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Localization on Hopf surfaces

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ABSTRACT: We discuss localization of the path integral for supersymmetric gauge theories with an R-symmetry on Hermitian four-manifolds. After presenting the localization locus equations for the general case, we focus on backgrounds with $S^1 \times S^3$ topology, admitting two supercharges of opposite R-charge. These are Hopf surfaces, with two complex structure moduli p, q . We compute the localized partition function on such Hopf surfaces, allowing for a very large class of Hermitian metrics, and prove that this is proportional to the supersymmetric index with fugacities p, q . Using zeta function regularisation, we determine the exact proportionality factor, finding that it depends only on p, q , and on the anomaly coefficients a, c of the field theory. This may be interpreted as a supersymmetric Casimir energy, and provides the leading order contribution to the partition function in a large N expansion.

KEYWORDS: Supersymmetric gauge theory, Matrix Models

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Contents

1	Introduction	2
2	Supersymmetric backgrounds and Lagrangians	4
2.1	Background geometry	4
2.2	Supersymmetry transformations and Lagrangians	7
2.2.1	Vector multiplet	8
2.2.2	Chiral multiplet	10
2.3	Supersymmetric locus equations	13
2.3.1	Vector multiplet	13
2.3.2	Chiral multiplet	14
3	Hopf surfaces	15
3.1	Generalities	15
3.2	Global properties	16
3.3	Complex structure	20
4	Localization	21
4.1	Localization locus	22
4.2	The matrix model	23
4.3	One-loop determinants	25
4.3.1	Vector multiplet	25
4.3.2	Chiral multiplet	28
5	The partition function	30
5.1	Anomaly cancellations and the supersymmetric index	30
5.2	Supersymmetric Casimir energy and large N limit	33
6	Conclusions	35
A	Conventions and identities	36
B	Weyl transformations	38
C	$S^1 \times S^3_\nu$ with arbitrary b_1, b_2	40
D	Non-direct product metric	43
E	Proof that $(z_1, z_2) \in \mathbb{C}^2 - (0, 0)$	44
F	Reduction of the 4d supersymmetry equations to 3d	46

1 Introduction

The complete information of a quantum field theory is contained in the generating functional of correlation functions; however, in an interacting theory this is very hard to compute exactly. In favourable situations the technique of supersymmetric localization [1] allows one to perform exact non-perturbative computations of special types of generating functionals and other observables. In particular, in certain supersymmetric field theories defined on compact Riemannian manifolds, it is possible to evaluate a class of BPS observables by reducing the functional integrals over all the field configurations to Gaussian integrals around a supersymmetric locus. In this paper we will present a detailed calculation of the partition function of $\mathcal{N} = 1$ supersymmetric field theories, defined on a four-dimensional complex manifold.

A systematic procedure for constructing supersymmetric field theories in a fixed background geometry has been put forward in [2]. In four dimensions, one way to obtain supersymmetric theories is by taking a suitable limit of new minimal supergravity [3–5], that contains two auxiliary vector fields, one of which is the gauge field for a local chiral symmetry. In such rigid limit, these, together with the metric, provide background fields coupled to a supersymmetric gauge theory with an R-symmetry, comprising ordinary vector and chiral multiplets. Explicit expressions for supersymmetric Lagrangians and supersymmetry transformations can be obtained from [3–5] and will be presented below.

Supersymmetric theories may be defined only on backgrounds admitting solutions to certain Killing spinor equations (see (2.1), (2.2) below), which in Euclidean signature are equivalent to the requirement that the four-dimensional manifold is complex and the metric Hermitian [6, 7]. In this paper we will construct Lagrangians that are total supersymmetry variations, and therefore can be utilised to implement the localization technique in $\mathcal{N} = 1$ field theories defined on arbitrary Hermitian manifolds. We will then employ these to compute in closed form the partition function of general supersymmetric gauge theories, in the case that the manifold admits at least two supercharges of opposite R-charge, and has the topology of $S^1 \times S^3$. These manifolds are then *Hopf surfaces*, with complex structure characterised by two parameters p, q , that we will denote as $\mathcal{H}_{p,q} \simeq S^1 \times S^3$.

The main result of this paper is the derivation of a formula for the partition function Z of an $\mathcal{N} = 1$ supersymmetric field theory with an R-symmetry, defined on a Hopf surface $\mathcal{H}_{p,q}$, endowed with a very general Hermitian metric. Namely, we will show that

$$Z[\mathcal{H}_{p,q}] = e^{-\mathcal{F}(p,q)} \mathcal{I}(p, q), \quad (1.1)$$

where $\mathcal{I}(p, q)$ is the *supersymmetric index* with p, q fugacities and $\mathcal{F}(p, q)$ is a function of the complex structure parameters given by

$$\mathcal{F}(p, q) = \frac{4\pi}{3} \left(|b_1| + |b_2| - \frac{|b_1| + |b_2|}{|b_1||b_2|} \right) (\mathbf{a} - \mathbf{c}) + \frac{4\pi}{27} \frac{(|b_1| + |b_2|)^3}{|b_1||b_2|} (3\mathbf{c} - 2\mathbf{a}), \quad (1.2)$$

where $p = e^{-2\pi|b_1|}$, $q = e^{-2\pi|b_2|}$, and \mathbf{a} , \mathbf{c} are the R-symmetry traces, appearing in the Weyl and R-symmetry anomalies of superconformal field theories [8, 9]. As we will explain, the real parameters b_1, b_2 characterise an almost contact structure in the three-dimensional theory obtained from dimensional reduction on S^1 , allowing us to make contact with the results of [10], where the localized partition function of three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories was computed. The supersymmetric index was introduced in [11–13] in the context of superconformal field theories, and has been used in [14–17] (and many others) to test non-perturbative dualities.

The authors of [18] have shown that very generally the path integral of a supersymmetric field theory defined on a Hermitian manifold can depend only on complex structure deformations of the background. Based on this result, they have conjectured that the partition function defined on a Hopf surface $\mathcal{H}_{p,q}$ is proportional to the supersymmetric index $\mathcal{I}(p, q)$, up to possible local counterterms. Our explicit computation confirms the validity of this conjecture,¹ although we expect that the ratio $e^{-\mathcal{F}}$ between these two quantities generically cannot be expressed in terms of local counterterms. This provides an interesting quantity characterising a four-dimensional supersymmetric field theory, that we will refer to as *supersymmetric Casimir energy*.

Some progress towards obtaining the partition function (1.1) using localization was made in [19], where the one-loop determinant of an $\mathcal{N} = 1$ chiral multiplet on a Hopf surface was computed. In particular, in this reference the authors considered a specific Hermitian metric compatible with $|p| = |q|$. Localization computations of supersymmetric gauge theories on $S^1 \times S^3$ with a conformally flat metric have appeared in [20, 21].

One of our motivations for computing the partition function from first principles arose from holography [22]. In situations where there exist simple AdS₅ gravity duals, the gravity side predicts that the logarithm of the partition function, at leading order in a large N expansion, should be proportional to N^2 . In one dimension lower, the analogous problem is well understood: the $N^{3/2}$ scaling of the on-shell action on the gravity side can be matched to the large N limit of the localized free energy [23]; it has been shown in [24] that this agreement can be extended to a broad class of $\mathcal{N} = 2$ gauge theories, whose partition function was computed in [10]. In four dimensions the supersymmetric index scales like N^0 at large N [12, 25], implying that the N^2 scaling of the logarithm of the partition function must arise as an extra contribution. We find that this contribution is contained in (1.2).

Thus, for superconformal field theories with Sasaki-Einstein gravity duals (so that $\mathbf{c} = \mathbf{a}$ at leading order in N), we obtain a prediction for the holographically renormalised action of five-dimensional gauged supergravity, evaluated on a solution dual to a supersymmetric field theory defined on a Hopf surface $\mathcal{H}_{p,q} = \partial M_5$. In particular, we expect that for a solution $M_5 \simeq S^1 \times \mathbb{R}^4$, the renormalised on-shell action will be given by

$$S_{5\text{d sugra}}[M_5] = \frac{\pi^2}{54G_5} \frac{(|b_1| + |b_2|)^3}{|b_1||b_2|}, \tag{1.3}$$

up to finite local counterterms.

¹For simplicity we will restrict attention to the case where the parameters p, q are real.

The rest of this paper is organized as follows. Section 2 contains a discussion of the background geometry of four-manifolds allowing for at least one supercharge, and sets the stage for implementing localization in general four-dimensional $\mathcal{N} = 1$ gauge theories with an R-symmetry. In section 3 we discuss the specific background geometry for Hopf surfaces with $S^1 \times S^3$ topology and $U(1)^3$ isometries. In section 4 we perform the localization computation on the Hopf surfaces. In section 5 we compare our result for the exact partition function with the supersymmetric index. We emphasize the presence of the extra pre-factor and define the supersymmetric Casimir energy. We also comment on the implications of our results for gravity duals. We conclude in section 6 by outlining some perspectives for future work. We also included several appendices. Appendix A contains our conventions. Appendix B provides a proof that the partition function is independent of the conformal factor of the metric. Appendix C describes familiar examples of the background geometries considered in section 3. Appendix D elaborates on possible generalizations of our results by considering non-direct product metrics, associated to complex values of the complex structure moduli. Appendix E includes computations used in section 3. Appendix F contains details of the reduction of four-dimensional backgrounds to three dimensions. Appendix G contains the details of the regularization of one-loop determinants.

2 Supersymmetric backgrounds and Lagrangians

We begin our analysis by reviewing and elaborating results about the new minimal formulation of rigid supersymmetry on curved space. Our considerations in this section will be entirely local, while global properties will be discussed in section 3.

2.1 Background geometry

As shown in [2], in the presence of an R-symmetry the supersymmetry transformations and the Lagrangian of a field theory defined on a curved manifold can be derived by coupling the theory to the new minimal formulation of off-shell supergravity [3–5] and freezing the fields in the gravity multiplet to background values, in such a way that the gravitino variation vanishes. The bosonic fields in the gravity multiplet are the metric and two auxiliary vector fields A_μ, V_μ ; after the rigid limit, these play the role of background fields. In Euclidean signature, A_μ and V_μ are allowed to take complex values, whereas for simplicity the metric will be constrained to be real.

The real part of A_μ is associated to $u(1)_R$ R-symmetry transformations, and transforms (locally) as a gauge field, while the imaginary part must be a well-defined one-form. Being the Hodge dual of a closed three-form, $V = *dB$ is assumed to be a globally defined one-form, constrained by $\nabla^\mu V_\mu = 0$. In Euclidean signature, the condition that the gravitino variation vanishes corresponds to two independent first-order differential equations

$$(\nabla_\mu - iA_\mu)\zeta + iV_\mu\zeta + iV^\nu\sigma_{\mu\nu}\zeta = 0, \tag{2.1}$$

$$(\nabla_\mu + iA_\mu)\tilde{\zeta} - iV_\mu\tilde{\zeta} - iV^\nu\tilde{\sigma}_{\mu\nu}\tilde{\zeta} = 0, \tag{2.2}$$

where ζ and $\tilde{\zeta}$ are two-component complex spinors of opposite chirality, and with opposite charge under the background gauge field A , associated with the R-symmetry. Solutions to

these equations are either identically zero or nowhere vanishing. Throughout the paper, spinors with no tilde transform in the $(\mathbf{2}, \mathbf{1})$ representation of the $\text{Spin}(4) = \text{SU}(2)_+ \times \text{SU}(2)_-$ Lorentz group, while spinors with a tilde transform in the $(\mathbf{1}, \mathbf{2})$. See appendix A for further details on our notation and conventions.

It was shown in [6, 7] that a necessary and sufficient condition for a Riemannian four-manifold to have a solution ζ to (2.1) is that it admits an integrable complex structure $J^\mu{}_\nu$. Lowering an index with the Hermitian metric, the corresponding fundamental two-form can be constructed as a spinor bilinear,

$$J_{\mu\nu} = \frac{2i}{|\zeta|^2} \zeta^\dagger \sigma_{\mu\nu} \zeta. \tag{2.3}$$

One can also introduce a complex two-form bilinear as $P_{\mu\nu} = \zeta \sigma_{\mu\nu} \zeta$, which is anti-holomorphic with respect to the complex structure $J^\mu{}_\nu$. Together these define a $\text{U}(2)$ structure on the four-manifold. The solution of (2.1) can be expressed in terms of a nowhere vanishing complex function s as $\zeta_\alpha = \sqrt{\frac{s}{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and the background fields are determined by

$$V_\mu = -\frac{1}{2} \nabla^\rho J_{\rho\mu} + U_\mu, \tag{2.4}$$

$$A_\mu = A_\mu^c - \frac{1}{4} (\delta_\mu^\nu - i J_\mu{}^\nu) \nabla^\rho J_{\rho\nu} + \frac{3}{2} U_\mu, \tag{2.5}$$

where A_μ^c is defined as

$$A_\mu^c = \frac{1}{4} J_\mu{}^\nu \partial_\nu \log \sqrt{g} - \frac{i}{2} \partial_\mu \log s, \tag{2.6}$$

with g the determinant of the metric in complex coordinates. The solution contains an arbitrariness parametrised by the vector field U^μ , which is constrained to be holomorphic, namely $J^\mu{}_\nu U^\nu = iU^\mu$, and to obey $\nabla_\mu U^\mu = 0$. Note that the combination $A_\mu^{\text{cs}} \equiv A_\mu - \frac{3}{2} V_\mu$ is independent of the choice of U_μ .² Of course a solution $\tilde{\zeta}$ to (2.2) is also equivalent to the existence of an integrable complex structure defined by

$$\tilde{J}_{\mu\nu} = \frac{2i}{|\tilde{\zeta}|^2} \tilde{\zeta}^\dagger \tilde{\sigma}_{\mu\nu} \tilde{\zeta}, \tag{2.7}$$

and leads to expressions for the background fields A_μ and V_μ analogous to the ones above, with a few sign changes; see [7] for the explicit formulae.

When there exist *both* a non-zero solution ζ to (2.1) and a non-zero solution $\tilde{\zeta}$ to (2.2), namely in the presence of two supercharges of opposite R-charge, the four-dimensional manifold is endowed with a pair of commuting complex structures $J^\mu{}_\nu, \tilde{J}^\mu{}_\nu$, inducing opposite orientations, and subject to certain compatibility conditions [7]. This means that the manifold admits a specific *ambihermitian* structure³ [26]. Solutions with two

²We denote this as A^{cs} as it is the background field arising when the theory is coupled to conformal supergravity.

³Note that the similar term “bihermitian” refers to the different case where the two commuting complex structures induce the same orientation on the manifold.

supercharges of opposite R-charge may be more efficiently characterised by a complex vector field K^μ , constructed as a spinor bilinear as

$$K^\mu = \zeta \sigma^\mu \tilde{\zeta}. \tag{2.8}$$

In particular, one can show that K^μ is holomorphic with respect to both complex structures and satisfies the algebraic property $K_\mu K^\mu = 0$ as well as the differential condition $\nabla_{(\mu} K_{\nu)} = 0$, therefore it comprises two real Killing vectors. If K^μ commutes with its complex conjugate, $K^\nu \nabla_\nu \bar{K}^\mu - \bar{K}^\nu \nabla_\nu K^\mu = 0$, then the vector field U^μ above is restricted to take the form $U^\mu = \kappa K^\mu$, where κ is a complex function such that $K^\mu \partial_\mu \kappa = 0$, but otherwise arbitrary [7].⁴ Moreover, introducing adapted complex coordinates w, z (holomorphic with respect to $J^\mu{}_\nu$) such that the complex Killing vector is $K = \partial_w$, the metric takes the form

$$ds^2 = \Omega^2[(dw + h dz)(d\bar{w} + \bar{h} d\bar{z}) + c^2 dz d\bar{z}], \tag{2.9}$$

where $\Omega(z, \bar{z})$ and $c(z, \bar{z})$ are real, positive functions, while $h(z, \bar{z})$ is a complex function. It is useful to introduce the complex frame⁵

$$e^1 = \Omega c dz, \quad e^2 = \Omega(dw + h dz). \tag{2.10}$$

We choose the orientation by fixing the volume form as $\text{vol}_4 = -\frac{1}{4}e^1 \wedge \bar{e}^1 \wedge e^2 \wedge \bar{e}^2$. Then, as a one-form, K reads

$$K = \frac{1}{2}\Omega^2(d\bar{w} + \bar{h} d\bar{z}) = \frac{1}{2}\Omega \bar{e}^2, \tag{2.11}$$

and the real two-forms associated with the commuting complex structures are

$$\begin{aligned} J &= \frac{2i}{\Omega^2} K \wedge \bar{K} - \frac{i}{2}\Omega^2 c^2 dz \wedge d\bar{z} = -\frac{i}{2} \left(e^1 \wedge \bar{e}^1 + e^2 \wedge \bar{e}^2 \right), \\ \tilde{J} &= \frac{2i}{\Omega^2} K \wedge \bar{K} + \frac{i}{2}\Omega^2 c^2 dz \wedge d\bar{z} = \frac{i}{2} \left(e^1 \wedge \bar{e}^1 - e^2 \wedge \bar{e}^2 \right). \end{aligned} \tag{2.12}$$

With our choice of orientation J is self-dual while \tilde{J} is anti-self-dual.⁶ Following [7], we will require also that

$$\bar{K}^\mu \partial_\mu \kappa = \bar{K}^\mu \partial_\mu |s| = K^\mu \partial_\mu |s| = 0, \tag{2.13}$$

so that both K and \bar{K} preserve A and V in addition to the metric. With these restrictions, the functions κ and $|s|$ do not depend on w, \bar{w} , but can still have an arbitrary dependence on z and \bar{z} . In the frame (2.10), the spinors ζ and $\tilde{\zeta}$ solving (2.1) and (2.2) read

$$\zeta_\alpha = \sqrt{\frac{s}{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \tilde{\zeta}^{\dot{\alpha}} = \frac{\Omega}{\sqrt{2s}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.14}$$

⁴It is shown in [7] that if $[K, \bar{K}] \neq 0$, then the manifold is locally isometric to $\mathbb{R} \times S^3$, with the standard round metric on S^3 .

⁵Here e^1 and e^2 are exchanged with respect to those appearing in [7]. This implies that the $\tilde{\zeta}$ given in (2.14) below has swapped components with respect to the one in [7].

⁶Our convention for the Hodge star is $*\theta^{a_1 \dots a_k} = \frac{1}{(4-k)!} \epsilon^{a_1 \dots a_k a_{k+1} \dots a_4} \theta^{a_{k+1} \dots a_4}$, where θ^a denotes a real frame. This is related to the complex frame as $e^1 = \theta^1 + i\theta^2$, $e^2 = \theta^3 + i\theta^4$; so the volume form introduced above is $\text{vol}_4 = \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4$.

Let us present more explicit formulae for A and V . Noting that $\nabla^\rho J_{\rho\mu} dx^\mu = *d*J = *dJ$ and using the expression for J in (2.12), simple manipulations show that (2.4) and (2.5) can be written as⁷

$$V = d^c \log \Omega + \frac{2}{\Omega^2 c^2} \text{Im}(\partial_{\bar{z}} h K) + \kappa K, \quad (2.15)$$

$$A = \frac{1}{2} d^c \log(\Omega^3 c) - \frac{i}{2} d \log(\Omega^{-1} s) + \left(\frac{3}{2} \kappa - \frac{i}{\Omega^2 c^2} \partial_{\bar{z}} h \right) K, \quad (2.16)$$

where we used $\sqrt{g} = \Omega^4 c^2$.

For later applications it is important to observe that we can use the freedom in choosing κ and s to arrange for A to be *real*. Indeed, requiring $\text{Im} A = 0$ in (2.16) and separating the different components, we obtain the conditions

$$|s| = \Omega, \quad \kappa = \frac{2i}{3\Omega^2 c^2} \partial_{\bar{z}} h, \quad (2.17)$$

where we fixed an irrelevant multiplicative constant in $|s|$. With these choices of κ and $|s|$, the gauge field A takes the simple form

$$A = \frac{1}{2} d^c \log(\Omega^3 c) + \frac{1}{2} d\omega, \quad (2.18)$$

where ω denotes the phase of s , i.e. $s = |s| e^{i\omega}$. Note that ω has not been fixed so far, while it will be determined by our global analysis in section 3. The one-form V in general remains complex

$$V = d^c \log \Omega - \frac{i}{3\Omega^2 c^2} \partial_{\bar{z}} h K + \frac{i}{\Omega^2 c^2} \partial_{\bar{z}} \bar{h} \bar{K}. \quad (2.19)$$

Recalling that Ω and c are real and depend only on the z, \bar{z} coordinates, we can also write more explicitly

$$\begin{aligned} A &= \text{Im}[\partial_z \log(\Omega^3 c) dz] + \frac{1}{2} d\omega, \\ V &= 2 \text{Im}[\partial_z \log \Omega dz] - \frac{i}{3\Omega^2 c^2} \partial_{\bar{z}} h K + \frac{i}{\Omega^2 c^2} \partial_{\bar{z}} \bar{h} \bar{K}. \end{aligned} \quad (2.20)$$

Finally, the spinors (2.14) take the form

$$\zeta_\alpha = \sqrt{\frac{\Omega}{2}} e^{i\frac{\omega}{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \tilde{\zeta}^{\dot{\alpha}} = \sqrt{\frac{\Omega}{2}} e^{-i\frac{\omega}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (2.21)$$

2.2 Supersymmetry transformations and Lagrangians

In this section we present the supersymmetry variations and relevant Lagrangians of the theories that we consider in this paper. In Euclidean signature, defining $\mathcal{N} = 1$ supersymmetry requires to double the number of degrees of freedom in each multiplet. This can be realized formally by thinking about a given field and its Hermitian conjugate as transforming independently under supersymmetry. To define the path integral over the fields of a multiplet, one then has to make a choice of reality conditions, reducing the number of degrees of freedom in a multiplet to the usual one. In the following, we will first consider a vector multiplet and then a chiral multiplet.

⁷For any function f we define $d^c f = J_\mu^\nu \partial_\nu f dx^\mu = -i(\partial - \bar{\partial})f$.

2.2.1 Vector multiplet

The $\mathcal{N} = 1$ vector multiplet contains a gauge field \mathcal{A}_μ , a pair of two-component complex spinors $\lambda, \tilde{\lambda}$ of opposite chirality and an auxiliary field D , all transforming in the adjoint representation of the gauge group G . As already noted, a priori in Euclidean signature the fermionic fields $\lambda, \tilde{\lambda}$ are independent, and the bosonic fields \mathcal{A}_μ, D are not Hermitian. We define a covariant derivative as

$$D_\mu = \nabla_\mu - i\mathcal{A}_\mu \cdot -iq_R A_\mu, \tag{2.22}$$

where \cdot denotes the action in the relevant representation, and the R-charges q_R of the fields $(\mathcal{A}_\mu, \lambda, \tilde{\lambda}, D)$ are given respectively by $(0, 1, -1, 0)$. The supersymmetry transformations of the fields in the multiplet are

$$\begin{aligned} \delta\mathcal{A}_\mu &= i\zeta\sigma_\mu\tilde{\lambda} + i\tilde{\zeta}\tilde{\sigma}_\mu\lambda, \\ \delta\lambda &= \mathcal{F}_{\mu\nu}\sigma^{\mu\nu}\zeta + iD\zeta, \\ \delta\tilde{\lambda} &= \mathcal{F}_{\mu\nu}\tilde{\sigma}^{\mu\nu}\tilde{\zeta} - iD\tilde{\zeta}, \\ \delta D &= -\zeta\sigma^\mu\left(D_\mu\tilde{\lambda} - \frac{3i}{2}V_\mu\tilde{\lambda}\right) + \tilde{\zeta}\tilde{\sigma}^\mu\left(D_\mu\lambda + \frac{3i}{2}V_\mu\lambda\right), \end{aligned} \tag{2.23}$$

where $\mathcal{F}_{\mu\nu} \equiv \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - i[\mathcal{A}_\mu, \mathcal{A}_\nu]$. Note that the two independent spinorial parameters $\zeta, \tilde{\zeta}$ need to be solutions to the equations (2.1), (2.2), and are commuting variables. It is understood that when one of the two equations only admits the trivial solution, the corresponding spinor is set to zero in the supersymmetry transformations. The fermionic fields $\lambda, \tilde{\lambda}$ are anti-commuting, and therefore correspondingly the supersymmetry variation δ is defined as a Grassmann-odd operator. Note also that in the above transformations only the conformal invariant and U^μ -independent combination of background fields $A_\mu^{\text{cs}} \equiv A_\mu - \frac{3}{2}V_\mu$ appears, in the covariant derivative $D_\mu^{\text{cs}} \equiv \nabla_\mu - i\mathcal{A}_\mu \cdot -iq_R A_\mu^{\text{cs}}$.

The supersymmetry algebra is given by

$$\begin{aligned} \{\delta_\zeta, \delta_{\tilde{\zeta}}\} &= \{\delta_{\tilde{\zeta}}, \delta_\zeta\} = 0, \\ [\delta_\zeta, \delta_K] &= [\delta_{\tilde{\zeta}}, \delta_K] = 0, \\ \{\delta_\zeta, \delta_{\tilde{\zeta}}\} &= 2i\delta_K, \end{aligned} \tag{2.24}$$

where δ_ζ (respectively, $\delta_{\tilde{\zeta}}$) means that $\tilde{\zeta}$ (respectively, ζ) is set to zero in the supersymmetry transformations (2.23), and on a field of R-charge q_R we have $\delta_K = \mathcal{L}_K - iK^\mu\mathcal{A}_\mu \cdot -iq_R K^\mu A_\mu$, where \mathcal{L}_K is the Lie derivative along K . If there is only one Killing spinor ζ , then one just has $\delta_\zeta^2 = 0$.

A tedious calculation shows that the Lagrangian

$$\mathcal{L}_{\text{vector}} = \text{Tr} \left[\frac{1}{4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} - \frac{1}{2}D^2 + \frac{i}{2}\lambda\sigma^\mu D_\mu^{\text{cs}}\tilde{\lambda} + \frac{i}{2}\tilde{\lambda}\tilde{\sigma}^\mu D_\mu^{\text{cs}}\lambda \right] \tag{2.25}$$

is invariant under the supersymmetry transformations (2.23). Here Tr is the trace in the adjoint representation of the gauge group. We will show momentarily that if both spinors

$\zeta, \tilde{\zeta}$ exist, then this Lagrangian is the sum of two supersymmetry variations; this will be important for applying the localization argument.

Given that in Euclidean signature the degrees of freedom are doubled, it is conceptually clearer to impose reality conditions on the fields only after computing the supersymmetry variations. Therefore, to define various supersymmetry-exact terms, we introduce an involution \ddagger acting as

$$(\mathcal{A}_\mu, D)^\ddagger = (\mathcal{A}_\mu, -D), \quad \zeta^\ddagger = \zeta^\dagger, \quad (2.26)$$

and as complex conjugation on numbers.⁸ Then we define

$$\begin{aligned} \mathcal{L}_{\text{vector}}^{(+)} &= \delta_\zeta V^{(+)} = -\delta_\zeta \left(\frac{1}{4|\zeta|^2} \text{Tr} (\delta_\zeta \lambda)^\ddagger \lambda \right) \\ &= \frac{1}{4|\zeta|^2} \text{Tr} (\delta_\zeta \lambda)^\ddagger \delta_\zeta \lambda - \frac{1}{4|\zeta|^2} \text{Tr} \delta_\zeta \left((\delta_\zeta \lambda)^\ddagger \right) \lambda \\ &\equiv \delta V_{\text{bos}}^{(+)} + \delta V_{\text{fer}}^{(+)}. \end{aligned} \quad (2.27)$$

The bosonic term is straightforward to evaluate and reads

$$\delta V_{\text{bos}}^{(+)} = \frac{1}{4} \text{Tr} \left(\mathcal{F}_{\mu\nu}^{(+)} \mathcal{F}^{(+)\mu\nu} - D^2 \right), \quad (2.28)$$

where $\mathcal{F}_{\mu\nu}^{(\pm)} = \frac{1}{2}(\mathcal{F} \pm *\mathcal{F})_{\mu\nu}$. The fermionic term reads

$$\delta V_{\text{fer}}^{(+)} = \frac{1}{4|\zeta|^2} \text{Tr} \left[-(\zeta^\dagger \sigma^{\mu\nu} \lambda) \delta_\zeta \mathcal{F}_{\mu\nu} + i(\zeta^\dagger \lambda) \delta_\zeta D \right] \quad (2.29)$$

and with some manipulations can be rewritten as

$$\delta V_{\text{fer}}^{(+)} = \text{Tr} \left[\frac{i}{2} \lambda \sigma^\mu \left(D_\mu \tilde{\lambda} - \frac{3i}{2} V_\mu \tilde{\lambda} \right) \right]. \quad (2.30)$$

To obtain this we used the following expression for the supersymmetry variation of the gauge field strength $\mathcal{F}_{\mu\nu}$

$$\begin{aligned} \delta \mathcal{F}_{\mu\nu} &= 2i \zeta \sigma_{[\nu} D_{\mu]} \tilde{\lambda} + V_{[\mu} \zeta \sigma_{\nu]} \tilde{\lambda} + \epsilon_{\mu\nu\kappa\lambda} V^\kappa (\zeta \sigma^\lambda \tilde{\lambda}) \\ &\quad + 2i \tilde{\zeta} \tilde{\sigma}_{[\nu} D_{\mu]} \lambda - V_{[\mu} \tilde{\zeta} \tilde{\sigma}_{\nu]} \lambda + \epsilon_{\mu\nu\kappa\lambda} V^\kappa (\tilde{\zeta} \tilde{\sigma}^\lambda \lambda). \end{aligned} \quad (2.31)$$

We have thus shown that

$$\mathcal{L}_{\text{vector}}^{(+)} = \text{Tr} \left(\frac{1}{4} \mathcal{F}_{\mu\nu}^{(+)} \mathcal{F}^{(+)\mu\nu} - \frac{1}{4} D^2 + \frac{i}{2} \lambda \sigma^\mu D_\mu^{\text{cs}} \tilde{\lambda} \right). \quad (2.32)$$

If there exists a second Killing spinor $\tilde{\zeta}$, then the previous computations can be repeated with trivial modifications. Namely, we can define

$$\mathcal{L}_{\text{vector}}^{(-)} = \delta_{\tilde{\zeta}} V^{(-)} = -\delta_{\tilde{\zeta}} \left(\frac{1}{4|\tilde{\zeta}|^2} \text{Tr} (\delta_{\tilde{\zeta}} \tilde{\lambda})^\ddagger \tilde{\lambda} \right)$$

⁸We will not need to define the action of \ddagger on λ and $\tilde{\lambda}$.

$$\begin{aligned}
 &= \frac{1}{4|\tilde{\zeta}|^2} \text{Tr} (\delta_{\tilde{\zeta}} \tilde{\lambda})^\dagger \delta_{\tilde{\zeta}} \tilde{\lambda} - \frac{1}{4|\tilde{\zeta}|^2} \text{Tr} \delta_{\tilde{\zeta}} \left((\delta_{\tilde{\zeta}} \tilde{\lambda})^\dagger \right) \tilde{\lambda} \\
 &\equiv \delta V_{\text{bos}}^{(-)} + \delta V_{\text{fer}}^{(-)},
 \end{aligned} \tag{2.33}$$

with

$$\mathcal{L}_{\text{vector}}^{(-)} = \text{Tr} \left(\frac{1}{4} \mathcal{F}_{\mu\nu}^{(-)} \mathcal{F}^{(-)\mu\nu} - \frac{1}{4} D^2 + \frac{i}{2} \tilde{\lambda} \tilde{\sigma}^\mu D_\mu^{\text{cs}} \lambda \right). \tag{2.34}$$

The sum of the two terms is

$$\mathcal{L}_{\text{vector}}^{(+)} + \mathcal{L}_{\text{vector}}^{(-)} = \text{Tr} \left[\frac{1}{4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} - \frac{1}{2} D^2 + \frac{i}{2} \lambda \sigma^\mu D_\mu^{\text{cs}} \tilde{\lambda} + \frac{i}{2} \tilde{\lambda} \tilde{\sigma}^\mu D_\mu^{\text{cs}} \lambda \right] = \mathcal{L}_{\text{vector}}. \tag{2.35}$$

Therefore, we have shown that the vector multiplet Lagrangian $\mathcal{L}_{\text{vector}}$ in (2.25) is the sum of a δ_ζ -exact term and a $\delta_{\tilde{\zeta}}$ -exact term. Note that to derive this result we have *not* imposed any reality condition, and correspondingly at this stage the bosonic part of the Lagrangian is not positive semi-definite.

In order to apply the localization arguments, it will be important that $\mathcal{L}_{\text{vector}}^{(+)}$ and $\mathcal{L}_{\text{vector}}^{(-)}$ are separately invariant under *both* supersymmetries associated with ζ and $\tilde{\zeta}$, so that

$$\begin{aligned}
 \delta_\zeta \mathcal{L}_{\text{vector}}^{(-)} &= \delta_\zeta \delta_{\tilde{\zeta}} V^{(-)} = \text{tot der}, \\
 \delta_{\tilde{\zeta}} \mathcal{L}_{\text{vector}}^{(+)} &= \delta_{\tilde{\zeta}} \delta_\zeta V^{(+)} = \text{tot der},
 \end{aligned} \tag{2.36}$$

where ‘‘tot der’’ denotes a total derivative. Recalling that $\delta_\zeta^2 = \delta_{\tilde{\zeta}}^2 = 0$, these are equivalent to the fact that the vector multiplet Lagrangian is invariant under both supersymmetry variations, namely $\delta_\zeta \mathcal{L}_{\text{vector}} = \delta_{\tilde{\zeta}} \mathcal{L}_{\text{vector}} = \text{tot der}$.

2.2.2 Chiral multiplet

The $\mathcal{N} = 1$ chiral multiplet contains two complex scalars $\phi, \tilde{\phi}$, a pair of two-component complex spinors $\psi, \tilde{\psi}$ of opposite chirality, and two complex auxiliary fields F, \tilde{F} . As for the fields of the vector multiplet, in Euclidean signature the fermionic fields $\psi, \tilde{\psi}$ and the complex scalars $\phi, \tilde{\phi}$, and F, \tilde{F} are all independent. The fields (ϕ, ψ, F) transform in a representation \mathcal{R} , while $(\tilde{\phi}, \tilde{\psi}, \tilde{F})$ transform in the conjugate representation \mathcal{R}^* . The R-charges q_R entering in (2.22) for the fields $(\phi, \psi, F, \tilde{\phi}, \tilde{\psi}, \tilde{F})$ are given by $(r, r-1, r-2, -r, -r+1, -r+2)$ respectively, with r arbitrary. The supersymmetry transformations of the fields in the multiplet can be read off from [2, 4, 27] and are

$$\begin{aligned}
 \delta\phi &= \sqrt{2} \zeta \psi, \\
 \delta\psi &= \sqrt{2} F \zeta + i\sqrt{2} (\sigma^\mu \tilde{\zeta}) D_\mu \phi, \\
 \delta F &= i\sqrt{2} \tilde{\zeta} \tilde{\sigma}^\mu \left(D_\mu \psi - \frac{i}{2} V_\mu \psi \right) - 2i (\tilde{\zeta} \lambda) \phi, \\
 \delta\tilde{\phi} &= \sqrt{2} \tilde{\zeta} \tilde{\psi}, \\
 \delta\tilde{\psi} &= \sqrt{2} \tilde{F} \tilde{\zeta} + i\sqrt{2} (\tilde{\sigma}^\mu \zeta) D_\mu \tilde{\phi}, \\
 \delta\tilde{F} &= i\sqrt{2} \zeta \sigma^\mu \left(D_\mu \tilde{\psi} + \frac{i}{2} V_\mu \tilde{\psi} \right) + 2i \tilde{\phi} (\zeta \lambda).
 \end{aligned} \tag{2.37}$$

These preserve the Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{chiral}} = & D_\mu \tilde{\phi} D^\mu \phi + V^\mu (i D_\mu \tilde{\phi} \phi - i \tilde{\phi} D_\mu \phi) + \frac{r}{4} (R + 6V_\mu V^\mu) \tilde{\phi} \phi + \tilde{\phi} D \phi - \tilde{F} F \\ & + i \tilde{\psi} \tilde{\sigma}^\mu D_\mu \psi + \frac{1}{2} V^\mu \tilde{\psi} \tilde{\sigma}_\mu \psi + i \sqrt{2} (\tilde{\phi} \lambda \psi - \tilde{\psi} \tilde{\lambda} \phi). \end{aligned} \quad (2.38)$$

This depends on both background fields A and V , except when the R-charge takes the value $r = 2/3$, in which case these only appear in the combination $A^{\text{cs}} = A - \frac{3}{2}V$ and the Lagrangian is conformal invariant. Below we will show that the existence of a single supersymmetry parameter ζ is enough to express $\mathcal{L}_{\text{chiral}}$ as a total supersymmetry variation, up to an irrelevant boundary term.

In general, one can consider several chiral multiplets with different R-charges r_I , with Lagrangian given by the sum of the (2.38) for each multiplet, and also add to this a superpotential term \mathcal{L}_W , as in flat space. The explicit expression in component notation is given in [2]. The superpotential W can be an arbitrary holomorphic function⁹ of the fields ϕ_I , and in order not to break the R-symmetry of the theory it must be homogeneous of degree two in the R-charges. This follows from the fact that the fermions ψ_I have R-charges $r_I - 1$ and in components the superpotential contains a fermionic piece

$$\frac{\partial^2 W}{\partial \phi_I \partial \phi_J} \psi_I \psi_J \in \mathcal{L}_W, \quad (2.39)$$

whose R-charge is $r[W] - r_I - r_J + (r_I - 1) + (r_J - 1)$. On integrating out the auxiliary fields F_I one obtains¹⁰

$$\tilde{F}_I = \frac{\partial W}{\partial \phi_I}, \quad F_I = \frac{\partial \tilde{W}}{\partial \tilde{\phi}_I}. \quad (2.40)$$

In order to write the supersymmetry-exact terms we extend the action of the involution \ddagger used for the vector multiplet to the bosonic fields of the chiral multiplet as

$$(\phi, F, \tilde{\phi}, \tilde{F})^\ddagger = (\tilde{\phi}, -\tilde{F}, \phi, -F). \quad (2.41)$$

While we will not need to define how \ddagger acts on $\psi, \tilde{\psi}$ and on V_μ , we will need its action on A_μ . There are two natural definitions we can take, which in general are not equivalent. If we define $A_\mu^\ddagger = A_\mu$, then the computation below shows that the Lagrangian $\mathcal{L}_{\text{chiral}}$ is δ_ζ -exact (up to a boundary term) without any restriction on A_μ . However, notice that this Lagrangian is *not* invariant under changes of U_μ , and its bosonic part is not positive semi-definite even after imposing reality conditions on the dynamical fields. If instead we define $A_\mu^\ddagger = A_\mu^\dagger$, then the localizing term that we will choose in the next section does not depend on U_μ and its bosonic part is positive semi-definite after choosing suitable reality conditions. However, for complex A_μ , this does not reconstruct the Lagrangian (2.38). In the following we will assume that A_μ is real, so that the two definitions are equivalent; as showed at the end of section 2.1, this is certainly possible in the presence of two supercharges of opposite R-charge. Later we will make some comments about relaxing this choice.

⁹In this paper we assume that W is a polynomial in the fields ϕ_I .

¹⁰A priori \tilde{W} is an arbitrary function of $\tilde{\phi}_I$, but reality conditions will relate this to W .

We consider

$$\begin{aligned}
 \delta V_{\text{chiral}} &= \delta_\zeta V_1 + \delta_\zeta V_2 + \delta_\zeta V_3 + \delta_\zeta V_U \\
 &= \delta_\zeta \left(\frac{1}{2|\zeta|^2} \left[(\delta_\zeta \psi)^\dagger \psi - \tilde{\psi} (\delta_\zeta \tilde{\psi})^\dagger \right] \right) + \delta_\zeta V_3 + \delta_\zeta V_U \\
 &= \frac{1}{2|\zeta|^2} \left[(\delta_\zeta \psi)^\dagger \delta_\zeta \psi + \delta_\zeta ((\delta_\zeta \psi)^\dagger) \psi + (\delta_\zeta \tilde{\psi})^\dagger \delta_\zeta \tilde{\psi} + \tilde{\psi} \delta_\zeta ((\delta_\zeta \tilde{\psi})^\dagger) + 2i \delta_\zeta (\tilde{\phi} \zeta^\dagger \lambda \phi) \right. \\
 &\quad \left. - \sqrt{2} \delta_\zeta (U^\mu \zeta^\dagger \sigma_\mu \tilde{\psi} \phi) \right] \\
 &\equiv \delta V_{\text{bos}1} + \delta V_{\text{fer}1} + \delta V_{\text{bos}2} + \delta V_{\text{fer}2} + \delta V_{\text{bos}3} + \delta V_{\text{fer}3} + \delta V_{\text{bos}U} + \delta V_{\text{fer}U}. \quad (2.42)
 \end{aligned}$$

For the bosonic part, the supersymmetry transformations (2.37) lead to

$$\begin{aligned}
 \delta V_{\text{bos}1} &= -\tilde{F}F, \\
 \delta V_{\text{bos}2} &= (g^{\mu\nu} - iJ^{\mu\nu}) D_\mu \tilde{\phi} D_\nu \phi \\
 &= D_\mu \tilde{\phi} D^\mu \phi - 2i(V^\mu - U^\mu) \tilde{\phi} D_\mu \phi + \frac{r}{4} (R + 6V_\mu V^\mu) \tilde{\phi} \phi + \frac{1}{2} J^{\mu\nu} \tilde{\phi} \mathcal{F}_{\mu\nu} \phi - i \nabla_\mu (J^{\mu\nu} \tilde{\phi} D_\nu \phi), \\
 \delta V_{\text{bos}3} &= -\frac{1}{2} J^{\mu\nu} \tilde{\phi} \mathcal{F}_{\mu\nu} \phi + \tilde{\phi} D \phi, \\
 \delta V_{\text{bos}U} &= 2i U^\mu D_\mu \tilde{\phi} \phi, \quad (2.43)
 \end{aligned}$$

where to go from the first to the second line in the second term we have used the identity (A.12), and in the last line we used the holomorphicity of U^μ , namely $J^\mu{}_\nu U^\nu = iU^\mu$. As for the fermionic terms, after some computations involving the Fierz identities in (A.8) we find

$$\begin{aligned}
 \delta V_{\text{fer}1} &= -\frac{i}{2} D_\mu \tilde{\psi} \tilde{\sigma}^\mu \psi - \frac{1}{2} J^{\mu\nu} D_\mu \tilde{\psi} \tilde{\sigma}_\nu \psi + \frac{1}{2|\zeta|^2} V^\mu (\zeta \sigma_\mu \tilde{\psi}) (\zeta^\dagger \psi) - i \frac{\sqrt{2}}{|\zeta|^2} \tilde{\phi} (\zeta \lambda) (\zeta^\dagger \psi), \\
 \delta V_{\text{fer}2} &= \frac{i}{2} \tilde{\psi} \tilde{\sigma}^\mu D_\mu \psi - \frac{1}{2} J^{\mu\nu} \tilde{\psi} \tilde{\sigma}_\nu D_\mu \psi + V^\mu \tilde{\psi} \tilde{\sigma}_\mu \psi - \frac{1}{2|\zeta|^2} V^\mu (\zeta^\dagger \sigma_\mu \tilde{\psi}) (\zeta \psi) - i\sqrt{2} \tilde{\psi} \tilde{\lambda} \phi, \\
 \delta V_{\text{fer}3} &= i \frac{\sqrt{2}}{|\zeta|^2} \tilde{\phi} (\zeta^\dagger \lambda) (\zeta \psi) \\
 \delta V_{\text{fer}U} &= -U^\mu \tilde{\psi} \tilde{\sigma}_\mu \psi, \quad (2.44)
 \end{aligned}$$

where in the last equality we used holomorphicity of U^μ , in the form $U^\mu \tilde{\sigma}_\mu \zeta = 0$. The total fermionic part can be written as

$$\begin{aligned}
 \delta V_{\text{fer}1} + \delta V_{\text{fer}2} + \delta V_{\text{fer}3} + \delta V_{\text{fer}U} &= i \tilde{\psi} \tilde{\sigma}^\mu D_\mu \psi + \frac{1}{2} V^\mu \tilde{\psi} \tilde{\sigma}_\mu \psi + i\sqrt{2} (\tilde{\phi} \lambda \psi - \tilde{\psi} \tilde{\lambda} \phi) \\
 &\quad - \frac{i}{2} D_\mu \left((\delta^\mu{}_\nu - iJ^\mu{}_\nu) \tilde{\psi} \tilde{\sigma}^\nu \psi \right). \quad (2.45)
 \end{aligned}$$

Adding everything up, we obtain

$$\delta V_{\text{chiral}} = \mathcal{L}_{\text{chiral}} + \nabla_\mu Y^\mu, \quad (2.46)$$

where $\mathcal{L}_{\text{chiral}}$ is the Lagrangian (2.38) and the total derivative term is

$$Y^\mu = -i J^{\mu\nu} \tilde{\phi} D_\nu \phi - i(V^\mu - 2U^\mu) \tilde{\phi} \phi - \frac{i}{2} (\delta^\mu{}_\nu - iJ^\mu{}_\nu) \tilde{\psi} \tilde{\sigma}^\nu \psi. \quad (2.47)$$

In a similar way, one can see that $\mathcal{L}_{\text{chiral}}$ is also exact under the variation generated by $\tilde{\zeta}$.

2.3 Supersymmetric locus equations

Let us now discuss how to use the results above to compute the path integral of supersymmetric field theories, using the localization method. The standard localization arguments require to deform the path integral defined by a supersymmetric action by adding a term that is a supersymmetry variation, and whose bosonic part is positive semi-definite. In this way the complete path integral is given by the one-loop determinant around the locus where this bosonic part vanishes. We will address the vector multiplet and chiral multiplet separately.

2.3.1 Vector multiplet

If the manifold admits one Killing spinor ζ , then we can deform the vector multiplet Lagrangian (2.25) by adding to it the δ_ζ -exact term (2.27) with an arbitrary parameter t , namely

$$S = \int d^4x \sqrt{g} \left(\mathcal{L}_{\text{vector}} + t \delta_\zeta V^{(+)} \right). \quad (2.48)$$

We see that imposing the reality conditions $\mathcal{A}_\mu^\dagger = \mathcal{A}_\mu$, $D^\dagger = -D$ implies that the bosonic part (2.28) of the deformation term is positive semi-definite.¹¹ The localization locus is given by $\delta V_{\text{bos}}^{(+)} = 0$, yielding the conditions

$$\mathcal{F}_{\mu\nu}^{(+)} = 0, \quad D = 0. \quad (2.49)$$

Of course this is also equivalent to $\delta\lambda = 0$, whose independent components give $J_{\mu\nu}\mathcal{F}^{\mu\nu} = P_{\mu\nu}\mathcal{F}^{\mu\nu} = 0 = D$. The conclusion is that when there exists only one supercharge associated with ζ , the localization locus is given by anti-instanton configurations. In the case of a supercharge associated with $\tilde{\zeta}$, the same argument works by considering the term $\delta_{\tilde{\zeta}} V^{(-)}$ in (2.33), with the conclusion being that the localization locus is given by instanton configurations.

If the manifold admits both ζ and $\tilde{\zeta}$, then we can deform the vector multiplet Lagrangian (2.25) by adding both the δ_ζ -exact and $\delta_{\tilde{\zeta}}$ -exact terms, namely

$$S = \int d^4x \sqrt{g} \left(\mathcal{L}_{\text{vector}} + t_+ \delta_\zeta V^{(+)} + t_- \delta_{\tilde{\zeta}} V^{(-)} \right). \quad (2.50)$$

To see that the path integral is independent of the parameter t_+ one notes that $\delta_\zeta \mathcal{L}_{\text{vector}} = \delta_\zeta \delta_{\tilde{\zeta}} V^{(-)} = \text{tot der}$. Similarly, the path integral is also independent of the parameter t_- . In the end one can take $t_+ = t_- = t$ and omit the first term, without affecting the conclusions. The localization locus then is given by $\delta V_{\text{bos}}^{(+)} = \delta V_{\text{bos}}^{(-)} = 0$, which is equivalent to the conditions

$$\mathcal{F}_{\mu\nu} = 0, \quad D = 0, \quad (2.51)$$

so that both the self-dual and the anti-self-dual parts of the gauge field strength vanish. We will discuss the solutions to these equations in section 4.1, after specializing the topology of the four-dimensional manifold.

¹¹We note that actually the weaker reality condition $\mathcal{F}_{\mu\nu}^{(+)\dagger} = \mathcal{F}_{\mu\nu}^{(+)}$ is sufficient to guarantee positivity of the deformation term. The condition $\mathcal{A}_\mu^\dagger = \mathcal{A}_\mu$ implies that also the original Lagrangian (2.25) has positive bosonic part, but this is not necessary for the localization argument.

Notice that the conclusions above are manifestly independent of the choice of the holomorphic vector field U^μ , as well as of the reality properties of the background fields A_μ , V_μ .

2.3.2 Chiral multiplet

If the manifold admits one Killing spinor ζ , then we can deform the chiral multiplet Lagrangian (2.38), possibly supplemented by a superpotential, by adding to it the δ_ζ -exact term $\delta_\zeta(V_1 + V_2)$ defined in section 2.2.2. Namely, we consider

$$S = \int d^4x \sqrt{g} [\mathcal{L}_{\text{chiral}} + \mathcal{L}_W + t \delta_\zeta(V_1 + V_2)] , \quad (2.52)$$

where t is an arbitrary parameter. We must then choose reality conditions such that $\delta V_{\text{bos}1}$ and $\delta V_{\text{bos}2}$ are positive semi-definite.¹² The former requirement is satisfied imposing $\tilde{F} = -F^\dagger$. In order to ensure that $2|\zeta|^2 \delta V_{\text{bos}2} = (\delta_\zeta \tilde{\psi})^\dagger \delta_\zeta \tilde{\psi}$ is positive we require $\tilde{\phi} = \phi^\dagger$ (hence the involution \ddagger acts as the Hermitian conjugation \dagger). Note that $\delta V_{\text{bos}2}$ does not depend on the background field V_μ , therefore there are no reality constraints to impose on the latter. On the other hand, it does depend on the background field A_μ , hence its choice may a priori affect positivity. When A is real,¹³ the localization locus is defined by the conditions $\delta V_{\text{bos}1} = \delta V_{\text{bos}2} = 0$, so that in particular $\delta_\zeta \psi = \delta_\zeta \tilde{\psi} = 0$. These are equivalent to

$$F = 0, \quad J^\mu{}_\nu D^\nu \tilde{\phi} = i D^\mu \tilde{\phi} . \quad (2.53)$$

The second equation means that $D^\mu \tilde{\phi}$ is a holomorphic vector, or equivalently that $\tilde{\phi}$ is a holomorphic section on a suitable line bundle. These configurations are still very complicated and in this paper we will not analyse them further. Before moving to the case of two supercharges, let us briefly comment on the role of U^μ . Since this is a holomorphic vector, it drops out from the supersymmetry transformations (2.37), and therefore, if we define $A_\mu^\ddagger = A_\mu^\dagger$, it also drops out from the localizing term and hence from the locus equations (2.53). In this case the positivity property of $\delta V_{\text{bos}2}$ is not affected by the choice of U^μ .

Let us now discuss the case when the manifold admits both ζ and $\tilde{\zeta}$. In this case, the same deformation term in (2.52) can be written *also* as $\delta_{\tilde{\zeta}}$ -exact term $\delta_{\tilde{\zeta}}(\tilde{V}_1 + \tilde{V}_2)$, with tilded and untilded objects appropriately swapped. Assuming the same reality conditions, and in particular choosing A_μ real (with again no reality condition on V_μ), the localization locus becomes $\delta_\zeta \psi = \delta_\zeta \tilde{\psi} = \delta_{\tilde{\zeta}} \psi = \delta_{\tilde{\zeta}} \tilde{\psi} = 0$. Contracting with appropriate spinors this can be recast into the equations

$$F = 0, \quad J^\mu{}_\nu D^\nu \tilde{\phi} = i D^\mu \tilde{\phi}, \quad \tilde{J}^\mu{}_\nu D^\nu \phi = i D^\mu \phi . \quad (2.54)$$

The last two equations imply $K^\mu D_\mu \phi = K^\mu D_\mu \tilde{\phi} = 0$. Notice that the locus equations $J^\mu{}_\nu D^\nu \tilde{\phi} = i D^\mu \tilde{\phi}$ and $\tilde{J}^\mu{}_\nu D^\nu \phi = i D^\mu \phi$ are derived from two deformation terms that

¹²The reason why we are not using simply $t \mathcal{L}_{\text{chiral}}$, which is also δ_ζ -exact, is that its bosonic part contains the terms $\delta V_{\text{bos}3}$ and δV_U , which are not positive after imposing the reality conditions.

¹³For example, on Kähler manifolds, the canonical choice is to take A real and $V = 0$.

are equal up to a total derivative (exactly equal when integrated over the compact four-manifold). This means that although the two equations may be different locally, they admit the same global solutions.

As in the case of the vector multiplet, the solutions to the locus equations (2.54) depend on the global structure of the four-manifold considered. In section 4.1 we will solve (2.54) in the case of $M_4 = S^1 \times M_3$, where M_3 is topologically a three-sphere, allowing for a very general class of metrics.

Before moving to the analysis of the localization on Hopf surfaces, it is interesting to note that, for manifolds admitting two Killing spinors of opposite R-charge, one can prove that the localization locus and one-loop determinants do not depend on the conformal factor Ω of the metric. This argument is presented in appendix B. It is in agreement with [19], that showed that the partition function is independent of small metric deformations that do not affect the complex structures. We will see in section 4.3 how indeed the dependence on Ω drops from the computation.

3 Hopf surfaces

In this section we focus on a particular class of geometries admitting two spinors of opposite R-charge, requiring that the four-dimensional manifold has the *topology* of $S^1 \times S^3$. This will play an important role in the calculation of the localized partition function in section 4. Furthermore, in order to make contact with the results of [10], we will assume that there exists a third Killing vector commuting with K , and that the metric is a direct product.

3.1 Generalities

A *Hopf surface* is essentially a four-dimensional complex manifold with the topology of $S^1 \times S^3$, and it may be defined as a compact complex surface whose universal covering is $\mathbb{C}^2 - (0, 0)$. Any such surface arises as the quotient by a finite group Γ of a *primary* Hopf surface, which is defined as having fundamental group isomorphic to \mathbb{Z} [28, 29]. In the following we will restrict our attention to primary Hopf surfaces, referring to them simply as Hopf surfaces. These are described as a quotient of $\mathbb{C}^2 - (0, 0)$, with coordinates z_1, z_2 by a cyclic group

$$(z_1, z_2) \sim (pz_1 + \lambda z_2^m, qz_2), \tag{3.1}$$

where \sim denotes identification of coordinates, $m \in \mathbb{N}$, and p, q, λ are complex parameters, such that $0 < |p| \leq |q| < 1$ and $(p - q^m)\lambda = 0$. See e.g. [30]. It was shown in [28, 29] that all primary Hopf surfaces are diffeomorphic to $S^1 \times S^3$. Moreover, it is shown in [18] that Hopf surfaces with¹⁴ $\lambda \neq 0$ admit only one Killing spinor ζ , and we will not consider them further. We will only consider Hopf surfaces with $\lambda = 0$, showing that these admit a very general class of metrics, compatible with both complex structures J and \tilde{J} , and hence both solutions ζ and $\tilde{\zeta}$.

From the geometric point of view, the question that usually arises is whether on a manifold there exists a particular type of metric. In the case of Hopf surfaces, a class of metric

¹⁴These are referred to as of “class 0” in [30], while those with $\lambda = 0$ are referred to as of “class 1”.

that appears to be of interest is that of *locally conformally Kähler* (LCK) metrics [30]. This means that there exists, at least locally, a conformal rescaling of the metric, to a Kähler one. A simple way to state this property is that the Lee form associated to the complex structure is closed: $d\theta = 0$. Indeed ref. [30] constructed a large class of LCK metrics on a Hopf surface. However, from the point of view of rigid supersymmetry, there is no natural condition on the curvature of a metric, and indeed the LCK property is too restrictive. From the expressions (2.4), (2.5) we see that this property is equivalent to the requirement that the curvature of the conformally invariant background field A^{cs} is purely real:

$$\text{Im}[dA^{\text{cs}}] = 0 \quad \Leftrightarrow \quad \text{LCK} . \tag{3.2}$$

Although the Hermitian metric discussed in [22] (see e.g. equation (5.38) of this reference) is indeed LCK, as can be seen from the expression of A^{cs} in (5.10), in general this property is not satisfied by Hermitian metrics admitting two Killing spinors of opposite R-charge.

Notice also that the metrics written in equation (4.7) of [18] arise from the particular choice of complex coordinates on $\mathbb{C}^2 - (0,0)$ made in this reference. Below we will present a different construction, where we will start with a smooth metric on $S^1 \times S^3$, containing arbitrary functional degrees of freedom. This will make transparent the fact that the constants p, q parameterise the complex structure of the Hopf surface, while the metric is largely independent of these.

3.2 Global properties

We will discuss the geometries of interest starting from a four-dimensional metric that is *by construction* a non-singular complete metric on $S^1 \times S^3$. Requiring that this is compatible with an integrable complex structure ensures that it is a metric on a Hopf surface [28, 29]. The existence of two Killing spinors $\zeta, \tilde{\zeta}$ is guaranteed imposing that the metric admits a complex Killing vector K commuting with its complex conjugate and satisfying $K_\mu K^\mu = 0$.

The global analysis of the geometry is facilitated if we assume that there exists an additional real Killing vector commuting with K , so that generically the isometry group of the metric is $U(1)^3$, with a $U(1)$ acting on S^1 and a $U(1) \times U(1)$ acting on a transverse metric on S^3 . The three-dimensional part is therefore *toric*, and in particular admits an almost contact structure and a dual Reeb vector field whose orbits in general do not close.¹⁵ In appendix D we analyse the most general metric with $U(1)^3$ isometry, while in the rest of the present section we will consider the following metric of direct product form¹⁶

$$ds^2 = \Omega^2 d\tau^2 + ds^2(M_3) \equiv \Omega^2 d\tau^2 + f^2 d\rho^2 + m_{IJ} d\varphi_I d\varphi_J \quad I, J = 1, 2. \tag{3.3}$$

¹⁵It would be straightforward to analyse the case where the isometry group of the four-dimensional metric is $U(1)^2$. Since a $U(1)$ factor acts on S^1 , the other $U(1)$ is generated by a Reeb vector field on $M_3 \simeq S^3$ of regular type. This case is however less interesting.

¹⁶Note that this Riemannian metric is related to a supersymmetric Lorentzian metric with time coordinate $t = i\tau$ [31]. This implies that the partition function we will compute in section 4 can also be thought of as arising from the Euclidean (and compactified) time path integral of a theory defined on $\mathbb{R}_t \times M_3$. This partially motivates our choice of restricting to a direct product metric. Other motivations are discussed in appendix D.

Here $\tau \sim \tau + 2\pi$ is a coordinate on S^1 , while for $M_3 \simeq S^3$ we take coordinates $\rho, \varphi_1, \varphi_2$ adapted to the description of S^3 as a $T^2 \simeq U(1)^2$ fibration over an interval. In these coordinates the Killing vectors generating the $U(1) \times U(1)$ isometry are $\partial/\partial\varphi_1$ and $\partial/\partial\varphi_2$. Without loss of generality we take canonical 2π periodicities for φ_1, φ_2 , and assume $0 \leq \rho \leq 1$, with the extrema of the interval corresponding to the north and south poles of the three-sphere. For $\rho \in [0, 1]$, we require that $\Omega = \Omega(\rho) > 0$, $f = f(\rho) > 0$ and that the torus metric $m_{IJ} = m_{IJ}(\rho)$ is positive-definite. Moreover, in order for the metric to be non-singular, some conditions need to be satisfied at the poles of S^3 , which we will spell out below.

Near to an end-point, one of the one-cycles of the torus remains finite, while the other one-cycle must shrink, in a way such that the associated angular coordinate locally describes, together with ρ , a copy of \mathbb{R}^2 . Let us assume that $\partial/\partial\varphi_1$ (respectively, $\partial/\partial\varphi_2$) generates the one-cycle that shrinks at $\rho \rightarrow 1$ (respectively, $\rho \rightarrow 0$). Then, as $\rho \rightarrow 0$ we require that

$$f \rightarrow f_2, \quad m_{11} \rightarrow m_{11}(0), \quad m_{22} = (f_2\rho)^2 + \mathcal{O}(\rho^3), \quad m_{12} = \mathcal{O}(\rho^2), \quad (3.4)$$

where $f_2 > 0$ and $m_{11}(0) > 0$ are constants. Similarly, as $\rho \rightarrow 1$ we require

$$f \rightarrow f_1, \quad m_{11} = f_1^2(1-\rho)^2 + \mathcal{O}((1-\rho)^3), \quad m_{22} \rightarrow m_{22}(1), \quad m_{12} = \mathcal{O}[(1-\rho)^2], \quad (3.5)$$

where $f_1 > 0$ and $m_{22}(1) > 0$ are constants. Note that m_{IJ} must degenerate at the poles, since either one of the vectors $\partial/\partial\varphi_I$ has vanishing norm there. Indeed, as $\rho \rightarrow 0$ we see that $\det(m_{IJ})$ goes to zero precisely as $m_{11}(0)(f_2\rho)^2$, while when $\rho \rightarrow 1$ it goes to zero as $m_{22}(1)f_1^2(1-\rho)^2$.

It is now simple to construct supersymmetric backgrounds preserving two supercharges of opposite R-charge, with metric given by (3.3). As reviewed in section 2.1, a solution ζ and a solution $\tilde{\zeta}$ to equations (2.1), (2.2) exist if the metric admits a complex Killing vector K commuting with its complex conjugate, $[K, \bar{K}] = 0$, and squaring to zero, $K_\mu K^\mu = 0$. We choose

$$K = \frac{1}{2} \left[b_1 \frac{\partial}{\partial\varphi_1} + b_2 \frac{\partial}{\partial\varphi_2} - i \frac{\partial}{\partial\tau} \right], \quad (3.6)$$

where b_1 and b_2 are two real parameters, so that the orbits of $\text{Re}K$ generically do not close. Notice that $\text{Re}K$ is a Reeb vector on M_3 , whose dual one-form defines an almost contact structure. This clearly satisfies $[K, \bar{K}] = 0$, while the condition $K_\mu K^\mu = 0$ is equivalent to

$$\Omega^2 = b^I m_{IJ} b^J \quad \text{for } \rho \in [0, 1]. \quad (3.7)$$

Note that this can be regarded as a constraint on the $g_{\tau\tau}$ component of the metric (3.3), hence the three-dimensional part of (3.3) is a non-singular metric on $M_3 \simeq S^3$, independent of the two parameters b_1, b_2 [10, 24]. In appendix D we discuss how this condition is generalised in the case of a non-direct product metric, showing that this is related to *complexifying* the parameters b_1, b_2 .

The background fields A and V can be determined using the formulae in section 2.1, which require first casting the metric in the canonical complex coordinates w, z . We will do this in two steps. Firstly, we will show that the metric can be written as

$$ds^2 = \Omega^2 \left[d\tau^2 + (d\psi + a)^2 + c^2 dz d\bar{z} \right], \quad (3.8)$$

where ψ is an angular coordinate such that

$$\frac{\partial}{\partial\psi} = b_1 \frac{\partial}{\partial\varphi_1} + b_2 \frac{\partial}{\partial\varphi_2}, \tag{3.9}$$

and z is a complex coordinate defined in terms of $\rho, \varphi_1, \varphi_2$. Moreover, $c = c(z, \bar{z})$ is a real, positive function of z , while $a = a_z(z, \bar{z})dz + \bar{a}_{\bar{z}}(z, \bar{z})d\bar{z}$ is a real one-form. Notice that the three-dimensional part of the metric (3.8) is precisely of the form implied by new minimal supersymmetry in three dimensions [27], and used in the analysis of [10]. Secondly, we will introduce another complex coordinate, w , thus arriving at the form (2.9).

A convenient¹⁷ choice of Killing vector on M_3 independent of (3.6) is

$$\frac{\partial}{\partial\chi} = b_1 \frac{\partial}{\partial\varphi_1} - b_2 \frac{\partial}{\partial\varphi_2}, \tag{3.10}$$

with the corresponding change of coordinates given by

$$\varphi_1 = b_1(\psi + \chi), \quad \varphi_2 = b_2(\psi - \chi). \tag{3.11}$$

In terms of the ψ, χ coordinates, the M_3 part of the metric (3.3) becomes

$$ds^2(M_3) = \Omega^2 [(d\psi + a)^2 + \Omega^{-2} f^2 d\rho^2 + c^2 d\chi^2], \tag{3.12}$$

where Ω^2 is given in (3.7), the function c reads

$$c = \frac{2|b_1 b_2|}{\Omega^2} \sqrt{\det(m_{IJ})}, \tag{3.13}$$

and the one-form $a = a_\chi d\chi$ is given by

$$a_\chi = \frac{1}{\Omega^2} (b_1^2 m_{11} - b_2^2 m_{22}). \tag{3.14}$$

Next, we define the complex coordinate z as $z = u(\rho) + i\chi$, where the real function $u(\rho)$ is a solution to

$$u' = \frac{f}{\Omega c}, \tag{3.15}$$

with prime denoting derivative with respect to ρ . This differential equation can be solved for $\rho \in (0, 1)$, so the complex coordinate z , together with ψ , covers S^3 everywhere except at the poles, which are found at $\text{Re } z \rightarrow \pm\infty$ (cf. the expansions in (3.25) below). We then see that the metric takes the desired form (3.8). In these coordinates, the vector K becomes

$$K = \frac{1}{2} \left(\frac{\partial}{\partial\psi} - i \frac{\partial}{\partial\tau} \right), \tag{3.16}$$

while as a one-forms it reads

$$K = \frac{1}{2} \Omega^2 (d\psi + a - i d\tau). \tag{3.17}$$

¹⁷The only requirement is that the change of coordinates should be invertible.

Note that although the metric components in (3.12) depend explicitly on b_1, b_2 , this is just an artefact of the choice of coordinates. In particular, global properties of the metric may be analysed only in the coordinates $\rho, \varphi_1, \varphi_2$, and not in the coordinates ψ, z , as neither ψ nor $\chi = \text{Im } z$ are period coordinates in general.

Let us now cast the metric (3.8) in the form (2.9), introducing a complex coordinate w in addition to z . We take

$$w = \psi + i\tau + P(z, \bar{z}), \tag{3.18}$$

where $P(z, \bar{z})$ is a complex function. With this definition, we have $K = \partial/\partial w$, and the two metrics match if we impose

$$\partial_z \bar{P} = a_z \quad \text{and} \quad h = \partial_z (\bar{P} - P), \tag{3.19}$$

where the first equation can be solved for P , while the second equation determines h . We can now discuss the background fields V and A given, for example, in (2.20), with the latter chosen real for convenience. Noting that (3.19) implies

$$\partial_{\bar{z}} h = \partial_{\bar{z}} a_z - \partial_z \bar{a}_{\bar{z}} = -\frac{i}{2} *_2(da), \tag{3.20}$$

where $*_2$ denotes the Hodge star of the 2d metric $dzd\bar{z}$, with volume form $\text{vol}_2 = \frac{i}{2} dz \wedge d\bar{z}$, we see that the choice of κ in (2.17), ensuring that A is real, reads

$$\kappa = \frac{*_2(da)}{3\Omega^2 c^2}, \tag{3.21}$$

so that κ is real and completely determined by the metric on M_3 . Then the formula for V in (2.20) can be written as

$$V = 2 \text{Im}[\partial_z \log \Omega dz] - \frac{1}{3c^2} *_2(da)(d\psi + a) - \frac{i}{6c^2} *_2(da) d\tau. \tag{3.22}$$

In the coordinates $\rho, \varphi_1, \varphi_2$, this becomes

$$V = \frac{1}{2f} \left[c\Omega' - \frac{\Omega}{6c} (a_\chi^2)' \right] \left(\frac{d\varphi_1}{b_1} - \frac{d\varphi_2}{b_2} \right) - \frac{\Omega}{6cf} (a_\chi)' \left(\frac{d\varphi_1}{b_1} + \frac{d\varphi_2}{b_2} + i d\tau \right), \tag{3.23}$$

where the functions $\Omega(\rho)$, $c(\rho)$ and $a_\chi(\rho)$ are those in (3.7), (3.13), (3.14). Similarly, the expression for the real gauge field A in (2.20) becomes

$$A = \frac{1}{4\Omega^2 f} (\Omega^3 c)' \left(\frac{d\varphi_1}{b_1} - \frac{d\varphi_2}{b_2} \right) + \frac{1}{2} d\omega. \tag{3.24}$$

Having obtained V and A in the $\rho, \varphi_1, \varphi_2$ coordinates, we can now discuss their global properties, in particular their regularity at the poles of S^3 . Recalling our assumptions on f and m_{IJ} , it is easy to see that for ρ close to zero the functions Ω , c and a_χ behave as

$$\Omega^2 = b_1^2 [m_{11}(0) + m'_{11}(0)\rho] + \mathcal{O}(\rho^2), \quad c = \frac{2f_2}{\sqrt{m_{11}(0)}} \frac{|b_2|}{|b_1|} \rho + \mathcal{O}(\rho^2), \quad a_\chi = 1 + \mathcal{O}(\rho^2), \tag{3.25}$$

with analogous expressions holding for $\rho \rightarrow 1$. Hence, at leading order in $\rho \rightarrow 0$, we see that V behaves as

$$V = k \left(\frac{1}{b_1} d\varphi_1 + \frac{i}{2} d\tau \right) + \mathcal{O}(\rho), \tag{3.26}$$

where k is a constant.¹⁸ This is regular, as neither the one-cycle dual to $d\varphi_1$ nor the one dual to $d\tau$ shrink to zero size at $\rho = 0$. Regularity of V at $\rho = 1$ is seen in a similar way.

On the other hand, regularity of A is not automatic; by imposing this we determine ω , namely the phase of s . At leading order in $\rho \rightarrow 0$ we have

$$A = \frac{|b_2|}{2} \left(\frac{d\varphi_1}{b_1} - \frac{d\varphi_2}{b_2} \right) + \frac{1}{2} d\omega + \mathcal{O}(\rho), \tag{3.27}$$

while at leading order in $(1 - \rho) \rightarrow 0$ we have

$$A = -\frac{|b_1|}{2} \left(\frac{d\varphi_1}{b_1} - \frac{d\varphi_2}{b_2} \right) + \frac{1}{2} d\omega + \mathcal{O}(1 - \rho). \tag{3.28}$$

In order to ensure that A does not have a component along the S^1 that shrinks at either poles, we must take

$$\omega = \text{sgn}(b_1) \varphi_1 + \text{sgn}(b_2) \varphi_2. \tag{3.29}$$

To summarise, starting with an arbitrary non-singular metric $ds^2(M_3)$ on S^3 , we have constructed a non-singular (direct-product) metric on $S^1 \times S^3$, compatible with two commuting complex structures, and thus admitting two supercharges with opposite R-charge $\zeta, \tilde{\zeta}$. The choice (3.29) guarantees that the background fields A, V are non-singular. In appendix C we illustrate the formulae above in an explicit example based on the Berger three-sphere.

3.3 Complex structure

The pair (ds^2, J) determines a Hopf surface, which must arise as a quotient of $\mathbb{C}^2 - (0, 0)$ as in (3.1). We now show this explicitly, by relating the complex coordinates w, z to complex coordinates z_1, z_2 on $\mathbb{C}^2 - (0, 0)$, and determining the complex structure parameters p, q in terms of the parameters b_1, b_2 introduced above. This will provide a relation between the complex structure in four dimensions, and the almost contact structure in the three-dimensional geometry obtained by reduction along the S^1 .

Using (3.15), and taking $P(z, \bar{z}) = iQ(\rho)$ with $Q(\rho)$ a real function, the first equation in (3.19) becomes

$$Q' = \frac{f a_\chi}{\Omega c}, \tag{3.30}$$

and we claim that an appropriate choice of complex coordinates on $\mathbb{C}^2 - (0, 0)$ is given by

$$\begin{aligned} z_1 &= e^{-|b_1|(iw+z)}, \\ z_2 &= e^{-|b_2|(iw-z)}. \end{aligned} \tag{3.31}$$

¹⁸This reads $k = -\frac{b_1^2}{3|b_2|f_2^2} \left[\frac{1}{2} m''_{11}(0) - (m'_{11}(0))^2 - (f_2 b_2 / b_1)^2 \right]$.

Since these are related to w, z by a holomorphic change of coordinates, they are automatically compatible with the complex structure induced by supersymmetry. In terms of the globally defined coordinates on $S^1 \times S^3$ we have

$$\begin{aligned} z_1 &= e^{|b_1|\tau} e^{|b_1|(Q-u)} e^{-i \operatorname{sgn}(b_1)\varphi_1} , \\ z_2 &= e^{|b_2|\tau} e^{|b_2|(Q+u)} e^{-i \operatorname{sgn}(b_2)\varphi_2} . \end{aligned} \tag{3.32}$$

If (z_1, z_2) are indeed coordinates on $\mathbb{C}^2 - (0, 0)$, it is immediate to see that the identification $\tau \sim \tau + 2\pi$ leads to

$$(z_1, z_2) \sim (e^{2\pi|b_1|} z_1, e^{2\pi|b_2|} z_2) , \tag{3.33}$$

corresponding to a Hopf surface with parameters $p = e^{-2\pi|b_1|}$ and $q = e^{-2\pi|b_2|}$.¹⁹ Note that the choice of p, q is *independent* of the metric on M_3 , and only affects the four-dimensional metric through Ω^2 .

It remains to show that z_1, z_2 are complex coordinates on $\mathbb{C}^2 - (0, 0)$ when τ is decompactified, so that $\tau \in \mathbb{R}$. From (3.32) it is clear that the phases $-\operatorname{sgn}(b_j)\varphi_j$ correspond to the angular directions in polar coordinates for the two copies of \mathbb{C} in $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$. Therefore we have to show that $|z_1|, |z_2|$ are appropriate radial directions, and that the point $(0, 0)$ is excluded. The proof is given in appendix E, while below we present a simple example where the function Q derived from (3.30) can be obtained explicitly.

Consider the Berger sphere $M_3 = S_v^3$ with metric

$$ds^2(S_v^3) = d\theta^2 + \sin^2 \theta d\varphi^2 + v^2(d\zeta + \cos \theta d\varphi)^2 , \tag{3.34}$$

discussed in detail in appendix C. In the special case $b_1 = -b_2 = \frac{1}{2v} > 0$ we have $\theta = \pi\rho$, $\Omega = 1$, $f = \pi$, $c = \frac{1}{v} \sin \theta$, $a_\chi = \cos \theta$. The equations (3.15) and (3.30) become $\partial_\theta u = v(\sin \theta)^{-1}$ and $\partial_\theta Q = v \cotan \theta$ and are solved by

$$u(\theta) = v \log \tan \frac{\theta}{2} , \quad Q(\theta) = v \log \sin \theta , \tag{3.35}$$

yielding the coordinates

$$\begin{aligned} z_1 &= \sqrt{2} e^{\frac{\tau}{2v}} \cos \frac{\theta}{2} e^{-i\varphi_1} , \\ z_2 &= \sqrt{2} e^{\frac{\tau}{2v}} \sin \frac{\theta}{2} e^{-i\varphi_2} , \end{aligned} \tag{3.36}$$

in agreement with [22]. It is straightforward to see that these indeed cover $\mathbb{C}^2 - (0, 0)$ when $\tau \in \mathbb{R}$.

4 Localization

In this section we will compute the partition function of a four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theory defined on a background geometry admitting two supercharges of opposite R-charge, comprising a Hopf surface with arbitrary (real) parameters p, q , and a very general Hermitian metric with $U(1)^3$ isometry. We will consider gauge theories with a vector multiplet transforming in the adjoint representation of a gauge group G , and chiral multiplets transforming in arbitrary representations of G .

¹⁹Strictly speaking, it is $p = e^{-2\pi|b_1|}$, $q = e^{-2\pi|b_2|}$ if $|b_2| \leq |b_1|$ and $p = e^{-2\pi|b_2|}$, $q = e^{-2\pi|b_1|}$ if $|b_1| \leq |b_2|$.

4.1 Localization locus

The vector multiplet supersymmetric locus given by (2.51) implies that \mathcal{A}_μ is a *flat connection*. After having specified an $S^1 \times S^3$ topology, the flat connections are characterized by the holonomy of constant gauge fields around S^1 . In particular, up to gauge transformations, the localized fields of the vector multiplet are

$$\mathcal{A}_\mu = (\mathcal{A}_i, \mathcal{A}_\tau) = (0, \mathcal{A}_0), \quad D = 0, \quad (4.1)$$

where \mathcal{A}_0 is constant. Notice that this result holds without any further assumption on the metric, therefore it is true also if the metric is not a direct product or/and it has only a $U(1)^2$ isometry.

Let us fix the vector multiplet fields at their locus values (4.1) and proceed to analyse the supersymmetric locus of a chiral multiplet with R-charge r , determined by the equations (2.54). Following the discussion of section 2.3.2, we will choose A_μ real and impose the reality conditions $\tilde{\phi} = \phi^\dagger$ and $\tilde{F} = -F^\dagger$ on the bosonic fields. Then the locus equations read

$$\begin{aligned} F &= 0, \\ (J_\mu^\nu + \tilde{J}_\mu^\nu) D_\nu \phi &= 0, \\ (J_\mu^\nu - \tilde{J}_\mu^\nu) D_\nu \phi &= -2i D_\mu \phi. \end{aligned} \quad (4.2)$$

Contracting the second equation with K^μ and \bar{K}^μ leads to $K^\mu D_\mu \phi = \bar{K}^\mu D_\mu \phi = 0$. Using the expressions for J , \tilde{J} and K given in section 2.1, the equations for ϕ become

$$\begin{aligned} D_\tau \phi &= \partial_\tau \phi - i\mathcal{A}_0 \phi = 0, \\ D_\psi \phi &= \partial_\psi \phi - ir A_\psi \phi = 0, \\ D_{\bar{z}} \phi &= \partial_{\bar{z}} \phi - ir A_{\bar{z}} \phi = 0, \end{aligned} \quad (4.3)$$

where we have used the fact that $A_\tau = 0$. The first equation implies that ϕ is proportional to $e^{i\mathcal{A}_0\tau}$, which is not globally defined on S^1 , except when $\mathcal{A}_0 = 0$ modulo large gauge transformations.²⁰ Therefore in this case we immediately conclude that $\phi = 0$. When $\mathcal{A}_0 = 0$ the analysis is slightly more subtle. The first equation implies that ϕ is independent of τ , and using (2.18) the two remaining equations are solved by

$$\phi = C(z) (\Omega^3 c)^{-\frac{r}{2}} e^{\frac{ir}{2} (\text{sgn}(b_1)\varphi_1 + \text{sgn}(b_2)\varphi_2)}, \quad (4.4)$$

with $C(z)$ a (locally) holomorphic function of z . In order to obtain a globally defined solution, we must impose periodicity around the two S^1 parametrized by φ_1 and φ_2 . Recalling that $z = u(\rho) + \frac{i}{2} \left(\frac{\varphi_1}{b_1} - \frac{\varphi_2}{b_2} \right)$, periodicity under the shift $\varphi_1 \rightarrow \varphi_1 + 2\pi \text{sgn}(b_1)$ yields

$$C\left(z + \frac{\pi i}{|b_1|}\right) e^{\pi ir} = C(z), \quad (4.5)$$

and similarly periodicity under $\varphi_2 \rightarrow \varphi_2 + 2\pi \text{sgn}(b_2)$ gives

$$C\left(z - \frac{\pi i}{|b_2|}\right) e^{\pi ir} = C(z), \quad (4.6)$$

so that in particular $C(z)$ is a periodic function in the imaginary direction²¹ $C(z +$

²⁰We discuss these large gauge transformations below.

²¹This is true, independently of whether χ is a periodic or a non-compact coordinate.

$i\pi \frac{|b_1|+|b_2|}{|b_1 b_2|}) = C(z)$. Since $|\phi| = |C(z)|(\Omega^3 c)^{-\frac{r}{2}}$, with $\Omega^3 c$ vanishing only at the poles $\rho = 0, \rho = 1$ (see appendix E), we see that in order to have a non-singular solution ϕ for $r > 0$, $C(z)$ must vanish at $\rho = 0, \rho = 1$, that is $\lim_{\text{Re } z \rightarrow \pm\infty} C(z) = 0$. Extending $C(z)$ to the complex (u, χ) plane, we see that it is a bounded entire function, and therefore Liouville's theorem implies it is a constant. The limits at the poles imply $C = 0$, thus showing that for $r > 0$, the localization locus is $\phi = 0$.

If $r \leq 0$ we get the following restriction. The general solution of (4.5) is $C(z) = \sum_{n \in \mathbb{Z}} C_n e^{-|b_1|(r+2n)z}$, where C_n are constants. Inserting this into (4.6), we see that for each $n \in \mathbb{Z}$, either $e^{\pi i |b_1|(r+2n)+\pi i r} = 1$ or $C_n = 0$. So there can be non-trivial solutions if and only if the R-charge r takes the very special form

$$r = -\frac{2|b_1|n + 2|b_2|m}{|b_1| + |b_2|} \leq 0 \quad , \quad n, m \in \mathbb{Z} . \tag{4.7}$$

Thus simply assuming that r is not one of the special values (4.7), the chiral multiplet localization locus is given by

$$F = \phi = 0 . \tag{4.8}$$

The full supersymmetric locus is thus completely characterized by the constant Lie algebra element \mathcal{A}_0 . Correspondingly, the path integral splits into a matrix integral over \mathcal{A}_0 , and a Gaussian integral over all the fluctuations about the saddle point locus (4.1), (4.8). Following a similar discussion in [32], we will now explain how to use the residual gauge freedom to extract the correct integration measure of the *matrix model*.

4.2 The matrix model

First of all, one can use constant gauge transformations to diagonalize \mathcal{A}_0 and reduce the integration to the Cartan subalgebra of the gauge group G , introducing a Vandermonde determinant

$$\Delta_0[\mathcal{A}_0] = \prod_{\alpha \in \Delta_+} (\alpha_{\mathcal{A}_0})^2 , \tag{4.9}$$

where Δ_+ denotes the set of positive roots and $\alpha_{\mathcal{A}_0} \equiv \alpha(\mathcal{A}_0)$. In a Cartan basis $\{H_k\}$ we have $\mathcal{A}_0 = \sum_{k=1}^{r_G} a_k H_k$, where r_G is the rank of the gauge group G . Then for a root $\alpha = \{\alpha_k\}$, we have $\alpha_{\mathcal{A}_0} = \sum_k a_k \alpha_k$. One also has to divide by the order of the Weyl group $|\mathcal{W}|$ in order to take care of gauge transformations that permute the elements of the Cartan basis.

Furthermore, the path integral must be invariant under large gauge transformations along the S^1 , that shift $\mathcal{A}_0 \rightarrow \mathcal{A}_0 + \sum_k d_k H_k$, where $d_k \in \mathbb{Z}$.²² Thus we can restrict the range of integration of the constants $\{a_k\}$ to be over the maximal torus T^{r_G} of G , parametrised by

$$z = \{z_k\} = \{e^{2\pi i a_k}\} \in T^{r_G} . \tag{4.10}$$

The localization argument then reduces the partition function to the form

$$Z = \frac{1}{|\mathcal{W}|} \int_{T^{r_G}} \frac{dz}{2\pi i z} \Delta_0[\mathcal{A}_0] Z_{\text{classic}}[\mathcal{A}_0] \tilde{Z}_{1\text{-loop}}^{\text{vector}}[\mathcal{A}_0] \prod_J Z_{1\text{-loop}}^{\text{chiral}(J)}[\mathcal{A}_0] , \tag{4.11}$$

²²We assume that the gauge field is normalized so that all the matter fields have integer charges.

where the integration measure $d\mathcal{A}_0$ has been replaced by

$$d\mathcal{A}_0 \equiv \prod_{k=1}^{r_G} da_k \rightarrow \frac{dz}{2\pi iz} \equiv \prod_{k=1}^{r_G} \frac{dz_k}{2\pi iz_k}. \quad (4.12)$$

Here $Z_{\text{classic}}[\mathcal{A}_0]$ is the classical contribution from the vector and chiral multiplets. However, for the theories that we consider, with Lagrangians (2.25), (2.38) (plus superpotential couplings), we have $Z_{\text{classic}} = e^{-S_{\text{classic}}} = 1$. The remaining factors $\tilde{Z}_{1\text{-loop}}^{\text{vector}}[\mathcal{A}_0]$ and $Z_{1\text{-loop}}^{\text{chiral}(J)}[\mathcal{A}_0]$ are the one-loop determinants of the vector multiplet and chiral multiplets fluctuations around the configurations (4.1) and (4.8).

Denoting by \mathcal{A}_τ and \mathcal{A}_i the components of the gauge field \mathcal{A}_μ along S^1 and M_3 , respectively, we will impose the following gauge-fixing conditions

$$\nabla_\tau a = 0, \quad \nabla^i \mathcal{A}_i = 0, \quad (4.13)$$

where $a \equiv \frac{1}{\text{vol}(M_3)} \int_{M_3} \mathcal{A}_\tau$. Let us discuss the first condition, while we will deal with the second condition later [33, 34]. The Faddeev-Popov determinant $\det'(\nabla^\tau D_\tau^{(0)})$ associated to $\nabla_\tau a = 0$ can be written in terms of ghost fields $\gamma, \bar{\gamma}$, yielding an integral over the following gauge-fixing term

$$S_a^{\text{gauge-fixing}} = \int d\tau \text{Tr} \left[\bar{\gamma} (\nabla^\tau D_\tau^{(0)}) \gamma + \xi \nabla_\tau a \right], \quad (4.14)$$

where $D_\tau^{(0)} = \nabla_\tau - i[\mathcal{A}_0, \cdot]$ and a prime on the determinant means that it does not contain the zero mode along S^1 . The second term is simply a rewriting of the delta function $\delta(\nabla_\tau a)$ enforcing the gauge-fixing condition, with ξ a Lagrange multiplier. The gauge fixing action (4.14) can be included in the deformation term by replacing $\delta V \rightarrow \delta' V'$, with $\delta' = \delta + \delta_B$, where δ_B is the BRST transformation, and $V' = V + \text{Tr} \bar{\gamma} \nabla_\tau a$ [35]. We refer to [1] for a more rigorous treatment of the ghosts.

Writing $a = \mathcal{A}_0 + \nabla_\tau \varphi$ and doing the path integral over φ introduces a Jacobian factor $(\det' \nabla_\tau^2)^{-1/2}$, which combined with the Faddeev-Popov determinant yields

$$\tilde{Z}_{1\text{-loop}}^{\text{vector}}[\mathcal{A}_0] = \Delta_2[\mathcal{A}_0] Z_{1\text{-loop}}^{\text{vector}}[\mathcal{A}_0], \quad (4.15)$$

where

$$\Delta_2[\mathcal{A}_0] \equiv \det' D_\tau^{(0)} = \prod_{\alpha \in \mathfrak{g}} \prod_{n \neq 0} (in - i\alpha \mathcal{A}_0), \quad (4.16)$$

and $\alpha \in \mathfrak{g}$ labels both non-zero roots and Cartan generators. A straightforward computation yields

$$\Delta_2[\mathcal{A}_0] = (2\pi)^{r_G} \prod_{\alpha \in \Delta_+} \frac{4 \sin^2(\pi \alpha \mathcal{A}_0)}{\alpha_{\mathcal{A}_0}^2}, \quad (4.17)$$

where we used the formula $\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$, and employed zeta function regularisation to regularise the infinite products. Finally, the matrix model becomes

$$Z = \frac{1}{|\mathcal{W}|} \int_{T^{r_G}} \frac{dz}{2\pi iz} \Delta_1[\mathcal{A}_0] Z_{1\text{-loop}}^{\text{vector}}[\mathcal{A}_0] \prod_J Z_{1\text{-loop}}^{\text{chiral}(J)}[\mathcal{A}_0], \quad (4.18)$$

with

$$\Delta_1[\mathcal{A}_0] = \Delta_0[\mathcal{A}_0] \Delta_2[\mathcal{A}_0] = (2\pi)^{r_G} \prod_{\alpha \in \Delta_+} 4 \sin^2(\pi \alpha_{\mathcal{A}_0}) . \quad (4.19)$$

4.3 One-loop determinants

Our strategy to compute the one-loop determinants on $S^1 \times M_3$ for the vector and chiral multiplets is to take advantage of the three-dimensional results²³ of [10]. First we expand the fields into Kaluza-Klein (KK) modes along the S^1 parametrized by τ . Denoting by Φ a generic field (bosonic or fermionic), we take

$$\Phi(x^i, \tau) = \sum_{n \in \mathbb{Z}} \Phi_n(x^i) e^{-in\tau} . \quad (4.20)$$

The four-dimensional one-loop determinant may be replaced by the product over one-loop determinants for the KK modes on M_3

$$Z_{1\text{-loop}}^{4d}[\Phi] = \prod_{n \in \mathbb{Z}} Z_{1\text{-loop}}^{3d}[\Phi_n] . \quad (4.21)$$

The one-loop determinants on M_3 were computed in [10] and our aim is to use the results therein for $Z_{1\text{-loop}}^{3d}[\Phi_n]$. For this to be possible we need to show that the Gaussian action for fluctuations around the localization locus, resulting from the deformation terms δV , matches the Gaussian action for the three-dimensional fluctuations of [10], with an appropriate mapping between fields. Instead of proving this directly, we will take an alternative route, which is to show that the four-dimensional supersymmetry transformations given by (2.23), (2.37) reduce under KK decomposition to the three-dimensional supersymmetry transformations of [10]. Then it will follow that the three-dimensional Gaussian actions for the KK multiplets are identical to the Gaussian actions of [10] by construction.

In order to proceed with the reduction to three dimensions, we need to relate the four-dimensional background fields to the three-dimensional ones. This analysis is presented in appendix F; the explicit relations between the four-dimensional background fields (A_μ, V_μ) and the three-dimensional background fields $(\check{A}_i, \check{V}_i, \check{h})$ are given in (F.23) (we use a $\check{}$ symbol to denote three-dimensional quantities). With our choice of real A_μ , the three-dimensional fields $\check{A}_i, \check{V}_i, \check{h}$ are also real, as it is assumed in [10].

4.3.1 Vector multiplet

We denote as \mathcal{B}_τ and \mathcal{B}_i the fluctuations of the gauge field \mathcal{A}_μ along S^1 and M_3 , respectively, $\sigma = \Omega^{-1} \mathcal{B}_\tau$ and consider the KK fields fluctuations $(\mathcal{B}_{nj}, \sigma_n, \lambda_n, \tilde{\lambda}_n, D_n)$ around the localization locus (4.1), where it is understood that $(\tilde{\lambda}_n)_\alpha = i(\sigma^4)_{\alpha\dot{\alpha}} \tilde{\lambda}_n^{\dot{\alpha}}$. The supersymmetry transformations (2.23) (with $\tilde{\zeta} = 0$) read for these KK fields

$$\begin{aligned} \delta \mathcal{B}_{nj} &= i\zeta \gamma_j \tilde{\lambda}_n , & \delta \sigma_n &= \zeta \tilde{\lambda}_n , \\ \delta \lambda_n &= -\frac{i}{2} \varepsilon^{ijk} \mathcal{F}_{nij} \gamma_k \zeta - i \left(\partial_j \sigma_n - i \check{V}_j \sigma_n + \frac{i}{\Omega} [\mathcal{A}_0, \mathcal{B}_{nj}] + \frac{i}{\Omega} n \mathcal{B}_{nj} \right) \gamma^j \zeta + (\check{D}_n - \check{h} \sigma_n) \zeta , \end{aligned}$$

²³Previous studies of relations between the index of four-dimensional gauge theories and the partition function in three dimensions include [36–40].

$$\delta\tilde{\lambda}_n = 0, \quad \delta\check{D}_n = -i\zeta\gamma^j \left(\check{\nabla}_j - i\check{A}_j + \frac{i}{2}\check{V}_j \right) \tilde{\lambda}_n + \frac{1}{2}\check{V}_j \zeta\gamma^j \tilde{\lambda}_n + \frac{i}{\Omega}\zeta[\mathcal{A}_0, \tilde{\lambda}_n] + \frac{i}{\Omega}n\zeta\tilde{\lambda}_n + \frac{\check{h}}{2}\zeta\tilde{\lambda}_n, \quad (4.22)$$

where we defined $\check{D}_n = iD_n + (\check{h} - \check{V}_\psi)\sigma_n$ and used the convention $\gamma^j = -i\sigma^4\check{\sigma}^j$ for the three-dimensional gamma matrices (see appendix F for more details about the 3d conventions).²⁴

These transformations correspond to the supersymmetry transformations of the three-dimensional $\mathcal{N} = 2$ vector multiplet fluctuations $(\mathcal{A}_j, \sigma, \lambda, \lambda^\dagger, D)_{3d}$ of [10] with respect to the three-dimensional spinor $\eta = \sqrt{2}\zeta$,²⁵ with the map

$$\left(\mathcal{B}_{nj}, \sigma_n, \sqrt{2}\lambda_n, \sqrt{2}\tilde{\lambda}_n, \check{D}_n \right) = \left(\mathcal{A}_j, -\sigma, \lambda^\dagger, \lambda, -D \right)_{3d}, \quad n + [\mathcal{A}_0, \cdot] = [\sigma_0, \cdot]. \quad (4.23)$$

The evaluation of the one-loop determinant is done by decomposing all KK fields, denoted generically Φ_n , into the Cartan basis of the gauge algebra

$$\Phi_n = \sum_{k=1}^{r_G} \Phi_{nk} H_k + \sum_{\alpha \in \text{roots}} \Phi_{n\alpha} E_\alpha, \quad (4.24)$$

where H_j generate the Cartan subalgebra and E_α are the ladder operators. The map (4.23) descends to the α -component multiplets, with

$$n + \alpha_{\mathcal{A}_0} = \alpha(\sigma_0). \quad (4.25)$$

The multiplets along the Cartan directions can be associated with “vanishing roots” $\alpha = 0$.

To be able to map the four-dimensional deformation terms to the three-dimensional ones, we note that on $S^1 \times M_3$ the deformation terms (2.27) and (2.33), expanded at quadratic order around the localization locus, are equal: $\delta_\zeta V^{(+)} = \delta_{\check{\zeta}} V^{(-)}$. For the fermionic part this is obvious, while for the bosonic part this follows from the identity

$$\text{Tr} \int_{S^1 \times M_3} \mathcal{F} \wedge \mathcal{F} = \text{Tr} \int_{S^1 \times M_3} d(\mathcal{B} \wedge d\mathcal{B} - 2i\mathcal{A}_0 \wedge \mathcal{B} \wedge \mathcal{B}) = 0. \quad (4.26)$$

Hence we have $\delta V_{4d} = -\frac{1}{2|\zeta|^2} \delta_\zeta (\text{Tr}(\delta_\zeta \lambda)^\dagger \lambda)$. In section 2.3.1, we saw that the reality conditions which, along with a real A_μ , ensure positivity of the bosonic deformation terms

²⁴In deriving the KK supersymmetry transformations, we have made use of the relation (F.24). We also point out the fact that the three-dimensional free parameter $\check{\kappa}$ of (F.23) drops from the supersymmetry transformations and does not affect the whole computation.

²⁵The authors of [10] performed localization using a spinor ϵ of positive charge under \check{A}_μ and wrote explicitly the supersymmetry transformations for ϵ . In our derivation, the relations between four-dimensional and three-dimensional background fields imply that the four-dimensional supersymmetry parameter ζ is mapped to a three-dimensional supersymmetry parameter η of negative charge under \check{A}_μ , see appendix F for details. Thus the supersymmetry transformations (4.22) are mapped to the three-dimensional supersymmetry transformations with respect to a negative charge spinor. These are not detailed in [10], but they can be derived from the ϵ transformations by changing (in our notations) $\epsilon_\alpha \rightarrow \eta_\alpha$, $(\check{A}_j, \check{V}_j, \mathcal{A}_j) \rightarrow -(\check{A}_j, \check{V}_j, \mathcal{A}_j)$ and $\tilde{\lambda} \leftrightarrow \lambda$ (also $\tilde{\Phi} \leftrightarrow \Phi$ for all fields for the chiral multiplet). They are also given in [27]. The fact that we have a negative charge spinor η in three dimensions does not prevent us from using the results of [10], since the localization computation is unchanged if η is used instead of ϵ .

are $\mathcal{A}_\mu^\dagger = \mathcal{A}_\mu$, $D^\dagger = -D$. For the fermions we choose $i\sigma_4 \tilde{\lambda} = \lambda^\dagger$. For the KK modes these translate into $\mathcal{A}_{n\mu} = \mathcal{A}_{-n\mu}^\dagger$, $D_n = -D_{-n}^\dagger$ and $\tilde{\lambda}_n = \lambda_{-n}^\dagger$.

Then, using the map (4.23) to three-dimensional fields, the Gaussian action for the n -th KK mode and α component fluctuations can be expressed as

$$\delta V_{(4d)}^{(n,\alpha)} = -\frac{1}{2|\zeta|^2} \text{Tr} \delta_\zeta \left((\delta_\zeta \lambda_{(n,\alpha)})^\dagger \lambda_{(n,\alpha)} \right) = -\frac{1}{4|\eta|^2} \text{Tr} \delta_\eta \left((\delta_\eta \lambda_{(\alpha)})^\dagger \lambda_{(\alpha)} \right)_{(3d)} = \delta V_{(3d)}[\sigma_0^{(n,\alpha)}], \quad (4.27)$$

where the action of \dagger on the KK modes is $\Phi_{(n,\alpha)}^\dagger = \Phi_{(-n,-\alpha)}$, and the constant scalar for the resulting three-dimensional deformation term is $\sigma_0^{(n,\alpha)} = n + \alpha_{\mathcal{A}_0}$. This three-dimensional deformation term is the same as the one considered in [10]. The reality conditions on the three-dimensional fields in α components obtained from this map are $\Phi_{(\alpha)}^{(3)} = \Phi_{(-\alpha)}^{(3)\dagger}$ for bosons and $\tilde{\lambda}_{(\alpha)}^{(3)} = \lambda_{(-\alpha)}^{(3)\dagger}$ for fermions, and match the reality conditions of [10]. Moreover, the three-dimensional gauge fixing condition $\nabla^j \mathcal{B}_j = 0$ chosen above becomes $\nabla^j \mathcal{A}_j^{(3)} = 0$, reproducing the gauge fixing condition of [10]. We can then use the result of [10] for the three-dimensional one-loop determinant for each (n, α) -component multiplet. Note that the contribution from the Faddeev-Popov determinant of the three-dimensional gauge fixing (namely the second in (4.13)) is included in the result of [10]. We obtain the expected relation

$$Z_{1\text{-loop}}^{\text{vector}}[\mathcal{A}_0] = \prod_{\alpha \in \mathfrak{g}} \prod_{n \in \mathbb{Z}} Z_{1\text{-loop}}^{\text{vector}}(3d)[\sigma_0^{(n,\alpha)}], \quad (4.28)$$

with $\sigma_0^{(n,\alpha)} = n + \alpha_{\mathcal{A}_0}$ and here $\alpha \in \mathfrak{g}$ labels both roots and Cartan components.

From [10], we extract

$$Z_{1\text{-loop}}^{\text{vector}}(3d)[\sigma_0^{(\alpha)}] = \frac{1}{i\alpha(\sigma_0)} \prod_{n_1, n_2 \geq 0} \frac{n_1 b_1 + n_2 b_2 + i\alpha(\sigma_0)}{-(n_1 + 1)b_1 - (n_2 + 1)b_2 + i\alpha(\sigma_0)}, \quad (4.29)$$

holding for $b_1, b_2 > 0$. A careful re-examination²⁶ of the three-dimensional one-loop computation in [10] shows that for arbitrary real b_1, b_2 , the one-loop determinant is given by the formula above with b_1, b_2 replaced by $|b_1|, |b_2|$.

Renaming $n \rightarrow n_0$, our one-loop determinant is expressed by the infinite product:

$$\begin{aligned} Z_{1\text{-loop}}^{\text{vector}} &= Z_{\text{Cartan}} \prod_{\alpha \in \text{roots}} \prod_{n_0 \in \mathbb{Z}} \left(\frac{1}{i(n_0 + \alpha_{\mathcal{A}_0})} \prod_{n_1, n_2 \geq 0} \frac{n_1 b_1 + n_2 b_2 + i(n_0 + \alpha_{\mathcal{A}_0})}{-(n_1 + 1)b_1 - (n_2 + 1)b_2 + i(n_0 + \alpha_{\mathcal{A}_0})} \right) \\ &= Z_{\text{Cartan}} \Delta_1^{-1} \prod_{\alpha \in \text{roots}} \left(\prod_{n_0 \in \mathbb{Z}} \prod_{n_1, n_2 \geq 0} \frac{n_1 b_1 + n_2 b_2 + i(n_0 + \alpha_{\mathcal{A}_0})}{-(n_1 + 1)b_1 - (n_2 + 1)b_2 + i(n_0 + \alpha_{\mathcal{A}_0})} \right). \end{aligned} \quad (4.30)$$

We see that the first factor cancels with the matrix model measure $\Delta_1[\mathcal{A}_0]$, while the second factor needs to be regularized. We perform this regularization in appendix G, using multiple Gamma functions. These manipulations yield the Jacobi theta function $\theta(z, p)$

²⁶We thank J. Sparks for discussions about this point.

and the Pochhammer symbol $(z; p)$, defined for $z, p \in \mathbb{C}$ and $|p| < 1$ respectively by

$$\theta(z, p) = \prod_{n \geq 0} (1 - zp^n) (1 - z^{-1}p^{n+1}), \quad (z; p) = \prod_{n \geq 0} (1 - zp^n). \quad (4.31)$$

The result is the following expression for the one-loop determinant

$$Z_{1\text{-loop}}^{\text{vector}} = e^{i\pi\Psi_{\text{vec}}^{(0)}} e^{i\pi\Psi_{\text{vec}}^{(1)}} (p; p)^{r_G} (q; q)^{r_G} \Delta_1^{-1} \prod_{\alpha \in \Delta_+} \theta(e^{2\pi i \alpha_{\mathcal{A}_0}}, p) \theta(e^{-2\pi i \alpha_{\mathcal{A}_0}}, q), \quad (4.32)$$

with

$$\Psi_{\text{vec}}^{(0)} = \frac{i}{12} \left(b_1 + b_2 - \frac{b_1 + b_2}{b_1 b_2} \right) |G|, \quad \Psi_{\text{vec}}^{(1)} = -i \frac{b_1 + b_2}{b_1 b_2} \sum_{\alpha \in \Delta_+} \alpha_{\mathcal{A}_0}^2, \quad (4.33)$$

where $p = e^{-2\pi b_1}$, $q = e^{-2\pi b_2}$, $|G|$ is the dimension of G , and we have split the prefactor into a part $\Psi_{\text{vec}}^{(0)}$ independent of $\alpha_{\mathcal{A}_0}$ and a part $\Psi_{\text{vec}}^{(1)}$ depending on $\alpha_{\mathcal{A}_0}$. This result looks puzzling, because the factor $e^{i\pi\Psi_{\text{vec}}^{(1)}}$ spoils the invariance under the shifts $\alpha_{\mathcal{A}_0} \rightarrow \alpha_{\mathcal{A}_0} + d$ for $d \in \mathbb{Z}$, associated to large gauge transformations $\mathcal{A}_0 \rightarrow \mathcal{A}_0 + \sum_k d_k H_k$, $d_k \in \mathbb{Z}$. In other words, $e^{i\pi\Psi_{\text{vec}}^{(1)}}$ is not a function of $z_\alpha = e^{2\pi i \alpha_{\mathcal{A}_0}}$ as it must be. For the final matrix model to be consistent, all such ‘‘anomalous’’ terms breaking the symmetry under large gauge transformations must cancel. We will see in section 5.1 that this is indeed what happens if the theory satisfies relevant physical constraints.

4.3.2 Chiral multiplet

The evaluation of the one-loop determinant for the chiral multiplet proceeds in a similar fashion. The KK fields $(\phi_n, \psi_n, F_n, \tilde{\phi}_n, \tilde{\psi}_n, \tilde{F}_n)$ all vanish on the localization locus (4.8), hence we can keep the same notations for their fluctuations around zero. The supersymmetry transformations (2.37), with respect to the spinor $\zeta = \frac{1}{\sqrt{2}}\eta$, and with the vector multiplet localized to (4.1), read for these KK fields:

$$\delta\phi_n = \eta\psi_n, \quad \delta\tilde{\phi}_n = 0, \quad \delta\psi_n = F_n\eta, \quad \delta F_n = 0, \quad (4.34)$$

$$\delta\tilde{\psi}_n = -i \left(\check{D}_j \tilde{\phi}_n + \frac{r}{2} \partial_j \log \Omega \right) \gamma^j \eta - \frac{i}{\Omega} (n + \mathcal{A}_0) \tilde{\phi}_n \eta - r \check{h} \tilde{\phi}_n \eta,$$

$$\delta\tilde{F}_n = i\eta\gamma^j \left(\check{D}_j - \frac{i}{2} \check{V}_j + \frac{r}{2} \partial_j \log \Omega \right) \tilde{\psi}_n - \frac{i}{\Omega} (n + \mathcal{A}_0) \eta \tilde{\psi}_n - \left(r - \frac{1}{2} \right) \check{h} \eta \tilde{\psi}_n, \quad (4.35)$$

with $\check{D}_j = \check{\nabla}_j + iq_R (\check{A}_j - \frac{1}{2} \check{V}_j)$ acting on a field fluctuation of R-charge q_R , and where $(\tilde{\psi}_n)_\alpha \equiv i(\sigma^4)_{\alpha\dot{\alpha}} \tilde{\psi}_n^{\dot{\alpha}}$. The match with the three-dimensional multiplet of [10] is given by

$$\begin{aligned} (\phi_n, -\psi_n, -iF_n, \tilde{\phi}_n, \tilde{\psi}_n, -i\tilde{F}_n) &= \Omega^{-r/2} (\phi, \psi, F, \phi^\dagger, \psi^\dagger, F)_{3d}, \\ n + \mathcal{A}_0 &= \sigma_0. \end{aligned} \quad (4.36)$$

The reality conditions ensuring the positivity of the four-dimensional deformation term are $\phi_n^\dagger = \tilde{\phi}_{-n}$ and $F_n^\dagger = -\tilde{F}_{-n}$ for bosons, while for the fermions we choose $\psi_n^\dagger = -\tilde{\psi}_{-n}$.

It follows that the Gaussian action around the locus solution for the n -th KK mode is

$$\delta V_{4d}^{(n)} = \delta_\zeta \left(\left[(\delta_\zeta \psi_n)^\dagger \psi_n - \tilde{\psi}_n (\delta_\zeta \tilde{\psi}_n)^\dagger \right] \right) = \delta_\eta \left(\left[(\delta_\eta \psi)^\dagger \psi + \psi^\dagger (\delta_\eta \psi^\dagger)^\dagger \right] \right)_{3d} = \delta V_{3d}[\sigma_0^{(n)}], \tag{4.37}$$

with $\sigma_0^{(n)} = n + \mathcal{A}_0$ and where we have dropped overall factors of Ω that can be cancelled by irrelevant redefinition of the deformation terms.²⁷ Again we recover the three-dimensional deformation term used in [10]. The reality conditions on three-dimensional fields following from our map are $(\Phi^{(3)\dagger})^\dagger = \Phi^{(3)}$ for bosons and $(\psi^{(3)\dagger})^\dagger = \psi^{(3)}$ for fermions, matching [10], so that we are able to use their three-dimensional one-loop determinant for each KK multiplet.

Decomposing the fields along the weight basis of their representation \mathcal{R} ,²⁸

$$\Phi_n = \sum_{\rho \text{ weight}} \Phi_{n,\rho}, \tag{4.38}$$

the 4d-3d map holds for the fields $\Phi_{(n,\rho)}$ with $\sigma_0^{(n,\rho)} = n + \rho_{\mathcal{A}_0}$, where $\rho_{\mathcal{A}_0} \equiv \rho(\mathcal{A}_0) = \sum_{k=1}^{r_G} \rho_k a_k$. We obtain the expected result

$$Z_{1\text{-loop}}^{\text{chiral}}[\mathcal{A}_0] = \prod_{\rho \in \text{weights}} \prod_{n \in \mathbb{Z}} Z_{1\text{-loop}}^{\text{chiral}}(3d)[\sigma_0^{(n,\rho)}], \tag{4.39}$$

where $\rho \in \text{weights}$ denotes a sum over the weights of the chiral multiplet representation \mathcal{R} . From [10], we extract the result (for $b_1, b_2 > 0$)

$$Z_{1\text{-loop}}^{\text{chiral}}(3d)[\sigma_0^{(\rho)}] = \prod_{n_1, n_2 \geq 0} \frac{n_1 b_1 + n_2 b_2 + i\rho(\sigma_0) - \frac{r-2}{2}(b_1 + b_2)}{n_1 b_1 + n_2 b_2 - i\rho(\sigma_0) + \frac{r}{2}(b_1 + b_2)}. \tag{4.40}$$

For arbitrary real b_1, b_2 , the one-loop determinant is given by the formula above with $|b_1|, |b_2|$ instead of b_1, b_2 .

Renaming $n \rightarrow n_0$, the one-loop determinant is

$$Z_{1\text{-loop}}^{\text{chiral}} = \prod_{\rho \in \text{weights}} \prod_{n_0 \in \mathbb{Z}} \prod_{n_1, n_2 \geq 0} \frac{\rho_{\mathcal{A}_0} + i\frac{r-2}{2}(b_1 + b_2) + n_0 - in_1 b_1 - in_2 b_2}{-\rho_{\mathcal{A}_0} - \frac{ir}{2}(b_1 + b_2) - n_0 - in_1 b_1 - in_2 b_2}, \tag{4.41}$$

Again the regularization of the infinite product is detailed in appendix G. This involves the elliptic gamma function, defined for $z, p, q \in \mathbb{C}$ and $|p| < 1, |q| < 1$ by

$$\Gamma_e(z, p, q) = \prod_{n_1, n_2 \geq 0} \frac{1 - z^{-1} p^{n_1+1} q^{n_2+1}}{1 - z p^{n_1} q^{n_2}}. \tag{4.42}$$

The result is

$$Z_{1\text{-loop}}^{\text{chiral}} = e^{i\pi\Psi_{\text{chi}}^{(0)}} e^{i\pi\Psi_{\text{chi}}^{(1)}} \prod_{\rho \in \Delta_{\mathcal{R}}} \Gamma_e \left(e^{2\pi i \rho_{\mathcal{A}_0}} (pq)^{\frac{r}{2}}, p, q \right), \tag{4.43}$$

²⁷See also appendix B, where an alternate way to see that Ω does not affect the result is given.

²⁸Note that the fields with a tilde transform in the complex conjugate representation \mathcal{R}^* , whose weights are opposite to the weights of \mathcal{R} .

with

$$\begin{aligned} \Psi_{\text{chi}}^{(0)} &= \frac{i}{24} \frac{b_1 + b_2}{b_1 b_2} \left[(r-1)^3 (b_1 + b_2)^2 - (r-1) (b_1^2 + b_2^2 + 2) \right] |\mathcal{R}|, \\ \Psi_{\text{chi}}^{(1)} &= \sum_{\rho \in \Delta_{\mathcal{R}}} -\frac{\rho_{\mathcal{A}_0}^3}{3 b_1 b_2} - i(r-1) \frac{b_1 + b_2}{2 b_1 b_2} \rho_{\mathcal{A}_0}^2 + [3(r-1)^2 (b_1 + b_2)^2 - 2 - b_1^2 - b_2^2] \frac{\rho_{\mathcal{A}_0}}{12 b_1 b_2}, \end{aligned} \tag{4.44}$$

where $p = e^{-2\pi b_1}$, $q = e^{-2\pi b_2}$, $\Delta_{\mathcal{R}}$ is the set of weights of the representation \mathcal{R} , and $|\mathcal{R}|$ is its dimension. As in the case of the vector multiplet, we have split the prefactor into a part $\Psi_{\text{chi}}^{(0)}$ independent of \mathcal{A}_0 , and an ‘‘anomalous’’ part $\Psi_{\text{chi}}^{(1)}$ carrying the inconsistent \mathcal{A}_0 dependence. To obtain a consistent result, we will require in the final matrix model that these ‘‘anomalous’’ terms vanish.

5 The partition function

In this section we present our final result for the exact partition function and compare it with the supersymmetric index. We find that the two quantities match, up to a prefactor that defines a Casimir energy for a supersymmetric gauge theory on a curved background.

5.1 Anomaly cancellations and the supersymmetric index

For the matrix model to be well-defined as an integral over the maximal torus T^{rG} , we have pointed out that the sum of the anomalous parts must cancel

$$\Psi_{\text{vec}}^{(1)}(\mathcal{A}_0) + \sum_J \Psi_{\text{chi},(J)}^{(1)}(\mathcal{A}_0) = 0, \tag{5.1}$$

where \sum_J is a sum over the chiral multiplets of the theory. From (4.33), (4.44), assuming arbitrary values of b_1, b_2 , this gives rise to four constraints on the gauge group and matter content of the theory.²⁹

$$\begin{aligned} \text{(i)} \quad & \sum_J \text{Tr}_{\mathcal{R}_J} (\mathcal{A}_0^3) = 0, \\ \text{(ii)} \quad & \text{Tr}_{\text{Adj}} (\mathcal{A}_0^2) + \sum_J (r_J - 1) \text{Tr}_{\mathcal{R}_J} (\mathcal{A}_0^2) = 0, \\ \text{(iii)} \quad & \sum_J (r_J - 1)^2 \text{Tr}_{\mathcal{R}_J} (\mathcal{A}_0) = 0, \\ \text{(iv)} \quad & \sum_J \text{Tr}_{\mathcal{R}_J} (\mathcal{A}_0) = 0, \end{aligned} \tag{5.2}$$

where Adj denotes the adjoint representation of the gauge group G . Using the Cartan decomposition $\mathcal{A}_0 = \sum_{k=1}^{rG} a_k H_k$, with $a_k \in \mathbb{R}$, and requiring (5.2) for all a_k leads to

$$\text{(i)} \quad \sum_J \text{Tr}_{\mathcal{R}_J} (H_{(k_1} H_{k_2} H_{k_3)}) = 0,$$

²⁹The translation into group theory language is the following: in a representation \mathcal{R} with weights $\{\rho^j\}$, the matrix representing \mathcal{A}_0 in a weight basis is $\mathcal{A}_0^{\mathcal{R}} = \text{diag}[\sum_k a_k \rho_k^j, 1 \leq j \leq |\mathcal{R}|] = \text{diag}[\rho_{\mathcal{A}_0}^j, 1 \leq j \leq |\mathcal{R}|]$. More generally $(\mathcal{A}_0^{\mathcal{R}})^n = \text{diag}[(\rho_{\mathcal{A}_0}^j)^n, 1 \leq j \leq |\mathcal{R}|]$ and the trace in the representation \mathcal{R} is $\text{Tr}_{\mathcal{R}} (\mathcal{A}_0^n) = \text{Tr}((\mathcal{A}_0^{\mathcal{R}})^n) = \sum_{j=1}^{|\mathcal{R}|} (\rho_{\mathcal{A}_0}^j)^n = \sum_{\rho \in \Delta_{\mathcal{R}}} (\rho_{\mathcal{A}_0})^n$.

$$\begin{aligned}
 \text{(ii)} \quad & \text{Tr}_{\text{Adj}}(H_{(k_1} H_{k_2)}) + \sum_J (r_J - 1) \text{Tr}_{\mathcal{R}_J}(H_{(k_1} H_{k_2)}) = 0, \\
 \text{(iii)} \quad & \sum_J (r_J - 1)^2 \text{Tr}_{\mathcal{R}_J}(H_k) = 0, \\
 \text{(iv)} \quad & \sum_J \text{Tr}_{\mathcal{R}_J}(H_k) = 0, \tag{5.3}
 \end{aligned}$$

where $k = 1, \dots, r_G$ for all k -indices. These conditions can all be interpreted in terms of vanishing of triangle Feynman diagrams contributing to various anomalies.³⁰ Condition (i) is implied by the requirement of the vanishing of the non-Abelian gauge anomaly; condition (ii) is implied by the vanishing of the ABJ anomaly, responsible for non-conservation of the R-symmetry current in an instanton background; condition (iii) holds requiring the vanishing of the mixed gauge-R symmetry anomaly $G \times \text{U}(1)_R^2$; condition (iv) is equivalent to the vanishing of the mixed gauge-gravitational anomaly. All these anomalies arise from chiral fermions with R-charge $r_J - 1$ in the \mathcal{R}_J representation. The contribution from the gauginos appears only in condition (ii), while it drops out from the other ones, because the adjoint representation is real.

All the conditions are necessary for the preservation of the dynamical gauge symmetry at the quantum level, in a generic background. Notice that the conditions (iii) and (iv) hold automatically when the gauge group G has no $\text{U}(1)$ factors. Moreover, the absence of the ABJ anomaly (condition (ii)), is equivalent to the vanishing of the NSVZ exact gauge beta functions of the theory [41, 42]. In particular, this is satisfied by all theories that flow to a SCFT in the infra-red (IR). However, one can also consider theories exhibiting confinement in the IR, obtained for instance by suitable superpotential deformations [16]. Pure $\mathcal{N} = 1$ super Yang-Mills (SYM) is an example of a theory for which the partition function (and hence the supersymmetric index) is ill-defined.

Gathering the results of the vector and chiral multiplets (4.33), (4.44), the partition function on $S^1 \times M_3$ is expressed by the exact formula

$$Z[\mathcal{H}_{p,q}] = e^{-\mathcal{F}(p,q)} \frac{(p;p)^{r_G} (q;q)^{r_G}}{|\mathcal{W}|} \int_{T^{r_G}} \frac{dz}{2\pi iz} \prod_{\alpha \in \Delta_+} \theta(z^\alpha, p) \theta(z^{-\alpha}, q) \prod_J \prod_{\rho \in \Delta_J} \Gamma_e(z^\rho (pq)^{\frac{r_J}{2}}, p, q), \tag{5.4}$$

where $z^{\pm\alpha} = e^{\pm 2\pi i \alpha \mathcal{A}_0}$, $z^\rho = e^{2\pi i \rho \mathcal{A}_0}$, J labels various chiral multiplets of R-charge r_J transforming in representation \mathcal{R}_J , Δ_J is the set of weights of \mathcal{R}_J , and

$$\begin{aligned}
 \mathcal{F}(p, q) &= \frac{\pi}{12} \left(|b_1| + |b_2| - \frac{|b_1| + |b_2|}{|b_1 b_2|} \right) \left(|G| + \sum_J (r_J - 1) |\mathcal{R}_J| \right) \\
 &+ \frac{\pi}{24} \frac{(|b_1| + |b_2|)^3}{|b_1 b_2|} \sum_J ((r_J - 1)^3 - (r_J - 1)) |\mathcal{R}_J| \\
 &= \frac{4\pi}{3} \left(|b_1| + |b_2| - \frac{|b_1| + |b_2|}{|b_1 b_2|} \right) (\mathbf{a} - \mathbf{c}) + \frac{4\pi}{27} \frac{(|b_1| + |b_2|)^3}{|b_1 b_2|} (3\mathbf{c} - 2\mathbf{a}), \tag{5.5}
 \end{aligned}$$

³⁰See [17] for a discussion of anomalies in relation to the supersymmetric index.

where in the second line we have used the following definitions

$$\begin{aligned} \mathbf{a} &= \frac{3}{32} (3 \operatorname{tr} R^3 - \operatorname{tr} R) = \frac{3}{32} \left[2|G| + \sum_J \left(3(r_J - 1)^3 - (r_J - 1) \right) |\mathcal{R}_J| \right], \\ \mathbf{c} &= \frac{1}{32} (9 \operatorname{tr} R^3 - 5 \operatorname{tr} R) = \frac{1}{32} \left[4|G| + \sum_J \left(9(r_J - 1)^3 - 5(r_J - 1) \right) |\mathcal{R}_J| \right], \end{aligned} \quad (5.6)$$

with R the R-symmetry charge and “tr” runs over the fermionic fields of the multiplets of the theory. When the theory flows to a fixed point, \mathbf{a} and \mathbf{c} are the central charges of the SCFT [8, 42, 43].

Comparing with the supersymmetric index $\mathcal{I}(p, q)$ with fugacities p, q given for instance in [40], we obtain the relation advertised in the introduction

$$Z[\mathcal{H}_{p,q}] = e^{-\mathcal{F}(p,q)} \mathcal{I}(p, q). \quad (5.7)$$

The partition function depends on the geometry of $S^1 \times M_3$ only through the complex structure parameters $p = e^{-2\pi|b_1|}$, $q = e^{-2\pi|b_2|}$, as predicted by [18]. More precisely, the authors of [18] have conjectured that the ratio $Z[\mathcal{H}_{p,q}]/\mathcal{I}(p, q) = e^{-\mathcal{F}(p,q)}$ can be set to one by a choice of local counterterms. However, by computing the partition function explicitly in a zeta function regularisation scheme, we have found that this ratio depends on the geometry only through the complex structure parameters, and thus generically it cannot be given in terms of integrals of densities local in the background fields. This is clear since generally such densities would depend on (functional) degrees of freedom in the metric.

Notice that for supersymmetric field theories defined on Hopf surfaces the integrated Weyl anomaly vanishes [9] and therefore the corresponding “logarithmic” term in the partition function, arising from conformal transformations of the functional measure [44], is absent. Thus (5.7) is the complete answer for the partition function.

In the remainder of this section we will discuss further the interpretation of $\mathcal{F}(p, q)$. Firstly, we will show that this plays a role in the reduction of the partition function to the partition function of a three-dimensional theory on M_3 , upon taking the limit of small S^1 . Following [40], the reduction along S^1 is performed by setting $b_1 = \beta \check{b}_1$, $b_2 = \beta \check{b}_2$, $\mathcal{A}_0 = \beta \sigma_0$ and taking the limit $\beta \rightarrow 0$ while keeping $\check{b}_1, \check{b}_2, \sigma_0$ fixed. In this limit the integration over T^{r_G} for $e^{2\pi i \mathcal{A}_0}$ becomes an integration over the Cartan sub-algebra \mathbb{R}^{r_G} for σ_0 . The limits of the various factors in the matrix model are discussed in [40], where it is shown that this reduces to the matrix model of the dimensionally reduced theory on M_3 . However, it was noticed that a potentially divergent overall factor appears in the reduction of the index $\mathcal{I}(p, q)$, given by

$$\exp \left[-\frac{\pi}{12\beta} \frac{|\check{b}_1| + |\check{b}_2|}{|\check{b}_1||\check{b}_2|} \left(|G| + \sum_J (r_J - 1) |\mathcal{R}_J| \right) \right] = \exp \left[\frac{4\pi}{3\beta} \frac{|\check{b}_1| + |\check{b}_2|}{|\check{b}_1||\check{b}_2|} (c - a) \right] \quad (5.8)$$

and this was dropped to recover the exact three-dimensional partition function. Our results imply that to complete the reduction one should take into account the contribution from the prefactor $e^{-\mathcal{F}(p,q)}$. The linear part in β vanishes when $\beta \rightarrow 0$ (we discuss this part below),

while the part proportional to $\frac{1}{\beta}$ precisely cancels (5.8). We conclude that the full four-dimensional partition function, computed using the regularization in appendix G, reduces to the exact three-dimensional partition function of the dimensionally reduced theory.

5.2 Supersymmetric Casimir energy and large N limit

We now discuss how the term linear in β appearing in \mathcal{F} may be interpreted as a Casimir energy, and then comment on the large N limit. In general, the vacuum energy of a field theory defined on $S^1 \times M_3$ may be defined from the path integral as

$$E_{\text{Casimir}} = - \lim_{\beta \rightarrow \infty} \frac{d}{d\beta} \log Z[\beta; M_3], \tag{5.9}$$

where one takes the limit of infinite radius of S^1 , keeping all other parameters fixed. Using this definition, our partition function computed with supersymmetric boundary conditions for the fermions gives:

$$E_{\text{susy}}(\check{b}_1, \check{b}_2) = \frac{4\pi}{3} (|\check{b}_1| + |\check{b}_2|) (\mathbf{a} - \mathbf{c}) + \frac{4\pi}{27} \frac{(|\check{b}_1| + |\check{b}_2|)^3}{|\check{b}_1||\check{b}_2|} (3\mathbf{c} - 2\mathbf{a}), \tag{5.10}$$

that we refer to as *supersymmetric Casimir energy*. This arises from the $\beta \rightarrow \infty$ limit of (5.5), and we used the fact that $\lim_{\beta \rightarrow \infty} \frac{d}{d\beta} \mathcal{I} = 0$. We see that E_{susy} depends on the complex structure parameters of the geometry, and on both the central charges \mathbf{a} and \mathbf{c} , characterising the field theory. Since the parameter β enters both in the $g_{\tau\tau}$ component of the metric and in V_τ , one can see that E_{susy} receives contributions both from the energy-momentum tensor and from the currents in the R-multiplet. When $p = q$, with $|b_1| = |b_2| \equiv \frac{\beta}{2\pi}$, this reduces to

$$E_{\text{susy}} = \frac{4}{27} (\mathbf{a} + 3\mathbf{c}), \tag{5.11}$$

which agrees³¹ with the expression for the “index Casimir energy” given in appendix B of [45]. The latter was defined as $\text{tr}[(-1)^F H]$, where H is the Hamiltonian commuting with the supercharges, and a particular supersymmetric regularisation was adopted. Extending to general p, q a prescription given therein for $p = q$, we find that our E_{susy} can be expressed in terms of the letter indices [12, 14, 25]

$$f_{\text{chiral}}(p, q) = \frac{(pq)^{\frac{r}{2}} - (pq)^{\frac{2-r}{2}}}{(1-p)(1-q)}, \quad f_{\text{vector}}(p, q) = \frac{2pq - p - q}{(1-p)(1-q)}, \tag{5.12}$$

with $p = e^{-2\pi\beta\check{b}_1}$, $q = e^{-2\pi\beta\check{b}_2}$, as

$$E_{\text{susy}}(\check{b}_1, \check{b}_2) = -\frac{1}{2} \lim_{\beta \rightarrow 0} \frac{d}{d\beta} \sum_{\text{all fields}} \left(f_{\text{chiral}}(p, q) + f_{\text{vector}}(p, q) \right) - \frac{4}{\pi\beta^2} \frac{|\check{b}_1| + |\check{b}_2|}{|\check{b}_1||\check{b}_2|} (\mathbf{a} - \mathbf{c}), \tag{5.13}$$

where the finite part reproduces E_{susy} and the $O(\beta^{-2})$ term is proportional to $\mathbf{a} - \mathbf{c}$.

³¹Up to a factor of 2/3 noted in [22]. Note that (5.11) holds for an *arbitrary* metric on $M_3 \simeq S^3$, as anticipated in [22].

In order to compare our $E_{\text{susy}}(\check{b}_1, \check{b}_2)$ with other Casimir energies in the literature we should restrict to the sub-space $p = q$, and assume that the metric is the round one on $S^1 \times S^3$. In this case, it was shown in [46, 47], that in a *conformal field theory* (not necessarily supersymmetric) the Casimir energy, defined as

$$E_0 = \int_{S^3} \langle T^{00} \rangle \text{vol}(S^3), \tag{5.14}$$

is proportional to the trace anomaly coefficient \mathbf{a} , namely

$$E_0 = \frac{3}{4} \mathbf{a} \quad \text{in a CFT.} \tag{5.15}$$

Note that this result is valid for an arbitrary CFT, where \mathbf{a} and \mathbf{c} are not necessarily related. For an $\mathcal{N} = 1$ SCFT defined on the round $S^1 \times S^3$, when both can be computed, E_0 and E_{susy} are two *different* measures of the vacuum energy of a theory.

Notice that in the particular case of $\mathcal{N} = 4$ SYM theory on $S^1 \times S^3$ with $G = \text{SU}(N)$, the Casimir energy, can be computed in the *free field* limit [48] and agrees with E_0 , while it differs from E_{susy} by a numerical factor, namely

$$E_{\text{free}} = \frac{3(N^2 - 1)}{16} = E_0, \quad E_{\text{susy}} = \frac{4(N^2 - 1)}{27} \quad \text{for } \mathcal{N} = 4 \text{ SYM.} \tag{5.16}$$

Although E_{susy} is valid for any value of the coupling constant (and for any N) and in particular at weak coupling in the $\mathcal{N} = 4$ SYM theory, a priori it does not have to coincide with E_{free} or E_0 . It would be interesting to understand precisely the relationships between these Casimir energies.

Finally, let us discuss the implications of our results for field theories that admit a gravity dual. For concreteness, we will now assume that the gauge theory is a quiver, with gauge group $G = \text{SU}(N)^k$ and chiral fields transforming in bi-fundamental representations $(\mathbf{N}, \bar{\mathbf{N}})$. We also assume that there is a non-trivial superpotential, and that the theory flows to an interacting fixed point in the IR, with $\mathbf{a} = \mathbf{c} + \mathcal{O}(1) = \mathcal{O}(N^2)$, in the limit $N \rightarrow \infty$. These theories are expected to admit a gravity dual solution in type IIB supergravity with geometry $M_5 \times Y_5$, where Y_5 is a Sasaki-Einstein manifold [49] and M_5 is a deformation of AdS_5 , supported by N units of five-form flux. Moreover, it should be possible to construct such solutions within the consistent truncation to minimal gauged supergravity and then uplift these to ten dimensions, as illustrated in [22]. In these cases, at leading order in a large N expansion, the prefactor (5.5) in the partition function simplifies to

$$\mathcal{F}(p, q) = \frac{4\pi}{27} \frac{(|b_1| + |b_2|)^3}{|b_1||b_2|} \mathbf{a}, \tag{5.17}$$

and using the AdS/CFT relation $\exp(-S_{\text{gravity}}[M_5]) = Z_{\text{QFT}}[\partial M_5]$, we obtain the following prediction for the five-dimensional holographically renormalised on-shell action:

$$S_{5\text{d sugra}}[M_5] = \frac{\pi^2}{54G_5} \frac{(|b_1| + |b_2|)^3}{|b_1||b_2|}. \tag{5.18}$$

Here we used the relation $\mathbf{a} = \mathbf{c} = \frac{\pi \ell^3}{8G_5}$ (at leading order in N), with G_5 denoting the Newton constant of the five-dimensional supergravity, and we have set the AdS₅ radius $\ell = 1$.

In the solution of [22] this formula was found valid, up to some local counterterms. In particular, in that solution $p = q = e^{-\beta}$, albeit the boundary metric comprises a biaxially squashed three-sphere (see appendix C) and hence it is not conformally flat. In the case of a solution of the form AdS₅ × Y₅, the expression (5.18) reduces to $S_{5\text{d sugra}}[\text{AdS}_5] = \frac{2\pi\beta}{27G_5}$ and again this should be contrasted with the computation in [48], giving $S_{5\text{d sugra}}[\text{AdS}_5] = \frac{3\pi\beta}{32G_5}$. When Y₅ = S⁵ the latter agrees with the large N limit of $E_{\text{free}} = E_{\text{CFT}}$ above, while the former gives a different value. We expect that this difference can be traced to the use of different holographic regularisation procedures. However, this interesting problem deserves to be studied in a future occasion.

Finally, it is tantalizing to compare (5.18) with analogous formulae for the on-shell actions in the case of four-dimensional and six-dimensional gauged supergravities,³²

$$S_{4\text{d sugra}}[M_4] = \frac{\pi}{8G_4} \frac{(|b_1| + |b_2|)^2}{|b_1||b_2|}, \quad S_{6\text{d sugra}}[M_6] = \frac{\pi^2}{4G_6} \frac{(|b_1| + |b_2| + |b_3|)^3}{|b_1||b_2||b_3|}, \quad (5.19)$$

put forward in [24] and [50], respectively. Here we simply note that these are expressions for the holographically renormalised on-shell action of supersymmetric solutions dual to field theories defined on backgrounds with topology of S³ and S⁵, respectively, referring to [24] and [50] for more details.

6 Conclusions

In this paper we have computed the partition function of $\mathcal{N} = 1$ supersymmetric gauge theories — comprising a vector multiplet for a general gauge group, chiral multiplets with generic R-charges and possibly a superpotential — defined on a primary Hopf surface $\mathcal{H}_{p,q}$. We have found that this depends on the background only through the complex structure moduli p, q of the Hopf surface, and is proportional to the supersymmetric index $\mathcal{I}(p, q)$ with fugacities p, q . We have carried out the computation reducing the path integral to a matrix integral over the holonomy of the gauge field around S¹, and evaluating explicitly the one-loop determinant using the method developed in [10].

Our result is essentially in agreement with the conjecture made in [18], but we have also determined the proportionality factor $e^{-\mathcal{F}}$ by performing a careful regularisation of the infinite products, employing generalised zeta function techniques. This factor defines a *supersymmetric Casimir energy*, depending on the anomaly coefficients \mathbf{a}, \mathbf{c} and containing the leading contribution of $\log Z$ in the large N limit. We believe that this term cannot be expressed as a supersymmetric local counterterm and therefore it should be independent of the details of the regularisation scheme. We plan to investigate this further, for example by classifying the possible supersymmetric counterterms.

³²The second formula was verified in several explicit examples in [50], and conjectured to hold for general solutions with the topology of the six-ball. In [50] it is presented in terms of positive coefficients b_1, b_2, b_3 , parameterising a contact structure on the five-sphere.

Perhaps a related question is that of clarifying the dependence of the partition function on the function κ , parametrising the freedom in choosing the background fields A_μ, V_μ [18]. Throughout this paper we have worked with the specific choice of κ in (2.17), dictated by requiring that A_μ is *real*. The general arguments presented in [18] imply that the partition function should not depend on κ , at least when the path integral is well defined. However, for a generic choice of κ the Lagrangian (2.38) does not have positive-definite bosonic part, so that the localization arguments become more formal. It would be nice to analyse the dependence on κ more explicitly.

There are several directions for future work. It would be interesting to apply our method to compute other BPS observables, such as a supersymmetric Wilson loop. It should also be possible to prove factorisation of the index [21, 51] using a generalisation of the arguments in section 5.2 of [10]. As a simple generalisation of our analysis, it should be possible to consider non-direct-product metrics, thus allowing for general complex parameters p, q (see appendix D). A more challenging extension is that of performing a localization computation on Hermitian manifolds with different topologies, requiring only the existence of one supercharge.

One of the motivations for this work was to clarify the results of [22], by obtaining a precise prediction for the holographically renormalised on-shell action in five-dimensional gauged supergravity, which we presented in (5.18). It would be interesting to reproduce this formula directly from the dual gravitational perspective. We have noted that in dimensions four, five, and six, the relevant on-shell actions appear to follow a precise pattern, and we expect that explaining this will improve our general understanding of the gauge/gravity duality.

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A Conventions and identities

In this appendix we spell out our conventions and give some identities, useful for the computations in the main text.

Our spinor conventions are as in [7]. A two-component notation is used: left-handed spinors carry an undotted index, as ζ_α , $\alpha = 1, 2$, while right-handed spinors are denoted by a tilde and carry a dotted index, as $\tilde{\zeta}^{\dot{\alpha}}$. These transform in the $(\mathbf{2}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2})$ representations of $\text{Spin}(4) = \text{SU}(2)_+ \times \text{SU}(2)_-$, respectively. The Hermitian conjugate spinors have index structure

$$(\zeta^\dagger)^\alpha = (\zeta_\alpha)^*, \quad (\tilde{\zeta}^\dagger)_{\dot{\alpha}} = (\tilde{\zeta}^{\dot{\alpha}})^*, \tag{A.1}$$

and the spinor norms are given by $|\zeta|^2 = \zeta^\dagger \zeta$ and $|\tilde{\zeta}|^2 = \tilde{\zeta}^\dagger \tilde{\zeta}$.

The Clifford algebra is generated by 2×2 sigma matrices

$$\sigma_{\alpha\dot{\alpha}}^a = (\vec{\sigma}, -i\mathbb{1}_2), \quad \tilde{\sigma}^{a\dot{\alpha}\alpha} = (-\vec{\sigma}, -i\mathbb{1}_2), \quad (\text{A.2})$$

where $a = 1, \dots, 4$ is a frame index, and $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices. The generators of $SU(2)_+$ and $SU(2)_-$ are given by

$$\sigma_{ab} = \frac{1}{4}(\sigma_a \tilde{\sigma}_b - \sigma_b \tilde{\sigma}_a), \quad \tilde{\sigma}_{ab} = \frac{1}{4}(\tilde{\sigma}_a \sigma_b - \tilde{\sigma}_b \sigma_a), \quad (\text{A.3})$$

and satisfy

$$\frac{1}{2}\epsilon_{abcd}\sigma^{cd} = \sigma_{ab}, \quad \frac{1}{2}\epsilon_{abcd}\tilde{\sigma}^{cd} = -\tilde{\sigma}_{ab}, \quad (\text{A.4})$$

with $\epsilon_{1234} = 1$, namely they are self-dual and anti-self-dual, respectively. The sigma matrices have the following hermiticity properties

$$(\sigma_a)^\dagger = -\tilde{\sigma}_a, \quad (\sigma_{ab})^\dagger = -\sigma_{ab}, \quad (\tilde{\sigma}_{ab})^\dagger = -\tilde{\sigma}_{ab}, \quad (\text{A.5})$$

and satisfy the relations

$$\begin{aligned} \sigma_a \tilde{\sigma}_b + \sigma_b \tilde{\sigma}_a &= -2\delta_{ab}, & \tilde{\sigma}_a \sigma_b + \tilde{\sigma}_b \sigma_a &= -2\delta_{ab}, \\ \sigma_a \tilde{\sigma}_b \sigma_c &= -\delta_{ab}\sigma_c + \delta_{ac}\sigma_b - \delta_{bc}\sigma_a + \epsilon_{abcd}\sigma^d, \\ \tilde{\sigma}_a \sigma_b \tilde{\sigma}_c &= -\delta_{ab}\tilde{\sigma}_c + \delta_{ac}\tilde{\sigma}_b - \delta_{bc}\tilde{\sigma}_a - \epsilon_{abcd}\tilde{\sigma}^d, \\ \sigma_{ab}\sigma_{cd} &= \frac{1}{4}(-\epsilon_{abcd} - 2\delta_{ad}\sigma_{bc} + 2\delta_{ac}\sigma_{bd} - 2\delta_{bc}\sigma_{ad} + 2\delta_{bd}\sigma_{ac} - \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}), \\ \tilde{\sigma}_{ab}\tilde{\sigma}_{cd} &= \frac{1}{4}(+\epsilon_{abcd} - 2\delta_{ad}\tilde{\sigma}_{bc} + 2\delta_{ac}\tilde{\sigma}_{bd} - 2\delta_{bc}\tilde{\sigma}_{ad} + 2\delta_{bd}\tilde{\sigma}_{ac} - \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}). \end{aligned} \quad (\text{A.6})$$

Our supersymmetry parameters $\zeta, \tilde{\zeta}$ are commuting spinors, with the supersymmetry variation $\delta_\zeta, \delta_{\tilde{\zeta}}$ being Grassmann-odd operators; on the other hand, the dynamical spinor fields are assumed anti-commuting. The spinor indices are raised or lowered acting from the left with the antisymmetric symbol $\varepsilon^{\alpha\beta} = -\varepsilon_{\alpha\beta} = \varepsilon^{\dot{\alpha}\dot{\beta}} = -\varepsilon_{\dot{\alpha}\dot{\beta}}$, chosen such that $\varepsilon^{12} = +1$. When constructing a spinor bilinear, the indices are contracted as $\zeta\chi = \zeta^\alpha\chi_\alpha$ and $\tilde{\zeta}\tilde{\chi} = \tilde{\zeta}_{\dot{\alpha}}\tilde{\chi}^{\dot{\alpha}}$. Then one has the following relations for commuting spinors

$$\begin{aligned} \zeta\chi &= -\chi\zeta, & \tilde{\zeta}\tilde{\chi} &= -\tilde{\chi}\tilde{\zeta}, \\ \zeta\sigma_a\tilde{\chi} &= \tilde{\chi}\tilde{\sigma}_a\zeta, & \zeta\sigma_{ab}\chi &= \chi\sigma_{ab}\zeta, \\ (\sigma_a\tilde{\zeta})\chi &= -\tilde{\zeta}\tilde{\sigma}_a\chi, & (\sigma_{ab}\zeta)\chi &= -\zeta\sigma_{ab}\chi, \\ (\zeta\chi)^\dagger &= \chi^\dagger\zeta^\dagger, & (\tilde{\zeta}\tilde{\chi})^\dagger &= \tilde{\chi}^\dagger\tilde{\zeta}^\dagger, \\ (\zeta\sigma_a\tilde{\chi})^\dagger &= -\tilde{\chi}^\dagger\tilde{\sigma}_a\zeta^\dagger, & (\zeta\sigma_{ab}\chi)^\dagger &= -\chi^\dagger\sigma_{ab}\zeta^\dagger, \end{aligned} \quad (\text{A.7})$$

as well as the Fierz identities

$$\begin{aligned} (\zeta\chi)(\tilde{\zeta}\tilde{\chi}) &= -\frac{1}{2}(\zeta\sigma_a\tilde{\chi})(\chi\sigma^a\tilde{\zeta}), \\ (\chi_1\chi_2)(\chi_3\chi_4) &= -(\chi_1\chi_3)(\chi_4\chi_2) - (\chi_1\chi_4)(\chi_2\chi_3). \end{aligned} \quad (\text{A.8})$$

When the spinors are anti-commuting one has to include an extra minus sign whenever the relation involves swapping two of them.

The spinor covariant derivative is given by

$$\nabla_\mu \zeta = \partial_\mu \zeta - \frac{1}{2} \omega_{\mu ab} \sigma^{ab} \zeta, \quad \nabla_\mu \tilde{\zeta} = \partial_\mu \tilde{\zeta} - \frac{1}{2} \omega_{\mu ab} \tilde{\sigma}^{ab} \tilde{\zeta}, \quad (\text{A.9})$$

where $\omega_{\mu ab}$ is the spin connection, defined from the vielbein e^a_μ and its inverse e^μ_a as

$$\omega_\mu^{ab} = 2 e^{\nu[a} \partial_{[\mu} e^{b]}_{\nu]} - e^{\nu[a} e^{b]\rho} e_{c\mu} \partial_\nu e^d_\rho. \quad (\text{A.10})$$

From the spin connection we can construct the Riemann tensor via³³

$$R_{\mu\nu ab} = \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} + \omega_{\mu a}^c \omega_{\nu cb} - \omega_{\nu a}^c \omega_{\mu cb}. \quad (\text{A.11})$$

The integrability condition of the supersymmetry equation (2.1) implies the following relations

$$\begin{aligned} (R + 6V^\mu V_\mu) \zeta &= 4i (\partial_\mu A_\nu - \partial_\nu A_\mu) \sigma^{\mu\nu} \zeta, \\ R + 6V^\mu V_\mu &= 2J^{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu). \end{aligned} \quad (\text{A.12})$$

The first is derived using $[\nabla_\mu, \nabla_\nu] \zeta = -\frac{1}{2} R_{\mu\nu ab} \sigma^{ab} \zeta$, contracted with $\sigma^{\mu\nu}$, and implies the second.

B Weyl transformations

In this appendix we discuss how the supersymmetry transformations and Lagrangians are affected by a conformal rescaling of the geometry and of the dynamical fields, in the case when there exist two supercharges of opposite R-charge. This will explicitly show that the conformal factor Ω can be rescaled away from the localizing terms, and therefore does not affect the result of the computation of the one-loop determinants.

We consider a Weyl rescaling of the general metric (2.9),

$$g_{\mu\nu} = \Lambda^2 \hat{g}_{\mu\nu}, \quad (\text{B.1})$$

corresponding to redefining the conformal factor Ω as

$$\Omega = \Lambda \hat{\Omega}, \quad (\text{B.2})$$

where here and below a hat denotes the transformed quantities. We assume that Λ is a real, positive function depending on z, \bar{z} only, so that rescaled background still admits two supercharges of opposite R-charge. If Λ is chosen equal to Ω , then the conformal factor of the new metric is simply $\hat{\Omega} = 1$. The vielbein and the spin connection transform as

$$e^a_\mu = \Lambda \hat{e}^a_\mu, \quad \omega_{\mu ab} = \hat{\omega}_{\mu ab} + \hat{e}^c_\mu (\delta_{ca} \hat{e}^\nu_b - \delta_{cb} \hat{e}^\nu_a) \partial_\nu \log \Lambda, \quad (\text{B.3})$$

³³Our spin connection and Riemann tensor differ by a sign from those of [7] (so our Ricci scalar is positive on a round sphere).

while the two-form $J_{\mu\nu}$ transforms in the same way as the metric, $J_{\mu\nu} = \Lambda^2 \hat{J}_{\mu\nu}$, and the complex structure $J^\mu{}_\nu$ remains invariant. As a vector, K is invariant, while as a one-form it transforms as $K_\mu = \Lambda^2 \hat{K}_\mu$. Starting from (2.15), (2.16), we can now deduce how the background fields A and V transform. We will also assign a weight to $|s|$ and κ ,

$$|s| = \Lambda |\hat{s}|, \quad \kappa = \Lambda^{-2} \hat{\kappa}, \quad (\text{B.4})$$

so that both the imaginary part of A and the one of V remain invariant.³⁴ Note that these conditions are consistent with those ensuring that A is real, given in (2.17). Then from (2.15) and (2.16) we obtain

$$V_\mu = \hat{V}_\mu + (d^c \log \Lambda)_\mu, \quad A_\mu = \hat{A}_\mu + \frac{3}{2} (d^c \log \Lambda)_\mu, \quad (\text{B.5})$$

where $(d^c \log \Lambda)_\mu = J_\mu{}^\nu \partial_\nu \log \Lambda = -\tilde{J}_\mu{}^\nu \partial_\nu \log \Lambda$. Finally, from (2.14) we see that the spinors transform as

$$\zeta = \Lambda^{1/2} \hat{\zeta}, \quad \tilde{\zeta} = \Lambda^{1/2} \hat{\tilde{\zeta}}. \quad (\text{B.6})$$

We now consider the variations of the fields in the supersymmetry multiplets, showing that these are covariant if the Weyl transformation is accompanied by suitable rescaling of the fields. Let us start with the gauge multiplet, where we assign the standard conformal weights

$$\mathcal{A}_\mu = \hat{\mathcal{A}}_\mu, \quad \lambda = \Lambda^{-3/2} \hat{\lambda}, \quad \tilde{\lambda} = \Lambda^{-3/2} \hat{\tilde{\lambda}}, \quad D = \Lambda^{-2} \hat{D}. \quad (\text{B.7})$$

It is easy to see that the supersymmetry variations (2.23) transform covariantly as

$$\delta \mathcal{A}_\mu = \hat{\delta} \hat{\mathcal{A}}_\mu, \quad \delta \lambda = \Lambda^{-3/2} \hat{\delta} \hat{\lambda}, \quad \delta \tilde{\lambda} = \Lambda^{-3/2} \hat{\delta} \hat{\tilde{\lambda}}, \quad \delta D = \Lambda^{-2} \hat{\delta} \hat{D}, \quad (\text{B.8})$$

where the variation $\hat{\delta}$ uses $\hat{\zeta}$, $\hat{\tilde{\zeta}}$, and is done on the transformed background defined by $\hat{g}_{\mu\nu}$, \hat{V} and \hat{A} . The only non-trivial check is for the relation involving D : this follows using the fact that the $A_\mu^{\text{cs}} = A_\mu - \frac{3}{2} V_\mu$ is invariant under the Weyl transformation, and the following identity

$$\tilde{\zeta} \tilde{\sigma}^\mu \nabla_\mu \lambda \equiv \Lambda^{-1/2} \hat{\tilde{\zeta}} \hat{\tilde{\sigma}}^\mu (\hat{\nabla}_\mu - \hat{\sigma}_\mu{}^\nu \partial_\nu \log \Lambda) (\Lambda^{-3/2} \hat{\lambda}) = \Lambda^{-2} \hat{\tilde{\zeta}} \hat{\tilde{\sigma}}^\mu \hat{\nabla}_\mu \hat{\lambda}, \quad (\text{B.9})$$

where we used $\tilde{\sigma}^a \sigma_{ab} = -\frac{3}{2} \tilde{\sigma}_b$.

It is also easy to see that the localizing terms, as well as the Lagrangian (2.25) for the vector multiplet scale as Λ^{-4} , so that the action is invariant, namely

$$\int d^4x \sqrt{g} \mathcal{L}_{\text{vector}} = \int d^4x \sqrt{\hat{g}} \hat{\mathcal{L}}_{\text{vector}}. \quad (\text{B.10})$$

We then pass to the chiral multiplet, whose supersymmetry variations were given in (2.37). For the scalar ϕ we take $\phi = \Lambda^{-w} \hat{\phi}$, and choose the conformal weight w such as

³⁴The transformation of $|s|$ is necessary to make sure that the spinors transform correctly and that the imaginary part of A does not transform. The transformation of κ is imposed for simplicity: as explained in section 2 any choice of $U_\mu = \kappa K_\mu$ drops from the supersymmetry variations and the localizing terms, as long as one defines $A_\mu^\dagger = A_\mu^\dagger$.

$w = 3r/2$. The conformal weight of $\psi, \tilde{\psi}$ is $w + 1/2$, while the one of F, \tilde{F} is $w + 1$. Again, one can show that the supersymmetry variations are covariant under the rescaling, namely

$$\delta\phi = \Lambda^{-w} \hat{\delta}\hat{\phi}, \quad \delta\psi = \Lambda^{-w-1/2} \hat{\delta}\hat{\psi}, \quad \delta F = \Lambda^{-w-1} \hat{\delta}\hat{F}, \quad (\text{B.11})$$

with exactly the same relations for $\tilde{\phi}, \tilde{\psi}$ and \tilde{F} . While this is straightforward for the variation of ϕ , it is less obvious for the others. For instance, in the variation of $\tilde{\psi}$ in (2.37) we have

$$\begin{aligned} \tilde{\sigma}^\mu \zeta D_\mu \tilde{\phi} &= \Lambda^{-1/2} \tilde{\sigma}^\mu \hat{\zeta} \left(\hat{D}_\mu + \frac{3}{2} i r J_\mu^\nu \partial_\nu \log \Lambda \right) \left(\Lambda^{-w} \tilde{\phi} \right) \\ &= \Lambda^{-w-1/2} \tilde{\sigma}^\mu \hat{\zeta} \left[\hat{D}^\mu \tilde{\phi} - \left(w \delta^\mu_\nu - \frac{3}{2} r i J^\mu_\nu \right) \partial^\nu \log \Lambda \tilde{\phi} \right]. \end{aligned} \quad (\text{B.12})$$

Since we set $w = \frac{3}{2}r$, the second term vanishes because the vector $X^\mu = (\delta^\mu_\nu - iJ^\mu_\nu) \partial^\nu \log \Lambda$ is holomorphic, and therefore satisfies $X^\mu \tilde{\sigma}_\mu \zeta = 0$. We can now discuss how the localizing term $\delta_\zeta(V_1 + V_2)$ for the chiral multiplet transforms. Given that this is constructed as a combination of supersymmetry variations, it is also covariant under the Weyl transformation. Specifically, it transforms as

$$\delta_\zeta(V_1 + V_2) = \Lambda^{-2w-2} \hat{\delta}_\zeta(\hat{V}_1 + \hat{V}_2). \quad (\text{B.13})$$

Now consider taking $\Lambda = \Omega$, so that $\hat{\Omega} = 1$. If as a localizing term we consider the following *modified* integral weighted by the suitable power of Ω

$$\int d^4x \sqrt{g} \Omega^{2w-2} \delta_\zeta(V_1 + V_2), \quad (\text{B.14})$$

then we see that this is precisely equal to the original localizing term, in a background with $\Omega = 1$, namely

$$\int d^4x \sqrt{g} \Omega^{2w-2} \delta_\zeta(V_1 + V_2) = \int d^4x \sqrt{\hat{g}} \hat{\delta}_\zeta(\hat{V}_1 + \hat{V}_2). \quad (\text{B.15})$$

In this way the background dependence on Ω in the localizing term can be reabsorbed by a redefinition of the dynamical fields.

In conclusion, we have shown that the localizing terms on the left hand side of (B.10) and (B.15) are equivalent upon rescaling the dynamical fields to the same localizing terms defined on a background having $\Omega = 1$. This is in agreement with the results of [18].

C $S^1 \times S_v^3$ with arbitrary b_1, b_2

In this appendix we apply the formulae of section 3.2 in a familiar example. We will consider a geometry comprising the Berger sphere S_v^3 , namely the biaxially squashed three-sphere with $SU(2) \times U(1)$ isometry and squashing parameter v . For any value of v , this yields a family of four-dimensional supersymmetric backgrounds $S^1 \times S_v^3$, depending on the two parameters b_1 and b_2 which define the Killing vector (3.6). The results of the present paper show that the partition function depends on b_1, b_2 , and not on v . A similar construction

of three-dimensional backgrounds, obtained from a dual holographic perspective, has been presented in [24].

We take a four-dimensional metric

$$ds^2 = \Omega^2 d\tau^2 + ds^2(S_v^3), \tag{C.1}$$

where the metric on the Berger sphere in standard form is

$$ds^2(S_v^3) = d\theta^2 + \sin^2 \theta d\varphi^2 + v^2(d\varsigma + \cos \theta d\varphi)^2, \tag{C.2}$$

with $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$, $\varsigma \in [0, 4\pi]$, and $v > 0$ being the squashing parameter. This can be written in the toric form (3.3) by changing coordinates as

$$\varphi = \varphi_1 + \varphi_2, \quad \varsigma = \varphi_1 - \varphi_2. \tag{C.3}$$

Identifying $\theta = \pi\rho$, so that $f = \pi$, the matrix m_{IJ} reads

$$\begin{aligned} m_{11} &= 4 \cos^2 \frac{\theta}{2} \left(\sin^2 \frac{\theta}{2} + v^2 \cos^2 \frac{\theta}{2} \right), & m_{12} &= (1 - v^2) \sin^2 \theta, \\ m_{22} &= 4 \sin^2 \frac{\theta}{2} \left(v^2 \sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} \right). \end{aligned} \tag{C.4}$$

Given the choice of Killing vector K in (3.6), the supersymmetry condition $K_\mu K^\mu = 0$ yields

$$\Omega^2 = b_+^2 \sin^2 \theta + v^2(b_- + b_+ \cos \theta)^2, \tag{C.5}$$

where $b_\pm = b_1 \pm b_2$. The background fields A and V are obtained from eqs. (3.23), (3.24) by first evaluating the functions c and a_χ appearing in the form (3.8) of the metric. We find

$$\begin{aligned} c &= \frac{4v|b_1 b_2|}{\Omega^2} \sin \theta, \\ a_\chi &= \frac{1}{\Omega^2} [b_+ b_- \sin^2 \theta + v^2(b_- + b_+ \cos \theta)(b_+ + b_- \cos \theta)], \end{aligned} \tag{C.6}$$

with the map to the ψ, χ coordinates being

$$\varphi = b_+ \psi + b_- \chi, \quad \varsigma = b_- \psi + b_+ \chi. \tag{C.7}$$

One can also determine the complex coordinate $z = u(\theta) + \frac{i}{4b_1 b_2}(b_+ \varsigma - b_- \varphi)$ entering in (3.8) by integrating (3.15), which takes the form

$$\frac{du}{d\theta} = \frac{\Omega(\theta)}{4v|b_1 b_2| \sin \theta}, \tag{C.8}$$

and can be solved in closed form. Then from (3.24) we obtain

$$\begin{aligned} A &= \frac{v \operatorname{sgn}(b_1 b_2)}{2\Omega^3} [2b_+^2 \cos \theta \sin^2 \theta + v^2(b_- + b_+ \cos \theta)(b_- \cos \theta + b_+ \cos(2\theta))] (b_+ d\varsigma - b_- d\varphi) \\ &\quad + \frac{1}{2} d\omega, \end{aligned} \tag{C.9}$$

with

$$\omega = \frac{1}{2} [\text{sgn}(b_1)(\varphi + \varsigma) + \text{sgn}(b_2)(\varphi - \varsigma)] , \quad (\text{C.10})$$

while (3.23) gives

$$\begin{aligned} V = & \frac{v}{48|b_1 b_2| \Omega^3} \left\{ -4v^4(b_+ + b_- \cos \theta)(b_- + b_+ \cos \theta)^3 + v^2 b_+ [8b_-^3 + 7b_- b_+^2 \right. \\ & + b_+ ((22b_-^2 + 4b_+^2) \cos \theta + 16b_- b_+ \cos(2\theta) + 2(b_-^2 + 2b_+^2) \cos(3\theta) + b_- b_+ \cos(4\theta))] \\ & + 2b_+^2 [2(3b_+^2 - b_-^2) \cos \theta + b_- b_+ (3 + \cos(2\theta))] \sin^2 \theta \left. \right\} (b_+ d\varsigma - b_- d\varphi) \\ & + \frac{v \text{sgn}(b_1 b_2)}{3\Omega} [b_+ (2b_- \cos \theta + b_+ (1 + \cos^2 \theta)) - v^2 (b_- + b_+ \cos \theta)^2] \left(\frac{b_+ d\varphi - b_- d\varsigma}{4b_1 b_2} + \frac{i}{2} d\tau \right) . \end{aligned} \quad (\text{C.11})$$

These expressions simplify in the following two special cases.

Case $b_1 = -b_2$, with v arbitrary

If we choose $b_1 = -b_2 = b/2 > 0$, we obtain

$$\Omega = b v , \quad c = \frac{\sin \theta}{v} , \quad a_\chi = \cos \theta . \quad (\text{C.12})$$

The complex coordinate z is given by $z = \frac{1}{b} (\log \tan \frac{\theta}{2} + i\varphi)$. The background fields A and V reduce to the $SU(2) \times U(1) \times U(1)$ invariant expressions

$$\begin{aligned} A &= \frac{1}{2} (d\varsigma + \cos \theta d\varphi) , \\ V &= \frac{v^2}{3} \left(d\varsigma + \cos \theta d\varphi + \frac{i}{2} b d\tau \right) , \end{aligned} \quad (\text{C.13})$$

with the conformally invariant combination being

$$A^{\text{cs}} = A - \frac{3}{2} V = \frac{1}{2} (1 - v^2) (d\varsigma + \cos \theta d\varphi) - \frac{i}{4} b v^2 d\tau . \quad (\text{C.14})$$

The gravity dual of superconformal field theories on $S^1 \times S_v^3$ with this $SU(2) \times U(1) \times U(1)$ invariant choice of background one-forms has been studied in [22].

Case $v = 1$, with b_1 and b_2 arbitrary

Let us keep b_1 and b_2 arbitrary, and set $v = 1$, so that the metric (C.2) becomes the one of the round three-sphere. Then Ω^2 , c and a_χ simplify to

$$\begin{aligned} \Omega^2 &= 4 \left(b_1^2 \cos^2 \frac{\theta}{2} + b_2^2 \sin^2 \frac{\theta}{2} \right) , \quad c = \frac{4|b_1 b_2| \sin \theta}{\Omega^2} , \\ a_\chi &= \frac{4}{\Omega^2} \left(b_1^2 \cos^2 \frac{\theta}{2} - b_2^2 \sin^2 \frac{\theta}{2} \right) , \end{aligned} \quad (\text{C.15})$$

and the background fields read

$$A = \frac{\text{sgn}(b_1 b_2)}{4\Omega^3} [4(b_1^2 + b_2^2) \cos \theta + (b_1^2 - b_2^2) (1 + 3 \cos(2\theta))] (b_+ d\varsigma - b_- d\varphi) + \frac{1}{2} d\omega , \quad (\text{C.16})$$

$$A^{\text{cs}} = A - \frac{3}{2}V = -\frac{\text{sgn}(b_1 b_2)}{2\Omega}(b_+ d\varphi - b_- d\varsigma) - i\frac{|b_1 b_2|}{\Omega} d\tau + \frac{1}{2}d\omega. \tag{C.17}$$

As a final remark, we observe that the class of three-sphere metrics (3.3) also comprises the elliptically squashed three-sphere with $U(1)^2$ isometry. This may be obtained redefining the coordinate ρ into a coordinate $\vartheta \in [0, \pi/2]$ such that $f d\rho = [\gamma_1^2 \sin^2 \vartheta + \gamma_2^2 \cos^2 \vartheta]^{1/2} d\vartheta$, and taking $m_{11} = \gamma_1^2 \cos^2 \vartheta$, $m_{22} = \gamma_2^2 \sin^2 \vartheta$, $m_{12} = 0$; here, γ_1 and γ_2 are real parameters, with the squashing being controlled by γ_2/γ_1 . The particular choice $\gamma_1 = 1/b_1$ and $\gamma_2 = 1/b_2$ leads to simpler expressions (for instance eq. (3.7) gives $\Omega = 1$ and the background fields also simplify), however we stress that this choice is not necessary; again, the partition function depends on b_1, b_2 and not on γ_1, γ_2 .

D Non-direct product metric

In this paper we consider supersymmetric backgrounds having $S^1 \times S^3$ topology and admitting two supercharges of opposite R-charge. In the main text we focused on direct product metrics with $U(1)^3$ isometry, together with a complex Killing vector K depending on two real parameters b_1, b_2 , cf. eqs. (3.3) and (3.6), respectively. We discussed how these data are sufficient to characterize the supersymmetric background. In this appendix, we relax the direct product condition and make a preliminary analysis of the more general case in which S^1 is fibered over S^3 , still preserving a $U(1)^3$ isometry. As we show below, this generalization allows to consider complex values of the moduli b_1 and b_2 parametrising the complex structure on the Hopf surface and appearing in the supersymmetric partition function.

The most general metric with $U(1)^3$ invariance on the topological product $S^1 \times S^3$ can be written as

$$ds^2 = \Omega^2 (d\tau + c_I d\varphi^I + \tilde{c} d\rho)^2 + f^2 d\rho^2 + m_{IJ} (d\varphi^I + n^I d\rho) (d\varphi^J + n^J d\rho), \tag{D.1}$$

where all the metric functions depend solely on the ρ coordinate. An immediate simplification occurs by noting that one can set $n^I = \tilde{c} = 0$ by a suitable redefinition of the angular coordinates φ^I and τ ; hence with no loss of generality we can restrict to the simpler metric

$$ds^2 = \Omega^2 (d\tau + c_I d\varphi^I)^2 + f^2 d\rho^2 + m_{IJ} d\varphi^I d\varphi^J. \tag{D.2}$$

Further, the Killing vector K in (3.6) can be generalised by analytically continuing the parameters b_1 and b_2 to complex values

$$K = \frac{1}{2} \left[\mathbb{b}_1 \frac{\partial}{\partial \varphi_1} + \mathbb{b}_2 \frac{\partial}{\partial \varphi_2} - i \frac{\partial}{\partial \tau} \right], \tag{D.3}$$

where $\mathbb{b}_I = b_I + ik_I$, with b_I and k_I real. Since $[K, \bar{K}] = 0$ is still satisfied, for the background to be supersymmetric we just need to solve the condition $K_\mu K^\mu = 0$. This constrains the metric as

$$\Omega^2 (1 + i c_I \mathbb{b}^I)^2 = \mathbb{b}^I m_{IJ} \mathbb{b}^J. \tag{D.4}$$

Separating the real and imaginary parts, we obtain

$$b^I c_I = \Omega^{-1} \text{Im} \sqrt{\mathbb{b}^I m_{IJ} \mathbb{b}^J},$$

$$k^I c_I = 1 - \Omega^{-1} \operatorname{Re} \sqrt{\mathbb{b}^I m_{IJ} \mathbb{b}^J}. \tag{D.5}$$

In the generic case where the 2×2 matrix $\begin{pmatrix} b_I \\ k_I \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \\ k_1 & k_2 \end{pmatrix}$ is invertible, these equations can be solved for the c_I . In the main text we considered instead the non-generic case $k_I = 0$, with the second equation solved by $\Omega^2 = b^I m_{IJ} b^J$, and the first satisfied by setting $c_I = 0$, namely assuming a direct product metric on $S^1 \times M_3$. Note that in the generic case one cannot set $c_I = 0$. In both cases, the metric on M_3 remains arbitrary, in particular independent of the \mathbb{b}_I .

Let us discuss regularity of the metric in the generic case. In addition to the conditions stated in section 3.2, ensuring regularity of the metric on M_3 , we need that the one-form describing the S^1 fibration be well-defined on M_3 . This amounts to requiring that $c_2 \rightarrow 0$ as $\rho \rightarrow 0$ (where the cycle dual to $d\varphi_2$ shrinks to zero size), and that $c_1 \rightarrow 0$ as $\rho \rightarrow 1$ (where the cycle dual to $d\varphi_1$ shrinks). Let us study the behavior at $\rho \rightarrow 0$, the case $\rho \rightarrow 1$ being completely analogous. Recalling the requirements (3.4), from (D.5) we see that as $\rho \rightarrow 0$,

$$\begin{aligned} c_1 &\rightarrow \frac{\Omega(0)^{-1} \sqrt{m_{11}(0)} (b_1 b_2 + k_1 k_2) - b_2}{b_1 k_2 - b_2 k_1}, \\ c_2 &\rightarrow \frac{-\Omega(0)^{-1} \sqrt{m_{11}(0)} |\mathbb{b}_1|^2 + b_1}{b_1 k_2 - b_2 k_1}. \end{aligned} \tag{D.6}$$

The regularity condition $c_2(0) = 0$ fixes $\Omega(0) = \sqrt{m_{11}(0)} \frac{|\mathbb{b}_1|^2}{b_1}$, which then gives $c_1 \rightarrow \frac{k_1}{|\mathbb{b}_1|^2}$. Apart for the behavior at the poles, in this generic case $\Omega(\rho)$ is arbitrary.

In order to complete the global analysis, and check regularity of the background fields A and V as well, we should proceed as done in the main text for the direct product case: define complex coordinates w, z and then use the formulae in section 2.1. Although straightforward, we will not pursue this in the present paper.

E Proof that $(z_1, z_2) \in \mathbb{C}^2 - (0, 0)$

Below we complete the proof that the coordinates (3.32), namely

$$\begin{aligned} z_1 &= e^{-|b_1|(iw+z)} = e^{|b_1|\tau} e^{|b_1|(Q-u)} e^{-i \operatorname{sgn}(b_1)\varphi_1}, \\ z_2 &= e^{-|b_2|(iw-z)} = e^{|b_2|\tau} e^{|b_2|(Q+u)} e^{-i \operatorname{sgn}(b_2)\varphi_2}, \end{aligned} \tag{E.1}$$

where the function $Q(\rho), u(\rho)$ obey

$$Q' = \frac{f a_\chi}{\Omega c}, \quad u' = \frac{f}{\Omega c}, \tag{E.2}$$

span $\mathbb{C}^2 - (0, 0)$. Recall that the functions appearing on the right hand side of these equations are given by

$$\begin{aligned} c &= \frac{2|b_1 b_2|}{\Omega^2} \sqrt{\det(m_{IJ})}, \\ a_\chi &= \frac{1}{\Omega^2} (b_1^2 m_{11} - b_2^2 m_{22}), \end{aligned}$$

$$\Omega^2 = b^I m_{IJ} b^J, \tag{E.3}$$

with f arbitrary, and obey certain boundary conditions near to the end-points of the interval $[0, 1]$. Fixing $|z_2| = e^{|b_2|\delta_2}$ for finite $\delta_2 \in \mathbb{R}$ and solving for $\tau = \delta_2 - Q - u$, we obtain

$$|z_1| = e^{|b_1|\delta_2} e^{-2|b_1|u}, \quad |z_2| = e^{|b_2|\delta_2}, \tag{E.4}$$

and similarly fixing $|z_1| = e^{|b_1|\delta_1}$ for finite $\delta_1 \in \mathbb{R}$ and solving for $\tau = \delta_1 - Q + u$ we obtain

$$|z_1| = e^{|b_1|\delta_1}, \quad |z_2| = e^{|b_2|\delta_1} e^{2|b_2|u}. \tag{E.5}$$

The expansion near to $\rho \rightarrow 0$ and $\rho \rightarrow 1$ of the various metric functions imply

$$\begin{aligned} u'(\rho) &= \frac{1}{2|b_2|\rho} + \mathcal{O}(\rho^0), \\ u'(\rho) &= \frac{1}{2|b_1|(1-\rho)} + \mathcal{O}((1-\rho)^0), \end{aligned} \tag{E.6}$$

leading to

$$\begin{aligned} u(\rho) &= \frac{1}{2|b_2|} \log \rho + \mathcal{O}(\rho^0), \\ u(\rho) &= -\frac{1}{2|b_1|} \log(1-\rho) + \mathcal{O}((1-\rho)^0). \end{aligned} \tag{E.7}$$

Using these, and noticing that $u(\rho)$ is a monotonically increasing function of ρ , since $u' = \frac{f}{\Omega c} \geq 0$, we see that $u(\rho)$ is a bijection $(0, 1) \rightarrow (-\infty, +\infty)$. Therefore, at fixed non-zero $|z_2|$, the radial coordinate $|z_1|$ covers $\mathbb{R}_{>0}$ (once) and at fixed non-zero $|z_1|$, the radial coordinate $|z_2|$ covers $\mathbb{R}_{>0}$ (once).

So far we have seen that for $(\tau, \varphi_1, \varphi_2, \rho) \in \mathbb{R} \times [0, 2\pi) \times [0, 2\pi) \times (0, 1)$, the coordinates (z_1, z_2) cover $\mathbb{C}^2 - \{(\mathbb{C}, 0)\} - \{(0, \mathbb{C})\}$. The cases $u = \pm\infty$, corresponding to $\rho = 0$ and $\rho = 1$, must be considered separately, since we may not be able to solve for $\tau \in \mathbb{R}$ in those cases ($\tau = \pm\infty \notin \mathbb{R}$!). Again solving for Q and u near to $\rho \rightarrow 0$ and $\rho \rightarrow 1$, we obtain

$$\begin{aligned} Q - u &= \mathcal{O}(\rho^2), & Q - u &= -\frac{1}{|b_1|} \log(1-\rho) + \mathcal{O}((1-\rho)^0), \\ Q + u &= \frac{1}{|b_2|} \log \rho + \mathcal{O}(\rho^0), & Q + u &= \mathcal{O}((1-\rho)^2). \end{aligned} \tag{E.8}$$

In the limit $\rho = 0$ we have

$$|z_1| = e^{|b_1|\tau}, \quad |z_2| = 0, \tag{E.9}$$

while in the limit $\rho = 1$ we have

$$|z_1| = 0, \quad |z_2| = e^{|b_2|\tau}. \tag{E.10}$$

Then we observe that at $|z_1| = 0$, $|z_2|$ covers $\mathbb{R}_{>0}$ (once) and at $|z_2| = 0$, $|z_1|$ covers $\mathbb{R}_{>0}$ (once). This concludes the proof that (z_1, z_2) covers $\mathbb{C}^2 - (0, 0)$.

F Reduction of the 4d supersymmetry equations to 3d

In this appendix we revisit the 4d \rightarrow 3d reduction of the supersymmetry equations (2.1), (2.2) discussed in [27, appendix D] (see also [6]), including a more general identification between the background fields as well as a non-trivial dilaton. Then we show that the 4d background described in section 3.2 reduces to the 3d background considered in [10].

General reduction

Similarly to the four-dimensional case, in three dimensions the supersymmetry equation arising from the rigid limit of “new minimal” supergravity contains different signs depending on whether the spinor parameter has R-charge +1 or -1 . In terms of a spinor ϵ with R-charge +1 and a spinor η with R-charge -1 , one has [27]

$$(\check{\nabla}_i - i\check{A}_i)\epsilon + \frac{i}{2}\check{h}\gamma_i\epsilon + i\check{V}_i\epsilon + \frac{1}{2}\epsilon_{ijk}\check{V}^j\gamma^k\epsilon = 0, \quad (\text{F.1})$$

$$(\check{\nabla}_i + i\check{A}_i)\eta + \frac{i}{2}\check{h}\gamma_i\eta - i\check{V}_i\eta - \frac{1}{2}\epsilon_{ijk}\check{V}^j\gamma^k\eta = 0, \quad (\text{F.2})$$

where i, j, k are 3d curved indices, and we append a $\check{}$ on 3d quantities that may be confused with 4d ones. The 3d spinor covariant derivative is defined as

$$\check{\nabla}_i\epsilon = \left(\partial_i + \frac{i}{4}\check{\omega}_{i\check{a}\check{b}}\epsilon^{\check{a}\check{b}\check{c}}\gamma_{\check{c}} \right)\epsilon, \quad (\text{F.3})$$

(same for η), where $\check{\omega}_{i\check{a}\check{b}}$ is the 3d spin connection, and $\check{a}, \check{b}, \check{c}$ are 3d flat indices. Moreover, \check{A}_i is the 3d background gauge field coupling to the R-current, while \check{V}_i and \check{h} are a background one-form and a background scalar, respectively. Our 3d gamma matrices are defined as $(\gamma^{\check{a}})_{\alpha}{}^{\beta} = \sigma_{\text{Pauli}}^{\check{a}}$. These are related to the 4d sigma matrices as

$$\sigma_{\alpha\check{\alpha}}^{\check{a}} = i(\gamma^{\check{a}})_{\alpha}{}^{\beta}\sigma_{\beta\check{\beta}}^4, \quad \check{\sigma}^{\check{a}\check{\alpha}\alpha} = -i\check{\sigma}^{4\check{\alpha}\beta}(\gamma^{\check{a}})_{\beta}{}^{\alpha}, \quad (\text{F.4})$$

which imply

$$\begin{aligned} \sigma_{\check{a}4} &= -\frac{i}{2}\gamma_{\check{a}}, & \sigma_{\check{a}\check{b}} &= -\frac{i}{2}\epsilon_{\check{a}\check{b}\check{c}}\gamma^{\check{c}}, \\ \check{\sigma}_{\check{a}4} &= -\frac{i}{2}\check{\sigma}_4\gamma_{\check{a}}\sigma_4, & \check{\sigma}_{\check{a}\check{b}} &= +\frac{i}{2}\epsilon_{\check{a}\check{b}\check{c}}\check{\sigma}_4\gamma^{\check{c}}\sigma_4. \end{aligned} \quad (\text{F.5})$$

In this way, a 4d left-handed spinor ζ_{α} directly reduces to a 3d spinor, while a 4d right-handed spinor $\check{\zeta}^{\check{\alpha}}$ is mapped to a 3d spinor via $i\sigma_{\alpha\check{\alpha}}^4\check{\zeta}^{\check{\alpha}}$.

Let us consider a 4d metric of the form

$$ds^2 = \check{g}_{ij}(x)dx^i dx^j + e^{2\Phi(x)}(d\tau + c_i(x)dx^i)^2, \quad (\text{F.6})$$

where we are splitting the 4d coordinates as $x^{\mu} = (x^i, \tau)$, and \check{g}_{ij} , c_i , Φ are a 3d metric, a 3d one-form and a dilaton function, respectively, depending on the 3d coordinates only. The 4d vielbein and its inverse can be written as

$$e^a{}_{\mu} = \begin{pmatrix} \check{e}^{\check{a}}_i & 0 \\ e^{\Phi}c_i & e^{\Phi} \end{pmatrix}, \quad e^{\mu}{}_a = \begin{pmatrix} \check{e}^i_{\check{a}} & 0 \\ -c_j\check{e}^j_{\check{a}} & e^{-\Phi} \end{pmatrix}, \quad (\text{F.7})$$

where $\check{e}^{\check{a}}_i$ is a vielbein for \check{g}_{ij} , with inverse $\check{e}^i_{\check{a}}$. The 4d spin connection ω_{cab} splits as

$$\begin{aligned}\omega_{\check{c}\check{a}\check{b}} &= \check{e}^i_{\check{c}} \check{\omega}_{i\check{a}\check{b}}, & \omega_{4\check{a}\check{b}} &= -e^\Phi \partial_{[i} c_{j]} \check{e}^i_{\check{a}} \check{e}^j_{\check{b}}, \\ \omega_{\check{c}4\check{b}} &= e^\Phi \partial_{[i} c_{j]} \check{e}^i_{\check{b}} \check{e}^j_{\check{c}}, & \omega_{44\check{b}} &= \check{e}^i_{\check{b}} \partial_i \Phi.\end{aligned}\tag{F.8}$$

We now reduce the 4d equation for ζ given in (2.1) along the Killing direction $\partial/\partial\tau$. Assuming that ζ is independent of τ , we obtain the following 3d equations

$$\left[\frac{1}{4} e^\Phi v^i \gamma_i - \frac{i}{2} \partial_i \Phi \gamma^i - i e^{-\Phi} A_\tau + i e^{-\Phi} V_\tau - \frac{1}{2} (V_i - c_i V_\tau) \gamma^i \right] \zeta = 0, \tag{F.9}$$

$$\left[\check{\nabla}_i + \frac{1}{4} e^\Phi \epsilon_{ijk} v^j \gamma^k - i (A_i - c_i A_\tau) + i (V_i - c_i V_\tau) + \frac{1}{2} e^{-\Phi} V_\tau \gamma_i + \frac{1}{2} \epsilon_i{}^{jk} (V_j - c_j V_\tau) \gamma_k \right] \zeta = 0, \tag{F.10}$$

where we introduced

$$v^i = -i \epsilon^{ijk} \partial_j c_k. \tag{F.11}$$

The first equation is solved by requiring that the 4d one-form

$$\mathcal{U}_\mu = (\mathcal{U}_i, \mathcal{U}_\tau) = \left(V_i - 3c_i V_\tau + 2c_i A_\tau - \frac{1}{2} e^\Phi v_i + i \partial_i \Phi, 2A_\tau - 2V_\tau \right) \tag{F.12}$$

satisfies

$$\mathcal{U}_\mu \tilde{\sigma}^\mu \zeta = 0 \quad \Leftrightarrow \quad (\mathcal{U}_i - c_i \mathcal{U}_\tau) \gamma^i \zeta + i e^{-\Phi} \mathcal{U}_\tau \zeta = 0, \tag{F.13}$$

which is equivalent to $J_\mu{}^\nu \mathcal{U}_\nu = i \mathcal{U}_\mu$, meaning that \mathcal{U}_μ is of type (0, 1) with respect to the complex structure J defined by ζ . Then eq. (F.10) can be matched with either one of the 3d supersymmetry conditions (F.1), (F.2). As we will need to precisely recover the solution studied in [10], we choose to match the equation (F.2) for η , although this leads to a map between 4d and 3d background fields containing some awkward minus signs. Identifying the spinor parameters as $\eta = \zeta$, the 3d background fields are given by

$$\check{A}_i = -(A_i - c_i A_\tau) - \frac{1}{2} e^\Phi v_i, \quad \check{V}_i = -(V_i - c_i V_\tau) - \frac{1}{2} e^\Phi v_i, \quad \check{h} = -i e^{-\Phi} V_\tau. \tag{F.14}$$

The reduction of the equation (2.2) for a spinor $\tilde{\zeta}$ works similarly. In this case, we need to require that the one-form

$$\tilde{\mathcal{U}}_\mu = (\tilde{\mathcal{U}}_i, \tilde{\mathcal{U}}_\tau) = \left(V_i - 3c_i V_\tau + 2c_i A_\tau - \frac{1}{2} e^\Phi v_i - i \partial_i \Phi, 2A_\tau - 2V_\tau \right) \tag{F.15}$$

(differing from \mathcal{U}_μ just by the sign of $\partial_i \Phi$) satisfies

$$\tilde{\mathcal{U}}_\mu \sigma^\mu \tilde{\zeta} = 0 \quad \Leftrightarrow \quad (\tilde{\mathcal{U}}_i - c_i \tilde{\mathcal{U}}_\tau) \gamma^i \sigma_4 \tilde{\zeta} - i e^{-\Phi} \tilde{\mathcal{U}}_\tau \sigma_4 \tilde{\zeta} = 0, \tag{F.16}$$

namely is of type (0, 1) with respect to the complex structure \tilde{J} defined by $\tilde{\zeta}$. Identifying the spinors as $\epsilon = i \sigma_4 \tilde{\zeta}$, eq. (F.1) is retrieved by taking exactly the same 3d background fields as in (F.14).

From (F.14), we see that if we want both the 4d A and the 3d \check{A} to be real, then the purely imaginary v has to vanish. In this case, it is possible to set c to zero by redefining the τ coordinate, so that the 4d metric takes a direct product form.

We observe that the 3d background fields are not uniquely determined though, as the 3d equations are invariant under certain shifts [27]. This remains true even if the analogous shift freedom in 4d has been fixed. For our purposes, it will be enough to discuss this for real 3d background fields $\check{A}^i, \check{V}^i, \check{h}$. In this case, given a solution ϵ to (F.1), one also has a solution to (F.2) by taking the charge conjugate, $\eta = \epsilon^c$. This implies the existence of a real Killing vector $\check{K}^i = \epsilon^\dagger \gamma^i \epsilon$. Then the equations (F.1), (F.2) are invariant under shifting the background fields as

$$\check{A} \rightarrow \check{A} + \frac{3}{2} \frac{\check{\kappa} \check{K}}{\sqrt{\check{K}^i \check{K}_i}}, \quad \check{V} \rightarrow \check{V} + \frac{\check{\kappa} \check{K}}{\sqrt{\check{K}^i \check{K}_i}}, \quad \check{h} \rightarrow \check{h} + \check{\kappa}, \quad (\text{F.17})$$

where $\check{\kappa}$ is a real function. The identifications (F.14) between 4d and 3d background fields for a general $\check{\kappa}$ become

$$\check{A} + \frac{3}{2} \frac{\check{\kappa} \check{K}}{\sqrt{\check{K}^i \check{K}_i}} = -A_i dx^i, \quad \check{V} + \frac{\check{\kappa} \check{K}}{\sqrt{\check{K}^i \check{K}_i}} = -V_i dx^i, \quad \check{h} + \check{\kappa} = -i e^{-\Phi} V_\tau, \quad (\text{F.18})$$

where we have assumed that the 4d metric is in a direct product form, i.e. $c_i = v^i = 0$, as this is the case that will be relevant below.

Reduction of our background

We now apply the formulae above and show that the $S^1 \times M_3$ background given in section 3.2 reduces to the 3d background studied in [10]. Here, neither the fact that the 3d metric admits $U(1)^2$ isometry, nor the global constraints discussed in section 3.2 will play any role. The solution in [10] has real background fields and supercharges related by charge conjugation. The metric takes the general form

$$ds_{\text{there}}^2 = \Omega^2 [(d\psi + a)^2 + c^2 dz d\bar{z}], \quad (\text{F.19})$$

where $\Omega = \Omega(z, \bar{z})$, $c = c(z, \bar{z})$, $a = a_z(z, \bar{z}) dz + \bar{a}_{\bar{z}}(z, \bar{z}) d\bar{z}$, and for the spinors we take

$$\epsilon = \sqrt{s_{\text{there}}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \eta = -i\sigma_2 \epsilon^* = \sqrt{s_{\text{there}}^*} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (\text{F.20})$$

where $|s_{\text{there}}| = \Omega$. Then the 3d Killing vector is $\check{K} = \partial/\partial\psi$, which as a one-forms reads $\check{K} = \Omega^2(d\psi + a)$. Finally, the background fields given in [10] read

$$\begin{aligned} A_{\text{there}} &= -\text{Im}[\partial_z \log(\Omega^3 c) dz] + \frac{1}{2} d\text{Arg}(s_{\text{there}}) + \frac{*_2(da)}{c^2} (d\psi + a), \\ V_{\text{there}} &= -2 \text{Im}[\partial_z \log \Omega dz] + \frac{*_2(da)}{c^2} (d\psi + a), \\ h_{\text{there}} &= \frac{*_2(da)}{2\Omega c^2}. \end{aligned} \quad (\text{F.21})$$

These expressions are obtained expanding eqs. (2.11)–(2.16) therein and translating to our notation (in particular $c_{\text{there}} = \Omega c$).

Reducing our 4d metric (3.8) along $\partial/\partial\tau$ clearly matches (F.19). In order to match the spinors in (F.20) with our spinors (2.21), we need to identify

$$\zeta_\alpha = \frac{1}{\sqrt{2}} \eta_\alpha, \quad i\sigma_{\alpha\dot{\alpha}}^4 \tilde{\zeta}^{\dot{\alpha}} = \frac{1}{\sqrt{2}} \epsilon_\alpha \quad \Rightarrow \quad \omega = -\text{Arg}(s_{\text{there}}). \quad (\text{F.22})$$

Using the formulae derived above, we can also check that the background fields reduce as needed. Since the 4d metric is a direct product, we set $c_i = v^i = 0$; in addition, we take $\Phi = \log \Omega$. Starting from our expressions (2.18), (3.22) for the 4d fields A and V , it is easy to check that the conditions on \mathcal{U} and $\tilde{\mathcal{U}}$ are indeed satisfied. Then (F.18) gives

$$\begin{aligned} \check{A} + \frac{3}{2} \check{\kappa} \Omega (d\psi + a) &= -A_i dx^i = -\text{Im}[\partial_z \log(\Omega^3 c) dz] - \frac{1}{2} d\omega, \\ \check{V} + \check{\kappa} \Omega (d\psi + a) &= -V_i dx^i = -2 \text{Im}[\partial_z \log \Omega dz] + \frac{1}{3c^2} *_2(da)(d\psi + a), \\ \check{h} + \check{\kappa} &= -\frac{i}{\Omega} V_\tau = -\frac{*_2(da)}{6\Omega c^2}. \end{aligned} \quad (\text{F.23})$$

Comparing (F.21) and (F.23), we see that \check{A} , \check{V} and \check{h} agree with A_{there} , V_{there} and h_{there} if we pick $\check{\kappa} = -\frac{2}{3\Omega c^2} *_2(da)$. However, in the main text it will be not necessary to fix $\check{\kappa}$, as it actually drops from the 3d supersymmetry transformations.

The condition on \mathcal{U} translates into the relation

$$(\check{V}_i - i \partial_i \log \Omega) \gamma^i \eta + \frac{1}{\Omega} \check{V}_\psi \eta = 0. \quad (\text{F.24})$$

This is useful in the 4d \rightarrow 3d reduction of the supersymmetry variations in section 4.3.

G Regularization of one-loop determinants

In this appendix we proceed with the regularization of one-loop determinants for the vector multiplet and for the chiral multiplet.

For the vector multiplet the one-loop determinant is given by the infinite product (4.30)

$$\begin{aligned} Z_{1\text{-loop}}^{\text{vector}} &= Z_{\text{Cartan}} \Delta_1^{-1} \prod_{\alpha \in \text{roots}} \left(\prod_{n_0 \in \mathbb{Z}} \prod_{n_1, n_2 \geq 0} \frac{n_1 b_1 + n_2 b_2 + i(n_0 + \alpha_{\mathcal{A}_0})}{-(n_1 + 1)b_1 - (n_2 + 1)b_2 + i(n_0 + \alpha_{\mathcal{A}_0})} \right) \\ &= Z_{\text{Cartan}} \Delta_1^{-1} \prod_{\alpha \in \text{roots}} F[\alpha_{\mathcal{A}_0}, ib_1, ib_2], \end{aligned} \quad (\text{G.1})$$

with $b_1 > 0, b_2 > 0$.

A natural regularization is to use the Barnes multiple zeta/gamma functions and we refer the reader to [52, 53] for definitions, notations and useful formulae, in particular for the function Γ_3 and ζ_3 . The first step is to rewrite the infinite product above, labelled by a root α , with triple gamma functions:

$$F[w_\alpha, \tau, \sigma] \equiv \prod_{n_0 \in \mathbb{Z}} \prod_{n_1, n_2 \geq 0} \frac{w_\alpha + n_0 - n_1 \tau - n_2 \sigma}{w_\alpha + \tau + \sigma + n_0 + n_1 \tau + n_2 \sigma} \quad (\text{G.2})$$

$$\begin{aligned}
&= \prod_{n_0, n_1, n_2 \geq 0} \frac{1 + w_\alpha + n_0 - n_1\tau - n_2\sigma}{w_\alpha + \tau + \sigma + n_0 + n_1\tau + n_2\sigma} \\
&\quad \times \prod_{n_0, n_1, n_2 \geq 0} \frac{-w_\alpha + n_0 + n_1\tau + n_2\sigma}{1 - w_\alpha - \tau - \sigma + n_0 - n_1\tau - n_2\sigma} \\
&= \frac{\Gamma_3(w_\alpha + \tau + \sigma | 1, \tau, \sigma) \Gamma_3(1 - w_\alpha - \tau - \sigma | 1, -\tau, -\sigma)}{\Gamma_3(1 + w_\alpha | 1, -\tau, -\sigma) \Gamma_3(-w_\alpha | 1, \tau, \sigma)}
\end{aligned}$$

where $w_\alpha = \alpha_{\mathcal{A}_0}$, and we renamed the parameters b_1, b_2 into $\tau = ib_1$ and $\sigma = ib_2$ for the ease of comparison with references [52, 53]. Then using formula (6.4) in [52] we get:

$$\begin{aligned}
F[w_\alpha, \tau, \sigma] &= e^{i\pi\{\zeta_3(0, -w_\alpha | 1, \tau, \sigma) - \zeta_3(0, w_\alpha + \tau + \sigma | 1, \tau, \sigma)\}} \prod_{n_1, n_2 \geq 0} \frac{1 - e^{2\pi i(-w_\alpha + n_1\tau + n_2\sigma)}}{1 - e^{2\pi i(w_\alpha + (n_1+1)\tau + (n_2+1)\sigma)}} \\
&= e^{i\pi\{\zeta_3(0, -w_\alpha | 1, \tau, \sigma) - \zeta_3(0, w_\alpha + \tau + \sigma | 1, \tau, \sigma)\}} \frac{1}{\tilde{\Gamma}_e(-w_\alpha, \tau, \sigma)}, \tag{G.3}
\end{aligned}$$

where $\tilde{\Gamma}_e$ is the elliptic gamma function defined for $x, \tau, \sigma \in \mathbb{C}$ and $\text{Im}(\tau), \text{Im}(\sigma) > 0$ by:

$$\tilde{\Gamma}_e(x, \tau, \sigma) = \prod_{n_1, n_2 \geq 0} \frac{1 - e^{2\pi i(-x + (n_1+1)\tau + (n_2+1)\sigma)}}{1 - e^{2\pi i(x + n_1\tau + n_2\sigma)}}. \tag{G.4}$$

In the product over roots α we can combine the factors for the roots α and $-\alpha$ and use some formulae in [53]:

$$F[w_\alpha, \tau, \sigma] F[-w_\alpha, \tau, \sigma] = \frac{e^{i\pi\Psi(w_\alpha, \tau, \sigma)}}{\tilde{\Gamma}_e(-w_\alpha, \tau, \sigma) \tilde{\Gamma}_e(w_\alpha, \tau, \sigma)} = e^{i\pi\Psi(w_\alpha, \tau, \sigma)} \theta_0(w_\alpha, \tau) \theta_0(-w_\alpha, \sigma), \tag{G.5}$$

where

$$\begin{aligned}
\Psi(w_\alpha, \tau, \sigma) &= \zeta_3(0, -w_\alpha | 1, \tau, \sigma) - \zeta_3(0, w_\alpha + \tau + \sigma | 1, \tau, \sigma) \\
&\quad + \zeta_3(0, w_\alpha | 1, \tau, \sigma) - \zeta_3(0, -w_\alpha + \tau + \sigma | 1, \tau, \sigma)
\end{aligned} \tag{G.6}$$

and θ_0 is the Jacobi theta function, defined for $w_\alpha, \tau \in \mathbb{C}, \text{Im}(\tau) > 0$ by

$$\theta_0(w_\alpha, \tau) = \prod_{n \geq 0} \left(1 - e^{2\pi i(n\tau + w_\alpha)}\right) \left(1 - e^{2\pi i((n+1)\tau - w_\alpha)}\right). \tag{G.7}$$

Formula (5.24) in [52] gives:

$$\Psi(w_\alpha, \tau, \sigma) = w_\alpha^2 \left(\frac{1}{\tau} + \frac{1}{\sigma}\right) + \frac{1}{6} \left(\tau + \sigma + \frac{1}{\tau} + \frac{1}{\sigma}\right). \tag{G.8}$$

In total we have

$$Z_{1\text{-loop}}^{\text{vector}} = Z_{\text{Cartan}} \Delta_2^{-1} \prod_{\alpha \in \Delta_+} e^{i\pi\Psi_\alpha} \theta_0(\alpha_{\mathcal{A}_0}, ib_1) \theta_0(-\alpha_{\mathcal{A}_0}, ib_2). \tag{G.9}$$

The contribution of a Cartan component corresponds to the contribution of a root $\alpha = 0$. To evaluate it we can simply take the square root of the contribution of a positive root α and send α to zero,³⁵

$$Z_{\text{Cartan}} = \left(\lim_{\alpha \rightarrow 0} \sqrt{Z_\alpha} \right)^{r_G}, \quad Z_\alpha = \frac{e^{i\pi\Psi_\alpha} \theta_0(\alpha_{\mathcal{A}_0}, ib_1) \theta_0(-\alpha_{\mathcal{A}_0}, ib_2)}{4 \sin(\pi\alpha_{\mathcal{A}_0})^2}, \quad (\text{G.10})$$

where r_G is the rank of G (i.e. the number of Cartan generators). This yields

$$Z_{\text{Cartan}} = e^{\frac{i\pi}{2}\Psi(0,\tau,\sigma)r_G} (e^{-2\pi b_1}; e^{-2\pi b_1})^{r_G} (e^{-2\pi b_2}; e^{-2\pi b_2})^{r_G}, \quad (\text{G.11})$$

with the Pochhammer symbol defined for $x, q \in \mathbb{C}$, $|q| < 1$, by $(x; q) = \prod_{n \geq 0} (1 - xq^n)$. With the change of notation $\theta_0(x, y) = \theta(e^{2\pi i x}, e^{2\pi i y})$, we have

$$\begin{aligned} Z_{1\text{-loop}}^{\text{vector}} &= e^{i\pi\Psi_{\text{vec}}^{(0)}} e^{i\pi\Psi_{\text{vec}}^{(1)}} (p; p)^{r_G} (q; q)^{r_G} \Delta_2^{-1} \prod_{\alpha \in \Delta_+} \theta(e^{2\pi i \alpha_{\mathcal{A}_0}}, p) \theta(e^{-2\pi i \alpha_{\mathcal{A}_0}}, q), \\ \Psi_{\text{vec}}^{(0)} &= \frac{i}{12} \left(b_1 + b_2 - \frac{b_1 + b_2}{b_1 b_2} \right) |G|, \\ \Psi_{\text{vec}}^{(1)} &= -i \frac{b_1 + b_2}{b_1 b_2} \sum_{\alpha \in \Delta_+} \alpha_{\mathcal{A}_0}^2, \end{aligned} \quad (\text{G.12})$$

with $p = e^{-2\pi b_1}$, $q = e^{-2\pi b_2}$ and $|G|$ is the dimension of G , and we have split the prefactor into a part $\Psi_{\text{vec}}^{(1)}$ depending on $\alpha_{\mathcal{A}_0}$ and a part $\Psi_{\text{vec}}^{(0)}$ independent of $\alpha_{\mathcal{A}_0}$.

The regularization of the chiral multiplet one-loop determinant proceeds similarly

$$\begin{aligned} Z_{1\text{-loop}}^{\text{chiral}} &= \prod_{\rho \in \text{weights}} \prod_{n_0 \in \mathbb{Z}} \prod_{n_1, n_2 \geq 0} \frac{\rho_{\mathcal{A}_0} + i\frac{r-2}{2}(b_1 + b_2) + n_0 - in_1 b_1 - in_2 b_2}{-\rho_{\mathcal{A}_0} - \frac{i\tilde{r}}{2}(b_1 + b_2) - n_0 - in_1 b_1 - in_2 b_2} \\ &= \prod_{\rho \in \text{weights}} \frac{\Gamma_3(u_\rho | 1, \tau, \sigma) \Gamma_3(1 - u_\rho | 1, -\tau, -\sigma)}{\Gamma_3(1 + u_\rho - \tau - \sigma | 1, -\tau, -\sigma) \Gamma_3(-u_\rho + \tau + \sigma | 1, \tau, \sigma)}, \end{aligned} \quad (\text{G.13})$$

where we have regularized the infinite product using triple Gamma functions,³⁶ and we have defined $\rho_{\mathcal{A}_0} \equiv \rho(\mathcal{A}_0)$, $u_\rho = \rho_{\mathcal{A}_0} + \frac{r}{2}(\tau + \sigma)$, and again $\tau = ib_1$, $\sigma = ib_2$.

Using formula (6.4) of [52] leads to

$$\begin{aligned} Z_{1\text{-loop}}^{\text{chiral}} &= \prod_{\rho \in \text{weights}} \left(e^{i\pi\Psi(u_\rho, \tau, \sigma)} \prod_{n_1, n_2 \geq 0} \frac{1 - e^{2\pi i(-u_\rho + (n_1+1)\tau + (n_2+1)\sigma)}}{1 - e^{2\pi i(u_\rho + n_1\tau + n_2\sigma)}} \right) \\ &= \prod_{\rho \in \text{weights}} e^{i\pi\Psi(u_\rho, \tau, \sigma)} \tilde{\Gamma}_e(u_\rho, \tau, \sigma), \end{aligned} \quad (\text{G.14})$$

with

$$\Psi(u_\rho, \tau, \sigma) = \zeta_3(0, \tau + \sigma - u_\rho | 1, \tau, \sigma) - \zeta_3(0, u_\rho | 1, \tau, \sigma) = \frac{(u'_\rho)^3}{3\tau\sigma} + \frac{2 - \tau^2 - \sigma^2}{12\tau\sigma} u'_\rho, \quad (\text{G.15})$$

³⁵We consider the square root because the α -factor contains both the contribution of the roots α and $-\alpha$.

³⁶The product $\prod_{n_0 \in \mathbb{Z}}$ has been split into $\prod_{n_0 > 0} \times \prod_{n_0 \leq 0}$ in the numerator and $\prod_{n_0 \geq 0} \times \prod_{n_0 < 0}$ in the denominator.

and $u'_\rho = u_\rho - \frac{\tau+\sigma}{2} = \rho_{\mathcal{A}_0} + \frac{r-1}{2}(\tau + \sigma)$. The full chiral multiplet one-loop determinant is

$$\begin{aligned}
 Z_{1\text{-loop}}^{\text{chiral}} &= e^{i\pi\Psi_{\text{chi}}^{(0)}} e^{i\pi\Psi_{\text{chi}}^{(1)}} \prod_{\rho \in \Delta_{\mathcal{R}}} \Gamma_e \left(e^{2\pi i \rho_{\mathcal{A}_0}} (pq)^{\frac{\tau}{2}}, p, q \right), \\
 \Psi_{\text{chi}}^{(0)} &= \frac{i}{24} \frac{b_1 + b_2}{b_1 b_2} \left[(r-1)^3 (b_1 + b_2)^2 - (r-1) (b_1^2 + b_2^2 + 2) \right] |\mathcal{R}|, \\
 \Psi_{\text{chi}}^{(1)} &= \sum_{\rho \in \Delta_{\mathcal{R}}} -\frac{\rho_{\mathcal{A}_0}^3}{3 b_1 b_2} - i(r-1) \frac{b_1 + b_2}{2 b_1 b_2} \rho_{\mathcal{A}_0}^2 + [3(r-1)^2 (b_1 + b_2)^2 - 2 - b_1^2 - b_2^2] \frac{\rho_{\mathcal{A}_0}}{12 b_1 b_2}, \quad (\text{G.16})
 \end{aligned}$$

where $p = e^{-2\pi b_1}$, $q = e^{-2\pi b_2}$, $|\mathcal{R}| = \dim(\mathcal{R})$, $\Delta_{\mathcal{R}}$ is the set of weights of \mathcal{R} and we have redefined the $\tilde{\Gamma}_e$ function as $\tilde{\Gamma}_e(x, \tau, \sigma) = \Gamma_e(e^{2\pi i x}, e^{2\pi i \tau}, e^{2\pi i \sigma})$.

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