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Comments on the Casimir energy in supersymmetric field theories

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ABSTRACT: We study the Casimir energy of four-dimensional supersymmetric gauge theories in the context of the rigid limit of new minimal supergravity. Firstly, revisiting the computation of the localized partition function on $S^1 \times S^3$, we recover the supersymmetric Casimir energy from its path integral definition. Secondly, we consider the same theories in the Hamiltonian formalism on $\mathbb{R} \times S^3$, focussing on the free limit and including a one-parameter family of background gauge fields along $\mathbb{R}$. We compute the vacuum expectation value of the canonical Hamiltonian using zeta function regularization, and show that this interpolates between the supersymmetric Casimir energy and the ordinary Casimir energy of a supersymmetric free field theory.

KEYWORDS: Supersymmetric gauge theory, Renormalization Regularization and Renormalons

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1 Introduction

It is remarkable that in certain situations the path integral of an interacting supersymmetric field theory can be reduced to the computation of one-loop determinants, using the idea of localization [1]. Employing this method, the partition function Z of $\mathcal{N}=1$ supersymmetric gauge theories with an R-symmetry, defined on a Hopf surface $S^1 \times M_3$, with periodic boundary conditions for the fermions on S^1, has been computed in [2]. This is proportional [3] to the supersymmetric index [4]

$$\mathcal{I}(\beta) = \mathrm{Tr}\,(-1)^F \mathrm{e}^{-\beta H_{\mathrm{susy}}}\,, \tag{1.1}$$

where F is the fermion number and H_{susy} is the Hamiltonian generating translations on S^1, and commuting with at least one supercharge Q, which exists on the manifold $S^1 \times M_3$. Here the trace is over the Hilbert space of states on M_3. Although this index can be refined introducing additional fugacities, associated to other conserved charges commuting with Q, below we consider the simplest case with only one fugacity β, proportional to the radius of the S^1.

In analogy with the standard path integral definition of the vacuum energy of a field theory, from the localized partition function Z, one can define a quantity dubbed *supersymmetric Casimir energy* [2, 5] as

$$E_{\text{susy}} \equiv - \lim_{\beta\to\infty} \frac{\mathrm{d}}{\mathrm{d}\beta} \log Z \ . \tag{1.2}$$

In [2] this was shown to be given by a linear combination of the central charges $\mathbf{a}$ and $\mathbf{c}$ of the supersymmetric theory, which in the simplest case reads

$$E_{\text{susy}} = \frac{4}{27}(\mathbf{a} + 3\mathbf{c}) \ . \tag{1.3}$$

In two dimensions, the fact that the Casimir energy is proportional to the unique central charge $\mathbf{c}$ has been known for long time [6], and attempts to generalize this result to higher dimensions have been discussed in several places [7–9]. The usual strategy consists in relating the expectation value $\langle T_{tt}\rangle$ of the energy-momentum tensor to its anomalous trace $\langle T^{\mu}_{\mu}\rangle$. However, this approach has some limitations. First of all, the resulting Casimir energy is *ambiguous*, with the ambiguities being related to the ambiguities in the trace anomaly. Moreover, previous results focussed on the case of conformally flat geometries.[1] Finally, in a generic background with non-dynamical fields, the energy-momentum tensor is not conserved, even classically.

One may be concerned that the definition (1.2) leads to an ambiguous result, due to the existence of local counterterms, that can shift arbitrarily the value of $\log Z$. However, it has been shown in [10] that in the context of new minimal supergravity [11], on $S^1 \times M_3$ all the possible supersymmetric counterterms vanish, strongly suggesting that the supersymmetric Casimir energy is not ambiguous. Thus, for supersymmetric theories, we regard (1.3) as a natural generalization of the results [6] in two dimensions. Notice that this is an exact result, thus valid for any value of the coupling constants. For example, it is insensitive to the superpotential of the theory.

The linear combination of central charges $\mathbf{a}$, $\mathbf{c}$ in (1.3) had previously appeared in [12], in the context of studies of the supersymmetric index. Recent papers exploring relations of the central charges $\mathbf{a}$, $\mathbf{c}$ with the supersymmetric index include [13–15].

In this note we investigate further the supersymmetric Casimir energy. In particular, we show that it coincides with the vacuum expectation value (vev) of the Hamiltonian H_{susy}, appearing in the definition of the supersymmetric index

$$\langle H_{\text{susy}}\rangle = E_{\text{susy}} \ , \tag{1.4}$$

computed using zeta function regularization.

Since the results (1.3) and (1.4) are also valid at weak (or zero) coupling, one may compare these with the Casimir energy in free supersymmetric field theories, computed

[1] While here we also restrict ourselves to a round metric on S^3, the methods we use do not rely on conformal invariance and can be extended to the more general definition of the supersymmetric Casimir energy on a Hopf surface [2].

using zeta function techniques. In particular, for a free theory of N_v vector multiplets and N_χ chiral multiplets, this reads [7, 16]

$$E_{\text{free}} = \langle H_{\text{free}} \rangle = \frac{1}{192}\left(21N_v + 5N_\chi\right) \ . \tag{1.5}$$

This is the free field value of the Casimir energy of N_χ conformally coupled complex scalar fields, $N_\chi + N_v$ Weyl spinors, and N_v Abelian gauge fields, and does *not* agree with (1.4) [2, 5]. See e.g. [17] for a concise derivation. Working in the framework of rigid new minimal supergravity, we will consider the canonical Hamiltonian H of a multi-parameter family of supersymmetric theories defined on the round $\mathbb{R} \times S^3$ or its compactification to $S^1 \times S^3$. We will show that this interpolates continuously between H_{susy} and H_{free}, thus resolving the apparent tension between (1.4) and (1.5). Generically, the Hamiltonian is not a BPS quantity, and we find that its vev, computed using zeta function regularization, cannot be expressed as a linear combination of the anomaly coefficients **a**, **c**.

The rest of this note is organized as follows. In section 2 we introduce the background geometries and the field theories. In section 3 we reconsider the computation of the supersymmetric Casimir energy from its path integral definition. Section 4 contains the main results of this note. We examine the field theories in the canonical formalism and derive (1.4). Our conclusions are presented in section 5. Three appendices are included. In appendix A we collect details of the relevant spherical harmonics on the three-sphere. Appendix B contains the definition and some useful properties of the Hurwitz zeta function. In appendix C we write expressions for the energy-momentum tensor and other useful formulas.

2 Supersymmetric field theories

In this section we present the background geometry, that we view as a solution to the rigid limit of new supergravity and then introduce the relevant supersymmetric Lagrangians. We follow verbatim the notation of [2], to which we refer for more details.

2.1 Background geometry

We begin with the background in Euclidean signature, discussing the differences in Lorentzian signature later. We consider a background comprising the following metric

$$\begin{aligned} \mathrm{d}s^2(S^1 \times S^3) &= r_1^2 \mathrm{d}\tau^2 + \mathrm{d}s^2(S^3) \\ &= r_1^2 \mathrm{d}\tau^2 + \frac{r_3^2}{4}\left(\mathrm{d}\theta^2 + \sin\theta^2 \mathrm{d}\varphi^2 + (\mathrm{d}\varsigma + \cos\theta \mathrm{d}\varphi)^2\right) , \end{aligned} \tag{2.1}$$

where τ is a coordinate on S^1 and $\theta, \varphi, \varsigma$ with $0 \leq \theta < \pi\,, \varphi \sim \varphi + 2\pi\,, \varsigma \sim \varsigma + 4\pi$ are coordinates on a round three-sphere.[2] We note the Ricci scalar of this metric is given

[2]For $r_1 = 1$ and $r_3 = 2$ this metric and the other background fields can be obtained specializing the background discussed in appendix C of [2] to $v = 1$, $b_1 = -b_2 = 1/2$. Below we will set $r_1 = 1$ and $r_3 = 2$, but these can be easily restored by dimensional analysis.

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by $R = 6/r_3^2$, which is the same as the Ricci scalar of the 3d metric. We introduce the following orthonormal frame[3]

$$\begin{aligned} e^1 &= \frac{r_3}{2}\left(\cos\varsigma \mathrm{d}\theta + \sin\theta \sin\varsigma \mathrm{d}\varphi\right) , \\ e^2 &= \frac{r_3}{2}\left(-\sin\varsigma \mathrm{d}\theta + \sin\theta \cos\varsigma \mathrm{d}\varphi\right) , \\ e^3 &= \frac{r_3}{2}\left(\mathrm{d}\varsigma + \cos\theta \mathrm{d}\varphi\right) , \\ e^4 &= r_1 \mathrm{d}\tau , \end{aligned} \tag{2.2}$$

which corresponds to a *left-invariant* frame $\{e^1, e^2, e^3\}$ on S^3. We will consider a class of backgrounds admitting a solution to the new minimal supersymmetry equation

$$(\nabla_\mu - \mathrm{i}A_\mu + \mathrm{i}V_\mu + \mathrm{i}V^\nu \sigma_{\mu\nu})\,\zeta \;=\; 0 \,, \tag{2.3}$$

with the metric (2.1). Let us set $r_1 = 1$ and $r_3 = 2$ below. In our coordinates the supersymmetric complex Killing vector K reads

$$K = \frac{1}{2}\left(\partial_\varsigma - \mathrm{i}\partial_\tau\right) , \tag{2.4}$$

and the dual one-form is

$$K = \frac{1}{2}(e^3 - \mathrm{i}e^4) \,. \tag{2.5}$$

We define the following "reference" values of the background fields

$$\mathring{A} = \frac{3}{4}e^3 + \frac{\mathrm{i}}{2}\left(\mathfrak{q} - \frac{1}{2}\right)e^4 \,, \qquad\qquad \mathring{V} = \frac{1}{2}e^3 \,, \tag{2.6}$$

where we have included a constant $\mathfrak{q}$, which can be obtained performing a (large) gauge transformation $A \to A + \frac{\mathrm{i}}{2}\mathfrak{q}\mathrm{d}\tau$ starting from the gauge choice adopted in [2]. Although in Euclidean signature this yields an ill-defined spinor, this is not true in Lorentzian signature and later in the paper this parameter will play a role. Assuming that $U = \kappa K$, where κ is a *constant*, we can write the background fields as

$$A = \mathring{A} + \frac{3}{2}\kappa K \,, \qquad\qquad V = \mathring{V} + \kappa K \,. \tag{2.7}$$

We also note the value of the combination

$$A^{\mathrm{cs}} = A - \frac{3}{2}V \;=\; \mathring{A} - \frac{3}{2}\mathring{V} \;=\; \frac{\mathrm{i}}{2}\left(\mathfrak{q} - \frac{1}{2}\right)e^4 \,, \tag{2.8}$$

that is independent of κ. For generic values of κ, the solution ζ to (2.3) reads

$$\zeta = \frac{1}{\sqrt{2}}\mathrm{e}^{-\frac{1}{2}\mathfrak{q}\tau}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \,, \tag{2.9}$$

[3] Note that this frame is *different* from the frame used in [2].

where the normalization is chosen such that for $\mathfrak{q} = 0$ one has $|\zeta|^2 = 1/2$ as in [2]. Indeed, because τ is a periodic coordinate, this does not make sense unless[4] $\mathfrak{q} = 0$.

For generic values of κ this background preserves only a $\mathrm{SU}(2) \times \mathrm{U}(1)$ subgroup of the isometry group $\mathrm{SO}(4)$ of the round three-sphere. In the following, we will be interested in two special choices of κ. In particular, the choice $\kappa = \kappa^{\mathrm{ACM}} \equiv -1/3$ corresponds (for $\mathfrak{q} = 0$) to the values in [2], namely

$$A^{\mathrm{ACM}} = \frac{1}{2} e^3 \,, \qquad\qquad V^{\mathrm{ACM}} = \frac{1}{3}\left(e^3 + \frac{\mathrm{i}}{2} e^4\right) , \tag{2.10}$$

where notice that A^{ACM} is *real*. Another distinguished choice is $\kappa = \kappa^{\mathrm{st}} \equiv -1$, where the superscript stands for "standard", giving

$$A^{\mathrm{st}} = \frac{\mathrm{i}}{2}(1+\mathfrak{q}) e^4 \,, \qquad\qquad V^{\mathrm{st}} = \frac{\mathrm{i}}{2} e^4 \,. \tag{2.11}$$

For this choice, the full $\mathrm{SO}(4)$ symmetry of the three-sphere is restored and (2.3) has the more general solution

$$\zeta = \mathrm{e}^{-\frac{1}{2}\mathfrak{q}\tau} \zeta_0 \,, \tag{2.12}$$

with ζ_0 any constant spinor [4, 19].

Notice that, in addition to $\mathfrak{q} = 0$ [2], there are two other special values of the parameter $\mathfrak{q}$. Namely, for $\mathfrak{q} = -1$ the background field A^{st} in (2.11) vanishes, while for $\mathfrak{q} = 1/2$ we have $A^{\mathrm{cs}} = 0$. The significance of these three values will become clearer in later sections.

2.2 Lagrangians

We consider an $\mathcal{N} = 1$ supersymmetric field theory with a vector multiplet transforming in the adjoint representation of a gauge group G, and a chiral multiplet transforming in a representation $\mathcal{R}$, with Lagrangians given in section 2.2 of [2], evaluated in the background described in the previous section. These Lagrangians then depend on the two constant parameters $\mathfrak{q}$ and κ, as well as on the R-charges r_I of the scalar fields in the chiral multiplet, that below will be simply denoted r. We will restrict attention to Lagrangians expanded up to *quadratic* order in the fluctuations around a configuration where all fields vanish.

Adopting the notation of [2], we will therefore consider the following Lagrangian of a chiral multiplet

$$\begin{aligned}
\mathcal{L}^{\mathrm{chiral}} &= (\delta_\zeta V_1 + \delta_\zeta V_2 + \epsilon\, \delta_\zeta V_U)\,|_{\mathrm{quadratic}} \\
&= D_\mu \widetilde{\phi} D^\mu \phi + (V^\mu + (\epsilon - 1) U^\mu)\left(\mathrm{i} D_\mu \widetilde{\phi}\, \phi - \mathrm{i} \widetilde{\phi} D_\mu \phi\right) + \frac{r}{4}\left(R + 6 V_\mu V^\mu\right) \widetilde{\phi}\phi \\
&\quad + \mathrm{i} \widetilde{\psi}\, \widetilde{\sigma}^\mu D_\mu \psi + \left(\frac{1}{2} V^\mu + (1-\epsilon) U^\mu\right) \widetilde{\psi}\, \widetilde{\sigma}_\mu \psi \,,
\end{aligned} \tag{2.13}$$

where $D_\mu = \nabla_\mu - \mathrm{i} q_R A_\mu$ and q_R denotes the R-charges of the fields [2]. This is therefore the Lagrangian of $|\mathcal{R}| \equiv N_\chi$ free chiral multiplets, each with R-charge r, where we will denote as N_χ the dimension of the representation $\mathcal{R}$.

[4]Although an appropriately quantized pure imaginary value of $\mathfrak{q}$ would be allowed in (2.9), for generic R-charges we must have $\mathfrak{q} = 0$ for the correct periodicity of the matter fields [10].

The three terms in the first line of (2.13) are separately δ_ζ-exact, and the parameter ϵ allows us to continuously interpolate between the localizing Lagrangian in [2], obtained for $\epsilon = 0$, and the usual chiral multiplet Lagrangian [11], obtained for $\epsilon = 1$. Notice that at quadratic order the term $\delta_\zeta V_3$ [2] vanishes. Inserting the values of the background fields, and writing

$$\mathcal{L}^{\text{chiral}}(\mathfrak{q},\kappa,\epsilon,r) = \mathcal{L}^{\text{chiral}}_{\text{bos}}(\mathfrak{q},\kappa,\epsilon,r) + \mathcal{L}^{\text{chiral}}_{\text{fer}}(\mathfrak{q},\kappa,\epsilon,r)\,, \tag{2.14}$$

the bosonic part of the Lagrangian reads

$$\begin{aligned}\mathcal{L}^{\text{chiral}}_{\text{bos}}(\mathfrak{q},\kappa,\epsilon,r) \;=\; & -\widetilde{\phi}\partial_\tau^2\phi + \left[\frac{r}{2}\left(1-2\mathfrak{q}\right) + \kappa\left(\frac{3}{2}r-\epsilon\right)\right]\widetilde{\phi}\partial_\tau\phi - \widetilde{\phi}\nabla^i\nabla_i\phi \\ & +\mathrm{i}\left[\frac{3}{2}r - 1 + \kappa\left(\frac{3}{2}r-\epsilon\right)\right]\widetilde{\phi}\nabla_\varsigma\phi \\ & +\frac{r}{2}(1+\mathfrak{q})\left[\frac{r}{2}(2-\mathfrak{q}) + \kappa\left(\frac{3}{2}r-\epsilon\right)\right]\widetilde{\phi}\phi\,,\end{aligned} \tag{2.15}$$

where ∇_i, is the covariant derivative on the three-sphere, and we have omitted a total derivative. The fermionic part of the Lagrangian reads

$$\begin{aligned}\mathcal{L}^{\text{chiral}}_{\text{fer}}(\mathfrak{q},\kappa,\epsilon,r) \;=\; & \widetilde{\psi}\partial_\tau\psi - \mathrm{i}\widetilde{\psi}\gamma^a\partial_a\psi - \frac{1}{2}\left[\frac{3}{2}r - 1 + \kappa\left(\frac{3}{2}r-\epsilon\right)\right]\widetilde{\psi}\gamma_\varsigma\psi \\ & -\frac{1}{2}\left[\frac{1}{2}(r-1)(1-2\mathfrak{q}) + \frac{3}{2} + \kappa\left(\frac{3}{2}r-\epsilon\right)\right]\widetilde{\psi}\psi\,,\end{aligned} \tag{2.16}$$

where $a = 1,2,3$ are frame indices on the three-sphere and γ^a denote the Pauli matrices, generating the three-dimensional Clifford algebra. This expression is frame-dependent, and we used the left-invariant frame (2.2), which is useful for applying the angular momentum formalism. In particular, we used the identity

$$\mathrm{i}\widetilde{\sigma}^\mu\nabla_\mu\psi \;=\; \partial_\tau\psi - \mathrm{i}\gamma^a\nabla_a\psi = \partial_\tau\psi - \mathrm{i}\gamma^a\partial_a\psi - \frac{3}{4}\psi\,. \tag{2.17}$$

Notice that the Lagrangians in [4, 20] correspond to the values $\epsilon = 1$, $\kappa = -1$, and $\mathfrak{q} = 1/2$. Notice also that for $r = 2/3$ and $\epsilon = 1$ the total chiral multiplet Lagrangian does not depend on κ.

Let us introduce a compact notation, writing the Lagrangians above in terms of differential operators. Denoting ℓ_a the Killing vectors dual to the left-invariant frame e^a, and defining the "orbital" angular momentum operators as $L_a = \frac{\mathrm{i}}{2}\ell_a$, one finds these satisfy the SU(2) commutation relations

$$[L_a, L_b] = \mathrm{i}\epsilon_{abc}L_c\,, \tag{2.18}$$

and we have[5] $-\nabla^i\nabla_i = \vec{L}^2$ and $\nabla_\varsigma = -\mathrm{i}L_3$. Similarly, we identify the Pauli matrices with the spin operator as $S^a = \frac{1}{2}\gamma^a$, satisfying the same SU(2) algebra. Thus the Lagrangians

[5] Recall that here we have set $r_3 = 2$. In general, the three-dimensional Laplace operator is $r_3^2\nabla^i\nabla_i = \sum_a(\ell_a)^2$.

can be written as

$$\begin{aligned}\mathcal{L}^{\text{chiral}}_{\text{bos}} &= \widetilde{\phi}\,\widetilde{\mathcal{O}}_b\,\phi = \widetilde{\phi}\left(-\partial_\tau^2 + 2\mu\partial_\tau + \mathcal{O}_b\right)\phi\,,\\ \mathcal{L}^{\text{chiral}}_{\text{fer}} &= \widetilde{\psi}\,\widetilde{\mathcal{O}}_f\,\psi = \widetilde{\psi}\left(\partial_\tau + \mathcal{O}_f\right)\psi\,,\end{aligned} \tag{2.19}$$

where

$$\begin{aligned}\mathcal{O}_b &= 2\alpha_b\vec{L}^2 + 2\beta_b L_3 + \gamma_b\,,\\ \mathcal{O}_f &= 2\alpha_f\vec{L}\cdot\vec{S} + 2\beta_f S_3 + \gamma_f\,,\end{aligned} \tag{2.20}$$

with the constants taking the values $\alpha_b = \frac{1}{2}$,

$$\begin{aligned}\beta_b &= -\frac{1}{2} + \frac{3}{4}r + \frac{\kappa}{2}\left(\frac{3}{2}r - \epsilon\right)\,,\\ \gamma_b &= \frac{r}{2}(1+\mathfrak{q})\left[\frac{r}{2}(2-\mathfrak{q}) + \kappa\left(\frac{3}{2}r - \epsilon\right)\right]\,,\\ \mu &= \frac{1}{2}\left[\frac{r}{2}\left(1-2\mathfrak{q}\right) + \kappa\left(\frac{3}{2}r - \epsilon\right)\right]\,,\end{aligned} \tag{2.21}$$

and $\alpha_f = -1$, $\beta_f = -\beta_b$,

$$\gamma_f = -\left[\frac{1}{4}(r-1)(1-2\mathfrak{q}) + \frac{3}{4} + \frac{\kappa}{2}\left(\frac{3}{2}r - \epsilon\right)\right]\,, \tag{2.22}$$

respectively.

For the vector multiplet the quadratic Lagrangian is

$$\mathcal{L}^{\text{vector}} = \mathrm{Tr}\left[\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{\mathrm{i}}{2}\lambda\,\sigma^\mu D^{\text{cs}}_\mu\widetilde{\lambda} + \frac{\mathrm{i}}{2}\widetilde{\lambda}\,\widetilde{\sigma}^\mu D^{\text{cs}}_\mu\lambda\right]_{\text{quadratic}}\,, \tag{2.23}$$

where $D^{\text{cs}}_\mu = \nabla_\mu - \mathrm{i}q_R A^{\text{cs}}_\mu$. $\mathcal{F}$ is the linearized field strength of the gauge field $\mathcal{A}$ and λ is the gaugino, both transforming in the adjoint representation of the gauge group G. This is therefore the Lagrangian of $|G| \equiv N_v$ free vector multiplets, where we will denote N_v the dimension of the gauge group G.

The fermionic part of this Lagrangian can be put in the same form as the fermionic part of the chiral multiplet Lagrangian, namely

$$\mathcal{L}^{\text{vector}}_{\text{fer}} = \widetilde{\lambda}\,\widetilde{\mathcal{O}}^{\text{vec}}_f\,\lambda = \widetilde{\lambda}\left(\partial_\tau + \mathcal{O}^{\text{vec}}_f\right)\lambda\,, \tag{2.24}$$

where

$$\mathcal{O}^{\text{vec}}_f = 2\alpha_v\vec{L}\cdot\vec{S} + 2\beta_v S_3 + \gamma_v\,, \tag{2.25}$$

with $\alpha_v = -1$, $\beta_v = 0$, $\gamma_v = \frac{\mathfrak{q}}{2} - 1$. Notice that for $\mathfrak{q} = 1/2$, corresponding to $A^{\text{cs}} = 0$, this reduces to the standard massless Dirac operator on the three-sphere.

3 Supersymmetric Casimir energy

In this section we will recover in our set-up the supersymmetric Casimir energy defined in [2] as

$$E_{\rm susy} = -\lim_{\beta\to\infty}\frac{\mathrm{d}}{\mathrm{d}\beta}\log Z(\beta)\,, \tag{3.1}$$

where Z is the *supersymmetric partition function*, namely the path integral on $S^1\times S^3$ with periodic boundary conditions for the fermions on S^1, computed using localization. Restoring the radii of S^1 and S^3, the dimensionless parameter β in [2] is given by

$$\beta = \frac{2\pi r_1}{r_3}\,. \tag{3.2}$$

Differently from [2], here we will not fix the value of κ, showing that owing to the pairing of bosonic and fermionic eigenvalues in the one-loop determinant, the final result will be *independent* of κ. Although the computation in Euclidean signature requires to fix $\mathfrak{q}=0$, we will start presenting the explicit eigenvalues for generic values of $\mathfrak{q}$. We will then demonstrate that the pairing occurs if and only if $\mathfrak{q}=0$.

The partition function takes the form [2]

$$Z(\beta) = \mathrm{e}^{-\mathcal{F}(\beta)}\,\mathcal{I}(\beta)\,, \tag{3.3}$$

where $\mathcal{I}(\beta)$ is the supersymmetric index, and the pre-factor $\mathcal{F}(\beta) = -\mathrm{i}\pi\left(\Psi^{(0)}_{\rm chi}+\Psi^{(0)}_{\rm vec}\right)$ arises from the regularization of one-loop determinants in the chiral multiplets and vector multiplets, respectively [2] (see also [21]). Since the index $\mathcal{I}(\beta)$ does not contribute to (3.1), in order to compute $E_{\rm susy}$ we can restrict attention to $\Psi^{(0)}_{\rm chi}$ and $\Psi^{(0)}_{\rm vec}$, and thus effectively set the constant gauge field $\mathcal{A}_0=0$ in the one-loop determinants around the localization locus in [2]. In particular, as the vector multiplet Lagrangian does not depend on κ and ϵ, its contribution to $E_{\rm susy}$ can be simply borrowed from [2]. For example, by setting $|b_1|=|b_2|=r_1/r_3$ in eq. (4.33) of [2], one obtains

$$\Psi^{(0)}_{\rm vec} = \frac{\mathrm{i}}{6}\left(\frac{r_1}{r_3}-\frac{r_3}{r_1}\right)N_v\,. \tag{3.4}$$

For the chiral multiplet, we revisit the computation of the one-loop determinant (with $\mathcal{A}_0=0$) by working out the explicit eigenvalues for an arbitrary choice of the parameters κ and ϵ. The eigenvalues of the operators $\mathcal{O}_b$ and $\mathcal{O}_f$ can be obtained with elementary methods from the theory of angular momentum in quantum mechanics [22]. See appendix A for a summary of the relevant spherical harmonics. Thus, writing

$$\begin{aligned}\mathcal{O}_b\phi &= E_b^2\phi\,,\\ \mathcal{O}_f\psi &= \lambda^{\pm}\psi\,,\end{aligned} \tag{3.5}$$

for the scalar harmonics we have

$$E_b^2 = \frac{\alpha_b}{2}\ell(\ell+2)+2\beta_b m+\gamma_b\,, \tag{3.6}$$

where $\frac{\ell}{2}(\frac{\ell}{2}+1)$ for $\ell = 0, 1, 2, \ldots$ are the eigenvalues of $\vec{L}^2$, and $m = -\frac{\ell}{2}, \ldots, \frac{\ell}{2}$, are the eigenvalues of L_3. Each eigenvalue has degeneracy $(\ell+1)$, due to the $\mathrm{SU}(2)_R$ symmetry.

We distinguish two types of eigenvalues of $\mathcal{O}_f$. For any $\ell = 1, 2, 3, \ldots$ we have

$$\lambda^{\pm}_{\ell m} = -\frac{\alpha_f}{2} + \gamma_f \pm \sqrt{\frac{\alpha_f^2}{4}(\ell+1)^2 + \alpha_f\beta_f(1+2m) + \beta_f^2}\,, \tag{3.7}$$

where here the quantum number m takes the values $m = -\frac{\ell}{2}, \ldots, \frac{\ell}{2} - 1$. Furthermore, for any $\ell = 0, 1, 2, \ldots,$ we have the two special eigenvalues

$$\lambda_\ell^{\mathrm{special}\pm} = \frac{\alpha_f}{2}\ell \pm \beta_f + \gamma_f\ . \tag{3.8}$$

Again, each eigenvalue has degeneracy $(\ell+1)$, due to the $\mathrm{SU}(2)_R$ symmetry. Expanding the fields in Kaluza-Klein modes on S^1 as

$$\phi(x) = \sum_{k\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}k\tau}\phi_k(\theta, \varphi, \varsigma) \tag{3.9}$$

and similarly for ψ, we obtain the following eigenvalues for each mode

$$\begin{aligned} \widetilde{\mathcal{O}}_b\phi_k &= \left(k^2 - 2\mathrm{i}\mu k + E_b^2\right)\phi_k\,, \\ \widetilde{\mathcal{O}}_f\psi_k &= \left(-\mathrm{i}k + \lambda^{\pm}\right)\psi_k\ . \end{aligned} \tag{3.10}$$

For generic values of the quantum numbers ℓ, m, we say that the eigenvalues of the operators $\widetilde{\mathcal{O}}_b$ and $\widetilde{\mathcal{O}}_f$ are *paired*, if for all k we have

$$\left(-\mathrm{i}k + \lambda^+\right)\left(-\mathrm{i}k + \lambda^-\right) = -\left(k^2 - 2\mathrm{i}\mu k + E_b^2\right)\ . \tag{3.11}$$

Inserting the values of the parameters given in (2.21) and (2.22) we find that this is satisfied if and only if $\mathfrak{q} = 0$, and for any value of κ, ϵ, r. Let us then set $\mathfrak{q} = 0$ in the rest of this section. Restoring generic values of the radius r_3 of the S^3, the one-loop determinant for a fixed k is

$$Z^{(k)}_{1-\mathrm{loop}} = \frac{\det\widetilde{\mathcal{O}}_f}{\det\widetilde{\mathcal{O}}_b} = \frac{\prod_{\lambda^-}\left(-\mathrm{i}k + \frac{2}{r_3}\lambda^-\right)\prod_{\lambda^+}\left(-\mathrm{i}k + \frac{2}{r_3}\lambda^+\right)}{\prod_{E_b}\left(k^2 - \frac{4}{r_3}\mathrm{i}\mu k + \frac{4}{r_3^2}E_b^2\right)}\,, \tag{3.12}$$

where the products are over all the bosonic and fermionic eigenvalues, including the special ones. However, using the condition (3.11) all the paired eigenvalues cancel out.[6] For $m = \ell/2$ the generic fermionic eigenvalues do not exist, thus there are *unpaired* bosonic eigenvalues, obtained setting $m = \ell/2$ in (3.6), which read

$$(E_b^2)^{\mathrm{unpaired}} = \left(\frac{\alpha_f}{2}(\ell+1) + \beta_f\right)^2 - \mu^2 \qquad \ell = 0, 1, 2, \ldots\,, \tag{3.13}$$

and remain in the denominator of (3.12). Therefore, taking into account the contribution of the special fermionic eigenvalues in the numerator, and including the degeneracies, we obtain

$$Z^{(k)}_{1-\mathrm{loop}} = \prod_{n_0=1}^{\infty}\left(\frac{n_0 + 1 + r_3\mathrm{i}k - r}{n_0 - 1 - r_3\mathrm{i}k + r}\right)^{n_0}\,, \tag{3.14}$$

[6] Up to an irrelevant overall sign.

where we defined $n_0 = \ell + 1$ and used that $\alpha_f = -1$ and

$$\beta_f + \mu = \frac{1}{2}(1 - r) \ . \tag{3.15}$$

Upon obvious identifications, this coincides with the one-loop determinant of a $d = 3$, $\mathcal{N} = 2$ chiral multiplet on the round three-sphere, originally derived in [23] and [24], although our operators $\mathcal{O}_b$ and $\mathcal{O}_f$ are slightly more general and interpolate between those used in these two references. In particular, the Lagrangians used in [23] correspond to $\kappa = -1/3$ and $\epsilon = 0$, precisely as in [2], while those used in [24] correspond to $\kappa = -1$ and $\epsilon = 1$. Recall that in all cases we have $\mathfrak{q} = 0$.

Defining

$$z = 1 - r + \frac{r_3 \mathrm{i} k}{r_1} \,, \tag{3.16}$$

where r_1 can be restored simply rescaling the coordinate $\tau \to r_1 \tau$, one finds $Z^{(k)}_{1-\text{loop}}(z) = s_{b=1}(\mathrm{i}z)$, where $s_b(x)$ is the double sine function [24]. Alternatively, (3.14) can be written in terms of special functions by integrating the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}z} \log Z^{(k)}_{1-\text{loop}}(z) \ = \ -\pi z \cot(\pi z) \ , \tag{3.17}$$

where the Hurwitz zeta function has been used to regularize the infinite sum [23] (see appendix B).

In order to take the limit $\beta \to \infty$ it is more convenient to write (3.14) as an infinite product over two integers, namely

$$Z^{(k)}_{1-\text{loop}} = \prod_{n_1=0}^{\infty} \prod_{n_2=0}^{\infty} \frac{n_1 + n_2 + 1 + z}{n_1 + n_2 + 1 - z} \ . \tag{3.18}$$

Regularizing the infinite product over the Kaluza-Klein modes as in [2], one obtains $Z_{1-\text{loop}}$ in terms of Barnes triple-gamma functions, which eventually can be written as

$$Z_{1-\text{loop}} = \mathrm{e}^{\mathrm{i}\pi \Psi^{(0)}_{\text{chi}}} \, \widetilde{\Gamma}_e \left(\frac{\mathrm{i} r_1 r}{r_3}, \frac{\mathrm{i} r_1}{r_3}, \frac{\mathrm{i} r_1}{r_3} \right) \,, \tag{3.19}$$

where $\widetilde{\Gamma}_e$ is the elliptic gamma function [2] and[7]

$$\Psi^{(0)}_{\text{chi}} = \frac{\mathrm{i}}{6} \left(\frac{2 r_1}{r_3}(r-1)^3 - \left(\frac{r_1}{r_3} + \frac{r_3}{r_1} \right)(r-1) \right) \ . \tag{3.20}$$

From this, one finds the contribution of a chiral multiplet to (3.1) to be

$$E^{\text{chiral}}_{\text{susy}} = \frac{1}{12} \left(2(r-1)^3 - (r-1) \right) \ . \tag{3.21}$$

[7] Actually, the terms proportional to r_3/r_1 in (3.20) and (3.4) are not present if one uses a slightly different regularization, consistent with the results of [14]. However, this does not affect the terms proportional to r_1/r_3, which are relevant for the computation of E_{susy}. We thank B. Assel, D. Cassani, L. Di Pietro, and Z. Komargodski for discussions on this issue. See [18].

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This is exactly the contribution of a chiral multiplet with R-charge r to the total supersymmetric Casimir energy computed in [2], although we emphasize that here this has been derived for arbitrary values of the parameters κ and ϵ.

Combining the contributions of the chiral multiplets and the vector multiplets we recover the result

$$E_{\text{susy}} = \frac{4}{27}(\mathbf{a} + 3\mathbf{c}) , \tag{3.22}$$

with the anomaly coefficients defined as

$$\mathbf{a} = \frac{3}{32}(3\,\mathrm{tr}\mathbf{R}^3 - \mathrm{tr}\mathbf{R}) , \qquad \mathbf{c} = \frac{1}{32}(9\,\mathrm{tr}\mathbf{R}^3 - 5\,\mathrm{tr}\mathbf{R}) , \tag{3.23}$$

where $\mathbf{R}$ denotes the R-symmetry charge, and "tr" runs over the fermionic fields in the multiplets.

In the remainder of the paper we will show that (3.22) is also equal to the expectation value of the BPS Hamiltonian H_{susy}, appearing in supersymmetric index

$$\mathcal{I}(\beta) = \mathrm{Tr}\,(-1)^F \mathrm{e}^{-\beta H_{\text{susy}}} . \tag{3.24}$$

Therefore we now turn to the Hamiltonian formalism, working in a background with a non-compact time direction, thus with $\beta \to \infty$ from the outset.

4 Hamiltonian formalism

In this section we will study the theories defined in section 2.2 in a background $\mathbb{R} \times S^3$ in Lorentzian signature, obtained from the geometry in section 2.1 by a simple analytic continuation. In particular, we take the metric

$$\mathrm{d}s^2(\mathbb{R} \times S^3) = -\mathrm{d}t^2 + \mathrm{d}s^2(S^3) , \tag{4.1}$$

where t denotes the time coordinate on $\mathbb{R}$, and $\mathrm{d}s^2(S^3)$ is the metric on S^3, given in equation (2.1). Below we continue to set $r_3 = 2$. The background fields are obtained setting $A_t = -\mathrm{i}A_\tau$, $V_t = -\mathrm{i}V_\tau$, and $K_t = -\mathrm{i}K_\tau$, where here we must take $\kappa \in \mathbb{R}$. Moreover, the dynamical fields obey $\widetilde{\phi} = \phi^\dagger$ and $\widetilde{\psi} = \psi^\dagger$ from the start. The σ-matrices generating the appropriate Clifford algebra are obtained setting $\sigma^0_{\alpha\dot\alpha} = \mathrm{i}\sigma^4_{\alpha\dot\alpha} = \mathbb{1}_{\alpha\dot\alpha}$ and $\widetilde{\sigma}^0_{\alpha\dot\alpha} = \mathrm{i}\widetilde{\sigma}^4_{\alpha\dot\alpha} = \mathbb{1}_{\alpha\dot\alpha}$, with the remaining components unchanged, such that

$$\sigma_a\widetilde{\sigma}_b + \sigma_b\widetilde{\sigma}_a = -2\eta_{ab} , \qquad \widetilde{\sigma}_a\sigma_b + \widetilde{\sigma}_b\sigma_a = -2\eta_{ab} , \tag{4.2}$$

with $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$. The Lorentzian spinor ζ solving equation (2.3) for generic κ is then

$$\zeta = \frac{\mathrm{e}^{\frac{1}{2}\mathrm{i}qt}}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \tag{4.3}$$

again with a more general solution for the special value $\kappa = \kappa^{\text{st}} = -1$ [4, 19].

4.1 Conserved charges

The Hamiltonian[8] density $\mathcal{H} = \mathcal{H}_{\rm bos} + \mathcal{H}_{\rm fer}$, associated to the chiral multiplet Lagrangian (2.13), is obtained as usual, by defining the canonical momenta

$$\Pi = \partial_t \widetilde{\phi} - \mathrm{i}\mu\widetilde{\phi}\,, \qquad \widetilde{\Pi} = \partial_t \phi + \mathrm{i}\mu\phi\,, \qquad \pi^\alpha = \mathrm{i}\widetilde{\psi}_{\dot\alpha}\widetilde{\sigma}^{0\,\dot\alpha\alpha}\,, \qquad \widetilde{\pi}^\alpha = 0\,, \tag{4.4}$$

and its bosonic and fermionic parts read

$$\begin{aligned} \mathcal{H}_{\rm bos} &= \Pi\partial_t\phi + \widetilde{\Pi}\partial_t\widetilde{\phi} - \mathcal{L}^{\rm chiral}_{\rm bos}\,, \\ \mathcal{H}_{\rm fer} &= \pi\partial_t\psi + \widetilde{\pi}\partial_t\widetilde{\psi} - \mathcal{L}^{\rm chiral}_{\rm fer}\,, \end{aligned} \tag{4.5}$$

respectively. In terms of the operators $\mathcal{O}_b$ and $\mathcal{O}_f$ defined in equations (2.20) and (2.20), we have

$$\begin{aligned} \mathcal{H}_{\rm bos} &= \widetilde{\Pi}\Pi - \mathrm{i}\mu\big(\Pi\phi - \widetilde{\Pi}\widetilde{\phi}\big) + \widetilde{\phi}\left(\mathcal{O}_b + \mu^2\right)\phi\,, \\ \mathcal{H}_{\rm fer} &= -\widetilde{\psi}\mathcal{O}_f\psi\,. \end{aligned} \tag{4.6}$$

The Hamiltonian is then obtained by integrating[9] over the spatial S^3,

$$H = \int \sqrt{g_3}\mathrm{d}^3x\ \mathcal{H}\,. \tag{4.7}$$

The R-symmetry current $J^\mu_{\rm R}$ can be derived either from the Noether procedure or as the functional derivative of the action with respect to A_μ, namely

$$J^\mu_{\rm R} = \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta A_\mu}\,, \tag{4.8}$$

and it reads

$$J^\mu_{\rm R} = \mathrm{i}r\big(D^\mu\widetilde{\phi}\,\phi - \widetilde{\phi}D^\mu\phi\big) + 2r\big(V^\mu + \kappa(\epsilon-1)K^\mu\big)\widetilde{\phi}\phi + (r-1)\widetilde{\psi}\widetilde{\sigma}^\mu\psi\,. \tag{4.9}$$

This is conserved, i.e. $\nabla_\mu J^\mu_{\rm R} = 0$, and the corresponding conserved charge R is obtained by contracting it with the time-like Killing vector ∂_t, and integrating on the S^3, which yields

$$R = \int \sqrt{g_3}\mathrm{d}^3x\left(\mathrm{i}r\big(\widetilde{\phi}\widetilde{\Pi} - \phi\Pi\big) + (r-1)\widetilde{\psi}\widetilde{\sigma}^t\psi\right). \tag{4.10}$$

Rotational symmetry along the Killing vector ∂_ς gives rise to a conserved current with the corresponding conserved angular momentum

$$J_3 = -\mathrm{i}\int \sqrt{g_3}\mathrm{d}^3x\left((L_3\phi)\,\Pi + (L_3\widetilde{\phi})\,\widetilde{\Pi} + \mathrm{i}\widetilde{\psi}\,(L_3 + S_3)\,\psi\right). \tag{4.11}$$

Finally, supersymmetry gives rise to the conserved supercurrent

$$\zeta^\alpha J^\mu_{\mathrm{susy}\,\alpha} = -\sqrt{2}\zeta\sigma^\nu\widetilde{\sigma}^\mu\psi D_\nu\widetilde{\phi}\,. \tag{4.12}$$

[8] In this section we will consider mainly the chiral multiplet Lagrangian, therefore we will drop the superscript "chiral" from all the quantities.

[9] The integral is over the spatial S^3 with the metric $\mathrm{d}s^2(S^3)$ in (2.1). We define $\mathrm{d}^3x = \mathrm{d}\theta\mathrm{d}\varsigma\mathrm{d}\varphi$ and $g_3 = \sin^2\theta$ denotes the determinant of this metric.

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Using the equations of motion for the dynamical fields, after some calculations, one can verify that

$$\nabla_\mu(\zeta J^\mu_{\rm susy}) = 0 \ . \tag{4.13}$$

Note that $\nabla_\mu\zeta \neq 0$, and therefore $J^\mu_{\rm susy}$ is not conserved by itself, as is the case in the standard flat-space computation. Contracting $\zeta J^\mu_{\rm susy}$ with the time-like Killing vector ∂_t, we obtain the conserved supercharge

$$Q = -\sqrt{2}\int \mathrm{d}^3x\sqrt{g_3}\left(\zeta\psi\,\Pi - \mathrm{i}\widetilde{\phi}\zeta\widehat{\mathcal{O}}_f\psi\right) \ , \tag{4.14}$$

where we defined

$$\widehat{\mathcal{O}}_f \equiv 2\widehat{\alpha}\vec{S}\cdot\vec{L} + 2\widehat{\beta}S_3 + \widehat{\gamma} \ , \tag{4.15}$$

with

$$\widehat{\alpha} = -1 \ , \qquad \widehat{\beta} = \frac{3}{4}(1-r)\,, \qquad \widehat{\gamma} = -\frac{\kappa}{2}\left(\frac{3}{2}r - \epsilon\right) - \frac{3}{4} \ . \tag{4.16}$$

In summary, applying the Noether procedure to the Lagrangian (2.13), we have derived expressions for the Hamiltonian H, R-charge R, angular momentum J_3, and supercharge Q. These will provide the relevant operators in the quantized theory.

Let us briefly discuss other currents that can be considered, which however are not conserved generically. In particular, the usual energy-momentum tensor, defined as

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \ , \tag{4.17}$$

is *not conserved* in the presence of non-dynamical fields. This remains true even if $T_{\mu\nu}$ is contracted with a vector field that generates a symmetry of the metric and the other background fields. Thus, for example, T_{tt} does not define a conserved quantity, and in particular it does not coincide with the canonical Hamiltonian. Denoting the non-dynamical vector fields as A^I_μ, with $F^I = \mathrm{d}A^I$, and the associated currents as J^μ_I, in general the energy-momentum tensor (4.17) obeys the Ward identity

$$\nabla^\mu T_{\mu\nu} = \sum_I\left(F^I_{\mu\nu}J^\mu_I - A^I_\nu\nabla_\mu J^\mu_I\right) \ . \tag{4.18}$$

In the present case, after a tedious computation, one finds that the energy-momentum tensor satisfies

$$\begin{aligned}\nabla^\mu T_{\mu\nu} = {} & (\mathrm{d}A)_{\nu\mu}J^\mu_{\rm R} - \frac{3}{2}(\mathrm{d}V)_{\nu\mu}J^\mu_{\rm FZ} + (\mathrm{d}K)_{\nu\mu}J^\mu_K \\ & + \frac{3}{2}V_\nu\nabla_\mu J^\mu_{\rm FZ} - K_\nu\nabla_\mu J^\mu_K \ ,\end{aligned} \tag{4.19}$$

where $J^\mu_{\rm FZ}$ is the Ferrara-Zumino current

$$J^\mu_{\rm FZ} = -\frac{2}{3}\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta V_\mu} \ , \tag{4.20}$$

and

$$J^\mu_K = \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta K_\mu} \ . \tag{4.21}$$

Neither $J^\mu_{\rm FZ}$ nor J^μ_K are conserved. Explicit expressions for $T_{\mu\nu}$, $J^\mu_{\rm FZ}$, and J^μ_K are given in appendix C. Note that in this context, we must formally treat K_μ as a background field, although it was introduced in the Lagrangian as a shift of the original fields A_μ and V_μ. For the usual chiral multiplet Lagrangian with $\epsilon = 1$, however, one has $J^\mu_K = 0$.

For a generic Killing vector ξ, that is also a symmetry of the background fields, $\mathcal{L}_\xi A = \mathcal{L}_\xi V = 0$, we can define a conserved current as

$$Y^\mu_\xi = \xi_\nu \left(T^{\mu\nu} + J^\mu_{\rm R} A^\nu - \frac{3}{2} J^\mu_{\rm FZ} V^\nu + J^\mu_K K^\nu \right) \ . \tag{4.22}$$

One can show that indeed $\nabla_\mu Y^\mu_\xi = 0$. In particular, for $\xi = \partial_t$, one finds that the conserved charge is the Hamiltonian density

$$\mathcal{H} \ = \ -Y^t_{\partial_t} \ , \tag{4.23}$$

up to a total derivative on the S^3.

4.2 Canonical quantization

We now expand the dynamical fields in terms of creation and annihilation operators. Let us first focus on the scalar field. In order for the field ϕ to solve its equation of motion, we expand it as

$$\phi(x) = \sum_{\ell=0}^{\infty} \sum_{m,n=-\frac{\ell}{2}}^{\frac{\ell}{2}} \left(a_{\ell mn} u^{(+)}_{\ell mn}(x) + b^\dagger_{\ell mn} (u^{(-)}_{\ell mn})^*(x) \right) \ , \tag{4.24}$$

with[10]

$$u^{(\pm)}_{\ell mn}(x) \ \equiv \ \frac{1}{4\sqrt{\omega^\pm_{\ell m} \mp \mu}} \mathrm{e}^{-\mathrm{i}\omega^\pm_{\ell m} t} Y^{mn}_\ell(\vec{x}) \ , \tag{4.25}$$

where $Y^{mn}_\ell(\vec{x})$ are the scalar spherical harmonics on a three-sphere of unit radius (see appendix A for further details), and

$$\omega^\pm_{\ell m} = \pm\mu + \sqrt{\frac{\alpha_b}{2}\ell(\ell+2) \pm 2\beta_b m + \gamma_b + \mu^2} \ . \tag{4.26}$$

The canonical commutation relations

$$\begin{aligned} [\phi(t,\vec{x}), \Pi(t,\vec{x}\,')] \ &= \ \frac{\mathrm{i}}{\sqrt{-g}} \delta^{(3)}(\vec{x} - \vec{x}\,') \ , \\ [\phi(t,\vec{x}), \phi(t,\vec{x}\,')] \ &= \ [\Pi(t,\vec{x}), \Pi(t,\vec{x}\,')] \ = \ 0 \ , \end{aligned} \tag{4.27}$$

with $\delta^{(3)}(\vec{x} - \vec{x}\,') = \delta(\theta-\theta')\delta(\varphi-\varphi')\delta(\varsigma-\varsigma')$, hold by taking the oscillators to satisfy the usual

$$[a_{\ell mn}, a^\dagger_{\ell' m' n'}] = [b_{\ell mn}, b^\dagger_{\ell' m' n'}] = \delta_{\ell,\ell'}\delta_{m,m'}\delta_{n,n'} \ . \tag{4.28}$$

[10]Although all the eigenvalues in this paper never depend on the $\mathrm{SU}(2)_R \subset \mathrm{SO}(4)$ quantum number n, we keep track of this in the spherical harmonics and in the expansions.

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From (4.6) it follows that the Hamiltonian of the scalar field reads

$$
\begin{aligned}
H_{\text{bos}} = & \frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{m,n=-\frac{\ell}{2}}^{\frac{\ell}{2}} \omega^{+}_{\ell m}\left(a_{\ell mn}a^{\dagger}_{\ell mn} + a^{\dagger}_{\ell mn}a_{\ell mn}\right) \\
& + \frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{m,n=-\frac{\ell}{2}}^{\frac{\ell}{2}} \omega^{-}_{\ell m}\left(b_{\ell mn}b^{\dagger}_{\ell mn} + b^{\dagger}_{\ell mn}b_{\ell mn}\right) \ .
\end{aligned} \tag{4.29}
$$

Notice that we have used the *Weyl ordering* prescription, as this is the correct one for comparison with the path integral approach.

For the fermion, we expand the field ψ in terms of the spinor spherical harmonics $\mathbf{S}^{\pm}_{\ell mn}$. As discussed in appendix A, these are eigenspinors of the operator $\mathcal{O}_f$,

$$
\mathcal{O}_f \mathbf{S}^{\pm}_{\ell mn} = \lambda^{\pm}_{\ell m}\mathbf{S}^{\pm}_{\ell mn} \ , \tag{4.30}
$$

with the eigenvalues $\lambda^{\pm}_{\ell m}$ given in equation (3.7). In addition, there are the "special" spherical harmonics,

$$
\mathcal{O}_f \mathbf{S}^{\text{special}\pm}_{\ell n} = \lambda^{\text{special}\pm}_{\ell}\mathbf{S}^{\text{special}\pm}_{\ell n} \ , \tag{4.31}
$$

with $\lambda^{\text{special}\pm}_{\ell}$ given in equation (3.8). We expand the field ψ as

$$
\psi_\alpha = \sum_{\ell=0}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sum_{m=-\frac{\ell}{2}-1}^{\frac{\ell}{2}} c_{\ell mn}u_{\ell mn\,\alpha} + \sum_{\ell=1}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sum_{m=-\frac{\ell}{2}}^{\frac{\ell}{2}-1} d^{\dagger}_{\ell mn}v_{\ell mn\,\alpha} \ , \tag{4.32}
$$

with

$$
u_{\ell mn\,\alpha}(x) = \frac{1}{2\sqrt{2}}\mathrm{e}^{\mathrm{i}t\lambda^{-}_{\ell m}}\mathbf{S}^{-}_{\ell mn\,\alpha}(\vec{x}) \ , \qquad v_{\ell mn\,\alpha}(x) = \frac{1}{2\sqrt{2}}\mathrm{e}^{\mathrm{i}t\lambda^{+}_{\ell m}}\mathbf{S}^{+}_{\ell mn\,\alpha}(\vec{x}) \ . \tag{4.33}
$$

Here we included $\mathbf{S}^{\text{special}\pm}$ in the sums by defining

$$
\begin{aligned}
\mathbf{S}^{-}_{\ell,\frac{\ell}{2},n} &\equiv \mathbf{S}^{\text{special}+}_{\ell n} \ , & \lambda^{-}_{\ell,\frac{\ell}{2},n} &\equiv \lambda^{\text{special}+}_{\ell n} \ , \\
\mathbf{S}^{-}_{\ell,-\frac{\ell}{2}-1,n} &\equiv \mathbf{S}^{\text{special}-}_{\ell n} \ , & \lambda^{-}_{\ell,-\frac{\ell}{2}-1,n} &\equiv \lambda^{\text{special}-}_{\ell n} \ .
\end{aligned} \tag{4.34}
$$

Of course, by imposing the anti-commutation relations

$$
\{c_{\ell mn}, c^{\dagger}_{\ell mn}\} = \{d_{\ell mn}, d^{\dagger}_{\ell mn}\} = \delta_{\ell,\ell'}\delta_{m,m'}\delta_{n,n'} \ , \tag{4.35}
$$

one finds the field ψ_α and the conjugate momentum $\pi^\alpha = \mathrm{i}\widetilde{\psi}_{\dot\alpha}\widetilde{\sigma}^{0\,\dot\alpha\alpha}$ satisfy the canonical relations

$$
\begin{aligned}
\{\psi_\alpha(t,\vec{x}), \pi^\beta(t,\vec{x}\,')\} &= \frac{\mathrm{i}}{\sqrt{-g}}\delta^{(3)}(\vec{x}-\vec{x}\,')\,\delta_\alpha{}^\beta \ , \\
\{\psi_\alpha(t,\vec{x}), \psi_\beta(t,\vec{x}\,')\} &= \{\pi^\alpha(t,\vec{x}), \pi^\beta(t,\vec{x}\,')\} = 0 \ .
\end{aligned} \tag{4.36}
$$

The mode expansion (4.32) can now be inserted into the conserved charges of section 4.1, recalling that these have to be Weyl ordered. For example, the Hamiltonian density in (4.6) becomes

$$\mathcal{H}_{\rm fer} = \frac{1}{2}\big((\mathcal{O}_f\psi)\widetilde{\psi} - \widetilde{\psi}\mathcal{O}_f\psi\big)\ . \tag{4.37}$$

Inserting the mode expansion and integrating over the S^3 yields the quantized Hamiltonian

$$\begin{aligned} H_{\rm fer} \ =\ & \frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sum_{m=-\frac{\ell}{2}-1}^{\frac{\ell}{2}} \lambda^-_{\ell m}\left(c_{\ell mn}c^\dagger_{\ell mn} - c^\dagger_{\ell mn}c_{\ell mn}\right) \\ & -\frac{1}{2}\sum_{\ell=1}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sum_{m=-\frac{\ell}{2}}^{\frac{\ell}{2}-1} \lambda^+_{\ell m}\left(d_{\ell mn}d^\dagger_{\ell mn} - d^\dagger_{\ell mn}d_{\ell mn}\right)\ . \end{aligned} \tag{4.38}$$

In the next section we will turn to the computation of the expectation values of these Hamiltonians, and we will show that the infinite sums can be evaluated with (Hurwitz) zeta function regularization in two special cases. One case is obtained for $\mathfrak{q}=0$, for which we can use the pairing of bosonic and fermionic eigenvalues discussed in section 3 to evaluate the vev of $H = H_{\rm bos} + H_{\rm fer}$. Another case is obtained for $\beta_f = \beta_b = 0$, where we will be able to evaluate the vevs of $H_{\rm bos}$ and $H_{\rm fer}$ separately.

Thus, for simplicity in the remainder of this section we restrict to $\beta_f = \beta_b = 0$. Using the mode expansions of the fields, and after Weyl ordering, we obtain expressions for the remaining conserved charges of section 4.1. For the R-charge, equation (4.10), this leads to

$$\begin{aligned} R \ =\ & \frac{r}{2}\sum_{\ell=0}\sum_{m,n=-\frac{\ell}{2}}^{\frac{\ell}{2}}\left(a_{\ell mn}a^\dagger_{\ell mn} + a^\dagger_{\ell mn}a_{\ell mn}\right) - \frac{r}{2}\sum_{\ell=0}\sum_{m,n=-\frac{\ell}{2}}^{\frac{\ell}{2}}\left(b^\dagger_{\ell mn}b_{\ell mn} + b_{\ell mn}b^\dagger_{\ell mn}\right) \\ & -\frac{r-1}{2}\sum_{\ell=0}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sum_{m=-\frac{\ell}{2}-1}^{\frac{\ell}{2}}\left(c_{\ell mn}c^\dagger_{\ell mn} - c^\dagger_{\ell mn}c_{\ell mn}\right) \\ & +\frac{r-1}{2}\sum_{\ell=1}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sum_{m=-\frac{\ell}{2}}^{\frac{\ell}{2}-1}\left(d_{\ell mn}d^\dagger_{\ell mn} - d^\dagger_{\ell mn}d_{\ell mn}\right)\ . \end{aligned} \tag{4.39}$$

For the J_3 angular momentum, equation (4.11), we get

$$\begin{aligned} J_3 \ =\ & \frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{m=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}} m\big(a_{\ell mn}a^\dagger_{\ell mn} + a^\dagger_{\ell mn}a_{\ell mn}\big) \\ & +\frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{m=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}} m\big(b^\dagger_{\ell mn}b_{\ell mn} + b_{\ell mn}b^\dagger_{\ell mn}\big) \\ & -\frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sum_{m=-\frac{\ell}{2}-1}^{\frac{\ell}{2}}\left(m+\frac{1}{2}\right)\left(c_{\ell mn}c^\dagger_{\ell mn} - c^\dagger_{\ell mn}c_{\ell mn}\right) \\ & +\frac{1}{2}\sum_{\ell=1}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sum_{m=-\frac{\ell}{2}}^{\frac{\ell}{2}-1}\left(m+\frac{1}{2}\right)\left(d_{\ell mn}d^\dagger_{\ell mn} - d^\dagger_{\ell mn}d_{\ell mn}\right)\ , \end{aligned} \tag{4.40}$$

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and finally the supercharge (4.14) reads

$$
\begin{aligned}
Q = -\mathrm{i}\sum_{\ell=0}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sum_{m=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sqrt{\frac{\ell}{2}+m+1}\; a^{\dagger}_{\ell mn}c_{\ell mn} \\
-\mathrm{i}\sum_{\ell=1}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sum_{m=-\frac{\ell}{2}}^{\frac{\ell}{2}-1}(-1)^{-m-n}\sqrt{\frac{\ell}{2}-m}\; b_{\ell,-m,-n}d^{\dagger}_{\ell mn}\,.
\end{aligned}
\tag{4.41}
$$

By direct computation, one can now verify the following commutation relations

$$
[H,Q] = -\frac{\mathfrak{q}}{r_3}Q\,, \qquad [R,Q]=Q\,, \qquad [J_3,Q] = -\frac{1}{2}Q\,, \tag{4.42}
$$

where we restored the radius r_3 of the S^3. Note that the Hamiltonian commutes with Q only for $\mathfrak{q}=0$, which from equation (3.11) is the value required for the pairing of eigenvalues. By conjugating equation (4.41), one can further verify that

$$
\frac{r_3}{2}\{Q,Q^{\dagger}\} =: H + \frac{1}{r_3}(1+\mathfrak{q})R + \frac{2}{r_3}J_3 :\;\;, \tag{4.43}
$$

where $:\;\;:$ denotes normal ordering.

Let us set $r_3 = 1$ in (4.43) and (4.42) and comment on the special values of the parameter $\mathfrak{q}$ discussed in the literature. Setting $\mathfrak{q}=0$ we have

$$
\begin{aligned}
\frac{1}{2}\{Q,Q^{\dagger}\} &= :H+R+2J_3:\;\;, \\
[H,Q] &= 0\,,
\end{aligned}
\tag{4.44}
$$

corresponding[11] to the relations in eq. (5.9) of [25], where $H|_{\mathfrak{q}=0}$ coincides with H in that reference. For this reason we refer to $H|_{\mathfrak{q}=0} \equiv H_{\mathrm{susy}}$ as the BPS Hamiltonian.

Setting $\mathfrak{q}=1/2$ we have

$$
\begin{aligned}
\frac{1}{2}\{Q,Q^{\dagger}\} &= :H+\frac{3}{2}R+2J_3:\;\;, \\
[H,Q] &= -\frac{1}{2}Q\,,
\end{aligned}
\tag{4.45}
$$

which coincide for example with eq. in (7) of [20] as well as with eq. (6.11) in [25], where $H|_{\mathfrak{q}=1/2}$ corresponds to Δ in the latter reference.

Finally, setting $\mathfrak{q}=-1$ we have

$$
\begin{aligned}
\frac{1}{2}\{Q,Q^{\dagger}\} &= :H+2J_3:\;\;, \\
[H,Q] &= Q\,,
\end{aligned}
\tag{4.46}
$$

corresponding to eq. (5.6) of [25], where $H|_{\mathfrak{q}=-1}$ corresponds to P_0 in that reference.

[11]Here and below, the equations correspond up to convention dependent signs of R and J_3, as well a possible factor $\sqrt{2}$ in the supercharge Q, descending from the definition of the supersymmetry variations.

Although these commutation relations are here written for the chiral multiplet, it is straightforward to verify that they hold also for the vector multiplet, and hence for the total $H_{\text{tot}} = H + H_{\text{vec}}$, and similarly for the other operators. It was noticed in [5] that these may be formally derived from the abstract supersymmetry algebra of new minimal supergravity.

4.3 Casimir energy

We are now ready to compute the vacuum expectation value of the Hamiltonian. This yields infinite sums which we regularize using the zeta function method. Thus, for an operator A, we define its vacuum expectation value as

$$\langle A \rangle \equiv \lim_{s \to -1} \zeta_A(s) , \tag{4.47}$$

where, denoting with λ_n^A the set of all the eigenvalues (here n is a multi-index) of A and with d_n^A their degeneracies, the generalised zeta function is defined as

$$\zeta_A(s) = \mathrm{Tr}\, A^{-s} = \sum_n d_n^A (\lambda_n^A)^{-s} . \tag{4.48}$$

Notice that if $A = B + C$, with corresponding eigenvalues denoted as λ_n^B and λ_n^C, then

$$\lim_{s \to -1} \left(\sum_n (\lambda_n^B)^{-s} + \sum_n (\lambda_n^C)^{-s} \right) \neq \lim_{s \to -1} \sum_n (\lambda_n^B + \lambda_n^C)^{-s} . \tag{4.49}$$

This lack of additivity is related to the lack of associativity of functional determinants, $\det(BC) \neq \det(B) \cdot \det(C)$, which is known as "multiplicative anomaly". See e.g. [26].

In the present context, we use the following prescription for dealing with the infinite sums: for each given operator, we sum independently the eigenvalues corresponding to every different field. In particular, we define the vev of each operator as the sum of the vevs of the terms containing the fields ϕ, ψ, $\mathcal{A}$, and λ, respectively. Therefore, for example,

$$\langle H \rangle \equiv \langle H_{\text{bos}} \rangle + \langle H_{\text{fer}} \rangle , \tag{4.50}$$

and similarly for R, J_3, and Q. This recipe is in accordance with [27] and yields to the supersymmetric Casimir energy computed in [2].

The vevs of the scalar and fermion Hamiltonians, (4.29) and (4.38), are

$$\begin{aligned}
\langle H_{\text{bos}} \rangle &= \lim_{s \to -1} \left[\frac{1}{2} \sum_{\ell=0}^{\infty} \sum_{m,n=-\frac{\ell}{2}}^{\frac{\ell}{2}} (\omega_{\ell m}^{+})^{-s} + \frac{1}{2} \sum_{\ell=0}^{\infty} \sum_{m,n=-\frac{\ell}{2}}^{\frac{\ell}{2}} (\omega_{\ell m}^{-})^{-s} \right] , \\
\langle H_{\text{fer}} \rangle &= \lim_{s \to -1} \left[\frac{1}{2} \sum_{\ell=0}^{\infty} \sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}} \sum_{m=-\frac{\ell}{2}-1}^{\frac{\ell}{2}} (\lambda_{\ell m}^{-})^{-s} - \frac{1}{2} \sum_{\ell=1}^{\infty} \sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}} \sum_{m=-\frac{\ell}{2}}^{\frac{\ell}{2}-1} (\lambda_{\ell m}^{+})^{-s} \right] ,
\end{aligned} \tag{4.51}$$

respectively. However, due to the square roots appearing in both sets of eigenvalues $\omega_{\ell m}^{\pm}$ and $\lambda_{\ell m}^{\pm}$, the vevs in (4.51) cannot in general be separately regularized with any[12] zeta function and written in closed form.

[12] E.g. Hurwitz, Barnes, Shintani, Epstein zeta functions.

In the special case $\mathfrak{q}=0$, we can take advantage of the pairing as discussed in section 3 to compute the vev of the Hamiltonian of the chiral multiplet, $H = H_{\text{bos}} + H_{\text{fer}}$. Thus, setting $\mathfrak{q} = 0$ one has

$$\omega^+_{\ell m} = -\lambda^-_{\ell m}\,, \qquad \omega^-_{\ell,-m} = \lambda^+_{\ell m}\,, \qquad \text{for} \quad \ell \geq 1\,, \quad -\frac{\ell}{2} \leq m \leq \frac{\ell}{2} - 1\,. \tag{4.52}$$

The eigenvalues not included in equation (4.52) are the "special" fermion eigenvalues, which we can write as

$$\lambda_\ell^{\text{special}\pm} = -\frac{1}{2}(\ell+1) \pm \beta_f - \mu\,, \qquad \ell \geq 0\,, \tag{4.53}$$

and the "unpaired" bosonic eigenvalues

$$\omega^+_{\ell,\frac{\ell}{2}} = \frac{1}{2}(\ell+1) - \beta_f + \mu\,, \qquad \omega^-_{\ell,-\frac{\ell}{2}} = \frac{1}{2}(\ell+1) - \beta_f - \mu\,, \qquad \ell \geq 0\,. \tag{4.54}$$

Here we used that $\alpha_f < 0$ and assumed $\beta_f \leq -\frac{\alpha_f}{2}$ in order simplify the square roots in $\omega^+_{\ell,\frac{\ell}{2}}$ and $\omega^-_{\ell,-\frac{\ell}{2}}$. Due to the pairing, equation (4.52), all eigenvalues containing square roots exactly cancel against each other in (4.50), and we are left with

$$\begin{aligned}
\langle H\rangle_{\mathfrak{q}=0} = \lim_{s\to-1} & \left[\frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}} (\omega^+_{\ell,\frac{\ell}{2}})^{-s} + \frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}} (\omega^-_{\ell,-\frac{\ell}{2}})^{-s}\right.\\
& \left. + \frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}} (\lambda_\ell^{\text{special}+})^{-s} + \frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}} (\lambda_\ell^{\text{special}-})^{-s}\right]\\
= \lim_{s\to-1} & \left[\frac{1}{4}\sum_{k=1}^{\infty} k\,(k - 2(\beta_f+\mu))^{-s} - \frac{1}{4}\sum_{k=1}^{\infty} k\,(k + 2(\beta_f+\mu))^{-s}\right]\\
= & \frac{1}{12}(\beta_f+\mu)\left(1 - 8(\beta_f+\mu)^2\right)\,.
\end{aligned} \tag{4.55}$$

Notice that the first and third term in the first line further exactly cancelled and in the last step we regularized *separately* the two remaining sums using the Hurwitz zeta function. See appendix B. To summarize, since for $\mathfrak{q}=0$ one has $2(\beta_f+\mu) = 1-r$, the vev of the Hamiltonian of a chiral multiplet with R-charge r is

$$\langle H\rangle_{\mathfrak{q}=0} = \frac{1}{12 r_3}(1-r)\left(1 - 2(1-r)^2\right)\,, \tag{4.56}$$

where we restored r_3. This result is valid for any value of r, κ, ϵ. Notice that if we were to combine the two sums in the middle line of (4.55), before regularization, we would get a different result.

It is now simple to combine this with the contributions of the fields in the vector multiplet, and recover the supersymmetric Casimir energy E_{susy} of section 3. The Casimir energy of the gauge field does not depend on any of our parameters and is simply given by the result for an Abelian gauge field $\langle H_{\text{gauge}}\rangle = \frac{11}{120 r_3}$ (see e.g. [7]) multiplied by the dimension of the gauge group N_v. For the gaugino λ, the Casimir energy is computed as for the fermion ψ above, but using the operator $\mathcal{O}_f^{\text{vec}}$ in equation (2.25), giving simply $\langle H_{\text{gaugino}}\rangle = -\frac{1}{120 r_3}$, again multiplied by N_v. Adding everything together, we obtain

$$\langle H_{\text{tot}}\rangle_{\mathfrak{q}=0} = \frac{1}{12 r_3}\left(2\text{tr}\mathbf{R}^3 - \text{tr}\mathbf{R}\right) = \frac{4}{27 r_3}(\mathbf{a}+3\mathbf{c}) = \frac{1}{r_3}E_{\text{susy}}\ . \tag{4.57}$$

This result is valid for arbitrary values of κ and ϵ, in agreement[13] with (3.22). Indeed this is exactly the same BPS Hamiltonian defining the path integral, and therefore the free field result should have agreed with the localization result, that is valid for any value of the couplings.

Next, we consider the special case $\beta_b = \beta_f = 0$. This corresponds to setting $\kappa = -1$ and $\epsilon = 1$, but leaving arbitrary $\mathfrak{q}$. In this case, both sums in (4.51) can be separately regularized using Hurwitz zeta function, as the square roots in $\omega^{\pm}_{\ell m}$ and $\lambda^{\pm}_{\ell m}$ are absent, namely

$$\omega^+_\ell = \frac{1}{2}(\ell + 2 - r(1+\mathfrak{q}))\,, \qquad \omega^-_\ell = \frac{1}{2}(\ell + r(1+\mathfrak{q}))\ , \tag{4.58}$$

and

$$\lambda^-_\ell = \lambda^{\text{special}\pm}_\ell = -\frac{1}{2}(\ell + 2 - r(1+\mathfrak{q}) + \mathfrak{q})\,, \qquad \lambda^+_\ell = \frac{1}{2}(\ell + r(1+\mathfrak{q}) - \mathfrak{q})\ , \tag{4.59}$$

where we dropped the subscript m, as this quantum number becomes degenerate. Thus, regularizing the sums as described at the beginning of this subsection using the Hurwitz zeta function, we obtain the finite Casimir energies

$$\langle H_{\text{bos}}\rangle = \frac{1}{240}\left[1 - 10\big(r(1+\mathfrak{q}) - 1\big)^4\right]\ , \tag{4.60}$$

and

$$\begin{aligned}\langle H_{\text{fer}}\rangle = \frac{1}{240}\Big[&10(\mathfrak{q}+1)^3\,(1+\mathfrak{q})\,(r-1)^4 \\ &+20(\mathfrak{q}+1)^3(r-1)^3 - 10(\mathfrak{q}+1)(r-1) - 1\Big]\ .\end{aligned} \tag{4.61}$$

Adding (4.60) and (4.61), we obtain the Casimir energy of a chiral multiplet with R-charge r

$$\begin{aligned}\langle H\rangle = -\frac{1}{24}\Big[&\mathfrak{q}^4 + 2(\mathfrak{q}+1)^3(2\mathfrak{q}-1)(r-1)^3 \\ &+6\mathfrak{q}^2(\mathfrak{q}+1)^2(r-1)^2 + (\mathfrak{q}+1)\left(4\mathfrak{q}^3+1\right)(r-1)\Big]\ .\end{aligned} \tag{4.62}$$

This generalizes straightforwardly to an arbitrary number of chiral multiplets. As before, we can include easily an arbitrary number N_v of vector multiplets as well. In this case, for the gaugino, the Casimir energy can be obtained by formally setting $r = \frac{2\mathfrak{q}}{1+\mathfrak{q}}$ in equation (4.61), and reads

$$\langle H_{\text{gaugino}}\rangle = \frac{1}{240}\left(10\left(\mathfrak{q}^4 - 2\mathfrak{q}^3 + \mathfrak{q}\right) - 1\right). \tag{4.63}$$

[13]The quantity E_{susy} defined in [2] is dimensionless. Therefore, if writing the radius of the three-sphere explicitly, this has to be compared with the dimensionless combination $r_3\langle H_{\text{tot}}\rangle_{\mathfrak{q}=0}$.

Combining these results, we find that (for $\kappa = -1$, $\epsilon = 1$) using our regularization, the Casimir energy of a supersymmetric gauge theory with N_v vector multiplets and N_χ chiral multiplets with R-charges r_I is given by the following expression

$$\begin{aligned}\langle H_{\rm tot}\rangle = \frac{N_v}{12r_3}\left(\mathfrak{q}^4-2\mathfrak{q}^3+\mathfrak{q}+1\right)-\frac{1}{12r_3}\sum_{I=1}^{N_\chi}\Big(\mathfrak{q}^4+\left(4\mathfrak{q}^3+1\right)(\mathfrak{q}+1)(r_I-1)\\ +6\mathfrak{q}^2(\mathfrak{q}+1)^2(r_I-1)^2+2(2\mathfrak{q}-1)(\mathfrak{q}+1)^3(r_I-1)^3\Big)\,,\end{aligned}\tag{4.64}$$

where we restored the radius r_3 of the three-sphere. Setting $\mathfrak{q} = 0$ as in [2], we see that (4.64) reduce to

$$\langle H_{\rm tot}\rangle_{\mathfrak{q}=0} = \frac{4}{27r_3}(\mathbf{a}+3\mathbf{c})\,,\tag{4.65}$$

in agreement with (4.57).

In general, however, equation (4.64) cannot be written as a linear combination of $\mathbf{a}$ and $\mathbf{c}$. In the special case $\mathfrak{q} = 1/2$ and $r_I = 2/3$, corresponding to the usual conformally coupled scalars, Weyl spinors, and gauge fields, the Casimir energy (4.64) reduces to

$$\langle H_{\rm tot}\rangle_{\mathfrak{q}=\frac{1}{2},r_I=\frac{2}{3}} = \frac{1}{192r_3}(21N_v+5N_\chi) = \frac{1}{4r_3}(\mathbf{a}+2\mathbf{c})\,,\tag{4.66}$$

in accordance with standard zeta function computations (see e.g. [17]). In particular, notice that for theories with $N_\chi = 3N_v$ (so that $\mathbf{a} = \mathbf{c}$), such as $\mathcal{N} = 4$ super-Yang Mills, this becomes simply $\frac{3}{4r_3}\mathbf{a}$. However, the agreement with the CFT result of [8] for the Casimir energy is accidental [7, 8]. Finally, we note that for $\mathfrak{q} = -1$, the Casimir energy is independent of the R-charges and reads

$$\langle H_{\rm tot}\rangle_{\mathfrak{q}=-1} = \frac{1}{12r_3}\left(3N_v-N_\chi\right)\,.\tag{4.67}$$

This is simply because in this case $A = 0$, and therefore the Lagrangian does not depend on the R-charges, as we observed at the end of section 2.1.

We can also compute the vev[14] of the R-charge operator R, by using the regularization described above. For a single chiral multiplet of R-charge r we get $\langle R\rangle = \frac{1}{12}(r-1)$, where only the fermion ψ contributes, while for an (Abelian) vector multiplet we get $\langle R_{\rm vec}\rangle = \frac{1}{12}$, where again only the gaugino contributes. Thus, for the total R-charge operator $R_{\rm tot} = R + R_{\rm vec}$, we find

$$\langle R_{\rm tot}\rangle = \frac{4}{3}(\mathbf{a}-\mathbf{c})\,.\tag{4.68}$$

The results discussed in this section rely on the fact that the operators we are using are not normal ordered. See also [28] for a similar discussion.

[14] The vev of the supercharge Q is zero. For the chiral multiplet this follows from (4.14) or its mode expansion (4.41). Similarly, for the vector multiplet it follows from the explicit expression of $Q_{\rm vec}$, which is $Q_{\rm vec} = \frac{i}{2}\int\sqrt{g_3}\mathrm{d}^3x\,\zeta\sigma^\mu\tilde{\sigma}^\nu\sigma^0\widetilde{\lambda}\mathcal{F}_{\mu\nu}$.

5 Conclusions

The main purpose of this note was to examine aspects of the supersymmetric Casimir energy of $\mathcal{N}=1$ gauge theories introduced in [2], in the simplest case where the partition function depends only on one fugacity. It has been argued recently in [10] that this quantity does not suffer from ambiguities, and therefore it should be a good physical observable.

Firstly, by revisiting the localization computation in [2], we have verified explicitly that its value does not depend on the choice of the parameter κ, characterizing the background fields A and V, as expected. Secondly, we reproduced it by evaluating the expectation value of the BPS Hamiltonian that appears in the definition of the supersymmetric index, as anticipated in [12]. Our computations also clarify the relation of the supersymmetric Casimir energy with the Casimir energy of free conformal fields theories, demonstrating that these two quantities arise as the expectation values of two *different* Hamiltonians, evaluated using the *same* zeta function regularization method.

An obvious extension of this note is to compute the vev of the BPS Hamiltonian in the more general case of supersymmetric theories on a Hopf surface with two complex structure parameters, and to check that this agrees with the general form of the supersymmetric Casimir energy [2]. It should also be rewarding to study this problem from the viewpoint of [7–9, 29], or using the effective action approach of [14].

The discrepancy between the supersymmetric Casimir energy and the renormalized on-shell action in AdS_5, pointed out in [5], remains unexplained. We think it is imperative to elucidate how to reproduce the supersymmetric Casimir energy from a holographic computation.

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A Spherical harmonics

A.1 Scalar spherical harmonics

In this appendix we give some details on the scalar and spinor spherical harmonics on the three-sphere, following [19, 30]. We can obtain the metric on the unit three-sphere by considering a parametrization on $\mathbb{R}^4 \simeq \mathbb{C}^2$ with the metric,

$$\mathrm{d}s^2_{\mathbb{C}^2} = \mathrm{d}u\mathrm{d}\bar{u} + \mathrm{d}v\mathrm{d}\bar{v} \ . \tag{A.1}$$

The three-sphere of unit radius is then defined by

$$u\bar{u} + v\bar{v} = 1 \ . \tag{A.2}$$

The isometry group is SO(4) $\simeq$ SU(2)$_L$ $\times$ SU(2)$_R$, with generators[15] L_a^L and L_a^R, with $a = 1, 2, 3$, satisfying

$$\left[L_a^L, L_b^L\right] = \mathrm{i}\epsilon_{abc}L_c^L\,, \qquad \left[L_a^R, L_b^R\right] = \mathrm{i}\epsilon_{abc}L_c^R\,, \qquad \left[L_a^L, L_b^R\right] = 0\;. \tag{A.3}$$

As usual, we define raising and lowering operators,

$$L_\pm^L = L_1^L \pm \mathrm{i}L_2^L\,, \qquad L_\pm^R = L_1^R \pm \mathrm{i}L_2^R\;. \tag{A.4}$$

In the (u,v)-coordinates, these are represented by

$$\begin{aligned} L_+^L &= -u\partial_{\bar{v}} + v\partial_{\bar{u}}\,, \quad L_-^L = \bar{u}\partial_v - \bar{v}\partial_u\,,\\ L_+^R &= -u\partial_v + \bar{v}\partial_{\bar{u}}\,, \quad L_-^R = \bar{u}\partial_{\bar{v}} - v\partial_u\,, \end{aligned} \tag{A.5}$$

while

$$L_3^L = \frac{1}{2}\left(u\partial_u + v\partial_v - \bar{u}\partial_{\bar{u}} - \bar{v}\partial_{\bar{v}}\right)\,, \qquad L_3^R = \frac{1}{2}\left(u\partial_u - v\partial_v - \bar{u}\partial_{\bar{u}} + \bar{v}\partial_{\bar{v}}\right)\;. \tag{A.6}$$

In terms of these operators, the scalar Laplacian is

$$-\nabla_i\nabla^i = 4L_a^L L_a^L = 4L_a^R L_a^R\;. \tag{A.7}$$

The spherical harmonics Y_ℓ^{mn} are constructed starting from the highest weight state,

$$Y_\ell^{\frac{\ell}{2}\frac{\ell}{2}} = \sqrt{\frac{\ell+1}{2\pi^2}}\;u^\ell\,, \tag{A.8}$$

which is annihilated by the raising operators L_+^L and L_+^R. The number m (n) can be lowered by L_-^L (L_-^R), so that

$$Y_\ell^{mn} \;\propto\; \left(L_-^L\right)^{\frac{\ell}{2}-m}\left(L_-^R\right)^{\frac{\ell}{2}-n} Y_\ell^{\frac{\ell}{2}\frac{\ell}{2}}\,, \tag{A.9}$$

and take values $-\frac{\ell}{2} \le m, n \le \frac{\ell}{2}$. The spherical harmonics are eigenfunctions of the operator $\mathcal{O}_b$, equation (2.20),

$$\mathcal{O}_b Y_\ell^{mn} \;=\; E_b^2 Y_\ell^{mn}\,, \qquad E_b^2 = \frac{\alpha_b}{2}\ell(\ell+2) + 2\beta_b m + \gamma_b\,, \tag{A.10}$$

and also

$$L_3^L Y_\ell^{mn} = mY_\ell^{mn}\,, \qquad L_3^R Y_\ell^{mn} = nY_\ell^{mn}\;. \tag{A.11}$$

We normalize the spherical harmonics as [30]

$$Y_\ell^{\frac{\ell}{2}-a,\frac{\ell}{2}-b} = N_{\ell ab}\sum_k \frac{(-u)^{\ell+k-a-b}\bar{u}^k v^{b-k}\bar{v}^{a-k}}{k!(\ell+k-a-b)!(a-k)!(b-k)!}\,, \tag{A.12}$$

where the sum is over all integer values of k for which the exponents are non-negative, and

$$N_{\ell ab} = \sqrt{\frac{(\ell+1)a!b!(\ell-a)!(\ell-b)!}{2\pi^2}}\;. \tag{A.13}$$

[15] In the main text, we use the operators L_a^L, but drop the superscript L.

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Now specifically taking $u = \mathrm{i}\sin\frac{\theta}{2}\mathrm{e}^{\mathrm{i}(\varphi-\varsigma)/2}$ and $v = \cos\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}(\varphi+\varsigma)/2}$, one finds the metric

$$\mathrm{d}s^2_{S^3} = \frac{1}{4}\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2 + (\mathrm{d}\varsigma + \cos\theta \mathrm{d}\varphi)^2\right) , \tag{A.14}$$

and

$$L_3^L = \mathrm{i}\partial_\varsigma \qquad L_3^R = -\mathrm{i}\partial_\varphi . \tag{A.15}$$

With the above normalization, the spherical harmonics satisfy

$$\int \sqrt{g_3}\mathrm{d}^3x Y_\ell^{mn}\left(Y_{\ell'}^{m'n'}\right)^* = \delta_{\ell,\ell'}\delta^{m,m'}\delta^{n,n'} , \tag{A.16}$$

and

$$(Y_\ell^{mn})^* = (-1)^{m+n}Y_\ell^{-m,-n} , \tag{A.17}$$

as well as the completeness relation,

$$\sum_{\ell,m,n} Y_\ell^{mn}(\theta,\varphi,\varsigma)\left(Y_\ell^{mn}(\theta',\varphi',\varsigma')\right)^* = \frac{1}{\sin\theta}\delta^{(3)}(\vec{x}-\vec{x}\,') , \tag{A.18}$$

where $\delta^{(3)}(\vec{x}-\vec{x}\,') = \delta(\theta-\theta')\delta(\varphi-\varphi')\delta(\varsigma-\varsigma')$.

A.2 Spinor spherical harmonics

The spinor spherical harmonics can be constructed from the scalar harmonics. These are eigenspinors of the operator

$$\mathcal{O}_f = 2\alpha_f\vec{L}\cdot\vec{S} + 2\beta_f S_3 + \gamma_f , \tag{A.19}$$

where L_a are the left-invariant operators of the previous subsection, and $S_a = \frac{\gamma_a}{2}$, where γ_a are the Pauli matrices. For $\beta_f = 0$, the spinor spherical harmonics can be constructed as [19]

$$S^\pm_{\ell mn} = \begin{pmatrix} \cos\nu^\pm_{\ell m} Y_\ell^{mn} \\ \sin\nu^\pm_{\ell m} Y_\ell^{m+1,n}\end{pmatrix} , \tag{A.20}$$

where

$$\sin\nu^\pm_{\ell m} = \mp\sqrt{\frac{\ell+1\pm(2m+1)}{2(\ell+1)}} , \qquad \cos\nu^\pm_{\ell m} = \sqrt{\frac{\ell+1\mp(2m+1)}{2(\ell+1)}} . \tag{A.21}$$

For $S^+_{\ell mn}$, one has $\ell \geq 1$ and $-\frac{\ell}{2} \leq m \leq \frac{\ell}{2}-1$, while for $S^-_{\ell mn}$ one has $\ell \geq 0$ and $-\frac{\ell}{2}-1 \leq m \leq \frac{\ell}{2}$. In both cases $-\frac{\ell}{2} \leq n \leq \frac{\ell}{2}$. The spinor spherical harmonics satisfy the completeness relation

$$\sum_{m,n} S^\pm_{\ell mn\,\alpha}(x)\left(S^\pm_{\ell mn}(x)\right)^\dagger_{\dot\alpha} = \frac{1}{4\pi^2}n_\ell^\pm \mathbb{1}_{\alpha\dot\alpha} , \tag{A.22}$$

with $n_\ell^+ = \ell(\ell+1)$ and $n_\ell^- = (\ell+2)(\ell+1)$. Further, using the properties of Y_ℓ^{mn}, one can show the identities

$$\sum_{\ell,m,n}\left[S^+_{\ell mn\,\alpha}(x)\left(S^+_{\ell mn}(x')\right)^\dagger_{\dot\alpha} + S^-_{\ell mn\,\alpha}(x)\left(S^-_{\ell mn}(x')\right)^\dagger_{\dot\alpha}\right] = \frac{1}{\sin\theta}\delta^{(3)}(\vec{x}-\vec{x}\,')\,\mathbb{1}_{\alpha\dot\alpha} , \tag{A.23}$$

and

$$\int d^3x\sqrt{g_3}\, S^{\pm}_{\ell mn\,\alpha}(x)\left(S^{\pm'}_{\ell'm'n'}(x)\right)^{\dagger}_{\dot\alpha}\mathbb{1}^{\alpha\dot\alpha} = \delta_{\ell,\ell'}\delta_{m,m'}\delta_{n,n'}\delta^{\pm,\pm'}\,, \tag{A.24}$$

where the integral is on the unit three-sphere. Using that

$$\begin{aligned} L_+Y_\ell^{mn} &= \frac{1}{2}\sqrt{\ell(\ell+2)-4m(m+1)}Y_\ell^{m+1,n}\,,\\ L_-Y_\ell^{m+1,n} &= \frac{1}{2}\sqrt{\ell(\ell+2)-4m(m+1)}Y_\ell^{mn}\,, \end{aligned}\tag{A.25}$$

one can verify that

$$\mathcal{O}_f S^{\pm}_{\ell mn} = \lambda^{\pm}_\ell S^{\pm}_{\ell mn}\,, \tag{A.26}$$

with

$$\lambda^+_\ell = -\frac{\alpha_f}{2}(\ell+2)+\gamma_f\,,\qquad \lambda^-_\ell = \frac{\alpha_f}{2}\ell+\gamma_f\,. \tag{A.27}$$

When $\beta_f \neq 0$, the spinor spherical harmonics given by (A.20) are not eigenspinors of the operator $\mathcal{O}_f$, except the special cases

$$\begin{aligned} \mathbf{S}^{\text{special}+}_{\ell n} &\equiv S^-_{\ell,\frac{\ell}{2},n} = \begin{pmatrix} Y_\ell^{\frac{\ell}{2},n}\\ 0\end{pmatrix}, &\qquad \mathbf{S}^{\text{special}-}_{\ell n} &\equiv S^-_{\ell,-\frac{\ell}{2}-1,n} = \begin{pmatrix} 0\\ Y_\ell^{-\frac{\ell}{2},n}\end{pmatrix},\\ \mathcal{O}_f\mathbf{S}^{\text{special}\pm}_{\ell n} &= \lambda^{\text{special}\pm}_\ell\mathbf{S}^{\text{special}\pm}_{\ell n}\,, &\qquad \lambda^{\text{special}\pm}_\ell &= \left(\frac{\alpha_f}{2}\ell\pm\beta_f+\gamma_f\right). \end{aligned}\tag{A.28}$$

For the generic harmonics, the eigenspinors of $\mathcal{O}_f$ for general β_f are obtained by an SO(2) rotation,

$$\begin{pmatrix}\mathbf{S}^+_{\ell mn}\\ \mathbf{S}^-_{\ell mn}\end{pmatrix} \equiv \begin{pmatrix}\mathcal{R}_{11} & \mathcal{R}_{12}\\ \mathcal{R}_{21} & \mathcal{R}_{22}\end{pmatrix}\begin{pmatrix}S^+_{\ell mn}\\ S^-_{\ell mn}\end{pmatrix}. \tag{A.29}$$

The rotation matrix is given by

$$\begin{aligned} \mathcal{R}_{12} &= \mathcal{R}_{11}\frac{(\frac{\alpha_f}{2}(\ell+2)+\lambda^+_{\ell m}-\beta_f-\gamma_f)}{(\frac{\alpha_f}{2}\ell-\lambda^+_{\ell m}+\beta_f+\gamma_f)}\frac{\cos\nu^+_{\ell m}}{\cos\nu^-_{\ell m}}\,,\\ \mathcal{R}_{21} &= \mathcal{R}_{22}\frac{(-\frac{\alpha_f}{2}\ell+\lambda^-_{\ell m}-\beta_f-\gamma_f)}{(-\frac{\alpha_f}{2}(\ell+2)-\lambda^-_{\ell m}+\beta_f+\gamma_f)}\frac{\cos\nu^-_{\ell m}}{\cos\nu^+_{\ell m}}\,, \end{aligned}\tag{A.30}$$

with

$$\lambda^{\pm}_{\ell m} = -\frac{\alpha_f}{2}+\gamma_f\pm\sqrt{\frac{\alpha_f^2}{4}(\ell+1)^2+\alpha_f\beta_f(1+2m)+\beta_f^2}\,. \tag{A.31}$$

Requiring the matrix to be SO(2) fixes all the $\mathcal{R}_{ij}$, with a choice of overall sign fixed by requiring the matrix to be the identity matrix for $\beta_f = 0$. We then have

$$\mathcal{O}_f\mathbf{S}^{\pm}_{\ell mn} = \lambda^{\pm}_{\ell m}\mathbf{S}^{\pm}_{\ell mn}\,, \tag{A.32}$$

for $\ell \geq 1$, $-\frac{\ell}{2}\leq m\leq -\frac{\ell}{2}-1$, and $-\frac{\ell}{2}\leq m\leq\frac{\ell}{2}$.

B Hurwitz zeta function

In this appendix we include the definition of the Hurwitz zeta function and some useful properties. This is defined as the analytic continuation to complex $s \neq 1$, of the following series

$$\zeta_H(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}\,, \tag{B.1}$$

which is convergent for any $\mathrm{Re}(s) > 1$. Notice that

$$\zeta_H(s,1) = \zeta(s)\,, \tag{B.2}$$

corresponds to the Riemann zeta function.

For $s = -k$, where $k = 0, 1, 2, \ldots$, the Hurwitz zeta function reduces to the Bernoulli polynomials

$$\zeta_H(-k,a) = -\frac{B_{k+1}(a)}{k+1}\,, \tag{B.3}$$

defined as

$$B_k(a) = \sum_{n=0}^{k} \binom{k}{n} b_{k-n} a^n\,, \tag{B.4}$$

where b_n are the Bernoulli numbers. The first few ones read

$$\begin{aligned}
B_0(a) &= 1\,,\\
B_1(a) &= a - \frac{1}{2}\,,\\
B_2(a) &= a^2 - a + \frac{1}{6}\,,\\
B_3(a) &= a^3 - \frac{3}{2}a^2 + \frac{1}{2}a\,,\\
B_4(a) &= a^4 - 2a^3 + a^2 - \frac{1}{30}\,.
\end{aligned} \tag{B.5}$$

The following formulas used in the text are easily proved

$$\sum_{k=1}^{\infty} \frac{k}{(k+a)^s} = \zeta_H(s-1,a) - a\,\zeta_H(s,a)\,, \tag{B.6}$$

and

$$\sum_{k=1}^{\infty} \frac{k(k+1)}{(k+a)^s} = \zeta_H(s-2,a) + (1-2a)\zeta_H(s-1,a) + a(a-1)\zeta_H(s,a)\,. \tag{B.7}$$

C Energy-momentum tensor and other currents

In this appendix we provide explicit expressions for the energy-momentum tensor and other currents obtained from the (quadratic) chiral multiplet Lagrangian (2.13). Denoting with S the corresponding action, the energy-momentum tensor is defined as

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} \; . \tag{C.1}$$

A straightforward but tedious computation yields

$$\begin{aligned} T_{\mu\nu} = & \,(2\delta^{\rho}_{(\mu}\delta^{\lambda}_{\nu)} - g_{\mu\nu}g^{\rho\lambda})\Big[D_\rho\widetilde{\phi}D_\lambda\phi + \frac{3}{2}rV_\rho V_\lambda\widetilde{\phi}\phi \\ & +(V_\rho + \kappa\,(\epsilon-1)\,K_\rho)\left(\mathrm{i}D_\lambda\widetilde{\phi}\,\phi - \mathrm{i}\widetilde{\phi}D_\lambda\phi\right)\Big] \\ & +\frac{r}{2}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\widetilde{\phi}\phi + \frac{r}{2}\Big[g_{\mu\nu}\nabla_\rho\nabla^\rho(\widetilde{\phi}\phi) - \nabla_\mu\nabla_\nu(\widetilde{\phi}\phi)\Big] \\ & +\frac{\mathrm{i}}{2}D_{(\mu}\widetilde{\psi}\widetilde{\sigma}_{\nu)}\psi - \frac{\mathrm{i}}{2}\widetilde{\psi}\widetilde{\sigma}_{(\mu}D_{\nu)}\psi - \left(\frac{1}{2}V_{(\mu} + \kappa(1-\epsilon)K_{(\mu}\right)\widetilde{\psi}\widetilde{\sigma}_{\nu)}\psi \; , \end{aligned} \tag{C.2}$$

where the lower parenthesis denote symmetrization of the indices. Recall that we defined $D_\mu = \nabla_\mu - \mathrm{i}q_R A_\mu$, with q_R the R charges of the fields [2].

Below we collect some useful formulas for deriving this expression. For the bosonic part we used the variation of the Ricci tensor,

$$g^{\mu\nu}\delta R_{\mu\nu} = g_{\mu\nu}\nabla^\rho\nabla_\rho(\delta g^{\mu\nu}) - \nabla_\mu\nabla_\nu(\delta g^{\mu\nu}), \tag{C.3}$$

and we note that for any vector field X^μ,

$$[\nabla_\mu, \nabla_\nu]\, X^\mu = R_{\mu\nu}X^\mu \; . \tag{C.4}$$

For the femionic part, the variation of the action with respect to the metric gives

$$\delta S^{\text{chiral}}_{\text{fer}} = \int \mathrm{d}^4x \left[\delta\sqrt{-g}\,\mathcal{L}^{\text{chiral}}_{\text{fer}} + \sqrt{-g}\,\delta\mathcal{L}^{\text{chiral}}_{\text{fer}}\right] = \int \mathrm{d}^4x\sqrt{-g}\;\delta\mathcal{L}^{\text{chiral}}_{\text{fer}} \; , \tag{C.5}$$

where in the second equality we used that $\mathcal{L}^{\text{chiral}}_{\text{fer}}$ vanishes on-shell. The variation of the Lagrangian can be expressed in terms of variations of the vielbein and of the spin connection and reads

$$\delta\mathcal{L}^{\text{chiral}}_{\text{fer}} = \widetilde{\psi}\widetilde{\sigma}^a\left(\mathrm{i}D_\mu + \frac{1}{2}V_\mu + \kappa(1-\epsilon)K_\mu\right)\psi\,\delta e_a{}^\mu - \frac{\mathrm{i}}{2}\widetilde{\psi}\widetilde{\sigma}^\mu\sigma^{ab}\psi\,\delta\omega_{\mu ab} \; . \tag{C.6}$$

Using the property that the vielbein is covariantly constant,

$$0 = \nabla_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a - \Gamma^\rho_{\mu\nu}e_\rho{}^a + \omega_\mu{}^a{}_b e_\nu{}^b \; , \tag{C.7}$$

we read off the variation of the spin connection

$$\delta\omega_{\mu ab} = \delta\Gamma^\rho_{\mu\nu}e_{a\rho}e_b{}^\nu - e_b{}^\nu\nabla_\mu(\delta e_{a\nu}) \; . \tag{C.8}$$

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Further, using the variation of the Christoffel symbol

$$\delta\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g_{\mu\lambda} g_{\nu\rho} \nabla^{\sigma}(\delta g^{\lambda\rho}) - g_{\lambda(\mu}\nabla_{\nu)}(\delta g^{\lambda\sigma}) \ , \tag{C.9}$$

and

$$\delta e_{a\nu} = -g_{\nu\beta} e_{a\alpha} \delta g^{\alpha\beta} + g_{\mu\nu} \delta e_a{}^{\mu} \ , \tag{C.10}$$

we can write the variation of the spin connection as

$$\delta\omega_{\mu ab} = \nabla_{\nu} \left(g_{\mu\lambda} e_{[a}{}^{\nu} e_{b]\rho} \delta g^{\lambda\rho} + \frac{1}{2} e_{a\lambda} e_{b\rho} \delta^{\nu}_{\mu} \delta g^{\lambda\rho} - e_{b\rho} \delta^{\nu}_{\mu} \delta e_a{}^{\rho} \right) . \tag{C.11}$$

Using this, the second term of (C.6) can be written as,

$$\begin{aligned} -\frac{\mathrm{i}}{2} \widetilde{\psi} \widetilde{\sigma}^{\mu} \sigma^{ab} \psi \, \delta\omega_{\mu ab} = & -\widetilde{\psi} \widetilde{\sigma}^{a} \left(\mathrm{i} D_{\mu} + \frac{1}{2} V_{\mu} + \kappa(1-\epsilon) K_{\mu} \right) \psi \, \delta e_a{}^{\mu} \\ & + \frac{\mathrm{i}}{4} \left[D_{\mu} \widetilde{\psi} \widetilde{\sigma}_{\nu} \psi - \widetilde{\psi} \widetilde{\sigma}_{\mu} D_{\nu} \psi \right] \delta g^{\mu\nu} \\ & - \frac{1}{2} \left(\frac{1}{2} V_{\mu} + \kappa(1-\epsilon) K_{\mu} \right) \widetilde{\psi} \widetilde{\sigma}_{\nu} \psi \, \delta g^{\mu\nu} \ , \end{aligned} \tag{C.12}$$

up to a total divergence. Substituting this back into (C.6), the terms containing $\delta e_a{}^{\mu}$ cancel. The remaining terms are all proportional to $\delta g^{\mu\nu}$ and give the fermionic part of the energy-momentum tensor (C.2).

We also used the following identities for the σ-matrices in Lorentzian signature

$$\begin{aligned} \sigma^a \widetilde{\sigma}^b \sigma^c &= -\eta^{ab} \sigma^c + \eta^{ac} \sigma^b - \eta^{bc} \sigma^a + \mathrm{i} \epsilon^{abcd} \sigma_d \, , \\ \widetilde{\sigma}^a \sigma^b \widetilde{\sigma}^c &= -\eta^{ab} \widetilde{\sigma}^c + \eta^{ac} \widetilde{\sigma}^b - \eta^{bc} \widetilde{\sigma}^a - \mathrm{i} \epsilon^{abcd} \widetilde{\sigma}_d \, , \end{aligned} \tag{C.13}$$

with $\epsilon^{0123} = -1$, and the identities

$$[\nabla_{\mu}, \nabla_{\nu}] \, \psi = -\frac{1}{2} R_{\mu\nu ab} \sigma^{ab} \psi \ , \qquad [\nabla_{\mu}, \nabla_{\nu}] \, \widetilde{\psi} = -\frac{1}{2} R_{\mu\nu ab} \widetilde{\sigma}^{ab} \widetilde{\psi} \ , \tag{C.14}$$

valid for generic spinors ψ, $\widetilde{\psi}$.

One can easily compute the Ferrara-Zumino current

$$J^{\mu}_{\mathrm{FZ}} = -\frac{2}{3} \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta V_{\mu}} \ , \tag{C.15}$$

and the current

$$J^{\mu}_{K} = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta K_{\mu}} \ . \tag{C.16}$$

These read

$$J^{\mu}_{\mathrm{FZ}} = -\frac{2}{3} \left(\mathrm{i} \widetilde{\phi} D^{\mu} \phi - \mathrm{i} D^{\mu} \widetilde{\phi} \, \phi - 3 r V^{\mu} \widetilde{\phi} \phi + \frac{1}{2} \widetilde{\psi} \widetilde{\sigma}^{\mu} \psi \right) , \tag{C.17}$$

$$J^{\mu}_{K} = \kappa(1-\epsilon) \Big(\mathrm{i} D^{\mu} \widetilde{\phi} \, \phi - \mathrm{i} \widetilde{\phi} D^{\mu} \phi + \widetilde{\psi} \widetilde{\sigma}^{\mu} \psi \Big) , \tag{C.18}$$

and are not conserved. Starting with the expressions above, a further computation yields (4.19).

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