

June 19, 2017

Supersymmetry on curved spaces and superconformal anomalies

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Abstract

We study the consequences of unbroken rigid supersymmetry of four-dimensional field theories placed on curved manifolds. We show that in Lorentzian signature the background vector field coupling to the R-current is determined by the Weyl tensor of the background metric. In Euclidean signature, the same holds if two supercharges of opposite R-charge are preserved, otherwise the (anti-)self-dual part of the vector field-strength is fixed by the Weyl tensor. As a result of this relation, the trace and R-current anomalies of superconformal field theories simplify, with the trace anomaly becoming purely topological. In particular, in Lorentzian signature, or in the presence of two Euclidean supercharges of opposite R-charge, supersymmetry of the background implies that the term proportional to the central charge c vanishes, both in the trace and R-current anomalies. This is equivalent to the vanishing of a superspace Weyl invariant. We comment on the implications of our results for holography.

arXiv:1307.6567v4 [hep-th] 16 Jun 2017

1 Introduction

The Weyl anomaly has a very rich history [1, 2] and many different applications. This arises when a conformal field theory is placed on a curved space, and is measured by certain curvature invariants contributing to the trace of the energy-momentum tensor. Four-dimensional superconformal field theories possess an R-symmetry, which also becomes anomalous on generic curved backgrounds. The underlying supersymmetry of the theory relates the terms appearing in the trace and R-symmetry anomalies, which in fact belong to the same supermultiplet. For an $\mathcal{N} = 1$ superconformal theory coupled to an arbitrary background metric g_{mn} and gauge field A_m (sourcing the R-symmetry current), the anomalies read [3]¹ (see also [4])

$$T_m^m = \frac{c}{16\pi^2} C^2 - \frac{a}{16\pi^2} E - \frac{c}{6\pi^2} F^2, \quad (1.1)$$

$$\nabla_m J^m = \frac{c-a}{24\pi^2} P + \frac{5a-3c}{27\pi^2} F\tilde{F}, \quad (1.2)$$

where c and a are the central charges of the theory, C^2 , E and P denote the Weyl, Euler and Pontryagin invariants respectively (whose precise definition will be given later), while $F = dA$ and \tilde{F} is its Hodge dual. In the present paper, we will show that these anomaly formulae simplify when the background preserves some supersymmetry.

In *a priori* unrelated recent developments, there has been interest in coupling supersymmetric field theories to a curved metric and other background fields, in such a way that some suitably defined rigid supersymmetry is preserved. The main motivation for considering these deformations is that supersymmetric field theories on compact manifolds are often amenable to localization techniques, which may be used to compute exactly various observables, at any value of the couplings [5]. Starting with [6], this led to systematic studies of the conditions for a background to support rigid supersymmetry in different models and dimensions, both in Euclidean [7, 8, 9, 10, 11, 12, 13, 14] and in Lorentzian [15, 10, 16, 17] signature (see also [18] for some earlier work, and [19] for a superspace perspective). In this paper we will focus on four-dimensional spaces, either Lorentzian or Riemannian.

Three approaches to rigid supersymmetry in four dimensions have been discussed so far in the literature, based on old minimal [6, 7, 10, 11], new minimal [8, 9, 15] and conformal [8, 15] supergravity. Although these encode slightly different notions of supersymmetry, it turns out that in all cases, and both in Euclidean [8] and Lorentzian [15] signature, unbroken supersymmetry implies the existence of a solution ζ_+ to the *charged conformal Killing spinor*

¹Note that (1.1) and (1.2) correct errors in the formulae presented in reference [3]. We thank Dan Freedman for correspondence on this point.

(CKS) equation:

$$\nabla_m^A \zeta_+ = \frac{1}{4} \gamma_m \nabla^A \cdot \zeta_+. \quad (1.3)$$

Here ζ_+ is a chiral spinor charged under the background gauge field A_m coupling to the field theory R-current, with $\nabla_m^A \zeta_+ \equiv (\nabla_m - iA_m) \zeta_+$. The term $\nabla^A \cdot \zeta_+ \equiv \gamma^n \nabla_n^A \zeta_+$ makes the equation traceless and conformal invariant. This is also known as the (charged) twistor spinor equation [20]. In references [8, 9] it was shown that in Euclidean signature a nowhere vanishing solution to (1.3) exists if and only if there exists a Hermitian metric, while in reference [15] it was found that in Lorentzian signature a solution is equivalent to the existence of a null conformal Killing vector. In both cases, the gauge field A can be determined explicitly from the metric data, although in a rather intricate way.

In this paper, we will revisit the analysis of the CKS equation by studying its integrability condition. Combining the technique of G -structures with the Newman–Penrose formalism, we will demonstrate that in Lorentzian signature the gauge curvature can be extracted completely from the conformally invariant part of the spacetime curvature, namely from the *Weyl tensor* of the metric. This also holds in Euclidean signature when two supercharges of opposite R-charge are preserved.

Our characterization leads to precise relations among the curvature invariants which appear in the superconformal anomalies, implying that the latter simplify on supersymmetric backgrounds, with (1.1) becoming purely topological, as is (1.2). In particular, in Lorentzian signature, or in Euclidean backgrounds possessing two conformal Killing spinors with opposite R-charge, we obtain

$$C^2 - \frac{8}{3} F^2 = 0 = P - \frac{8}{3} F \tilde{F}, \quad (1.4)$$

so the contributions proportional to the central charge c cancel out both in (1.1) and (1.2). These relations are equivalent to the vanishing of the bosonic part of a super-Weyl invariant, whose real part is also the Lagrangian of conformal supergravity [21].

The paper is organised as follows. In section 2 we discuss the integrability condition of the CKS equation (1.3), yielding the relation between the Maxwell and Weyl background tensors. In section 3 we discuss the implications on the superconformal anomalies. In section 4 we illustrate the results by presenting backgrounds with $SU(2) \times U(1)^2$ symmetry that are free of anomalies. In section 5 we review the holographic realisation of the superconformal anomalies. Section 6 concludes. Appendix A gives our conventions, Appendix B derives the integrability condition, while in Appendix C we comment on the anomaly formulae (1.1) and (1.2).

2 Integrability of the CKS equation

In this section we study the integrability of the CKS equation. The analysis is performed by combining the method of G -structures with the Newman–Penrose tetrad formalism. The result will be a set of conditions relating the different components of the Weyl and Maxwell tensors.

Before starting our technical analysis, let us recall the different notions of rigid supersymmetry on four-dimensional curved manifolds that have been studied recently. These differ in the choice of the supercurrent multiplet being employed to couple the theory to background supergravity. If the flat-space field theory is superconformal, then it is naturally coupled to conformal supergravity. This is also seen from holography, as bulk gauged supergravity on an asymptotically locally AdS space induces conformal supergravity on the boundary [22] (with the conformal supergravity action being mapped into the field theory trace anomaly [23, 24], see section 5 for more details). The condition ensuring that the field theory preserves at least one supercharge arises from requiring that the gravitino variation vanishes on the background, and in this case is found to be the CKS equation (1.3) [8, 15]. In Euclidean signature, the supersymmetry parameters ζ_+ and $\tilde{\zeta}_-$ that in Lorentzian signature would be charge conjugate to each other transform under different $SU(2)$ subgroups of $Spin(4)$ and are thus independent. For this reason, one should regard the equation

$$\nabla_m^A \tilde{\zeta}_- = \frac{1}{4} \gamma_m \nabla^A \cdot \tilde{\zeta}_-, \quad (2.1)$$

which in Lorentzian signature would merely be the charge conjugate of (1.3), as independent. Here, $\tilde{\zeta}_-$ is a spinor of negative chirality with R-charge opposite to ζ_+ : $\nabla^A \tilde{\zeta}_- = (\nabla + iA)\tilde{\zeta}_-$. Moreover, in Euclidean signature the gauge field A is allowed to take complex values (while we will keep the metric real).

If the field theory is supersymmetric but not necessarily superconformal, it may be coupled to old minimal supergravity through its Ferrara–Zumino supercurrent or, if it has an R-symmetry, to new minimal supergravity via its R-multiplet. Although the requirements for a supersymmetric background are slightly different in these cases, the CKS equation turns out to always be a necessary condition. Therefore, even if in this paper we are mainly interested in superconformal theories, our results will also apply to these more general setups. Let us see the necessity of the CKS equation, working in Lorentzian signature for simplicity (the Euclidean case being analogous, with a doubling of the equations). When the field theory is coupled to background new minimal supergravity, the supersymmetry condition is

$$\left(\nabla_m + \frac{i}{2} v^n \gamma_{nm} + i v_m - i a_m \right) \zeta_+ = 0, \quad (2.2)$$

where the supergravity auxiliary fields a and v are a gauge connection and a well-defined one-form, respectively, with the latter satisfying $d*v = 0$. Redefining $a = A + \frac{3}{2}v$, and contracting the equation with γ^m , one obtains $v \cdot \zeta_+ = -\frac{i}{2}\nabla^A \cdot \zeta_+$. Plugging this back into (2.2), one arrives at the CKS equation (1.3). In fact, in a neighborhood where the spinor is non-vanishing, one can see that the new minimal equation (2.2) and the CKS equation are equivalent in Euclidean signature [8, 9], while they are equivalent modulo a conformal transformation in Lorentzian signature [15]. The condition stemming from coupling the field theory to old minimal supergravity reads

$$\left(\nabla_m + \frac{i}{3}b_m + \frac{i}{6}b^n\gamma_{nm}\right)\zeta_+ + \frac{1}{6}M\gamma_m\tilde{\zeta}_- = 0, \quad (2.3)$$

where the auxiliary fields b and M are a globally well-defined one-form and a complex scalar, respectively. It follows that $M\tilde{\zeta}_- = -\frac{3}{2}(\nabla - \frac{i}{6}b) \cdot \zeta_+$. Plugging this back into the equation and identifying $b = -2A$, again reproduces the CKS equation. However, (2.3) is a more restrictive condition, in part because one is not allowed to gauge-transform b [11].

Having highlighted the universality of the CKS equation as a necessary requirement for supersymmetric backgrounds, we now pass to study its integrability condition in detail. This is derived in appendix B, and reads

$$\left(\frac{1}{4}C_{mnpq} - \frac{i}{3}g_{p[m}F_{n]q}\right)\gamma^{pq}\zeta_+ - \frac{i}{3}\left(F_{mn} - \frac{1}{2}\gamma_{mnpq}F^{pq}\right)\zeta_+ = 0, \quad (2.4)$$

where

$$C_{mnpq} = R_{mnpq} - g_{m[p}R_{q]n} + g_{n[p}R_{q]m} + \frac{1}{3}R g_{m[p}g_{q]n} \quad (2.5)$$

is the Weyl tensor of the background metric, with R_{mnpq} , R_{mn} and R denoting the Riemann tensor, Ricci tensor and Ricci scalar, respectively. The geometric information contained in this condition can be extracted by contracting it with a basis of chiral spinors. In the following, we study the Lorentzian and Euclidean cases in turn. In Euclidean signature, we will also need to consider the integrability condition of the CKS equation (2.1) for $\tilde{\zeta}_-$. This takes the same form as (2.4), with the substitutions $\zeta_+ \rightarrow \tilde{\zeta}_-$ and $F \rightarrow -F$.

2.1 Lorentzian signature

In Lorentzian signature, the chiral spinor ζ_+ defines an \mathbb{R}^2 structure. This is characterized by the bilinears [15]

$$z_m = \frac{1}{4}\overline{\zeta_+}\gamma_m\zeta_+, \quad \omega_{mn} = -\frac{1}{4}\overline{(\zeta_+)^c}\gamma_{mn}\zeta_+, \quad (2.6)$$

see appendix A for our spinor conventions. By construction, the one-form z is real and null, $z_m z^m = 0$, while ω is a complex two-form satisfying $z \wedge \omega = 0$, which implies $\omega = z \wedge w$

for some complex one-form w . The one-forms z and w can be seen as elements of a frame, which may be completed by introducing an additional real one-form e^- (that however does not follow from the spinor ζ_+). Hence we can write $z = e^0 + e^1$, $e^- = -e^0 + e^1$, $w = e^2 - ie^3$. These satisfy the null tetrad relations

$$z_m z^m = e_m^- e^{-m} = w_m w^m = z_m w^m = e_m^- w^m = 0, \quad e_m^- z^m = w_m \bar{w}^m = 2, \quad (2.7)$$

and the metric reads

$$ds^2 = z e^- + w \bar{w}. \quad (2.8)$$

Note that we are assuming that ζ_+ is nowhere vanishing, otherwise our analysis will only be valid outside its zero locus. The two-form ω is anti-self-dual, $i * \omega = -\omega$.² A convenient way to parameterize the two-forms is to introduce the following anti-self-dual basis (generating the (0,1) representation of $SO(1,3)$):

$$\begin{aligned} \omega &= z \wedge w, \\ \theta &= e^- \wedge \bar{w}, \\ j &= \frac{i}{2}(z \wedge e^- + w \wedge \bar{w}). \end{aligned} \quad (2.9)$$

A basis of self-dual two-forms is obtained by simply complex-conjugating the anti-self-dual basis. The forms (2.9) satisfy the relations

$$\begin{aligned} j_m^p j_p^n &= -\delta_m^n, & \omega_m^p \omega_{pn} &= \theta_m^p \theta_{pn} = 0, \\ j_m^p \omega_{pn} &= i \omega_{mn}, & j_m^p \theta_{pn} &= -i \theta_{mn}, & \theta_m^p \omega_p^n &= -2(\delta_m^n + i j_m^n), \end{aligned} \quad (2.10)$$

which imply

$$j_{mn} j^{mn} = 4, \quad \omega_{mn} \theta^{mn} = 8, \quad \omega_{mn} \omega^{mn} = \theta_{mn} \theta^{mn} = \omega_{mn} j^{mn} = \theta_{mn} j^{mn} = 0. \quad (2.11)$$

As for the products of a self-dual with an anti-self-dual form, for the purpose of our computation it will be sufficient to note that their anti-symmetric parts all vanish ($j_{[m}^p \bar{j}_{n]p} = 0$, etc.) and that the same is true for the contraction of both indices ($j_{mn} \bar{j}^{mn} = 0$, etc.).

In Lorentzian signature, a basis of positive-chirality spinors is given by ζ_+ and $e_m^- \gamma^m (\zeta_+)^c$. Hence a basis of positive-chirality Dirac conjugates is given by $\overline{(\zeta_+)^c}$ and $\bar{\zeta}_+ e_m^- \gamma^m$. Contracting the integrability condition (2.4) with either one of these barred spinors, we obtain

$$\begin{aligned} \left(\frac{1}{4} C_{mnpq} - \frac{i}{3} g_{p[m} F_{n]q} \right) \omega^{pq} &= 0, \\ \left(\frac{1}{4} C_{mnpq} - \frac{i}{3} g_{p[m} F_{n]q} \right) j^{pq} &= \frac{2}{3} F_{mn}^-, \end{aligned} \quad (2.12)$$

²In [15], a form satisfying the very same property was called imaginary *self*-dual. Here we changed nomenclature for consistency with the Euclidean case, where the bilinears of a positive-chirality spinor turn out to be anti-self-dual (see below).

where we introduced the anti-self-dual part $F_{mn}^- = \frac{1}{2}(F_{mn} - i\tilde{F}_{mn})$ of the gauge field-strength, with $\tilde{F}_{mn} = \frac{1}{2}\epsilon_{mn}{}^{pq}F_{pq}$.

In order to efficiently work out the contractions in (2.12), we exploit the well-known fact that the Weyl tensor can be seen as a linear operator acting on the space of two-forms. The analysis of the algebraic properties of this operator leads to the Petrov classification, see e.g. Chapters 3, 4 of [25] for an extensive discussion. Defining the dual of the Weyl tensor as

$$\tilde{C}_{mnpq} = \frac{1}{2}\epsilon_{mn}{}^{rs}C_{rspq} = \frac{1}{2}\epsilon_{pq}{}^{rs}C_{mnrs}, \quad (2.13)$$

we can introduce its self-dual and anti-self-dual parts by

$$C_{mnpq}^\pm = \frac{1}{2}(C_{mnpq} \pm i\tilde{C}_{mnpq}). \quad (2.14)$$

Note that both C^- and $C^+ = \overline{C^-}$ are complex and contain the same degrees of freedom as C , i.e. ten real components. By contraction of the last pair of indices, C^+ acts on the self-dual two-forms, while C^- acts on the anti-self-dual ones. Adopting the Newman–Penrose tetrad formalism [26], we can use the basis ω, θ, j defined above to write the anti-self-dual part of the Weyl tensor as

$$\begin{aligned} C_{mnpq}^- &= \frac{1}{2}\Psi_0\theta_{mn}\theta_{pq} + \Psi_1(\theta_{mn}j_{pq} + j_{mn}\theta_{pq}) + \frac{1}{2}\Psi_2(\omega_{mn}\theta_{pq} + \theta_{mn}\omega_{pq} - 4j_{mn}j_{pq}) \\ &\quad + \Psi_3(\omega_{mn}j_{pq} + j_{mn}\omega_{pq}) + \frac{1}{2}\Psi_4\omega_{mn}\omega_{pq}, \end{aligned} \quad (2.15)$$

where the Weyl scalars parameterizing the tensor components are

$$\begin{aligned} \Psi_0 &= \frac{1}{32}C_{mnpq}\omega^{mn}\omega^{pq}, & \Psi_3 &= \frac{1}{32}C_{mnpq}\theta^{mn}j^{pq}, \\ \Psi_1 &= \frac{1}{32}C_{mnpq}\omega^{mn}j^{pq}, & \Psi_4 &= \frac{1}{32}C_{mnpq}\theta^{mn}\theta^{pq}, \\ \Psi_2 &= \frac{1}{32}C_{mnpq}\theta^{mn}\omega^{pq} = -\frac{1}{32}C_{mnpq}j^{mn}j^{pq}. \end{aligned} \quad (2.16)$$

This parameterization has the appealing feature of automatically incorporating all the algebraic symmetries of the Weyl tensor: $C_{mnpq} = C_{[mn]pq} = C_{mn[pq]} = C_{pqmn}$ and, slightly less obviously, $C^m{}_{nmp} = 0$ and $C_{m[npq]} = 0$. The five Weyl scalars are complex and independent, and exactly parameterize the ten real components of the Weyl tensor. We also record that

$$C_{mnpq}^- C^{-mnpq} \equiv \frac{1}{2}(C_{mnpq}C^{mnpq} - iC_{mnpq}\tilde{C}^{mnpq}) = 32(\Psi_0\Psi_4 + 4\Psi_1\Psi_3 + 3\Psi_2^2). \quad (2.17)$$

The forms j, ω, θ can also be used to parameterize the gauge field $F = F^+ + F^-$:

$$F^- = \Phi_j j + \Phi_\omega \omega + \Phi_\theta \theta, \quad (2.18)$$

with the Φ 's being arbitrary complex functions, and F^+ being given by $F^+ = \overline{F^-}$. Note that

$$F_{mn}^- F^{-mn} \equiv \frac{1}{2} (F_{mn} F^{mn} - i F_{mn} \tilde{F}^{mn}) = 16 \Phi_\omega \Phi_\theta + 4 \Phi_j^2. \quad (2.19)$$

Using this parameterization of the Weyl tensor and the gauge field, and recalling the relations (2.10), it is straightforward to show that equations (2.12) arising from the integrability condition of the CKS equation are equivalent to

$$\Psi_0 = 0, \quad \Psi_1 = \frac{1}{3} \Phi_\theta, \quad \Psi_2 = -\frac{1}{3} \Phi_j, \quad \Psi_3 = \Phi_\omega, \quad (2.20)$$

with no constraints being imposed on Ψ_4 . Thus, the field-strength F_{mn} has been determined completely in terms of the Weyl tensor.³ An immediate consequence of these relations, which is seen recalling (2.17) and (2.19), is

$$C_{mnpq}^- C^{-mnpq} = \frac{8}{3} F_{mn}^- F^{-mn}. \quad (2.21)$$

Separating the real and imaginary parts, we obtain

$$C_{mnpq} C^{mnpq} = \frac{8}{3} F_{mn} F^{mn}, \quad C_{mnpq} \tilde{C}^{mnpq} = \frac{8}{3} F_{mn} \tilde{F}^{mn}. \quad (2.22)$$

It is interesting to remark that the charged conformal Killing spinor equation (1.3) coincides with the unbroken supersymmetry condition for bosonic backgrounds of minimal conformal supergravity, see e.g. [21, eq. (2.37)]. Therefore, the first equation in (2.22) shows that on such supersymmetric backgrounds the conformal supergravity action, whose bosonic part in our conventions is given by $S \sim \int d^4x \sqrt{g} (C_{mnpq} C^{mnpq} - \frac{8}{3} F_{mn} F^{mn})$, vanishes. However, note that, differently from ordinary supergravity, this supersymmetry condition, supplemented with the Maxwell equation, does not imply the equation of motion of conformal supergravity.

Vanishing of some of the Weyl scalars corresponds to different types in the Petrov algebraic classification of the Weyl tensor (see e.g. [25]). The generic supersymmetric background with just $\Psi_0 = 0$ is algebraically general (Petrov type I). The property $\Psi_0 = 0$ means that our vector z built from ζ_+ is a principal null direction of the Weyl tensor, i.e. $z^n z_{[r} C_{m]np[q} z_s] z^p = 0$. If any Weyl scalar beyond Ψ_0 vanishes, then the Weyl tensor is said algebraically special; depending on which Weyl scalars are zero, it can be of Petrov type II, III, D, N or O. In particular, when the gauge field vanishes we have $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$ with Ψ_4 generically non-vanishing, which corresponds to a Petrov N spacetime.⁴ If in addition $\Psi_4 = 0$,

³We believe that (2.20), supplemented with the existence of a null conformal Killing vector, should be equivalent to a solution of the CKS equation. However, we have not verified this.

⁴The uncharged ($A = 0$) CKS equation had been studied prior to [15], in [27]. It was found that it admits solutions on Fefferman and on pp-wave-type spacetimes (depending on whether z is twisting or non-twisting, respectively). That the uncharged CKS equation implies Petrov N can be found e.g. in [20], around eqs. (6.1.6) and (8.1.4).

the full Weyl tensor vanishes and the spacetime is conformally flat (Petrov type O). Other algebraically special cases are also possible, for instance if F^- is proportional to ω then we have $\Psi_0 = \Psi_1 = \Psi_2 = 0$, corresponding to a Petrov III spacetime. This implies that the two sides of both equations (2.22) vanish separately, that is $C^2 = F^2 = C\tilde{C} = F\tilde{F} = 0$.

Note that our analysis is purely local, and the Petrov type can change at different space-time points. We will comment more on the implications of these results in section 3.

2.2 Euclidean signature

A completely parallel analysis can be done in Euclidean signature, albeit for the self-dual and anti-self-dual parts of the Weyl and Maxwell tensors separately. This is due to the fact that contrarily to their Lorentzian analogues, Euclidean forms of opposite chiralities, defining the (1,0) and (0,1) representations of $SO(4)$, are not related by complex conjugation.

We will assume that the chiral spinor ζ_+ does not vanish anywhere, being understood that if it does our local analysis will still be valid outside its zero locus. In Euclidean signature, ζ_+ defines a $U(2)$ structure, and we can construct the bilinears (see e.g. [8]):

$$\zeta_+^\dagger \zeta_+ = \|\zeta_+\|^2, \quad j_{mn}^- = \frac{-i}{\|\zeta_+\|^2} \zeta_+^\dagger \gamma_{mn} \zeta_+, \quad \omega_{mn}^- = \frac{-1}{\|\zeta_+\|^2} (\zeta_+^c)^\dagger \gamma_{mn} \zeta_+. \quad (2.23)$$

The two-form j^- is real while ω^- is complex and decomposable (namely, it can be written as the wedge of two complex one-forms). In our conventions they turn out to be anti-self-dual,

$$*j^- = -j^-, \quad *\omega^- = -\omega^-. \quad (2.24)$$

Starting from $\tilde{\zeta}_-$, we can construct similar bilinears j^+ , ω^+ , that are instead self-dual. The forms are related to the metric via $g_{mn} = j_{mp}^\pm I^{\pm p}{}_n$, where I^\pm is the almost complex structure defined by ω^\pm . Moreover, they are non-degenerate and related to the volume form as

$$\frac{1}{2} j^\pm \wedge j^\pm = \frac{1}{4} \omega^\pm \wedge \overline{\omega^\pm} = \pm \text{vol}_4, \quad (2.25)$$

with all other wedgings between them vanishing. Calling $\theta^\pm = \overline{\omega^\pm}$, one can show that for each chirality exactly the same properties (2.10) found in Lorentzian signature are satisfied (this explains why we are giving the same names to the Lorentzian and Euclidean two-forms). We also note that both the commutator and the contraction of a self-dual with an anti-self-dual form vanish ($j_{[m}^- j_{n]p}^+ = 0$, $j_{mn}^- j^{+mn} = 0$, etc.).

A basis of positive chirality spinors is given by ζ_+ together with its charge conjugate ζ_+^c . Contracting the integrability condition (2.4) with the hermitian conjugate of either one of these spinors, we find formally the same equations (2.12) obtained in Lorentzian signature.

If on the other hand we start from the negative chirality spinor $\tilde{\zeta}_-$ and its charge conjugate, we obtain similar equations, with a few sign changes. In detail, we have

$$\begin{aligned} \left(\frac{1}{4}C_{mnpq} \pm \frac{i}{3}g_{p[m}F_{n]q}\right)\omega_{\pm}^{pq} &= 0, \\ \left(\frac{1}{4}C_{mnpq} \pm \frac{i}{3}g_{p[m}F_{n]q}\right)j_{\pm}^{pq} &= \mp \frac{2}{3}F_{mn}^{\pm}, \end{aligned} \quad (2.26)$$

where the lower sign is associated with the ζ_+ equation (1.3), while the upper sign is associated with the $\tilde{\zeta}_-$ equation (2.1). F^{\pm} are the (anti)-self-dual parts of F : $F_{mn}^{\pm} = \frac{1}{2}(F_{mn} \pm \tilde{F}_{mn})$.

These equations can be analysed with the same approach used in Lorentzian signature. Namely, we can introduce the Euclidean self-dual and anti-self-dual parts of the Weyl tensor

$$C_{mnpq}^{\pm} = \frac{1}{2}(C_{mnpq} \pm \tilde{C}_{mnpq}), \quad (2.27)$$

Note that, in contrast to the Lorentzian case, where they are conjugate to each other, in Euclidean signature C^{\pm} are real, a priori independent tensors. Using the anti-self-dual basis j^-, ω^-, θ^- , the anti-self-dual Weyl tensor C^- can be parameterized as in (2.15). The same can be done for the self-dual tensor C^+ by using the basis j^+, ω^+, θ^+ . We thus have two sets of Weyl scalars defined as in (2.16): $\Psi_0^-, \dots, \Psi_4^-$ defined by means of the anti-self-dual basis, and $\Psi_0^+, \dots, \Psi_4^+$ defined in terms of the self-dual basis. The relations $\theta^{\pm} = \overline{\omega^{\pm}}$ and $\overline{j^{\pm}} = j^{\pm}$, together with the reality of the Weyl tensor, imply the following constraints between Weyl scalars of same chirality:

$$\overline{\Psi_4^{\pm}} = \Psi_0^{\pm}, \quad \overline{\Psi_3^{\pm}} = \Psi_1^{\pm}, \quad \overline{\Psi_2^{\pm}} = \Psi_2^{\pm}. \quad (2.28)$$

We see that the Ψ^+ (respectively, Ψ^-) Weyl scalars parameterize the five degrees of freedom in the anti-self-dual (respectively, self-dual) parts of the Weyl tensor. Altogether, they parameterize the ten independent real components of the Weyl tensor.⁵ The square of the chiral parts of the Weyl tensor read

$$C_{mnpq}^{\pm} C^{\pm mnpq} \equiv \frac{1}{2}(C_{mnpq} C^{mnpq} \pm C_{mnpq} \tilde{C}^{mnpq}) = 32 [\Psi_0^{\pm} \Psi_4^{\pm} + 4\Psi_1^{\pm} \Psi_3^{\pm} + 3(\Psi_2^{\pm})^2]. \quad (2.29)$$

We can use $j^{\pm}, \omega^{\pm}, \theta^{\pm}$ to also parameterize the gauge field $F = F^+ + F^-$:

$$F^{\pm} = \Phi_j^{\pm} j^{\pm} + \Phi_{\omega}^{\pm} \omega^{\pm} + \Phi_{\theta}^{\pm} \theta^{\pm}. \quad (2.30)$$

⁵This parameterization of the Weyl tensor in Euclidean signature was given in [28], where a complex frame was used rather than the $U(2)$ structure two-forms. Since our form ω is decomposable, the two descriptions are locally equivalent.

Since in Euclidean signature we are allowing for a complex gauge field, the components $\Phi_j^\pm, \Phi_\omega^\pm, \Phi_\theta^\pm$ are a priori arbitrary complex functions. Note that

$$F_{mn}^\pm F^{\pm mn} \equiv \frac{1}{2}(F_{mn}F^{mn} \pm F_{mn}\tilde{F}^{mn}) = 16\Phi_\omega^\pm\Phi_\theta^\pm + 4(\Phi_j^\pm)^2. \quad (2.31)$$

Applying this formalism to eq. (2.26), it is easy to see that the integrability condition of the CKS equation for ζ_+ leads to

$$\Psi_0^- = 0, \quad \Psi_1^- = \frac{1}{3}\Phi_\theta^-, \quad \Psi_2^- = -\frac{1}{3}\Phi_j^-, \quad \Psi_3^- = \Phi_\omega^-, \quad (2.32)$$

with no constraints on the Ψ^+ Weyl scalars nor on the Φ^+ components of F . Taking into account the reality conditions (2.28), we arrive at

$$\begin{aligned} \Psi_0^- = \Psi_4^- = 0, \quad \overline{\Psi_1^-} = \Psi_3^- = \Phi_\omega^-, \quad \Psi_2^- = -\frac{1}{3}\text{Re}\Phi_j^-, \\ \text{Im}\Phi_j^- = 0, \quad \Phi_\theta^- = 3\overline{\Phi_\omega^-}. \end{aligned} \quad (2.33)$$

Note that the second line imposes some ‘‘reality’’ constraints on the anti-self-dual part of the gauge field. In particular, recalling (2.31) we have that although F is complex, $(F^-)^2$ is real. Also recalling (2.29), we find that

$$C_{mnpq}^- C^{-mnpq} = \frac{8}{3} F_{mn}^- F^{-mn}, \quad (2.34)$$

which is the same as

$$C_{mnpq} C^{mnpq} - \frac{8}{3} F_{mn} F^{mn} = C_{mnpq} \tilde{C}^{mnpq} - \frac{8}{3} F_{mn} \tilde{F}^{mn}. \quad (2.35)$$

This is a weaker condition than the one found in Lorentzian signature, where the two sides of the equation were vanishing separately.

Similarly, integrability of the CKS equation for $\tilde{\zeta}_-$, together with (2.28), yields a set of conditions like (2.33), with the replacements $\Psi^- \rightarrow \Psi^+, \Phi^- \rightarrow -\Phi^+$, and no constraints on the Ψ^- and Φ^- components. This implies

$$C_{mnpq}^+ C^{+mnpq} = \frac{8}{3} F_{mn}^+ F^{+mn}, \quad (2.36)$$

which is the same as

$$C_{mnpq} C^{mnpq} - \frac{8}{3} F_{mn} F^{mn} = -C_{mnpq} \tilde{C}^{mnpq} + \frac{8}{3} F_{mn} \tilde{F}^{mn}. \quad (2.37)$$

A classification similar to the one of Petrov exists in Euclidean signature for the self-dual and anti-self-dual parts of the Weyl tensor independently, with the only allowed types being

I, D and O for each chirality, see e.g. [28]. Here, referring for definiteness to a solution to the CKS equation for ζ_+ , we observe that the conditions (2.33) imply that if F^- is real (i.e. $\text{Im } \Phi_j^- = 0$, $\Phi_\theta = \overline{\Phi_\omega^-}$), then it can only be proportional to j^- , and enforces $\Psi_0^- = \Psi_1^- = \Psi_3^- = \Psi_4^- = 0$, meaning that the anti-self-dual part of the Weyl tensor is of type D. On the other hand, when F^- is purely imaginary (i.e. $\text{Re } \Phi_j = 0$, $\Phi_\theta = -\overline{\Phi_\omega}$), all the Weyl scalars Ψ^- and the gauge field components Φ^- have to vanish, hence both the Weyl and the Maxwell tensors are self-dual. Note that for a purely imaginary gauge field, if ζ_+ is a solution to the CKS equation, its charge conjugate ζ_+^c is an independent solution to the same equation; hence we obtain two supercharges with the same R-charge. These observations about a purely imaginary gauge field are consistent with the results of [9, sect. 5].

3 Taming the superconformal anomalies

Below we will discuss the implications of the results in the previous section on the superconformal anomaly. For the sake of clarity we will keep the discussion of the Lorentzian and Euclidean signatures separated.

3.1 Lorentzian signature

Our results establish the absence of c -terms in the trace and R-symmetry anomalies for $\mathcal{N} = 1$ superconformal field theories coupled to a curved supersymmetric background. Let us write the anomaly formulae (1.1), (1.2) by separating the terms multiplied by the central charges a and c :

$$\begin{aligned} T_m^m &= \frac{c}{16\pi^2} \left(C^2 - \frac{8}{3} F^2 \right) - \frac{a}{16\pi^2} E, \\ \nabla_m J^m &= \frac{c}{24\pi^2} \left(P - \frac{8}{3} F \tilde{F} \right) + \frac{a}{24\pi^2} \left(-P + \frac{40}{9} F \tilde{F} \right). \end{aligned} \quad (3.1)$$

The quadratic Weyl, Euler and Pontryagin⁶ densities are defined as

$$\begin{aligned} C^2 &\equiv C_{mnpq} C^{mnpq} = R_{mnpq} R^{mnpq} - 2R_{mn} R^{mn} + \frac{1}{3} R^2, \\ E &\equiv \frac{1}{4} \epsilon^{mnpq} \epsilon^{rsuv} R_{mnr s} R_{pquv} = R_{mnpq} R^{mnpq} - 4R_{mn} R^{mn} + R^2, \\ P &\equiv \frac{1}{2} \epsilon^{mnpq} R_{mnr s} R_{pq}{}^{rs} = \frac{1}{2} \epsilon^{mnpq} C_{mnr s} C_{pq}{}^{rs}, \end{aligned} \quad (3.2)$$

⁶The density P is related to the first Pontryagin class of the manifold M as $p_1(M) = \frac{P}{16\pi^2} \text{vol}(M)$, where $\text{vol}(M)$ is the volume form of M .

where the last equalities in the first and in the third lines follow straightforwardly from the definition (2.5) of the Weyl tensor. We have also introduced the short-hand notation $F^2 \equiv F_{mn}F^{mn}$ and $F\tilde{F} \equiv \frac{1}{2}\epsilon^{mnpq}F_{mn}F_{pq}$. The conditions (2.22) show that for supersymmetric field theories in supersymmetric backgrounds *the terms multiplied by c vanish in the trace and R -current anomalies*, hence both anomalies are purely topological invariants.⁷ Specifically, the anomaly formulae reduce to

$$\begin{aligned} T_m^m &= -\frac{a}{16\pi^2}E, \\ \nabla_m J^m &= \frac{a}{36\pi^2}P = \frac{2a}{27\pi^2}F\tilde{F}. \end{aligned} \tag{3.3}$$

In [15] it was shown that a necessary and sufficient condition for the existence of a solution to the CKS equation is that there exists a null conformal Killing vector, with a natural coordinate system adapted to this. It is straightforward to compute the Weyl scalars Ψ_0, \dots, Ψ_4 in the explicit coordinates given in [15], and obtain F using the relations (2.20). In particular, computing the Euler and Pontryagin densities explicitly, we have checked that generically they do not vanish. However, in certain classes of geometries the Pontryagin density does vanish, implying that the $U(1)_R$ current is not anomalous on these backgrounds. The following are sufficient criteria for the vanishing of P :

- Using (2.17) we have

$$P = -64 \operatorname{Im} (4\Psi_1\Psi_3 + 3\Psi_2^2), \tag{3.4}$$

which vanishes for space-times of Petrov type O (where $C_{mnpq} = 0$), N (where $F_{mn} = 0$), and III (where $C^2 = 0 = P$). An example of Petrov type III is the $SU(2) \times U(1)$ -invariant background on $\mathbb{R} \times S^3$, arising as the boundary of a deformation of AdS_5 , discussed in [29]. Due to the split topology of the space and the homogeneity of the metric, in this specific example one also has $E = 0$, so the full superconformal anomaly vanishes.

- If there exists a (conformal) Killing vector k that is also hypersurface-orthogonal, namely the dual one-form satisfies $k \wedge dk = 0$, then $P = 0$. This may be verified noting that since P is a conformal invariant density, in the conformal class of metrics we can consider one where k is a Killing vector. If the Killing vector $k = \partial_t$ is time-like or space-like, then the metric is a Riemannian product $ds^2 = \pm dt^2 + ds^2(M_3)$,

⁷In Lorentzian signature, the manifolds of physical relevance are non-compact, hence defining topological invariants is more subtle than on compact spaces. Here we will refrain from integrating the anomalies.

where M_3 is a three-dimensional manifold.⁸ By direct computation of the curvature one can verify that $P = 0$. If k is null, then this corresponds to the case of non-twisting geometries discussed in [15], and again a direct computation shows that $P = 0$. An example possessing such Killing vectors is given by the boundary of the five-dimensional magnetic string solutions of [30, 31]. These comprise a metric on $\mathbb{R}^{1,1} \times S^2$ or $\mathbb{R}^{1,1} \times \mathbb{H}^2$ of Petrov type D (which has all Weyl scalars but Ψ_2 equal zero), and a non-trivial gauge field. Moreover, on this background $E = 0$, so the superconformal anomaly fully vanishes. Another Petrov D example with similar properties will be discussed later in section 4.

3.2 Euclidean signature

In Euclidean signature, the integrability equation (2.4) for ζ_+ does not contain informations about the self-dual parts of the Weyl and Maxwell tensors. Although these are completely determined by the CKS equation [9, 8], the weaker condition (2.35) is sufficient to show that the c -contribution to the conformal anomaly, on the left hand side of (2.35), is a topological density, on the right hand side of (2.35). Therefore, as in the Lorentzian case, the trace anomaly is purely topological, although generically the c -contribution to the anomalies does not vanish. Namely, the anomalies read

$$\begin{aligned} T_m^m &= \frac{c}{16\pi^2} \left[P - \frac{8}{3} \text{Re}(F\tilde{F}) \right] - \frac{a}{16\pi^2} E - i \frac{c}{6\pi^2} \text{Im}(F\tilde{F}), \\ \nabla_m J^m &= \frac{c-a}{24\pi^2} P + \frac{5a-3c}{27\pi^2} \text{Re}(F\tilde{F}) + i \frac{5a-3c}{27\pi^2} \text{Im}(F\tilde{F}). \end{aligned} \quad (3.5)$$

Although there is no new information on the R-current anomaly, we have emphasized that generically both anomalies include an *imaginary* part.

On a compact Riemannian manifold M without boundary, it is interesting to consider the integrated version of eqs. (3.5), and express these in terms of the *Euler characteristic* χ and *signature* σ of M , given by

$$\mathbb{Z} \ni \chi(M) = \frac{1}{32\pi^2} \int_M d^4x \sqrt{g} E, \quad (3.6)$$

$$\mathbb{Z} \ni \sigma(M) = \frac{1}{3} \int_M p_1(M) = \frac{1}{48\pi^2} \int_M d^4x \sqrt{g} P. \quad (3.7)$$

Introducing the real and imaginary parts of the gauge field as⁹

$$F = \frac{1}{2}(\mathcal{F} + i\mathcal{G}), \quad (3.8)$$

⁸Note that the existence of a time-like hypersurface-orthogonal Killing vector is also a sufficient condition for Wick-rotating the background to Euclidean signature.

⁹The factor of $\frac{1}{2}$ is introduced for convenience, as it will become clear momentarily.

we also define

$$\begin{aligned} \nu &\equiv \frac{1}{2\pi^2} \int_M d^4x \sqrt{g} F \tilde{F} = \\ &= \int_M \frac{\mathcal{F}}{2\pi} \wedge \frac{\mathcal{F}}{2\pi} - \int_M \frac{\mathcal{G}}{2\pi} \wedge \frac{\mathcal{G}}{2\pi} + 2i \int_M \frac{\mathcal{F}}{2\pi} \wedge \frac{\mathcal{G}}{2\pi} . \end{aligned} \quad (3.9)$$

In order to evaluate the integrals in (3.9), we pause to recall some of the *global* properties of the backgrounds, discussed in [9, 8]. Here, it is important to assume that the spinor ζ_+ is well-defined and non-vanishing everywhere on M . Firstly, notice that our gauge field A is related to the gauge field in [9] as $A^{\text{here}} = A^{\text{DFS}} - \frac{3}{2}V^{\text{DFS}}$, where A^{DFS} and V^{DFS} are the auxiliary fields of new minimal supergravity (called a and v at the beginning of section 2). However, assuming that V^{DFS} is a globally defined one-form on M , this does not affect the discussion of the global properties of A . In particular, using eq. (3.17) in [9] we have (in our notation):

$$A_m = A_m^c + \frac{1}{4}(\delta_m^n + 2i j^-{}_m{}^n) \nabla_p j^{-p}{}_n , \quad (3.10)$$

where $j^-{}_m{}^n$ is the integrable complex structure on M , obtained raising an index on the two-form defined in (2.23), and

$$A^c = -\frac{i}{4}(\partial - \bar{\partial}) \log \sqrt{g} - \frac{i}{2} d \log s . \quad (3.11)$$

Here s is a complex function that may be defined as $\zeta_+ = \sqrt{s} \zeta_+^0$, where ζ_+^0 is a constant chiral spinor. The important point to note is that $A_m - A_m^c$ is a global one-form on M , therefore \mathcal{G} is an exact two-form, and it drops out of (3.9). The real part of A_m , however, transforms as a gauge field for the local $U(1)_R$ symmetry, being a connection on the line bundle $\mathcal{L} = \bar{\mathcal{K}}^{-1/2}$, where $\bar{\mathcal{K}} = \Lambda^{(0,2)}(M)$ is the anti-canonical bundle of $(0, 2)$ -forms on M . Correspondingly, the spinor ζ_+ is a section of the spin ^{c} bundle $\mathcal{V} = \bar{\mathcal{K}}^{-1/2} \otimes \mathcal{S}_+$, where \mathcal{S}_+ is the bundle of positive-chirality spinors. Although neither bundle may exist separately due to the second Stiefel–Whitney class $w_2(M) \in H_2(M; \mathbb{Z}_2)$ being non-trivial, the tensor product \mathcal{V} always exists as a genuine complex vector bundle.

Recall that on any Hermitian manifold, the metric g_{mn} and complex structure $(j^-)_m{}^n$ induce a connection on the anti-canonical bundle $\bar{\mathcal{K}}$, given by

$$\rho = -\frac{i}{2}(\partial - \bar{\partial}) \log \sqrt{g} , \quad (3.12)$$

whose curvature is the Ricci two-form $\mathcal{R}_{mn} = \frac{1}{2}(j^-)^{pq} R_{pqmn}$. The latter defines the first Chern class $c_1(M) = [\frac{\mathcal{R}}{2\pi}] \in H^2(M; \mathbb{R})$. Comparing this with the definition of A^c in (3.11),

we have the following chain of identities:

$$\frac{1}{2} \left[\frac{\mathcal{F}}{2\pi} \right] = \left[\frac{dA^c}{2\pi} \right] = \frac{1}{2} \left[\frac{d\rho}{2\pi} \right] = \frac{1}{2} \left[\frac{\mathcal{R}}{2\pi} \right]. \quad (3.13)$$

This is of course consistent with the fact that $2 \cdot \text{Re}A$ is a connection on the anti-canonical bundle, and justifies the factor of $1/2$ in the definition (3.8). In particular, the quantization condition

$$\int_{\Sigma_a} \frac{\mathcal{F}}{2\pi} \in \mathbb{Z}, \quad (3.14)$$

where Σ_a is a basis of generators of $H_2(M; \mathbb{Z})$, is automatically satisfied. Note that on *Kähler* manifolds the identities in (3.13) hold more strongly as identities of two-forms, rather than in co-homology. Using these, we see that the integral defining ν is the self-intersection of the first Chern class

$$\mathbb{N} \ni \nu(M) = \frac{1}{(2\pi)^2} \int_M \mathcal{F} \wedge \mathcal{F} = \int_M c_1(M) \wedge c_1(M), \quad (3.15)$$

which on a complex manifold is given by

$$\nu(M) = 3\sigma(M) + 2\chi(M). \quad (3.16)$$

Therefore the integrated anomalies are determined completely by the signature and Euler characteristic of M and read

$$\begin{aligned} \int_M d^4x \sqrt{g} T_m^m &= 3c\sigma(M) - 2a\chi(M) - \frac{c}{3}\nu(M), \\ \int_M d^4x \sqrt{g} \nabla_m J^m &= 2(c-a)\sigma(M) + \frac{2}{27}(5a-3c)\nu(M). \end{aligned} \quad (3.17)$$

In general, all the topological invariants are non-zero, implying that the corresponding densities are non-trivial. Simple examples are provided by Kähler manifolds, as we will discuss momentarily. Sufficient criteria for the vanishing of the Pontryagin density P are analogous to the ones discussed in Lorentzian signature. Moreover, we note that if there exists a non-trivial¹⁰ Codazzi tensor B_{mn} , *i.e.* a symmetric tensor satisfying the equation $\nabla_{[m} B_{n]p} = 0$, then the Pontryagin density vanishes by Theorem 2 of [32].¹¹ A necessary and sufficient condition for the signature $\sigma(M)$ to be zero is that the complex manifold M is the boundary of an oriented five-dimensional manifold M_5 : $M = \partial M_5$. This follows from

¹⁰Not proportional to the metric.

¹¹Reference [32] discusses Riemannian manifolds, however this criterion may as well remain valid in Lorentzian signature.

a theorem by Rohlin (recalling that any complex manifold is oriented). In particular, the signature vanishes if the manifold is topologically $S^1 \times M_3$, even if the metric is not the direct-product metric. Indeed, this is the boundary of $D^1 \times M_3$, where D^1 is a disk. Note that $\sigma(M) = 0$ is needed in order to construct a five-dimensional holographic dual solution on a manifold M_5 “filling in” the background $M = \partial M_5$.

Manifolds admitting two CKS of opposite R-charge

When there exist two supercharges of opposite R-charge, namely a solution ζ_+ to the CKS equation (1.3) together with a solution $\tilde{\zeta}_-$ to eq. (2.1), the implications on the anomalies are stronger, and analogous to the Lorentzian signature case. In the following, we assume that ζ_+ and $\tilde{\zeta}_-$ are nowhere vanishing. This case was analysed in [9, 8], where it was shown that there exist two (generically) commuting Killing vectors, constraining the form of the metric to a \mathbb{T}^2 fibered over a Riemann surface Σ . Namely, in complex coordinates adapted to the complex structure j^- , this takes the form

$$ds^2 = \Omega^2(z, \bar{z}) [(dw + h(z, \bar{z})dz)(d\bar{w} + \bar{h}(z, \bar{z})d\bar{z}) + c^2(z, \bar{z})dzd\bar{z}] . \quad (3.18)$$

Here we note that two solutions to the CKS equations of opposite chiralities impose the constraints we discussed above, for *both* self-dual and anti-self-dual parts of the Maxwell and Weyl tensors. In particular, we have

$$\begin{aligned} C^2 - \frac{8}{3}\text{Re } F^2 &= 0 , \\ P - \frac{8}{3}\text{Re } F\tilde{F} &= 0 , \\ \text{Im } F^2 = \text{Im } F\tilde{F} &= 0 . \end{aligned} \quad (3.19)$$

These imply that the anomalies take the same form as in the Lorentzian case, namely (3.3), and are automatically *real*. Integrating on a compact manifold without boundary, gives the simple relations

$$\begin{aligned} \int_M d^4x \sqrt{g} T_m^m &= -2a \chi(M) , \\ \int_M d^4x \sqrt{g} \nabla_m J^m &= \frac{4}{3}a \sigma(M) , \end{aligned} \quad (3.20)$$

as well as the topological relation $\sigma(M) = \frac{1}{9}\nu(M)$. By evaluating the Ricci form of the metric (3.18),

$$\mathcal{R} = i \partial \bar{\partial} \log[\Omega^2(z, \bar{z})c(z, \bar{z})] , \quad (3.21)$$

it is straightforward to see that this is basic with respect to the (complex) Killing vector ∂_w , namely $\partial_w \lrcorner \mathcal{R} = 0$. It immediately follows that $\nu(M)$ vanishes. So, recalling (3.16), on backgrounds preserving two supercharges of opposite R-charges we have

$$\sigma(M) = \nu(M) = \chi(M) = 0, \quad (3.22)$$

hence both the integrated anomalies vanish.

Examples

We conclude this section discussing briefly simple examples of complex manifolds, which may be used as supersymmetric backgrounds. Note that although the manifolds below all admit Kähler metrics (in some cases there exist also Einstein metrics), one is not restricted to these. The considerations we make below are purely topological.

- A class of examples is provided by $M = dP_k$ the k th *del Pezzo* surface, with $k = 0, \dots, 8$, obtained blowing up k points at generic positions in $\mathbb{C}P^2$. In particular, $dP_0 = \mathbb{C}P^2$. For $k = 1, 2, 3$ these are toric manifolds, and for all $k \neq 1, 2$ there exist Kähler–Einstein metrics, although for $k \neq 0$ these are not known in explicit form. The non-zero Betti numbers are $b_0 = b_4 = 1$, and $b_2 = k + 1$, thus the Euler characteristic is $\chi(dP_k) = k + 3$. Using $K^2 = K \cdot K = 9 - k$, where K is the anti-canonical class, we obtain $\nu(dP_k) = 9 - k$ and, by eq. (3.16), $\sigma(dP_k) = 1 - k$. Inserting these expressions in the formulae for the anomalies we see that for generic values of the central charges a and c , and for all values of k , these are different from zero. Note that the only del Pezzo surface having vanishing signature, and so the only one arising as the boundary of a five-dimensional manifold, is dP_1 . Explicit Hermitian metrics on dP_1 were presented in [33] and [34].
- Another class of simple examples is given by the product of two Riemann surfaces $\Sigma_1 \times \Sigma_2$. The signature is zero since any oriented Riemann surface bounds an oriented 3-manifold N_3 (called handlebody), and so $\Sigma_1 \times \Sigma_2 = \partial(N_3 \times \Sigma_2)$. Hence $\sigma(\Sigma_1 \times \Sigma_2) = 0$. The Euler characteristic is given by the product of the Euler characteristics of the two surfaces, hence $\chi(\Sigma_1 \times \Sigma_2) = 2(1 - g_1) \cdot 2(1 - g_2)$, where g_1, g_2 are their genera. By formula (3.16), we have $\nu(\Sigma_1 \times \Sigma_2) = 2\chi(\Sigma_1 \times \Sigma_2)$. A special case is that of $\Sigma_1 \times \Sigma_2 = \mathbb{T}^2 \times \Sigma_2$, which is an example admitting two supercharges of opposite R-charges. As we discussed above, these necessarily have $\sigma(M) = \chi(M) = 0$.

4 A background with vanishing anomalies

In section 3 we mentioned a few non conformally-flat backgrounds where the full trace and R-symmetry anomalies vanish, which also have a known gravity dual [29, 30, 31]. In this section, we present in some detail another example of this type, whose gravity dual will be the object of a separate publication. This example has both a Lorentzian and an Euclidean avatar (as [30, 31] but differently from [29], which appears to be just Lorentzian), and will also serve as an illustration of the general results in the previous sections. We focus on its Euclidean version, commenting on the Lorentzian case at the end.

We consider an $S^3 \times S^1$ topology, and impose that $SU(2)_{\text{left}} \times U(1)_{\text{right}} \times U(1)_t$ symmetry is preserved. Then the metric reads

$$ds^2 = \frac{r^2}{4} (\sigma_1^2 + \sigma_2^2 + 4s^2 \sigma_3^2) + (dt + k \sigma_3)^2 \quad (4.1)$$

and the general form of the gauge potential is

$$A = p dt + q \sigma_3. \quad (4.2)$$

Here, the σ 's are $SU(2)$ left-invariant one-forms on S^3 parameterized by Euler angles ϑ, ϕ, ψ :

$$\begin{aligned} \sigma_1 &= -\sin \psi d\vartheta + \cos \psi \sin \vartheta d\phi, \\ \sigma_2 &= \cos \psi d\vartheta + \sin \psi \sin \vartheta d\phi, \\ \sigma_3 &= d\psi + \cos \vartheta d\phi, \end{aligned} \quad (4.3)$$

r is the radius of S^3 , s is a squashing parameter, and we are fixing the modulus of the g_{tt} metric component to one (which can always be done by rescaling k and t). In Euclidean signature, we allow p and q to be complex parameters, while the parameters in the metric are assumed real. We will work in the frame

$$e^1 = \frac{r}{2} \sigma_1, \quad e^2 = \frac{r}{2} \sigma_2, \quad e^3 = r s \sigma_3, \quad e^4 = dt + k \sigma_3. \quad (4.4)$$

We observe that the terms proportional to k in the metric (4.1) can be removed by performing the change of coordinates

$$\psi' = \psi + \frac{k}{s^2 r^2 + k^2} t, \quad t' = \frac{1}{\lambda} t, \quad (4.5)$$

accompanied by the redefinition of the parameters

$$s' = \lambda s, \quad p' = \lambda p - \frac{k}{s^2 r^2 \lambda} q, \quad (4.6)$$

where $\lambda = \sqrt{1 + \frac{k^2}{s^2 r^2}}$. One can check that in the primed variables the metric takes the direct product form

$$ds^2 = \frac{r^2}{4} (\sigma_1'^2 + \sigma_2'^2 + 4s'^2 \sigma_3'^2) + dt'^2 \quad (4.7)$$

and the gauge field is still in the original form

$$A = p' dt + q \sigma_3'. \quad (4.8)$$

Here, $\sigma_1', \sigma_2', \sigma_3'$ are $SU(2)$ left-invariant one-forms constructed using ψ' at the place of ψ .

For any value of the radius r and squashing s , the metric (4.1) allows for solutions to the CKS equation and thus yields a good supersymmetric background, provided an appropriate gauge field is chosen. Indeed, the metric describes a torus fibration over S^2 (with an S^1 in the torus being the Hopf fiber and the other being generated by ∂_t), hence recalling the results in [9] the background allows for at least two solutions to the CKS equation with opposite R-charge. The gauge field can be determined using the general formulae given in [8, 9]. Here, we apply our alternative method exploiting the integrability condition. This completely determines the field-strength F , and thus the potential A modulo a closed one-form.

Before presenting the details, let us observe that both the Euler and Pontryagin densities E and P associated with the metric (4.1) vanish. This can be checked by evaluating (3.2). Another way to see it is first to observe that since (4.1) is a left-invariant metric on a homogeneous space, both E and P must be constant, and then to recall that the respective integrals have to vanish, as noted in section 3.2 while discussing backgrounds with two CKS of opposite R-charge. Together with the relation between Weyl and Maxwell tensors imposed by supersymmetry, this shows that the whole Weyl and R-symmetry anomalies vanish.

Four supercharges

Let us first review the case in which the gauge field-strength vanishes, which was already studied in [35, 9]. From the analysis in section 2.2, we know that when there are two conformal Killing spinors of opposite chirality and $F = 0$, then the Weyl tensor has to vanish, meaning that the space is conformally flat. For the metric (4.1), this translates into

$$k^2 = \frac{r^2}{4}(1 - 4s^2), \quad (4.9)$$

hence (choosing the positive root for k)

$$ds^2 = \frac{r^2}{4} (\sigma_1^2 + \sigma_2^2 + 4s^2 \sigma_3^2) + \left(dt + \frac{r}{2} \sqrt{1 - 4s^2} \sigma_3 \right)^2. \quad (4.10)$$

Here, the squashing is restricted to $0 < s \leq \frac{1}{2}$, namely the size of the S^1 Hopf fiber inside S^3 is not larger than the one of the base S^2 . Having satisfied the integrability condition, one

can solve the CKS equation explicitly. If we require the spinors not to depend on t , so that they are well-defined on S^1 , then we need to introduce a purely imaginary Wilson line gauge potential

$$A = \frac{is}{r} dt. \quad (4.11)$$

The solution for ζ_+ is

$$\zeta_+ = \begin{pmatrix} \cos \frac{\vartheta}{2} e^{-\frac{i}{2}(\psi+\phi)} & -\sin \frac{\vartheta}{2} e^{-\frac{i}{2}(\psi-\phi)} \\ \mu \sin \frac{\vartheta}{2} e^{\frac{i}{2}(\psi-\phi)} & \mu \cos \frac{\vartheta}{2} e^{\frac{i}{2}(\psi+\phi)} \end{pmatrix} \zeta_+^0, \quad (4.12)$$

where $\mu = 2s - i\sqrt{1 - 4s^2}$, and ζ_+^0 is any constant spinor of positive chirality. Hence we have two independent solutions to the CKS equation (1.3).¹² This was expected from the fact that the integrability condition has not imposed any algebraic restrictions on the spinors. On the same background, the $\tilde{\zeta}_-$ equation (2.1) is solved by negative-chirality spinors having the same form as (4.12), with μ replaced by $\bar{\mu}$. So any such conformally-flat background allows two independent solutions to the CKS equation for each choice of chirality, thus preserving a total of four supercharges.

When $s = \frac{1}{2}$, the metric becomes the standard, round one on the direct product $S^3 \times S^1$.¹³

$$ds^2 = \frac{r^2}{4} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + dt^2. \quad (4.13)$$

Actually, for any allowed value of s the metric (4.10) is related to (4.13) by the transformation (4.5), (4.6). Indeed, specializing to $k = \frac{r}{2}\sqrt{1 - 4s^2}$, $p = \frac{is}{r}$, $q = 0$, the transformation reads

$$\psi' = \psi + \frac{2}{r}\sqrt{1 - 4s^2}t, \quad t' = 2st, \quad s' = \frac{1}{2}, \quad p' = \frac{i}{2r}. \quad (4.14)$$

Since $s' = \frac{1}{2}$, we obtain the round S^3 metric on the direct product $S^3 \times S^1$. This was expected from the classification of solutions to the new minimal equation given in [9], where it was found that the only compact background admitting two supercharges with same chirality and including an S^3 topology is (a discrete quotient of) the direct product $S^3 \times S^1$, with the round metric on S^3 .

Finally, we consider a reduction of this background to three dimensions. Starting from the metric (4.10) and reducing along ∂_t , we obtain the squashed S^3 background of [35]. From the arguments above it is also clear that the same three-dimensional background can

¹²The new minimal equation (2.2) is also satisfied, with background fields $a = v = -\frac{2is}{r} dt$.

¹³For $s = \frac{1}{2}$, there is also a solution having an opposite gauge potential, $A = -\frac{i}{2r} dt$, and any constant ζ_+ or $\tilde{\zeta}_-$ being allowed; these are $SU(2)_{\text{left}}$ invariant spinors on the round S^3 , each transforming as a doublet under $SU(2)_{\text{right}}$. On the other hand, spinors of the form (4.12) are $SU(2)_{\text{right}}$ invariant, and transform as a doublet under $SU(2)_{\text{left}}$.

equally well be obtained starting from the round metric (4.13) and reducing along a $U(1)$ generated by a combination of ∂_ψ and ∂_t [35]. On the other hand, reducing (4.13) along the trivial direction ∂_t leads to a round S^3 in three dimensions. Hence we see that different three-dimensional backgrounds can be obtained starting from the same four-dimensional configuration and reducing along different directions.

Two supercharges

Let us now assume that F is not zero. In order to evaluate the constraints (2.33) following from the integrability condition of the CKS equation for ζ_+ , and the analogous ones from the $\tilde{\zeta}_-$ equation, as a first thing we construct a basis of two-forms $j^\pm, \omega^\pm, \theta^\pm$ from generic chiral spinors $\zeta_+, \tilde{\zeta}_-$ as described in section 2.2. Then, studying the constraints involving exclusively the gauge field components Φ , we find that the imaginary part of F has to vanish, and also obtain some algebraic constraints on the two-forms, which can be rephrased as constraints on the spinors. Taking these into account, the only non-vanishing Weyl scalars are $\Psi_2^- = \Psi_2^+ = \frac{4k^2 - r^2(1 - 4s^2)}{3r^4}$. The conditions relating the Weyl and Maxwell tensors then fix the real part of F . In formulae, we find that the integrability condition of the CKS equations for ζ_+ and $\tilde{\zeta}_-$ is solved by

$$F = \pm \left(\frac{4s^2 - 1}{2} + \frac{2k^2}{r^2} \right) \sigma_1 \wedge \sigma_2, \quad \gamma^{12}\zeta_+ = \mp i\zeta_+, \quad \gamma^{12}\tilde{\zeta}_- = \pm i\tilde{\zeta}_-, \quad (4.15)$$

where either the upper or the lower signs have to be chosen, and $k^2 \neq \frac{r^2}{4}(1 - 4s^2)$. Note that the projections being imposed on the chiral spinors imply that the background admits a single solution ζ_+ and a single solution $\tilde{\zeta}_-$ to the respective CKS equations, meaning that exactly two supercharges are preserved. To complete the solution, we look directly at the CKS equation, and find that any constant chiral spinor satisfying the projections above is a solution, provided one takes¹⁴

$$A = \pm \left(\frac{4s^2 - 1}{2} + \frac{2k^2}{r^2} \right) \sigma_3 + \left(\pm \frac{2k}{r^2} - \frac{is}{r} \right) dt. \quad (4.16)$$

This supersymmetric background was already mentioned (for $k = 0$) in [8], where it was identified with the lift of the three-dimensional $SU(2) \times U(1)$ invariant background of [36]. Of course, the change of coordinates (4.5) and redefinition of parameters (4.6) can be used to set $k = 0$. However, the three-dimensional background obtained by reducing along ∂_t with generic parameters s and k is different from that of [36].

¹⁴In this case the new minimal equation (2.2) is solved by taking $v = \frac{2is}{r} dt$ and $a = A + \frac{3}{2}v$.

Lorentzian signature

Finally, let us briefly comment on the background in Lorentzian signature. In this case, the topology is $S^3 \times \mathbb{R}$. The Wick rotation of the metric (4.1) is performed by setting $t_E = -i t_L$, $k_E = -i k_L$. Using the results of [15], it is immediate to see that supersymmetry is preserved because there is a globally-defined null Killing vector given by a combination of the time-like vector ∂_t and the space-like vector ∂_ψ . Then one can follow the same steps as in the Euclidean case, implementing the analytic continuation $t_E = -i t_L$, $k_E = -i k_L$, $p_E = i p_L$ everywhere in the equations above. Note that in this way the gauge potential A becomes real, as it has to be in Lorentzian signature.¹⁵ Also, in Lorentzian signature $\tilde{\zeta}_-$ is the charge conjugate of ζ_+ . When $F = 0$, the spacetime is conformally flat and the CKS equation has two independent solutions (four real supercharges). When $F \neq 0$, the only non-vanishing Weyl scalar is Ψ_2 , hence our spacetime is of Petrov type D and we have one solution to the CKS equation (two real supercharges). As in the Euclidean setup, the topological densities E and P vanish, so there are no trace and R-symmetry anomalies.

5 Holographic superconformal anomalies

Our discussion so far was valid for general supersymmetric field theories, whereas in this section we will discuss the special situation when a SCFT has an asymptotically locally AdS gravity dual.¹⁶ Although the formulae we report below are review of known results, we emphasise that the agreement of the holographic computations with the combined equations (1.1) and (1.2) was not noted before.

It is well-known that the trace anomaly of a CFT (not necessarily supersymmetric) is related to a logarithmic¹⁷ divergence of the on-shell bulk gravity action, and can be obtained using the method of holographic renormalisation [24]. On the other hand, the chiral anomaly of an axial current arises from a Chern–Simons term [37] in the bulk action. For superconformal field theories, the trace and R-symmetry holographic anomalies are obtained from five-dimensional minimal gauged supergravity. The latter arises as a consistent truncation of *e.g.* type IIB supergravity on any five-dimensional Sasaki–Einstein manifold [38]. Its bosonic action reads

$$S = \frac{1}{16\pi G_5} \int \left[d^5x \sqrt{-\hat{g}} \left(\hat{R} + \frac{12}{\ell^2} - \frac{\ell^2}{3} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} \right) - \frac{8\ell^3}{27} \hat{A} \wedge \hat{F} \wedge \hat{F} \right], \quad (5.1)$$

where μ, ν are five-dimensional indices, \hat{R} denotes the Ricci scalar of a five-dimensional metric

¹⁵To determine A one could also use the general formulae given in [15].

¹⁶For definiteness, throughout this section we will work in Lorentzian signature.

¹⁷In a specific coordinate system.

$\hat{g}_{\mu\nu}$, \hat{A} is the graviphoton gauge field, with field-strength $\hat{F} = d\hat{A}$, and ℓ is the radius of the AdS solution. We have decorated all five-dimensional quantities with a hat, to distinguish them from the corresponding four-dimensional objects, that arise as their boundary values. The reason for choosing this unconventional normalisation of the gauge field is that its boundary value can be identified with the background gauge field A_m that we considered in the previous sections.¹⁸ When evaluated in an asymptotically locally AdS solution the action (5.1) contains the following logarithmically divergent term

$$I_{\log} = \frac{\ell^3}{128\pi G_5} \int d^4x \sqrt{-g} \left(R_{mn} R^{mn} - \frac{1}{3} R^2 - \frac{4}{3} F_{mn} F^{mn} \right), \quad (5.3)$$

where R_{mn} denotes the four-dimensional Ricci tensor of the boundary metric g_{mn} , R its Ricci scalar, and F_{mn} the field-strength of the boundary value of the field graviphoton, denoted A_m . Notice that the Chern–Simons term in (5.1) is not divergent [39]. This result was first obtained in [39],¹⁹ followed by [40, 41, 42].

We can then use the identity

$$C_{mnpq} C^{mnpq} - E = 2R_{mn} R^{mn} - \frac{2}{3} R^2 \quad (5.4)$$

to rewrite the divergent action (5.3) as

$$I_{\log} = \frac{\ell^3}{256\pi G_5} \int d^4x \sqrt{-g} \left(C_{mnpq} C^{mnpq} - \frac{8}{3} F_{mn} F^{mn} - E \right). \quad (5.5)$$

This is proportional to the bosonic action of conformal supergravity [21], up to the topological term [23, 22]. Moreover, the integrand of the logarithmically divergent part of the action is proportional to the trace anomaly [24]. More precisely, in the radial variable used in [24] the proportionality factor is -2 , hence we have

$$T_m^m|_{\text{hol}} = \frac{\ell^3}{128\pi G_5} \left(C_{mnpq} C^{mnpq} - \frac{8}{3} F_{mn} F^{mn} - E \right). \quad (5.6)$$

Comparing this with (1.1) yields the well-known large N result [24]

$$a = c = \frac{\ell^3 \pi}{8G_5}. \quad (5.7)$$

¹⁸To compare with a more standard normalisation in the literature, one should set $\hat{A} = \frac{\sqrt{3}}{\ell} A^s$ (where “s” stands for “standard”). Using this, the action of minimal gauged supergravity reads

$$S = \frac{1}{16\pi G_5} \int \left[d^5x \sqrt{-\hat{g}} \left(\hat{R} + \frac{12}{\ell^2} - F_{\mu\nu}^s F^{s\mu\nu} \right) - \frac{8}{3\sqrt{3}} A^s \wedge F^s \wedge F^s \right]. \quad (5.2)$$

¹⁹The normalisation of the gauge field in [39] corresponds to $\frac{\ell}{\sqrt{3}} \hat{A} = \frac{1}{2} A_{\text{Taylor}}$.

For supersymmetric backgrounds, the relations (2.22) imply that the divergent part of the on-shell action is purely topological, namely

$$I_{\log}^{\text{susy}} = -\frac{\ell^3}{256\pi G_5} \int d^4x \sqrt{-g} E . \quad (5.8)$$

Note that this explains the observation made in [39], that in the magnetic string solution of [30, 31] the on-shell action does not have logarithmic divergences. For completeness, let us recall how the well-known formula for the central charge a is obtained from type IIB supergravity on $\text{AdS}_5 \times M_5$, where M_5 is a Sasaki–Einstein manifold. Using $\frac{1}{G_5} = \frac{\ell^5 \text{vol}'(M_5)}{G_{10}}$, where $\text{vol}'(M_5)$ is the dimension-less volume of the manifold M_5 , together with the expression for the AdS_5 radius in terms of N units of five-form flux,

$$\ell^4 = \frac{\sqrt{2G_{10}} \pi N}{\text{vol}'(M_5)} , \quad (5.9)$$

eq. (5.7) can be written as

$$a = c = \frac{\pi^3 N^2}{4 \text{vol}'(M_5)} . \quad (5.10)$$

Let us now discuss the holographic R-symmetry anomaly. A calculation of the holographic chiral anomaly induced by a bulk Chern–Simons term for an Abelian gauge field has appeared in [43], and it is straightforward to extract the R-symmetry anomaly from the results of this reference. The authors of [43] consider a five-dimensional Einstein–Maxwell model, whose action, in the normalisations of (5.2), contains a Chern–Simons term

$$S_\kappa = \frac{\kappa}{2\pi G_5} \int d^5x \sqrt{-g} \epsilon^{\mu\nu\rho\sigma\lambda} A_\mu^s F_{\nu\rho}^s F_{\sigma\lambda}^s . \quad (5.11)$$

This leads to the following expression for holographic chiral anomaly [37]:

$$\nabla^m J_m^s|_{\text{hol}} = -\frac{\kappa}{6\pi G_5} \epsilon^{mnpq} F_{mn}^s F_{pq}^s . \quad (5.12)$$

To obtain the R-symmetry anomaly we need to use the value $\kappa = -\frac{1}{4\sqrt{3}}$ of the Chern–Simons coupling fixed by (5.2), and to convert to the normalisation of the gauge field adopted in (5.1). Namely, we need to rescale the field $A_m^s = \frac{\ell}{\sqrt{3}} A_m$, and of course also the current J_m^s as follows from

$$J_m^s = \frac{\delta S}{\delta A_m^s} = \frac{\sqrt{3}}{\ell} J_m . \quad (5.13)$$

Finally, using (5.7) we obtain

$$\nabla_m J^m|_{\text{hol}} = \frac{2a}{27\pi^2} F \tilde{F} , \quad (5.14)$$

which agrees exactly with (1.2) upon setting $a = c$.

6 Discussion

In this paper, we have elaborated on properties of four-dimensional field theories in curved backgrounds preserving rigid supersymmetry. In particular, we have described a method for determining the field-strength of the background gauge field, using the integrability of the (charged) conformal Killing spinor equation. Noting that backgrounds solving the old or new minimal rigid supersymmetry equation must satisfy the CKS equation, our results are valid for all the supersymmetric backgrounds considered so far in the literature, both in Lorentzian and Euclidean signature. Using our characterisation of the gauge field, we have shown that on any supersymmetric background there exist precise relations between various curvature invariants. In Lorentzian signature, these relations imply that the terms multiplying the central charge c in the superconformal anomalies *vanish*. In Euclidean signature, the existence of a chiral solution to the CKS equation is a slightly weaker condition, and it implies that the c -term in the trace anomaly becomes topological, rather than zero. However, if there exist two solutions with opposite R-charges, then the c -anomalies vanish, as in Lorentzian signature; moreover, all the integrated anomalies vanish, including the terms involving the central charge a . A converse version of this result might be true, so that perhaps if in a compact complex manifold the signature and Euler characteristic vanish, then a second CKS with opposite chirality exists. It would be interesting to see whether such statement holds.

One of the motivations for our work was to study further the relationship of rigid supersymmetry with local supersymmetry, through holography [8, 15]. For example, we have shown that, when evaluated on supersymmetric asymptotically locally AdS solutions, the logarithmically divergent part of the on-shell action of five-dimensional minimal gauged supergravity is a topological density. In particular, this explains why the on-shell action does not contain the logarithmic divergence in all the (few) supersymmetric solutions in the literature (which also turn out to have $E = 0$). We have also shown that the coefficient of the holographic R-symmetry anomaly agrees nicely with that of the field theory, filling a small gap in the literature. Armed with the results of [8, 15], together with those presented in this paper, we would like to address the problem of constructing non-trivial examples of five-dimensional supergravity solutions, dual to superconformal field theories on curved backgrounds.²⁰ There exist well-known topological obstructions for the existence of a five-dimensional “filling” of four-dimensional manifolds. Specifically, an oriented four-manifold bounds an oriented five-manifold if and only if its signature vanishes. As we noted, supersymmetric backgrounds with two supercharges of opposite R-charge automatically satisfy

²⁰These would be analogous to the four-dimensional gravity duals in [44, 45, 46, 47].

this requirement, and therefore are natural candidates for admitting (smooth) gravity duals. Such five-dimensional gravity-duals will be studied in a separate paper [48].

It will be interesting to explore the possibility that the simplification of the anomalies could be used to devise new methods for computing the central charges a and c [49]. Furthermore, it is natural to extend our approach to other dimensions, in particular this could lead to analogous simplifications of the superconformal anomalies in six-dimensional SCFTs. One could also consider currents of non-R-symmetries coupled to additional background gauge fields, and study how supersymmetry of the background affects their chiral anomalies as well as the new terms arising in the trace and R-current anomalies.

Acknowledgments

We would like to thank M. Bianchi, D. Freedman, K. Intriligator, D. Panov, K. Skenderis, J. Sparks, A. Tomasiello, A. Tseytlin, B. Wecht and L. Wulff for useful comments and correspondence, as well as C. Closset and Z. Komargodski for stimulating discussions. D.C. is supported by an STFC grant ST/J002798/1. D.M. is supported by an ERC Starting Grant – Grant Agreement N. 304806 – Gauge-Gravity, and also acknowledges partial support from an EPSRC Advanced Fellowship EP/D07150X/3 and from the STFC grant ST/J002798/1.

A Lorentzian vs Euclidean conventions

In Lorentzian signature $(-+++)$, our orientation is fixed by $\epsilon_{0123} = -\epsilon^{0123} = 1$ and $\text{vol}_4 = e^0 \wedge e^1 \wedge e^2 \wedge e^3$. In Euclidean signature $(++++)$, we take $\epsilon_{1234} = 1$ and $\text{vol}_4 = e^1 \wedge e^2 \wedge e^3 \wedge e^4$. In both signatures, the charge conjugate of a spinor ζ is defined by $\zeta^c = B\zeta^*$, where B is the intertwiner such that $BB^\dagger = 1$, $BB^* = -1$ and $\gamma_m^* = (-)^\tau B^{-1}\gamma_m B$, with $\tau = 0$ in Euclidean signature and $\tau = 1$ in Lorentzian signature. In Lorentzian signature, $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, while in Euclidean signature $\gamma_5 = \gamma^1\gamma^2\gamma^3\gamma^4$. Note that the charge conjugate of a Lorentzian chiral spinor has opposite chirality, while the charge conjugate of an Euclidean chiral spinor has the same chirality. In Lorentzian signature we also define the Dirac conjugate $\bar{\zeta} = \zeta^\dagger\gamma^0$.

To pass from Lorentzian to Euclidean signature, we set $x^0 = ix^4$ and therefore $\gamma^0 = i\gamma^4$. Since in Lorentzian signature $** = -1$ on two-forms, (anti)-self-dual forms ϕ^\pm are necessarily complex. In our conventions, $i*\phi^\pm = \pm\phi^\pm$. Note that the complex conjugate of a Lorentzian chiral form has the opposite chirality. In Euclidean signature, $** = +1$ and we define (anti)-self-dual forms by $*\phi^\pm = \pm\phi^\pm$. Euclidean chiral forms can be complex or real, and complex conjugation does not flip the chirality.

B Derivation of the integrability condition

In this appendix, we derive the integrability condition (2.4) of the CKS equation (1.3).

Taking the commutator of two covariant derivatives on ζ_+ and using (1.3) we obtain

$$\frac{1}{4}R_{mnpq}\gamma^{pq}\zeta_+ - iF_{mn}\zeta_+ = (\gamma_n\nabla_m^A - \gamma_m\nabla_n^A)\eta_-, \quad (\text{B.1})$$

where we set

$$\eta_- = \frac{1}{4}\nabla^A\cdot\zeta_+, \quad (\text{B.2})$$

and $\nabla_m^A\eta_- = (\nabla_m - iA_m)\eta_-$. Contracting with γ^n we get

$$-\frac{1}{2}R_{mn}\gamma^n\zeta_+ - iF_{mn}\gamma^n\zeta_+ = 2\nabla_m^A\eta_- + \gamma_m\nabla^A\cdot\eta_-. \quad (\text{B.3})$$

Contracting this with γ^m we determine

$$\nabla^A\cdot\eta_- = -\frac{1}{12}R\zeta_+ - \frac{i}{6}F_{mn}\gamma^{mn}\zeta_+, \quad (\text{B.4})$$

which substituted back into (B.3) gives an equation for $\nabla_m^A\eta_-$:

$$\nabla_m^A\eta_- = \left(-\frac{1}{2}S_{mp} - \frac{i}{3}F_{mp}\right)\gamma^p\zeta_+ + \frac{i}{12}\gamma_{mpq}F^{pq}\zeta_+, \quad (\text{B.5})$$

where $S_{mn} = \frac{1}{2}(R_{mn} - \frac{1}{6}Rg_{mn})$ is the Schouten tensor. This can be used to eliminate η_- from the original equation (B.1). We observe that the Riemann tensor combines with its contractions in the Schouten tensor to give the Weyl tensor, defined in (2.5). After some gamma-matrix algebra, we arrive at (2.4). Note that contracting (2.4) with γ^n the equation trivializes, as one may expect observing that the same is true for the CKS equation.

It would be interesting to clarify whether eq. (B.5) for η_- implies additional constraints on the background, which are known to be encoded in the CKS equation [8, 9, 15]. For instance, one can consider the integrability condition of (B.5), which yields (in Lorentzian signature for definiteness)

$$\left(\frac{1}{4}C_{mnpq} - i g_{p[m}F_{n]q}\right)\gamma^{pq}\eta_- = \left(-\nabla_{[m}S_{n]p} - \frac{2i}{3}\nabla_{[m}F_{n]p} + \frac{1}{3}\nabla_{[m}\tilde{F}_{n]p}\right)\gamma^p\zeta_+ + \frac{i}{3}\left(5F_{mn} + i\tilde{F}_{mn}\right)\eta_-. \quad (\text{B.6})$$

Contracting with γ^n and using the twice-contracted Bianchi identity of the Riemann tensor (written in the form $\nabla_{[n}S_m]^n = 0$) we obtain

$$\frac{4i}{3}F_{mn}\gamma^n\eta_- + \left(-\frac{i}{3}\nabla^n F_{nm} + \frac{1}{6}\nabla^n\tilde{F}_{nm}\right)\zeta_+ + \left(\frac{1}{2}\nabla_n S_{mp} - \frac{i}{2}\nabla_{[m}F_{n]p} - \frac{1}{6}\nabla_n\tilde{F}_{mp}\right)\gamma^{np}\zeta_+ = 0. \quad (\text{B.7})$$

Contracting again with γ^m , we eventually arrive at

$$(dF)_{mnp}\gamma^{mnp}\zeta_+ = 0, \quad (\text{B.8})$$

which is automatic if the Bianchi identity for F is satisfied.

C Review of anomaly formulae

In this appendix, we briefly review how the superconformal anomaly formulae are obtained from a superspace computation [3] and give some arguments in support of the corrected expressions (1.1), (1.2). In the limit $a = c$, the validity of these expressions is also confirmed by large N holography as discussed in section 5. An introduction to the relevant superspace formalism can be found in [50, 51, 52].

In superspace, the stress-energy tensor T_{mn} and the R-current J_m are components of the Ferrara–Zumino supercurrent $\mathcal{J}_{\alpha\dot{\alpha}}$, which is a real vector superfield satisfying the conservation equation

$$\bar{D}^{\dot{\alpha}} \mathcal{J}_{\alpha\dot{\alpha}} = D_{\alpha} \mathcal{T}. \quad (\text{C.1})$$

The chiral superfield \mathcal{T} is called the supertrace, and its $\theta\theta$ component contains both the trace of the stress-energy tensor and the divergence of the R-current:

$$-\frac{1}{4}(D^{\alpha} D_{\alpha} \mathcal{T})|_{\theta=0} = \frac{2}{3} T_m^m + i \nabla_m J^m, \quad (\text{C.2})$$

where we use the conventions of [50]. More complete expressions for these superfields can be found e.g. in [53, sect. 2.1]. The supertrace vanishes for a superconformal field theory on flat space, but acquires anomalous contributions when the theory is put on a non-trivial background.²¹ One has (see [54] for the general form and [55, 56] for an explicit evaluation)

$$\mathcal{T} = \frac{1}{24\pi^2} (c W^2 - a \Xi_c), \quad (\text{C.3})$$

where $W^2 = W^{\alpha\beta\gamma} W_{\alpha\beta\gamma}$ is the square of the super-Weyl tensor, while Ξ_c is the chirally projected super-Euler density. Extracting the $\theta\theta$ component gives the relevant anomaly formulae,

$$\frac{2}{3} T_m^m + i \nabla_m J^m = \frac{1}{24\pi^2} \left(-\frac{c}{4} D^{\alpha} D_{\alpha} W^2|_{\theta=0} + \frac{a}{4} D^{\alpha} D_{\alpha} \Xi_c|_{\theta=0} \right). \quad (\text{C.4})$$

The authors of [3] evaluate this expression in their appendix A, obtaining their equations (4.3) and (4.7). In the following we motivate why that result should be amended, by putting together some expressions appeared elsewhere in the literature. The super-Weyl invariant $D^{\alpha} D_{\alpha} W^2$ has been independently evaluated in [57], and is given by eq. (B.23) therein. In our notation, its bosonic part reads²²

$$-\frac{1}{4}(D^{\alpha} D_{\alpha} W^2)|_{\theta=0} = C_{mnpq} C^{mnpq} - \frac{8}{3} F_{mn} F^{mn} + i R_{mnpq} \tilde{R}^{mnpq} - \frac{8}{3} i F_{mn} \tilde{F}^{mn}. \quad (\text{C.5})$$

²¹If the classical theory is not conformally invariant then in addition to the quantum contribution (C.3) the supertrace also includes a classical term.

²²In the conventions of [57] (LPSW), $G_{ab} = 2D_{[a} G_{b]}$, with $G_a| = \frac{1}{6} A_a^{\text{LPSW}}$. Moreover, their vector field A^{LPSW} is related to our A by $A^{\text{LPSW}} = -2A$. To see this, compare the gravitino variation in their eq. (2.9) with our old minimal equation (2.3), note that A^{LPSW} is the same as b , and recall the discussion below our (2.3).

Note that the expression on the right hand side is exactly the one that is set to zero by the integrability of the Lorentzian CKS equation (indeed $R_{mnpq}\tilde{R}^{mnpq} = C_{mnpq}\tilde{C}^{mnpq}$). Hence we have found that, at least in Lorentzian signature, the CKS equation implies vanishing of the bosonic part of the super-Weyl invariant. Plugging the expression above into (C.4), we arrive at

$$\frac{2}{3}T_m^m + i\nabla_m J^m = \frac{c}{24\pi^2} \left(C_{mnpq}C^{mnpq} - \frac{8}{3}F_{mn}F^{mn} + iR_{mnpq}\tilde{R}^{mnpq} - \frac{8}{3}iF_{mn}\tilde{F}^{mn} \right) + a\text{-terms.} \quad (\text{C.6})$$

Separating the real and imaginary parts, we obtain precisely the c -terms in our eqs. (1.1), (1.2). Note the symmetry between the relative coefficients in the real and imaginary parts. These coefficients also agree with the independent superspace computation done in [58]. Comparing with eqs. (4.3) and (4.7) of [3], if we assume that our field-strength F is the same as their V , then we find a sign mismatch between the F^2 and V^2 terms as well as a mismatch by a factor of $1/3$ between the $F\tilde{F}$ and $V\tilde{V}$ terms.

Finally, we compare the expression for the R-current anomaly with the one given in [59] (IW), which was not obtained from superspace. Specializing eq. (2.8) in [59] to the R-current, we have

$$\nabla_m J_{\text{IW}}^m = \frac{1}{384\pi^2} \left(k_R R_{mnpq}\tilde{R}^{mnpq} + 8k_{RRR} F_{mn}^{\text{IW}} \tilde{F}^{\text{IW}mn} \right). \quad (\text{C.7})$$

The anomaly coefficients k_R and k_{RRR} can be eliminated in favour of the SCFT central charges a and c using the relations $a = \frac{3}{32}(3k_{RRR} - k_R)$ and $c = \frac{1}{32}(9k_{RRR} - 5k_R)$ found in [3]. In this way one obtains

$$\nabla_m J_{\text{IW}}^m = \frac{a-c}{24\pi^2} R_{mnpq}\tilde{R}^{mnpq} + \frac{5a-3c}{27\pi^2} F_{mn}^{\text{IW}} \tilde{F}^{\text{IW}mn}, \quad (\text{C.8})$$

which agrees with our eq. (1.2) provided one redefines $J_m = -J_m^{\text{IW}}$ and $F_{mn} = iF_{mn}^{\text{IW}}$.²³

References

- [1] M. J. Duff, *Twenty years of the Weyl anomaly*, Class. Quant. Grav. **11**, 1387 (1994) [hep-th/9308075].
- [2] M. J. Duff, *40 years of the Weyl anomaly*, seminar slides available at the URL <http://cft.physics.tamu.edu/Slides/Duff.pdf>
- [3] D. Anselmi, D. Z. Freedman, M. T. Grisaru and A. A. Johansen, *Nonperturbative formulas for central functions of supersymmetric gauge theories*, Nucl. Phys. B **526**, 543 (1998) [hep-th/9708042].

²³We thank Ken Intriligator and Brian Wecht for communications on consistency of (C.7) with (1.2).

- [4] H. Osborn, *N=1 superconformal symmetry in four-dimensional quantum field theory*, Annals Phys. **272** (1999) 243 [hep-th/9808041].
- [5] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, Commun. Math. Phys. **313**, 71 (2012) [arXiv:0712.2824 [hep-th]].
- [6] G. Festuccia and N. Seiberg, *Rigid Supersymmetric Theories in Curved Superspace*, JHEP **1106** (2011) 114 [arXiv:1105.0689 [hep-th]].
- [7] H. Samtleben and D. Tsimpis, *Rigid supersymmetric theories in 4d Riemannian space*, JHEP **1205** (2012) 132 [arXiv:1203.3420 [hep-th]].
- [8] C. Klare, A. Tomasiello and A. Zaffaroni, *Supersymmetry on Curved Spaces and Holography*, JHEP **1208** (2012) 061 [arXiv:1205.1062 [hep-th]].
- [9] T. T. Dumitrescu, G. Festuccia and N. Seiberg, *Exploring Curved Superspace*, JHEP **1208**, 141 (2012) [arXiv:1205.1115 [hep-th]].
- [10] J. T. Liu, L. A. Pando Zayas and D. Reichmann, *Rigid Supersymmetric Backgrounds of Minimal Off-Shell Supergravity*, JHEP **1210** (2012) 034 [arXiv:1207.2785 [hep-th]].
- [11] T. T. Dumitrescu and G. Festuccia, *Exploring Curved Superspace (II)*, JHEP **1301** (2013) 072 [arXiv:1209.5408 [hep-th]].
- [12] A. Kehagias and J. G. Russo, *Global Supersymmetry on Curved Spaces in Various Dimensions*, Nucl. Phys. B **873** (2013) 116 [arXiv:1211.1367 [hep-th]].
- [13] C. Closset, T. T. Dumitrescu, G. Festuccia and Z. Komargodski, *Supersymmetric Field Theories on Three-Manifolds*, JHEP **1305** (2013) 017 [arXiv:1212.3388 [hep-th]].
- [14] H. Samtleben, E. Sezgin and D. Tsimpis, *Rigid 6D supersymmetry and localization*, JHEP **1303** (2013) 137 [arXiv:1212.4706 [hep-th]].
- [15] D. Cassani, C. Klare, D. Martelli, A. Tomasiello and A. Zaffaroni, *Supersymmetry in Lorentzian Curved Spaces and Holography*, arXiv:1207.2181 [hep-th].
- [16] P. de Medeiros, *Rigid supersymmetry, conformal coupling and twistor spinors*, arXiv:1209.4043 [hep-th].
- [17] K. Hristov, A. Tomasiello and A. Zaffaroni, *Supersymmetry on Three-dimensional Lorentzian Curved Spaces and Black Hole Holography*, JHEP **1305** (2013) 057 [arXiv:1302.5228 [hep-th]].

- [18] M. Blau, *Killing spinors and SYM on curved spaces*, JHEP **0011** (2000) 023 [hep-th/0005098].
- [19] S. M. Kuzenko, *Symmetries of curved superspace*, JHEP **1303** (2013) 024 [arXiv:1212.6179 [hep-th]].
- [20] R. Penrose and W. Rindler, *Spinors And Space-time. Vol. 2: Spinor And Twistor Methods In Space-time Geometry*, Cambridge University Press (1988).
- [21] E. S. Fradkin and A. A. Tseytlin, *Conformal Supergravity*, Phys. Rept. **119**, 233 (1985).
- [22] V. Balasubramanian, E. G. Gimon, D. Minic and J. Rahmfeld, *Four-dimensional conformal supergravity from AdS space*, Phys. Rev. D **63**, 104009 (2001) [hep-th/0007211].
- [23] H. Liu and A. A. Tseytlin, *$D = 4$ superYang-Mills, $D = 5$ gauged supergravity, and $D = 4$ conformal supergravity*, Nucl. Phys. B **533**, 88 (1998) [hep-th/9804083].
- [24] M. Henningson and K. Skenderis, *The Holographic Weyl anomaly*, JHEP **9807**, 023 (1998) [hep-th/9806087].
- [25] H. Stephani, D. Kramer, M. A. H. MacCallum, C. Hoenselaers and E. Herlt, *Exact solutions of Einstein's field equations*, Cambridge University Press (2003).
- [26] E. Newman and R. Penrose, *An Approach to gravitational radiation by a method of spin coefficients*, J. Math. Phys. **3** (1962) 566.
- [27] J. Lewandowski, *Twistor equation in a curved spacetime*, Class. Quant. Grav. **8** (1991), no. 1, L11–L17.
- [28] C. Batista, *Weyl Tensor Classification in Four-dimensional Manifolds of All Signatures*, Gen. Rel. Grav. (2013) [arXiv:1204.5133 [gr-qc]].
- [29] J. P. Gauntlett, J. B. Gutowski and N. V. Suryanarayana, *A Deformation of $AdS_5 \times S^5$* , Class. Quant. Grav. **21** (2004) 5021 [hep-th/0406188].
- [30] A. H. Chamseddine and W. A. Sabra, *Magnetic strings in five-dimensional gauged supergravity theories*, Phys. Lett. B **477** (2000) 329 [hep-th/9911195].
- [31] D. Klemm and W. A. Sabra, *Supersymmetry of black strings in $D = 5$ gauged supergravities*, Phys. Rev. D **62** (2000) 024003 [hep-th/0001131].
- [32] A. Derdzinski, C. L. Shen, *Codazzi tensor fields, curvature and Pontryagin forms*, Proc. London Math. Soc. **47** (3) (1983) 15-26.

- [33] D. N. Page, *A Compact Rotating Gravitational Instanton*, Phys. Lett. B **79**, 235 (1978).
- [34] D. Martelli and J. Sparks, *Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals*, Commun. Math. Phys. **262**, 51 (2006) [hep-th/0411238].
- [35] Y. Imamura and D. Yokoyama, *$N=2$ supersymmetric theories on squashed three-sphere*, Phys. Rev. D **85** (2012) 025015 [arXiv:1109.4734 [hep-th]].
- [36] N. Hama, K. Hosomichi and S. Lee, *SUSY Gauge Theories on Squashed Three-Spheres*, JHEP **1105** (2011) 014 [arXiv:1102.4716 [hep-th]].
- [37] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2**, 253 (1998) [hep-th/9802150].
- [38] A. Buchel and J. T. Liu, *Gauged supergravity from type IIB string theory on $Y^{p,q}$ manifolds*, Nucl. Phys. B **771** (2007) 93 [hep-th/0608002].
- [39] M. Taylor, *More on counterterms in the gravitational action and anomalies*, hep-th/0002125.
- [40] J. Kalkkinen and D. Martelli, *Holographic renormalization group with fermions and form fields*, Nucl. Phys. B **596**, 415 (2001) [hep-th/0007234].
- [41] M. Bianchi, D. Z. Freedman and K. Skenderis, *Holographic renormalization*, Nucl. Phys. B **631**, 159 (2002) [hep-th/0112119].
- [42] D. Martelli and W. Mueck, *Holographic renormalization and Ward identities with the Hamilton-Jacobi method*, Nucl. Phys. B **654** (2003) 248 [hep-th/0205061].
- [43] K. Landsteiner, E. Megias, L. Melgar and F. Pena-Benitez, *Holographic Gravitational Anomaly and Chiral Vortical Effect*, JHEP **1109**, 121 (2011) [arXiv:1107.0368 [hep-th]].
- [44] D. Martelli, A. Passias and J. Sparks, *The gravity dual of supersymmetric gauge theories on a squashed three-sphere*, Nucl. Phys. B **864**, 840 (2012) [arXiv:1110.6400 [hep-th]].
- [45] D. Martelli and J. Sparks, *The gravity dual of supersymmetric gauge theories on a biaxially squashed three-sphere*, Nucl. Phys. B **866**, 72 (2013) [arXiv:1111.6930 [hep-th]].
- [46] D. Martelli, A. Passias and J. Sparks, *The supersymmetric NUTs and bolts of holography*, arXiv:1212.4618 [hep-th].

- [47] D. Martelli and A. Passias, *The gravity dual of supersymmetric gauge theories on a two-parameter deformed three-sphere*, arXiv:1306.3893 [hep-th].
- [48] D. Cassani and D. Martelli *The gravity dual of supersymmetric gauge theories on curved four-manifolds*, to appear
- [49] A. D. Shapere and Y. Tachikawa, *Central charges of $N=2$ superconformal field theories in four dimensions*, JHEP **0809**, 109 (2008) [arXiv:0804.1957 [hep-th]].
- [50] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton Univ. Press (1992).
- [51] S. J. Gates, M. T. Grisaru, M. Rocek and W. Siegel, *Superspace Or One Thousand and One Lessons in Supersymmetry*, Front. Phys. **58** (1983) 1 [hep-th/0108200].
- [52] I. L. Buchbinder and S. M. Kuzenko, *Ideas and methods of supersymmetry and supergravity: Or a walk through superspace*, IOP (1998).
- [53] Z. Komargodski and N. Seiberg, *From Linear SUSY to Constrained Superfields*, JHEP **0909** (2009) 066 [arXiv:0907.2441 [hep-th]].
- [54] L. Bonora, P. Pasti and M. Tonin, *Cohomologies And Anomalies In Supersymmetric Theories*, Nucl. Phys. B **252** (1985) 458.
- [55] I. N. McArthur, *Super- b_4 coefficients in supergravity*, Class. Quant. Grav. **1** (1984) 245.
- [56] I. L. Buchbinder and S. M. Kuzenko, *Matter Superfields In External Supergravity: Green Functions, Effective Action And Superconformal Anomalies*, Nucl. Phys. B **274** (1986) 653.
- [57] H. Lu, C. N. Pope, E. Sezgin and L. Wulff, *Critical and Non-Critical Einstein-Weyl Supergravity*, JHEP **1110**, 131 (2011) [arXiv:1107.2480 [hep-th]].
- [58] A. Schwimmer and S. Theisen, *Spontaneous Breaking of Conformal Invariance and Trace Anomaly Matching*, Nucl. Phys. B **847** (2011) 590 [arXiv:1011.0696 [hep-th]].
- [59] K. A. Intriligator and B. Wecht, *The Exact superconformal R symmetry maximizes a* , Nucl. Phys. B **667**, 183 (2003) [hep-th/0304128].