POLISH TOPOLOGIES FOR GRAPH PRODUCTS OF CYCLIC GROUPS

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Abstract. We give a complete characterization of the graph products of cyclic groups admitting a Polish group topology, and show that they are all realizable as the group of automorphisms of a countable structure. In particular, we characterize the right-angled Coxeter groups (resp. Artin groups) admitting a Polish group topology. This generalizes results from [7], [9] and [4].

1. Introduction

Definition 1. *Let $\Gamma = (V, E)$ be a graph and $\mathfrak{p} : V \to \{p^n : p$ prime and $1 \leqslant n\} \cup \{\infty\}$ a graph colouring. We define a group $G(\Gamma, \mathfrak{p})$ with the following presentation:*

$$\langle V \mid a^{\mathfrak{p}(a)} = 1,\ bc = cb : \mathfrak{p}(a) \neq \infty \text{ and } bEc \rangle.$$

We call the group $G(\Gamma, \mathfrak{p})$ the Γ-*product*[1] *of the cyclic groups* $\{C_{\mathfrak{p}(v)} : v \in \Gamma\}$, or simply the *graph product of* $(\Gamma, \mathfrak{p})$. The groups $G(\Gamma, \mathfrak{p})$ where $\mathfrak{p}$ is constant of value ∞ (resp. of value 2) are known as *right-angled Artin groups* $A(\Gamma)$ (resp. *right-angled Coxeter groups* $C(\Gamma)$). These groups have received much attention in combinatorial and geometric group theory. In the present paper we tackle the following problem:

Problem 2. *Characterize the graph products of cyclic groups admitting a Polish group topology, and which among these are realizable as the group of automorphisms of a countable structure.*

This problem is motivated by the work of Shelah [7] and Solecki [10], who showed that no uncountable Polish group can be free or free abelian (notice that for Γ discrete (resp. complete) $A(\Gamma)$ is a free group (resp. a free abelian group)). These negative results have been later generalized by the authors to the class of uncountable right-angled Artin groups [4]. In this paper we give a complete solution to Problem 2 proving the following theorem:

Theorem 3. *Let $G = G(\Gamma, \mathfrak{p})$, and recall that $\mathfrak{p}$ is a graph colouring (cf. Definition 1), and so we refer to the elements in the range of $\mathfrak{p}$ as colors. Then G admits a Polish group topology if only if $(\Gamma, \mathfrak{p})$ satisfies the following four conditions:*

(a) there exists a countable $A \subseteq \Gamma$ such that for every $a \in \Gamma$ and $a \neq b \in \Gamma - A$, a is adjacent to b;

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[1] Notice that this is consistent with the general definition of graph products of groups from [2]. In fact every graph product of cyclic groups can be represented as $G(\Gamma, \mathfrak{p})$ for some Γ and $\mathfrak{p}$ as above.

(b) there are only finitely many colors c such that the set of vertices of color c is uncountable;
(c) there are only countably many vertices of color ∞;
(d) if there are uncountably many vertices of color c, then the set of vertices of color c has the size of the continuum.

Furthermore, if $(\Gamma, \mathfrak{p})$ satisfies conditions (a)-(d) above, then G can be realized as the group of automorphisms of a countable structure.

Thus, the only graph products of cyclic groups admitting a Polish group topology are the direct sums $G_1 \oplus G_2$ with G_1 a countable graph product of cyclic groups and G_2 a direct sum of finitely many continuum sized vector spaces over a finite field. From our general result we deduce a solution to Problem 2 in the particular case of right-angled Artin groups (already proved in [4]) and right-angled Coxeter groups.

Corollary 4. *No uncountable Polish group can be a right-angled Artin group.*

Corollary 5. *An uncountable right-angled Coxeter group $C(\Gamma)$ admits a Polish group topology if and only if it is realizable as the group of automorphisms of a countable structure if and only if $|\Gamma| = 2^\omega$ and there exists a countable $A \subseteq \Gamma$ such that for every $a \in \Gamma$ and $a \neq b \in \Gamma - A$, a is adjacent to b.*

In works in preparation we deal with the characterization problem faced here in the more general setting of graph products of general groups [6], and with questions of embeddability of graph products of groups into Polish groups [5].

2. Preliminaries

We will make a crucial use of the following special case of [9, 3.1].

Notation 6. *By a group term $\sigma(\bar{x})$ we mean a word in the alphabet $\{x : x \in \bar{x}\}$, i.e. an expression of the form $x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$, where $x_1, ..., x_n$ are from $\bar{x}$ and each ε_i is either 1 or -1. The number n is known as the length of the group term $\sigma(\bar{x})$.*

Fact 7 ([9]). *Let $G = (G, \mathfrak{d})$ be a Polish group and $\bar{g} = (\bar{g}_n : n < \omega)$, with $\bar{g}_n \in G^{\ell(n)}$ and $\ell(n) < \omega$.*

(1) For every non-decreasing $f \in \omega^\omega$ with $f(n) \geqslant 1$ and $(\varepsilon_n)_{n<\omega} \in (0,1)_{\mathbb{R}}^\omega$ there is a sequence $(\zeta_n)_{n<\omega}$ (which we call an f-continuity sequence for $(G, \mathfrak{d}, \bar{g})$, or simply an f-continuity sequence) satisfying the following conditions:
  (A) for every $n < \omega$:
    (a) $\zeta_n \in (0,1)_{\mathbb{R}}$ and $\zeta_n < \varepsilon_n$;
    (b) $\zeta_{n+1} < \zeta_n/2$;
  (B) for every $n < \omega$, group term $\sigma(x_0, ..., x_{m-1}, \bar{y}_n)$ and $(h_{(\ell,1)})_{\ell<m}, (h_{(\ell,2)})_{\ell<m} \in G^m$, the $\mathfrak{d}$-distance from $\sigma(h_{(0,1)}, ..., h_{(m-1,1)}, \bar{g}_n)$ to $\sigma(h_{(0,2)}, ..., h_{(m-1,2)}, \bar{g}_n)$ is $< \zeta_n$, when:
    (a) $m \leqslant n+1$;
    (b) $\sigma(x_0, ..., x_{m-1}, \bar{y}_n)$ has length $\leqslant f(n)+1$;
    (c) $h_{(\ell,1)}, h_{(\ell,2)} \in Ball(e; \zeta_{n+1})$;
    (d) $G \models \sigma(e, ..., e, \bar{g}_n) = e$.

(2) The set of equations $\Gamma = \{x_n = (x_{n+1})^{k(n)} d_n : n < \omega\}$ is solvable in G when for every $n < \omega$:
  (a) $f \in \omega^\omega$ is non-decreasing and $f(n) \geqslant 1$;
  (b) $1 \leqslant k(n) < f(n)$;

(c) $(\zeta_n)_{n<\omega}$ *is an* f*-continuity sequence;*
(d) $\mathfrak{d}(d_n, e) < \zeta_{n+1}$.

Convention 8. *If we apply Fact 7(1) without mentioning* $\bar{g}$ *it means that we apply Fact 7(1) for* $\bar{g}_n = \emptyset$*, for every* $n < \omega$.

We shall use the following observation freely throughout the paper.

Observation 9. *Suppose that* $(G, \mathfrak{d})$ *is Polish,* $A \subseteq G$ *is uncountable and* $\zeta > 0$*. Then for some* $g_1 \neq g_2 \in A$ *we have* $\mathfrak{d}((g_1)^{-1}g_2, e) < \zeta$.

Proof. First of all, notice that we can find $g_1 \in A$ such that g_1 is an accumulation point of A, because otherwise we contradict the separability of $(G, \mathfrak{d})$. Furthermore, the function $(x, y) \mapsto x^{-1}y$ is continuous and so for every $(x_1, y_1) \in G^2$ and $\zeta > 0$ there is $\delta > 0$ such that, for every $(x_2, y_2) \in G^2$, if $\mathfrak{d}(x_1, x_2), \mathfrak{d}(y_1, y_2) < \delta$, then $\mathfrak{d}((x_1)^{-1}y_1, (x_2)^{-1}y_2) < \zeta$. Let now $g_2 \in Ball(g_1; \delta) \cap A - \{g_1\}$, then $\mathfrak{d}((g_1)^{-1}g_2, (g_1)^{-1}g_1) = \mathfrak{d}((g_1)^{-1}g_2, e) < \zeta$, and so we are done. ■

Before proving Lemma 20 we need some preliminary work. Given $A \subseteq \Gamma$ we denote the induced subgraph of Γ on vertex set A as Γ_A.

Fact 10. *Let* $G = G(\Gamma, \mathfrak{p})$*,* $A \subseteq \Gamma$ *and* $G_A = (\Gamma_A, \mathfrak{p} \restriction A)$*. Then there exists a unique homomorphism* $\mathbf{p} = \mathbf{p}_A : G \to G_A$ *such that* $\mathbf{p}(c) = c$ *if* $c \in A$*, and* $\mathbf{p}(c) = e$ *if* $c \notin A$.

Proof. For arbitrary $G = G(\Gamma, \mathfrak{p})$, let $\Omega_{(\Gamma,\mathfrak{p})}$ be the set of equations from Definition 1 defining $G(\Gamma, \mathfrak{p})$. Then for the $\Omega_{(\Gamma,\mathfrak{p})}$ of the statement of the fact we have $\Omega_{(\Gamma,\mathfrak{p})} = \Omega_1 \cup \Omega_2 \cup \Omega_3$, where:
(a) $\Omega_1 = \Omega_{(\Gamma_A, \mathfrak{p} \restriction A)}$;
(b) $\Omega_2 = \Omega_{(\Gamma_{\Gamma - A}, \mathfrak{p} \restriction \Gamma - A)}$;
(c) $\Omega_3 = \{bc = cb : bE_\Gamma c \text{ and } \{b, c\} \not\subseteq A\}$.
Notice now that $\mathbf{p}$ maps each equation in Ω_1 to itself and each equation in $\Omega_2 \cup \Omega_3$ to a trivial equation, and so p is an homomorphism (clearly unique). ■

Definition 11. *Let* $(\Gamma, \mathfrak{p})$ *be as usual and* $G = G(\Gamma, \mathfrak{p})$.
(1) A word w *in the alphabet* Γ *is a sequence* $(a_1^{\alpha_1}, ..., a_k^{\alpha_k})$*, with* $a_1 \neq a_2 \neq \cdots \neq a_k \in \Gamma$ *and* $\alpha_1, ..., \alpha_k \in \mathbb{Z} - \{0\}$.
(2) We denote words simply as $a_1^{\alpha_1} \cdots a_k^{\alpha_k}$ *instead of* $(a_1^{\alpha_1}, ..., a_k^{\alpha_k})$.
(3) We call each $a_i^{\alpha_i}$ *a syllable of the word* $a_1^{\alpha_1} \cdots a_k^{\alpha_k}$.
(4) We say that the word $a_1^{\alpha_1} \cdots a_k^{\alpha_k}$ *spells the element* $g \in G$ *if* $G \models g = a_1^{\alpha_1} \cdots a_k^{\alpha_k}$.
(5) We say that the word w *is reduced if there is no word with fewer syllables which spells the same element of* G.
(6) We say that the consecutive syllables $a_i^{\alpha_i}$ *and* $a_{i+1}^{\alpha_{i+1}}$ *are adjacent if* $a_i E_\Gamma a_{i+1}$.
(7) We say that the word w *is a normal form for* g *if it spells* g *and it is reduced.*
(8) We say that two normal forms are equivalent if there they spell the same element $g \in G$.

As usual, when useful we identify words with the elements they spell.

Fact 12 ([3, Lemmas 2.2 and 2.3])**.** *Let* $G = G(\Gamma, \mathfrak{p})$.
(1) If the word $a_1^{\alpha_1} \cdots a_k^{\alpha_k}$ *spelling the element* $g \in G$ *is not reduced, then there exist* $1 \leqslant p < q \leqslant k$ *such that* $a_p = a_q$ *and* a_p *is adjacent to each vertex* $a_{p+1}, a_{p+2}, ..., a_{q-1}$.

(2) If $w_1 = a_1^{\alpha_1} \cdots a_k^{\alpha_k}$ and $w_2 = b_1^{\beta_1} \cdots b_k^{\beta_k}$ are normal forms for $g \in G$, then w_1 can be transformed into w_2 by repetedly swapping the order of adjacent syllables.

Definition 13. *Let $g \in G(\Gamma, \mathfrak{p})$. We define:*
(1) $sp(g) = \{a_i \in \Gamma : a_1^{\alpha_1} \cdots a_i^{\alpha_i} \cdots a_k^{\alpha_k}$ is a normal form for $g\}$;
(2) $F(g) = \{a_1^{\alpha_1} : a_1^{\alpha_1} \cdots a_k^{\alpha_k}$ is a normal form for $g\}$;
(3) $L(g) = \{a_k^{\alpha_k} : a_1^{\alpha_1} \cdots a_k^{\alpha_k}$ is a normal form for $g\}$;
(4) $\hat{L}(g) = \{a_k^{-\alpha_k} : a_k^{\alpha_k} \in L(g)\}$.

Definition 14. *We say that the normal form $a_1^{\alpha_1} \cdots a_k^{\alpha_k}$ is cyclically normal if either $k = 1$ or there is no equivalent normal form $b_1^{\beta_1} \cdots b_k^{\beta_k}$ with $b_1 = b_k$.*

Observation 15. *(1) Notice that if $g \in G(\Gamma, G_a)$ is spelled by a cyclically normal form, then any of the normal forms spelling g are cyclically normal.*
(2) We say that the group element $g \in G(\Gamma, G_a)$ is cyclically normal if any of the normal forms (which are words) spelling g are cyclically normal.

Notation 16. *Given a sequence of words $w_1, ..., w_k$ with some of them possibly empty, we say that the word $w_1 \cdots w_k$ is a normal form (resp. a cyclically normal form) if after deleting the empty words the resulting word is a normal form (resp. a cyclically normal form).*

Recall that given $A \subseteq \Gamma$ we denote the induced subgraph of Γ on vertex set A as Γ_A.

Fact 17 ([1, Corollary 24])**.** *Any element $g \in G(\Gamma, \mathfrak{p})$ can be written in the form $w_1 w_2 w_3 w_2' w_1^{-1}$, where:*
(1) $w_1 w_2 w_3 w_2' w_1^{-1}$ is a normal form;
(2) $w_3 w_2' w_2$ is cyclically normal;
(3) $sp(w_2) = sp(w_2')$;
(4) if $w_2 \neq e$, then $\Gamma_{sp(w_2)}$ is a complete graph;
(5) $F(w_2) \cap \hat{L}(w_2') = \emptyset$.

Proposition 18. *Let $G = G(\Gamma, \mathfrak{p})$, and assume that $\mathfrak{p}$ has finite range $\{c_1, ..., c_t\}$. Let p be a prime such that if $c_i \neq \infty$ then $p > c_i$, for $i = 1, ..., t$. Then for every $g \in G$ we have $sp(g) \subseteq sp(g^p)$.*

Proof. Let g be written as $w_1 w_2 w_3 w_2' w_1^{-1}$ as in Fact 17, and assume $g \neq e$. We make a case distinction.
Case 1. $w_3 = e$.
Notice that $w_2 w_2' \neq e$, because by assumption $g \neq e$, and that $w_2 w_2'$ is a normal form (recall Notation 16). Let $a_1^{\alpha_1} \cdots a_k^{\alpha_k}$ be a normal form for $w_2 w_2'$. Then by items (3) and (4) of Fact 17 we have:

$$g^p = w_1 (a_1^{\alpha_1} \cdots a_k^{\alpha_k})^p w_1^{-1} = w_1 a_1^{p\alpha_1} \cdots a_k^{p\alpha_k} w_1^{-1}.$$

Now, necessarily, for every $\ell \in \{1, ..., k\}$, $a_\ell^{p\alpha_\ell} \neq e$, since the order of a_ℓ does not divide α_ℓ and p is a prime. Thus, we are done.
Case 2. $w_2 = e$.
By item (3) of Fact 17 also $w_2' = e$, and so, by item (2) of Fact 17, $w_3 w_2' w_2 = w_3 \neq e$ is cyclically normal. Let $a_1^{\alpha_1} \cdots a_k^{\alpha_k}$ be a normal form for w_3.
Case 2.1. $k = 1$.
In this case, letting $a_k^{\alpha_k} = a^\alpha$, we have $g^p = w_1 a^{p\alpha} w_1^{-1}$, and so, arguing as in Case

1, we are done.
<u>*Case 2.2*</u>. $k > 1$.
In this case g^p is spelled by the following normal form:

$$w_1 \underbrace{w_3 \cdots w_3}_{p} w_1^{-1},$$

and so, clearly, we are done.
<u>*Case 3*</u>. $w_3 \neq e$ and $w_2 \neq e$.
In this case, letting w_0' stand for a normal form for $w_3 w_2' w_2$, g^p is spelled by the following normal form:

$$g^p = w_1 w_2 \underbrace{w_0' \cdots w_0'}_{p-1} w_3 w_2' w_1^{-1},$$

Furthermore, by item (3) and (5) of Fact 17, $sp(w_0') = sp(w_3) \cup sp(w_2) = sp(w_3) \cup sp(w_2') = sp(w_3) \cup sp(w_2) \cup sp(w_2')$, and so we are done. ∎

Proposition 19. *Let $G = G(\Gamma, \mathfrak{p})$ and $g \in G$.*
(1) If $a_1, a_2, b_1, b_2 \in \Gamma - sp(g)$ are distinct and a_i is not adjacent to b_i ($i = 1, 2$), then for every $n \geqslant 2$ the element $g a_1^{-1} a_2 b_1^{-1} b_2$ has no n-th root.
(2) If $a, b_1, b_2, b_3, b_4 \in \Gamma$ are distinct, a is not adjacent to b_i ($i = 1, 2, 3, 4$), and $\{b_1, b_2, b_3, b_4\} \cap sp(g) = \emptyset$, then for every $n \geqslant 2$ the element $g a^{-1} b_1^{-1} b_2 a b_3^{-1} b_4$ has no n-th root.

Proof. We prove (1). Let $g_* = g a_1^{-1} a_2 b_1^{-1} b_2$, $A = \{a_2, b_2\}$ and $\mathbf{p} = \mathbf{p}_A$ the homomorphism from Fact 10. Then $\mathbf{p}(g_*) = a_2 b_2$. Since a_2 is not adjacent to b_2, for every $n \geqslant 2$ the element $a_2 b_2$ does not have an n-th root. As $\mathbf{p}_A$ is an homomorphism, we are done.

We prove (2). Let $g_* = g a^{-1} b_1^{-1} b_2 a b_3^{-1} b_4$, $A = \{a, b_1, b_2, b_3, b_4\}$ and $\mathbf{p} = \mathbf{p}_A$ the homomorphism from Fact 10. There are two cases:
<u>*Case 1*</u>. $\mathbf{p}(g) = e$.
Then $\mathbf{p}(g_*) = a^{-1} b_1^{-1} b_2 a b_3^{-1} b_4$. Since a is not adjacent to b_i ($i = 1, 2, 3, 4$), for every $n \geqslant 2$ the element $a^{-1} b_1^{-1} b_2 a b_3^{-1} b_4$ does not have an n-th root.
<u>*Case 2*</u>. $\mathbf{p}(g) \neq e$.
Since $sp(\mathbf{p}(g)) \subseteq sp(g) \cap \{a, b_1, b_2, b_3, b_4\} \subseteq \{a\}$ and $\mathbf{p}(g) \neq e$, we must have $sp(\mathbf{p}(g)) = \{a\}$. Hence, $\mathbf{p}(g_*) = a^{\alpha} b_1^{-1} b_2 a b_3^{-1} b_4$, for $\alpha \in \mathbb{Z} - \{0\}$. Since a is not adjacent to b_i ($i = 1, 2, 3, 4$), for every $n \geqslant 2$ the element $a^{\alpha} b_1^{-1} b_2 a b_3^{-1} b_4$ does not have an n-th root. ∎

3. Negative Side

In this section we show that conditions (a)-(d) of Theorem 3 are necessary. Concerning conditions (a)-(c) we prove three separate lemmas: Lemmas 20, 21 and 22. Lemmas 21 and 22 are more general that needed for the proof of Theorem 3, and of independent interest. Concerning condition (d), it follows from Lemma 23 and Observation 24, which are also more general that needed for our purposes.

We denote the cyclic groups by C_n, C_∞ (or $\mathbb{Z}_n, \mathbb{Z}_\infty = \mathbb{Z}$ in additive notation).

Lemma 20. *Let $G = G(\Gamma, \mathfrak{p})$, with $|\Gamma| = 2^\omega$. Suppose that there does not exist a countable $A \subseteq \Gamma$ such that for every $a \in \Gamma$ and $a \neq b \in \Gamma - A$, a is adjacent to b. Then G does not admit a Polish group topology.*

Proof. Suppose that $G = (\Gamma, \mathfrak{p})$ is as in the assumptions of the theorem, and that $G = (G, \mathfrak{d})$ is Polish. Then either of the following cases happens:

(i) in Γ there are $\{a_i : i < \omega_1\}$ and $\{b_i : i < \omega_1\}$ such that if $i < j < \omega_1$, then $a_i \neq a_j$, $b_i \neq b_j$, $|\{a_i, a_j, b_i, b_j\}| = 4$ and a_i is not adjacent to b_i;
(ii) in Γ there are a_* and $\{b_i : i < \omega_1\}$ such that if $i < j < \omega_1$, then $|\{a_*, b_i, b_j\}| = 3$ and a_* is not adjacent to b_i.

<u>*Case 1*</u>. There are $\{a_i : i < \omega_1\}$ and $\{b_i : i < \omega_1\}$ as in (i) above.
Without loss of generality we can assume that all the $\{a_i : i < \omega_1\}$ have fixed color k^*_1 and all the $\{b_i : i < \omega_1\}$ have fixed color k^*_2, for some $k^*_1, k^*_2 \in \{p^n : p \text{ prime and } 1 \leqslant n\} \cup \{\infty\}$. Let p be a prime such that if $k^*_\ell \neq \infty$ then $p > k^*_\ell$, for $\ell = 1, 2$. Recalling Convention 8, let $(\zeta_n)_{n<\omega} \in (0,1)^\omega_{\mathbb{R}}$ be as in Fact 7 for $f \in \omega^\omega$ constantly $p + 10$. Using Observation 9, by induction on $n < \omega$, choose $(i_n = i(n), j_n = j(n))$ such that:

(a) if $m < n$, then $j_m < i_n$;
(b) $i_n < j_n < \omega_1$;
(c) $\mathfrak{d}(a^{-1}_{j(n)} a_{i(n)}, e), \mathfrak{d}(b^{-1}_{j(n)} b_{i(n)}, e) < \zeta_{n+8}$.

Consider now the following set of equations:

$$\Delta = \{x_n = (x_{n+1})^p h^{-1}_n : n < \omega\},$$

where $h_n = b^{-1}_{i(n)} b_{j(n)} a^{-1}_{i(n)} a_{j(n)}$. By (c) above and Fact 7(1)(B) we have $\mathfrak{d}(h^{-1}_n, e) < \zeta_{n+1}$, and so by Fact 7(2) the set Δ is solvable in G. Let $(g'_n)_{n<\omega}$ witness this. Let A the set of vertices of color k^*_1 or k^*_2, $\mathbf{p} = \mathbf{p}_A$ the homomorphism from Fact 10 and let $g_n = \mathbf{p}(g'_n)$. Then for every $n < \omega$ we have:

$$G \models (g_{n+1})^p = g_n h_n,$$

and so by Proposition 18 we have:

$$sp(g_n) \subseteq sp(g_0) \cup \{b_{i(\ell)}, b_{j(\ell)}, a_{i(\ell)}, a_{j(\ell)} : \ell < n\}.$$

Let $n < \omega$ be such that $sp(g_0) \cap \{b_{i(n)}, b_{j(n)}, a_{i(n)}, a_{j(n)}\} = \emptyset$. Then:

$$(g_{n+1})^p = g_n b^{-1}_{i(n)} b_{j(n)} a^{-1}_{i(n)} a_{j(n)} \text{ and } sp(g_n) \cap \{b_{i(n)}, b_{j(n)}, a_{i(n)}, a_{j(n)}\} = \emptyset,$$

which contradicts Proposition 19(1).
<u>*Case 2*</u>. There are a_* and $\{b_i : i < \omega_1\}$ as in (ii) above.
Let $k^*_1 = \mathfrak{p}(a_*)$. Without loss of generality, we can assume that all the $\{b_i : i < \omega_1\}$ have fixed color k^*_2, for some $k^*_2 \in \{p^n : p \text{ prime and } 1 \leqslant n\} \cup \{\infty\}$. Let p be a prime such that if $k^*_\ell \neq \infty$ then $p > k^*_\ell$, for $\ell = 1, 2$. Let $(\zeta_n)_{n<\omega} \in (0,1)^\omega_{\mathbb{R}}$ be as in Fact 7 for $\bar{g}_n = (a_*)$ (and so in particular $\ell(n) = 1$) and $f \in \omega^\omega$ constantly $p + 10$. Using Observation 9, by induction on $n < \omega$, choose $(i_n = i(n), j_n = j(n), i'_n = i'(n), j'_n = j'(n))$ such that:

(a) if $m < n$, then $j'_m < i_n$;
(b) $i_n < j_n < i'_n < j'_n < \omega_1$;
(c) $\mathfrak{d}(b^{-1}_{j(n)} b_{i(n)}, e), \mathfrak{d}(b^{-1}_{j'(n)} b_{i'(n)}, e) < \zeta_{n+8}$.

Consider now the following set of equations:

$$\Delta = \{x_n = (x_{n+1})^p h^{-1}_n : n < \omega\},$$

where $h_n = a^{-1}_* b^{-1}_{i(n)} b_{j(n)} a_* b^{-1}_{i'(n)} b_{j'(n)}$. By (c) above and Fact 7(1)(B) we have $\mathfrak{d}(h^{-1}_n, e) < \zeta_{n+1}$, and so by Fact 7(2) the set Δ is solvable in G. Let $(g'_n)_{n<\omega}$ witness this. Let A be the set of vertices of color k_1^* or k_2^*, $\mathbf{p} = \mathbf{p}_A$ the homomorphism from Fact 10 and let $g_n = \mathbf{p}(g'_n)$. Then for every $n < \omega$ we have:

$$G \models (g_{n+1})^p = g_n h_n,$$

and so by Proposition 18 we have:

$$sp(g_n) \subseteq sp(g_0) \cup \{a_*\} \cup \{b_{i(\ell)}, b_{j(\ell)}, b_{i'(\ell)}, b_{j'(\ell)} : \ell < n\}.$$

Let $n < \omega$ be such that $sp(g_0) \cap \{b_{i(n)}, b_{j(n)}, b_{i'(\ell)}, b_{j'(\ell)}\} = \emptyset$. Then:

$$(g_{n+1})^p = g_n a_*^{-1} b_{i(n)}^{-1} b_{j(n)} a_* b_{i'(n)}^{-1} b_{j'(n)} \text{ and } sp(g_n) \cap \{b_{i(n)}, b_{j(n)}, b_{i'(n)}^{-1} b_{j'(n)}\} = \emptyset,$$

which contradicts Proposition 19(2). ■

Recall that we denote the cyclic group of order n by C_n.

Lemma 21. *Let $G = G' \oplus G''$, with $G'' = \bigoplus_{n<\omega} G_n$, $G_n = \bigoplus_{\alpha<\lambda_n} C_{k(n)}$, $\aleph_0 < \lambda_n$, $k(n) = p_n^{t(n)}$, for p_n prime and $1 \leqslant t(n)$, and the $k(n)$ pairwise distinct. Then G does not admit a Polish group topology.*

Proof. Suppose that $G = (G, \mathfrak{d})$ is Polish and let $(\zeta_n)_{n<\omega} \in (0,1)^\omega_\mathbb{R}$ be as in Fact 7 for $f \in \omega^\omega$ such that $f(n) = k(n) + 2$. Assume that $G = G' \oplus G''$ is as in the assumptions of the lemma. Without loss of generality we can assume that either of the following cases happens:

(i) for every $n < m < \omega$, $p_n < p_m$;
(ii) for every $n < \omega$, $p_n = p$ and $\prod_{i<n} p^{t(i)}$ is not divisible by $p^{t(n)}$.

Using Observation 9, by induction on $n < \omega$, choose $g_n, h_n \in G_n$ such that g_n, h_n and $h_n^{-1} g_n$ have order $k(n)$ and $\mathfrak{d}(h_n^{-1} g_n, e) < \zeta_{n+1}$. Consider now the following set of equations:

$$\Gamma = \{x_n = (x_{n+1})^{k(n)} h_n^{-1} g_n : n < \omega\}.$$

By Fact 7(2) the set Γ is solvable in G. Let $(d_n)_{n<\omega}$ witness this. Let then $n < \omega$ be such that $d_0 \in G' \oplus \bigoplus_{i<n} G_i$. Notice now that:

$$\begin{aligned} d_0 &= (d_1)^{k(0)} h_0^{-1} g_0 \\ &= ((d_2)^{k(1)} h_1^{-1} g_1)^{k(0)} h_0^{-1} g_0 \\ &= (...((d_{n+1})^{k(n)} h_n^{-1} g_n)^{k(n-1)} \cdots h_0^{-1} g_0. \end{aligned}$$

Let $\mathbf{p} = \mathbf{p}_n$ be the projection of G onto G_n. Then we have:

$$G_n \models e = d_0 = (\mathbf{p}(d_{n+1})^{k(n)} h_n^{-1} g_n)^{\prod_{i<n} k(i)} = (h_n^{-1} g_n)^{\prod_{i<n} k(i)},$$

which is absurd. ■

When we write $G = \bigoplus_{\alpha<\lambda} \mathbb{Z} x_\alpha$ we mean that x_α is the generator of the α-th copy of $\mathbb{Z}$. This convention is used in Lemmas 22 and 23, and Observation 24.

Lemma 22. *Let $G = G_1 \oplus G_2$, with $G_2 = \bigoplus_{\alpha<\lambda} \mathbb{Z} x_\alpha$ and $\lambda > \aleph_0$. Then G does not admit a Polish group topology.*

Proof. Suppose that $G = (G, \mathfrak{d})$ is Polish and let $(\zeta_n)_{n<\omega} \in (0,1)^\omega_\mathbb{R}$ be as in Fact 7 for $f \in \omega^\omega$ constantly $2 + 10$. Assume that $G = G_1 \oplus G_2$ is as in the assumptions of the lemma. Using Observation 9, by induction on $n < \omega$, choose (i_n, j_n) such that:

(i) if $m < n$, then $j_m < i_n$;
(ii) $i_n < j_n < \omega_1 \leqslant \lambda$;
(iii) $\mathfrak{d}(x_{i_n}, x_{j_n}) < \zeta_{n+1}$.

For every $n < \omega$ let:

(a) $x_{i_n} = h_n$;
(b) $x_{j_n} = g_n$;
(c) $\mathbb{Z}x_{i_n} \oplus \mathbb{Z}x_{j_n} = H_n$.

Consider now the following set of equations:

$$\Gamma = \{x_n = (x_{n+1})^2 h_n^{-1} g_n : n < \omega\}.$$

By Fact 7(2) the set Γ is solvable in G. Let $(d_n)_{n<\omega}$ witness this. Let then $n < \omega$ be such that $d_0 \in G_1 \oplus \bigoplus_{i<n} H_n$. Notice now that:

$$\begin{array}{rcl} d_0 & = & (d_1)^2 h_0^{-1} g_0 \\ & = & ((d_2)^2 h_1^{-1} g_1)^2 h_0^{-1} g_0 \\ & = & (...((d_{n+1})^2 h_n^{-1} g_n)^2 \cdots h_0^{-1} g_0. \end{array}$$

Let $\mathbf{p}$ be the projection of G onto H_n. Then we have:

$$H_n \models e = d_0 = (\mathbf{p}(d_{n+1})^2 h_n^{-1} g_n)^{2^n} = (h_n^{-1} g_n)^{2^n},$$

which is absurd, since $H_n = \mathbb{Z}x_{i_n} \oplus \mathbb{Z}x_{j_n}$ is torsion-free and $h_n^{-1} g_n \neq e$. ■

In the rest of this section we use additive notation.

Lemma 23. *Let $G = (G, \mathfrak{d})$ be an uncountable Polish group, p a prime and $1 \leqslant t < \omega$. Suppose that $G = G_1 \oplus G_2$, with $G_2 = \bigoplus_{\alpha<\lambda} \mathbb{Z}_{p^t} x_\alpha$. If $\lambda > \aleph_0$, then there is $\bar{y} \subseteq G$ such that:*

(a) $\bar{y} = (y_\alpha : \alpha < 2^{\aleph_0})$;
(b) $p^t y_\alpha = 0$ and, for $\ell < t$, $p^\ell y_\alpha \neq 0$;
(c) if $\alpha < \beta$, then $p^t(y_\alpha - y_\beta) = 0$, and, for $\ell < t$, $p^\ell(y_\alpha - y_\beta) \neq 0$;
(d) if $\alpha < \beta$, then $y_\alpha - y_\beta$ is not divisible by p in G.

Proof. By induction on $n < \omega$, choose (i_n, j_n) such that:

(i) if $m < n$, then $j_m < i_n$;
(ii) $i_n < j_n < \omega_1$;
(iii) $\mathfrak{d}(x_{i_n}, x_{j_n}) < 2^{-2^n}$.

For $A \subseteq \omega$ and $n < \omega$, let:

$$y_{A,n} = \sum \{x_{i_k} - x_{j_k} : k \in A,\, k < n\}.$$

Then for every $A \subseteq \omega$, $(y_{A,n})_{n<\omega}$ is Cauchy. Let $y_A \in G$ be its limit. Then by continuity we have:

(a) $p^t y_A = 0$, and, for $\ell < t$, $p^\ell y_A \neq 0$;
(b) if $A \neq B \subseteq \omega$, then y_A and y_B commute, $p^t(y_A - y_B) = 0$ and, for $\ell < t$, $p^\ell(y_A - y_B) \neq 0$.

We define the following equivalence relation E on $\mathcal{P}(\omega)$:

$$A_1 E A_2 \;\Leftrightarrow\; \exists x \in G(y_{A_1} - y_{A_2} = px).$$

We then have:

(I) E is analytic;
(II) if $B \subseteq \omega$, $n \in B$ and $A = B - \{n\}$, then $\neg(y_A E y_B)$;
(III) by [8, Lemma 13] we have $|\mathcal{P}(\omega)/E| = 2^\omega$.

Hence, we can find $(y_\alpha : \alpha < 2^{\aleph_0})$ as wanted. ■

Observation 24. *Let $G = (G, \mathfrak{d})$ be an uncountable Polish group, p a prime and $1 \leqslant t < \omega$. Suppose that $G = G_0 \oplus G_1 \oplus G_2$, with G_0 countable, G_1 abelian, $\lambda > \aleph_0$ and $G_2 = \bigoplus_{\alpha<\lambda} \mathbb{Z}_{p^t} x_\alpha$. Let $(y_\alpha : \alpha < 2^{\aleph_0})$ be as in Lemma 23 with respect to the decomposition $G'_1 \oplus G'_2$ for $G'_1 = G_0 \oplus G_1$ and $G'_2 = G_2$. Then there is a pure embedding of $H = \bigoplus_{\alpha<2^{\aleph_0}} \mathbb{Z}_{p^t} y_\alpha$ into the abelian group $G_1 \oplus G_2$.*

Proof. Define:

$$\mathcal{U}_1 = \{\alpha < 2^{\aleph_0} : \text{ for no } \xi \in \bigoplus_{\beta<\alpha} \mathbb{Z}_{p^t} y_\beta \text{ we have } y_\alpha - \xi \text{ is divisible by } p \text{ in } G_1 \oplus G_2\},$$

$$\mathcal{U}_2 = \{\alpha < 2^{\aleph_0} : \text{ for no } \xi \in \bigoplus_{\beta<\alpha} \mathbb{Z}_{p^t} y_\beta \text{ and } \ell < t \text{ we have } p^\ell(y_\alpha - \xi) = 0\}.$$

Let $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$. For $\alpha \notin \mathcal{U}$, let $(\xi_\alpha, \ell_\alpha)$ be witnesses of $\alpha \notin \mathcal{U}$, with $\ell = t$ if $\alpha \notin \mathcal{U}_1$.
Claim 24.1. $|\mathcal{U}| = 2^{\aleph_0}$.
Proof. Suppose that $|\mathcal{U}| < 2^{\aleph_0}$ and let $\mu = \aleph_0 + |\mathcal{U}|$. Hence $\mathcal{U} \cap \mu^+$ is bounded. Let $\alpha_* = sup(\mathcal{U} \cap \mu^+)$. By Fodor's lemma for some stationary set $S \subseteq \mu^+ - (\alpha_* + 1)$ we have $\alpha \in S$ implies $(\xi_\alpha, \ell_\alpha) = (\xi_*, \ell_*)$. Let $\alpha_1 < \alpha_2 \in S$. Then if $\ell_* = t$ we have that $y_{\alpha_2} - y_{\alpha_1}$ is divisible by p in $G_1 \oplus G_2$, and if $\ell_* < t$ we have that $p^\ell(y_{\alpha_2} - y_{\alpha_1}) = 0$. In both cases we reach a contradiction, and so $|\mathcal{U}| = 2^{\aleph_0}$. ■
Let now $\mathbf{p}_\ell$ be the canonical projection of G onto G_ℓ ($\ell = 1, 2$). Then, by the claim above, $\{(\mathbf{p}_1 + \mathbf{p}_2)(y_\alpha) : \alpha \in \mathcal{U}\}$ is a basis of a pure subgroup of $G_1 \oplus G_2$ isomorphic to H, and so we are done. ■

4. Positive Side

In this section we prove the the sufficiency of conditions (a)-(d) of Theorem 3.

Lemma 25. *Suppose that $G = G(\Gamma, \mathfrak{p})$ satisfies conditions (a)-(d) of Theorem 3 and $|\Gamma| = 2^\omega$. Then G is realizable as the group of automorphisms of a countable structure.*

Proof. Let $G = G(\Gamma, \mathfrak{p})$ be as in the assumptions of the theorem. Then we have:

$$G \cong H \oplus \bigoplus_{p^n | n_*} \bigoplus_{\alpha < \lambda_{(p,n)}} \mathbb{Z}_{p^n},$$

for some countable group H, natural number $n_* < \omega$, and $\lambda_{(p,n)} \in \{0, 2^{\aleph_0}\}$ (here we are crucially using conditions (a)-(d) of the statement of the theorem, of course). Since finite sums of groups realizable as groups of automorphisms of countable structures are realizable as groups of automorphisms of countable structures, it suffices to show that for given p^n the group:

$$H_1 = \bigoplus_{\alpha<2^{\aleph_0}} \mathbb{Z}_{p^n} \cong \mathbb{Z}_{p^n}^\omega$$

is realizable as the group of automorphisms of countable structure. To this extent, let A be a countable first-order structure such that $Aut(A) = \mathbb{Z}_{p^n}$. Let B be the disjoint union of $\aleph_0$ copies of A, then $\mathbb{Z}_{p^n}^\omega \cong Aut(B)$, and so we are done. ■

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