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An L^p-Approach to the Well-Posedness of Transport Equations Associated with a Regular Field: Part I

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Abstract. Transport equations associated with a Lipschitz field $\mathscr{F}$ on some subspace of $\mathbb{R}^N$ endowed with some general measure μ are considered. Our aim is to extend the results obtained in two previous contributions (Arlotti et al. in Mediterr J Math 6:367–402, 2009, Mediterr J Math 8:1–35, 2011) in the L^1-context to L^p-spaces $1 < p < \infty$. This is the first part of a two-part contribution (see in Arlotti and Lods An L^p-approach to the well-posedness of transport equations associated with a regular field—part II, Mediterr. J. Math. 16:145, 2019, for the second part) and we here establish the general mathematical framework we are dealing with and notably prove trace formula and uniqueness of boundary value transport problems with abstract boundary conditions. The abstract results of this first part will be used in the Part II of this work (Arlotti and Lods in Meditter J Math 16:145, 2019) to deal with general initial and boundary value problems and semigroup generation properties.

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Contents

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1. Introduction

This paper deals with the study in an L^p-setting ($p > 1$) of the general transport equation

$$\partial_t f(\mathbf{x},t) + \mathscr{F}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} f(\mathbf{x},t) = 0 \quad (\mathbf{x} \in \boldsymbol{\Omega},\ t > 0) \tag{1.1a}$$

supplemented by the abstract boundary condition

$$f_{|\Gamma_-}(\mathbf{y},t) = H(f_{|\Gamma_+})(\mathbf{y},t), \quad (\mathbf{y} \in \Gamma_-, t > 0) \tag{1.1b}$$

and the initial condition

$$f(\mathbf{x},0) = f_0(\mathbf{x}), \quad (\mathbf{x} \in \boldsymbol{\Omega}). \tag{1.1c}$$

The above problem was already examined by the authors in an L^1-setting in a series of papers [3,4] (in collaboration with Banasiak), and [1]. Aim of the present paper is to show that the theory and tools introduced in [3,4] can be extended to L^p-spaces with $1 < p < \infty$.

Let us make precise the setting we are considering in the present paper, which is somehow the one considered earlier in [3,4]. The set $\boldsymbol{\Omega}$ is a sufficiently smooth open subset of $\mathbb{R}^N$. We assume that $\mathbb{R}^N$ is endowed with a general positive Radon measure μ and that $\mathscr{F}$ is a restriction to $\boldsymbol{\Omega}$ of a time-independent globally Lipschitz vector field $\mathscr{F} : \mathbb{R}^N \to \mathbb{R}^N$. With this field we associate a flow $(T_t)_{t\in\mathbb{R}}$ (with the notations of Sect. 2.1, $T_t = \Phi(\cdot,t)$) and, as in [3], we assume the measure μ to be invariant under the flow $(T_t)_{t\in\mathbb{R}}$, i.e.

$$\mu(T_t A) = \mu(A) \text{ for any measurable subset } A \subset \mathbb{R}^N \text{ and any } t \in \mathbb{R}. \tag{1.2}$$

The sets $\Gamma_\pm$ appearing in (1.1b) are suitable boundaries of the phase space and the boundary operator H is a linear, but not necessarily *bounded*, operator between trace spaces $L^p_\pm$ corresponding to the boundaries $\Gamma_\pm$ (see Sect. 2 for details).

We refer to the papers [3,4] for the importance of the above transport equation in mathematical physics (Vlasov-like equations) and the link of assumption (1.2) with some generalized divergence free assumptions on $\mathscr{F}$. We also refer to the introduction of [4] and the references therein for an account of the relevant literature on the subject.

Here we only recall that transport problems of type (1.1) with general fields $\mathscr{F}$ have been studied in an L^1 context, in the special case in which the measure μ is the Lebesgue measure over $\mathbb{R}^N$, by Bardos [6] when the boundary conditions are the "no re-entry" boundary conditions (i.e. $H = 0$), by Beals–Protopopescu [8] when the boundary conditions are dissipative.

Furthermore, an optimal trace theory in L^p spaces has been developed by Cessenat [9,10] for the so-called free transport equation i. e. when μ is the Lebesgue measure and

$$\mathscr{F}(x) = (v, 0), \quad \mathbf{x} = (r, v) \in \mathbf{\Omega},$$

where $\mathbf{\Omega}$ is a cylindrical domain of the type $\mathbf{\Omega} = D \times \mathbb{R}^3 \subset \mathbb{R}^6$ (D being a sufficiently smooth open subset of $\mathbb{R}^3$).

For more general fields and for more general and abstract measures, the mathematical treatment of (1.1) is much more delicate. It requires an understanding of the intricate interplay between the geometry of the domain and the flow as well as their relation to the properties of the measure μ. Problems with a general measure μ and general fields have been addressed only very recently in [3,4] in an L^1-context.

The study of transport operators with L^p-spaces for $1 < p < \infty$ for abstract vector fields and abstract measure μ is new to our knowledge. As already mentioned in [4], the motivation of this abstract approach is to provide an unified treatment of first-order linear problems. This should allow to apply the same formalism to transport equations on an open subset of the Euclidian space $\mathbb{R}^N$ (in such a case μ is a restriction of the Lebesgue measure over $\mathbb{R}^N$) and to transport equations associated with flows on networks (where the measure μ is then supported on graphs, see, e.g. [5,7,12,14] and the reference therein). Several examples appearing in the literature will be dealt with in the second part of the paper.

Besides showing the robustness of the theory developed in [3,4], the present contribution—together with [2] which is its second part—provides a thorough analysis of a large variety of boundary operators arising in first-order partial equations—including unbounded boundary operators, dissipative, conservative and multiplicative boundary operators.

The organization of the paper is as follows. In Sect. 2, we recall the relevant results of [3]: the definition of the measures $\mu_\pm$ over $\Gamma_\pm$, the integration along the characteristic curves associated with $\mathscr{F}$. This allows us to give the precise definition of the transport operator $\mathcal{T}_{\max,\,p}$ in the L^p-context and gives the crucial link between $\mathcal{T}_{\max,\,p}$ and the operator $\mathcal{T}_{\max,\,1}$ which was thoroughly investigated in [3] (see Theorem 2.8). In Sect. 3, we apply the results of Sect. 2 to prove well-posedness of the time-dependent transport problem with no reentry boundary conditions and we generalize the trace theory of Cessenat [9,10] to more general fields and measures. The generalization is based on the construction of suitable trace spaces which are related to $L^p(\Gamma_\pm, \mathrm{d}\mu_\pm)$.

The abstract results obtained in the present paper is aimed in providing a thorough analysis of the semigroups associated with $\mathcal{T}_{\max}$ with general boundary operator in the second part [2].

2. Preliminary Results

We recall here the construction of characteristic curves, boundary measures $\mu_\pm$ and the maximal transport operator associated with (1.1) as established in

[3]. Most of the results in this section can be seen as technical generalizations to those of [3] and the proof of the main result of this section (Theorem 2.6) is deferred to Appendix A.

2.1. Integration Along Characteristic Curves

The definition of the transport operator (and the corresponding trace) involved in (1.1), [3], relies heavily on the characteristic curves associated with the field $\mathscr{F}$. Precisely, define the flow Φ: $\mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$, such that, for $(\mathbf{x}, t) \in \mathbb{R}^N \times \mathbb{R}$, the mapping $t \in \mathbb{R} \longmapsto \Phi(\mathbf{x}, t)$ is the only solution to the initial-value problem

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t}(t) = \mathscr{F}(\mathbf{X}(t)), \quad \forall t \in \mathbb{R}\,; \quad \mathbf{X}(0) = \mathbf{x} \in \mathbf{\Omega}. \tag{2.1}$$

Of course, solutions to (2.1) do not necessarily belong to $\mathbf{\Omega}$ for all times, leading to the definition of stay times of the characteristic curves in $\mathbf{\Omega}$ as well as the related *incoming* and *outgoing* parts of the boundary $\partial\mathbf{\Omega}$.

Definition 2.1. For any $\mathbf{x} \in \mathbf{\Omega}$, define $\tau_\pm(\mathbf{x}) = \inf\{s > 0; \Phi(\mathbf{x}, \pm s) \notin \mathbf{\Omega}\}$, with the convention that $\inf \varnothing = \infty$. Moreover, set

$$\Gamma_\pm := \{\mathbf{y} \in \partial\mathbf{\Omega}; \exists \mathbf{x} \in \mathbf{\Omega},\ \tau_\pm(\mathbf{x}) < \infty \text{ and } \mathbf{y} = \Phi(\mathbf{x}, \pm\tau_\pm(\mathbf{x}))\}\,. \tag{2.2}$$

Notice that the characteristic curves of the vector field $\mathscr{F}$ are not assumed to be of finite length and hence we introduce the sets

$$\mathbf{\Omega}_\pm = \{\mathbf{x} \in \mathbf{\Omega};\ \tau_\pm(\mathbf{x}) < \infty\}, \quad \mathbf{\Omega}_{\pm\infty} = \{\mathbf{x} \in \mathbf{\Omega};\ \tau_\pm(\mathbf{x}) = \infty\},$$

and $\Gamma_{\pm\infty} = \{\mathbf{y} \in \Gamma_\pm;\ \tau_\mp(\mathbf{y}) = \infty\}$. Then one can prove (see [3, Section 2]).

Proposition 2.2. *There are unique positive Borel measures* $\mathrm{d}\mu_\pm$ *on* $\Gamma_\pm$ *such that, for any* $h \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$,

$$\int_{\mathbf{\Omega}_\pm} h(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) = \int_{\Gamma_\pm} \mathrm{d}\mu_\pm(\mathbf{y}) \int_0^{\tau_\mp(\mathbf{y})} h\left(\Phi(\mathbf{y}, \mp s)\right)\mathrm{d}s, \tag{2.3}$$

and

$$\int_{\mathbf{\Omega}_\pm \cap \mathbf{\Omega}_{\mp\infty}} h(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) = \int_{\Gamma_{\pm\infty}} \mathrm{d}\mu_\pm(\mathbf{y}) \int_0^{\infty} h\left(\Phi(\mathbf{y}, \mp s)\right)\mathrm{d}s. \tag{2.4}$$

Moreover, for any $\psi \in L^1(\Gamma_-, \mathrm{d}\mu_-)$,

$$\int_{\Gamma_- \setminus \Gamma_{-\infty}} \psi(\mathbf{y})\mathrm{d}\mu_-(\mathbf{y}) = \int_{\Gamma_+ \setminus \Gamma_{+\infty}} \psi(\Phi(\mathbf{z}, -\tau_-(\mathbf{z})))\mathrm{d}\mu_+(\mathbf{z}). \tag{2.5}$$

2.2. The Maximal Transport Operator and Trace Results

The results of the previous section allow us to define the (maximal) transport operator $\mathcal{T}_{\max,\,p}$ as the weak derivative along the characteristic curves. To be precise, let us define the space of *test functions* $\mathfrak{Y}$ as follows:

Definition 2.3 (*Test functions*). Let $\mathfrak{Y}$ be the set of all measurable and bounded functions $\psi : \mathbf{\Omega} \to \mathbb{R}$ with compact support in $\mathbf{\Omega}$ and such that, for any $\mathbf{x} \in \mathbf{\Omega}$, the mapping

$$s \in (-\tau_-(\mathbf{x}), \tau_+(\mathbf{x})) \longmapsto \psi(\Phi(\mathbf{x}, s))$$

is continuously differentiable with

$$\mathbf{x} \in \boldsymbol{\Omega} \longmapsto \frac{\mathrm{d}}{\mathrm{d}s}\psi(\Phi(\mathbf{x},s))\Big|_{s=0} \quad \text{measurable and bounded.} \tag{2.6}$$

In the next step, we define the transport operator $(\mathcal{T}_{\max,p}, \mathscr{D}(\mathcal{T}_{\max,p}))$ in $L^p(\boldsymbol{\Omega}, \mathrm{d}\mu)$, $p \geqslant 1$.

Definition 2.4 (*Transport operator* $\mathcal{T}_{\max,p}$). Given $p \geqslant 1$, the domain of the maximal transport operator $\mathcal{T}_{\max,p}$ is the set $\mathscr{D}(\mathcal{T}_{\max,p})$ of all $f \in L^p(\boldsymbol{\Omega}, \mathrm{d}\mu)$ for which there exists $g \in L^p(\boldsymbol{\Omega}, \mathrm{d}\mu)$ such that

$$\int_{\boldsymbol{\Omega}} g(\mathbf{x})\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) = \int_{\boldsymbol{\Omega}} f(\mathbf{x})\frac{\mathrm{d}}{\mathrm{d}s}\psi(\Phi(\mathbf{x},s))\Big|_{s=0}\mathrm{d}\mu(\mathbf{x}), \quad \forall \psi \in \mathfrak{Y}.$$

In this case, $g =: \mathcal{T}_{\max,p} f$.

Remark 2.5. It is easily seen that, with this definition, $(\mathcal{T}_{\max,p}, \mathscr{D}(\mathcal{T}_{\max,p}))$ is a closed operator in $L^p(\boldsymbol{\Omega}, \mathrm{d}\mu)$. Indeed, if $(f_n)_n \subset \mathscr{D}(\mathcal{T}_{\max,p})$ is such that

$$\lim_n \|f_n - f\|_{L^p(\boldsymbol{\Omega},\mathrm{d}\mu)} = \lim_n \|\mathcal{T}_{\max,p} f_n - g\|_{L^p(\boldsymbol{\Omega},\mathrm{d}\mu)} = 0$$

for some $f, g \in L^p(\boldsymbol{\Omega}, \mathrm{d}\mu)$, then for any test function $\psi \in \mathfrak{Y}$, the identity

$$\int_{\boldsymbol{\Omega}} \mathcal{T}_{\max,p} f_n(\mathbf{x})\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) = \int_{\boldsymbol{\Omega}} f_n(\mathbf{x})\frac{\mathrm{d}}{\mathrm{d}s}\psi(\Phi(\mathbf{x},s))\Big|_{s=0}\mathrm{d}\mu(\mathbf{x})$$

holds for any $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, we obtain the identity

$$\int_{\boldsymbol{\Omega}} g(\mathbf{x})\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) = \int_{\boldsymbol{\Omega}} f(\mathbf{x})\frac{\mathrm{d}}{\mathrm{d}s}\psi(\Phi(\mathbf{x},s))\Big|_{s=0}\mathrm{d}\mu(\mathbf{x})$$

which proves that $f \in \mathscr{D}(\mathcal{T}_{\max,p})$ with $g = \mathcal{T}_{\max,p} f$.

2.3. Fundamental Representation Formula: Mild Formulation

Recall that, if f_1 and f_2 are two functions defined over $\boldsymbol{\Omega}$, we say that f_2 is a *representative of* f_1 if $\mu\{\mathbf{x} \in \boldsymbol{\Omega};\ f_1(\mathbf{x}) \neq f_2(\mathbf{x})\} = 0$, i.e. when $f_1(\mathbf{x}) = f_2(\mathbf{x})$ for μ-almost every $\mathbf{x} \in \boldsymbol{\Omega}$. The following fundamental result provides a characterization of the domain of $\mathscr{D}(\mathcal{T}_{\max,p})$:

Theorem 2.6. *Let $f \in L^p(\boldsymbol{\Omega}, \mu)$. The following are equivalent:*

(1) *There exists $g \in L^p(\boldsymbol{\Omega}, \mu)$ and a representative $f^\sharp$ of f such that, for μ-almost every $\mathbf{x} \in \boldsymbol{\Omega}$ and any $-\tau_-(\mathbf{x}) < t_1 \leqslant t_2 < \tau_+(\mathbf{x})$*

$$f^\sharp(\Phi(\mathbf{x},t_1)) - f^\sharp(\Phi(\mathbf{x},t_2)) = \int_{t_1}^{t_2} g(\Phi(\mathbf{x},s))\mathrm{d}s. \tag{2.7}$$

(2) *$f \in \mathscr{D}(\mathcal{T}_{\max,p})$. In this case, $g = \mathcal{T}_{\max,p} f$.*

Moreover, if f satisfies one of these equivalent conditions, then

$$\lim_{t \to 0+} f^\sharp(\Phi(\mathbf{y},t)) \tag{2.8}$$

exists for almost every $\mathbf{y} \in \Gamma_-$. Similarly, $\lim_{t\to 0+} f^\sharp(\Phi(\mathbf{y},-t))$ exists for almost every $\mathbf{y} \in \Gamma_+$.

The proof of the theorem is made of several steps following the approach developed in [3] in the L^1-context. The extension to L^p-space with $1 < p < \infty$ is somehow a technical generalization and we refer to Appendix A for details of the proof. Notice that the existence of the limit (2.8) can be proven exactly as in [3, Proposition 3.16]

The above representation theorem allows to define the trace operators.

Definition 2.7. For any $f \in \mathscr{D}(\mathcal{T}_{\max,p})$, define the *traces* $\mathsf{B}^{\pm}f$ by

$$\mathsf{B}^{+}f(\mathbf{y}) := \lim_{t \to 0+} f^{\sharp}(\Phi(\mathbf{y},-t)) \quad \text{and} \quad \mathsf{B}^{-}f(\mathbf{y}) := \lim_{t \to 0+} f^{\sharp}(\Phi(\mathbf{y},t))$$

for any $\mathbf{y} \in \Gamma_{\pm}$ for which the limits exist, where $f^{\sharp}$ is a suitable representative of f.

Notice that, by virtue of (2.7), it is clear that, for any $f \in \mathscr{D}(\mathcal{T}_{\max,p})$, the traces $\mathsf{B}^{\pm}f$ on $\Gamma_{\pm}$ are well defined and, for $\mu_{\pm}$-a.e. $\mathbf{z} \in \Gamma_{\pm}$,

$$\mathsf{B}^{\pm}f(\mathbf{z}) = f^{\sharp}(\Phi(\mathbf{z},\mp t)) \mp \int_0^t [\mathcal{T}_{\max,p}f](\Phi(\mathbf{z},\mp s))\mathrm{d}s, \quad \forall t \in (0,\tau_{\mp}(\mathbf{z})). \tag{2.9}$$

2.4. Additional Properties

An important general property of $\mathcal{T}_{\max,p}$ we shall need in the sequel is given by the following proposition, which makes the link between $\mathcal{T}_{\max,p}$ and the operator $\mathcal{T}_{\max,1}$ studied in [3].

Theorem 2.8. *Let $p \geqslant 1$ and $f \in \mathscr{D}(\mathcal{T}_{\max,p})$. Then $|f|^p \in \mathscr{D}(\mathcal{T}_{\max,1})$ and*

$$\mathcal{T}_{\max,1}|f|^p = p\,\mathrm{sign}(f)\,|f|^{p-1}\,\mathcal{T}_{\max,p}f \tag{2.10}$$

where $\mathrm{sign}(f)(\mathbf{x}) = 1$ *if* $f(\mathbf{x}) > 0$ *and* $\mathrm{sign}(f)(\mathbf{x}) = -1$ *if* $f(\mathbf{x}) < 0$ $(\mathbf{x} \in \mathbf{\Omega})$.

Remark 2.9. Observe that, since both f and $\mathcal{T}_{\max p}f$ belong to $L^p(\mathbf{\Omega},\mathrm{d}\mu)$, one sees that the right-hand side of (2.10) indeed belongs to $L^1(\mathbf{\Omega},\mathrm{d}\mu)$.

Proof. The proof follows the path of the version $p=1$ given in [4, Proposition 2.2]. Let $f \in \mathscr{D}(\mathcal{T}_{\max,p})$ and $\psi \in \mathfrak{Y}$ be fixed. We shall denote by $f^{\sharp}$ the representative of f given by Theorem 2.6. Using (2.3), one has

$$\begin{aligned}
&\int_{\mathbf{\Omega}_-} |f(\mathbf{x})|^p \frac{\mathrm{d}}{\mathrm{d}s}\psi(\Phi(\mathbf{x},s))\big|_{s=0}\mathrm{d}\mu(\mathbf{x})\\
&\quad= \int_{\Gamma_-}\mathrm{d}\mu_-(\mathbf{y})\int_0^{\tau_+(\mathbf{y})} |f(\Phi(\mathbf{y},t))|^p\frac{\mathrm{d}}{\mathrm{d}t}\psi(\Phi(\mathbf{y},t))\mathrm{d}t\\
&\quad= \int_{\Gamma_-}\mathrm{d}\mu_-(\mathbf{y})\int_0^{\tau_+(\mathbf{y})} |f^{\sharp}(\Phi(\mathbf{y},t))|^p\frac{\mathrm{d}}{\mathrm{d}t}\psi(\Phi(\mathbf{y},t))\mathrm{d}t.
\end{aligned}$$

Let us fix $\mathbf{y} \in \Gamma_-$ and introduce $I_{\mathbf{y}} := \{t \in (0,\tau_+(\mathbf{y}));\ f^{\sharp}(\Phi(\mathbf{y},t)) > 0\}$. As in [4, Proposition 2.2], there exists a sequence of mutually disjoint intervals $(I_k(\mathbf{y}))_k = (s_k^-(\mathbf{y}),s_k^+(\mathbf{y}))_k \subset (0,\tau_+(\mathbf{y}))$ such that

$$I_{\mathbf{y}} = \bigcup_{k \in \mathbb{N}} \left(s_k^-(\mathbf{y})\,,\,s_k^+(\mathbf{y})\right).$$

We have

$$\int_{I_{\mathbf{y}}} |f^\sharp(\Phi(\mathbf{y},t))|^p \frac{\mathrm{d}}{\mathrm{d}t}\psi(\Phi(\mathbf{y},t))\mathrm{d}t = \sum_k \int_{s_k^-(\mathbf{y})}^{s_k^+(\mathbf{y})} \left(f^\sharp(\Phi(\mathbf{y},t))\right)^p \frac{\mathrm{d}}{\mathrm{d}t}\psi(\Phi(\mathbf{y},t))\mathrm{d}t.$$

Let us distinguish several cases. If $s_k^-(\mathbf{y}) \neq 0$ and $s_k^+(\mathbf{y}) \neq \tau_+(\mathbf{y})$ then, from the continuity of $t \mapsto f^\sharp(\Phi(\mathbf{y},t))$, we see that $f^\sharp\left(\Phi(\mathbf{y},s_k^-(\mathbf{y}))\right) = f^\sharp\left(\Phi(\mathbf{y},s_k^+(\mathbf{y}))\right) = 0$. Using (2.7) on the interval $(s_k^-(\mathbf{y}), s_k^+(\mathbf{y}))$, a simple integration by parts leads to

$$\int_{s_k^-(\mathbf{y})}^{s_k^+(\mathbf{y})} \left(f^\sharp(\Phi(\mathbf{y},t))\right)^p \frac{\mathrm{d}}{\mathrm{d}t}\psi(\Phi(\mathbf{y},t))\mathrm{d}t = \int_{s_k^-(\mathbf{y})}^{s_k^+(\mathbf{y})} \psi(\Phi(\mathbf{y},t))F(\Phi(\mathbf{y},t))\mathrm{d}t, \tag{2.11}$$

where

$$F := p\,\mathrm{sign}(f^\sharp)\,|f^\sharp|^{p-1}\,\mathcal{T}_{\max,\,p}f.$$

As already observed, $F \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$. Next, we consider the case when $s_k^-(\mathbf{y}) = 0$ or $s_k^+(\mathbf{y}) = \tau_+(\mathbf{y}) < \infty$ for some k. Using the fact that ψ is of compact support in $\mathbf{\Omega}$ while $\Phi(\mathbf{y}, s_k^+(\mathbf{y})) \in \partial\mathbf{\Omega}$, one proves again (2.11) integrating by parts. The last case to consider is $s_k^+(\mathbf{y}) = \tau_+(\mathbf{y}) = \infty$ for some k. We shall use [3, Lemma 3.3] according to which for μ_- almost every $\mathbf{y} \in \Gamma_-$ there is a sequence $(t_n)_n$ such that $t_n \to \infty$ and $\psi(\Phi(t_n, \mathbf{y})) = 0$. Thus, focusing our attention on such $\mathbf{y}$s, as in the proof of [3, Theorem 3.6], integration by parts gives

$$\int_{s_k^-(\mathbf{y})}^{t_n} \left(f^\sharp(\Phi(\mathbf{y},t))\right)^p \frac{\mathrm{d}}{\mathrm{d}t}\psi(\Phi(\mathbf{y},t))\mathrm{d}t = \int_{s_k^-(\mathbf{y})}^{t_n} \psi(\Phi(\mathbf{y},t))F(\Phi(\mathbf{y},t))\mathrm{d}t$$

for any n and, by integrability of both sides, we prove Formula (2.11) for $s_k^+(\mathbf{y}) = \tau_+(\mathbf{y}) = \infty$. In other words, (2.11) is true for any $k \in \mathbb{N}$, and summing up over $\mathbb{N}$, we finally get

$$\int_{I_{\mathbf{y}}} |f^\sharp(\Phi(\mathbf{y},t))|^p \frac{\mathrm{d}}{\mathrm{d}t}\psi(\Phi(\mathbf{y},t))\mathrm{d}t = \int_{I_{\mathbf{y}}} \psi(\Phi(\mathbf{y},t))F(\Phi(\mathbf{y},t))\mathrm{d}t.$$

Arguing in the same way on $J_{\mathbf{y}} = \{t \in (0, \tau_+(\mathbf{y}));\ f^\sharp(\Phi(\mathbf{y},t)) < 0\}$, we get

$$\int_{J_{\mathbf{y}}} |f^\sharp(\Phi(\mathbf{y},t))|^p \frac{\mathrm{d}}{\mathrm{d}t}\psi(\Phi(\mathbf{y},t))\mathrm{d}t = \int_{J_{\mathbf{y}}} \psi(\Phi(\mathbf{y},t))F(\Phi(\mathbf{y},t))\mathrm{d}t,$$

where, obviously, $|f^\sharp(\Phi(\mathbf{y},t))| = -f^\sharp(\Phi(\mathbf{y},t))$ for any $t \in J_{\mathbf{y}}$. Now, integration over Γ_- leads to

$$\int_{\mathbf{\Omega}_-} |f(\mathbf{x})|^p \frac{\mathrm{d}}{\mathrm{d}s}\psi(\Phi(\mathbf{x},s))\big|_{s=0}\mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}_-} F(\mathbf{x})\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}).$$

Using now parametrization over Γ_+, we prove in the same way that

$$\int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_{-\infty}} |f(\mathbf{x})|^p \frac{\mathrm{d}}{\mathrm{d}s}\psi(\Phi(\mathbf{x},s))\big|_{s=0}\mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_{-\infty}} F(\mathbf{x})\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}).$$

In the same way, following the proof of [4, Proposition 2.2], one gets that

$$\int_{\mathbf{\Omega}_{+\infty}\cap\mathbf{\Omega}_{-\infty}} |f(\mathbf{x})|^p \frac{\mathrm{d}}{\mathrm{d}s}\psi(\Phi(\mathbf{x},s))\big|_{s=0}\mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}} F(\mathbf{x})\,\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}),$$

where we notice that assumption (1.2) is crucial at this stage. Therefore, one sees that

$$\int_{\mathbf{\Omega}} |f(\mathbf{x})|^p \frac{\mathrm{d}}{\mathrm{d}s}\psi(\Phi(\mathbf{x},s))\big|_{s=0}\mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}} F(\mathbf{x})\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) \quad \forall \psi \in \mathfrak{Y}.$$

Since $F \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$, this exactly means that $|f|^p \in \mathscr{D}(\mathcal{T}_{\max,\,1})$ with $\mathcal{T}_{\max,\,1}$ $|f|^p = F$ and the proof is complete. □

We can now generalize Green's formula:

Proposition 2.10 (Green's formula). *Let $f \in \mathscr{D}(\mathcal{T}_{\max,\,p})$ satisfies $\mathsf{B}^-f \in L^p_-$. Then $\mathsf{B}^+f \in L^p_+$ and*

$$\|\mathsf{B}^-f\|^p_{L^p_-} - \|\mathsf{B}^+f\|^p_{L^p_+} = p\int_{\mathbf{\Omega}} \mathrm{sign}(f)\,|f|^{p-1}\,\mathcal{T}_{\max,\,p}f\,\mathrm{d}\mu. \tag{2.12}$$

Proof. Let $f \in \mathscr{D}(\mathcal{T}_{\max,\,p})$ with $\mathsf{B}^-f \in L^p_-$, be given. Let $F = |f|^p$. One checks without difficulty that $|\mathsf{B}^{\pm}f|^p = \mathsf{B}^{\pm}|f|^p$ while, from the previous result, $F \in \mathscr{D}(\mathcal{T}_{\max,\,1})$. Since $\mathsf{B}^-F \in L^1_-$, applying the L^1-version of Green's formula [3, Proposition 4.4], we get

$$\int_{\mathbf{\Omega}} \mathcal{T}_{\max,1}|f|^p\mathrm{d}\mu = \int_{\Gamma_-} \mathsf{B}^-|f|^p\mathrm{d}\mu_- - \int_{\Gamma_+} \mathsf{B}^+|f|^p\mathrm{d}\mu_+$$

which gives exactly the result thanks to Theorem 2.8. □

3. Well-Posedness for Initial and Boundary Value Problems

3.1. Absorption Semigroup

From now on, we fix $p > 1$ and we will denote $X = L^p(\mathbf{\Omega}, \mathrm{d}\mu)$ endowed with its natural norm $\|\cdot\|_p$. The conjugate exponent will always be denoted by q, i.e. $1/p + 1/q = 1$. Let $\mathcal{T}_{0,\,p}$ be the free streaming operator with *no re-entry boundary conditions*:

$$\mathcal{T}_{0,\,p}\psi = \mathcal{T}_{\max,\,p}\psi, \qquad \text{for any } \psi \in \mathscr{D}(\mathcal{T}_{0,\,p}),$$

where the domain $\mathscr{D}(\mathcal{T}_{0,\,p})$ is defined by

$$\mathscr{D}(\mathcal{T}_{0,\,p}) = \{\psi \in \mathscr{D}(\mathcal{T}_{\max,\,p});\ \mathsf{B}^-\psi = 0\}.$$

We state the following generation result:

Theorem 3.1. *The operator $(\mathcal{T}_{0,\,p}, \mathscr{D}(\mathcal{T}_{0,\,p}))$ is the generator of a nonnegative C_0-semigroup of contractions $(U_0(t))_{t\geqslant 0}$ in $L^p(\mathbf{\Omega}, \mathrm{d}\mu)$ given by*

$$U_0(t)f(\mathbf{x}) = f(\Phi(\mathbf{x},-t))\chi_{\{t<\tau_-(\mathbf{x})\}}(\mathbf{x}), \quad (\mathbf{x} \in \mathbf{\Omega},\ f \in X), \tag{3.1}$$

where χ_A denotes the characteristic function of a set A.

Proof. Let us first check that the family of operators $(U_0(t))_{t\geqslant 0}$ is a nonnegative contractive C_0-semigroup in X. As in [3, Theorem 4.1], for any $f \in X$ and any $t \geqslant 0$, the mapping $U_0(t)f : \mathbf{\Omega} \to \mathbb{R}$ is measurable and the semigroup properties $U_0(0)f = f$ and $U_0(t)U_0(s)f = U_0(t+s)f$ $(t, s \geqslant 0)$ hold. Let us now show that $\|U_0(t)f\|_p \leqslant \|f\|_p$. We have

$$\|U_0(t)f\|_p^p = \int_{\mathbf{\Omega}_+} |U_0(t)f|^p \mathrm{d}\mu + \int_{\mathbf{\Omega}_- \cap \mathbf{\Omega}_{+\infty}} |U_0(t)f|^p \mathrm{d}\mu + \int_{\mathbf{\Omega}_{-\infty} \cap \mathbf{\Omega}_{+\infty}} |U_0(t)f|^p \mathrm{d}\mu.$$

As in [3, Theorem 4.1], one checks that

$$\int_{\mathbf{\Omega}_- \cap \mathbf{\Omega}_{+\infty}} |U_0(t)f|^p \mathrm{d}\mu = \int_{\mathbf{\Omega}_- \cap \mathbf{\Omega}_{+\infty}} |f|^p \mathrm{d}\mu, \quad \int_{\mathbf{\Omega}_{-\infty} \cap \mathbf{\Omega}_{+\infty}} |U_0(t)f|^p \mathrm{d}\mu = \int_{\mathbf{\Omega}_{-\infty} \cap \mathbf{\Omega}_{+\infty}} |f|^p \mathrm{d}\mu.$$

Therefore,

$$\|f\|_p^p - \|U_0(t)f\|_p^p = \int_{\mathbf{\Omega}_+} |f|^p \mathrm{d}\mu - \int_{\mathbf{\Omega}_+} |U_0(t)|^p \mathrm{d}\mu.$$

Now, using (2.3) together with the expression of $U_0(t)f$ in (3.1), we get

$$\int_{\mathbf{\Omega}_+} |U_0(t)f|^p \mathrm{d}\mu = \int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{z}) \int_t^{\max(t, \tau_-(\mathbf{z}))} |f(\Phi(\mathbf{z}, -s))|^p \mathrm{d}s,$$

so that

$$\|f\|_p^p - \|U_0(t)f\|_p^p = \int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{z}) \int_0^t |f(\Phi(\mathbf{z}, -s))|^p \chi_{\{s < \tau_-(\mathbf{z})\}} \mathrm{d}s. \quad (3.2)$$

This proves that $\|U_0(t)f\|_p \leqslant \|f\|_p$, i.e. $(U_0(t))_{t\geqslant 0}$ is a contraction semigroup. The rest of the proof is as in [3, Theorem 4.1] since it involves only "pointwise" estimates. □

3.2. Some Useful Operators

We introduce here some linear operators which will turn useful in the study of boundary value problem. We start with

$$C_\lambda := (\lambda - \mathcal{T}_{0,\,p})^{-1}, \quad \forall \lambda > 0.$$

Since

$$C_\lambda f = \int_0^\infty \exp(-\lambda\, t) U_0(t) f \mathrm{d}t, \quad \forall f \in X, \ \ \lambda > 0$$

one sees that

$$\begin{cases} C_\lambda \,:\, X \longrightarrow \mathscr{D}(\mathcal{T}_{0,\,p}) \subset X \\ \quad f \longmapsto [C_\lambda f]\,(\mathbf{x}) = \displaystyle\int_0^{\tau_-(\mathbf{x})} f(\Phi(\mathbf{x}, -s)) \exp(-\lambda s) \mathrm{d}s, \quad \mathbf{x} \in \mathbf{\Omega}. \end{cases}$$

In particular, $\|C_\lambda f\|_p \leqslant \frac{1}{\lambda}\|f\|_p$ for all $\lambda > 0$, $f \in X$. Introduce then, for all $f \in X$,

$$G_\lambda f = \mathsf{B}^+ C_\lambda f, \quad \forall \lambda > 0, f \in X.$$

According to Green's formula, $G_\lambda f \in L^p_+$ and one has

$$\begin{cases} G_\lambda \, : \, X \longrightarrow L^p_+ \\ \quad f \longmapsto \left[G_\lambda f\right](\mathbf{z}) = \displaystyle\int_0^{\tau_-(\mathbf{z})} f(\Phi(\mathbf{z}, -s)) \exp(-\lambda s) \mathrm{d}s, \quad \mathbf{z} \in \Gamma_+ \, . \end{cases}$$

One has then the following:

Lemma 3.2. *For any $\lambda > 0$ and any $f \in X$, one has*

$$\|G_\lambda f\|^p_{L^p_+} + \lambda\, p \|C_\lambda f\|^p_p = p \int_{\mathbf{\Omega}} \mathrm{sign}(C_\lambda f)\, |C_\lambda f|^{p-1}\, f \mathrm{d}\mu. \tag{3.3}$$

In particular,

$$\|G_\lambda f\|^p_{L^p_+} + \lambda\, p \|C_\lambda f\|^p_p \leqslant p \|C_\lambda f\|^{p-1}_p \|f\|_p.$$

Proof. Given $f \in X$ and $\lambda > 0$, let $g = C_\lambda f = (\lambda - \mathcal{T}_{0,\,p})^{-1} f$. One has $g \in \mathscr{D}(\mathcal{T}_{0,\,p})$, i.e. $\mathsf{B}^- g = 0$. Green's formula (Proposition 2.10) gives

$$\|G_\lambda f\|^p_{L^p_+} = \|\mathsf{B}^+ g\|^p_{L^p_+} = -p \int_{\mathbf{\Omega}} \mathrm{sign}(g) |g|^{p-1} \mathcal{T}_{\max,\,p} g \mathrm{d}\mu$$

and, since $\mathcal{T}_{\max,\,p} g = \mathcal{T}_{0,\,p} g = \lambda\, g - f$, we get

$$\|G_\lambda f\|^p_{L^p_+} = -\lambda\, p \int_{\mathbf{\Omega}} |g|^p \mathrm{d}\mu + p \int_{\mathbf{\Omega}} \mathrm{sign}(g) |g|^{p-1}\, f \mathrm{d}\mu$$

which gives (3.3) since $g = C_\lambda f$. The second part of the result comes from Hölder's inequality since $\mathrm{sign}(g)|g|^{p-1} \in L^q(\mathbf{\Omega}, \mathrm{d}\mu)$ with $1/q + 1/p = 1$. □

3.3. Generalized Cessenat's Theorems

The theory and tools we have recalled in the previous section allow us to carry out a more detailed study of the trace operators. First of all, we show that Cessenat's trace result [9,10] can be generalized to our case:

Theorem 3.3. *Define the following measures over $\Gamma_\pm$:*

$$\mathrm{d}\xi_\pm(\mathbf{y}) = \min\left(\tau_\mp(\mathbf{y}), 1\right) \mathrm{d}\mu_\pm(\mathbf{y}), \quad \mathbf{y} \in \Gamma_\pm$$

and set

$$Y^\pm_p := L^p(\Gamma_\pm, \mathrm{d}\xi_\pm)$$

with usual norm. Then, for any $f \in \mathscr{D}(\mathcal{T}_{\max,\,p})$, the trace $\mathsf{B}^\pm f$ belongs to $Y^\pm_p$ with

$$\|\mathsf{B}^\pm f\|^p_{Y^\pm_p} \leqslant 2^{p-1} \left(\|f\|^p_p + \|\mathcal{T}_{\max,\,p} f\|^p_p\right), \quad f \in \mathscr{D}(\mathcal{T}_{\max,\,p}).$$

Proof. The proof is an almost straightforward application of the representation formula (2.9). The proof is similar to the one given in [4, Theorem 3.1] for $p = 1$, namely let $f \in \mathscr{D}(\mathcal{T}_{\max,\,p})$ be fixed. It is clear from (2.9) that the mapping $\mathbf{y} \in \Gamma_- \mapsto \mathsf{B}^- f(\mathbf{y})$ is measurable. Now, for μ_--almost every $\mathbf{y} \in \Gamma_-$, one has

$$\left|\mathsf{B}^- f(\mathbf{y})\right|^p \leqslant 2^{p-1}|f^\sharp(\Phi(\mathbf{y},s))|^p + 2^{p-1}\left(\int_0^s |\mathcal{T}_{\max,\,p} f(\Phi(\mathbf{y},r))|\mathrm{d}r\right)^p,$$
$$\forall 0 < s < \tau_+(\mathbf{y}).$$

Now, for any $0 < s < t < \min(1, \tau_+(\mathbf{y}))$, using first Hölder inequality, we get

$$\left(\int_0^s |\mathcal{T}_{\max,\,p} f(\Phi(\mathbf{y},r))|\mathrm{d}r\right)^p \leqslant s^{\frac{p}{q}} \int_0^s |\mathcal{T}_{\max,\,p} f(\Phi(\mathbf{y},r))|^p\mathrm{d}r$$
$$\leqslant \int_0^s |\mathcal{T}_{\max,\,p} f(\Phi(\mathbf{y},r))|^p\mathrm{d}r.$$

Integrating the above inequality with respect to s over $(0,t)$ leads to

$$t\left|\mathsf{B}^- f(\mathbf{y})\right|^p = \int_0^t \left|\mathsf{B}^- f(\mathbf{y})\right|^p \mathrm{d}s \leqslant 2^{p-1}\int_0^t |f^\sharp(\Phi(\mathbf{y},s))|^p\mathrm{d}s$$
$$+ 2^{p-1}\int_0^t |\mathcal{T}_{\max,\,p} f(\Phi(\mathbf{y},s))|^p\mathrm{d}s$$

since $t \leqslant 1$. We conclude exactly as in the proof of [4, Theorem 3.1]. □

Remark 3.4. A simple consequence of the above continuity result is the following: if $(f_n)_n \subset \mathscr{D}(\mathcal{T}_{\max,\,p})$ is such that

$$\lim_n \left(\|f_n - f\|_p + \|\mathcal{T}_{\max,\,p} f_n - \mathcal{T}_{\max,\,p} f\|_p\right) = 0$$

then $(\mathsf{B}^\pm f_n)_n$ converges to $\mathsf{B}^\pm f$ in $Y_p^\pm$.

Clearly,

$$L^p_\pm = L^p(\Gamma_\pm, \mathrm{d}\mu_\pm) \hookrightarrow Y_p^\pm, \tag{3.4}$$

where the embedding is continuous (it is a contraction). Define then, for all $\lambda > 0$ and any $u \in Y_p^-$:

$$\begin{cases} [M_\lambda u]\,(\mathbf{z}) = u(\Phi(\mathbf{z}, -\tau_-(\mathbf{z})))\exp\left(-\lambda\tau_-(\mathbf{z})\right)\chi_{\{\tau_-(\mathbf{z})<\infty\}}, & \mathbf{z} \in \Gamma_+,\\ [\Xi_\lambda u]\,(\mathbf{x}) = u(\Phi(\mathbf{x}, -\tau_-(\mathbf{x})))\exp\left(-\lambda\tau_-(\mathbf{x})\right)\chi_{\{\tau_-(\mathbf{x})<\infty\}}, & \mathbf{x} \in \boldsymbol{\Omega}. \end{cases}$$

We also introduce the following measures on $\Gamma_\pm$:

$$\mathrm{d}\tilde{\mu}_{\pm,p}(\mathbf{y}) := \left(\min(\tau_\mp(\mathbf{y}),\, 1)\right)^{1-p}\,\mathrm{d}\mu_\pm(\mathbf{y}), \quad \mathbf{y} \in \Gamma_\pm$$

and set $\widetilde{\mathcal{Y}}_{\pm,\,p} = L^p(\Gamma_\pm,\, \mathrm{d}\tilde{\mu}_{\pm,p})$ with the usual norm. Notice that $\mathrm{d}\tilde{\mu}_{\pm\,p}$ is absolutely continuous with respect to $\mathrm{d}\mu_\pm$ and the embedding

$$\widetilde{\mathcal{Y}}_{\pm,\,p} \hookrightarrow L^p(\Gamma_\pm, \mathrm{d}\mu_\pm) =: L^p_\pm \tag{3.5}$$

is continuous since it is a contraction. One has the following result:

Lemma 3.5. *Let $\lambda > 0$ be given. Then*

$$M_\lambda \in \mathscr{B}(Y_p^-, Y_p^+) \quad \text{and} \quad \Xi_\lambda \in \mathscr{B}(Y_p^-, X).$$

Moreover, given $u \in Y_p^-$ it holds:

(1) $\Xi_\lambda u \in \mathscr{D}(\mathcal{T}_{\max,\,\mathrm{p}})$ *with*

$$\mathcal{T}_{\max,\,p}\Xi_\lambda u = \lambda \Xi_\lambda u, \quad \mathsf{B}^-\Xi_\lambda u = u, \;\; \mathsf{B}^+\Xi_\lambda u = M_\lambda u. \tag{3.6}$$

(2) $M_\lambda u \in L^p_+$ *if and only if* $u \in L^p_-$.

(3) $M_\lambda u \in \widetilde{\mathscr{Y}}_{+,\,p}$ *if and only if* $u \in \widetilde{\mathscr{Y}}_{-,\,p}$.

Proof. Let $\lambda > 0$ and $u \in Y_p^-$ be fixed. From the definition of Ξ_λ, one sees that

$$|[\Xi_\lambda u](\Phi(\mathbf{y}, t))|^p = |u(\mathbf{y})|^p \exp(-\lambda\, p\, t), \quad \forall \mathbf{y} \in \Gamma_-, \quad 0 < t < \tau_+(\mathbf{y}), \tag{3.7}$$

and, thanks to Proposition 2.2:

$$\begin{aligned}\int_{\mathbf{\Omega}} |[\Xi_\lambda u](\mathbf{x})|^p \mathrm{d}\mu(\mathbf{x}) &= \int_{\Gamma_-} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} |u(\mathbf{y})|^p \exp(-\lambda\, p\, t)\mathrm{d}t \\ &= \frac{1}{\lambda\, p} \int_{\Gamma_-} (1 - \exp(-\lambda\, p\, \tau_+(\mathbf{y})))\, |u(\mathbf{y})|^p \mathrm{d}\mu_-(\mathbf{y}) \\ &\leqslant \max\left(1, \frac{1}{\lambda\, p}\right) \int_{\Gamma_-} |u(\mathbf{y})|^p \mathrm{d}\xi_-(\mathbf{y}),\end{aligned}$$

where, as in [4, Theorem 3.2], we used that $(1 - \exp(-s)) \leqslant \min(1, s)$ for all $s \geqslant 0$. This shows, in particular, that

$$\|\Xi_\lambda u\|_p^p \leqslant \max\left(1, \frac{1}{\lambda\, p}\right) \|u\|^p_{Y_p^-}.$$

Moreover, arguing as in [4, Lemma 3.1], one has

$$\begin{aligned}\int_{\Gamma_+} |[M_\lambda u](\mathbf{z})|^p \mathrm{d}\xi_+(\mathbf{z}) &= \int_{\Gamma_+ \setminus \Gamma_{+\infty}} \exp(-\lambda\, p \tau_-(\mathbf{z}))|u(\Phi(\mathbf{z}, -\tau_-(\mathbf{z}))|^p \mathrm{d}\xi_+(\mathbf{z}) \\ &= \int_{\Gamma_- \setminus \Gamma_{-\infty}} \exp(-\lambda\, p \tau_+(\mathbf{y}))|u(\mathbf{y})|^p \mathrm{d}\xi_-(\mathbf{y}) \leqslant \|u\|^p_{Y_p^-}.\end{aligned} \tag{3.8}$$

This shows the first part of the Lemma. To prove that $\Xi_\lambda u \in \mathscr{D}(\mathcal{T}_{\max,\,p})$ one argues as in the proof of [3, Theorem 4.2] to get that $f_2 = \Xi_\lambda u$ satisfies $\mathcal{T}_{\max,\,p} f_2 = \lambda\, f_2$. Moreover, it is clear from the definition of B^+ that $\mathsf{B}^+\Xi_\lambda u = M_\lambda u$. This shows point (1). To prove (2), we first notice that, for $u \in L^p_-$, as in (3.8), one sees that

$$\int_{\Gamma_+} |[M_\lambda u](\mathbf{z})|^p \mathrm{d}\xi_+(\mathbf{z}) = \int_{\Gamma_- \setminus \Gamma_{-\infty}} |u(\mathbf{y})|^p \exp(-p\lambda \tau_+(\mathbf{y}))\mathrm{d}\mu_-(\mathbf{y}) \leqslant \|u\|^p_{L^p_-}. \tag{3.9}$$

This together with the embedding (3.5) shows that $M_\lambda u \in L^p_+$. Conversely, assume that $M_\lambda u \in L^p_+$ and define

$$\Gamma_{-,1} = \{\mathbf{y} \in \Gamma_-;\, \tau_+(\mathbf{y}) \leqslant 1\}, \quad \Gamma_{-,2} = \Gamma_- \backslash \Gamma_{-,1}.$$

One has $\int_{\Gamma_{-,2}} |u(\mathbf{y})|^p \mathrm{d}\xi_-(\mathbf{y}) = \int_{\Gamma_{-,2}} |u(\mathbf{y})|^p \mathrm{d}\mu_-(\mathbf{y}) < \infty$. Moreover, since $\lambda s + \exp(-\lambda s) \geqslant 1$ for any $s \geqslant 0$, one has

$$\begin{aligned}\int_{\Gamma_{-,1}} |u(\mathbf{y})|^p \mathrm{d}\mu_-(\mathbf{y}) &\leqslant \int_{\Gamma_{-,1}} (\lambda p \tau_+(\mathbf{y}) + \exp(-\lambda p \tau_+(\mathbf{y}))\, |u(\mathbf{y})|^p \mathrm{d}\mu_-(\mathbf{y}) \\ &\leqslant \lambda p \int_{\Gamma_{-,1}} |u(\mathbf{y})|^p \mathrm{d}\xi_-(\mathbf{y}) \\ &\quad + \int_{\Gamma_- \backslash \Gamma_{-\infty}} \exp(-\lambda p \tau_+(\mathbf{y}))|u(\mathbf{y})|^p \mathrm{d}\mu_-(\mathbf{y}) \\ &= \lambda p \int_{\Gamma_{-,1}} |u(\mathbf{y})|^p \mathrm{d}\xi_-(\mathbf{y}) + \int_{\Gamma_+} |[M_\lambda u](\mathbf{z})|^p \mathrm{d}\mu_+(\mathbf{z}),\end{aligned}$$

according to (3.8). This shows that $u \in L^p_-$ and proves the second point.

It is clear now that, if $u \in \widetilde{Y}_{-,\,p}$, then $M_\lambda u \in \widetilde{Y}_{+,\,p}$. Conversely, assume that $M_\lambda u \in \widetilde{Y}_{+,\,p}$. To prove that $u \in \widetilde{Y}_{-,\,p}$, we only have to focus on the integral over $\Gamma_{+,1}$ since the measures $\mathrm{d}\tilde{\mu}_{+,p}$ and $\mathrm{d}\xi_+$ coincide on $\Gamma_{+,2}$. Then, by assumption, it holds $I_1 < \infty$ with

$$I_1 := \int_{\Gamma_{+,1}} |u(\Phi(\mathbf{z}, -\tau_-(\mathbf{z}))|^p \exp\left(-p\lambda\, \tau_-(\mathbf{z})\right) \tau_-(\mathbf{z})^{1-p} \mathrm{d}\mu_+(\mathbf{z}).$$

Notice that I_1 can be written as

$$I_1 = \int_{\Gamma_{-,1}} |u(\mathbf{y})|^p \exp\left(-p\lambda\, \tau_+(\mathbf{y})\right) \tau_+(\mathbf{y})^{1-p} \mathrm{d}\mu_-(\mathbf{y}),$$

and, since $\exp\left(-p\lambda\, \tau_+(\mathbf{y})\right) \geqslant \exp(-\lambda p)$ for any $\mathbf{y} \in \Gamma_{-,1}$, we get

$$\int_{\Gamma_{-,1}} |u(\mathbf{y})|^p \mathrm{d}\tilde{\mu}_{-,\,p}(\mathbf{y}) = \int_{\Gamma_{-,1}} |u(\mathbf{y})|^p \tau_+(\mathbf{y})^{1-p} \mathrm{d}\mu_-(\mathbf{y}) \leqslant \exp(\lambda p) I_1 < \infty.$$

As above, since $\mathrm{d}\tilde{\mu}_{-,\,p}$ coincides with $\mathrm{d}\xi_-$ on $\Gamma_{-,2}$, this shows that $u \in \widetilde{Y}_{-,\,p}$. □

3.4. Boundary Value Problem

The above results allow us to treat more general boundary value problems:

Theorem 3.6. *Let $u \in Y_p^-$ and $g \in X$ be given. Then the function*

$$\begin{aligned} f(\mathbf{x}) = &\int_0^{\tau_-(\mathbf{x})} \exp(-\lambda t)\, g(\Phi(\mathbf{x}, -t))\mathrm{d}t \\ &+ \chi_{\{\tau_-(\mathbf{x})<\infty\}} \exp(-\lambda \tau_-(\mathbf{x})) u(\Phi(\mathbf{x}, -\tau_-(\mathbf{x})))\end{aligned}$$

*is a **unique** solution $f \in \mathscr{D}(\mathcal{T}_{\max,\,p})$ of the boundary value problem:*

$$\begin{cases} (\lambda - \mathcal{T}_{\max,\,p}) f = g, \\ \mathsf{B}^- f = u, \end{cases} \tag{3.10}$$

where $\lambda > 0$. *Moreover, if* $u \in L^p_-$ *then*

$$\lambda\, p\, \|f\|_p^p + \|\mathsf{B}^+ f\|^p_{L^p_+} \leqslant \|u\|^p_{L^p_-} + p\|g\|_p\, \|f\|_p^{p-1}. \tag{3.11}$$

Proof. The fact that f is the unique solution to (3.10) is proven as in [3, Theorem 4.2]. We recall here the main steps since we need the notations introduce therein. Write $f = f_1 + f_2$ with $f_1 = C_\lambda g$ and $f_2 = \Xi_\lambda u$. Since $f_1 = (\lambda - \mathcal{T}_{0,\,p})^{-1} g$, one has $f_1 \in \mathscr{D}(\mathcal{T}_{\max,\,p})$ with $(\lambda - \mathcal{T}_{\max,\,p}) f_1 = g$ and $\mathsf{B}^- f_1 = 0$. Moreover, from Proposition 2.10, $\mathsf{B}^+ f_1 \in L^p_+$. On the other hand, according to Lemma 3.5, $f_2 \in \mathscr{D}(\mathcal{T}_{\max,\,p})$ with $(\lambda - \mathcal{T}_{\max,\,p}) f_2 = 0$ and $\mathsf{B}^- f_2 = u$. We get then that f is a solution to (3.10). The uniqueness also follows the line of [3, Theorem 4.2]. Finally, it remains to prove (3.11). Recall that f is a solution to (3.10) and, applying Green's formula (2.12), we obtain

$$\begin{aligned}\|u\|^p_{L^p_-} - \|\mathsf{B}^+ f\|^p_{L^p_+} &= p \int_{\mathbf{\Omega}} \mathrm{sign}(f)\, |f|^{p-1}\, \mathcal{T}_{\max,\,p} f \,\mathrm{d}\mu \\ &= p \int_{\mathbf{\Omega}} \mathrm{sign}(f)\, |f|^{p-1}\, (\lambda\, f - g) \,\mathrm{d}\mu,\end{aligned}$$

i.e.

$$\|u\|^p_{L^p_-} - \|\mathsf{B}^+ f\|^p_{L^p_+} = p\,\lambda \|f\|_p^p - p \int_{\mathbf{\Omega}} \mathrm{sign}(f)\, |f|^{p-1}\, g \,\mathrm{d}\mu \tag{3.12}$$

which results easily in (3.11). □

Remark 3.7. Notice that, for $g = 0$ and using the above notations, we have $f = f_2$ and (3.12) reads

$$\lambda\, p \|f_2\|_p^p + \|\mathsf{B}^+ f_2\|^p_{L^p_+} = \|u\|^p_{L^p_-} < \infty. \tag{3.13}$$

Conversely, assuming $u = 0$, we get $f = f_1 = C_\lambda g$ and $\mathsf{B}^+ f = G_\lambda g$ and (3.12) is nothing but Lemma 3.2.

3.5. Additional Properties of the Traces

The generalization of [4, Proposition 2.3] to the case $p > 1$ is the following:

Proposition 3.8. *Given* $h \in \widetilde{\mathscr{Y}}_{+\,p}$, *let*

$$f(\mathbf{x}) = \begin{cases} h(\Phi(\mathbf{x}, \tau_+(\mathbf{x})))\dfrac{\tau_-(\mathbf{x}) e^{-\tau_+(\mathbf{x})}}{\tau_-(\mathbf{x}) + \tau_+(\mathbf{x})} & \textit{if } \tau_-(\mathbf{x}) + \tau_+(\mathbf{x}) < \infty, \\ h(\Phi(\mathbf{x}, \tau_+(\mathbf{x}))) e^{-\tau_+(\mathbf{x})} & \textit{if } \tau_-(\mathbf{x}) = \infty \textit{ and } \tau_+(\mathbf{x}) < \infty, \\ 0 & \textit{if } \tau_+(\mathbf{x}) = \infty. \end{cases}$$

Then $f \in \mathscr{D}(\mathcal{T}_{\max,\,p})$, $\mathsf{B}^- f = 0$, *and* $\mathsf{B}^+ f = h$. *Moreover,* $\|f\|_p + \|\mathcal{T}_{0,\,p} f\|_p \leqslant 3\|h\|_{\widetilde{\mathscr{Y}}_{+,\,p}}$.

Proof. Let us first show that $f \in X$ with $\|f\|_p^p \leqslant \frac{1}{p}\|h\|^p_{L^p_+}$. We begin with noticing that

$$\begin{aligned}\int_{\mathbf{\Omega}} |f(\mathbf{x})|^p \mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}_+} |f(\mathbf{x})|^p \mathrm{d}\mu(\mathbf{x}) &= \int_{\mathbf{\Omega}_+ \cap \mathbf{\Omega}_-} |f(\mathbf{x})|^p \mathrm{d}\mu(\mathbf{x}) \\ &\quad + \int_{\mathbf{\Omega}_+ \cap \mathbf{\Omega}_{-\infty}} |f(\mathbf{x})|^p \mathrm{d}\mu(\mathbf{x}),\end{aligned}$$

since $f(\mathbf{x}) = 0$ whenever $\tau_+(\mathbf{x}) = \infty$. Now, according to the integration formula (2.3),

$$\int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_-} |f(\mathbf{x})|^p \mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_-} |h(\Phi(\mathbf{x},\tau_+(\mathbf{x})))|^p \frac{\tau_-(\mathbf{x})^p e^{-p\tau_+(\mathbf{x})}}{(\tau_-(\mathbf{x})+\tau_+(\mathbf{x}))^p}\mathrm{d}\mu(x)$$
$$= \int_{\Gamma_+\setminus\Gamma_{+\infty}} \mathrm{d}\mu_+(\mathbf{z}) \int_0^{\tau_-(\mathbf{z})} \frac{|h(\mathbf{z})|^p}{\tau_-(\mathbf{z})^p}(\tau_-(\mathbf{z})-s)^p e^{-ps}\mathrm{d}s.$$

Since, for any $\tau > 0$ and any $s \in (0,\tau)$, $0 \leqslant 1 - \frac{s}{\tau} \leqslant 1$, we have

$$\frac{1}{\tau^p}\int_0^\tau (\tau - s)^p e^{-ps}\mathrm{d}s = \int_0^\tau \left(1-\frac{s}{\tau}\right)^p e^{-ps}\mathrm{d}s \leqslant \int_0^\tau e^{-ps}\mathrm{d}s \leqslant \frac{1}{p}$$

we get that

$$\int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_-} |f(\mathbf{x})|^p \mathrm{d}\mu(\mathbf{x}) \leqslant \frac{1}{p}\int_{\Gamma_+\setminus\Gamma_{+\infty}} |h(\mathbf{z})|^p \mathrm{d}\mu_+(\mathbf{z}).$$

In the same way, according to Eq. (2.4),

$$\int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_{-\infty}} |f(\mathbf{x})|^p \mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_{-\infty}} |h(\Phi(\mathbf{x},\tau_+(\mathbf{x})))|^p e^{-p\tau_+(\mathbf{x})}\mathrm{d}\mu(\mathbf{x})$$
$$= \int_{\Gamma_{+\infty}} \mathrm{d}\mu_+(\mathbf{z}) \int_0^\infty |h(\mathbf{z})|^p e^{-ps}\mathrm{d}s$$
$$= \frac{1}{p}\int_{\Gamma_{+\infty}} |h(\mathbf{z})|^p \mathrm{d}\mu_+(\mathbf{z}).$$

Thus,

$$\|f\|_p^p \leqslant \tfrac{1}{p}\|h\|_{L^p_+}^p \leqslant \tfrac{1}{p}\|h\|_{\widetilde{\mathscr{Y}}_{+,\,p}}^p \leqslant \|h\|_{\widetilde{\mathscr{Y}}_{+,\,p}}^p$$

and $f \in X$. Setting

$$g(\mathbf{x}) = \begin{cases} -h(\Phi(\mathbf{x},\tau_+(\mathbf{x})))\, e^{-\tau_+(\mathbf{x})} \dfrac{1+\tau_-(\mathbf{x})}{\tau_-(\mathbf{x})+\tau_+(\mathbf{x})} & \text{if } \tau_-(\mathbf{x})+\tau_+(\mathbf{x}) < \infty, \\ -h(\Phi(\mathbf{x},\tau_+(\mathbf{x})))\, e^{-\tau_+(\mathbf{x})} & \text{if } \tau_-(\mathbf{x}) = \infty \text{ and } \tau_+(\mathbf{x}) < \infty, \\ 0 & \text{if } \tau_+(\mathbf{x}) = \infty, \end{cases} \tag{3.14}$$

it is easily seen that, if $g \in X$, then $f \in \mathscr{D}(\mathcal{T}_{\max,\,p})$ with $\mathcal{T}_{\max,\,p} f = g$. Let us then prove that $g \in X$. Clearly, as before,

$$\int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_{-\infty}} |g(\mathbf{x})|^p \mathrm{d}\mu(\mathbf{x}) = \frac{1}{p}\int_{\Gamma_{+\infty}} |h(\mathbf{z})|^p \mathrm{d}\mu_+(\mathbf{z}).$$

Moreover,

$$\int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_-} |g(\mathbf{x})|^p \mathrm{d}\mu(\mathbf{x})$$
$$= \int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_-} |h(\Phi(\mathbf{x},\tau_+(\mathbf{x})))|^p e^{-p\tau_+(\mathbf{x})} \frac{(1+\tau_-(\mathbf{x}))^p}{(\tau_-(\mathbf{x})+\tau_+(\mathbf{x}))^p}\mathrm{d}\mu(\mathbf{x})$$
$$= \int_{\Gamma_+\setminus\Gamma_{+\infty}} \mathrm{d}\mu_+(\mathbf{z}) \int_0^{\tau_-(\mathbf{z})} \frac{|h(\mathbf{z})|^p}{\tau_-(\mathbf{z})^p} e^{-ps}(1+\tau_-(\mathbf{z})-s)^p \mathrm{d}s.$$

Now, it is easy to check that, for any $\tau > 0$,

$$\frac{1}{\tau^p}\int_0^\tau e^{-ps}(1+\tau-s)^p\mathrm{d}s = \int_0^\tau \left(\frac{1+s}{\tau}\right)^p e^{-p(\tau-s)}\mathrm{d}s \leqslant 2^p\left[\min(\tau,1)\right]^{1-p}$$

so that

$$\begin{aligned}\int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_-}|g(\mathbf{x})|^p\mathrm{d}\mu(\mathbf{x}) &\leqslant 2^p\int_{\Gamma_+\setminus\Gamma_{+\infty}}\left[\min(\tau_-(\mathbf{z}),1)\right]^{1-p}|h(\mathbf{z})|^p\mathrm{d}\mu_+(\mathbf{z})\\ &\leqslant 2^p\|h\|^p_{\widetilde{\mathscr{Y}}_{+,\,p}}.\end{aligned}$$

Consequently, we obtain

$$\int_{\mathbf{\Omega}}|g(\mathbf{x})|^p\mathrm{d}\mu(\mathbf{x}) \leqslant 2^p\|h\|^p_{\widetilde{\mathscr{Y}}_{+,\,p}} < \infty$$

which proves that $f \in \mathscr{D}(\mathcal{T}_{\max,\,p})$. To compute the traces B^+f and B^-f, one proceeds as in [4, Proposition 2.3]. The inequality $\|f\|_p + \|\mathcal{T}_{\max,\,p}f\|_p = \|f\|_p + \|g\|_p \leqslant 3\|h\|_{\widetilde{\mathscr{Y}}_{+,\,p}}$ is immediate. $\square$

Remark 3.9. Notice that, for $p = 1$, $\widetilde{\mathscr{Y}}_{\pm,1} = L^1(\Gamma_+\,,\mathrm{d}\mu_\pm)$ and the above proposition is nothing but [4, Proposition 2.3].

One also has the following:

Lemma 3.10. *For any $\lambda > 0$ and $f \in X$, one has $G_\lambda f \in \widetilde{\mathscr{Y}}_{+,\,p}$ and*

$$\|G_\lambda f\|_{\widetilde{\mathscr{Y}}_{+,\,p}} \leqslant \left(1 + (\lambda\, q)^{-1/q}\right)\|f\|_p, \quad \frac{1}{p}+\frac{1}{q} = 1. \tag{3.15}$$

Moreover, for any $\lambda > 0$, the mapping $G_\lambda\,:\,X \to \widetilde{\mathscr{Y}}_{+,\,p}$ is surjective.

Proof. Let $g \in \widetilde{\mathscr{Y}}_{+,\,p}$ and $\lambda > 0$. According to Proposition 3.8, there is an $f \in \mathscr{D}(\mathcal{T}_{\max,\,p})$, such that $\mathsf{B}^+f = g$ and $\mathsf{B}^-f = 0$. In particular, since $f \in \mathscr{D}(\mathcal{T}_{0,\,p})$, there is $\psi \in X$ such that $f = (\lambda - \mathcal{T}_{0,\,p})^{-1}\psi = C_\lambda\psi$. In this case, $g = \mathsf{B}^+f = G_\lambda\psi$. This proves that $G_\lambda\,:\,X \to \widetilde{\mathscr{Y}}_{+,\,p}$ is surjective. Let us now prove (3.15): for $\lambda > 0$ and $f \in X$ it holds

$$\begin{aligned}\|G_\lambda f\|^p_{\widetilde{\mathscr{Y}}_{+,\,p}} &= \int_{\Gamma_{+,1}}\left|\int_0^{\tau_-(\mathbf{z})}e^{-\lambda\,t}f(\Phi(z,-t))\mathrm{d}t\right|^p\tau_-(\mathbf{z})^{1-p}\mathrm{d}\mu_+(\mathbf{z})\\ &\quad+\int_{\Gamma_{+,2}}\left|\int_0^{\tau_-(\mathbf{z})}e^{-\lambda\,t}f(\Phi(z,-t))\mathrm{d}t\right|^p\mathrm{d}\mu_+(\mathbf{z}),\end{aligned}$$

where $\Gamma_{+,1} = \{\mathbf{z} \in \Gamma_+;\,\tau_-(\mathbf{z}) \leqslant 1\}$ and $\Gamma_{+,2} = \{\mathbf{z} \in \Gamma_+;\,\tau_-(\mathbf{z}) > 1\}$. For the first integral, we use Hölder inequality to get, for any $\mathbf{z} \in \Gamma_{+,1}$:

$$\left|\int_0^{\tau_-(\mathbf{z})}e^{-\lambda\,t}f(\Phi(z,-t))\mathrm{d}t\right|^p \leqslant \left(\int_0^{\tau_-(\mathbf{z})}|f(\Phi(\mathbf{z},-t))|^p e^{-\lambda\,pt}\mathrm{d}t\right)\tau_-(\mathbf{z})^{p/q},$$

i.e.

$$\left|\int_0^{\tau_-(\mathbf{z})}e^{-\lambda\,t}f(\Phi(z,-t))\mathrm{d}t\right|^p \leqslant \tau_-(\mathbf{z})^{p-1}\int_0^{\tau_-(\mathbf{z})}|f(\Phi(\mathbf{z},-t))|^p\mathrm{d}t$$

and, using (2.3),

$$\int_{\Gamma_{+,1}} \left|\int_0^{\tau_-(\mathbf{z})} e^{-\lambda t} f(\Phi(z,-t))\mathrm{d}t\right|^p \tau_-(\mathbf{z})^{1-p}\mathrm{d}\mu_+(\mathbf{z})$$
$$\leqslant \int_{\Gamma_{+,1}} \mathrm{d}\mu_+(\mathbf{z}) \int_0^{\tau_-(\mathbf{z})} |f(\Phi(\mathbf{z},-t))|^p\mathrm{d}t \leqslant \|f\|_p^p.$$

In the same way, we see that for all $\mathbf{z} \in \Gamma_{+,2}$,

$$\left|\int_0^{\tau_-(\mathbf{z})} e^{-\lambda t} f(\Phi(z,-t))\mathrm{d}t\right|^p \leqslant \left(\int_0^{\tau_-(\mathbf{z})} |f(\Phi(\mathbf{z},-t))|^p\mathrm{d}t\right)\left(\int_0^{\tau_-(\mathbf{z})} e^{-\lambda qt}\mathrm{d}t\right)^{p/q}$$
$$\leqslant \left(\frac{1}{\lambda q}\right)^{p-1} \int_0^{\tau_-(\mathbf{z})} |f(\Phi(\mathbf{z},-t)|^p\mathrm{d}t$$

from which we deduce as above that

$$\int_{\Gamma_{+,2}} \left|\int_0^{\tau_-(\mathbf{z})} e^{-\lambda t} f(\Phi(z,-t))\mathrm{d}t\right|^p \mathrm{d}\mu_+(\mathbf{z}) \leqslant \left(\frac{1}{\lambda q}\right)^{p-1} \|f\|_p^p.$$

Combining both the estimates, we obtain (3.15). □

Remark 3.11. A clear consequence of the above Lemma is the following: if $\varphi \in \mathscr{D}(\mathcal{T}_{\max,p})$ and $\mathsf{B}^-\varphi = 0$ then $\mathsf{B}^+\varphi \in \widetilde{\mathscr{Y}}_{+,p}$.

The following result generalizes [11, Theorem 2, p. 253]:

Proposition 3.12. *Let $\psi_\pm \in Y_p^\pm$ be given. There exists $\varphi \in \mathscr{D}(\mathcal{T}_{\max,p})$ such that $\mathsf{B}^\pm\varphi = \psi_\pm$ if and only if*

$$\psi_+ - M_\lambda\psi_- \in \widetilde{\mathscr{Y}}_{+,p} \quad \text{for some/all} \quad \lambda > 0.$$

Proof. Assume first there exists $\varphi \in \mathscr{D}(\mathcal{T}_{\max,p})$ such that $\mathsf{B}^\pm\varphi = \psi_\pm$. Set $g = \varphi - \Xi_\lambda\psi_-$. Clearly, one can deduce from Eq. (3.6) that $g \in \mathscr{D}(\mathcal{T}_{\max,p})$ with $(\lambda - \mathcal{T}_{\max,p})g = (\lambda - \mathcal{T}_{\max,p})\varphi$ and $\mathsf{B}^-g = 0$. Moreover, $\mathsf{B}^+g = \psi_+ - M_\lambda\psi_-$. Since $\mathsf{B}^-g = 0$, one has $\mathsf{B}^+g \in \widetilde{\mathscr{Y}}_{+,p}$ (see Remark 3.11). Notice also that

$$\mathsf{B}^+g = G_\lambda(\lambda - \mathcal{T}_{\max,p})\varphi$$

so that, from (3.15),

$$\|\mathsf{B}^+g\|_{\widetilde{\mathscr{Y}}_{+,p}} \leqslant (1 + (\lambda q)^{-1/q})\|(\lambda - \mathcal{T}_{\max,p})\varphi\|_p$$
$$\leqslant (1 + (\lambda q)^{-1/q}) \max(1,\lambda)\left(\|\varphi\|_p + \|\mathcal{T}_{\max,p}\varphi\|_p\right). \quad (3.16)$$

Conversely, let $h = \psi_+ - M_\lambda\psi_- \in \widetilde{\mathscr{Y}}_{+,p}$. Then, thanks to Proposition 3.8, one can find a function $f \in \mathscr{D}(\mathcal{T}_{\max,p})$ such that $\mathsf{B}^-f = 0$ and $\mathsf{B}^+f = h$. Setting $\varphi = f + \Xi_\lambda\psi_-$, one sees from (3.6) that $\varphi \in \mathscr{D}(\mathcal{T}_{\max,p})$ with $\mathsf{B}^-\varphi = \mathsf{B}^-f + \mathsf{B}^-\Xi_\lambda\psi_- = \psi_-$ and $\mathsf{B}^+\varphi = h + \mathsf{B}^+\Xi_\lambda\psi_- = h + M_\lambda\psi_- = \psi_+$. □

A consequence of the above Proposition is the following:

Corollary 3.13. *Given $\psi_\pm \in Y_p^\pm$, there exists $\varphi \in \mathscr{D}(\mathcal{T}_{\max,p})$ such that $\mathsf{B}^\pm\varphi = \psi_\pm$ and $\mathsf{B}^\mp\varphi = 0$ if and only if $\psi_\pm \in \widetilde{\mathscr{Y}}_{\pm,p}$.*

Proof. The proof of the result is an obvious consequence of the above Proposition (see also Remark 3.11) and point (3) of Lemma 3.5. □

With this, we can prove the following:

Proposition 3.14. *One has*

$$\mathscr{W} := \left\{f \in \mathscr{D}(\mathcal{T}_{\max,\,p});\ \mathsf{B}^- f \in L^p_-\right\} = \left\{f \in \mathscr{D}(\mathcal{T}_{\max,\,p});\ \mathsf{B}^+ f \in L^p_+\right\}.$$

In particular, Green's formula (2.12) *holds for any* $f \in \mathscr{W}$.

Proof. We have already seen that, given $f \in \mathscr{D}(\mathcal{T}_{\max,\,p})$ with $\mathsf{B}^- f \in L^p_-$, it holds that $\mathsf{B}^+ f \in L^p_+$. Now, let $f \in \mathscr{D}(\mathcal{T}_{\max,\,p})$ be such that $\mathsf{B}^+ f \in L^p_+$. Set $u = \mathsf{B}^- f$. According to Proposition 3.12, $\mathsf{B}^+ f - M_\lambda u \in \widetilde{\mathscr{Y}}_{+,\,p} \subset L^p_+$. Therefore, $M_\lambda u \in L^p_+$ and since $u \in Y^-_p$, one deduces from Lemma 3.5 that $u \in L^p_-$. □

Define now $\mathscr{E}$ as the space of elements $(\psi_+, \psi_-) \in Y^+_p \times Y^-_p$ such that $\psi_+ - M_\lambda \psi_- \in \widetilde{\mathscr{Y}}_{+,\,p}$ for some/all $\lambda > 0$. We equip $\mathscr{E}$ with the norm

$$\|(\psi_+, \psi_-)\|_{\mathscr{E}} := \left[\|\psi_+\|^p_{Y^+_p} + \|\psi_-\|^p_{Y^-_p} + \|\psi_+ - M_1\psi_-\|^p_{\widetilde{\mathscr{Y}}_{+,\,p}}\right]^{1/p} \tag{3.17}$$

that makes it a Banach space. In the following, $\mathscr{D}(\mathcal{T}_{\max,\,p})$ is endowed with the graph norm: $\|f\|_{\mathscr{D}} := \|f\|_p + \|\mathcal{T}_{\max,\,p} f\|_p$, $f \in \mathscr{D}(\mathcal{T}_{\max,\,p})$.

Corollary 3.15. *The trace mapping* $\mathbb{B} : \varphi \in \mathscr{D}(\mathcal{T}_{\max,\,p}) \longmapsto (\mathsf{B}^+\varphi, \mathsf{B}^-\varphi) \in \mathscr{E}$ *is continuous, surjective with continuous inverse.*

Proof. The fact that the trace mapping is surjective follows from Proposition 3.12. Moreover, for any $\varphi \in \mathscr{D}(\mathcal{T}_{\max,\,p})$, Theorem 3.3 yields $\|\mathsf{B}^\pm\varphi\|_{Y^\pm_p} \leqslant 2^{1-\frac{1}{p}}\|\varphi\|_{\mathscr{D}}$. Moreover, according to (3.16)

$$\|\psi_+ - M_\lambda\psi_-\|^p_{\widetilde{\mathscr{Y}}_{+,\,p}} = \|\mathsf{B}^+ g\|^p_{\widetilde{\mathscr{Y}}_{+,\,p}} \leqslant (1 + (\lambda\, q)^{-1/q})^p \max(1, \lambda^p)\|\varphi\|^p_{\mathscr{D}}$$

for any $\lambda > 0$. Choosing $\lambda = 1$, this proves that $\mathbb{B}$ is continuous. Conversely, suppose $(\psi_+, \psi_-) \in \mathscr{E}$. From the proof of Proposition 3.12 with $\lambda = 1$, the inverse operator may be defined by

$$\mathbb{B}^{-1}\ :\ (\psi_-, \psi_+) \mapsto \varphi = f + \Xi_1\psi_-,$$

where $f \in \mathscr{D}(\mathcal{T}_{\max,\,p})$ satisfies $\mathsf{B}^- f = 0$ and $\mathsf{B}^+ f = h = \psi_+ - M_1\psi_-$. Now, by Proposition 3.8,

$$\|f\|_{\mathscr{D}} \leqslant 3\|h\|_{\widetilde{\mathscr{Y}}_{+,\,p}} = 3\|\psi_+ - M_1\psi_-\|_{\widetilde{\mathscr{Y}}_{+,\,p}}$$

and one deduces easily the continuity of $\mathbb{B}^{-1}$. □

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Appendix A: Proof of Theorem 2.6

The scope here is to prove Theorem 2.6 in Sect. 2.2. The difficult part of the proof is the implication (2) $\implies$ (1). It is carried out through several technical lemmas based upon *mollification along the characteristic curves* (recall that, whenever μ is not absolutely continuous with respect to the Lebesgue measure, no global convolution argument is available). Let us make precise what this is all about. Consider a sequence $(\varrho_n)_n$ of one-dimensional mollifiers supported in $[0,1]$, i.e. for any $n \in \mathbb{N}$, $\varrho_n \in \mathscr{C}_0^\infty(\mathbb{R})$, $\varrho_n(s) = 0$ if $s \notin [0, 1/n]$, $\varrho_n(s) \geqslant 0$ and $\int_0^{1/n} \varrho_n(s)\mathrm{d}s = 1$. Then, for any $f \in L^p(\mathbf{\Omega}, \mathrm{d}\mu)$, define the (extended) mollification:

$$\varrho_n \diamond f(\mathbf{x}) = \int_0^{\tau_-(\mathbf{x})} \varrho_n(s) f(\Phi(\mathbf{x}, -s))\mathrm{d}s.$$

Note that, with such a definition, it is not clear *a priori* that $\varrho_n \diamond f$ defines a measurable function, finite almost everywhere. It is proved in the following that such a function does actually belong to $L^p(\mathbf{\Omega}, \mathrm{d}\mu)$.

Lemma A.1. *Given $f \in L^p(\mathbf{\Omega}, \mathrm{d}\mu)$, $\varrho_n \diamond f \in L^p(\mathbf{\Omega}, \mathrm{d}\mu)$ for any $n \in \mathbb{N}$. Moreover,*

$$\|\varrho_n \diamond f\|_p \leqslant \|f\|_p, \quad \forall f \in L^p(\mathbf{\Omega}, \mathrm{d}\mu), n \in \mathbb{N}. \tag{A.1}$$

Proof. One considers, for a given $f \in L^p(\mathbf{\Omega}, \mathrm{d}\mu)$, the extension of f by zero outside $\mathbf{\Omega}$:

$$\overline{f}(\mathbf{x}) = f(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbf{\Omega}, \quad \overline{f}(\mathbf{x}) = 0 \quad \forall \mathbf{x} \in \mathbb{R}^N \backslash \mathbf{\Omega}.$$

Then $\overline{f} \in L^p(\mathbb{R}^N, \mathrm{d}\mu)$. Let us consider the transformation:

$$\Upsilon \;:\; (\mathbf{x}, s) \in \mathbb{R}^N \times \mathbb{R} \mapsto \Upsilon(\mathbf{x}, s) = (\Phi(\mathbf{x}, -s), -s) \in \mathbb{R}^N \times \mathbb{R}.$$

As a homeomorphism, Υ is measure preserving for pure Borel measures. It is also measure preserving for completions of Borel measures (such as a Lebesgue measure) since it is measure-preserving on Borel sets and the completion of a measure is obtained by adding to the Borel σ-algebra all sets contained in measure-zero Borel sets, see [13, Theorem 13.B, p. 55]. Then, according to [13, Theorem 39.B, p. 162], the mapping

$$(\mathbf{x}, s) \in \mathbb{R}^N \times \mathbb{R} \mapsto \overline{f}(\Phi(\mathbf{x}, -s))$$

is measurable as the composition of Υ with the measurable function $(\mathbf{x}, s) \mapsto \overline{f}(\mathbf{x})$. Define now $\Lambda = \{(\mathbf{x}, s);\, \mathbf{x} \in \mathbf{\Omega},\, 0 < s < \tau_-(\mathbf{x})\}$, Λ is a measurable subset of $\mathbb{R}^N \times \mathbb{R}$. Therefore, the mapping

$$(\mathbf{x}, s) \in \mathbb{R}^N \times \mathbb{R} \longmapsto \overline{f}(\Phi(\mathbf{x}, -s))\chi_\Lambda(\mathbf{x}, s)\varrho_n(s)$$

is measurable. In the same way,

$$(\mathbf{x}, s) \in \mathbb{R}^N \times \mathbb{R} \longmapsto \left|\overline{f}(\Phi(\mathbf{x}, -s))\right|^p \chi_\Lambda(\mathbf{x}, s)\varrho_n(s)$$

is measurable. For almost every $\mathbf{x} \in \mathbf{\Omega}$, it holds

$$\int_0^\infty \varrho_n(s) \left|\overline{f}(\Phi(\mathbf{x}, -s))\right|^p \chi_\Lambda(\mathbf{x}, s)\mathrm{d}s = \int_0^{\min(\tau_-(\mathbf{x}), 1/n)} \varrho_n(s) \left|f(\Phi(\mathbf{x}, -s))\right|^p \mathrm{d}s.$$

Setting $q = \frac{p}{p-1}$, observe now that $\varrho_n^{1/q} \in L^q(0, 1/n)$ while, for a.e. $\mathbf{x} \in \mathbf{\Omega}$,

$$s \mapsto \varrho_n(s)^{1/p} f(\Phi(\mathbf{x}, -s)) \in L^p(0, \min(\tau_-(\mathbf{x}), 1/n)).$$

Therefore, for almost every $\mathbf{x} \in \mathbf{\Omega}$

$$\varrho_n(s) f(\Phi(\mathbf{x}, -s)) \in L^1(0, \min(\tau_-(\mathbf{x}), 1/n))$$

thanks to Hölder's inequality. Thus,

$$[\varrho_n \diamond f](\mathbf{x}) := \int_0^{\tau_-(\mathbf{x})} \varrho_n(s) f(\Phi(\mathbf{x}, -s)) \mathrm{d}s$$

is finite for almost every $\mathbf{x} \in \mathbf{\Omega}$ with

$$\begin{aligned} |[\varrho_n \diamond f](\mathbf{x})| &\leqslant \left(\int_0^{1/n} \varrho_n(s) \mathrm{d}s\right)^{1/q} \left(\int_0^{\tau_-(\mathbf{x})} \varrho_n(s) \left|f(\Phi(\mathbf{x}, -s))\right|^p \mathrm{d}s\right)^{1/p} \\ &= \left(\int_0^{\tau_-(\mathbf{x})} \varrho_n(s) \left|f(\Phi(\mathbf{x}, -s))\right|^p \mathrm{d}s\right)^{1/p}. \end{aligned}$$

From this, one sees that

$$|[\varrho_n \diamond f](\mathbf{x})|^p \leqslant [\varrho_n \diamond |f|^p](\mathbf{x}). \tag{A.2}$$

Now, since $|f|^p \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$, one can use [3, Theorem 3.7] to get first that $\varrho_n \diamond |f|^p \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$ with

$$\|[\varrho_n \diamond |f|^p\|_1 \leqslant \| \, |f|^p \|_1 = \|f\|_p^p.$$

One deduces from this and (A.2) that $[\varrho_n \diamond f] \in L^p(\mathbf{\Omega}, \mathrm{d}\mu)$ and (A.1) holds true. □

As it is the case for classical convolution, the family $(\varrho_n \diamond f)_n$ approximates f in L^p-norm:

Proposition A.2. *Given $f \in L^p(\mathbf{\Omega}, \mathrm{d}\mu)$,*

$$\lim_{n \to \infty} \int_{\mathbf{\Omega}} \left|(\varrho_n \diamond f)(\mathbf{x}) - f(\mathbf{x})\right|^p \mathrm{d}\mu(\mathbf{x}) = 0. \tag{A.3}$$

Proof. The proof is very similar to that of [3, Proposition 3.8]. Let us fix a nonnegative f continuous over $\mathbf{\Omega}$ and compactly supported. We introduce, for any $n \in \mathbb{N}$, $\mathcal{O}_n := \mathrm{supp}(\varrho_n \diamond f) \cup \mathrm{supp}(f)$ and $\mathcal{O}_n^- = \{\mathbf{x} \in \mathcal{O}_n \,;\, \tau_-(\mathbf{x}) < 1/n\}$. Since $\sup_{\mathbf{x} \in \mathbf{\Omega}} |\varrho_n \diamond f(\mathbf{x})| \leqslant \sup_{\mathbf{x} \in \mathbf{\Omega}} |f(\mathbf{x})|$, for any $\varepsilon > 0$, there exists $n_0 \geqslant 1$ such that

$$\int_{\mathcal{O}_n^-} |f(\mathbf{x})|^p \mathrm{d}\mu(\mathbf{x}) \leqslant \varepsilon, \quad \text{and} \quad \int_{\mathcal{O}_n^-} |\varrho_n \diamond f(\mathbf{x})|^p \mathrm{d}\mu(\mathbf{x}) \leqslant \varepsilon \quad \forall n \geqslant n_0.$$

Now, noticing that $\mathrm{Supp}(\varrho_n \diamond f - f) \subset \mathcal{O}_n$, one has for any $n \geqslant n_0$,

$$\int_{\mathbf{\Omega}} |\varrho_n \diamond f - f|^p \mathrm{d}\mu = \int_{\mathcal{O}_n} |\varrho_n \diamond f - f|^p d\mu \leqslant 2\varepsilon + \int_{\mathcal{O}_n \setminus \mathcal{O}_n^-} |\varrho_n \diamond f - f|^p \mathrm{d}\mu. \tag{A.4}$$

For any $\mathbf{x} \in \mathcal{O}_n \backslash \mathcal{O}_n^-$, since ϱ is supported in $[0, 1/n]$, one has

$$\begin{aligned}[\varrho_n \diamond f](\mathbf{x}) - f(\mathbf{x}) &= \int_0^{1/n} \varrho_n(s) f(\Phi(\mathbf{x}, -s)) \mathrm{d}s - f(\mathbf{x}) \\ &= \int_0^{1/n} \varrho_n(s) \left(f(\Phi(\mathbf{x}, -s)) - f(\mathbf{x}) \right) \mathrm{d}s.\end{aligned}$$

Then, as in the previous Lemma, one deduces from Hölder's inequality that

$$|[\varrho_n \diamond f](\mathbf{x}) - f(\mathbf{x})|^p \leqslant \int_0^{1/n} \varrho_n(s) \, |f(\Phi(\mathbf{x}, -s) - f(\mathbf{x})|^p \, \mathrm{d}s.$$

As in the proof of [3, Proposition 3.8], one sees that, because f is uniformly continuous on $\mathcal{O}_1$, there exists some $n_1 \geqslant 0$, such that

$$\sup_{\substack{s \in (0,1/n) \\ \mathbf{x} \in \mathcal{O}_1}} |f(\Phi(\mathbf{x}, -s)) - f(\mathbf{x})|^p \leqslant \varepsilon \quad \forall n \geqslant n_1$$

which results in $|[\varrho_n \diamond f](\mathbf{x}) - f(\mathbf{x})|^p \leqslant \varepsilon$ for any $\mathbf{x} \in \mathcal{O}_n \backslash \mathcal{O}_n^-$ and any $n \geqslant n_1$. One obtains then, for any $n \geqslant n_1$,

$$\int_{\mathbf{\Omega}} |\varrho_n \diamond f - f|^p \mathrm{d}\mu \leqslant 2\varepsilon + \varepsilon \mu(\mathcal{O}_n \backslash \mathcal{O}_n^-) \leqslant 2\varepsilon + \varepsilon \mu(\mathcal{O}_1)$$

which proves the result. □

As in [3, Lemma 3.9, Proposition 3.11], one has the following:

Lemma A.3. *Given $f \in L^p(\mathbf{\Omega}, \mathrm{d}\mu)$, set $f_n = \varrho_n \diamond f$, $n \in \mathbb{N}$. Then, $f_n \in \mathscr{D}(\mathcal{T}_{\max,\,p})$ with*

$$[\mathcal{T}_{\max,\,p} f_n](\mathbf{x}) = -\int_0^{\tau_-(\mathbf{x})} \varrho_n'(s) f(\Phi(\mathbf{x}, -s)) \mathrm{d}s, \quad \mathbf{x} \in \mathbf{\Omega}. \tag{A.5}$$

Moreover, for $f \in \mathscr{D}(\mathcal{T}_{\max,\,p})$, then

$$[\mathcal{T}_{\max,\,p}(\varrho_n \diamond f)](\mathbf{x}) = [\varrho_n \diamond \mathcal{T}_{\max,\,p} f](\mathbf{x}), \quad (\mathbf{x} \in \mathbf{\Omega}\,, \, n \in \mathbb{N}). \tag{A.6}$$

We are in position to prove the following:

Proposition A.4. *Let $f \in L^p(\mathbf{\Omega}, \mathrm{d}\mu)$ and $f_n = \varrho_n \diamond f$, $n \in \mathbb{N}$. Then for μ_--a. e. $\mathbf{y} \in \Gamma_-$,*

$$\begin{aligned} f_n(\Phi(\mathbf{y}, s)) - f_n(\Phi(\mathbf{y}, t)) = \int_s^t [\mathcal{T}_{\max,\,p} f_n](\Phi(\mathbf{y}, r)) \mathrm{d}r \\ \forall 0 < s < t < \tau_+(\mathbf{y}). \end{aligned} \tag{A.7}$$

In the same way, for almost every $\mathbf{z} \in \Gamma_+$,

$$\begin{aligned} &f_n(\Phi(\mathbf{z}, -s)) - f_n(\Phi(\mathbf{z}, -t)) \\ &\quad = -\int_s^t [\mathcal{T}_{\max,\,p} f_n](\Phi(\mathbf{z}, -r)) \mathrm{d}r, \quad \forall 0 < s < t < \tau_-(\mathbf{z}). \end{aligned} \tag{A.8}$$

Moreover, for any $f \in \mathscr{D}(\mathcal{T}_{\max,\,p})$, there exist some functions $\widetilde{f}_{\pm} \in L^p(\mathbf{\Omega}_{\pm}, \mathrm{d}\mu)$ such that $\widetilde{f}_{\pm}(\mathbf{x}) = f(\mathbf{x})$ for μ-almost every $\mathbf{x} \in \mathbf{\Omega}_{\pm}$ and, for μ_-–almost every $\mathbf{y} \in \Gamma_-$:

$$\widetilde{f}_-(\Phi(\mathbf{y},s)) - \widetilde{f}_-(\Phi(\mathbf{y},t)) = \int_s^t [\mathcal{T}_{\max,\,p}f](\Phi(\mathbf{y},r))\mathrm{d}r \quad \forall 0 < s < t < \tau_+(\mathbf{y}), \tag{A.9}$$

while, for μ_+-almost every $\mathbf{z} \in \Gamma_+$:

$$\begin{aligned}\widetilde{f}_+(\Phi(\mathbf{z},-s)) &- \widetilde{f}_+(\Phi(\mathbf{z},-t)) \\ &= -\int_s^t [\mathcal{T}_{\max,\,p}f](\Phi(\mathbf{z},-r))\mathrm{d}r \quad \forall 0 < s < t < \tau_-(\mathbf{z}).\end{aligned} \tag{A.10}$$

Proof. The proof is very similar to that of [3, Propositions 3.13 & 3.14]. Because $f \in L^p(\mathbf{\Omega}, \mathrm{d}\mu)$, the integral

$$\int_0^{\tau_+(\mathbf{y})} |f(\Phi(\mathbf{y},r))|^p \mathrm{d}r$$

exists and is finite for μ_--almost every $\mathbf{y} \in \Gamma_-$. As such, for μ_--almost every $\mathbf{y} \in \Gamma_-$ and any $0 < t < \tau_+(\mathbf{y})$, one has

$$\begin{aligned}&\left|\int_0^t \varrho_n(t-s) f(\Phi(\mathbf{y},s))\mathrm{d}s\right| \\ &\quad\leqslant \int_0^t \varrho_n(t-s)|f(\Phi(\mathbf{y},s))|\mathrm{d}s \\ &\quad\leqslant \left(\int_0^t \varrho_n(t-s)\mathrm{d}s\right)^{1/q} \left(\int_0^t \varrho_n(t-s)|f(\Phi(\mathbf{y},s))|^p\mathrm{d}s\right)^{1/p} < \infty.\end{aligned}$$

This shows that, for μ_--almost every $\mathbf{y} \in \Gamma_-$ and any $0 < t < \tau_+(\mathbf{y})$, the quantity $\int_0^t \varrho_n(t-s)f(\Phi(\mathbf{y},s))\mathrm{d}s$ is well defined and finite. The same argument shows that also $\int_0^t \varrho_n'(t-s)f(\Phi(\mathbf{y},s))\mathrm{d}s$ is well defined and finite for μ_--almost every $\mathbf{y} \in \Gamma_-$ and any $0 < t < \tau_+(\mathbf{y})$. The rest of the proof is exactly as in [3, Proposition 3.13] by virtue of Lemma A.3 yielding (A.7)–(A.8).

The proof of (A.9)–(A.10) is deduced from [3, Proposition 3.14], namely, for any $n \geqslant 1$, set $f_n = \varrho_n \diamond f$, so that, from Proposition A.2 and (A.6), $\lim_{n\to\infty} \left(\|f_n - f\|_p + \|\mathcal{T}_{\max,\,p}f_n - \mathcal{T}_{\max,\,p}f\|_p\right) = 0$. In particular, from Eq. (2.3)

$$\begin{aligned}&\int_{\Gamma_-} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} |f_n(\Phi(\mathbf{y},s)) - f(\Phi(\mathbf{y},s))|^p \,\mathrm{d}s \\ &\quad + \int_{\Gamma_-} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} |[\mathcal{T}_{\max,\,p}f_n](\Phi(\mathbf{y},s)) - [\mathcal{T}_{\max,\,p}f](\Phi(\mathbf{y},s))|^p \,\mathrm{d}s \underset{n\to\infty}{\longrightarrow} 0\end{aligned}$$

since $\mathcal{T}_{\max,\,p}f$ and $\mathcal{T}_{\max,\,p}f_n$ both belong to $L^p(\mathbf{\Omega}, \mathrm{d}\mu)$. Consequently, for almost every $\mathbf{y} \in \Gamma_-$ (up to a subsequence, still denoted by f_n), we get

$$\begin{cases} f_n(\Phi(\mathbf{y},\cdot)) \longrightarrow f(\Phi(\mathbf{y},\cdot)) \\ [\mathcal{T}_{\max,\,p}f_n](\Phi(\mathbf{y},\cdot)) \longrightarrow [\mathcal{T}_{\max,\,p}f](\Phi(\mathbf{y},\cdot)) \quad \text{in} \quad L^p((0,\tau_+(\mathbf{y}))\,,\mathrm{d}s) \end{cases} \tag{A.11}$$

as $n \to \infty$. Let us fix $\mathbf{y} \in \Gamma_-$ for which this holds. Passing again to a subsequence, we may assume that $f_n(\Phi(\mathbf{y},s))$ converges (pointwise) to $f(\Phi(\mathbf{y},s))$ for almost every $s \in (0,\tau_+(\mathbf{y}))$. Let us fix such a s_0. From (A.7),

$$f_n(\Phi(\mathbf{y},s_0)) - f_n(\Phi(\mathbf{y},s)) = \int_{s_0}^{s} [\mathcal{T}_{\max,\,p}f_n](\Phi(\mathbf{y},r))\mathrm{d}r \quad \forall s \in (0,\tau_+(\mathbf{y})). \tag{A.12}$$

Now, from Hölder's inequality,

$$\left| \int_{s_0}^{s} [\mathcal{T}_{\max,\,p}f_n](\Phi(\mathbf{y},r))\mathrm{d}r - \int_{s_0}^{s} [\mathcal{T}_{\max,\,p}f](\Phi(\mathbf{y},r))\mathrm{d}r \right| \leqslant |s-s_0|^{1/q} \left(\int_{s_0}^{s} \left|[\mathcal{T}_{\max,\,p}f_n](\Phi(\mathbf{y},r)) - [\mathcal{T}_{\max,\,p}f](\Phi(\mathbf{y},r))\right|^p \mathrm{d}r \right)^{1/p}$$

and the last term goes to zero as $n \to \infty$ from (A.11). Hence, one sees that the right-hand side of (A.12) converges for any $s \in (0,\tau_+(\mathbf{y}))$ as $n \to \infty$. Therefore, the second term on the left-hand side also must converge as $n \to \infty$. Thus, for any $s \in (0,\tau_+(\mathbf{y}))$, the limit

$$\lim_{n\to\infty} f_n(\Phi(\mathbf{y},s)) = \widetilde{f}_-(\Phi(\mathbf{y},s))$$

exists, and for any $0 < s < \tau_+(\mathbf{y})$

$$\widetilde{f}_-(\Phi(\mathbf{y},s)) = \widetilde{f}_-(\Phi(\mathbf{y},s_0)) - \int_{s_0}^{s} [\mathcal{T}_{\max,\,p}f](\Phi(\mathbf{y},r))\mathrm{d}r.$$

It is easy to check then that $\widetilde{f}_-(\mathbf{x}) = f(\mathbf{x})$ for almost every $\mathbf{x} \in \mathbf{\Omega}_-$. The same arguments lead to the existence of $\widetilde{f}_+$. □

The above result shows that the mild formulation of Theorem 2.6 is fulfilled for any $\mathbf{x} \in \mathbf{\Omega}_- \cup \mathbf{\Omega}_+$. It remains to deal with $\mathbf{\Omega}_\infty := \mathbf{\Omega}_{-\infty} \cap \mathbf{\Omega}_{+\infty}$ and one has the following whose proof is a slight modification of the one of [3, Proposition 3.15] as in the above proof:

Proposition A.5. *Let $f \in \mathscr{D}(\mathcal{T}_{\max,\,p})$. Then there exists a set $\mathcal{O} \subset \mathbf{\Omega}_\infty$ with $\mu(\mathcal{O}) = 0$ and a function $\widetilde{f}$ defined on $\{\mathbf{z} = \Phi(\mathbf{x},t),\ \mathbf{x} \in \mathbf{\Omega}_\infty \setminus \mathcal{O},\ t \in \mathbb{R}\}$ such that $f(\mathbf{x}) = \widetilde{f}(\mathbf{x})$ μ-almost every $\mathbf{x} \in \mathbf{\Omega}_\infty$ and*

$$\widetilde{f}(\Phi(\mathbf{x},s)) - \widetilde{f}(\Phi(\mathbf{x},t)) = \int_s^t [\mathcal{T}_{\max,\,p}f](\Phi(\mathbf{x},r))\mathrm{d}r, \quad \forall\, \mathbf{x} \in \mathbf{\Omega}_\infty \setminus \mathcal{O},\ s < t.$$

Combining all the above results, the proof of Theorem 2.6 becomes exactly the same as that of [3, Theorem 3.6].

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