Since the S^2 is large compared to the other dimensions, we can neglect the fact that the S^3 is fibered. This metric looks like the metric of a fivebrane in flat space where r and Ω_3 are the four directions transverse to the fivebrane. We have also specified the regime where the solution is valid. In this regime the size of the S^2 is constant and very big. The upper bound on t_5 in (A.21) comes from equating $t \sim \mu(t) \sim t_5$. It is also convenient to estimate the value of γ that we would obtain from this solution. We do this by simply extrapolating μ to the origin, where (A.17) is not really valid. That then gives the estimate

$$\gamma^2 = c'(0) = 1 + e^{-t_5} \ , \qquad \gamma^2 - 1 \propto e^{-t_5} \ . \tag{A.22}$$

We now solve the equation in the region where $t \sim t_5$. In this region we approximate the function $k(\tau) \sim t_5^2$. Then the energy (A.8) becomes conserved we get

$$H = \frac{1}{16}\frac{(\partial_\tau f)^2}{\sqrt{f}} - \frac{k(\tau)}{\sqrt{f}} = e^{2(\phi-\phi_0)} = E \ . \tag{A.23}$$

We rewrite this in terms of c to obtain

$$(c^2 - t_5^2)\partial_\tau c = E^2 \ , \qquad \tau = \frac{e^{2t}}{8} \tag{A.24}$$

$$\frac{c^3}{3} - t_5^2 c = E^2\tau - \frac{2}{3}t_5^3 \tag{A.25}$$

$$E = e^{2\phi_\infty - 2\phi_0} \tag{A.26}$$

where we used the large t limit of (A.5). E is an integration constant equal to the energy. In addition we fixed another integration constant by saying that $c = t_5$ for $\tau = 0$. We can now determine the integration constant E by matching to the previous expression (A.18). We expand (A.25) for small τ by writing $c = t_5 + \mu$, as we did before. We then find that

$$t_5\mu^2 = E^2\tau \ , \qquad \mu = \frac{E}{\sqrt{t_5 8}}e^t = e^{t-t_5} \ , \quad E = \sqrt{8t_5}e^{-t_5} \ . \tag{A.27}$$

where we solved for μ using (A.25) and equated it to our previous value (A.17). Once we have determined E, we can now determine the value of t_∞ in this solution by looking at the large t behavior

$$c \sim \frac{1}{6}e^{\frac{2(t-t_\infty)}{3}} \sim (E^2\tau 3)^{1/3} = 3^{1/3}t_5^{1/3}e^{\frac{2}{3}(t-t_5)} \ . \tag{A.28}$$

We see that

$$E \sim \frac{e^{-t_\infty}}{9} \quad \text{and} \quad e^{-2t_\infty} = 3^4 \, 8 \, t_5 \, e^{-2t_5} \ . \tag{A.29}$$

We see that $t_5 = t_\infty + o(\log t_\infty)$, thus we can replace t_5 by t_∞ in some of the above formulas.

We can see that (A.25) reduces to the resolved conifold as follows. We introduce ρ through

$$c = t_\infty + \frac{\rho^2}{6} \tag{A.30}$$

and we write

$$\begin{aligned}
\partial_t c &= \frac{\rho^2}{9}\frac{\rho^2 + 18t_\infty}{\rho^2 + 12t_\infty} = \frac{\rho^2}{9}\kappa(\rho) \ , \qquad \alpha^2 = 2t_\infty && \text{(A.31)}\\
\partial_t c dt^2 &= \frac{\rho^2 d\rho^2}{9\partial_t c} = \frac{d\rho^2}{\kappa(\rho)} && \text{(A.32)}\\
& -\log t_\infty \ll \log\frac{\rho^2}{\alpha^2} \ll (\log t_\infty) \ . && \text{(A.33)}
\end{aligned}$$

In this way we see that we recover the resolved conifold metric (2.11). We have also indicated the range of ρ were we can trust the resolved conifold metric. In the lower bound we encounter the near brane region of the fivebranes and in the upper bound we need to start taking into account the "running" of α^2. The region of validity is very large for large t_∞.

In summary, the solution has various regions. The transition between various regions happens at $t \sim t_\infty$, or within are region of size $\log t_\infty$ around this value. For $t \ll t_\infty$ we have the CV-MN solution, which can be viewed as the near brane geometry of M fivebranes. When $t \sim t_5$ we leave the near brane geometry and the dilaton becomes constant. For larger values of t, but still within t/t_∞ of order one, we can view the solution as the resolved conifold plus some branes on the S^2. Notice that the metric behaves as the metric of the resolved conifold up to $\rho^2 \sim 1$ at $t \sim t_5$, see (A.27), (A.30). This is much smaller than t_∞ which is setting the size of the sphere of the resolved conifold. Furthermore, the resolved conifold metric is accurate up to a value of t where the "running" of the size starts to matter. In other words, the full metric (2.1) has a ratio of sizes of two-spheres going like t. Thus, we can approximate that ratio as a constant for values of t such that t/t_∞ is of order one. On the other hand, since the relation between ρ and t is exponential, we see that we can trust the metric of the conifold up to a value of ρ such that $\log(\rho/\alpha) \sim o(t_\infty)$, see (A.33). Thus, there is a large region of the geometry that is accurately given by the resolved conifold. For larger values of t, then we should take into account this "running", but by this stage, the resolution parameter is a small (but non-normalizable) deformation of the metric.

The above formulas give us an accurate description of the solution in (2.1) before we perform the boosting procedure. However, in order to do the boosting, we will also need the behavior of the dilaton to the next order. In order words, we want $e^{2(\phi-\phi_\infty)}-1$. It is convenient to write the expression for f from (A.25) as

$$f = \frac{c^2 - t_\infty^2}{\partial_\tau c} \ , \qquad f^{1/2} = \frac{E}{\partial_\tau c} \tag{A.34}$$

We can find the expression for the variation of the dilaton by using the energy non-conservation equation (A.10)

$$
\begin{aligned}
e^{2(\phi-\phi_\infty)} - 1 &= e^{-2(\phi_\infty-\phi_0)} \int_t^\infty dt' \frac{\partial_t h}{\sqrt{f}} = \frac{1}{E^2}\int_t^\infty dt' \partial_t h \partial_\tau c \\
&= \frac{2}{E^2}\int_t^\infty dt'\, t' \partial_\tau c \ . \qquad \text{(A.35)}
\end{aligned}
$$

We now change variable from t to ρ. We substitute $c = t_\infty + \rho^2/6$ into (A.25), obtaining

$$
e^{2t} = \frac{1}{81E^2}\rho^4(\rho^2 + 18t_\infty) \ . \qquad \text{(A.36)}
$$

Then we have

$$
\begin{aligned}
\hat{h} = e^{2(\phi-\phi_\infty)} - 1 &= \frac{8}{E^2}\int_\rho^\infty d\rho' \frac{\rho'}{3} t' e^{-2t'} \\
&= 8\int_\rho^\infty d\rho \frac{\rho}{6} \log\left[\frac{\rho^4(\rho^2+18t_\infty)}{81E^2}\right] \frac{81}{\rho^4(\rho^2+18t_\infty)} \ . \qquad \text{(A.37)}
\end{aligned}
$$

We see that the only remaining E dependence is inside the log. We can now estimate $\hat{h}$ in the range $\rho^2 \sim t_\infty$, but $1 \ll \rho^2$. The argument of the logarithm in (A.37) has a denominator that varies exponentially with t_∞ (A.29). Thus, we can approximate the logarithm as $\log[\,] \propto t_\infty$. Then all factors of ρ outside the logarithm are approximated via $\rho^2 \propto t_\infty$. In this range we see that $\hat{h} \sim 1/t_\infty$. Thus, $\hat{h}^{1/2} t_\infty \sim \sqrt{t_\infty}$ which justifies the approximation in (4.27).

In order to turn the asymptotics to precisely KS, we need to further redefine coordinates, introducing a new coordinate r defined via

$$
\rho = E^{1/3} r \qquad \text{(A.38)}
$$

Now we see that we get

$$
\hat{h} = \frac{8}{E^{4/3}}\int_r^\infty dr \frac{r}{6}\log\left[\frac{r^4(r^2+18\hat{t}_\infty)}{81}\right]\frac{81}{r^4(r^2+18\hat{t}_\infty)} \ , \qquad \hat{t}_\infty = t_\infty E^{-2/3} \ . \qquad \text{(A.39)}
$$

Now we have the standard expression for the KS asymptotics. Furthermore, this agrees with the the expression of the warp factor of the solution by Pando-Zayas and Tseytlin [28], up to terms that are important at small t and probably arise once we take into account the leading order variation of the dilaton at small t, $t \sim t_5$.

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