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An L^p-Approach to the Well-Posedness of Transport Equations Associated to a Regular Field: Part II

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Abstract. This work represents the continuation of our paper (Arlotti and Lods, in An L^p-approach to the well-posedness of transport equations associated to a regular field—part I. Mediterr J Math. 2018. https://doi.org/10.1007/s00009-019-1425-8). In L^p-spaces $1 < p < \infty$ we investigate well-posedness of transport equations with general external Lipschitz fields and general measures associated to a large variety of boundary conditions modelled by abstract boundary operators H. In particular, a new explicit formula for the corresponding transport semigroup is given. Some applications are also presented.

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Contents

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1. Introduction

This paper is the second part of a general contribution on the thorough study of transport equations in L^p-spaces with $1 < p < \infty$. We refer to the first part [5] of the contribution for the physical motivation and relevant references for the study of general transport equation in both L^1 and L^p spaces. We shall also use most of the abstract results and notations introduced in the first part [5]. We just recall here that our aim is to investigate the transport equation associated to a general Lipschitz field $\mathscr{F}$ and a general Radon measure μ on a sufficiently smooth open subset $\mathbf{\Omega}$ of $\mathbb{R}^N$. To the time independent globally Lipschitz vector field $\mathscr{F} : \mathbb{R}^N \to \mathbb{R}^N$, we can associate a flow $(T_t)_{t\in\mathbb{R}}$ (with the notations of [5, Section 2.1]) and we assume the measure μ to be invariant under the flow $(T_t)_{t\in\mathbb{R}}$. The incoming and outgoing boundaries $\Gamma_\pm$ of the phase space are defined in [5, Section 2].

The first part [5] of the present contribution was aimed to introduce the mathematical framework of the analysis with namely

(a) The precise definition of the characteristic curves associated to $\mathscr{F}$ and μ (as in [2,3]);
(b) A precise definition of the maximal transport $\mathcal{T}_{\max,\,p}$ in $L^p(\mathbf{\Omega}, \mu)$ associated to the field $\mathscr{F}$;
(c) The definition of the trace operators $\mathsf{B}^{\pm}$ and the trace spaces $L^p_\pm$ and, in particular, a proof of Green's formula.

Besides this general framework, we also initiated in [5] the investigation of initial and boundary value problems associated to $\mathcal{T}_{\max,\,p}$ showing in particular that

(1) The maximal transport operator associated to no-reentry boundary conditions $\mathcal{T}_{0,\,p}$ is the generator of a C_0-semigroup $(U_0(t))_{t\geqslant 0}$ in $L^p(\mathbf{\Omega}, \mathrm{d}\mu)$;
(2) the general boundary value problem of the form

$$\begin{cases}(\lambda - \mathcal{T}_{\max,\,p})f = g,\\ \mathsf{B}^- f = u,\end{cases} \tag{1.1}$$

with $g \in X = L^p(\mathbf{\Omega}, \mathrm{d}\mu)$ and u belonging to the Cessenat trace space Y_p^- (see [5, Section 3.3] for definition) admits a unique solution $f \in \mathscr{D}(\mathcal{T}_{\max,\,p})$ for any $\lambda > 0$.

All these abstract results will be used in the present second part to provide a thorough analysis of a large variety of boundary operators arising in first-order partial differential equations—including unbounded boundary operators, dissipative, conservative and multiplicative boundary operators. Roughly speaking, we aim here to show that there is a C_0-semigroup associated to the initial and boundary value problem

$$\partial_t f(\mathbf{x}, t) + \mathscr{F}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} f(\mathbf{x}, t) = 0 \quad (\mathbf{x} \in \mathbf{\Omega},\ t > 0), \tag{1.2a}$$

supplemented by the abstract boundary condition

$$f_{|\Gamma_-}(\mathbf{y}, t) = H(f_{|\Gamma_+})(\mathbf{y}, t), \quad (\mathbf{y} \in \Gamma_-, t > 0), \tag{1.2b}$$

and the initial condition

$$f(\mathbf{x},0)=f_0(\mathbf{x}), \quad (\mathbf{x}\in\mathbf{\Omega}) \tag{1.2c}$$

for a large class of boundary operators H.

To obtain criteria ensuring the well-posedness of transport equations for conservative and multiplicative boundary conditions, we use a series representation of the solution to the associated initial and boundary value problem introduced by the first author [4] in the L^1 setting. The construction of such series representation is somehow reminiscent of the Dyson–Phillips representation of perturbed semigroups (see [6]) and supports the conjecture that boundary conditions can be seen as "boundary" perturbation of the transport operator with no-reentry boundary conditions (see [1] where we adapted the substochastic theory of additive perturbations of C_0-semigroups to boundary perturbations). We refer to the seminal paper [13] where boundary conditions were already considered as perturbations of semigroup generators. The series approach to Eq. (1.2) allows us to get some semi-explicit expression of the solution to (1.2) like the following (see Corollary 3.14):

$$U_H(t)f(\mathbf{x})=\begin{cases}U_0(t)f(\mathbf{x})=f(\Phi(\mathbf{x},-t)) & \text{if } t<\tau_-(\mathbf{x})\\ \left[H\left(\mathsf{B}^+U_H(t-\tau_-(\mathbf{x}))\right)f\right](\Phi(\mathbf{x},-\tau_-(\mathbf{x}))) & \text{if } t\geqslant\tau_-(\mathbf{x}).\end{cases}$$

which holds for any $f\in\mathscr{D}(\mathcal{T}_{H,p})$ (see the subsequent section for notations). Notice that such a representation was conjectured already by Voigt (see [16, p. 103]) for the free transport case. It has also been proved for a one-dimensional population dynamics problem in L^1 with contractive boundary conditions (see [10, Theorem 2. 3]). We also refer to [9] for the study of a one-dimensional free transport equation with multiplicative boundary conditions in L^p—but the criteria ensuring the well-posedness of the problem obtained there depend on p. We will revisit the result of [9] at the end of the paper and we will also the theoretical results established here to several models of interest for transport on network dealt with in the recent papers [7,8,12]. To the best of our knowledge, for general fields and measures, multiplicative boundary conditions, in the L^p-setting the representation is new and has to be seen as one of the major contributions of the present paper.

The organization of the paper is as follows. In Sect. 2 we deal with a very large class of boundary operators H, notably operators which are not necessarily bounded in the trace spaces $L^p(\Gamma_\pm,\mathrm{d}\mu_\pm)$. This allows in particular to prove the well-posedness of (1.2) for general dissipative operators. Section 3 contains the construction of the series associated to (1.2) in the case of bounded H. This is done through some generalization of the Dyson–Phillips iterates. Notice that if the series is convergent, then a C_0-semigroup solution to (1.2) can be defined. This happens for dissipative boundary operators and for some particular conservative and multiplicative boundary operators. For a multiplicative boundary operator, the sufficient condition ensuring that a C_0-semigroup solution can be defined is the same as in the L^1-setting, and therefore independent of p. Several examples are given at the end of the paper to illustrate the theoretical results obtained in the paper.

2. Generation Properties for Unbounded Boundary Operators

As mentioned in the introduction, the various notations and functional spaces we are dealing with in the present work have been introduced in the first part of the paper [5]. We will deal here with general boundary unbounded operator H acting in the Cessenat trace spaces, namely

$$H\ : \mathscr{D}(H) \subset Y_p^+ \to Y_p^-$$

is a linear unbounded operator with domain $\mathscr{D}(H)$. The graph of H will be denoted $\mathscr{G}(H)$. We assume in this section that

$$\mathscr{G}(H) \subset \mathscr{E} \tag{2.1}$$

where $\mathscr{E}$ has been defined in [5, Section 3.5] as the space of elements $(\psi_+, \psi_-) \in Y_p^+ \times Y_p^-$ such that $\psi_+ - M_\lambda \psi_- \in \widetilde{\mathscr{Y}}_{+,p}$ for some/all $\lambda > 0$. A norm on $\mathscr{E}$ which makes it a Banach space is defined in [5, Eq. (3.17)]. We define now $\mathcal{T}_{H,\,p}$ as $\mathcal{T}_{H,\,p} f = \mathcal{T}_{\max,\,p} f$ for any $f \in \mathscr{D}(\mathcal{T}_{H,\,p})$, where

$$\mathscr{D}(\mathcal{T}_{H,\,p}) = \left\{ f \in \mathscr{D}(\mathcal{T}_{\max,\,p});\, \mathbb{B}f = (\mathsf{B}^+ f, \mathsf{B}^- f) \in \mathscr{G}(H) \right\}.$$

Notice that, thanks to [5, Corollary 3.15], for any $\psi \in \mathscr{D}(H)$, there exists $f \in \mathscr{D}(\mathcal{T}_{H,\,p})$ such that $\mathsf{B}^+ f = \psi$.

From now on, we equip $\mathscr{D}(H)$ with the norm:

$$\|\psi\|_{\mathscr{D}(H)} := \|\psi\|_{Y_p^+} + \|H\psi\|_{Y_p^-} + \|(I - M_\lambda H)\psi\|_{\widetilde{\mathscr{Y}}_{+,\,p}}, \quad \psi \in \mathscr{D}(H), \tag{2.2}$$

which is well defined by (2.1). Then, one has the following result whose proof is exactly the same as [3, Lemma 4.1] where we recall that $\mathscr{W}$ is defined as

$$\mathscr{W} = \{f \in \mathscr{D}(\mathcal{T}_{\max,\,p})\ ;\ \mathsf{B}^- f \in L^p_-\} = \{f \in \mathscr{D}(\mathcal{T}_{\max,\,p})\ ;\ \mathsf{B}^+ f \in L^p_+\}.$$

Lemma 2.1. *The set $\mathscr{D}(\mathcal{T}_{H,\,p}) \cap \mathscr{W}$ is dense in $\mathscr{D}(\mathcal{T}_{H,\,p})$ endowed with the graph norm if and only if $\mathscr{D}(H) \cap L^p_+$ is dense in $\mathscr{D}(H)$. Moreover, for any $\lambda > 0$, the following are equivalent:*

(1) $[I - M_\lambda H]\, \mathscr{D}(H) = \widetilde{\mathscr{Y}}_{+,\,p}$;
(2) $\mathrm{Ran}(\lambda - \mathcal{T}_{H,\,p}) = X$.

Proof. The proof of the first point is exactly the same as the one of [3, Lemma 4.1] while the proof of the second one is exactly the one of [3, Lemma 4.2]. □

We provide now necessary and sufficient conditions on H so that $\mathcal{T}_{H,\,p}$ generates a C_0-semigroup of contractions in X. Our result generalizes [11, Theorem 3, p. 254] in the context of L^p-spaces but with general external field $\mathscr{F}$ and Radon measure μ.

Theorem 2.2. *Let $H : \mathscr{D}(H) \subset Y_p^+ \to Y_p^-$ be such that*

(1) *The graph $\mathscr{G}(H)$ of H is a closed subspace of $\mathscr{E}$.*
(2) *The range $Ran(I - M_\lambda H)$ is a dense subset of $\widetilde{\mathscr{Y}}_{+,\,p}$.*

(3) *There is some positive constant* $C>0$ *such that*

$$\|(I-M_\lambda H)\psi_+\|_{\widetilde{\mathcal{Y}}_{+,\,p}} \geqslant C\left(\|\psi_+\|_{Y_p^+} + \|H\psi_+\|_{Y_p^-}\right), \quad \forall \psi_+ \in \mathscr{D}(H). \tag{2.3}$$

(4) $\mathscr{D}(H)\cap L^p_+$ *is dense in* $\mathscr{D}(H)$ *endowed with the norm* (2.2).
(5) *The restriction of* H *to* L^p_+ *is a contraction, i.e.*

$$\|H\psi\|_{L^p_-} \leqslant \|\psi\|_{L^p_+}, \quad \forall \psi \in \mathscr{D}(H)\cap L^p_+. \tag{2.4}$$

Then, $\mathcal{T}_{H,\,p}$ *generates a* C_0*-semigroup of contractions in* X*. Conversely, if* $\mathcal{T}_{H,\,p}$ *generates a* C_0*-semigroup of contractions and* $\mathscr{D}(\mathcal{T}_{H,\,p})\cap\mathscr{W}$ *is dense in* $\mathscr{D}(\mathcal{T}_{H,\,p})$ *endowed with the graph norm, then* H *satisfies assumptions (1)–(5).*

Proof. Assume (1)–(5) to hold. Let $f\in\mathscr{D}(\mathcal{T}_{H,\,p})\cap\mathscr{W}$. Setting $g=(\lambda-\mathcal{T}_{H,\,p})f$, one sees that f solves the boundary value problem (1.1) with $u=H\mathsf{B}^+f$ and, from [5, Eq. (3.11)],

$$\begin{aligned}\lambda p\|f\|_p^p - p\|(\lambda-\mathcal{T}_{H,\,p})f\|_p\,\|f\|_X^{p-1} &\leqslant \|\mathsf{B}^-f\|_{L^p_-} - \|\mathsf{B}^+f\|_{L^p_+}\\ &= \|H(\mathsf{B}^+f)\|_{L^p_-} - \|\mathsf{B}^+f\|_{L^p_+} \leqslant 0.\end{aligned}$$

Thus, $\lambda\|f\|_p^p \leqslant \|(\lambda-\mathcal{T}_{H,\,p})f\|_p\,\|f\|_p^{p-1}$. This shows that $\lambda\|f\|_p \leqslant \|(\lambda-\mathcal{T}_{H,\,p})f\|_p$, i.e., $\mathcal{T}_{H,\,p}$ is dissipative over $\mathscr{W}$. From (4) and Lemma 2.1, it is clear that $\mathcal{T}_H$ is dissipative over $\mathscr{D}(\mathcal{T}_{H,\,p})$. Now, according to (1), one sees that $\mathscr{D}(H)$ equipped with the norm (2.2) is a Banach space. Moreover, for any $\lambda>0$, $I-M_\lambda H$ is continuous from $\mathscr{D}(H)$ into $\widetilde{\mathcal{Y}}_{+,\,p}$ and (2)–(3) imply that it is invertible with continuous inverse. In particular, since $\mathrm{Ran}(I-M_\lambda H)=\widetilde{\mathcal{Y}}_{+,\,p}$, Lemma 2.1 implies that $\mathrm{Ran}(\lambda-\mathcal{T}_{H,\,p})=X$ so that the Lumer–Phillips Theorem [15, p. 14] can be applied to state that $\mathcal{T}_{H,\,p}$ generates a C_0-semigroup of contractions in X.

Conversely, assume that $\mathcal{T}_{H,\,p}$ generates a C_0-semigroup of contractions and $\mathscr{D}(\mathcal{T}_{H,\,p})\cap\mathscr{W}$ is dense in $\mathscr{D}(\mathcal{T}_{H,\,p})$ endowed with the graph norm. According to the Lumer–Phillips Theorem, for any $f\in\mathscr{D}(\mathcal{T}_{H,\,p})$ and any $g\in L^q(\mathbf{\Omega},\mathrm{d}\mu)$ with $\int_{\mathbf{\Omega}} f(\mathbf{x})g(\mathbf{x})\mathrm{d}\mu(\mathbf{x})=\|f\|_p^p$, one has

$$\int_{\mathbf{\Omega}} g(\mathbf{x})\mathcal{T}_{H,\,p}f(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) \leqslant 0.$$

Then, for any $f\in\mathscr{D}(\mathcal{T}_{H,\,p})\cap\mathscr{W}$, choosing $g=\mathrm{sign}f\,|f|^{p-1}$, [5, Theorem 2.8] ensures that $g\,\mathcal{T}_{H,\,p}f=\frac{1}{p}\mathcal{T}_{H,\,1}(|f|^p)$ so that

$$\begin{aligned}0\geqslant p\int_{\mathbf{\Omega}}\mathcal{T}_{H,\,p}f(\mathbf{x})g(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) &= \int_{\mathbf{\Omega}}\mathcal{T}_{H,\,1}(|f|^p)(\mathbf{x})\mathrm{d}\mu(\mathbf{x})\\ &= \int_{\Gamma_-}|\mathsf{B}^-f(\mathbf{x})|^p\mathrm{d}\mu_-(\mathbf{x}) - \int_{\Gamma_+}|\mathsf{B}^+f(\mathbf{x})|^p\mathrm{d}\mu_+(\mathbf{x})\\ &= \|H\left(\mathsf{B}^+f\right)\|_{L^p_-}^p - \|\mathsf{B}^+f\|_{L^p_+}^p\end{aligned}$$

where we used Green's formula and the fact that $\mathsf{B}^\pm|f|^p=|\mathsf{B}^\pm f|^p$. This proves that (2.4) holds for all $f\in\mathscr{D}(\mathcal{T}_{H,\,p})\cap\mathscr{W}$. The rest of the proof is as in [3, Theorem 4.1]. $\square$

Remark 2.3. As will be seen later, if $H \in \mathscr{B}(L^p_+, L^p_-)$ is a bounded boundary operator, a practical criterion ensuring that H satisfies the above properties (1)–(5) is simply that $\|H\|_{\mathscr{B}(L^p_+, L^p_-)} < 1$ (see Proposition 3.2 hereafter). We wish to point out, however, that the above Theorem 2.2 is more general since it allows to treat the case of an unbounded boundary operator $H : \mathscr{D}(H) \subset L^p_+ \to L^p_-$.

3. Explicit Transport Semigroup for Bounded Boundary Operators

We provide in this section a general and explicit construction of the transport operator associated to bounded boundary operators. Namely, we shall analyse from now on transport equations with boundary operators $H \in \mathscr{B}(L^p_+, L^p_-)$. We denote for simplicity

$$|||H||| = \|H\|_{\mathscr{B}(L^p_+, L^p_-)}.$$

We introduce the associated transport operator $\mathcal{T}_{H,\,p}$:

$$\mathcal{T}_{H,\,p}\psi = \mathcal{T}_{\max,\,p}\psi, \qquad \text{for any } \psi \in \mathscr{D}(\mathcal{T}_{H,\,p}),$$

where the domain $\mathscr{D}(\mathcal{T}_{H,\,p})$ is defined by

$$\mathscr{D}(\mathcal{T}_{H,\,p}) = \{\psi \in \mathscr{D}(\mathcal{T}_{\max,\,p});\ \mathsf{B}^+\psi \in L^p_+ \quad \text{and} \quad \mathsf{B}^-\psi = H\mathsf{B}^+\psi\}.$$

3.1. About the Resolvent of $\mathcal{T}_H$

In the case we are considering, $M_\lambda H \in \mathscr{B}(L^p_+)$ for any $\lambda > 0$. We begin with the following result:

Proposition 3.1. *Assume that $H \in \mathscr{B}(L^p_+, L^p_-)$. Let $\lambda > 0$ be given such that $I - M_\lambda H \in \mathscr{B}(L^p_+)$ is invertible. Then, $(\lambda - \mathcal{T}_{H,\,p})$ is invertible and*

$$(\lambda - \mathcal{T}_{H,\,p})^{-1} = C_\lambda + \Xi_\lambda H\,(I - M_\lambda H)^{-1}\,G_\lambda. \tag{3.1}$$

Proof. Let $\lambda > 0$ be such that $I - M_\lambda H$ is invertible. Given $g \in X$, we wish to solve the resolvent equation

$$(\lambda - \mathcal{T}_{H,\,p})f = g \tag{3.2}$$

for $f \in \mathscr{D}(\mathcal{T}_{H,\,p})$. This means that f solves the boundary value problem $(\lambda - \mathcal{T}_{\max,\,p})f = g$ with $\mathsf{B}^-f = H\mathsf{B}^+f$. If such a solution exists, it is given by

$$f = C_\lambda g + \Xi_\lambda \mathsf{B}^- f \tag{3.3}$$

and therefore, taking the trace over Γ_+:

$$\mathsf{B}^+f = \mathsf{B}^+C_\lambda g + \mathsf{B}^+\Xi_\lambda\mathsf{B}^-f = G_\lambda g + M_\lambda\mathsf{B}^-f = G_\lambda g + M_\lambda H\mathsf{B}^+f.$$

Since $\mathsf{B}^+f \in L^p_+$ and $I - M_\lambda H$ is invertible, we get that B^+f is given by

$$\mathsf{B}^+f = (I - M_\lambda H)^{-1}G_\lambda g. \tag{3.4}$$

Then, inserting $\mathsf{B}^-f = H\mathsf{B}^+f$ into (3.3), we get that, if the resolvent equation (3.2) admits a solution, this solution is necessarily $f = C_\lambda g + \Xi_\lambda H(I - M_\lambda H)^{-1}G_\lambda g$. Now, for any $g \in X$ and $\lambda > 0$, we know that $f_1 := C_\lambda g \in$

$\mathscr{D}(\mathcal{T}_{\max,p})$ with $(\lambda - \mathcal{T}_{\max,p})f_1 = g$. Since $G_\lambda g \in L^p_+$ one has $f_2 := \Xi_\lambda H(I - M_\lambda H)^{-1}G_\lambda g$ is well defined and belongs to X. Moreover, according to [5, Lemma 3.5], $f_2 \in \mathscr{D}(\mathcal{T}_{\max,p})$ with $\mathcal{T}_{\max,p} f_2 = \lambda f_2$. This shows that, $f_0 := f_1 + f_2 \in \mathscr{D}(\mathcal{T}_{\max,p})$ with $(\lambda - \mathcal{T}_{\max,p})f_0 = g$. Moreover, still using [5, Lemma 3.5], $\mathsf{B}^- f_1 = 0$ while $\mathsf{B}^- f_2 = H(I - M_\lambda H)^{-1}G_\lambda g \in L^p_-$, i.e.,

$$\mathsf{B}^- f_0 = H(I - M_\lambda H)^{-1}G_\lambda g.$$

On the other side, $\mathsf{B}^+ f_1 = G_\lambda g$ while $\mathsf{B}^+ f_2 = M_\lambda H(I - M_\lambda H)^{-1}G_\lambda g$ where we used again [5, Lemma 3.5]. Thus, $\mathsf{B}^+ f_0 = G_\lambda g + M_\lambda H(I - M_\lambda H)^{-1}G_\lambda g$ from which we deduce easily that

$$\mathsf{B}^+ f_0 = (I - M_\lambda H)^{-1}G_\lambda g.$$

Hence, $\mathsf{B}^- f_0 = H\mathsf{B}^+ f_0$ and $f_0 \in \mathscr{D}(\mathcal{T}_{H,p})$. This proves that f_0 is indeed a solution to (3.2) and the proof is achieved. □

Now we can prove the following

Proposition 3.2. *If $\|\!|H|\!\| < 1$, then the operator $(\mathcal{T}_{H,p}, \mathscr{D}(\mathcal{T}_{H,p}))$ is the generator of a C_0-semigroup of contractions $(U_H(t))_{t\geqslant 0}$ in X.*

Proof. First of all we prove that $(\mathcal{T}_{H,p}, \mathscr{D}(\mathcal{T}_{H,p}))$ is a closed operator. Let $(f_n)_n \subset \mathscr{D}(\mathcal{T}_{H,p})$ and $f, g \in X$ be such that $\lim_n \|f - f_n\|_p = \lim_n \|\mathcal{T}_{H,p} f_n - g\|_p = 0$. We have to prove that $f \in \mathscr{D}(\mathcal{T}_{H,p})$ with $\mathcal{T}_{H,p} f = g$. Using the fact that $\mathcal{T}_{\max,p}$ is closed (see [5, Remark 2.5]) and $\mathscr{D}(T_{H,p}) \subset \mathscr{D}(\mathcal{T}_{\max,p})$ we get that $f \in \mathscr{D}(\mathcal{T}_{\max,p})$ with $\mathcal{T}_{\max,p} f = g$. To prove the result, we "only" have to prove that $\mathsf{B}^- f \in L^p_-$, $\mathsf{B}^+ f \in L^p_+$ and $\mathsf{B}^- f = H\mathsf{B}^+ f$. First, according to Green's formula one has, for any $n, m \geqslant 1$

$$\begin{aligned}&\left|\|\mathsf{B}^- f_n - \mathsf{B}^- f_m\|^p_{L^p_-} - \|\mathsf{B}^+ f_n - \mathsf{B}^+ f_m\|^p_{L^p_+}\right| \\ &\qquad \leqslant p\|f_n - f_m\|_p^{p-1}\,\|\mathcal{T}_{H,p} f_n - \mathcal{T}_{H,p} f_m\|_p.\end{aligned}$$

Since $\|\!|H|\!\| < 1$, we have

$$\begin{aligned}&\left|\|\mathsf{B}^- f_n - \mathsf{B}^- f_m\|^p_{L^p_-} - \|\mathsf{B}^+ f_n - \mathsf{B}^+ f_m\|^p_{L^p_+}\right| \\ &\qquad \geqslant (1 - \|\!|H|\!\|^p)\|\mathsf{B}^+ f_n - \mathsf{B}^+ f_m\|^p_{L^p_+}.\end{aligned}$$

In particular, $(\mathsf{B}^+ f_n)_n$ is a Cauchy sequence in L^p_+ so it converges in L^p_+. But, according to [5, Remark 3.4], if $(f_n)_n \subset \mathscr{D}(\mathcal{T}_{\max,p})$ is such that $\lim_n (\|f_n - f\|_p + \|\mathcal{T}_{\max,p} f_n - \mathcal{T}_{\max,p} f\|_p) = 0$ then $(\mathsf{B}^+ f_n)_n$ converges to $\mathsf{B}^+ f$ in Y^+_p. Therefore the only possible limit of $(\mathsf{B}^+ f_n)_n$ in $L^p_+$is $\mathsf{B}^+ f$, i.e $\lim_n \|\mathsf{B}^+ f_n - \mathsf{B}^+ f\|_{L^p_+} = 0$. Since H is a bounded operator, we deduce that $\lim_n \|\mathsf{B}^- f_n - \mathsf{B}^- f\|_{L^p_-} = 0$ and $H\mathsf{B}^+ f = \mathsf{B}^- f$, i.e. $f \in \mathscr{D}(\mathcal{T}_{H,p})$.

Let us now prove that $\mathscr{D}(\mathcal{T}_{H,p})$ is dense in X. Notice that

$$\mathscr{D}_0 := \{\psi \in \mathscr{D}(\mathcal{T}_{\max,p});\ \mathsf{B}^-\psi = \mathsf{B}^+\psi = 0\} \subset \mathscr{D}(\mathcal{T}_{H,p}).$$

Now, since the set of continuously differentiable and compactly supported functions $\mathcal{C}^1_0(\mathbf{\Omega})$ is dense in X and $\mathcal{C}^1_0(\mathbf{\Omega}) \subset \mathscr{D}_0$, we get the desired result.

Finally when H is a strict contraction, one has $\|M_\lambda H\|_{\mathscr{B}(L^p_+)} < 1$ for any $\lambda > 0$ which, thanks to Hadamard's criterion, ensures that $(I - M_\lambda H)$

is invertible with inverse $(I - M_\lambda H)^{-1} = \sum_{n=0}^{\infty}(M_\lambda H)^n$ for any $\lambda > 0$. Then according to Proposition 3.1 $(\lambda - \mathcal{T}_{H,\,p})$ is invertible and Equation 3.1 holds. Furthermore thanks to [5, Eq. (3.11)], exactly as in the proof of Theorem 2.2 we can state $\lambda \|f\|_p^p \leqslant \|(\lambda - \mathcal{T}_{H,\,p})f\|_p \, \|f\|_p^{p-1}$. This implies $\|(\lambda - \mathcal{T}_{H,\,p})^{-1}f\|_p \leqslant \frac{1}{\lambda}\|f\|_p$ for any $\lambda > 0$ so that the proof is achieved. □

3.2. Boundary Dyson–Phillips Iterated Operators

Introduce as earlier the set

$$\mathscr{D}_0 := \{\psi \in \mathscr{D}(\mathcal{T}_{\max,\,p});\ \mathsf{B}^-\psi = \mathsf{B}^+\psi = 0\} \subset \mathscr{D}(\mathcal{T}_{H,\,p}).$$

Recall now that, from [5, Theorem 3.1], $(\mathcal{T}_{0,\,p}, \mathscr{D}(\mathcal{T}_{0,\,p}))$ generates a C_0-semigroup $(U_0(t))_{t\geqslant 0}$ in X given by

$$U_0(t)f(\mathbf{x}) = f(\Phi(\mathbf{x}, -t))\chi_{\{t<\tau_-(\mathbf{x})\}}(\mathbf{x}), \quad (\mathbf{x} \in \mathbf{\Omega},\ f \in X), \tag{3.5}$$

where χ_A is the characteristic function of the measurable set A. Notice that, for any $f \in \mathscr{D}_0$,

$$U_0(t)f \in \mathscr{D}(\mathcal{T}_{0,\,p}) \quad \forall t \geqslant 0.$$

In particular, $\mathsf{B}^\pm U_0(t)f \in Y_p^\pm$. We set

$$\mathcal{I}_t^0[f] = \int_0^t U_0(s)f\mathrm{d}s, \quad \forall t \geqslant 0,\ f \in X.$$

Recall that, as a general property of C_0-semigroups (see for instance [15, Theorem 2.4.]):

$$\mathcal{I}_t^0[f] \in \mathscr{D}(\mathcal{T}_{0,\,p}) \quad \text{with} \quad \mathcal{T}_{0,\,p}\mathcal{I}_t^0[f] = U_0(t)f - f.$$

One has the following

Proposition 3.3. *For any $f \in X$ and any $t > 0$, the traces $\mathsf{B}^\pm\mathcal{I}_t^0[f] \in L_\pm^p$ and the mappings $t \geqslant 0 \mapsto \mathsf{B}^\pm\mathcal{I}_t^0[f] \in L_\pm^p$ are continuous. Moreover*

$$\left\|\mathsf{B}^+\left(\mathcal{I}_{t+h}^0[f] - \mathcal{I}_t^0[f]\right)\right\|_{L_+^p}^p \leqslant h^{p-1}\left|\,\|U_0(t+h)f\|_p^p - \|U_0(t)f\|_p^p\,\right| \quad \forall h > 0. \tag{3.6}$$

Proof. For $f \in X$, since $\mathcal{I}_t^0[f] \in \mathscr{D}(\mathcal{T}_{0,\,p})$, one has $\mathsf{B}^-\mathcal{I}_t^0[f] = 0$. In particular, the trace $\mathsf{B}^-\mathcal{I}_t^0[f]$ belongs to L_-^p with the mapping

$$0 \leqslant t \mapsto \mathsf{B}^-\mathcal{I}_t^0[f] \in L_-^p$$

continuous. According to Proposition A.1, $\mathsf{B}^+\mathcal{I}_t^0[f] \in L_+^p$ for all $t \geqslant 0$ and the mapping $t \geqslant 0 \mapsto \mathsf{B}^+\mathcal{I}_t^0[f] \in L_+^p$ is continuous. Given $0 \leqslant t < t+h$, one has

$$\begin{aligned}\left\|\mathsf{B}^+\left(\mathcal{I}_{t+h}^0[f] - \mathcal{I}_t^0[f]\right)\right\|_{L_+^p}^p &= \int_{\Gamma_+}\left|\int_t^{t+h} f(\Phi(\mathbf{z}, -s))\chi_{\{s<\tau_-(\mathbf{z})\}}\mathrm{d}s\right|^p \mathrm{d}\mu_+(\mathbf{z})\\ &\leqslant h^{p-1}\int_{\Gamma_+}\int_t^{t+h} |f(\Phi(\mathbf{z}, -s))|^p\chi_{\{s<\tau_-(\mathbf{z})\}}\mathrm{d}s\end{aligned} \tag{3.7}$$

where we used Hölder's inequality in the last inequality. One recognizes then, thanks to [5, Equation (3.2)], that the last integral in the above inequality coincides with $\|U_0(t)f\|_p^p - \|U_0(t+h)f\|_p^p$. This proves (3.6). □

Remark 3.4. Notice that, for $t=0$, (3.6) becomes

$$\|\mathsf{B}^+\mathcal{I}_h^0[f]\|_{L^p_+} \leqslant h^{1-1/p}\|f\|_p, \quad \forall h>0. \tag{3.8}$$

One also has

Proposition 3.5. *For any $f \in \mathscr{D}_0$, any $t \geqslant 0$, the traces $\mathsf{B}^\pm U_0(t)f \in L^p_\pm$ and the mappings $t \geqslant 0 \mapsto \mathsf{B}^\pm U_0(t)f \in L^p_\pm$ are continuous with*

$$\int_0^t \|\mathsf{B}^+U_0(s)f\|^p_{L^p_+}\mathrm{d}s = \|f\|_p^p - \|U_0(t)f\|_p^p, \quad \forall t \geqslant 0.$$

In particular,

$$\int_0^t \|\mathsf{B}^+U_0(s)f\|^p_{L^p_+}\mathrm{d}s \leqslant \frac{t^p}{p}\int_0^t\left(\int_{\Gamma_+}|[\mathcal{T}_{\max,\,p}f](\Phi(\mathbf{z},-s))|^p\,\chi_{\{s<\tau_-(\mathbf{z})\}}\mathrm{d}\mu_+(\mathbf{z})\right)\mathrm{d}s. \tag{3.9}$$

Proof. For $f \in \mathscr{D}_0$, we have $U_0(t)f - f = \mathcal{I}_t^0[\mathcal{T}_{0,\,p}f]$, and therefore $\mathsf{B}^\pm U_0(t)f = \mathsf{B}^\pm\mathcal{I}_t^0[\mathcal{T}_{0,\,p}f]$. Thanks to Proposition 3.3 we can state that the traces $\mathsf{B}^\pm U_0(t)f \in L^p_\pm$ and the mappings $0 \leqslant t \mapsto \mathsf{B}^\pm U_0(t)f \in L^p_\pm$ are continuous. Then

$$\int_0^t \|\mathsf{B}^+U_0(s)f\|^p_{L^p_+}\mathrm{d}s = \int_0^t\left(\int_{\Gamma_+}|f(\Phi(\mathbf{z},-s))|^p\chi_{\{s<\tau_-(\mathbf{z})\}}\mathrm{d}\mu_+(\mathbf{z})\right)\mathrm{d}s$$

so that, thanks to Fubini's Theorem

$$\int_0^t \|\mathsf{B}^+U_0(s)f\|^p_{L^p_+}\mathrm{d}s = \int_{\Gamma_+}\left(\int_0^t|f(\Phi(\mathbf{z},-s))|^p\chi_{\{s<\tau_-(\mathbf{z})\}}\mathrm{d}s\right)\mathrm{d}\mu_+(\mathbf{z})$$

which, using again [5, Eq. (3.2)], gives the first part of the Proposition. Let us now prove (3.9). Using [5, Eq. (2.9)], one has from the previous identity that

$$\int_{\Gamma_+}\left(\int_0^t|f(\Phi(\mathbf{z},-s))|^p\chi_{\{s<\tau_-(\mathbf{z})\}}\mathrm{d}s\right)\mathrm{d}\mu_+(\mathbf{z}) = \int_{\Gamma_+}\left(\int_0^t\left|\int_0^s\mathcal{T}_{\max,\,p}f(\Phi(\mathbf{z},-r))\mathrm{d}r\right|^p\chi_{\{s<\tau_-(\mathbf{z})\}}\mathrm{d}s\right)\mathrm{d}\mu_+(\mathbf{z}).$$

Then, since, for almost every $\mathbf{z} \in \Gamma_+$ and any $s \in (0,t)$:

$$\left|\int_0^s\mathcal{T}_{\max,\,p}f(\Phi(\mathbf{z},-r))\mathrm{d}r\right|^p \leqslant s^{p-1}\int_0^s|\mathcal{T}_{\max,\,p}f(\Phi(\mathbf{z},-r))|^p\,\mathrm{d}r \leqslant s^{p-1}\int_0^t|\mathcal{T}_{\max,\,p}f(\Phi(\mathbf{z},-r))|^p\,\mathrm{d}r$$

we get the result after integrating with respect to s over $(0,t)$. □

We are now in position to define inductively the following:

Definition 3.6. Let $t \geqslant 0$, $k \geqslant 1$ and $f \in \mathscr{D}_0$ be given. For $\mathbf{x} \in \mathbf{\Omega}$, we define

$$[U_k(t)f](\mathbf{x}) = \left[H\mathsf{B}^+U_{k-1}(t-s)f\right](\mathbf{y}) \qquad \text{if } t \geqslant \tau_-(\mathbf{x}),$$
$$[U_k(t)f](\mathbf{x}) = 0 \qquad \text{if } 0 < t \leqslant \tau_-(\mathbf{x}),$$

and

$$U_k(0)f = 0,$$

where $\mathbf{y} \in \Gamma_-$ and $s \in (0, \min(t, \tau_+(\mathbf{y}))$ are the unique elements such that $\mathbf{x} = \Phi(\mathbf{y}, s)$ where $t \geqslant \tau_-(\mathbf{x})$.

Remark 3.7. Clearly, for $\mathbf{x} \in \mathbf{\Omega}$ with $\tau_-(\mathbf{x}) < t$, the unique $\mathbf{y} \in \Gamma_-$ and $s \in (0, \min(t, \tau_+(\mathbf{y}))$ such that $\mathbf{x} = \Phi(\mathbf{y}, s)$ are

$$\mathbf{y} = \Phi(\mathbf{x}, -\tau_-(\mathbf{x})), \quad s = \tau_-(\mathbf{x})$$

so that the above definition reads

$$[U_k(t)f](\mathbf{x}) = \left[H(\mathsf{B}^+U_{k-1}(t-\tau_-(\mathbf{x}))f\right](\Phi(\mathbf{x}, -\tau_-(\mathbf{x}))).$$

The fact that, with this definition, $(U_k(t))_{t\geqslant 0}$ is a well-defined family which extends to a family of operators in $\mathscr{B}(X)$ satisfying the following, is given in the Appendix.

Theorem 3.8. *For any $k \geqslant 1$, $f \in \mathscr{D}_0$ one has $U_k(t)f \in X$ for any $t \geqslant 0$ with*

$$\|U_k(t)f\|_p \leqslant |||H|||^k \, \|f\|_p.$$

In particular, $U_k(t)$ can be extended to a bounded linear operator, still denoted $U_k(t) \in \mathscr{B}(X)$ with

$$\|U_k(t)\|_{\mathscr{B}(X)} \leqslant |||H|||^k \quad \forall t \geqslant 0, k \geqslant 1.$$

Moreover, the following holds for any $k \geqslant 1$

(1) *$(U_k(t))_{t\geqslant 0}$ is a strongly continuous family of $\mathscr{B}(X)$.*
(2) *For any $f \in X$ and any $t, s \geqslant 0$, it holds*

$$U_k(t+s)f = \sum_{j=0}^{k} U_j(t)U_{k-j}(s)f.$$

(3) *For any $f \in \mathscr{D}_0$, the mapping $0 \leqslant t \mapsto U_k(t)f$ is differentiable with*

$$\frac{\mathrm{d}}{\mathrm{d}t}U_k(t)f = U_k(t)\mathcal{T}_{\max,\,p}f \quad \forall t \geqslant 0.$$

(4) *For any $f \in \mathscr{D}_0$, one has $U_k(t)f \in \mathscr{D}(\mathcal{T}_{\max,\,p})$ for all $t \geqslant 0$ with $\mathcal{T}_{\max,\,p}U_k(t)f = U_k(t)\mathcal{T}_{\max,\,p}f$.*
(5) *For any $f \in X$ and any $t > 0$, one has*

$$\mathcal{I}_t^k[f] := \int_0^t U_k(s)f\mathrm{d}s \in \mathscr{D}(\mathcal{T}_{\max,\,p}) \quad \textit{with} \quad \mathcal{T}_{\max,\,p}\mathcal{I}_t^k[f] = U_k(t)f.$$

(6) *For any* $f \in \mathscr{D}_0$ *and any* $t \geqslant 0$, *the traces* $\mathsf{B}^{\pm}U_k(t)f \in L^p_{\pm}$ *and the mappings* $0 \leqslant t \mapsto \mathsf{B}^{\pm}U_k(t)f \in L^p_{\pm}$ *are continuous. Moreover, for all* $f \in X$ *and* $t > 0$, *one has*

$$\mathsf{B}^{\pm}\int_0^t U_k(s)f\mathrm{d}s \in L^p_{\pm} \quad \textit{with} \quad \mathsf{B}^{-}\int_0^t U_k(s)f\mathrm{d}s = H\mathsf{B}^{+}\int_0^t U_{k-1}(s)f\mathrm{d}s.$$

(7) *For any* $f \in \mathscr{D}_0$, *it holds*

$$\int_0^t \|\mathsf{B}^{+}U_k(s)f\|^p_{L^p_+}\mathrm{d}s \leqslant \|\!|H|\!\|^p \int_0^t \|\mathsf{B}^{+}U_{k-1}(s)f\|^p_{L^p_+}\mathrm{d}s, \quad \forall t \geqslant 0.$$

(8) *For any* $f \in X$ *and* $\lambda > 0$, *setting* $F_k = \int_0^\infty \exp(-\lambda t)U_k(t)f\mathrm{d}t$ *one has*

$$F_k \in \mathscr{D}(\mathcal{T}_{\max,\,p}) \quad \textit{with} \quad \mathcal{T}_{\max,\,p}F_k = \lambda F_k$$

and $\mathsf{B}^{\pm}F_k \in L^p_{\pm}$ *with*

$$\mathsf{B}^{-}F_k = H\mathsf{B}^{+}F_{k-1} \quad \mathsf{B}^{+}F_k = (M_\lambda H)^k G_\lambda f.$$

3.3. Generation Theorem

Introduce the following truncation operator

Definition 3.9. For any $\delta > 0$, introduce

$$\Gamma_+^\delta := \{\mathbf{z} \in \Gamma_+ \ ; \ \tau_-(\mathbf{z}) > \delta\}$$

and define the following truncation operator $\chi_\delta \in \mathscr{B}(L^p_+)$ given by

$$[\chi_\delta \psi](\mathbf{z}) = \psi(\mathbf{z})\chi_{\Gamma_+ \setminus \Gamma_+^\delta}(\mathbf{z}), \quad \forall \mathbf{z} \in \Gamma_+;\ \psi \in L^p_+.$$

One has then the following

Lemma 3.10. *Assume that* $H \in \mathscr{B}(L^p_+, L^p_-)$. *Then, with the notations of Theorem* 3.8,

$$\left(\int_0^t \|\mathsf{B}^{+}U_k(s)f\|^p_{L^p_+}\mathrm{d}s\right)^{1/p} \leqslant \sum_{j=0}^{\min(k,[t/\delta]+1)} \binom{k}{j} \|\!|H\chi_\delta|\!\|^{k-j}\|\!|H|\!\|^j \,\|f\|_p$$

for any $f \in \mathscr{D}_0$, *and any* $t > 0$. (3.10)

while

$$\|U_k(t)\|_{\mathscr{B}(X)} \leqslant \sum_{j=0}^{\min(k,[t/\delta]+1)} \binom{k}{j} \|\!|H\chi_\delta|\!\|^{k-j}\|\!|H|\!\|^j. \tag{3.11}$$

Proof. For $k = 0$, both inequalities are clearly true. Let $k \geqslant 1$ and $f \in \mathscr{D}_0, t > 0$ be given. For $0 \leqslant s \leqslant t$, one has for μ_+-a.e. $\mathbf{z} \in \Gamma_+$:

$$\left[\mathsf{B}^{+}U_k(s)f\right](\mathbf{z}) = \left[H\left(\mathsf{B}^{+}U_{k-1}(s-\tau_-(\mathbf{z}))\right)f\right](\Phi(\mathbf{z},-\tau_-(\mathbf{z})))\chi_{(0,s)}(\tau_-(\mathbf{z}))$$

Thus, writing $H = H\chi_\delta + H(\mathbf{I}-\chi_\delta)$, we can estimate

$$\left(\int_0^t \|\mathsf{B}^{+}U_k(s)f\|^p_{L^p_+}\mathrm{d}s\right)^{\frac{1}{p}} \leqslant J_1 + J_2$$

with

$$J_1^p = \int_0^t \left(\int_{\Gamma_+} \left| \left[H\chi_\delta \left(\mathsf{B}^+ U_{k-1}(s-\tau_-(\mathbf{z})) \right) f \right] \right. \right.$$

$$\left. \left. (\Phi(\mathbf{z}, -\tau_-(\mathbf{z}))) \right|^p \chi_{(0,s)}(\tau_-(\mathbf{z})) \mathrm{d}\mu_+(\mathbf{z}) \right) \mathrm{d}s$$

$$J_2^p = \int_0^t \left(\int_{\Gamma_+} \left| \left[H(\mathbf{I} - \chi_\delta) \left(\mathsf{B}^+ U_{k-1}(s-\tau_-(\mathbf{z})) \right) f \right] \right. \right.$$

$$\left. \left. (\Phi(\mathbf{z}, -\tau_-(\mathbf{z}))) \right|^p \chi_{(0,s)}(\tau_-(\mathbf{z})) \mathrm{d}\mu_+(\mathbf{z}) \right) \mathrm{d}s.$$

As in the proof of Lemma A.8, one has

$$J_1^p \leqslant |||H\chi_\delta|||^p \int_0^t \|\mathsf{B}^+ U_{k-1}(s) f\|_{L^p_+}^p \mathrm{d}s$$

and

$$\begin{aligned} J_2^p &\leqslant \int_0^t \left(\int_{\Gamma_-} \left| \left[H(\mathbf{I} - \chi_\delta) \left(\mathsf{B}^+ U_{k-1}(s) \right) f \right] (\mathbf{y}) \right|^p \mathrm{d}\mu_-(\mathbf{y}) \right) \mathrm{d}s \\ &\leqslant |||H|||^p \int_0^t \left(\int_{\Gamma_+^\delta} \left| \left[\mathsf{B}^+ U_{k-1}(s) f \right] (\mathbf{z}) \right|^p \mathrm{d}\mu_+(\mathbf{z}) \right) \mathrm{d}s \\ &= |||H|||^p \int_0^t \|\mathsf{B}^+ U_{k-1}(s) f\|_{L^p(\Gamma_+^\delta, \mathrm{d}\mu_+)}^p \mathrm{d}s. \end{aligned}$$

Introduce now the following quantities, where we recall that $\delta > 0$ is fixed: let $C_\delta = |||H\chi_\delta|||$, $A = |||H|||$ and, for any $k \geqslant 1$,

$$S_k(t) = \left(\int_0^t \|\mathsf{B}^+ U_k(s) f\|_{L^p_+}^p \mathrm{d}s \right)^{1/p}, \quad Z_k(t) = \left(\int_0^t \|\mathsf{B}^+ U_k(s) f\|_{L^p(\Gamma_+^\delta, \mathrm{d}\mu_+)}^p \mathrm{d}s \right)^{1/p}.$$

One proved already that

$$S_k(t) \leqslant C_\delta\, S_{k-1}(t) + A\, Z_{k-1}(t), \quad \forall k \geqslant 1. \tag{3.12}$$

Let us now estimate inductively $Z_k(t)$. Assume $t > \delta$. One has, as before, splitting H as $H = H\chi_\delta + H(\mathbf{I} - \chi_\delta)$:

$$Z_k(t) \leqslant \mathcal{J}_1 + \mathcal{J}_2$$

with

$$\mathcal{J}_1^p = \int_0^t \left(\int_{\Gamma_+^\delta} \left| \left[H\chi_\delta \left(\mathsf{B}^+ U_{k-1}(s-\tau_-(\mathbf{z})) \right) f \right] \right. \right.$$

$$\left. \left. (\Phi(\mathbf{z}, -\tau_-(\mathbf{z}))) \right|^p \chi_{(0,s)}(\tau_-(\mathbf{z})) \mathrm{d}\ \mu_+(\mathbf{z}) \right) \mathrm{d}s$$

$$\mathcal{J}_2^p = \int_0^t \left(\int_{\Gamma_+^\delta} \left| \left[H(\mathbf{I} - \chi_\delta) \left(\mathsf{B}^+ U_{k-1}(s-\tau_-(\mathbf{z})) \right) f \right] \right. \right.$$

$$\left. \left. (\Phi(\mathbf{z}, -\tau_-(\mathbf{z}))) \right|^p \chi_{(0,s)}(\tau_-(\mathbf{z})) \mathrm{d}\mu_+(\mathbf{z}) \right) \mathrm{d}s.$$

Now, as in the previous computation

$$\mathscr{J}_1^p = \int_{\Gamma_+^\delta}\left(\int_0^{\max(0,t-\tau_-(\mathbf{z}))}\left|\left[H\chi_\delta\left(\mathsf{B}^+U_{k-1}(s)\right)f\right](\Phi(\mathbf{z},-\tau_-(\mathbf{z})))\right|^p \mathrm{d}s\right)\mathrm{d}\mu_+(\mathbf{z})$$
$$\leqslant \int_0^{t-\delta}\left(\int_{\Gamma_+}\left|\left[H\chi_\delta\left(\mathsf{B}^+U_{k-1}(s)\right)f\right](\Phi(\mathbf{z},-\tau_-(\mathbf{z})))\right|^p \mathrm{d}\mu_+(\mathbf{z})\right)\mathrm{d}s$$

where we used that $\tau_-(\cdot) > \delta$ on Γ_+^δ. We obtain then easily that

$$\mathscr{J}_1^p \leqslant C_\delta^p\, S_{k-1}^p(t-\delta).$$

In the same way $\mathscr{J}_2^p \leqslant A^p\, Z_{k-1}^p(t-\delta)$ which results in

$$Z_k(t) \leqslant C_\delta\, S_{k-1}(t-\delta) + A\, Z_{k-1}(t-\delta), \quad \forall t \geqslant \delta. \tag{3.13}$$

Combining this with (3.12), one obtains easily by induction that

$$S_k(t) \leqslant \sum_{j=0}^{k-1}\binom{k-1}{j} C_\delta^{k-1-j}\, A^j\left(C_\delta\, S_0(t-j\delta) + A\, Z_0(t-j\delta)\right)$$

and

$$Z_k(t) \leqslant \sum_{j=0}^{k-1}\binom{k-1}{j} C_\delta^{k-1-j}\, A^j\left(C_\delta\, S_0(t-(j+1)\delta) + A\, Z_0(t-(j+1)\delta)\right)$$

with the convention that $S_k(r) = Z_k(r) = 0$ for $r < 0$. Since $Z_0(t) \leqslant S_0(t) \leqslant \|f\|_p$ (see Proposition 3.5) and setting $k_\delta(t) = \min(k-1, [\frac{t}{\delta}])$ we get

$$S_k(t) \leqslant \|f\|_p \sum_{j=0}^{k_\delta(t)}\binom{k-1}{j} C_\delta^{k-1-j}\, A^j(C_\delta + A)$$

since $Z_0(t-j\delta) = S_0(t-j\delta) = 0$ for $j \geqslant t/\delta$. Now, it is not difficult to check that

$$\sum_{j=0}^{k_\delta(t)}\binom{k-1}{j} C_\delta^{k-1-j}\, A^j\,(C_\delta + A) \leqslant \sum_{j=0}^{k_\delta(t)+1}\binom{k}{j} C_\delta^{k-j}\, A^j$$

which gives (3.10). Now, from the definition of $U_k(t)$, one has

$$\|U_k(t)f\|_p^p = \int_{\Gamma_-}\left(\int_0^{\tau_+(\mathbf{y})}\left|\left[H\left(\mathsf{B}^+U_{k-1}(t-s)\right)f\right](\mathbf{y})\right|^p \mathrm{d}s\right)\mathrm{d}\mu_-(\mathbf{y})$$

and, writing again $H = H\chi_\delta + H(\mathbf{I}-\chi_\delta)$ one arrives without difficulty to

$$\|U_k(t)f\|_p \leqslant C_\delta S_{k-1}(t) + AZ_{k-1}(t)$$

and, as before, this gives

$$\|U_k(t)f\|_p \leqslant \sum_{j=0}^{\min(k,[t/\delta]+1)}\binom{k}{j} \left|\!\left|\!\left|H\chi_\delta\right|\!\right|\!\right|^{k-j}\left|\!\left|\!\left|H\right|\!\right|\!\right|^j\,\|f\|_p$$

for any $f \in \mathscr{D}_0$ and we obtain the result by density. □

This allows to prove the following

Theorem 3.11. *Assume that* $H \in \mathscr{B}(L^p_+, L^p_-)$ *is such that*

$$\limsup_{\delta \to 0^+} |||H\chi_\delta||| < 1. \tag{3.14}$$

Then, the series $\sum_{k=0}^{\infty} \|U_k(t)\|_{\mathscr{B}(X)}$ *is convergent for any* $t \geqslant 0$ *and, setting*

$$U_H(t) = \sum_{k=0}^{\infty} U_k(t), \quad t \geqslant 0$$

it holds that $(U_H(t))_{t \geqslant 0}$ *is a* C_0*-semigroup in* X *with generator* $(\mathcal{T}_{H,p}, \mathscr{D}(\mathcal{T}_{H,p}))$.

Proof. Let $\delta_0 > 0$ be such that $C := \sup_{\delta \in (0,\delta_0)} |||H\chi_\delta||| < 1$ and let us consider from now on $\delta < \delta_0$. Fix $t \geqslant 0$ and, with the notations of the above Lemma, let

$$\boldsymbol{u}_k = \sum_{j=0}^{\min(k,[t/\delta]+1)} \binom{k}{j} |||H\chi_\delta|||^{k-j} |||H|||^j, \quad k \geqslant 0.$$

The series $\sum_k \boldsymbol{u}_k$ is convergent. Indeed, setting $A := |||H|||$, one has

$$\sum_{k=0}^{\infty} \boldsymbol{u}_k \leqslant \sum_{j=0}^{[t/\delta]+1} A^j \sum_{k=j}^{\infty} \binom{k}{j} C^{k-j}.$$

Using the well-known identity, valid for $0 \leqslant C < 1$:

$$\sum_{k=j}^{\infty} \binom{k}{j} C^{k-j} = \frac{1}{(1-C)^{j+1}}$$

we obtain that

$$\sum_{k=0}^{\infty} \boldsymbol{u}_k \leqslant \frac{1}{1-C} \sum_{j=0}^{[t/\delta]+1} \left(\frac{A}{1-C}\right)^j \leqslant M \exp(\omega t) \tag{3.15}$$

with

$$\begin{aligned} M &= \frac{A^2}{(1-C)^2(A+C-1)} \quad \text{and} \quad \omega = \frac{1}{\delta}\log\left(\frac{A}{1-C}\right) \quad \text{if} \quad A > 1-C, \\ M &= \frac{2}{1-C} \quad \text{and} \quad \omega = \frac{1}{\delta} \quad \text{if} \quad A = 1-C \\ M &= \frac{1}{1-A-C} \quad \text{and} \quad \omega = 0 \quad \text{if} \quad A < 1-C \end{aligned} \tag{3.16}$$

According to Lemma 3.10, one sees that, for any $\delta \in (0, \delta_0)$,

$$\sum_{k=0}^{\infty} \|U_k(t)\|_{\mathscr{B}(X)} \leqslant M \exp(\omega t), \quad \forall t \geqslant 0.$$

This proves that, for any $t \geqslant 0$, the series $\sum_{k=0}^{\infty} U_k(t)$ converges in $\mathscr{B}(X)$ and, denoting its sum by $U_H(t)$, one has

$$\|U_H(t)\|_{\mathscr{B}(X)} \leqslant M \exp(\omega t), \quad t \geqslant 0.$$

According to Theorem 3.8, $(U_H(t))_{t\geqslant 0}$ is a strongly continuous family of $\mathscr{B}(X)$. Moreover

$$\lim_{t\to 0^+} U_H(t)f = \lim_{t\to 0^+} U_0(t)f = f$$

for any $f \in X$. Finally, using point (2) of Theorem 3.8, one sees that $(U_H(t))_{t\geqslant 0}$ is a C_0-semigroup in X. Let us denote by $\mathcal{A}$ its generator. We prove exactly as in [4, Theorem 4.1] that $\mathcal{A} = \mathcal{T}_{H,\,p}$.

First step: $\mathcal{T}_{H,\,p}$ is an extension of $\mathcal{A}$. Let $g \in \mathscr{D}(\mathcal{A})$ and $\lambda > \omega$ be given. Set $f = (\lambda - \mathcal{A})g$. As known, and using the notations of Theorem 3.8:

$$g = \int_0^\infty \exp(-\lambda\, t)U_H(t)f\mathrm{d}t = \sum_{k=0}^\infty \int_0^\infty \exp(-\lambda\, t)U_k(t)f\mathrm{d}t = \sum_{k=0}^\infty F_k.$$

According to (8) in Theorem 3.8, one has $g \in \mathscr{D}(\mathcal{T}_{\max,\,p})$ with $\mathcal{T}_{\max,\,p}g = \lambda\, g - f$, i.e. $\mathcal{T}_{\max,\,p}g = \mathcal{A}\, g$. Moreover, $\mathsf{B}^+g = \mathsf{B}^+(\sum_{k=0}^\infty F_k)$. By virtue of (8) of Theorem 3.8,

$$\begin{aligned} \|\mathsf{B}^+F_k\|_{L^p_+} &= \|\,(M_\lambda H)^k\, G_\lambda f\|_{L^p_+} = \|M_\lambda (HM_\lambda)^{k-1}HG_\lambda f\|_{L^p_+} \\ &\leqslant \|(HM_\lambda)^{k-1}HG_\lambda f\|_{L^p_-}. \end{aligned} \tag{3.17}$$

Now,

$$\begin{aligned} \|HM_\lambda\|_{\mathscr{B}(L^p_-)} &\leqslant \|H\chi_\delta M_\lambda\|_{\mathscr{B}(L^p_-)} + \|H(\mathbf{I}-\chi_\delta)M_\lambda\|_{\mathscr{B}(L^p_-)} \\ &\leqslant \|H\chi_\delta\|_{\mathscr{B}(L^p_+,L^p_-)} + \|\!|H|\!\|\exp(-\lambda\delta) \end{aligned}$$

where we used that, as in [5, Eq. (3.9)], for all $u \in L^p_-$

$$\|(\mathbf{I}-\chi_\delta)M_\lambda\|^p_{L^p_+} = \int_{\Gamma^\delta_+} |[M_\lambda u](\mathbf{z})|^p \mathrm{d}\mu_+(\mathbf{z}) \leqslant \exp(-p\lambda\delta)\|u\|^p_{L^p_-}.$$

In other words, with the above notations, for $\lambda > \delta$

$$\|HM_\lambda\|_{\mathscr{B}(L^p_-)} \leqslant C + A\exp(-\lambda\delta) < C + A\exp(-\omega\delta) \leqslant 1 \tag{3.18}$$

by definition of ω (see Eq. (3.16)). This, together with (3.17) shows that the series $\sum_{k=0}^\infty \|\mathsf{B}^+F_k\|_{L^p_+}$ converges and therefore,

$$\mathsf{B}^+g = \sum_{k=0}^\infty \mathsf{B}^+F_k \in L^p_+.$$

Then, being H continuous, by (8) of Theorem 3.8, we get

$$\mathsf{B}^-g = \sum_{k=0}^\infty H\mathsf{B}^+F_k = H\mathsf{B}^+g.$$

This proves that $g \in \mathscr{D}(\mathcal{T}_{H,\,p})$, i.e.,

$$\mathscr{D}(\mathcal{A}) \subset \mathscr{D}(\mathcal{T}_{H,\,p}) \quad \text{and} \quad \mathcal{A}\, g = \mathcal{T}_{\max,\,p}g = \mathcal{T}_{H,\,p}g, \quad \forall g \in \mathscr{D}(\mathcal{A}).$$

Second step: $\mathcal{A}$ is an extension of $\mathcal{T}_{H,\,p}$. Conversely, let $g \in \mathscr{D}(\mathcal{T}_{H,\,p})$ and $\lambda > \omega$ be given. Set $f = (\lambda - \mathcal{T}_{H,\,p})g$. We define

$$F = \int_0^\infty \exp(-\lambda\, t)U_H(t)f\mathrm{d}t = (\lambda - \mathcal{A})^{-1}f, \quad \text{and} \quad G = g - F.$$

From the first point, $G \in \mathscr{D}(\mathcal{T}_{H,p})$ with $\mathcal{T}_{H,p}G = \lambda G$. In particular, $G = \Xi_\lambda \mathsf{B}^- G$ while

$$\mathsf{B}^- G = H\mathsf{B}^+ G = HM_\lambda \mathsf{B}^- G.$$

But, if $\|\mathsf{B}^- G\|_{L^p_-} \neq 0$, then (3.18) implies $\|\mathsf{B}^- G\|_{L^p_-} < \|\mathsf{B}^- G\|_{L^p_-}$. Hence, $\mathsf{B}^- G = 0$. Since $G = \Xi_\lambda \mathsf{B}^- G$, we get $G = 0$ and $g = F$ and this proves $\mathcal{A} = \mathcal{T}_{H,p}$. □

Remark 3.12. The novelty of the above approach, with respect to [3, Theorem 5.1], is that the above proof is constructive and we give a precise and explicit expression of the semigroup $(U_H(t))_{t\geqslant 0}$. The proof presented here is similar to the one in [4] and differs from the one in our previous contribution [3]. Notice however that it would be possible to adapt in a simple way the proof given in [3, Section 5] which is based on some suitable change of variables.

Remark 3.13. The above estimate (3.15) with M, ω given by (3.16) allowed us to prove the convergence (in $\mathscr{B}(X)$) of the series $\sum_{k=0}^{\infty} U_k(t)$ but does not yield the optimal estimate for the limit $\|U_H(t)\|_{\mathscr{B}(X)}$. For instance, if $A = \|\|H\|\| < 1$ then the semigroup $(U_H(t))_{t\geqslant 0}$ is a contraction semigroup while the above estimate yields $\|U_H(t)\|_{\mathscr{B}(X)} \leqslant \frac{1}{1-A-C}$ with $\frac{1}{1-A-C} > 1$.

A useful consequence of the above is the following more tractable expression of the semigroup $(U_H(t))_{t\geqslant 0}$:

Corollary 3.14. *For any $f \in \mathscr{D}(\mathcal{T}_{H,p})$ and any $t \geqslant 0$, the following holds for μ-a.e. $\mathbf{x} \in \mathbf{\Omega}$:*

$$U_H(t)f(\mathbf{x}) = \begin{cases} U_0(t)f(\mathbf{x}) = f(\Phi(\mathbf{x}, -t)) & \text{if } t < \tau_-(\mathbf{x}) \\ \left[H\left(\mathsf{B}^+ U_H(t - \tau_-(\mathbf{x}))f\right)\right](\Phi(\mathbf{x}, -\tau_-(\mathbf{x}))) & \text{if } t \geqslant \tau_-(\mathbf{x}). \end{cases}$$

Remark 3.15. Notice that, if $f \in \mathscr{D}(T_{H,p})$, for any $t \geqslant 0$, $\psi(t) = U_H(t)f$ is the unique classical solution (see [15]) to the Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \mathcal{T}_{H,p}\psi(t), \quad \psi(0) = f.$$

The above Corollary provides therefore the (semi)-explicit expression of the solution to this Cauchy problem.

Proof. Let us consider first $f \in \mathscr{D}_0$. Then, for all $k \geqslant 0$, $t > 0$

$$\mathsf{B}^+ U_k(t)f = \mathsf{B}^+ \left(\int_0^t U_k(s)\mathcal{T}_{\max,p} f \mathrm{d}s\right).$$

In particular, from Theorem 3.11, the series $\sum_{k=0}^{\infty} \|\mathsf{B}^+ U_k(t)f\|_{L^p_+}$ is convergent and therefore

$$\mathsf{B}^+ U_H(t)f = \sum_{k=0}^{\infty} \mathsf{B}^+ U_k(t)f$$

where the series converges in L^p_+. Given $0 \leqslant s < t$, we get then

$$HB^+U_H(t-s)f = \sum_{k=0}^{\infty} H\mathsf{B}^+U_k(t-s)f.$$

Pick then $\mathbf{x} \in \mathbf{\Omega}$ and $\mathbf{y} \in \Gamma_-$ such that $\mathbf{x} = \Phi(\mathbf{y}, s)$. We have, by definition,

$$\left[H\mathsf{B}^+U_k(t-s)f\right](\mathbf{y}) = [U_{k+1}(t)f](\mathbf{x}).$$

Therefore,

$$[H\mathsf{B}^+U_H(t-s)f](\mathbf{y}) = \sum_{k=1}^{\infty}\left[U_{k+1}(t)f\right](\mathbf{x}) = [U_H(t)f](\mathbf{x}) - [U_0(t)f](\mathbf{x}).$$

To summarize, for almost every $\mathbf{x} \in \mathbf{\Omega}$, there exist a unique $\mathbf{y} \in \Gamma_-$ and a unique $0 < s < \tau_+(\mathbf{y})$ such that $\mathbf{x} = \Phi(\mathbf{y}, s)$ and we proved that

$$[U_H(t)f](\mathbf{x}) = [U_0(t)f](\mathbf{x}) + [H\mathsf{B}^+U_H(t-s)f](\mathbf{y})$$

which proves the result for $f \in \mathscr{D}_0$.

Consider now $f \in \mathscr{D}(\mathcal{T}_{H,p})$. Then, for any $t > 0$, $U_H(t)f \in \mathscr{D}(\mathcal{T}_{H,p})$. Introduce then the mapping

$$\mathbf{x} \in \mathbf{\Omega}_- \longmapsto \left[H\mathsf{B}^+U_H(t-s)f\right](\mathbf{y}) = g(t,\mathbf{x})$$

where $\mathbf{x} = \Phi(\mathbf{y}, s)$ for some unique $\mathbf{y} \in \Gamma_-$ and $s \in (0, \tau_+(\mathbf{y}))$. It holds, for any $\lambda > \omega$ (with ω introduced in the proof of Theorem 3.11):

$$\begin{aligned}\int_0^{\infty} \exp(-\lambda\, t) g(t,\mathbf{x})\mathrm{d}t &= \int_0^{\infty} \exp(-\lambda\, t)\left[H\mathsf{B}^+U_H(t-s)f\right](\mathbf{y})\mathrm{d}t\\ &= \exp(-\lambda\, s)\int_0^{\infty}\exp(-\lambda\, t)\left[H\mathsf{B}^+U_H(t)f\right](\mathbf{y})\mathrm{d}t\\ &= \exp(-\lambda\, s)\left[H\mathsf{B}^+\int_0^{\infty}\exp(-\lambda\, t)U_H(t)f\mathrm{d}t\right](\mathbf{y})\end{aligned}$$

Therefore, using point (8) of Theorem 3.8

$$\begin{aligned}\int_0^{\infty}\exp(-\lambda\, t)g(t,\mathbf{x})\mathrm{d}t &= \exp(-\lambda\, s)\left[H\sum_{k=0}^{\infty}\left(M_\lambda H\right)^k G_\lambda f\right](\mathbf{y})\\ &= \exp(-\lambda\, s)\left[H(I - M_\lambda H)^{-1}G_\lambda f\right](\mathbf{y}).\end{aligned}$$

Since moreover

$$\int_0^{\infty}\exp(-\lambda\, t)U_H(t)f\mathrm{d}t = \int_0^{\infty}\exp(-\lambda\, t)U_0(t)f\mathrm{d}t + \Xi_\lambda H(I - M_\lambda H)^{-1}G_\lambda f$$

the result follows. □

3.4. Examples

We illustrate the results obtained so far with several examples of interest, in particular in the context of transport equation on network [8,12].

Example 3.16. We begin with the simplest case of a transport equation with one-velocity in dimension one. Namely, consider the transport equation

$$\partial_t f(x,t) - \partial_x f(x,t) = 0, \quad x \in (0,1), t > 0$$

with initial condition $f(x,0) = f_0(x)$ and boundary condition

$$f(1,t) = Hf(0,t)$$

where here H is a constant. For such a case, with the above notations, one has $\mathbf{\Omega} = [0,1]$ endowed with the Lebesgue measure μ. One checks that

$$\Phi(x,t) = x - t \quad 0 < x < 1, \quad t \in \mathbb{R}$$

and

$$\tau_+(x) = x, \quad \tau_-(x) = 1 - x.$$

Moreover, one checks that $\Gamma_- = \{1\}$, and $\Gamma_+ = \{0\}$ with $\tau_-(0) = \tau_+(1) = 1$. The boundary condition reads then $\mathsf{B}^- f = H\mathsf{B}^+ f$ where of course H is a constant. Notice that, since $\tau_- = 1$ on Γ_+, with the notations of Theorem 3.11 it holds

$$H\chi_{\Gamma_+ \setminus \Gamma_+^\delta} = 0 \quad \forall 0 < \delta < 1$$

so that (3.14) is valid for any $H \in \mathbb{R}$. The above equation is then governed by a C_0-semigroup $U_H(t)$ in $L^p([0,1])$ and we obviously have

$$U_0(t)f(x) = f(x+t) \quad \forall 0 \leqslant t < 1 - x, \quad U_0(t)f(x) = 0 = f(x+t) \quad \forall t \geqslant 1 - x.$$

Then

$$\mathsf{B}^+ U_0(t) f = f(t) \quad \forall t \in [0,1), \quad \mathsf{B}^+ U_0(t) f = 0 \quad \forall t \geqslant 1.$$

Consequently,

$$U_1(t)f(x) = H\mathsf{B}^+ U_0(t - \tau_-(x))f = f(t + x - 1) \quad \forall 0 \leqslant t - \tau_-(x) < 1$$

which corresponds to $1 \leqslant t + x < 2$ whereas $U_1(t)f(x) = 0$ for any $t \in (-\infty, 1-x) \cup [2-x, \infty)$. Iterating this procedure, we find again the explicit solution obtained in [12, Eq. (5.2)] (see [8, Prop. 18.17]) for a scalar function f corresponding to $m = 1$ with the notations of [12].

Example 3.17. The above example, as mentioned, is a particular case of a more general model of transport in the network which can be described by the following system of equations:

$$\begin{cases} \partial_t f_j(x,t) = c_j \partial_x f_j(x,t), & x \in [0,1], \ t > 0, \\ f_j(x,0) = f_j^0(x), & x \in [0,1], \ j = 1, \dots, m \end{cases} \tag{3.19a}$$

subject to the general boundary condition

$$\sigma_{ij}^- c_j f_j(1,t) = \omega_{ij} \sum_{k=1}^m \sigma_{ik}^+ c_k f_k(0,t) \tag{3.19b}$$

for $i = 1, \dots, n$ and $j = 1, \dots, m$.

The above model corresponds to transport on a simple, directed and connected graph $G = (V, E)$ with vertices $V = \{v_1, \dots, v_n\}$ and directed edges $E = e_1, \dots, e_m$ where, on every edge e_i particles are flowing in only one direction with constant velocity $c_i > 0$ ($i = 1, \dots, n$) while in every vertex v_i the incoming material is distributed into the outgoing edges e_j according to weights $\omega_{ij} \geqslant 0$ satisfying

$$\sum_{j \in A_i^-} \omega_{ij} = 1, \quad \forall i = 1, \dots, n$$

where A_i^- represents the set of indices $j \in \{1, \dots, m\}$ such that v_i is connected to the edge e_j and e_j is an outgoing edge. The graph structure is described by the following $n \times m$ matrices: the outgoing incidence matrix $\sigma^- = (\sigma_{ij}^-)_{ij}$ is given by

$$\sigma_{ij}^- = 1 \quad \text{if} \quad v_i \in A_j^-$$

and is zero otherwise. The incoming incidence matrix $\sigma^+ = \left(\sigma_{ij}^+\right)_{ij}$ is defined as

$$\sigma_{ij}^+ = 1 \quad \text{if} \quad v_i \in A_j^+$$

and is zero otherwise where A_j^+ represents the set of vertices for which e_j is an incoming edge. The above model (3.19) has been investigated in a L^1-framework in [7,8] and in [12] in L^p-spaces $p > 1$ (actually a more general system has been considered in [12] including various additional control terms). We refer to [8, Section 18.1] and [7] for more details on the physical motivation of the model.

As mentioned earlier, for $n = m = 1$, the semigroup associated to the above model is described in Example 3.16. The above method is robust enough to deal with the general case $m > 1$ just by considering vector-valued functions instead of scalar functions. We consider a vector $\boldsymbol{f} = (f_1, \dots, f_m) \in \mathbb{R}^m$ and, for any $j = 1, \dots, m$ and any $x \in (0,1)$, we set

$$\Phi_j(x,t) = x - c_j t, \qquad t \in \mathbb{R}, \quad \tau_+^j(x) = \frac{x}{c_j}, \quad \tau_-^j(x) = \frac{1-x}{c_j}$$

and $\Gamma_- = \bigcup_{j=1}^m \Gamma_-^j$, $\Gamma_+ = \bigcup_{j=1}^m \Gamma_+^j$ where $\Gamma_-^j = \{1\}$ and $\Gamma_+^j = \{0\}$ for any $j = 1, \dots, m$. As explained in [8, Section 18.1], the boundary conditions (3.19b) can be reformulated as

$$\boldsymbol{f}(1) = H\boldsymbol{f}(0)$$

for some suitable matrix $H = (h_{jk})_{j,k=1,\dots,m}$ (this matrix is denoted $\mathbb{B}_C$ in [8] and we refer to [8, Eqs. (18.3)–(18.4)] for its exact expression which is irrelevant here). The semigroup $(U_0(t))_{t \geqslant 0}$ is then defined as a vector whose components are given, for $t \geqslant 0$ and $\boldsymbol{f} = (f_1, \dots, f_m)$ by

$$\left[U_0(t)\boldsymbol{f}\right]_j(x,t) = f_j(x + c_j t) \quad \text{if} \quad 0 < x + c_j t < 1,$$

with $\left[U_0(t)\boldsymbol{f}\right]_j(x,t) = 0$ otherwise. Then, the Dyson–Phillips iterations are defined by induction with $U_k(t)$ having components

$$\left[U_k(t)\boldsymbol{f}\right]_j(x,t) = \chi_{\{t \geqslant \tau_-^j(x)\}} \left[H\mathsf{B}^+ U_{k-1}(s)\boldsymbol{f}\right]_j \Big|_{s = t - \tau_-^j(x)}.$$

For instance, since $[\mathsf{B}^{+}U_0(t)\boldsymbol{f}]_j = f_j(c_j t)$ we have

$$\left[H\mathsf{B}^{+}U_0(s)\boldsymbol{f}\right]_j = \sum_{k=1}^{m} h_{jk} f_k(c_k t)$$

and

$$\left[U_1(t)f\right]_j(x) = \sum_{k=1}^{m} h_{jk} f_k\left(c_k t + \frac{c_k}{c_j}x - \frac{c_k}{c_j}\right), \quad \text{if } 0 < c_k t + \frac{c_k}{c_j}x - \frac{c_k}{c_j} < 1$$

and is zero else. Iterating this procedure, we find a general expression for the semigroup $(U_H(t))_{t\geqslant 0}$ associated to (3.19). Notice that, as in Example 3.16, the semigroup $(U_H(t))_{t\geqslant 0}$ is well defined thanks to Theorem 3.11 since

$$H\chi_{\Gamma_+\setminus\Gamma_+^{\delta}} = 0 \quad \text{for any} \quad 0 < \delta < \min\left(c_1^{-1}, \ldots, c_m^{-1}\right).$$

Example 3.18. We revisit here the model described in [9] which deals with a structured cell population for which each cell is distinguished by its cell cycle length $\ell \in (\ell_1, \ell_2)$ and by its age $a \in [0, \ell)$. Here, we consider the general case

$$0 \leqslant \ell_1 < \ell_2 \leqslant \infty.$$

If $f(t, a, \ell)$ denotes, at time $t \geqslant 0$, the cell density with respect to age a and cell cycle length ℓ, the cell population is then governed by the transport equation

$$\begin{cases} \partial_t f(t,a,\ell) & = -\partial_a f(t,a,\ell) - \mu(a,\ell) f(t,a,\ell) \\ & \quad + \displaystyle\int_0^{\ell} \mathrm{d}\ell' \int_0^{a} \eta(a,\ell,a',\ell') f(t,a',\ell') \mathrm{d}a' \mathrm{d}\ell' \\ f(0,a,\ell) & = f_0(a,\ell) \end{cases} \tag{3.20a}$$

subject to boundary condition describing the birth of cell

$$f(t,0,\ell) = \alpha\, f(t,\ell,\ell) + \beta \int_{\ell_1}^{\ell_2} k(\ell,\ell') f(t,\ell',\ell') \mathrm{d}\ell' \tag{3.20b}$$

Here, $\mu \geqslant 0$ is the cell mortality rate, $\eta(a, \ell, a', \ell')$ denotes the transition rate at which cells change their cell cycle length from ℓ' to ℓ and its age from a' to a. The nonnegative kernel $k(\ell, \ell')$ represents the correlation, during mitosis, between the cell cycle length of a mother cell l' and that of a daughter cell ℓ. The parameters α, β are nonnegative constants. We refer to [9,10] for more details about the model and relevant bibliography on the subject. Since in the present paper we deal with collisionless transport problems we shall assume $\mu = \eta = 0$. With our notations we have

$$\boldsymbol{\Omega} = \{\mathbf{x} = (a,\ell) : \ \ \ell \in (\ell_1,\ell_2), \ \ 0 < a < \ell\} \subset \mathbb{R}^2$$

and

$$\mathscr{F}(\mathbf{x}) = (1,0) \quad \forall \mathbf{x} = (a,\ell) \in \boldsymbol{\Omega}.$$

The measure we consider here is the Lebesgue measure $\mathrm{d}\mu(\mathbf{x}) = \mathrm{d}a\mathrm{d}\ell$. Consequently,

$$\Phi(\mathbf{x},t) = (a+t,\ell), \quad \tau_-(\mathbf{x}) = a, \quad \tau_+(\mathbf{x}) = \ell - a \quad \forall \mathbf{x} = (a,\ell) \in \boldsymbol{\Omega}$$

and

$$\Gamma_- = \{\mathbf{y} = (0,\ell)\ ;\ \ell_1 < \ell < \ell_2\} \quad \Gamma_+ = \{\mathbf{z} = (\ell,\ell)\ ;\ \ell_1 < \ell < \ell_2\}$$

with moreover

$$\Phi(\mathbf{y}, s) = (s,\ell) \quad \forall \mathbf{y} = (0,\ell) \in \Gamma_-, \quad 0 < s < \tau_+(\mathbf{y})$$

and

$$\Phi(\mathbf{z}, -s) = (\ell - s,\ell) \quad \forall \mathbf{z} = (\ell,\ell) \in \Gamma_+, \quad 0 < s < \tau_-(\mathbf{z}).$$

One has then easily

$$U_0(t)f(\mathbf{x}) = U_0(t)f(a,\ell) = f(a-t,\ell)\chi_{\{t<a\}}(a,\ell), \quad t \geqslant 0.$$

To treat the above equation (3.20) in $L^p(\mathbf{\Omega},\mu)$ with $p > 1$ we introduce, as in [9], the quantity

$$\overline{\kappa}_\infty := \left[\sup_{\ell_1 \leqslant \ell \leqslant \ell_2} \int_{\ell_1}^{\ell_2} k(\ell,\ell')\mathrm{d}\ell'\right]^{\frac{p-1}{p}} \left[\sup_{\ell_1 \leqslant \ell' \leqslant \ell_2} \int_{\ell_1}^{\ell_2} k(\ell,\ell')\mathrm{d}\ell\right]^{\frac{1}{p}}$$

and assuming that $\overline{\kappa}_\infty < \infty$ we see that the boundary operator $H : L^p_+ \to L^p_-$ defined as

$$H\varphi(\mathbf{y}) = H\varphi(0,\ell) = \alpha\,\varphi(\ell,\ell) + \beta \int_{\ell_1}^{\ell_2} k(\ell,\ell')\varphi(\ell',\ell')\mathrm{d}\ell', \quad \varphi \in L^p_+$$

is such that

$$\|H\|_{\mathscr{B}(L^p_+,L^p_-)} \leqslant \alpha + \beta\,\overline{\kappa}_\infty$$

and, in particular, under assumption $(\mathbf{A'_k})$ of [9], one sees that H is a contraction and Theorem 3.11 applies directly giving an explicit expression of the solution to (3.20). Moreover, we can also deal with the more general case in which H satisfies (3.14). Notice that, since $\tau_-(\mathbf{z}) = \tau_-(\ell,\ell) = \ell$ for all $\mathbf{z} \in \Gamma_+$ one has

$$[H\chi_{\Gamma_+\setminus\Gamma_+^\delta}\varphi](0,\ell) = \alpha\chi_{[0,\delta]\cap[\ell_1,\ell_2]}(\ell)\varphi(\ell,\ell) + \beta \int_{\ell_1}^{\ell_2} \chi_{[0,\delta]}(\ell')k(\ell,\ell')\varphi(\ell',\ell')\mathrm{d}\ell'$$

for any $\varphi \in L^p_+$. In particular, if $\ell_1 > 0$, one sees that

$$\limsup_{\delta\to 0} \left\|H\chi_{\Gamma_+\setminus\Gamma_+^\delta}\right\|_{\mathscr{B}(L^p_+,L^p_-)} = 0$$

independently of the coefficients α, β. The case $\ell_1 = 0$ has been studied in L^p-spaces in more details in [14] but no explicit description of the semigroup is provided there.

Appendix A: On the Family $(U_k(t))_{t\geqslant 0}$

We prove that the family of operators $(U_k(t))_{t\geqslant 0}$ introduced in Definition 3.6 is well defined and satisfies Theorem 3.8. We first need to establish general properties of $\mathcal{T}_{\max,\,p}$.

A.1 Additional Properties of $\mathcal{T}_{\max,\,p}$

We establish here several results, reminiscent of [4] about how $\mathcal{T}_{\max,\,p}$ and some strongly continuous family of operators can interplay. We start with the following where we recall that $\mathscr{D}_0$ has been defined in the beginning of Section 3.

Proposition A.1. *Let $(U(t))_{t\geqslant 0}$ be a strongly continuous family of $\mathscr{B}(X)$. For any $f\in X$, set*

$$I_t[f]=\int_0^t U(s)f\mathrm{d}s,\quad \forall t\geqslant 0.$$

Assume that

(i) *For any $f\in\mathscr{D}_0$, the mapping $t\in[0,\infty)\mapsto U(t)f\in X$ is differentiable with*

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t)f=U(t)\mathcal{T}_{\max,\,p}f,\quad t\geqslant 0.$$

(ii) *For any $f\in\mathscr{D}_0$ and any $t\geqslant 0$, it holds that $U(t)f\in\mathscr{D}(\mathcal{T}_{\max,\,p})$ with $\mathcal{T}_{\max,\,p}U(t)f=U(t)\mathcal{T}_{\max,\,p}f$.*

Then, the following holds

(1) *for any $f\in X$ and $t>0$, $I_t[f]\in\mathscr{D}(\mathcal{T}_{\max,\,p})$ with*

$$\mathcal{T}_{\max,\,p}I_t[f]=U(t)f-U(0)f.$$

(2) *for any $f\in\mathscr{D}_0$ the mapping $t\in[0,\infty)\mapsto\mathsf{B}^{\pm}U(t)f\in Y_p^{\pm}$ is continuous and,*

$$\mathsf{B}^{\pm}I_t[f]=\int_0^t\mathsf{B}^{\pm}U(s)f\mathrm{d}s\quad\forall t>0.$$

Let now $f\in X$ be such that $\mathsf{B}^-I_t[f]\in L^p_-$ for all $t>0$ with $t\in[0,\infty)\mapsto\mathsf{B}^-I_t[f]\in L^p_-$ continuous, then,

$$\mathsf{B}^+I_t[f]\in L^p_+\quad \textit{and}\quad t\in[0,\infty)\mapsto\mathsf{B}^+I_t[f]\in L^p_+\ \textit{continuous.}$$

Proof. Under assumptions $i)-ii)$, for any $f\in\mathscr{D}_0$, since both the mappings $t\mapsto U(t)f$ and $t\mapsto\mathcal{T}_{\max,\,p}U(t)f$ are continuous and $\mathcal{T}_{\max,\,p}$ is closed one has $I_t[f]\in\mathscr{D}(\mathcal{T}_{\max,\,p})$ with

$$\mathcal{T}_{\max,\,p}I_t[f]=\int_0^t\mathcal{T}_{\max,\,p}U(s)f\mathrm{d}s=\int_0^t\frac{\mathrm{d}}{\mathrm{d}s}U(s)f\mathrm{d}s=U(t)f-U(0)f.$$

This proves that (1) holds for $f\in\mathscr{D}_0$ and, since $\mathscr{D}_0$ is dense in X, the result holds for any $f\in X$.

Let us prove (2). Pick $f\in\mathscr{D}_0$. Since the mapping $t\geqslant 0\mapsto U(t)f\in\mathscr{D}(\mathcal{T}_{\max,\,p})$ is continuous for the graph norm on $\mathscr{D}(\mathcal{T}_{\max,\,p})$ while $\mathsf{B}^{\pm}:\mathscr{D}(\mathcal{T}_{\max,\,p})\mapsto Y_p^{\pm}$ is continuous (see [5, Remark 3.4]), the mapping

$t \geqslant 0 \mapsto \mathsf{B}^{\pm}U(t)f \in Y_p^{\pm}$ is continuous. Moreover, $\mathsf{B}^{\pm}I_t[f] = \int_0^t \mathsf{B}^{\pm}U(s)f\mathrm{d}s$ still thanks to the continuity on $\mathscr{D}(\mathcal{T}_{\max,\,p})$ with the graph norm and point (1).

Let now $f \in X$ be given such that $\mathsf{B}^{-}I_t[f] \in L^p_-$ for all $t > 0$ with $t \geqslant 0 \mapsto \mathsf{B}^{-}I_t[f] \in L^p_-$ continuous. For all $t \geqslant 0$, $h \geqslant -t$, denote now

$$I_{t,t+h}[f] = \int_t^{t+h} U(s)f\mathrm{d}s = I_{t+h}[f] - I_t[f].$$

Since $I_t[f] \in \mathscr{D}(\mathcal{T}_{\max,\,p})$ and $\mathsf{B}^{-}I_t[f] \in L^p_-$, one has clearly $I_{t,t+h}[f] \in \mathscr{D}(\mathcal{T}_{\max,\,p})$ and $\mathsf{B}^{-}I_{t,t+h}[f] \in L^p_-$ and Green's formula (see [5, Equation (2.12)]) yields

$$\begin{aligned}\|\mathsf{B}^{+}I_{t,t+h}[f]\|^p_{L^p_+} &= \|\mathsf{B}^{-}I_{t,t+h}[f]\|^p_{L^p_-}\\ &\quad -p\int_{\mathbf{\Omega}} |I_{t,t+h}[f]|^{p-1}\,\mathrm{sign}(I_{t,t+h}[f])\mathcal{T}_{\max,\,p}I_{t,t+h}[f]\mathrm{d}\mu\\ &\leqslant \|\mathsf{B}^{-}I_{t,t+h}[f]\|^p_{L^p_-} + p\|I_{t,t+h}[f]\|_p^{p-1}\,\|\mathcal{T}_{\max,\,p}I_{t,t+h}[f]\|_p.\end{aligned}$$

Since $\mathcal{T}_{\max,\,p}I_{t,t+h}[f] = U(t+h)f - U(t)f$, one gets

$$\begin{aligned}\|\mathsf{B}^{+}\left(I_{t+h}[f] - I_t[f]\right)\|^p_{L^p_+} &\leqslant \|\mathsf{B}^{-}\left(I_{t+h}[f] - I_t[f]\right)\|^p_{L^p_-}\\ &\quad +p\|I_{t+h}[f] - I_t[f]\|_p^{p-1}\,\|U(t+h)f - U(t)f\|_p.\end{aligned} \tag{A.1}$$

The continuity of $s \geqslant 0 \mapsto U(s)f \in X$ together with the one of $s \geqslant 0 \mapsto \mathsf{B}^{-}I_s[f] \in L^p_-$ gives then that

$$\lim_{h\to 0} \|\mathsf{B}^{+}\left(I_{t+h}[f] - I_t[f]\right)\|_{L^p_+} = 0$$

i.e. $t \geqslant 0 \mapsto \mathsf{B}^{+}I_t[f] \in L^p_+$ is continuous. □

We can complement the above with the following whose proof is exactly as that of [4, Proposition 3] and is omitted here:

Proposition A.2. *Let $(U(t))_{t\geqslant 0}$ be a strongly continuous family of $\mathscr{B}(X)$ satisfying the following, for any $f \in \mathscr{D}_0$:*

(i) *For any $t \geqslant 0$,*

$$[U(t)f](\mathbf{x}) = 0 \quad \forall \mathbf{x} \in \mathbf{\Omega} \text{ such that } \tau_-(\mathbf{x}) \geqslant t.$$

(ii) *For any $\mathbf{y} \in \Gamma_-$, $t > 0$, $0 < r < s < \tau_+(\mathbf{y})$, it holds*

$$[U(t)f]\left(\Phi(\mathbf{y},s)\right) = [U(t-s+r)f]\left(\Phi(\mathbf{y},r)\right).$$

(iii) *the mapping $t \geqslant 0 \mapsto U(t)f \in X$ is differentiable with $\frac{\mathrm{d}}{\mathrm{d}t}U(t)f = U(t)\mathcal{T}_{\max,\,p}f$ for any $t \geqslant 0$.*

Then, the following properties hold

(1) *For any $f \in X$ and any $t \geqslant 0$ and μ_--a.e. $\mathbf{y} \in \Gamma_-$, given $0 < s_1 < s_2 < \tau_+(\mathbf{y})$, there exists $0 < r < s_1$ such that*

$$\int_{s_1}^{s_2} [U(t)f]\left(\Phi(\mathbf{y},s)\right)\mathrm{d}s = \int_{t-s_2+r}^{t-s_1+r} [U(\tau)f]\left(\Phi(\mathbf{y},r)\right)\mathrm{d}\tau.$$

(2) *For any* $f \in \mathscr{D}_0$ *and* $t \geqslant 0$, *one has* $U(t)f \in \mathscr{D}(\mathcal{T}_{\max,\,p})$ *with* $\mathcal{T}_{\max,\,p}U(t)f = U(t)\mathcal{T}_{\max,\,p}f$.

A.2 Proof of Theorem 3.8

We now come to the proof of Theorem 3.8 which will consist of showing that the family of operators $(U_k(t))_{t\geqslant 0}$ introduced in Definition 3.6 is well defined and satisfies the properties listed in Theorem 3.8. The proof is made by induction and we start with a series of Lemmas (one for each of the above properties in Theorem 3.8) showing that $U_1(t)$ enjoys all the listed properties.

As already mentioned, the fact that the mapping

$$\Phi \;:\; \{(\mathbf{y},s)\in\Gamma_-\times(0,\infty)\;;\;0<s<\tau_+(\mathbf{y})\}\to\mathbf{\Omega}_-$$

is a measure isomorphism, for any $f\in\mathscr{D}_0$ and $t>0$, the function $U_1(t)f$ is well defined and measurable on $\mathbf{\Omega}$. Moreover, using [5, Proposition 2.2] and Fubini's Theorem:

$$\begin{aligned}\|U_1(t)f\|_p^p &= \int_{\Gamma_-}\mathrm{d}\mu_-(\mathbf{y})\int_0^{\tau_+(\mathbf{y})}|[U_1(t)f](\Phi(\mathbf{y},s))|^p\,\mathrm{d}s\\ &= \int_{\Gamma_-}\mathrm{d}\mu_-(\mathbf{y})\int_0^{\min(t,\tau_+(\mathbf{y}))}\left|\left[H(\mathsf{B}^+U_0(t-s)f)\right](\mathbf{y})\right|^p\mathrm{d}s\\ &\leqslant \int_0^t\|H(\mathsf{B}^+U_0(t-s)f)\|_{L^p_-}^p\,\mathrm{d}s.\end{aligned}$$

Therefore,

$$\|U_1(t)f\|_p^p\leqslant|||H|||^p\int_0^t\|\mathsf{B}^+U_0(t-s)f\|_{L^p_+}^p\,\mathrm{d}s=|||H|||^p\left(\|f\|_p^p-\|U_0(t)f\|_p^p\right)$$

thanks to Proposition 3.5. Therefore $\|U_1(t)f\|_p\leqslant|||H|||\,\|f\|_p$ for all $f\in\mathscr{D}_0$ with moreover

$$\lim_{t\to0^+}\|U_1(t)f\|_p=0\quad\forall f\in\mathscr{D}_0. \tag{A.2}$$

Since $\mathscr{D}_0$ is dense in X, this allows to define a unique extension operator, still denoted by $U_1(t)\in\mathscr{B}(X)$ with

$$\|U_1(t)\|_{\mathscr{B}(X)}\leqslant|||H|||,\quad\forall t\geqslant0.$$

Now, one has the following

Lemma A.3. *The family* $(U_1(t))_{t\geqslant0}$ *is strongly continuous on* X.

Proof. Let $t>0$ be fixed. Set $\mathbf{\Omega}_t=\{\mathbf{x}\in\mathbf{\Omega}_-\;;\;\tau_-(\mathbf{x})\leqslant t\}$. One has $[U_1(t)f](\mathbf{x})=0$ for any $\mathbf{x}\in\mathbf{\Omega}\backslash\mathbf{\Omega}_t$ and any $f\in X$. Let us fix $f\in\mathscr{D}_0$ and $h>0$. One has

$$\begin{aligned}\|U_1(t+h)f-U_1(t)f\|_p^p=&\int_{\mathbf{\Omega}_t}|U_1(t+h)f-U_1(t)f|^p\,\mathrm{d}\mu\\ &+\int_{\mathbf{\Omega}_{t+h}\backslash\mathbf{\Omega}_t}|U_1(t+h)f|^p\mathrm{d}\mu.\end{aligned} \tag{A.3}$$

Now,

$$\begin{aligned}\int_{\mathbf{\Omega}_t} &|U_1(t+h)f - U_1(t)f|^p \,\mathrm{d}\mu \\ &= \int_{\Gamma_-} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} |[U_1(t+h)f - U_1(t)f]\,(\Phi(\mathbf{y},s))|^p \,\mathrm{d}s\end{aligned}$$

and, repeating the reasoning before Lemma A.3 one gets

$$\int_{\mathbf{\Omega}_t} |U_1(t+h)f - U_1(t)f|^p \,\mathrm{d}\mu \leqslant |||H|||^p \int_0^t \|\mathsf{B}^+\left(U_0(s+h)f - U_0(s)f\right)\|_{L^p_+}^p \,\mathrm{d}s.$$

Since $U_0(s+h)f - U_0(s)f = U_0(s)\left(U_0(h)f - f\right)$ one gets from Proposition 3.5 that

$$\int_{\mathbf{\Omega}_t} |U_1(t+h)f - U_1(t)f|^p \,\mathrm{d}\mu \leqslant |||H|||^p \left(\|U_0(h)f - f\|_p^p - \|U_0(t)\left(U_0(h)f - f\right)\|_p^p\right).$$

This proves that

$$\lim_{h\to 0^+} \int_{\mathbf{\Omega}_t} |U_1(t+h)f - U_1(t)f|^p \,\mathrm{d}\mu = 0.$$

Let us investigate the second integral in (A.3). One first notices that, given $\mathbf{x} = \Phi(\mathbf{y},s)$ with $\mathbf{y} \in \Gamma_-$, $0 < s < \min(t, \tau_+(\mathbf{y}))$, it holds

$$\begin{aligned}\left[U_0(t)U_1(h)f\right](\mathbf{x}) &= \chi_{\{t<\tau_-(\mathbf{x})\}}\left[U_1(h)f\right](\Phi(\mathbf{x},-t)) \\ &= \chi_{(t,\infty)}(s)\;\left[U_1(h)f\right](\Phi(\mathbf{y},s-t)) \\ &= \chi_{(t,t+h]}(s)\left[H\left(\mathsf{B}^+U_0(t+h-s)f\right)\right](\mathbf{y}) \\ &= \chi_{(t,t+h]}(s)\left[U_1(t+h)f\right](\Phi(\mathbf{y},s)) \\ &= \chi_{\{t<\tau_-(\mathbf{x})\}}\left[U_1(t+h)f\right](\mathbf{x}).\end{aligned} \tag{A.4}$$

Therefore

$$\int_{\mathbf{\Omega}_{t+h}\setminus\mathbf{\Omega}_t} |U_1(t+h)f|^p \mathrm{d}\mu = \|U_0(t)U_1(h)f\|_p^p$$

and, since $(U_0(t))_{t\geqslant 0}$ is a contraction semigroup, we get

$$\int_{\mathbf{\Omega}_{t+h}\setminus\mathbf{\Omega}_t} |U_1(t+h)f|^p \mathrm{d}\mu \leqslant \|U_1(h)f\|_p^p.$$

Using (A.2), we get $\lim_{h\to 0^+} \int_{\mathbf{\Omega}_{t+h}\setminus\mathbf{\Omega}_t} |U_1(t+h)f|^p \mathrm{d}\mu = 0$ and we obtain finally that $\lim_{h\to 0^+} \|U_1(t+h)f - U_1(t)f\|_p^p = 0$. One argues in a similar way for negative h and gets

$$\lim_{h\to 0} \|U_1(t+h)f - U_1(t)f\|_p = 0, \quad \forall f \in \mathscr{D}_0.$$

Since $\mathscr{D}_0$ is dense in X and $\|U_1(t)\|_{\mathscr{B}(X)} \leqslant |||H|||$ we deduce that above limit vanishes for all $f \in X$. This proves the result. □

One has also the following

Lemma A.4. *For all $t \geqslant 0$, $h \geqslant 0$ and $f \in X$ it holds*

$$U_1(t+h)f = U_0(t)U_1(h)f + U_1(t)U_0(h)f.$$

Proof. It is clearly enough to consider $t > 0$, $h > 0$ since $U_1(0)f = 0$ while $U_0(0)$ is the identity operator. Notice that, for any $f \in \mathscr{D}_0$ and any $0 \leqslant t_1 \leqslant t_2$, for $\mathbf{x} = \Phi(\mathbf{y}, s) \in \mathbf{\Omega}_{t_1}$ we have

$$\int_{t_1}^{t_2} [U_1(\tau)f](\mathbf{x})\mathrm{d}\tau = \left[H\mathsf{B}^+ \int_{t_1-s}^{t_2-s} U_0(\tau)f\mathrm{d}\tau\right](\mathbf{y}).$$

Now, given $f \in X$ and $0 \leqslant t_1 \leqslant t_2$ the above formula is true for almost every $\mathbf{x} = \Phi(\mathbf{y}, s) \in \mathbf{\Omega}_{t_1}$ by a density argument. Therefore, for almost every $\mathbf{x} = \Phi(\mathbf{y}, s) \in \mathbf{\Omega}_t$ and any $\delta > 0$ it holds

$$\begin{aligned}\int_t^{t+\delta} [U_1(r+h)f](\mathbf{x})\mathrm{d}r &= \left[H\mathsf{B}^+ \int_{t-s}^{t+\delta-s} U_0(r+h)f\mathrm{d}r\right](\mathbf{y})\\ &= \left[H\mathsf{B}^+ \int_{t-s}^{t+\delta-s} U_0(r)U_0(h)f\mathrm{d}r\right](\mathbf{y})\end{aligned}$$

so that, using the definition of $U_1(r)$ again

$$\int_t^{t+\delta} [U_1(r+h)f](\mathbf{x})\mathrm{d}r = \int_t^{t+\delta} [U_1(r)U_0(h)f](\mathbf{x})\mathrm{d}r \quad \forall \delta > 0$$

from which we deduce that $U_1(t+h)f(\mathbf{x}) = U_1(t)U_0(h)f(\mathbf{x})$ for almost any $\mathbf{x} \in \mathbf{\Omega}_t$.With the notations of the previous proof, one has from (A.4) that

$$U_1(t+h)f(\mathbf{x}) = [U_0(t)U_1(h)f](\mathbf{x}) \quad \text{for a.e. } \mathbf{x} \in \mathbf{\Omega}_{t+h}\setminus\mathbf{\Omega}_t$$

This proves the result, since $U_1(t)f$ vanishes on $\mathbf{\Omega}_{t+h}\setminus\mathbf{\Omega}_t$ while $U_0(t)f$ vanishes on $\mathbf{\Omega}_t$. □

One has now the following

Lemma A.5. *For any $f \in \mathscr{D}_0$, the mapping $t \geqslant 0 \mapsto U_1(t)f \in X$ is differentiable with $\frac{\mathrm{d}}{\mathrm{d}t}U_1(t)f = U_1(t)\mathcal{T}_{\max,p}f$ for any $t \geqslant 0$.*

Proof. In virtue of the previous Lemma, it is enough to prove that $t \geqslant 0 \mapsto U_1(t)f \in X$ is differentiable at $t = 0$ with

$$\frac{\mathrm{d}}{\mathrm{d}t}U_1(t)f|_{t=0} = U_1(0)\mathcal{T}_{\max,p}f = 0.$$

Consider $t > 0$. One has

$$\|U_1(t)f\|_p^p \leqslant \|\!|\!|H|\!|\!\|^p \int_0^t \|\mathsf{B}^+U_0(s)f\|_{L^p_+}^p \mathrm{d}s.$$

Now, since $f \in \mathscr{D}(\mathcal{T}_{0,p})$, one has from (3.9)

$$\int_0^t \|\mathsf{B}^+U_0(s)f\|_{L^p_+}^p \mathrm{d}s \leqslant \frac{t^p}{p}\int_0^t \left(\int_{\Gamma_+} |[\mathcal{T}_{\max,p}f](\Phi(\mathbf{z},-s))|^p \chi_{\{s<\tau_-(\mathbf{z})\}}\mathrm{d}\mu_+(\mathbf{z})\right)\mathrm{d}s$$

so that

$$\frac{\|U_1(t)f\|_p^p}{t^p} \leqslant \frac{\|\!|\!|H|\!|\!\|^p}{p}\int_0^t \left(\int_{\Gamma_+} |[\mathcal{T}_{\max,p}f](\Phi(\mathbf{z},-s))|^p \chi_{\{s<\tau_-(\mathbf{z})\}}\mathrm{d}\mu_+(\mathbf{z})\right)\mathrm{d}s. \tag{A.5}$$

Using [5, Proposition 2.2], one has

$$\int_0^\infty \left(\int_{\Gamma_+} |[\mathcal{T}_{\max,\,p} f](\Phi(\mathbf{z}, -s))|^p \, \chi_{\{s<\tau_-(\mathbf{z})\}} \mathrm{d}\mu_+(\mathbf{z}) \right) \mathrm{d}s = \|\mathcal{T}_{\max,\,p} f\|_p^p < \infty$$

so that (A.5) yields

$$\lim_{t \to 0^+} \frac{\|U_1(t)f\|_p}{t} = 0.$$

This proves the result. □

Lemma A.6. *For any $f \in \mathscr{D}_0$ and any $t > 0$, one has $U_1(t)f \in \mathscr{D}(\mathcal{T}_{\max,\,p})$ with $\mathcal{T}_{\max,\,p} U_1(t)f = U_1(t)\mathcal{T}_{\max,\,p} f$.*

Proof. The proof follows from a simple application of Proposition A.2 where the assumptions (i)–(iii) are met thanks to the previous Lemmas. □

Let us now establish the following

Lemma A.7. *For any $f \in X$ and any $t > 0$, one has $\mathcal{I}_t^1[f] := \int_0^t U_1(s)f\mathrm{d}s \in \mathscr{D}(\mathcal{T}_{\max,\,p})$ with*

$$\mathcal{T}_{\max,\,p} \mathcal{I}_t^1[f] = U_1(t)f,$$

and $\mathsf{B}^{\pm}\mathcal{I}_t^1[f]\mathrm{d}s \in L^p_{\pm}$,

$$\mathsf{B}^- \mathcal{I}_t^1[f] = H\mathsf{B}^+ \int_0^t U_0(s)f\mathrm{d}s. \tag{A.6}$$

Moreover the mappings $t \geqslant 0 \mapsto \mathsf{B}^{\pm}\mathcal{I}_t^1[f]\mathrm{d}s \in L^p_{\pm}$ are continuous. Finally, for any $f \in \mathscr{D}_0$ and any $t \geqslant 0$, the traces $\mathsf{B}^{\pm}U_1(t)f \in L^p_{\pm}$ and the mappings $t \geqslant 0 \mapsto \mathsf{B}^{\pm}U_1(t)f \in L^p_{\pm}$ are continuous.

Proof. Thanks to the previous Lemmas, the family $(U_1(t))_{t\geqslant 0}$ satisfies assumptions (i)–(ii) of Proposition A.1. One deduces then from the same Proposition (point (1)) that, for any $f \in X$ and any $t > 0$, $\mathcal{I}_t^1[f] \in \mathscr{D}(\mathcal{T}_{\max,\,p})$ with $\mathcal{T}_{\max,\,p}\mathcal{I}_t^1[f] = U_1(t)f - U_1(0)f = U_1(t)f$.

To show that $\mathsf{B}^-\mathcal{I}_t^1[f]$ can be expressed through formula (A.6) we first suppose $f \in \mathscr{D}_0$. For such an f both $U_0(t)f$ and $U_1(t)f$ belong to $\mathscr{D}(\mathcal{T}_{\max,\,p})$ for any $t \geqslant 0$ with $\mathsf{B}^-U_1(t)f = H\mathsf{B}^+U_0(t)f \in L^p_-$. Using this equality, the continuity of H and Proposition A.1 (point 2) applied both to $(U_1(t))_{t\geqslant 0}$ and $(U_0(t))_{t\geqslant 0}$ one gets

$$\begin{aligned} \mathsf{B}^-\mathcal{I}_t^1[f] &= \int_0^t \mathsf{B}^-U_1(s)f\mathrm{d}s = \int_0^t H\mathsf{B}^+U_0(s)f\mathrm{d}s \\ &= H\left(\int_0^t \mathsf{B}^+U_0(s)f\mathrm{d}s\right) = H\mathsf{B}^+\mathcal{I}_t^0[f] \end{aligned}$$

i.e., (A.6) for $f \in \mathscr{D}_0$.

Consider now $f \in X$ and let $(f_n)_n \in \mathscr{D}_0$ be such that $\lim_n \|f_n - f\|_p = 0$. According to Eq. (3.8), the sequence $(\mathsf{B}^+\mathcal{I}_t^0[f_n])_n$ converges in L^p_+ towards $\mathsf{B}^+\mathcal{I}_t^0[f]$. Since (A.6) holds true for f_n, and H is continuous, then the sequence $(\mathsf{B}^-\mathcal{I}_t^1[f_n])_n$ converges in L^p_- to $H\mathsf{B}^+\mathcal{I}_t^0[f]$. One deduces from this that $\mathsf{B}^-\mathcal{I}_t^1[f] \in L^p_-$ with (A.6).

Moreover the mapping $t \geqslant 0 \mapsto \mathsf{B}^- I_t^1[f] \in L^p_-$ is continuous since both H and the mapping $t \geqslant 0 \mapsto \mathsf{B}^+\mathcal{I}_t^0[f] \in L^p_+$ are continuous (see Proposition 3.3). This property and Proposition A.1 imply that $\mathsf{B}^+\mathcal{I}_t^1[f] \in L^p_+$ and that the mapping $t \mapsto \mathsf{B}^+\mathcal{I}_t^1[f] \in L^p_+$ is continuous too.

Finally observe that, if $f \in \mathscr{D}_0$, then $f \in \mathscr{D}(\mathcal{T}_{\max,\,p})$ and for any $t \geqslant 0$ one has

$$\mathcal{I}_t^1\left[\mathcal{T}_{\max,\,p} f\right] = U_1(t)f.$$

Thus one can state that for any $f \in \mathscr{D}_0$ and any $t \geqslant 0$, the traces $\mathsf{B}^{\pm}U_1(t)f \in L^p_\pm$ and the mappings $t \geqslant 0 \mapsto \mathsf{B}^{\pm}U_1(t)f \in L^p_\pm$ are continuous. □

Let us now investigate Property (7):

Lemma A.8. *One has*

$$\int_0^t \|\mathsf{B}^+U_1(s)f\|^p_{L^p_+}\mathrm{d}s \leqslant \|\!|H|\!\|^p \int_0^t \|\mathsf{B}^+U_0(s)f\|^p_{L^p_+}\mathrm{d}s, \quad \forall t \geqslant 0, \forall f \in \mathscr{D}_0.$$

Proof. Given $f \in \mathscr{D}_0$, for any $s > 0$ and μ_+-a.e. $\mathbf{z} \in \Gamma_+$:

$$\left[\mathsf{B}^+U_1(s)f\right](\mathbf{z}) = \left[H\left(\mathsf{B}^+U_0(s-\tau_-(\mathbf{z}))f\right)\right](\Phi(\mathbf{z},-\tau_-(\mathbf{z}))\chi_{(0,s)}(\tau_-(\mathbf{z}))$$

Thus,

$$\begin{aligned} J &:= \int_0^t \|\mathsf{B}^+U_1(s)f\|^p_{L^p_+}\mathrm{d}s \\ &= \int_0^t \left(\int_{\Gamma_+} \left|\left[H\left(\mathsf{B}^+U_0(s-\tau_-(\mathbf{z}))\right)f\right](\Phi(\mathbf{z},-\tau_-(\mathbf{z})))\right|^p \chi_{(0,s)}(\tau_-(\mathbf{z}))\mathrm{d}\mu_+(\mathbf{z})\right)\mathrm{d}s. \end{aligned}$$

Now, using Fubini's Theorem and, for a given $\mathbf{z} \in \Gamma_+$, the change of variable $s \mapsto s - \tau_-(\mathbf{z})$, we get

$$J = \int_{\Gamma_+}\left(\int_0^{\max(0,t-\tau_-(\mathbf{z}))} \left|\left[H\left(\mathsf{B}^+U_0(s)\right)f\right](\Phi(\mathbf{z},-\tau_-(\mathbf{z})))\right|^p \mathrm{d}s\right)\mathrm{d}\mu_+(\mathbf{z}).$$

Using Fubini's Theorem again

$$\begin{aligned} J &\leqslant \int_0^t \left(\int_{\Gamma_+} \left|\left[H\left(\mathsf{B}^+U_0(s)\right)f\right](\Phi(\mathbf{z},-\tau_-(\mathbf{z})))\right|^p \mathrm{d}\mu_+(\mathbf{z})\right)\mathrm{d}s \\ &\leqslant \int_0^t \left(\int_{\Gamma_-} \left|\left[H\left(\mathsf{B}^+U_0(s)\right)f\right](\mathbf{y})\right|^p \mathrm{d}\mu_-(\mathbf{y})\right)\mathrm{d}s \end{aligned}$$

where we used [5, Eq. (2.5) in Prop. 2.2]. Therefore, it is easy to check that

$$J \leqslant \|H\|^p_{\mathscr{B}(L^p_+,L^p_+)} \int_0^t \|\mathsf{B}^+U_0(s)f\|^p_{L^p_+}\mathrm{d}s.$$

which is the desired result. □

We finally have the following

Lemma A.9. *Given $\lambda > 0$ and $f \in X$, set $F_1 = \int_0^\infty \exp(-\lambda t)U_1(t)f\mathrm{d}t$. Then $F_1 \in \mathscr{D}(\mathcal{T}_{\max,\,p})$ with $\mathcal{T}_{\max,\,p}F_1 = \lambda F_1$ and $\mathsf{B}^{\pm}F_1 \in L^p_\pm$ with*

$$\mathsf{B}^-F_1 = H\mathsf{B}^+C_\lambda f = HG_\lambda f \quad \mathsf{B}^+F_1 = (M_\lambda H)G_\lambda f.$$

Proof. Let us first assume $f \in \mathscr{D}_0$. Then, for any $\mathbf{y} \in \Gamma_-$, $s \in (0, \tau_+(\mathbf{y}))$:

$$F_1(\Phi(\mathbf{y},s)) = \int_s^\infty \exp(-\lambda t)\left[H\mathsf{B}^+U_0(t-s)f\right](\mathbf{y})\mathrm{d}t$$
$$= \exp(-\lambda s)\int_0^\infty \exp(-\lambda t)\left[H\mathsf{B}^+U_0(t)f\right](\mathbf{y})\mathrm{d}t$$
$$= \exp(-\lambda s)\left[H\mathsf{B}^+\left(\int_0^\infty \exp(-\lambda t)U_0(t)f\mathrm{d}t\right)\right](\mathbf{y})$$

i.e. $F_1(\Phi(\mathbf{y},s)) = \exp(-\lambda s)\left[H\mathsf{B}^+C_\lambda f\right](\mathbf{y})$. This exactly means that $F_1 = \Xi_\lambda H G_\lambda f$. By a density argument, this still holds for $f \in X$ and we get the desired result easily using the properties of Ξ_λ and G_λ. □

The above lemmas prove that the conclusion of Theorem 3.8 is true for $k = 1$. One proves then by induction that the conclusion is true for any $k \geqslant 1$ exactly as above. Details are left to the reader.

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