THE CLASS OF NON-DESARGUESIAN PROJECTIVE PLANES IS BOREL COMPLETE

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ABSTRACT. For every infinite graph Γ we construct a non-Desarguesian projective plane P^*_Γ of the same size as Γ such that $Aut(\Gamma) \cong Aut(P^*_\Gamma)$ and $\Gamma_1 \cong \Gamma_2$ iff $P^*_{\Gamma_1} \cong P^*_{\Gamma_2}$. Furthermore, restricted to structures with domain ω, the map $\Gamma \mapsto P^*_\Gamma$ is Borel. On one side, this shows that the class of countable non-Desarguesian projective planes is Borel complete, and thus not admitting a Ulm type system of invariants. On the other side, we rediscover the main result of [16] on the realizability of every group as the group of collineations of some projective plane. Finally, we use classical results of projective geometry to prove that the class of countable Pappian projective planes is Borel complete.

1. Introduction

Definition 1. *A* plane *is a system of points and lines satisfying:*

(A) every pair of distinct points determines a unique line;
(B) every pair of distinct lines intersects in at most one point;
(C) every line contains at least two points;
(D) there exist at least three non-collinear points.

A plane is projective *if in addition:*

(B') every pair of lines intersects in exactly one point.

As well-known (see e.g. [3] and [15, pg. 148]), the class of planes (resp. projective planes) corresponds canonically to the class of simple rank 3 matroids (resp. simple modular rank 3 matroids), or, equivalently, to the class of geometric lattices of rank 3 (resp. modular geometric lattices of rank 3). We prove:

Theorem 2. *For every graph $\Gamma = (V, E)$ there exists a plane P_Γ such that:*

(1) if Γ is finite, then P_Γ has size $3|V| + |E| + 17$;
(2) if Γ is infinite, then P_Γ has the same size of Γ;
(3) except for 17 points, every point of P_Γ is incident with at most two non-trivial lines;
(4) $Aut(\Gamma) \cong Aut(P_\Gamma)$;
(5) $\Gamma_1 \cong \Gamma_2$ if and only if $P_{\Gamma_1} \cong P_{\Gamma_2}$;
(6) restricted to structures with domain ω, the map $\Gamma \mapsto P_\Gamma$ is Borel (with respect to the naturally associated Polish topologies).

We then combine (a modification of) the construction $\Gamma \mapsto P_\Gamma$ of Theorem 2 with the the map $P \mapsto F(P)$ associating to each plane its free projective extension (in the sense of [10], cf. also Definition 11), and prove:

Theorem 3. *For every infinite graph Γ there exists a projective plane P^*_Γ such that:*

Partially supported by European Research Council grant 338821.

(1) P^*_Γ *has the same size of* Γ*;*
(2) P^*_Γ *is non-Desarguesian;*
(3) $Aut(\Gamma) \cong Aut(P^*_\Gamma)$*;*
(4) $\Gamma_1 \cong \Gamma_2$ *if and only if* $P^*_{\Gamma_1} \cong P^*_{\Gamma_2}$*;*
(5) restricted to structures with domain ω*, the map* $\Gamma \mapsto P^*_\Gamma$ *is Borel (with respect to the naturally associated Polish topologies).*

As a first consequence we get:

Definition 4. *(1) We say that a plane is simple (or* 17*-simple) if except for* 17 *points every point is incident with at most two non-trivial lines.*
(2) We denote by $\mathbf{K}_1$ *the class of countable simple planes.*
(3) We denote by $\mathbf{K}_2$ *the class of countable non-Desarguesian projective planes.*

Corollary 5. *Let* $\mathbf{K}$ *be either* $\mathbf{K}_1$ *or* $\mathbf{K}_2$ *(cf. Definition 4). Then:*
(1) $\mathbf{K}$ *is Borel complete (i.e. the isomorphism relation on* $\mathbf{K}$ *is* $Sym(\omega)$*-complete);*
(2) $\mathbf{K}$ *does not admit a Ulm type classification (cf.* [12] *for this notion).*

In [7] and [8] Frucht showed that every finite group is the group of automorphisms of a finite graph. Later, Sabadussi [17] and, independently, de Groot [5] proved that every group is the group of automorphisms of a graph. Using this, Harary, Piff, and Welsh [11] proved that every group is the group of automorphisms of a graphic matroid, possibly of infinite rank. In [1], Bonin and Kung showed that every infinite group is the group of automorphisms of a Dowling plane of the same cardinality. In [16], Mendelsohn proved that every group is the group of collineations of some projective plane. Using Theorems 2 and 3 we rediscover and improve these results:

Corollary 6. *(1) For every finite structure* M *(in the sense of model theory) there exists a simple plane* P_M *such that* P_M *is finite and* $Aut(P_M) \cong Aut(M)$*.*
(2) For every infinite structure M *(in the sense of model theory) there exists a simple plane* P_M *such that* $|M| = |P_M|$ *and* $Aut(P_M) \cong Aut(M)$*.*
(3) For every infinite structure M *there exists a non-Desarguesian projective plane* P_M *such that* $|M| = |P_M|$ *and* $Aut(P_M) \cong Aut(M)$*.*

Finally, we use classical results of projective geometry to prove:

Theorem 7. *Let* $\mathbf{K}_3$ *be the class of countable Pappian*[1] *projective planes. Then:*
(1) $\mathbf{K}_3$ *is Borel complete;*
(2) $\mathbf{K}_3$ *does not admit a Ulm type classification.*

We leave the following open problem:

Open Problem 8. *Characterize the Lenz-Barlotti classes of countable projective planes which are Borel complete.*

2. Preliminaries

Given a plane P we will freely refer to the canonically associated geometric lattice $G(P)$. On this see e.g. [3], or [14, Section 2], for an introduction directed to logicians. For our purposes the lattice-theoretic definitions in Definition 9(1-2) suffice.

Definition 9. *Let* P *be a plane.*

[1]Notice that Pappian planes are Desarguesian.

(1) *Given two distinct points a_1 and a_2 of P we let $a_1 \vee a_2$ be the unique line that they determine.*
(2) *Given two distinct lines ℓ_1 and ℓ_2 of P we let $\ell_1 \wedge \ell_2$ be the unique point in their intersection, if such a point exists, and 0 otherwise.*
(3) *The* size $|P|$ *of a plane P is the size of its set of points.*
(4) *We say that the point a (resp. the line ℓ) is* incident *with the line ℓ (resp. the point a) if the point a (resp. the line ℓ) is contained in the line ℓ (resp. contains the point a).*
(5) *We say that the line ℓ from P is* trivial *if ℓ is incident with exactly two points from P.*
(6) *We say that two lines ℓ_1 and ℓ_2 from P are* parallel *in P if $\ell_1 \wedge \ell_2 = 0$, i.e. there is no point $p \in P$ incident with both ℓ_1 and ℓ_2.*
(7) *We say that three distinct points a_1, a_2, a_3 of P are* collinear *if there is a line ℓ in P such that a_i is incident with ℓ for every $i = 1, 2, 3$ (in this case we also say that the set $\{a_1, a_2, a_3\}$ is dependent).*

We will use crucially the following fact from the theory of one-point extensions of matroids from [4] (see also [3, Chapter 10] and [14, Theorem 2.12]).

Fact 10. *Let P be a plane, L a set of parallel lines of P (in particular L can be empty or a singleton) and $p \notin P$. Then there exists a plane $P(L)$ (unique modulo isomorphism) such that its set of points is the set of points of P plus the point p, and p, q, r are collinear in $P(L)$ if and only if $q \vee r \in L$.*

We now introduce Hall's notion of free projective extension from [10]. In exposition and results we follow [13, Chapter XI].

Definition 11 (Cf. [13, Theorem 11.4])**.** *Given a plane P we define by induction on $n < \omega$ a chain of planes $(P_n : n < \omega)$ as follows:*
$n = 0$. Let $P_n = P$.
$n = m + 1$. For every pair of parallel lines $\ell \neq \ell'$ in P_m add a new point $\ell \wedge \ell'$ to P_m incident with only ℓ and ℓ'. Let P_n be the resulting plane.
We define the free projective extension *of P to be $F(P) := \bigcup_{n<\omega} P_n$.*

Definition 12. *Given two planes P_1 and P_2, we say that P_1 is a* subplane *of P_2 if $P_1 \subseteq P_2$, points of P_1 are points of P_2, lines of P_1 are lines of P_2, and the point p is on the line ℓ in P_1 if and only if the point p is on the line ℓ in P_2.*

Definition 13. *Let P be a plane.*
(1) *If P is* finite*, then we say that P is* confined *if every point of P is incident with at least three lines of P, and every line of P is non-trivial (cf. Definition 9(5)).*
(2) *We say that P is confined if every point and every line of P is contained in a finite confined subplane of P.*

We will make a crucial use of the following facts:

Definition 14 ([18, Definition 5.1.1])**.** *Let P be a projective plane. We say that P is* Desarguesian *if given two triples of distinct points p, q, r and p', q', r', if the lines $p \vee p'$, $q \vee q'$ and $r \vee r'$ are incident with a common point, then the points $(p \vee q) \wedge (p' \vee q')$, $(p \vee r) \wedge (p' \vee r')$ and $(q \vee r) \wedge (q' \vee r')$ are collinear.*

Fact 15 ([10, Theorem 4.6])**.** *Let P be a plane which is not a projective plane. Then $F(P)$ is non-Desarguesian.*

Fact 16 ([13, Theorem 11.11])**.** *Le P_1 and P_2 be confined planes. Then the following are equivalent:*
(1) $F(P_1) \cong F(P_2)$*;*
(2) $P_1 \cong P_2$*.*

Fact 17 ([13, Theorem 11.18])**.** *Let P be a confined plane. Then:*
$$Aut(P) \cong Aut(F(P)).$$

The following facts are classical results of projective geometry.

Definition 18 ([18, Definition 6.1.1])**.** *Let P be a projective plane. We say that P is* Pappian *if given two triples of distinct collinear points p, q, r and p', q', r' on distinct lines ℓ and ℓ', respectively, if $\ell \wedge \ell'$ is different from all six points, then the points $(p \vee q') \wedge (p' \vee q)$, $(p \vee r') \wedge (p' \vee r)$ and $(q \vee r') \wedge (q' \vee r)$ are collinear.*

Definition 19. *Given a field K we denote by $\mathfrak{P}(K)$ the corresponding projective plane (cf. e.g.* [13, Section 2]*).*

Fact 20 ([13, Theorem 2.6])**.** *Let K be a field. Then $\mathfrak{P}(K)$ is Pappian.*

Fact 21 ([13, Theorem 2.8])**.** *Let K and K' be fields. Then $\mathfrak{P}(K) \cong \mathfrak{P}(K')$ if and only if $K \cong K'$.*

Concerning the topological notions occurring in Theorem 2, they are in the sense of invariant descriptive set theory of $\mathfrak{L}_{\omega_1,\omega}$-classes, see e.g. [9, Chapter 11] for a thorough introduction. Notice that the classes of planes, simple planes, projective planes, (non-)Desarguesian projective planes (cf. Definition 14), and Pappian planes (cf. Definition 18) are first-order classes, considered e.g. in a language specifying points, lines and the point-line incidence relation.

3. Proof of Theorem 2

In this section we prove Theorem 2.

Notation 22. *We denote by P_* the plane represented in Figure 1. The plane P_* is taken from* [1]*, where it is denoted as T_S for $S = \{0, 1, 2, 3\}$.*

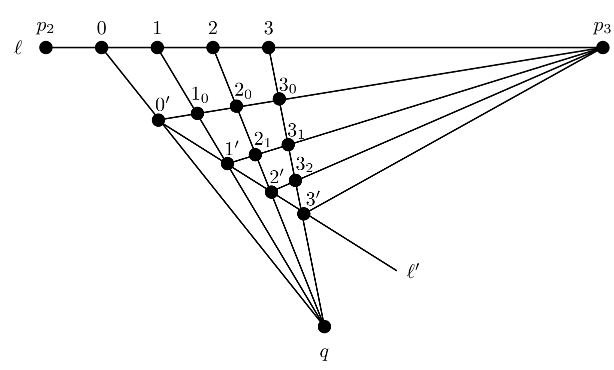

FIGURE 1. The plane P_*.

Proof of Theorem 2. Let $\Gamma = (V, E)$ be given and let $\{v_\alpha : \alpha < \lambda\}$ list V without repetitions. For $\gamma \leqslant \lambda$, let $\Gamma_\gamma = (V_\gamma, E_\gamma)$ be such that $V_\gamma = \{v_\beta : \beta < \gamma\}$ and for $\alpha < \beta < \gamma$ we have $v_\alpha E_\gamma v_\beta$ if and only if $v_\alpha E v_\beta$. Let P_* be the plane from Notation 22. Notice that $|P_*| = 17$ and, as proved in [1, Lemma 2], P_* is rigid, i.e. $Aut(P_*) = \{e\}$.

By induction on $\beta \leqslant \lambda$, we construct a plane $P_\Gamma(\beta)$ such that its set of points is:

$(*)\quad P_* \cup \{p_{(\alpha,0)} : \alpha < \beta\} \cup \{p_{(\alpha,1)} : \alpha < \beta\} \cup \{p_{(\alpha,2)} : \alpha < \beta\} \cup \{p_e : e \in E_\beta\}.$

For $\beta = 0$, let $P_\Gamma(\beta) = P_*$. For β limit ordinal, let $P_\Gamma(\beta) = \bigcup_{\alpha<\beta} P_\Gamma(\alpha)$. For $\beta = \alpha + 1$, we construct $P_\Gamma(\beta)$ from $P_\Gamma(\alpha)$ via a sequence of one-point extensions as follows. Firstly, add a new point $p_{(\alpha,0)}$ under the line $p_2 \vee 1'$ (using Fact 10 with $L = \{p_2 \vee 1'\}$). Secondly, add a new point $p_{(\alpha,1)}$ under the line $0 \vee 1'$ (using Fact 10 with $L = \{0 \vee 1'\}$). Thirdly, add a new point $p_{(\alpha,2)}$ under the line $p_{(\alpha,0)} \vee p_{(\alpha,1)}$ (using Fact 10 with $L = \{p_{(\alpha,0)} \vee p_{(\alpha,1)}\}$). Fourthly, for every $e = \{v_\delta, v_\alpha\} \in E_\beta$ add a point p_e under the parallel lines $p_{(\delta,0)} \vee p_{(\delta,1)}$ and $p_{(\alpha,0)} \vee p_{(\alpha,1)}$ (using Fact 10 with $L = \{p_{(\delta,0)} \vee p_{(\delta,1)}, p_{(\alpha,0)} \vee p_{(\alpha,1)}\}$). Let $P_\Gamma(\beta)$ be the resulting plane.

Let $P_\Gamma(\lambda) = P_\Gamma$. First of all, by $(*)$, the size of P_Γ is clearly as wanted. Also, if $p \notin P_*$, then, by construction, p is incident with at most two non-trivial lines. Furthermore, the construction of P_Γ from Γ is explicit, and so, restricted to structures with domain ω, the map $\Gamma \mapsto P_\Gamma$ is easily seen to be Borel, since to know a finite substructure of P_Γ it is enough to know a finite part of Γ. Thus, we are only left to show items (4) and (5) of the statement of the theorem. To this extent, first of all notice that, letting $p_{(\alpha,0)} \vee p_{(\alpha,1)} = \ell_\alpha$ (for $\alpha < \lambda$), we have:

($\star_1$) the set of lines $\{\ell_\alpha : \alpha < \lambda\}$ of P_Γ with edge relation $\ell_\alpha E \ell_\beta$ if and only if $\ell_\alpha \wedge \ell_\beta \neq 0$ (i.e. the two lines intersect) is isomorphic to Γ.

Now, for a point p let $\varphi(p)$ be the following statement:

(S) p is incident with exactly four distinct non-trivial lines, or p is incident with a non-trivial line ℓ which contains a point p' which is incident with four distinct non-trivial lines.

Notice that for a point $p \in P_\Gamma$ we have:

($\star_2$) $P_\Gamma \models \varphi(p)$ if and only if $p \in P_*$.

In fact, if the point $p \in P_*$, then either it is the point q, in which case there are four distinct non-trivial lines which are incident with it, or we can find a non-trivial line ℓ which is incident with the point p and contains the point p_3 (this is clear by inspection of Figure 1). On the other hand, if the point $p \notin P_*$, then it is either $p_{(\alpha,0)}$, $p_{(\alpha,1)}$, $p_{(\alpha,2)}$, or p_e, for some $\alpha < \lambda$ and $e \in E_\Gamma$. Notice now that:

($\star_3$) if $p = p_{(\alpha,0)}$, then p is incident with exactly two non-trivial lines, namely the lines $p_2 \vee 1'$ and $p_{(\alpha,0)} \vee p_{(\alpha,1)}$;

($\star_4$) if $p = p_{(\alpha,1)}$, then p is incident with exactly two non-trivial lines, namely the lines $0 \vee 1'$ and $p_{(\alpha,0)} \vee p_{(\alpha,1)}$;

($\star_5$) the point p_2 is incident with exactly two non-trivial lines, namely the line $p_2 \vee 0$ and the line $p_2 \vee 1'$; the point 0 is incident with exactly three non-trivial lines, namely the lines $p_2 \vee 0$, $0 \vee 0'$ and $0 \vee 1'$; the point $1'$ is incident with exactly three non trivial lines, namely the lines $1' \vee 0'$, $1' \vee 1_0$ and $1' \vee 2_1$;

($\star_6$) if $p = p_e$ and $e = \{v_\delta, v_\alpha\}$, then p_e is incident with exactly two non-trivial lines, namely the lines $p_{(\delta,0)} \vee p_{(\delta,1)}$ and $p_{(\alpha,0)} \vee p_{(\alpha,1)}$;

($\star_7$) for $\alpha < \lambda$, the set of points incident with the line $p_{(\alpha,0)} \vee p_{(\alpha,1)}$ is:

$$\{p_{(\alpha,0)}, p_{(\alpha,1)}, p_{(\alpha,2)}\} \cup \{p_e : p_\alpha \in e \in E_\Gamma\};$$

($\star_8$) if $\alpha \neq \beta < \lambda$, then $p_{(\alpha,0)} \vee p_{(\beta,1)}$ is a trivial line.

Thus, by ($\star_3$)-($\star_8$), it is clear that for $p \notin P_*$ we have that $P_\Gamma \not\models \varphi(p)$.

We now prove (5). Let $f : P_{\Gamma_1} \cong P_{\Gamma_2}$, $|\Gamma_1| = \lambda$ and, for $i = 1, 2$, let the set of points of P_{Γ_i} be:

$$\{(p, i) : p \in P_*\} \cup \{p^i_{(\alpha,j)} : j < 3, \alpha < \lambda\} \cup \{p^i_e : e \in E_{\Gamma_i}\},$$

(cf. ($*$) above). By $(\star)_2$, we have that f restricted to $\{(p, 1) : p \in P_*\}$ is an isomorphism from $\{(p, 1) : p \in P_*\}$ onto $\{(p, 2) : p \in P_*\}$, and so, as P_* is rigid, for every $p \in P_*$ we have that $f((p, 1)) = (p, 2)$. In particular, the line $(p_2, 1) \vee (1', 1)$ is mapped to the line $(p_2, 2) \vee (1', 2)$, and the line $(0, 1) \vee (1', 1)$ is mapped to the line $(0, 2) \vee (1', 2)$. Thus, f maps $\{p^1_{(\alpha,0)} : \alpha < \lambda\}$ onto $\{p^2_{(\alpha,0)} : \alpha < \lambda\}$ and $\{p^1_{(\alpha,1)} : \alpha < \lambda\}$ onto $\{p^2_{(\alpha,1)} : \alpha < \lambda\}$. Also, by ($\star_6$), f maps $\{p^1_{(\alpha,2)} : \alpha < \lambda\}$ onto $\{p^2_{(\alpha,2)} : \alpha < \lambda\}$. Finally, if $\alpha \neq \beta < \lambda$ and $f(p^1_{(\alpha,0)}) = p^1_{(\beta,0)}$, then $f(p^1_{(\alpha,1)}) = p^1_{(\beta,1)}$, since otherwise f would send the non-trivial line $p^1_{(\alpha,0)} \vee p^1_{(\alpha,1)}$ to a trivial line (cf. ($\star_8$)). Thus, f induces a bijection:

$$f_* : \{p^1_{(\alpha,0)} \vee p^1_{(\alpha,1)} : \alpha < \lambda\} \to \{p^2_{(\alpha,0)} \vee p^2_{(\alpha,1)} : \alpha < \lambda\}.$$

Hence, by $(\star)_1$, the map f_* induces an isomorphism from Γ_1 onto Γ_2, since clearly the isomorphism f sends pairs of intersecting lines to pairs of intersecting lines. Finally, item (4) is clear from the proof of item (5). ■

4. Proof of Theorem 3

In this section we prove Theorem 3.

Notation 23. *We denote by Q the plane represented in the matrix in Figure 2, where the letters occurring in the matrix represent the points of Q, and the columns of the matrix represent the lines of Q. The plane Q is taken from* [16] *(cf.* [16, Diagram 1]*), where it is attributed to S. Ditor.*

$$\begin{bmatrix} a & c & e & a & b & d & d & c & e & a & b \\ b & n & o & f & k & n & o & k & m & k & n \\ c & l & l & g & l & k & m & g & g & o & o \\ d & f & f & h & m & f & h & & & & \\ e & & & & & & & & & & \end{bmatrix}$$

Figure 2. The plane Q.

Strategy 24. *In proving Theorem 3 we will follow the following strategy:*

(1) for Γ an infinite graph, consider the P_Γ of Theorem 2 and extend it to a P^+_Γ adding independent copies of the plane Q (cf. Figure 2) at each point not in a finite confined subplane (cf. Definition 13(2)), and then adding independent copies of Q at each line not in a finite confined subplane, repeating this process for lines ω-many times (for points one application of the process suffices);

(2) *observe that, restricted to structures with domain ω, the set of P_Γ's is Borel and that the map $\Gamma \mapsto P_\Gamma \mapsto P^+_\Gamma$ is Borel;*
(3) *prove that $\Gamma \mapsto P^+_\Gamma$ is isomorphism invariant and that $Aut(\Gamma) \cong Aut(P^+_\Gamma)$;*
(4) *observe that, restricted to structures with domain ω, the map $P \mapsto F(P)$ (cf. Definition 11) is Borel;*
(5) *consider the free projective extension $F(P^+_\Gamma)$ of P^+_Γ, and use Fact 15 for non-Desarguesianess, Fact 16 for isomorphism invariance, and Fact 17 for:*

$$Aut(\Gamma) \cong Aut(F(P^+_\Gamma)).$$

First of all we deal with Strategy 24(4):

Lemma 25. *Restricted to structures with domain ω, the map $P \mapsto F(P)$ associating to each plane its free projective extension is a Borel map.*

Proof. Essentially as in the proof of Theorem 2. ■

Before proving Theorem 3 we isolate two constructions which will be crucially used in implementing Strategy 24(1).

Construction 26. *Let P be a plane and p a point of P. We define $P(p, Q, a)$ as the extension of P obtained by adding an independent copy of Q to P identifying the point p of P and the point a of Q, in such a way that if p' is a point of P different than p, and q is a point of Q different than a, then $p' \vee q$ is a trivial line.*

Construction 27. *Let P be a plane and ℓ a line of P. We define $P(\ell, Q, a \vee b)$ as the extension of P obtained by adding an independent copy of Q to P identifying the line ℓ of P and the line $a \vee b$ of Q, in such a way that if p' is a point of P not on ℓ, and q is a point of Q not on $a \vee b$, then $p' \vee q$ is a trivial line.*

Remark 28. *The construction of $P(p, Q, a)$ and $P(\ell, Q, a \vee b)$ from P can be formally justified using Fact 10. We elaborate on this:*

(i) *Concerning the case $P(p, Q, a)$. Add two generic points*[2] *b and f to P, corresponding to the points b and f of Q. Then $\langle p, b, f\rangle_P \cong \langle a, b, f\rangle_Q$ is a copy of the simple matroid of rank 3 and size 3. Now construct a copy of Q in P from $\{p, b, f\}$ point by point, following how Q is constructed from $\{a, b, f\}$ point by point. Notice that the order in which we do this does not matter.*
(ii) *Concerning the case $P(\ell, Q, a \vee b)$. Firs of all, let p and q be points of P such that $p \vee q = \ell$. Now, add one generic point*[3] *f to P, corresponding to the point f of Q. Then $\langle p, q, f\rangle_P \cong \langle a, b, f\rangle_Q$ is a copy of the simple matroid of rank 3 and size 3. Now construct a copy of Q in P from $\{p, q, f\}$ point by point, following how Q is constructed from $\{a, b, f\}$ point by point. Notice that the choice of p and q does not matter, as well as the order in which we construct the copy of Q in P from $\{p, q, f\}$, as observed also in (i).*

Proof of Theorem 3. We follow the strategy delineated in Strategy 24. Let Γ be an infinite graph and P_Γ be the respective plane from Theorem 2. We define P^+_Γ as the union of a chain of planes $(P^n_\Gamma : n < \omega)$, defined by induction on $n < \omega$.
$\underline{n = 0}$. Let $\{p_\alpha : 0 < \alpha < \kappa\}$ be an injective enumeration of the points of P_Γ not in a finite confined configuration (notice that there infinitely many such point in P_Γ). Let then:

[2] I.e. b and f are not incident with any line of P.
[3] I.e. f is not incident with any line of P.

(i) $P_\Gamma^{(0,0)} = P_\Gamma$;
(ii) $P_\Gamma^{(0,\alpha)} = P_\Gamma^{(0,\alpha-1)}(p_\alpha, Q, a)$, for $0 < \alpha < \kappa$ successor (cf. Construction 26);
(iii) $P_\Gamma^{(0,\alpha)} = \bigcup_{\beta<\alpha} P_\Gamma^{(0,\beta)}$, for α limit;
(iv) $P_\Gamma^0 = \bigcup_{\alpha<\kappa} P_\Gamma^{(0,\alpha)}$.

(Notice that the choice of the enumeration $\{p_\alpha : 0 < \alpha < \kappa\}$ does not matter, since the copies of Q that we add at every point are independent. In particular, in the countable case we can take the enumeration to be Borel. Furthermore, we now have that every point of P_Γ^0 is contained in a finite confined subplane of P_Γ^0.)
$\underline{n > 0}$. Let $\{\ell_\alpha : 0 < \alpha < \mu\}$ be an injective enumeration of the lines of P_Γ^{n-1} not in a finite confined configuration (notice that there infinitely many such lines in P_Γ^{n-1}, this is true for $n - 1 = 0$, and it is preserved by the induction). Let then:

(i) $P_\Gamma^{(n,0)} = P_\Gamma^{n-1}$;
(ii) $P_\Gamma^{(n,\alpha)} = P_\Gamma^{(0,\alpha-1)}(\ell_\alpha, Q, a \vee b)$, for $0 < \alpha < \mu$ successor (cf. Construction 27);
(iii) $P_\Gamma^{(n,\alpha)} = \bigcup_{\beta<\alpha} P_\Gamma^{(n,\beta)}$, for α limit;
(iv) $P_\Gamma^n = \bigcup_{\alpha<\mu} P_\Gamma^{(n,\alpha)}$.

(Notice that also in this case the choice of the enumeration $\{\ell_\alpha : 0 < \alpha < \mu\}$ does not matter, since the copies of Q that we add at every line are independent. In particular, in the countable case we can take the enumeration to be Borel. Furthermore, inductively, we maintain the condition that every point of P_Γ^n is contained in a finite confined subplane of P_Γ^n (although this is not true for lines).) Let then $P_\Gamma^+ = \bigcup_{n<\omega} P_\Gamma^n$. First of all, observe that the class of P_Γ's (P_Γ and Γ with domain ω) is Borel, since the appropriate restriction of the map $\Gamma \mapsto P_\Gamma$ is injective, in fact if $\Gamma \neq \Gamma'$, then there are $n \neq k \in \omega$ such that $nE_\Gamma k$ and $n \not\!E_{\Gamma'} k$ (by symmetry) and so in P_Γ the (codes of the) lines $p_{(n,0)} \vee p_{(n,1)}$ and $p_{(k,0)} \vee p_{(k,1)}$ are incident while in Γ' they are parallel. Furthermore, by the uniformity of the construction, the map P_Γ^+ from P_Γ is Borel, when restricted to structures with domain ω. Also, notice that the plane P_Γ^+ is confined and not projective, and so if we manage to complete Strategy 24(3), then by Lemma 25 and Facts 15, 16 and 17 we are done (as delineated in Strategy 24(4-5)). We are then only left with Strategy 24(3). To this extent notice that:

($\star_1$) the points from P_Γ^+ which are incident with at least four non-trivial lines are exactly the points of P_Γ.

Thus, from ($\star_1$) it is clear that if $P_{\Gamma_1}^+ \cong P_{\Gamma_2}^+$, then $P_{\Gamma_1} \cong P_{\Gamma_2}$, which in turn implies that $\Gamma_1 \cong \Gamma_2$ (cf. Theorem 2(5)). Furthermore, using again ($\star_1$), and the fact that by [16, Lemma 1] the plane Q has trivial automorphism group, it is easy to see that:

($\star_2$) every $f \in Aut(P_\Gamma^+)$ is induced by a $f^- \in Aut(P_\Gamma)$;

($\star_3$) every $f \in Aut(P_\Gamma)$ extends uniquely to a $f^+ \in Aut(P_\Gamma^+)$.

Thus, we have that $Aut(P_\Gamma^+) \cong Aut(P_\Gamma) \cong Aut(\Gamma)$, by Theorem 2(5). ■

5. Other proofs

Corollary 5 is a standard consequence of Theorems 2 and 3 (see e.g. [6] and [2] for an overview on Borel completeness, and [12] for Ulm invariants). Also, Corollary 6 follows from Theorems 2 and 3 and the following fact:

Fact 29. *(1) For every finite structure M (in the sense of model theory) there exists a finite graph Γ_M such that $Aut(\Gamma_M) \cong Aut(M)$.*
(2) For every infinite structure M (in the sense of model theory) there exists a graph Γ_M of the same cardinality of M such that $Aut(\Gamma_M) \cong Aut(M)$.

Finally, we prove Theorem 7. To this extent we need the following fact.

Fact 30 ([6, 3.2])**.** *The class of countable fields is Borel complete.*

Proof of Theorem 7. Immediate from Facts 20, 21 and 30. ■

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