# A note on the use of Rédei polynomials for solving the polynomial Pell equation and its generalization to higher degrees

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**Abstract**

The polynomial Pell equation is

$$P^2 - DQ^2 = 1$$

where $D$ is a given integer polynomial and the solutions $P, Q$ must be integer polynomials. A classical paper of Nathanson [11] solved it when $D(x) = x^2 + d$. We show that the Rédei polynomials can be used in a very simple and direct way for providing these solutions. Moreover, this approach allows to find all the integer polynomial solutions when $D(x) = f^2(x) + d$, for any $f \in \mathbb{Z}[X]$ and $d \in \mathbb{Z}$, generalizing the result of Nathanson. We are also able to find solutions of some generalized polynomial Pell equations introducing an extension of Rédei polynomials to higher degrees.

**Keywords:** Pell equation, polynomial Pell equation, Rédei polynomial
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## 1    Introduction

The Pell equation

$$x^2 - dy^2 = 1$$

is one of the most famous Diophantine equations. It has infinite integer solutions when $d$ is not a square and they can be determined using the continued fraction's expansion of $\sqrt{d}$. It is very interesting to study the polynomial Pell equation, i.e., find polynomials $P$ and $Q$ in $\mathbb{Z}[X]$ satisfying

$$P^2 - DQ^2 = 1 \tag{1}$$

where $D \in \mathbb{Z}[X]$ is a fixed polynomial. Nathanson [11] solved the polynomial Pell equation when $D = x^2 + d$. In particular, he proved that it has non-trivial solutions if and only if $d = \pm 1, \pm 2$ and provided explicitly the polynomial solutions. Pastor [13] showed that the solutions of Nathanson can be also expressed in terms of Chebyshev polynomials. Webb and Yokota [17] found a necessary and sufficient condition for which the polynomial Pell equation has non-trivial solutions when $D = A^2 + 2C$ monic polynomial, $A/C \in \mathbb{Z}[X]$ and $\deg C < 2$. In [18], such result is generalized when $pA/C \in \mathbb{Z}[X]$, for some prime $p$, without any condition on the degree of $C$. In this case, the authors also determined the solutions. Then, Yokota [19] found a necessary and sufficient condition for the solution of the polynomial Pell equation when $A/C \in \mathbb{Q}[X]$. Hazama [7] studied the polynomial Pell equation using the twist of a conic by another conic. Further studies can be found in [10] and [14]. Mc Laughlin [9] focused on the relations between polynomial solutions of the Pell equation and fundamental units of real quadratic fields. Some authors studied polynomial solutions of (1) in $\mathbb{C}[X]$, see [4], [6], [20]. Despite many studies on the polynomial Pell equation, there are no works that highlight its connection with the Rédei polynomials, which are also classical tools in number theory, see [15], [8]. In this paper, we show this connection and how using Rédei polynomials for solving (1) for polynomials $D$ more general than the polynomials studied in previous works.

The paper is structured as follows. We introduce the Rédei polynomials in section 2 where we show that solutions of (1) are the Rédei polynomials for some kind of polynomials $D$. The solutions

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of Nathanson arise as particular cases. The used method, even if simple and straightforward, is very general and allows to approach also polynomial Pell equations with higher degrees by means of a generalization of the Rédei polynomials, as we will see in section 3.

## 2 Solution of the polynomial Pell equation via Rédei polynomials

Rédei polynomials were introduced in [15] and they are a particular case of Dickson polynomials [8]. They arise from the development of

$$(\sqrt{\alpha} + z)^n = D_n(\alpha, z)\sqrt{\alpha} + N_n(\alpha, z), \quad n = 0, 1, \ldots$$

where $z, \alpha \in \mathbb{Z}$, $\alpha$ not square. They can be written in the following closed form:

$$N_n(\alpha, z) = \sum_{k=0}^{[n/2]} \binom{n}{2k} \alpha^k z^{n-2k}, \quad D_n(\alpha, z) = \sum_{k=0}^{[n/2]} \binom{n}{2k+1} \alpha^k z^{n-2k-1}.$$

It will be very useful the following matricial identity that can be easily proved by induction:

$$\begin{pmatrix} z & \alpha \\ 1 & z \end{pmatrix}^n = \begin{pmatrix} N_n(\alpha, z) & \alpha D_n(\alpha, z) \\ D_n(\alpha, z) & N_n(\alpha, z) \end{pmatrix}, \tag{2}$$

from which follows

$$\begin{pmatrix} z & \alpha \\ 1 & z \end{pmatrix} \begin{pmatrix} N_{n-1}(\alpha, z) \\ D_{n-1}(\alpha, z) \end{pmatrix} = \begin{pmatrix} N_n(\alpha, z) \\ D_n(\alpha, z) \end{pmatrix}, \quad \begin{pmatrix} N_{n-1}(\alpha, z) \\ D_{n-1}(\alpha, z) \end{pmatrix} = \begin{pmatrix} \dfrac{z}{z^2 - \alpha} & -\dfrac{\alpha}{z^2 - \alpha} \\ -\dfrac{1}{z^2 - \alpha} & \dfrac{z}{z^2 - \alpha} \end{pmatrix} \begin{pmatrix} N_n(\alpha, z) \\ D_n(\alpha, z) \end{pmatrix} \tag{3}$$

From (2), it also follows that the Rèdei polynomials are linear recurrent sequences with characteristic polynomial $t^2 - 2zt + z^2 - \alpha$, i.e.,

$$\begin{cases} N_n(\alpha, z) = 2z N_{n-1}(\alpha, z) - (z^2 - \alpha) N_{n-2}(\alpha, z), & n \geq 2 \\ N_0(\alpha, z) = 1, N_1(\alpha, z) = z \end{cases}, \tag{4}$$

$$\begin{cases} D_n(\alpha, z) = 2z D_{n-1}(\alpha, z) - (z^2 - \alpha) D_{n-2}(\alpha, z), & n \geq 2 \\ D_0(\alpha, z) = 0, D_1(\alpha, z) = 1 \end{cases} \tag{5}$$

Finally, we can observe that $N_n(\alpha, z)$ and $D_n(\alpha, z)$ are polynomials of degree $n$ and $n - 1$ in $z$, respectively. The Rédei rational functions $\dfrac{N_n(\alpha, z)}{D_n(\alpha, z)}$ have been studied and applied in several fields. For instance, they have been exploited to create public key cryptographic systems [3], [12] and to generate pseudorandom sequences [16]. Moreover, in [2], the authors used the Rédei rational functions for generating solutions of the classical Pell equation. For further properties of the Rédei polynomials, see [8].

In the classical paper of Nathanson [11], he proved that the polynomial Pell equation

$$P^2 - (x^2 + d)Q^2 = 1,$$

has non-trivial solutions $P(x), Q(x) \in \mathbb{Z}[X]$ if and only if $d = \pm 1, \pm 2$ and he gave these polynomials explicitly:

$$\begin{cases} A_n = \left(\frac{2}{d}x^2 + 1\right) A_{n-1} + \frac{2}{d}x(x^2 + d)B_{n-1}, & A_0 = 1 \\ B_n = \frac{2}{d}x A_{n-1} + \left(\frac{2}{d}x^2 + 1\right) B_{n-1}, & B_0 = 0 \end{cases}$$

for any $n \geq 1$, when $d = 1, \pm 2$ and

$$\begin{cases} A'_n = x A'_{n-1} + (x^2 - 1)B'_{n-1}, & A'_0 = 1 \\ B'_n = A'_{n-1} + x B'_{n-1}, & B'_0 = 0 \end{cases}$$

for any $n \geq 1$, when $d = -1$. Using the Rédei polynomials, we can solve a more general case of polynomial Pell equations, where we retrieve the results of Nathanson as a particular case. In the next theorem we see that the Rédei polynomials are polynomial solutions of

$$P^2 - (f^2(x) + d)Q^2 = 1, \tag{6}$$

where $f(x)$ is any integer polynomial and $d \in \mathbb{Z}$.

**Theorem 1.** *Given $d \in \mathbb{Z}$ and $f \in \mathbb{Z}[X]$, consider the Rédei polynomials $N_n(x) = N_n(f^2(x) + d, f(x))$ and $D_n(x) = D_n(f^2(x) + d, f(x))$, then*

$$\left( \frac{N_n(x)}{(-d)^{n/2}} \right)^2 - (f^2(x) + d) \left( \frac{D_n(x)}{(-d)^{n/2}} \right)^2 = 1.$$

*Moreover, the solutions are integer polynomials if and only if $d = 1, \pm 2$ and $n$ even, $d = -1$ for any $n$.*

*Proof.* From (2) we have

$$N_n^2(\alpha, z) - \alpha D_n^2(\alpha, z) = (z^2 - \alpha)^n,$$

for any $n \geq 1$. Considering $z = f(x)$ and $\alpha = f^2(x) + d$ we obtain

$$N_n^2(x) - (f^2(x) + d)D_n^2(x) = (-d)^n$$

i.e.,

$$\left( \frac{N_n(x)}{(-d)^{n/2}} \right)^2 - (f^2(x) + d) \left( \frac{D_n(x)}{(-d)^{n/2}} \right)^2 = 1.$$

Clearly, when $d = -1$, we have obtained solutions of the polynomial Pell equation which are integer polynomials. When $d = 1$, $\dfrac{N_n(x)}{(-d)^{n/2}}$ and $\dfrac{D_n(x)}{(-d)^{n/2}}$ are integer polynomials if and only if $n$ is even. We complete the proof showing that $d^{[n/2]} \mid N_n(x)$ if and only if $d = \pm 2$, for any $n \geq 1$. For $n = 1$ and $n = 2$, we have

$$N_1(x) = f(x), \quad N_2(x) = 2f^2(x) + d$$

and the statement is true. Let us observe that if $d \neq \pm 2$, then $d \!\not|\, N_2(x)$. Now, we proceed by induction. From (4), we have

$$N_n(x) = 2f(x)N_{n-1}(x) + dN_{n-2}(x) = 2f(x)d^{[(n-1)/2]} \cdot a + d \cdot d^{(n-2)/2} \cdot b$$

for certain integers $a$ and $b$. From the last equality, which holds by inductive hypothesis, we have that $d^{[n/2]} \mid N_n(x)$. Similarly, we can get the same result for $D_n(x)$. □

For proving that all the integer polynomial solutions of (6) are given in the previous theorem, we need the following lemma.

**Lemma 1.** *Let $f(x)$, $P(x)$ and $Q(x)$ be polynomials of degree $m$, $nm$ and $(n-1)m$, respectively, with coefficients of the highest degree that are positive and*

$$P^2 - (f^2(x) + d)Q^2 = (-d)^n, \tag{7}$$

*with $d \in \mathbb{Z}^*$. Given $P'(x) = -\dfrac{f(x)}{d}P(x) + \dfrac{f^2(x) + d}{d}Q(x)$ and $Q'(x) = \dfrac{1}{d}P(x) - \dfrac{f(x)}{d}Q(x)$ then*

1. $P'^2 - (f^2(x) + d)Q'^2 = (-d)^{n-1}$

2. $\deg P' < \deg P$ and $\deg Q' < \deg Q$

*Proof.* 1. We have

$$P'^2 - (f^2(x) + d)Q'^2 = \frac{f^2(x)}{d^2}P^2 + \frac{(f^2(x) + d)^2}{d^2}Q^2 - (f^2(x) + d)\frac{1}{d^2}P^2 - (f^2(x) + d)\frac{f^2(x)}{d^2}Q^2 =$$

$$= -(f^2(x) + d)\frac{1}{d^2}(P^2 - (f^2(x) + d)Q^2) + \frac{f^2(x)}{d^2}(P^2 - (f^2(x) + d)Q^2) =$$

$$= -(f^2(x) + d)(-d)^{n-2} + f^2(x)(-d)^{n-2} = (-d)^{n-1}.$$

2. Since $\deg P^2 = 2nm$, $\deg f^2Q^2 = 2nm$, $\deg Q^2 = 2nm - 2m$ and it holds

$$P^2 - f^2(x)Q^2 - dQ^2 = (-d)^n$$

we have that $P^2 - f^2Q^2$ has degree $2nm - 2m$. Moreover, we can observe that the degree of $P + fQ$ is $nm$ by the hypothesis that coefficients of the highest degree of $P, f, Q$ are positive. Thus, from $P^2 - f^2Q^2 = (P - fQ)(P + fQ)$, we have that the coefficients of degree $nm, nm - 1, \ldots nm - m$ of the polynomial $P - fQ$ are zero, i.e., $\deg Q' < \deg Q$. Similar considerations prove that $\deg P' < \deg P$. $\qquad\square$

**Theorem 2.** *All the integer polynomial solutions of*

$$P^2 - (f^2(x) + d)Q^2 = 1 \tag{8}$$

*are the polynomials $N_n(x)$ and $D_n(x)$ for $d = -1$ and any $n$, $\frac{N_n(x)}{(-d)^{n/2}}$ and $\frac{D_n(x)}{(-d)^{n/2}}$ for $d = 1, \pm 2$ and $n$ even.*

*Proof.* Let $P(x)$ and $Q(x)$ be solutions of (8), if $\deg f = m$, then must be $\deg P = mn$ and $\deg Q = m(n-1)$. Now we consider the polynomials $\tilde{P}(x) = d^{n/2} \cdot P(x)$ and $\tilde{Q}(x) = d^{n/2} \cdot Q(x)$ and we show that these polynomials coincide with the Rédei one's (unless the sign). We proceed by induction on $n$. The basis of the induction is straightforward to check. By Lemma 1 and the inductive hypothesis $-\dfrac{f(x)}{d}\tilde{P}(x) + \dfrac{f^2(x) + d}{d}\tilde{Q}(x) = N_{n-1}(x)$ and $\dfrac{1}{d}\tilde{P}(x) - \dfrac{f(x)}{d}\tilde{Q}(x) = D_{n-1}(x)$. Let us observe that we have assumed the positivity of the highest coefficients because of the form of the equation (8), in which only the squares $f^2, P^2, Q^2$ appear, hence hypotheses of Lemma 1 are satisfied. Thus, from (3), we have the thesis. $\qquad\square$

We have seen that the Rédei polynomials allow to study and solve a vast class of polynomial Pell equations in a very simple and direct way. This also generalizes the result of Nathanson [11] that we can retrieve when $f(x) = x$. Moreover, we can use the above approach for studying the most general case of polynomial Pell equations, i.e.,

$$P^2 - f(x)Q^2 = 1,$$

where $f$ is any integer polynomial. In this case, if we consider $\alpha = f(x)$ and $z = \pm\sqrt{f(x) + 1}$, surely we obtain that

$$N_n^2(x) - f(x)D_n^2(x) = 1$$

for any $n \geq 0$, where $N_n(x) = N_n(f(x), \pm\sqrt{f(x) + 1})$ and $D_n(x) = D_n(f(x), \pm\sqrt{f(x) + 1})$. If $f(x)$ is a polynomial such that $f(x) + 1$ is a square, then the Rédei polynomials $N_n(x)$ and $D_n(x)$ are integer polynomials for any $n \geq 0$.

**Question 1.** *Are all the solutions of the polynomial Pell equation (1), for any $D(x) \in \mathbb{Z}[X]$, the Rédei polynomials?*

**Example 1.** *In the following Table 1, we write the Rédei polynomials $N_n(x^4 - 1, x^2)$ and $D_n(x^4 - 1, x^2)$, by Theorem 1 we have that they are the solutions of the polynomial Pell equation $P^2 - (x^4 - 1)Q^2 = 1$.*

| $n$ | $N_n(x^4 - 1, x^2)$ | $D_n(x^4 - 1, x^2)$ |
|-----|---------------------|---------------------|
| 1 | $x^2$ | $1$ |
| 2 | $2x^4 - 1$ | $2x^2$ |
| 3 | $4x^6 - 3x^2$ | $4x^4 - 1$ |
| 4 | $8x^8 - 8x^4 + 1$ | $8x^6 - 4x^2$ |
| 5 | $16x^{10} - 20x^6 + 5x^2$ | $16x^8 - 12x^4 + 1$ |

Table 1: Polynomial solutions of $P^2 - (x^4 - 1)Q^2 = 1$

**Example 2.** *In Table 2, we summarize the Rèdei polynomials $N_n(x^4 + 2, x^2)$ and $D_n(x^4 + 2, x^2)$. For obtaining integer polynomials that are the solutions of the polynomial Pell equation $P^2 - (x^4 + 2)Q^2 = 1$, we have to consider the Rédei polynomials with even index $n$ and divide them by $-2^{n/2}$. In this way we the following solutions*

$$(-x^4 - 1, -x^2), \quad (2x^8 + 4x^4 + 1, 2x^6 + 2x^2), \quad (-4x^{12} - 12x^8 - 9x^4 - 1, -4x^{10} - 8x^6 - 3x^2), \dots$$

| $n$ | $N_n(x^4 + 2, x^2)$ | $D_n(x^4 + 2, x^2)$ |
|---|---|---|
| 1 | $x^2$ | $1$ |
| 2 | $2x^4 + 2$ | $2x^2$ |
| 3 | $4x^6 + 6x^2$ | $4x^4 + 2$ |
| 4 | $8x^8 + 16x^4 + 4$ | $8x^6 + 8x^2$ |
| 5 | $16x^{10} + 40x^6 + 20x^2$ | $16x^8 + 24x^4 + 4$ |
| 6 | $32x^{12} + 96x^8 + 72x^4 + 8$ | $32x^{10} + 64x^6 + 24x^2$ |

Table 2: Rèdei polynomials $N_n(x^4 + 2, x^2)$ and $D_n(x^4 + 2, x^2)$

**Example 3.** *If we consider the Rédei polynomials $N_n(x^2 + 3, x)$ and $D_n(x^2 + 3, x)$, summarized in Table 3, we have that $N_n(x^2 + 3, x)/(-d)^{n/2}$ and $D_n(x^2 + 3, x)/(-d)^{n/2}$ are the solutions of the polynomial Pell equation $P^2 - (x^2 + 3)Q^2 = 1$, however, in this case, we do not have integer polynomials as solutions.*

| $n$ | $N_n(x^2 + 3, x)$ | $D_n(x^2 + 3, x)$ |
|---|---|---|
| 1 | $x$ | $1$ |
| 2 | $2x^2 + 3$ | $2x$ |
| 3 | $4x^3 + 9x$ | $4x^2 + 3$ |
| 4 | $8x^4 + 24x^2 + 9$ | $8x^3 + 12x$ |
| 5 | $16x^5 + 60x^3 + 45x$ | $16x^4 + 36x^2 + 9$ |

Table 3: Rèdei polynomials $N_n(x^2 + 3, x)$ and $D_n(x^2 + 3, x)$

In the next scetion, we see that the Rédei polynomials can be generalized in a natural way and they are useful for studying polynomial Pell equations of higher degrees.

## 3   Polynomial Pell equation of higher degrees

The classical Pell equation can be generalized in a natural way to higher degrees. Indeed, we can observe that the Pell equation arises considering the unitary elements of the quotient filed $\mathbb{Q}[x]/(x^2 - d)$, where $x^2 - d$ is an irreducible polynomial over $\mathbb{Q}$. Thus, considering the unitary elements of $\mathbb{Q}[x]/(x^3 - c)$, where $c$ is not a cube, we get the cubic Pell equation

$$x^3 + cy^3 + cz^3 - 3cxyz = 1,$$

in the unknowns $x, y, z$. Similarly, we can construct the Pell equations of degree $m$ that is defined by

$$\det \begin{pmatrix} x_1 & rx_m & rx_{m-1} & \dots & rx_2 \\ x_2 & x_1 & rx_m & \dots & rx_3 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ x_{m-1} & x_{m-2} & \dots & x_1 & rx_m \\ x_m & x_{m-1} & x_{m-2} & \dots & x_1 \end{pmatrix} = 1 \tag{9}$$

in the unknowns $x_1, \dots x_m$, where $r$ is not a $m$-th power. For further details, see [1]. Thus, it is natural generalizing the study of the polynomial Pell equation to higher degrees, considering the

matrix

$$
P = \begin{pmatrix}
P_1 & RP_m & RP_{m-1} & \ldots & RP_2 \\
P_2 & P_1 & RP_m & \ldots & RP_3 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
P_{m-1} & P_{m-2} & \ldots & P_1 & RP_m \\
P_m & P_{m-1} & P_{m-2} & \ldots & P_1
\end{pmatrix}, \quad \det P = 1, \tag{10}
$$

where $P_1(x), \ldots, P_m(x)$ are unknown polynomials and $R(x)$ is a given integer polynomial. Gaunet [5] studied the polynomial cubic Pell equation

$$
P_1^3 + RP_2^3 + R^2 P_3^3 - 3RP_1P_2P_3 = 1 \tag{11}
$$

when $R(x) = x^3 + ax + b$, characterizing when the equation admits non-trivial solutions and finding them. Here, we study the general polynomial Pell equation of degree $m$, given by equation (10), where $P_1, \ldots P_m$ are unknown polynomial in $\mathbb{Z}[X]$ and $R(x) = f(x) + r$, with $f(x)$ is any integer polynomial and $r \in \mathbb{Z}$. As a particular case we obtain solutions of the equation (11) for a different class of polynomials $R$.

We define the *generalized Rédei polynomials* by means of

$$
(z + \sqrt[m]{\alpha})^n = A_n^{(0)}(z, \alpha) + A_n^{(1)}(z, \alpha) \sqrt[m]{\alpha} + \ldots + A_n^{(m-1)}(z, \alpha) \sqrt[m]{\alpha^{m-1}}.
$$

In the following, when there is no confusion, we omit the dependence on $z, \alpha$ when we write the generalized Rédei polynomials for the seek of simplicity. The generalized Rédei polynomials can be obtained by the powers of a particular $m \times m$ matrix. Indeed, by definition, we have

$$
A_{n+1}^{(0)} + A_{n+1}^{(1)} \sqrt[m]{\alpha} + \ldots + A_{n+1}^{(m-1)} \sqrt[m]{\alpha^{m-1}} = (A_n^{(0)} + A_n^{(1)} \sqrt[m]{\alpha} + \ldots + A_n^{(m-1)} \sqrt[m]{\alpha^{m-1}})(z + \sqrt[m]{\alpha})
$$

i.e.

$$
A_{n+1}^{(0)} = zA_n^{(0)} + \alpha A_n^{(m-1)}, \quad A_{n+1}^{(i)} = zA_n^i + A_n^{(i-1)}, \quad i = 1, \ldots, m - 1.
$$

Thus, given the matrix

$$
M = \begin{pmatrix}
z & 0 & 0 & \ldots & \alpha \\
1 & z & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \ldots & 1 & z & 0 \\
0 & \ldots & 0 & 1 & z
\end{pmatrix} \tag{12}
$$

we can write

$$
M \begin{pmatrix} A_n^{(0)} \\ \vdots \\ A_n^{(m-1)} \end{pmatrix} = \begin{pmatrix} A_{n+1}^{(0)} \\ \vdots \\ A_{n+1}^{(m-1)} \end{pmatrix}
$$

from which

$$
M^n = \begin{pmatrix}
A_n^{(0)} & \alpha A_n^{(m-1)} & \ldots & \alpha A_n^{(1)} \\
A_n^{(1)} & A_n^{(0)} & \ldots & \alpha A_n^{(2)} \\
\vdots & \vdots & \ddots & \vdots \\
A_n^{(m-1)} & A_n^{(m-2)} & \ldots & A_n^{(0)}
\end{pmatrix}. \tag{13}
$$

From (12), we can observe that $\det M^n = (z^m + (-1)^{m-1}\alpha)^n$, on the other hand, from (13) we have that $\det M^n = \det P$ when $P_1 = A_n^{(0)}, \ldots, P_m = A_n^{(m-1)}$ and $R = \alpha$. Thus, the generalized Rédei polynomials can be exploited for solving the polynomial Pell equation of higher degrees for convenient choices of $z$ and $\alpha$. Indeed, if $z = f(x) \in \mathbb{Z}[x]$ and $\alpha = (-f(x))^m + r$, with $r \in \mathbb{Z}$, then the Rédei polynomials satisfy the polynomial equation $\det P = ((-1)^{m-1}r)^n$, where $R(x) = (-f(x))^m + r$, and the polynomials $\dfrac{A_n^{(0)}(x)}{((-1)^{m-1}r)^{n/m}}, \ldots, \dfrac{A_n^{(m-1)}(x)}{((-1)^{m-1}r)^{n/m}}$ satisfy the polynomial Pell equation $\det X = 1$, for $R(x) = (-f(x))^m + r$. However, these solutions are not ever integer polynomials. They are integer polynomials in the following cases

1. $r = -1$ and any $n \geq 0$,

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2. $r = 1$ and any $n \equiv 0 \pmod{m}$,

3. $r = \pm m$ and any $n \equiv 0 \pmod{m}$, when $m$ is a prime number.

The situations 1 and 2 are immediate to verify. Let us focus on situation 3. We can observe that

$$A_1^{(0)}(x) = f(x), A_2^{(0)}(x) = f^2(x), \dots, A_{m-1}^{(0)}(x) = f^{m-1}(x), A_m^{(0)}(x) = \pm m$$

Moreover, the characteristic polynomial of $M$ is

$$x^m + \sum_{i=1}^{m-1} (-1)^i \binom{m}{i} x^{m-i} f^i(x) \pm m,$$

thus if $m$ is prime, we have

$$\pm m | A_m^{(0)}(x), \pm m | A_{m+1}^{(0)}(x), \dots, \pm m | A_{2m-1}^{(0)}(x)$$

and consequently $m^2 | A_{2m}^{(0)}(x)$. Thus we can prove by induction that the Rédei polynomials are in $\mathbb{Z}[x]$ for $r = \pm m$ and any $n \equiv 0 \pmod{m}$, when $m$ is a prime number.

**Question 2.** *Are all the integer polynomial solutions of* (10), *for* $R(x) = (-f(x))^m + r$, *the Rédei polynomials?*

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