Radial solutions of biharmonic equations with vanishing or singular radial potentials

(Article begins on next page)

Radial solutions for the bilaplacian equation with vanishing or singular radial potentials

Marino Badiale[a,b] - Stefano Greco - Sergio Rolando[c,b]

Abstract

Given three measurable functions $V(r) \geq 0$, $K(r) > 0$ and $Q(r) \geq 0$, $r > 0$, we consider the bilaplacian equation

$$\Delta^2 u + V(|x|)u = K(|x|)f(u) + Q(|x|) \quad \text{in } \mathbb{R}^N$$

and we find radial solutions thanks to compact embeddings of radial spaces of Sobolev functions into sum of weighted Lebesgue spaces.

Keywords. Bi-laplacian operator, weighted Sobolev spaces, compact embeddings, unbounded or decaying potentials

MSC (2010): Primary 35J91; Secondary 35J60, 46E35

1 Introduction

This paper is concerned with the following bilaplacian equation

$$\Delta^2 u + V(|x|)u = K(|x|)f(u) + Q(|x|) \quad \text{in } \mathbb{R}^N \tag{1.1}$$

where $N \geq 5$, $\Delta^2 u = \Delta(\Delta u)$ is the *bilaplacian* operator, the forcing term $Q \geq 0$ and the potentials $V \geq 0$ and $K > 0$ satisfy suitable hypotheses, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(0) = 0$. In particular we are interested in allowing the potential V to be singular at the origin and/or vanishing at infinity.

Bilaplacian equations arise in describing different physical phenomena, such as the propagation of laser beams in Kerr media or nonlinear oscillations in suspension bridges (see some references in [5, 13]), and have been extensively studied in the last decades (see e.g. [5–7,9,13,14] and the references therein). In spite of that, equations of type (1.1), namely with radial potentials possibly singular at the origin and vanishing at infinity, has been treated only in [6, 7] (at least to our knowledge), where the authors essentially consider power type potentials.

For problem (1.1) we will obtain several kinds of existence results of radial solutions. The main technical device for our results is given by some new theorems about compact embeddings of suitable Sobolev spaces into sum of weighted Lebesgue spaces. The natural approach in studying Eq. (1.1) is variational, since its weak solutions are (at least formally) critical points of a suitable Euler functional, as we will see. Then the problem

[a]Dipartimento di Matematica "Giuseppe Peano", Università degli Studi di Torino, Via Carlo Alberto 10, 10123 Torino, Italy. e-mail: marino.badiale@unito.it

[b]Partially supported by the MIUR 2015 PRIN project "Variational methods, with applications to problems in mathematical physics and geometry".

[c]Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca, Via Roberto Cozzi 53, 20125 Milano, Italy. e-mail: sergio.rolando@unito.it. Member of the Gruppo Nazionale di Alta Matematica (INdAM).

of existence is easily solved if V does not vanish at infinity and K is bounded, because standard embeddings theorems are available. As we will let V and K to vanish, or to go to infinity, as $|x| \to 0$ and $|x| \to +\infty$, the usual embeddings theorems for Sobolev spaces are not available anymore, and new embedding theorems must be proved. This kind of work has been started in [6, 7] (for the bilaplacian equation) and we continue it here, using some new ideas that has been introduced in [2–4].

The main novelty of our approach is two-folded. Firstly, we look for embeddings not into a Lebesgue space but into a sum of Lebesgue spaces. This allows us to study separately the behavior of V and K at 0 and ∞, and to assume different set of hypotheses about these behaviors. Secondly, we assume hypotheses not on V and K separately but on their ratio, so allowing general kind of asymptotic behavior for the two potentials.

Thanks to this second novelty we obtain embedding results, and thus existence results for Eq. (1.1), which extend the ones of [6, 7] to more general kinds of potentials. Moreover, thanks to the first novelty, we get new results also for power type potentials (cf. Example 2.10 below).

The paper is organized as follows. In Sections 2 and 6 we give our results on compact embeddings and existence of solutions to Eq. (1.1) respectively. The former will be proved in the Sections 3-5, the latter in Section 7.

Notations. We end this introductory section by collecting some notations used in the paper.
• We denote $\mathbb{R}_+ := (0, +\infty)$ and $\mathbb{R}_- := (-\infty, 0)$.
• For every $R > 0$, we set $B_R := \{x \in \mathbb{R}^N : |x| < r\}$.
• For any subset $A \subseteq \mathbb{R}^N$, we denote $A^c := \mathbb{R}^N \setminus A$.
• By $\to$ and $\rightharpoonup$ we respectively mean *strong* and *weak* convergence.
• $C_c^\infty(\Omega)$ is the space of the infinitely differentiable real functions with compact support in the open set $\Omega \subseteq \mathbb{R}^N$.
• If $1 \leq p \leq \infty$ then $L^p(A)$ and $L^p_{\mathrm{loc}}(A)$ are the usual real Lebesgue spaces (for any measurable set $A \subseteq \mathbb{R}^N$). If $\rho : A \to \mathbb{R}_+$ is a measurable function, then $L^p(A, \rho(z)\, dz)$ is the real Lebesgue space with respect to the measure $\rho(z)\, dz$ (dz stands for the Lebesgue measure on $\mathbb{R}^N$).
• $p' := p/(p-1)$ is the Hölder-conjugate exponent of p.

2 Main results

In this section we state our main results on compact embeddings, that we will prove in the following Sections 3-5. Firstly, we introduce some basic concepts and results. Assume $N \geq 5$ and define $2^{**} := \frac{2N}{N-4}$.

By usual Sobolev embeddings, there exists a suitable constant $C > 0$ such that for all $u \in C_c^\infty(\mathbb{R}^N)$ one has

$$\|u\|_{L^{2^{**}}} \leq C \left\|D^2 u\right\|_{L^2}$$

where

$$\|D^2 u\|_{L^2} := \left(\sum_{|\alpha|=2} \|D^\alpha u\|_{L^2}^2 \right)^{1/2}. \tag{2.1}$$

A basic space to work with is

$$D^{2,2}(\mathbb{R}^N) := \left\{ u \in L^{2^{**}}(\mathbb{R}^N) : \left\|D^2 u\right\|_{L^2} < +\infty \right\}.$$

It is the closure of $C_c^\infty(\mathbb{R}^N)$ in $L^{2^{**}}(\mathbb{R}^N)$ with respect to the norm $\left\|D^2 u\right\|_{L^2}$ and, endowed with such a norm, it is an Hilbert space. The bilinear form

$$(u, v) \mapsto \int_{\mathbb{R}^N} \Delta u \Delta v \, dx$$

defines a scalar product on $D^{2,2}(\mathbb{R}^N)$ and the associated norm, that is $\|u\|_{D^{2,2}} := \|\Delta u\|_{L^2}$, is equivalent to (2.1) (see for example [8]). Hence, one can also define $D^{2,2}(\mathbb{R}^N)$ as the closure of $C_c^\infty(\mathbb{R}^N)$ in $L^{2^{**}}(\mathbb{R}^N)$ with respect to the norm $\|\Delta u\|_{L^2}$. We will be particularly interested in the subspace of radial functions, i.e.,

$$D_r^{2,2} := D_r^{2,2}(\mathbb{R}^N) := \left\{ u \in D^{2,2}(\mathbb{R}^N) : u(x) = u(|x|) \right\},$$

for which the pointwise estimates given in the following lemma hold (see [12] for a proof).

Lemma 2.1. *For every $u \in D_r^{2,2}(\mathbb{R}^N)$ we have*

$$|u(x)| \leq \frac{2}{N-4} \frac{1}{\sqrt{N\sigma_N}} \frac{\|\Delta u\|_{L^2}}{|x|^{\frac{N-4}{2}}} \quad \text{almost everywhere in } \mathbb{R}^N \tag{2.2}$$

and

$$|\nabla u(x)| \leq \frac{1}{\sqrt{N\sigma_N}} \frac{\|\Delta u\|_{L^2}}{|x|^{\frac{N-2}{2}}} \quad \text{almost everywhere in } \mathbb{R}^N \tag{2.3}$$

where σ_N denotes the $(N-1)$-dimensional measure of the unit sphere of $\mathbb{R}^N$.

For any measurable function $V : \mathbb{R}_+ \to [0, +\infty]$, we define the space

$$H_V^2(\mathbb{R}^N) := \left\{ u \in D^{2,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(|x|)|u|^2 \, dx < \infty \right\}$$

This is an Hilbert space with scalar product

$$(u, v) := \int_{\mathbb{R}^N} \Delta u \Delta v \, dx + \int_{\mathbb{R}^N} V(|x|) uv \, dx \tag{2.4}$$

and associated norm

$$\|u\| := \left(\int_{\mathbb{R}^N} |\Delta u|^2 + V(|x|)|u|^2 \, dx \right)^{\frac{1}{2}}.$$

We are interested in finding solutions of (1.1) in the radial subspace of $H_V^2(\mathbb{R}^N)$, i.e.,

$$H_{V,r}^2 = H_{V,r}^2(\mathbb{R}^N) := \left\{ u \in H_V^2(\mathbb{R}^N) : u(x) = u(|x|) \right\}.$$

Remark 2.2. By the Sobolev embedding, there is a constant $S_N > 0$ such that

$$\forall u \in H_V^2(\mathbb{R}^N), \quad \|u\|_{L^{2^{**}}} \leq S_N \|u\|. \tag{2.5}$$

Remark 2.3. From the continuous embedding $H_{V,r}^2 \hookrightarrow D_r^{2,2}(\mathbb{R}^N)$ and inequality (2.2), we deduce that there exists a constant $C_N > 0$ such that

$$\forall u \in H_{V,r}^2(\mathbb{R}^N), \quad |u(x)| \leq C_N \frac{\|u\|}{|x|^{\frac{N-4}{2}}} \quad \text{almost everywhere in } \mathbb{R}^N. \tag{2.6}$$

We now introduce the sum of Lebesgue spaces. For any measurable function $K : \mathbb{R}_+ \to \mathbb{R}_+$ and $1 < q_1 \leq q_2 < \infty$, we define

$$L_K^{q_1} + L_K^{q_2} := L_K^{q_1}(\mathbb{R}^N) + L_K^{q_2}(\mathbb{R}^N) := \left\{ u_1 + u_2 : u_i \in L_K^{q_i}(\mathbb{R}^N), i = 1, 2 \right\}.$$

This space coincides with the set of measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ for which there exists a measurable set $E \subseteq \mathbb{R}^N$ such that $u \in L^{q_1}_K(E) \cap L^{q_2}_K(E^c)$ (where $L^{q_1}_K(E) := L^{q_1}(E, K(|x|)dx)$ and $L^{q_2}_K(E^c) := L^{q_2}(E^c, K(|x|)dx)$) and it is a Banach space when endowed with the norm

$$\|u\|_{L^{q_1}_K + L^{q_2}_K} := \inf_{u_1+u_2=u} \max \left\{ \|u_1\|_{L^{q_1}_K}, \|u_2\|_{L^{q_2}_K} \right\}$$

(see [1]). Note that L^q_K is continuously embedded into $L^{q_1}_K + L^{q_2}_K$ for all $q \in [q_1, q_2]$.

Our first result is Theorem 2.4 below. It provides sufficient condition to the embeddings we are interested in and uses the following assumptions:

(V) $V : \mathbb{R}_+ \to [0, +\infty]$ is a measurable function such that $V \in L^1_{loc}(\mathbb{R}_+)$

(K) $K : \mathbb{R}_+ \to \mathbb{R}_+$ is a measurable function such that $K \in L^s_{loc}(\mathbb{R}_+)$ for some $s > 1$

$\left(\mathcal{S}'_{q_1,q_2}\right)$ $\exists R_1, R_2 > 0$ such that $\mathcal{S}_0(q_1, R_1) < \infty$ and $\mathcal{S}_\infty(q_2, R_2) < \infty$

$\left(\mathcal{S}''_{q_1,q_2}\right)$ $\displaystyle\lim_{R\to 0^+} \mathcal{S}_0(q_1, R) = \lim_{R\to\infty} \mathcal{S}_\infty(q_2, R) = 0$

where q_1, q_2 will be specified each time and $\mathcal{S}_0, \mathcal{S}_\infty$ are the functions of $R > 0$ and $q > 1$ defined as follows:

$$\mathcal{S}_0(q, R) := \sup_{u \in H^2_{V,r},\, \|u\|=1} \int_{B_R} K(|x|)|u|^q \, dx, \tag{2.7}$$

$$\mathcal{S}_\infty(q, R) := \sup_{u \in H^2_{V,r},\, \|u\|=1} \int_{\mathbb{R}^N \setminus B_R} K(|x|)|u|^q \, dx. \tag{2.8}$$

Notice that $\mathcal{S}_0(q, \cdot)$ is increasing, while $\mathcal{S}_\infty(q, \cdot)$ is decreasing.

Theorem 2.4. *Assume* (**V**) *and* (**K**)*, and let* $q_1, q_2 > 1$.

1. *If* $\left(\mathcal{S}'_{q_1,q_2}\right)$ *holds, then* $H^2_{V,r}(\mathbb{R}^N)$ *is continuosly embedded into* $L^{q_1}_K(\mathbb{R}^N) + L^{q_2}_K(\mathbb{R}^N)$.
2. *If* $\left(\mathcal{S}''_{q_1,q_2}\right)$ *holds, then* $H^2_{V,r}(\mathbb{R}^N)$ *is compactly embedded into* $L^{q_1}_K(\mathbb{R}^N) + L^{q_2}_K(\mathbb{R}^N)$.

We now define two new functions of $R > 0$ and $q > 1$ as follows:

$$\mathcal{R}_0(q, R) := \sup_{u \in H^2_{V,r},\, h \in H^2_{V,r},\, \|u\|=\|h\|=1} \int_{B_R} K(|x|)|u|^{q-1}|h| \, dx, \tag{2.9}$$

$$\mathcal{R}_\infty(q, R) := \sup_{u \in H^2_{V,r},\, h \in H^2_{V,r},\, \|u\|=\|h\|=1} \int_{\mathbb{R}^N \setminus B_R} K(|x|)|u|^{q-1}|h| \, dx. \tag{2.10}$$

Note that $\mathcal{R}_0(q, \cdot)$ is increasing, while $\mathcal{R}_\infty(q, \cdot)$ is decreasing. Furthermore, for any (q, R) we have $\mathcal{S}_0(q, R) \leq \mathcal{R}_0(q, R)$ and $\mathcal{S}_\infty(q, R) \leq \mathcal{R}_\infty(q, R)$, so that $\left(\mathcal{S}''_{q_1,q_2}\right)$ is a consequence of the following stronger condition:

$\left(\mathcal{R}''_{q_1,q_2}\right)$ $\displaystyle\lim_{R\to 0^+} \mathcal{R}_0(q_1, R) = \lim_{R\to\infty} \mathcal{R}_\infty(q_2, R) = 0.$

In our next results we look for concrete conditions ensuring $\left(\mathcal{R}''_{q_1,q_2}\right)$ and thus the compactness of the embedding $H^2_{V,r}(\mathbb{R}^N) \hookrightarrow L^{q_1}_K(\mathbb{R}^N) + L^{q_2}_K(\mathbb{R}^N)$.

Our first results in this direction are Theorems 2.5 and 2.6 below. For any $\alpha \in \mathbb{R}$ and $\beta \in [0,1]$, define the functions

$$\alpha^*(\beta) := \max\left\{4\beta - 2 - \frac{N}{2}, -(1-\beta)N\right\} = \begin{cases} 4\beta - 2 - N/2 & \text{if } 0 \le \beta \le 1/2 \\ -(1-\beta)N & \text{if } 1/2 \le \beta \le 1 \end{cases}$$

and

$$q^*(\alpha,\beta) := 2\frac{\alpha - 4\beta + N}{N-4}.$$

Theorem 2.5. *Assume* (**V**) *and* (**K**). *Assume that* $\exists R_1 > 0$ *such that* $V(r) < +\infty$ *for almost every* $r \in (0, R_1)$ *and*

$$\operatorname*{ess\,sup}_{r\in(0,R_1)} \frac{K(r)}{r^{\alpha_0}V(r)^{\beta_0}} < +\infty \quad \text{for some } 0 \le \beta_0 \le 1 \text{ and } \alpha_0 > \alpha^*(\beta_0). \tag{2.11}$$

Then $\lim_{R\to 0^+} \mathcal{R}_0(q_1, R) = 0$ *for all* $q_1 \in \mathbb{R}$ *such that*

$$\max\{1, 2\beta_0\} < q_1 < q^*(\alpha_0, \beta_0). \tag{2.12}$$

Theorem 2.6. *Assume* (**V**) *and* (**K**). *Assume that* $\exists R_2 > 0$ *such that* $V(r) < +\infty$ *for almost every* $r > R_2$ *and*

$$\operatorname*{ess\,sup}_{r>R_2} \frac{K(r)}{r^{\alpha_\infty}V(r)^{\beta_\infty}} < +\infty \quad \text{for some } 0 \le \beta_\infty \le 1 \text{ and } \alpha_\infty \in \mathbb{R}. \tag{2.13}$$

Then $\lim_{R\to +\infty} \mathcal{R}_\infty(q_2, R) = 0$ *for all* $q_2 \in \mathbb{R}$ *such that*

$$q_2 > \max\{1, 2\beta_\infty, q^*(\alpha_\infty, \beta_\infty)\}. \tag{2.14}$$

Note that for all $(\alpha,\beta) \in \mathbb{R}\times[0,1]$, we have

$$\max\{1, 2\beta, q^*(\alpha,\beta)\} = \begin{cases} q^*(\alpha,\beta) & \text{if } \alpha \ge \alpha^*(\beta) \\ \max\{1, 2\beta\} & \text{if } \alpha \le \alpha^*(\beta) \end{cases}.$$

Remark 2.7. It is easy to check that the inequalities $\max\{1, 2\beta_0\} < q^*(\alpha_0, \beta_0)$ and $\alpha_0 > \alpha^*(\beta_0)$ are equivalent. Hence, in (2.12), the inequality $\max\{1, 2\beta_0\} < q^*(\alpha_0, \beta_0)$ is automatically satisfied.

In the next two theorems we assume stronger hypotheses than those of Theorems 2.5 and 2.6, and we get stronger results. For all $\alpha \in \mathbb{R}$, $\beta \le 1$ and $\gamma \in \mathbb{R}$, define

$$q_*(\alpha,\beta,\gamma) := 2\frac{\alpha - \gamma\beta + N}{N-\gamma} \quad \text{and} \quad q_{**}(\alpha,\beta,\gamma) := 2\frac{2\alpha + (1-2\beta)\gamma + 2(N-2)}{2(N-2)-\gamma}. \tag{2.15}$$

Notice that q_* is defined for $\gamma \ne N$, while q_{**} for $\gamma \ne 2(N-2)$.

Theorem 2.8. *Assume* (**V**) *and* (**K**). *Assume that* $\exists R_2 > 0$ *such that* $V(r) < +\infty$ *for almost every* $r > R_2$ *and*

$$\operatorname*{ess\,sup}_{r>R_2} \frac{K(r)}{r^{\alpha_\infty}V(r)^{\beta_\infty}} < +\infty \quad \text{for some } 0 \le \beta_\infty \le 1 \text{ and } \alpha_\infty \in \mathbb{R} \tag{2.16}$$

and

$$\operatorname*{ess\,inf}_{r>R_2} r^{\gamma_\infty} V(r) > 0 \quad \textit{for some } \gamma_\infty \leq 4. \tag{2.17}$$

Then $\lim_{R\to+\infty} \mathcal{R}_\infty(q_2, R) = 0$ *for all* $q_2 \in \mathbb{R}$ *such that*

$$q_2 > \max\{1, 2\beta_\infty, q_*, q_{**}\}, \tag{2.18}$$

where $q_* = q_*(\alpha_\infty, \beta_\infty, \gamma_\infty)$ *and* $q_{**} = q_{**}(\alpha_\infty, \beta_\infty, \gamma_\infty)$.

To give the statement of our last embedding result, we need to define a subset $\mathcal{A}_{\beta,\gamma}$ of the plane (α, q). Recalling the definitions of $q_* = q_*(\alpha, \beta, \gamma)$ and $q_{**} = q_{**}(\alpha, \beta, \gamma)$ in (2.15), we set

$$\begin{aligned}
&\mathcal{A}_{\beta,\gamma} := \{(\alpha, q) : \max\{1, 2\beta\} < q < \min\{q_*, q_{**}\}\} && \text{if } 4 \leq \gamma < N,\\
&\mathcal{A}_{\beta,\gamma} := \{(\alpha, q) : \max\{1, 2\beta\} < q < q_{**}, \alpha > -(1-\beta)N\} && \text{if } \gamma = N,\\
&\mathcal{A}_{\beta,\gamma} := \{(\alpha, q) : \max\{1, 2\beta, q_*\} < q < q_{**}\} && \text{if } N < \gamma < 2N-4,\\
&\mathcal{A}_{\beta,\gamma} := \{(\alpha, q) : \max\{1, 2\beta, q_*\} < q, \alpha > -(1-\beta)\gamma\} && \text{if } \gamma = 2N-4,\\
&\mathcal{A}_{\beta,\gamma} := \{(\alpha, q) : \max\{1, 2\beta, q_*, q_{**}\} < q\} && \text{if } \gamma > 2N-4.
\end{aligned} \tag{2.19}$$

Theorem 2.9. *Assume* (**V**) *and* (**K**). *Assume that* $\exists R_1 > 0$ *such that* $V(r) < +\infty$ *for almost every* $r \in (0, R_1)$ *and*

$$\operatorname*{ess\,sup}_{r\in(0,R_1)} \frac{K(r)}{r^{\alpha_0} V(r)^{\beta_0}} < +\infty \quad \textit{for some } 0 \leq \beta_0 \leq 1 \textit{ and } \alpha_0 \in \mathbb{R} \tag{2.20}$$

and Then $\lim_{R\to 0^+} \mathcal{R}_0(q_1, R) = 0$ *for all* $q_1 \in \mathbb{R}$ *such that*

$$(\alpha_0, q_1) \in \mathcal{A}_{\beta_0,\gamma_0}. \tag{2.21}$$

We end this section with an example that might clarify how to use our results and compare them with the ones of [6, 7]. Many other examples can be easily obtained by adapting the ones given in [2, Section 3].

Example 2.10. Consider the potentials

$$V(r) = \frac{1}{r^a}, \quad K(r) = \frac{1}{r^{a-1}}, \quad a \leq 4.$$

Since V satisfies (2.17) with $\gamma_\infty = a$, we apply Theorem 2.4 together with Theorems 2.5 and 2.8. Assumptions (2.11) and (2.16) hold if and only if $\alpha_0 \leq a\beta_0 - a + 1$ and $\alpha_\infty \geq a\beta_\infty - a + 1$. According to (2.12) and (2.18), it is convenient to choose α_0 as large as possible and α_∞ as small as possible, so we take

$$\alpha_0 = a\beta_0 - a + 1, \quad \alpha_\infty = a\beta_\infty - a + 1.$$

Then $q^* = q^*(\alpha_0, \beta_0)$, $q_* = q_*(\alpha_\infty, \beta_\infty, a)$ and $q_{**} = q_{**}(\alpha_\infty, \beta_\infty, a)$ are given by

$$q^* = 2\frac{N - a + 1 - (4-a)\beta_0}{N-4}, \quad q_* = 2\frac{N-a+1}{N-a} \quad \text{and} \quad q_{**} = 2\frac{2N-a-2}{2N-a-4}, \tag{2.22}$$

where $a \leq 4$ implies $q_* \leq q_{**}$. Note that $\alpha_0 > \alpha^*(\beta_0)$ for every β_0. Since q^* is decreasing in β_0 and q_{**} is independent of β_∞, it is convient to choose $\beta_0 = \beta_\infty = 0$, so that Theorems 2.5 and 2.8 yield to exponents q_1, q_2 such that

$$1 < q_1 < q^* = 2\frac{N-a+1}{N-4}, \quad q_2 > q_{**} = 2\frac{2N-a-2}{2N-a-4}.$$

If $a < 4$, one has $q_{**} < q^*$ and therefore we get the compact embedding

$$H^1_{V,\mathrm{r}} \hookrightarrow L^q_K \qquad \text{for} \quad 2\frac{2N-a-2}{2N-a-4} < q < 2\frac{N-a+1}{N-4}. \tag{2.23}$$

If $a = 4$, then $q_{**} = q^* = 2(N-3)/(N-4)$ and we have the compact embedding

$$H^1_{V,\mathrm{r}} \hookrightarrow L^{q_1}_K + L^{q_2}_K \qquad \text{for} \quad 1 < q_1 < 2\frac{N-3}{N-4} < q_2.$$

Since V and K are power potentials, one can also apply the results of [7], finding two exponents s_* and s^* such that the embedding $H^1_{V,\mathrm{r}} \hookrightarrow L^q_K$ is compact if $s_* < q < s^*$. These exponents are exactly q_* and q^* of (2.22) respectively, so that one obtains (2.23) again, provided that $a < 4$. If $a = 4$, instead, one gets $s_* = s^*$ and no result is avaliable in [7].

3 Proof of Theorem 2.4

This section is devoted to the proof of Theorem 2.4, so let $N \geq 5$, assume (**V**) and (**K**) and take $q_1, q_2 > 1$. We begin with some preliminary results.

Lemma 3.1. *Take $R > r > 0$ and $1 < q < \infty$. Then there exist two constants $\tilde{C} = \tilde{C}(N, r, R, q, s) > 0$ and $l = l(q, s) > 0$ such that $\forall u \in H^2_{V,r}$ one has*

$$\int_{B_R\setminus B_r} K(|x|)|u|^q\,dx \leq \tilde{C}\,\|K(|\cdot|)\|_{L^s(B_R\setminus B_r)}\,\|u\|^{q-2l}\left(\int_{B_R\setminus B_r}|u|^2\,dx\right)^l \tag{3.1}$$

*Furthermore, if $s > \frac{2N}{N+4}$ in assumption (**K**), then there exists $\tilde{C}_1 = \tilde{C}_1(N, r, R, q, s) > 0$ such that $\forall u \in H^2_{V,r}$ and $\forall h \in H^2_V$ we have*

$$\frac{\int_{B_R\setminus B_r} K(|x|)|u|^{q-1}|h|\,dx}{\tilde{C}_1\,\|K(|\cdot|)\|_{L^s(B_R\setminus B_r)}} \leq \begin{cases} \left(\int_{B_R\setminus B_r}|u|^2\,dx\right)^{\frac{q-1}{2}}\|h\| & if\ q \leq \tilde{q} \\ \left(\int_{B_R\setminus B_r}|u|^2\,dx\right)^{\frac{\tilde{q}-1}{2}}\|u\|^{q-\tilde{q}}\,\|h\| & if\ q > \tilde{q} \end{cases}$$

where $\tilde{q} := 2\left(1 + \frac{2}{N} - \frac{1}{s}\right)$.

Proof. Take $u \in H^2_{V,r}$ and fix $t \in (1, s)$ such that $t'q > 2$ (where $t' = t/(t-1)$). Then, by Hölder inequality and (2.6), we get

$$\begin{aligned}\int_{B_R\setminus B_r} K(|x|)|u|^q\,dx &\leq \left(\int_{B_R\setminus B_r} K(|x|)^t\,dx\right)^{\frac{1}{t}}\left(\int_{B_R\setminus B_r}|u|^{t'q}\,dx\right)^{\frac{1}{t'}} \\ &\leq |B_R\setminus B_r|^{\frac{1}{t}-\frac{1}{s}}\,\|K(|\cdot|)\|_{L^s(B_R\setminus B_r)}\left(\int_{B_R\setminus B_r}|u|^{t'q-2}|u|^2\,dx\right)^{\frac{1}{t'}} \\ &\leq |B_R\setminus B_r|^{\frac{1}{t}-\frac{1}{s}}\,\|K(|\cdot|)\|_{L^s(B_R\setminus B_r)}\left(\frac{C_N\,\|u\|}{r^{\frac{N-4}{2}}}\right)^{q-\frac{2}{t'}}\left(\int_{B_R\setminus B_r}|u|^2\,dx\right)^{\frac{1}{t'}}.\end{aligned}$$

This proves (3.1). To prove the second statement, take $u \in H^2_{V,r}$ and $h \in H^2_V$. Let $\sigma := \frac{2N}{N+4}$ be the Hölder conjugate exponent of 2^{**}. Notice that $\frac{s}{\sigma} > 1$. From Hölder inequality and (2.5) we deduce

$$\begin{aligned}\int_{B_R\setminus B_r} K(|x|)|u|^{q-1}|h|\,dx &\leq \left(\int_{B_R\setminus B_r} K(|x|)^\sigma |u|^{(q-1)\sigma}\,dx\right)^{\frac{1}{\sigma}} \left(\int_{B_R\setminus B_r} |h|^{2^{**}}\,dx\right)^{\frac{1}{2^{**}}} \\ &\leq \left(\left(\int_{B_R\setminus B_r} K(|x|)^s\,dx\right)^{\frac{\sigma}{s}} \left(\int_{B_R\setminus B_r} |u|^{(q-1)\sigma\left(\frac{s}{\sigma}\right)'}\,dx\right)^{\frac{1}{\left(\frac{s}{\sigma}\right)'}}\right)^{\frac{1}{\sigma}} S_N\,\|h\| \\ &\leq S_N\,\|K(|\cdot|)\|_{L^s(B_R\setminus B_r)}\,\|h\| \left(\int_{B_R\setminus B_r} |u|^{2\frac{q-1}{\tilde{q}-1}}\,dx\right)^{\frac{\tilde{q}-1}{2}},\end{aligned}$$

thanks to the fact that $\sigma\left(\frac{s}{\sigma}\right)' = \frac{2Ns}{(N+4)s-2N} = \frac{2}{\tilde{q}-1}$. If $q \leq \tilde{q}$ we have

$$\begin{aligned}\int_{B_R\setminus B_r} K(|x|)|u|^{q-1}|h|\,dx &\leq S_N\,\|K(|\cdot|)\|_{L^s(B_R\setminus B_r)}\,\|h\| \left(|B_R\setminus B_r|^{1-\frac{q-1}{\tilde{q}-1}} \left(\int_{B_R\setminus B_r} |u|^2\,dx\right)^{\frac{q-1}{\tilde{q}-1}}\right)^{\frac{\tilde{q}-1}{2}} \\ &= S_N\,\|K(|\cdot|)\|_{L^s(B_R\setminus B_r)}\,\|h\|\,|B_R\setminus B_r|^{\frac{\tilde{q}-q}{2}} \left(\int_{B_R\setminus B_r} |u|^2\,dx\right)^{\frac{q-1}{2}}.\end{aligned}$$

On the other hand, if $q > \tilde{q}$, from (2.6) we get

$$\begin{aligned}\int_{B_R\setminus B_r} K(|x|)|u|^{q-1}|h|\,dx &\leq S_N\,\|K(|\cdot|)\|_{L^s(B_R\setminus B_r)}\,\|h\| \left(\int_{B_R\setminus B_r} |u|^{2\frac{q-1}{\tilde{q}-1}-2}|u|^2\,dx\right)^{\frac{\tilde{q}-1}{2}} \\ &= S_N\,\|K(|\cdot|)\|_{L^s(B_R\setminus B_r)}\,\|h\| \left(\frac{C_N\,\|u\|}{r^{\frac{N-4}{2}}}\right)^{q-\tilde{q}} \left(\int_{B_R\setminus B_r} |u|^2\,dx\right)^{\frac{\tilde{q}-1}{2}}.\end{aligned}$$

Hence the thesis follows. □

We will also need the following result about the convergence in $L^{q_1}_K + L^{q_2}_K$. It is proved in [1].

Lemma 3.2. *Let $\{u_n\} \subseteq L^{q_1}_K + L^{q_2}_K$ be a sequence such that $\forall \epsilon > 0$ there are $n_\epsilon > 0$ and a sequence of measurable sets $E_{\epsilon,n} \subseteq \mathbb{R}^N$ satisfying*

$$\forall n > n_\epsilon, \quad \int_{E_{\epsilon,n}} K(|x|)|u_n|^{q_1}\,dx + \int_{E^c_{\epsilon,n}} K(|x|)|u_n|^{q_2}\,dx < \epsilon.$$

Then $u_n \to 0$ in $L^{q_1}_K + L^{q_2}_K$.

We can now prove Theorem 2.4. The arguments are similar to those of [2], so we will skip the details.

Proof of Theorem 2.4. 1. We can choose $R_1 < R_2$ in hypothesis $\left(\mathcal{S}'_{q_1,q_2}\right)$. If $u \in H^2_{V,r}$, $u \neq 0$, we get

$$\int_{B_{R_1}} K(|x|)|u|^{q_1}\,dx = \|u\|^{q_1} \int_{B_{R_1}} K(|x|)\frac{|u|^{q_1}}{\|u\|^{q_1}}\,dx \leq \|u\|^{q_1}\mathcal{S}_0(q_1,R_1) \tag{3.2}$$

and similarly

$$\int_{B_{R_2}^c} K(|x|)|u|^{q_2}\,dx \le \|u\|^{q_2}\mathcal{S}_\infty(q_2,R_2). \tag{3.3}$$

Furthemore, from Lemma 3.1 and the continuous embedding $D_r^{2,2}(\mathbb{R}^N) \hookrightarrow L^2_{\mathrm{loc}}(\mathbb{R}^N)$, we obtain that there is a constant $C_1 > 0$, independent from u, such that

$$\int_{B_{R_2}\setminus B_{R_1}} K(|x|)|u|^{q_1}\,dx \le C_1\|u\|^{q_1}. \tag{3.4}$$

Hence $u \in L_K^{q_1}(B_{R_2}) \cap L_K^{q_2}(B_{R_2}^c)$, so that $u \in L_K^{q_1} + L_K^{q_2}$. Moreover, by (3.2)-(3.4) and Lemma 3.2, one obtains that $u_n \to 0$ in $H^2_{V,r}$ implies $u_n \to 0$ in $L_K^{q_1} + L_K^{q_2}$.

2. Assume $\left(\mathcal{S}''_{q_1,q_2}\right)$, fix $\epsilon > 0$ and choose a sequence $u_n \rightharpoonup 0$ in $H^2_{V,r}$. From (3.2), (3.3), Lemma 3.1 and the compactness of the embedding $D_r^{2,2}(\mathbb{R}^N) \hookrightarrow L^2_{\mathrm{loc}}(\mathbb{R}^N)$, we obtain

$$\int_{B_{R_\epsilon}} K(|x|)|u|^{q_1}\,dx + \int_{B_{R_\epsilon^c}} K(|x|)|u|^{q_2}\,dx < \epsilon$$

for any n large enough. By Lemma 3.2, this implies that $u_n \to 0$ in $L_K^{q_1} + L_K^{q_2}$, which gives the compactness of the embedding.

□

4 Proofs of Theorems 2.5 and 2.6

In this section we let $N \ge 5$ and prove Theorems 2.5 and 2.6. The first step is the following lemma, which will be also useful in the proofs of Theorems 2.8 and 2.9.

Lemma 4.1. *Let $\Omega \subseteq \mathbb{R}^N$ a nonempty measurable set such that $V(|x|) < +\infty$ almost everywhere on Ω and assume that*

$$\Lambda := \operatorname*{ess\,sup}_{x\in\Omega} \frac{K(|x|)}{|x|^\alpha V(|x|)^\beta} < +\infty \quad \textit{for some } 0 \le \beta \le 1 \textit{ and } \alpha \in \mathbb{R}.$$

Take $u \in H^2_V$ and assume that there exist $\nu \in \mathbb{R}$ and $m > 0$ such that

$$|u(x)| \le \frac{m}{|x|^\nu} \quad \textit{for almost every } x \in \Omega.$$

Then $\forall h \in H^2_V$ and $\forall q > \max\{1, 2\beta\}$, we have

$$\int_\Omega K(|x|)|u|^{q-1}|h|\,dx \le \begin{cases} \Lambda m^{q-1} S_N^{1-2\beta}\left(\displaystyle\int_\Omega |x|^{\frac{\alpha-\nu(q-1)}{N+4(1-2\beta)}2N}\,dx\right)^{\frac{N+4(1-2\beta)}{2N}} \|h\| & \textit{if } 0 \le \beta \le \frac{1}{2} \\ \Lambda m^{q-2\beta}\left(\displaystyle\int_\Omega |x|^{\frac{\alpha-\nu(q-2\beta)}{1-\beta}}\,dx\right)^{1-\beta} \|u\|^{2\beta-1}\,\|h\| & \textit{if } \frac{1}{2} < \beta < 1 \\ \Lambda m^{q-2}\left(\displaystyle\int_\Omega |x|^{2\alpha-2\nu(q-2)} V(|x|)|u|^2\,dx\right)^{\frac{1}{2}} \|h\| & \textit{if } \beta = 1. \end{cases}$$

Proof. We consider several cases.

• Case $\beta = 0$. One has

$$\frac{1}{\Lambda}\int_\Omega K(|x|)|u|^{q-1}|h|\,dx \le \int_\Omega |x|^\alpha |u|^{q-1}|h|\,dx \le \left(\int_\Omega (|x|^\alpha |u|^{q-1})^{\frac{2N}{N+4}}\,dx\right)^{\frac{N+4}{2N}} \left(\int_\Omega |h|^{2^{**}}\,dx\right)^{\frac{1}{2^{**}}}$$
$$\le S_N m^{q-1}\left(\int_\Omega |x|^{\frac{\alpha-\nu(q-1)}{N+4}2N}\,dx\right)^{\frac{N+4}{2N}} \|h\|.$$

• Case $0 < \beta < \frac{1}{2}$. We have $\frac{1}{\beta} > 1$ and $\frac{1-\beta}{1-2\beta}2^{**} > 1$ with Hölder conjugate exponents $(\frac{1}{\beta})' = \frac{1}{1-\beta}$ and $\left(\frac{1-\beta}{1-2\beta}2^{**}\right)' = \frac{2N(1-\beta)}{N+4(1-2\beta)}$. Then we get

$$\begin{aligned}
\frac{1}{\Lambda}\int_\Omega K(|x|)|u|^{q-1}|h|\,dx &\le \int_\Omega |x|^\alpha V(|x|)^\beta |u|^{q-1}|h|\,dx \\
&\le \left(\int_\Omega \left(|x|^\alpha |u|^{q-1}|h|^{1-2\beta}\right)^{\frac{1}{1-\beta}}\,dx\right)^{1-\beta}\left(\int_\Omega V(|x|)|h|^2\,dx\right)^\beta \\
&\le \left[\left(\int_\Omega \left(|x|^\alpha|u|^{q-1}\right)^{\frac{2N}{N+4(1-2\beta)}}\,dx\right)^{\frac{N+4(1-2\beta)}{2N(1-\beta)}}\left(\int_\Omega |h|^{2^{**}}\,dx\right)^{\frac{(1-2\beta)}{(1-\beta)2^{**}}}\right]^{1-\beta}\|h\|^{2\beta} \\
&\le m^{q-1}\left(\int_\Omega |x|^{\frac{\alpha-\nu(q-1)}{N+4(1-2\beta)}2N}\,dx\right)^{\frac{N+4(1-2\beta)}{2N}} S_N^{1-2\beta}\|h\|^{1-2\beta}\|h\|^{2\beta} \\
&= m^{q-1}S_N^{1-2\beta}\left(\int_\Omega |x|^{\frac{\alpha-\nu(q-1)}{N+4(1-2\beta)}2N}\,dx\right)^{\frac{N+4(1-2\beta)}{2N}}\|h\|.
\end{aligned}$$

• Case $\beta = \frac{1}{2}$. We have

$$\frac{1}{\Lambda}\int_\Omega K(|x|)|u|^{q-1}|h|\,dx \le \int_\Omega |x|^\alpha|u|^{q-1}V(|x|)^{\frac{1}{2}}|h|\,dx \le \left(\int_\Omega |x|^{2\alpha}|u|^{2(q-1)}\,dx\right)^{\frac{1}{2}}\left(\int_\Omega V(|x|)|h|^2\,dx\right)^{\frac{1}{2}}$$
$$\le m^{q-1}\left(\int_\Omega |x|^{2\alpha-2\nu(q-1)}\,dx\right)^{\frac{1}{2}}\|h\|.$$

• Case $\frac{1}{2} < \beta < 1$. We have $\frac{1}{2\beta-1} > 1$ with Hölder conjugate exponent $\left(\frac{1}{2\beta-1}\right)' = \frac{1}{2(1-\beta)}$. Hence

$$\begin{aligned}
\frac{1}{\Lambda}\int_\Omega K(|x|)|u|^{q-1}|h|\,dx &\le \int_\Omega |x|^\alpha V(|x|)^\beta |u|^{q-1}|h|\,dx \\
&\le \left(\int_\Omega |x|^{2\alpha} V(|x|)^{2\beta-1}|u|^{2(q-1)}\,dx\right)^{\frac{1}{2}} \left(\int_\Omega V(|x|)|h|^2\,dx\right)^{\frac{1}{2}} \\
&\le \left(\int_\Omega |x|^{2\alpha}|u|^{2(q-2\beta)} V(|x|)^{2\beta-1}|u|^{2(2\beta-1)}\,dx\right)^{\frac{1}{2}} \|h\| \\
&\le \left[\left(\int_\Omega |x|^{\frac{\alpha}{1-\beta}}|u|^{\frac{q-2\beta}{1-\beta}}\,dx\right)^{2(1-\beta)} \left(\int_\Omega V(|x|)|u|^2\,dx\right)^{2\beta-1}\right]^{\frac{1}{2}} \|h\| \\
&\le m^{q-2\beta}\left[\left(\int_\Omega |x|^{\frac{\alpha}{1-\beta}-\nu\frac{q-2\beta}{1-\beta}}\,dx\right)^{2(1-\beta)} \left(\int_\Omega V(|x|)|u|^2\,dx\right)^{2\beta-1}\right]^{\frac{1}{2}} \|h\| \\
&\le m^{q-2\beta}\left(\int_\Omega |x|^{\frac{\alpha-\nu(q-2\beta)}{1-\beta}}\,dx\right)^{1-\beta} \|u\|^{2\beta-1}\|h\|\,.
\end{aligned}$$

• Case $\beta = 1$. As $q > \max\{1, 2\beta\}$, in this case we have $q > 2$. Hence

$$\begin{aligned}
\frac{1}{\Lambda}\int_\Omega K(|x|)|u|^{q-1}|h|\,dx &\le \int_\Omega |x|^\alpha V(|x|)|u|^{q-1}|h|\,dx \\
&\le \left(\int_\Omega |x|^{2\alpha} V(|x|)|u|^{2(q-1)}\,dx\right)^{\frac{1}{2}} \left(\int_\Omega V(|x|)|h|^2\,dx\right)^{\frac{1}{2}} \\
&\le \left(\int_\Omega |x|^{2\alpha}|u|^{2(q-2)} V(|x|)|u|^2\,dx\right)^{\frac{1}{2}} \|h\| \\
&\le m^{q-2}\left(\int_\Omega |x|^{2\alpha-2\nu(q-2)} V(|x|)|u|^2\,dx\right)^{\frac{1}{2}} \|h\|\,.
\end{aligned}$$

□

We can now give the proofs of Theorems 2.5 and 2.6.

Proof of Theorem 2.5. Let $u \in H^2_{V,r}$ and $h \in H^2_V$ such that $\|u\| = \|h\| = 1$. Let $0 < R \le R_1$. Thanks to (2.6) and

$$\operatorname*{ess\,sup}_{x\in B_R} \frac{K(|x|)}{|x|^{\alpha_0}V(|x|)^{\beta_0}} \le \operatorname*{ess\,sup}_{r\in(0,R_1)} \frac{K(r)}{r^{\alpha_0}V(r)^{\beta_0}} < +\infty,$$

we can apply Lemma 4.1 with $\Omega = B_R$, $\alpha = \alpha_0$, $\beta = \beta_0$, $m = C_N$ and $\nu = \frac{N-4}{2}$. In the following C is any positive constant independent from u, h and R. If $0 \le \beta_0 \le \frac{1}{2}$ we get

$$\int_{B_R} K(|x|)|u|^{q_1-1}|h|\,dx \le C\left(\int_{B_R} |x|^{\frac{\alpha_0-\frac{N-4}{2}(q_1-1)}{N+4(1-2\beta_0)}2N}\,dx\right)^{\frac{N+4(1-2\beta_0)}{2N}}$$

$$\le C\left(\int_0^R r^{\frac{2\alpha_0-(N-4)(q_1-1)}{N+4(1-2\beta_0)}N+N-1}\,dr\right)^{\frac{N+4(1-2\beta_0)}{2N}}$$

$$= C\left(R^{\frac{2\alpha_0-8\beta_0+2N-(N-4)q_1}{N+4(1-2\beta_0)}N}\right)^{\frac{N+4(1-2\beta_0)}{2N}},$$

because

$$2\alpha_0-8\beta_0+2N-(N-4)q_1 = (N-4)\left(2\frac{\alpha_0-4\beta_0+N}{N-4}-q_1\right) = (N-4)\left(q^*(\alpha_0,\beta_0)-q_1\right) > 0.$$

If $\frac{1}{2} < \beta_0 < 1$ we get

$$\int_{B_R} K(|x|)|u|^{q_1-1}|h|\,dx \le C\left(\int_{B_R} |x|^{\frac{\alpha_0-\frac{N-4}{2}(q_1-2\beta_0)}{1-\beta_0}}\,dx\right)^{1-\beta_0} \le C\left(\int_0^R r^{\frac{2\alpha_0-(N-4)(q_1-2\beta_0)}{2(1-\beta_0)}+N-1}\,dr\right)^{1-\beta_0}$$

$$= C\left(R^{\frac{2\alpha_0-(N-4)(q_1-2\beta_0)}{2(1-\beta_0)}+N}\right)^{1-\beta_0},$$

because

$$\frac{2\alpha_0-(N-4)(q_1-2\beta_0)}{2(1-\beta_0)}+N = \frac{(N-4)}{2(1-\beta_0)}\left(2\frac{\alpha_0-4\beta_0+N}{N-4}-q_1\right) = \frac{(N-4)}{2(1-\beta_0)}\left(q^*(\alpha_0,\beta_0)-q_1\right) > 0.$$

If $\beta_0 = 1$ we get

$$\int_{B_R} K(|x|)|u|^{q_1-1}|h|\,dx \le C\left(\int_{B_R} |x|^{2\alpha_0-(N-4)(q_1-2)}V(|x|)|u|^2\,dx\right)^{\frac{1}{2}}$$

$$\le C\left(R^{2\alpha_0-(N-4)(q_1-2)}\int_{B_R} V(|x|)|u|^2\,dx\right)^{\frac{1}{2}} \le C\,R^{\frac{2\alpha_0-(N-4)(q_1-2)}{2}},$$

because

$$2\alpha_0-8+2N-(N-4)q_1 = (N-4)\left(2\frac{\alpha_0-4+N}{N-4}-q_1\right) = (N-4)\left(q^*(\alpha_0,1)-q_1\right) > 0.$$

Hence, in any case, we get $\mathcal{R}_0(q_1,R) \le CR^\delta$ for some $\delta = \delta(N,\alpha_0,\beta_0,q_1) > 0$, which gives the result. □

Proof of Theorem 2.6. Let $u \in H^2_{V,r}$ and $h \in H^2_V$ such that $\|u\| = \|h\| = 1$. Let $R \ge R_2$. By (2.6) and

$$\operatorname*{ess\,sup}_{x\in B_R^C} \frac{K(|x|)}{|x|^{\alpha_\infty}V(|x|)^{\beta_\infty}} \le \operatorname*{ess\,sup}_{r>R_2} \frac{K(r)}{r^{\alpha_\infty}V(r)^{\beta_\infty}} < +\infty,$$

we can apply lemma 4.1 with $\Omega = B_R^c$, $\alpha = \alpha_\infty$, $\beta = \beta_\infty$, $m = C_N$ and $\nu = \frac{N-4}{2}$. Hereafter C denotes any positive constant independent form u, h and R. If $0 \le \beta_\infty \le \frac{1}{2}$ we get

$$\begin{aligned}\int_{B_R^c} K(|x|)|u|^{q_2-1}|h|\,dx &\le C\left(\int_{B_R^c} |x|^{\frac{\alpha_\infty - \frac{N-4}{2}(q_2-1)}{N+4(1-2\beta_\infty)}2N}\,dx\right)^{\frac{N+4(1-2\beta_\infty)}{2N}}\\ &\le C\left(\int_R^{+\infty} r^{\frac{2\alpha_\infty-(N-4)(q_2-1)}{N+4(1-2\beta_\infty)}N+N-1}\,dr\right)^{\frac{N+4(1-2\beta_\infty)}{2N}}\\ &= C\left(R^{\frac{2\alpha_\infty-8\beta_\infty+2N-(N-4)q_2}{N+4(1-2\beta_\infty)}N}\right)^{\frac{N+4(1-2\beta_\infty)}{2N}},\end{aligned}$$

because

$$2\alpha_\infty - 8\beta_\infty + 2N - (N-4)q_2 = (N-4)\left(2\frac{\alpha_\infty - 4\beta_\infty + N}{N-4} - q_2\right) = (N-4)\left(q^*(\alpha_\infty, \beta_\infty) - q_2\right) < 0.$$

If $\frac{1}{2} < \beta_\infty < 1$ we have

$$\begin{aligned}\int_{B_R^c} K(|x|)|u|^{q_2-1}|h|\,dx &\le C\left(\int_{B_R^c} |x|^{\frac{\alpha_\infty - \frac{N-4}{2}(q_2-2\beta_\infty)}{1-\beta_\infty}}\,dx\right)^{1-\beta_\infty}\\ &\le C\left(\int_R^{+\infty} r^{\frac{2\alpha_\infty-(N-4)(q_2-2\beta_\infty)}{2(1-\beta_\infty)}+N-1}\,dr\right)^{1-\beta_\infty}\\ &= C\left(R^{\frac{2\alpha_\infty-(N-4)(q_2-2\beta_\infty)}{2(1-\beta_\infty)}+N}\right)^{1-\beta_\infty},\end{aligned}$$

because

$$\begin{aligned}\frac{2\alpha_\infty-(N-4)(q_2-2\beta_\infty)}{2(1-\beta_\infty)} + N &= \frac{(N-4)}{2(1-\beta_\infty)}\left(2\frac{\alpha_\infty - 4\beta_\infty + N}{N-4} - q_2\right)\\ &= \frac{(N-4)}{2(1-\beta_\infty)}\left(q^*(\alpha_\infty, \beta_\infty) - q_2\right) < 0.\end{aligned}$$

If $\beta_\infty = 1$ we get

$$\begin{aligned}\int_{B_R^c} K(|x|)|u|^{q_2-1}|h|\,dx &\le C\left(\int_{B_R^c} |x|^{2\alpha_\infty-(N-4)(q_2-2)} V(|x|)|u|^2\,dx\right)^{\frac{1}{2}}\\ &\le C\left(R^{2\alpha_\infty-(N-4)(q_2-2)}\int_{B_R^C} V(|x|)|u|^2\,dx\right)^{\frac{1}{2}} \le C\,R^{\frac{2\alpha_\infty-(N-4)(q_2-2)}{2}},\end{aligned}$$

because

$$2\alpha_\infty - 8 + 2N - (N-4)q_2 = (N-4)\left(2\frac{\alpha_\infty - 4 + N}{N-4} - q_2\right) = (N-4)\left(q^*(\alpha_\infty, 1) - q_2\right) < 0.$$

So, in any case, we get $\mathcal{R}_\infty(q_2, R) \le CR^\delta$ for some $\delta = \delta(N, \alpha_0, \beta_0, q_1) < 0$. Hence the thesis follows. ◻

5 Proofs of Theorems 2.8 and 2.9

Let $N \geq 5$. To prove Theorems 2.8 and 2.9 we need some preliminary results about pointwise estimates of radial Sobolev functions.

For any open interval $\mathcal{I} \subset \mathbb{R}$, we will consider the space

$$W^{2,1}(\mathcal{I}) := \left\{ u \in L^1(\mathcal{I}) : D^\alpha u \in L^1(\mathcal{I}),\ \forall |\alpha| \leq 2 \right\}.$$

The proof of the following lemma can be easily derived from the arguments of [3, Appendix], so we skip it.

Lemma 5.1. *Let $u \in D_r^{2,2}(\mathbb{R}^N)$ and define $\tilde{u} : \mathbb{R}_+ \to \mathbb{R}$ such that $u(x) = \tilde{u}(|x|)$ for almost every $x \in \mathbb{R}^N$. Then $\tilde{u} \in W^{2,1}(\mathcal{I})$ for every open bounded interval $\mathcal{I} \subset \mathbb{R}_+$ such that* $\inf \mathcal{I} > 0$.

Proposition 5.2. *Assume that there exists $R_2 > 0$ such that $V(r) < +\infty$ for almost every $r > R_2$ and*

$$\lambda_\infty := \operatorname*{ess\,inf}_{r > R_2} r^{\gamma_\infty} V(r) > 0 \quad \textit{for some } \gamma_\infty \leq \frac{14}{3}.$$

Then $\forall u \in H^2_{V,r}$ we have

$$|u(x)| \leq c_\infty \lambda_\infty^{-\frac{1}{4}} \frac{\|u\|}{|x|^{\frac{2(N-2)-\gamma_\infty}{4}}} \quad \textit{almost everywhere in } B^c_{R_2}, \tag{5.1}$$

where $c_\infty = \dfrac{1}{\sqrt{\sigma_N}} \left(\dfrac{8}{N(2(N-2)-\gamma_\infty)} \right)^{\frac{1}{4}}$.

Notice that $2(N-2) - \gamma_\infty > 0$ in (5.1).

Proof. Let $u \in H^2_{V,r}$. Define $\tilde{u} : \mathbb{R}_+ \to \mathbb{R}$ as the continuous function such that $u(x) = \tilde{u}(|x|)$ for almost every $x \in \mathbb{R}^N$. Define

$$v(r) := r^{\frac{N}{2} - 1 - \frac{\gamma_\infty}{4}} \tilde{u}(r)^2 \quad \text{for every } r > 0.$$

If $\lambda := \liminf_{r \to +\infty} v(r) > 0$, then for r large enough we get

$$r^{N-1-\gamma_\infty} \tilde{u}(r)^2 \geq \frac{\lambda}{2 r^{\frac{3}{4}\gamma_\infty - \frac{N}{2}}}$$

and from this we get the following contradiction:

$$\int_{B^c_{R_2}} V(|x|) u^2 \, dx \geq \lambda_\infty \int_{B^c_{R_2}} \frac{u^2}{|x|^{\gamma_\infty}} \, dx = \lambda_\infty \sigma_N \int_{R_2}^{+\infty} \frac{\tilde{u}(r)^2}{r^{\gamma_\infty}} r^{N-1} \, dr \geq \lambda_\infty \sigma_N \int_{R_2}^{+\infty} \frac{\lambda}{2 r^{\frac{3}{4}\gamma_\infty - \frac{N}{2}}} \, dr = +\infty$$

where the last integral diverges because $N \geq 5$ and $\gamma_\infty \leq 14/3$. Hence, it must be $\lambda = 0$ and therefore there is a sequence $r_n \to +\infty$ such that $v(r_n) \to 0$. From lemma 5.1 we get $v \in W^{2,1}((r, r_n))$ for all $R_2 < r < r_n < +\infty$, whence

$$v(r_n) - v(r) = \int_r^{r_n} v'(s) \, ds.$$

Furthermore, for all $s \in (r, r_n)$ we have

$$v'(s) = \left(\frac{N}{2} - 1 - \frac{\gamma_\infty}{4}\right) s^{\frac{N}{2}-2-\frac{\gamma_\infty}{4}} \tilde{u}(s)^2 + 2s^{\frac{N}{2}-1-\frac{\gamma_\infty}{4}} \tilde{u}(s)\tilde{u}'(s) \geq 2s^{\frac{N}{2}-1-\frac{\gamma_\infty}{4}} \tilde{u}(s)\tilde{u}'(s)$$
$$\geq -2s^{\frac{N}{2}-1-\frac{\gamma_\infty}{4}} |\tilde{u}(s)||\tilde{u}'(s)|.$$

The first inequality derives from $\frac{N}{2} - 1 - \frac{\gamma_\infty}{4} > \frac{N}{2} - \frac{13}{6} > 0$. Then we get

$$v(r_n) - v(r) = \int_r^{r_n} v'(s)\, ds \geq -2 \int_r^{r_n} s^{\frac{N}{2}-1-\frac{\gamma_\infty}{4}} |\tilde{u}(s)||\tilde{u}'(s)|\, ds.$$

Now from (2.3) we deduce that

$$\begin{aligned}
v(r) - v(r_n) &\leq 2 \int_r^{r_n} s^{\frac{N}{2}-1-\frac{\gamma_\infty}{4}} |\tilde{u}(s)||\tilde{u}'(s)|\, ds \leq \frac{2}{\sqrt{N\sigma_N}} \int_r^{r_n} s^{\frac{N}{2}-1-\frac{\gamma_\infty}{4}} |\tilde{u}(s)| \frac{\|\Delta u\|_2}{s^{\frac{N-2}{2}}}\, ds \\
&= \frac{2\,\|\Delta u\|_2}{\sqrt{N\sigma_N}} \int_r^{r_n} \frac{|\tilde{u}(s)|}{s^{\frac{\gamma_\infty}{2}}} s^{\frac{N-1}{2}} \frac{1}{s^{\frac{N}{2}-\frac{1}{2}-\frac{\gamma_\infty}{4}}}\, ds \\
&\leq \frac{2\,\|\Delta u\|_2}{\sqrt{N\sigma_N}} \left(\int_r^{r_n} \frac{\tilde{u}(s)^2}{s^{\gamma_\infty}} s^{N-1}\, ds\right)^{\frac{1}{2}} \left(\int_r^{r_n} \frac{1}{s^{N-1-\frac{\gamma_\infty}{2}}}\, ds\right)^{\frac{1}{2}} \\
&\leq \frac{2\,\|\Delta u\|_2}{\sqrt{N\sigma_N}} \left(\int_{R_2}^{+\infty} \frac{\tilde{u}(s)^2}{s^{\gamma_\infty}} s^{N-1}\, ds\right)^{\frac{1}{2}} \left(\int_r^{+\infty} \frac{1}{s^{2\left(\frac{N}{2}-1-\frac{\gamma_\infty}{4}\right)+1}}\, ds\right)^{\frac{1}{2}} \\
&\leq \frac{2\,\|\Delta u\|_2}{\sqrt{N\sigma_N}} \left(\frac{1}{\lambda_\infty} \int_{R_2}^{+\infty} V(s)\tilde{u}(s)^2 s^{N-1}\, ds\right)^{\frac{1}{2}} \left(\left[\frac{s^{-2\left(\frac{N}{2}-1-\frac{\gamma_\infty}{4}\right)}}{-2\left(\frac{N}{2}-1-\frac{\gamma_\infty}{4}\right)}\right]_r^{+\infty}\right)^{\frac{1}{2}} \\
&\leq \frac{2\,\|\Delta u\|_2}{\sqrt{N\sigma_N}} \left(\frac{1}{\lambda_\infty \sigma_N} \int_{\mathbb{R}^N} V(|x|) u^2\, dx\right)^{\frac{1}{2}} \frac{r^{-\left(\frac{N}{2}-1-\frac{\gamma_\infty}{4}\right)}}{\sqrt{2\left(\frac{N}{2}-1-\frac{\gamma_\infty}{4}\right)}}.
\end{aligned}$$

Since it is easy to see that $\|\Delta u\|_2 \|u\|_V \leq \|u\|^2$, we obtain

$$v(r) - v(r_n) \leq \frac{1}{\sigma_N} \sqrt{\frac{8}{\lambda_\infty N[2(N-2) - \gamma_\infty]}} \frac{\|u\|^2}{r^{\left(\frac{N}{2}-1-\frac{\gamma_\infty}{4}\right)}}.$$

Finally, recalling the definition of $v(r)$ e the fact that $v(r_n) \to 0$, we conclude

$$|x|^{\left(\frac{N}{2}-1-\frac{\gamma_\infty}{4}\right)} |u(x)|^2 \leq \frac{1}{\sigma_N} \sqrt{\frac{8}{\lambda_\infty N[2(N-2) - \gamma_\infty]}} \frac{\|u\|^2}{|x|^{\left(\frac{N}{2}-1-\frac{\gamma_\infty}{4}\right)}}$$

and hence $|u(x)| \leq c_\infty \lambda_\infty^{-\frac{1}{4}} \|u\|\, |x|^{-\frac{2(N-2)-\gamma_\infty}{4}}$. □

We now prove a second pointwise estimate.

Proposition 5.3. *Assume there exists* $R > 0$ *sucht that* $V(r) < +\infty$ *almost everywhere on* $(0, R)$ *and*

$$\lambda_0 := \operatorname*{ess\,inf}_{r \in (0,R)} r^{\gamma_0} V(r) > 0 \quad \textit{for some } \gamma_0 \geq 4.$$

Then $\forall u \in H^2_{V,r}$ *we have*

$$|u(x)| \le c_0 \left(\frac{1}{\sqrt{\lambda_0}} + \frac{R^{\frac{\gamma_0-4}{2}}}{\lambda_0} \right)^{\frac{1}{2}} \frac{\|u\|}{|x|^{\frac{2N-4-\gamma_0}{2}}} \quad \textit{almost everywhere in } B_R, \tag{5.2}$$

where $c_0 = \sqrt{\dfrac{\max\{2/\sqrt{N}, N-7/2\}}{\sigma_N}}$.

Proof. Let $u \in H^2_{V,r}$ and define $\tilde{u} : \mathbb{R}_+ \to \mathbb{R}$ as the continuous function such that $u(x) = \tilde{u}(|x|)$ for almost every $x \in \mathbb{R}^N$. Define

$$v(r) := r^{\frac{2N-3-\gamma_0}{2}} \tilde{u}(r)^2 \quad \text{for all } r > 0.$$

If $\lambda := \liminf_{r\to+\infty} v(r) > 0$, then for all r large enough we have

$$r^{N-1-\gamma_0} \tilde{u}(r)^2 \ge \frac{\lambda}{2r^{\frac{\gamma_0}{2}-\frac{1}{2}}},$$

from which we derive a contradiction as follows:

$$\int_{B^R} V(|x|) u^2\, dx \ge \lambda_\infty \int_{B_R} \frac{u^2}{|x|^{\gamma_0}}\, dx = \lambda_0 \sigma_N \int_0^R \frac{\tilde{u}(r)^2}{r^{\gamma_0}} r^{N-1}\, dr \ge \lambda_0 \sigma_N \int_0^R \frac{\lambda}{2r^{\frac{\gamma_0-1}{2}}}\, dr = +\infty$$

where the last integral diverges because $\gamma_0 \ge 4$. This proves that $\lambda = 0$ and thus implies that there exists a sequence $r_n \to 0^+$ tale che $v(r_n) \to 0$. From lemma 5.1 we get $v \in W^{2,1}((r_n, r))$ for all $0 < r_n < r < R$, whence

$$v(r) - v(r_n) = \int_{r_n}^r v'(s)\, ds.$$

Furthemore for all $s \in (r_n, r)$ we have

$$v'(s) = \left(\frac{2N-3-\gamma_0}{2} \right) s^{\frac{2N-5-\gamma_0}{2}} \tilde{u}(s)^2 + 2s^{\frac{2N-3-\gamma_0}{2}} \tilde{u}(s)\tilde{u}'(s) = \left(\frac{2N-3-\gamma_0}{2} \right) I(s) + 2J(s)$$

with obvious definitions of $I(s)$ and $J(s)$, on which we obtain the following estimates:

$$\int_{r_n}^r I(s) ds = \int_{r_n}^r s^{\frac{2N-5-\gamma_0}{2}} \tilde{u}(s)^2\, ds = \int_{r_n}^r \frac{\tilde{u}(s)^2}{s^{\gamma_0}} s^{N-1} s^{\frac{\gamma_0-3}{2}}\, ds \le r^{\frac{\gamma_0-3}{2}} \int_0^R \frac{\tilde{u}(s)^2}{s^{\gamma_0}} s^{N-1}\, ds$$
$$\le r^{\frac{\gamma_0-3}{2}} \frac{1}{\lambda_0} \int_0^R V(s)\tilde{u}(s)^2 s^{N-1}\, ds \le r^{\frac{\gamma_0-3}{2}} \frac{\|u\|^2}{\lambda_0 \sigma_N} \le R^{\frac{\gamma_0-4}{2}} \frac{\|u\|^2}{\lambda_0 \sigma_N} r^{\frac{1}{2}}$$

and, by (2.3),

$$\begin{aligned}\int_{r_n}^{r} J(s)ds &= \int_{r_n}^{r} s^{\frac{2N-3-\gamma_0}{2}}\tilde{u}(s)\tilde{u}'(s)\,ds \le \int_{r_n}^{r} s^{\frac{2N-3-\gamma_0}{2}}|\tilde{u}(s)||\tilde{u}'(s)|\,ds\\
&\le \frac{\|\Delta u\|_2}{\sqrt{N\sigma_N}}\int_{r_n}^{r} s^{\frac{2N-3-\gamma_0}{2}}|\tilde{u}(s)|\frac{1}{s^{\frac{N-2}{2}}}\,ds = \frac{\|\Delta u\|_2}{\sqrt{N\sigma_N}}\int_{r_n}^{r}\frac{|\tilde{u}(s)|}{s^{\frac{\gamma_0}{2}}}s^{\frac{N-1}{2}}\,ds\\
&\le \frac{\|\Delta u\|_2}{\sqrt{N\sigma_N}}\left(\int_{r_n}^{r}\frac{\tilde{u}(s)^2}{s^{\gamma_0}}s^{N-1}\,ds\right)^{\frac{1}{2}}\left(\int_{r_n}^{r} ds\right)^{\frac{1}{2}}\\
&\le \frac{\|\Delta u\|_2}{\sqrt{N\sigma_N}}\left(\frac{1}{\lambda_0}\int_{0}^{R} V(s)\tilde{u}(s)^2 s^{N-1}\,ds\right)^{\frac{1}{2}}\left(\int_{0}^{r} ds\right)^{\frac{1}{2}}\\
&\le \frac{\|\Delta u\|_2}{\sigma_N\sqrt{N}}\frac{\|u\|_V}{\sqrt{\lambda_0}}r^{\frac{1}{2}} \le \frac{\|u\|^2}{\sigma_N\sqrt{N\lambda_0}}r^{\frac{1}{2}}.\end{aligned}$$

Now, if $4 \le \gamma_0 \le 2N-3$, we get

$$\begin{aligned}v(r)-v(r_n) &= \int_{r_n}^{r} v'(s)\,ds \le \left(\frac{2N-3-\gamma_0}{2}\right)\int_{r_n}^{r} I(s)ds + 2\int_{r_n}^{r} J(s)ds\\
&\le \left(\frac{2N-3-\gamma_0}{2}\right)R^{\frac{\gamma_0-4}{2}}\frac{\|u\|}{\lambda_0\sigma_N}r^{\frac{1}{2}} + \frac{2\|u\|^2}{\sigma_N\sqrt{N\lambda_0}}r^{\frac{1}{2}}\\
&\le \left(N-\frac{7}{2}\right)R^{\frac{\gamma_0-4}{2}}\frac{\|u\|^2}{\lambda_0\sigma_N}r^{\frac{1}{2}} + \frac{2\|u\|^2}{\sigma_N\sqrt{N\lambda_0}}r^{\frac{1}{2}}\end{aligned}$$

On the other hand, if $\gamma_0 \ge 2N-3$, we get

$$v'(s) = \left(\frac{2N-3-\gamma_0}{2}\right)I(s) + 2J(s) \le 2J(s)$$

and thus

$$v(r)-v(r_n) = \int_{r_n}^{r} v'(s)\,ds \le 2\int_{r_n}^{r} J(s)ds \le \frac{2\|u\|^2}{\sigma_N\sqrt{N\lambda_0}}r^{\frac{1}{2}}.$$

So, in any case, we have

$$v(r)-v(r_n) \le \frac{1}{\sigma_N}\left[\frac{2}{\sqrt{N\lambda_0}} + \left(N-\frac{7}{2}\right)\frac{R^{\frac{\gamma_0-4}{2}}}{\lambda_0}\right]\|u\|^2 r^{\frac{1}{2}}.$$

Hence, recalling the definition of $v(r)$ and the fact that $v(r_n) \to 0$, we deduce

$$|x|^{\frac{2N-3-\gamma_0}{2}}|u(x)|^2 \le \frac{1}{\sigma_N}\left[\frac{2}{\sqrt{N\lambda_0}} + \left(N-\frac{7}{2}\right)\frac{R^{\frac{\gamma_0-4}{2}}}{\lambda_0}\right]\|u\|^2 |x|^{\frac{1}{2}},$$

which gives (5.2). □

We will also need the following lemma.

Lemma 5.4. *Assume that there exists $R > 0$ be such that $V(r) < +\infty$ almost everywhere on $(0, R)$ and*

$$\Lambda_{\alpha,\beta}(R) := \operatorname*{ess\,sup}_{r\in(0,R)} \frac{K(r)}{r^\alpha V(r)^\beta} < +\infty \quad \textit{for some } \frac{1}{2} \leq \beta \leq 1 \textit{ and } \alpha \in \mathbb{R} \tag{5.3}$$

and

$$\lambda(R) := \operatorname*{ess\,inf}_{r\in(0,R)} r^{\gamma_0} V(r) > 0 \quad \textit{for some } \gamma_0 > 4.$$

Assume also that $\exists q > 2\beta$ such that $(2N - 4 - \gamma_0)q < 4\alpha + 4N - 2(\gamma_0 + 4)\beta$. Then $\forall u \in H^2_{V,r}$ and $\forall h \in H^2_V$ we have

$$\int_{B_R} K(|x|)|u|^{q-1}|h|\,dx \leq c_0^{q-2\beta} a(R) R^{\frac{4\alpha+4N-2(\gamma_0+4)\beta-(2N-4-\gamma_0)q}{4}} \|u\|^{q-1} \|h\|,$$

where $a(R) := \Lambda_{\alpha,\beta}(R) \left(\frac{1}{\sqrt{\lambda(R)}} + \frac{R^{\frac{\gamma_0-4}{2}}}{\lambda(R)} \right)^{(q-2\beta)/2}$ *and c_0 is given in Proposition 5.3.*

Proof. Take $u \in H^2_{V,r}$ and $h \in H^2_V$. Thanks to assumption (5.3) and Proposition 5.3, we can apply Lemma 4.1 with $\Omega = B_R$, $\Lambda = \Lambda_{\alpha,\beta}(R)$, $\nu = \frac{2N-4-\gamma_0}{4}$ and

$$m = c_0 \left(\frac{1}{\sqrt{\lambda(R)}} + \frac{R^{\frac{\gamma_0-4}{2}}}{\lambda(R)} \right)^{\frac{1}{2}} \|u\|.$$

If $\frac{1}{2} \leq \beta < 1$ we have

$$\begin{aligned}
\int_{B_R} K(|x|)|u|^{q-1}|h|\,dx &\leq \Lambda m^{q-2\beta} \left(\int_\Omega |x|^{\frac{\alpha-\nu(q-2\beta)}{1-\beta}}\,dx \right)^{1-\beta} \|u\|^{2\beta-1} \|h\| \\
&= c_0^{q-2\beta} a(R) \left(\int_{B_R} |x|^{\frac{4\alpha-(2N-4-\gamma_0)(q-2\beta)}{4(1-\beta)}}\,dx \right)^{1-\beta} \|u\|^{2\beta-1} \|u\|^{q-2\beta} \|h\| \\
&\leq c_0^{q-2\beta} a(R) \left(R^{\frac{4\alpha-(2N-4-\gamma_0)(q-2\beta)}{4(1-\beta)}+N} \right)^{1-\beta} \|u\|^{q-1} \|h\|,
\end{aligned}$$

because

$$\frac{4\alpha - (2N-4-\gamma_0)(q-2\beta)}{4(1-\beta)} + N = \frac{4\alpha + 4N - 2(\gamma_0+4)\beta - (2N-4-\gamma_0)q}{4(1-\beta)} > 0$$

If $\beta = 1$ we get

$$\begin{aligned}
\int_{B_R} K(|x|)|u|^{q-1}|h|\,dx &\leq \Lambda m^{q-2} \left(\int_\Omega |x|^{2\alpha-2\nu(q-2)} V(|x|)|u|^2\,dx \right)^{\frac{1}{2}} \|h\| \\
&= c_0^{q-2\beta} a(R) \left(\int_{B_R} |x|^{\frac{4\alpha-(2N-4-\gamma_0)(q-2)}{2}} V(|x|)|u|^2\,dx \right)^{\frac{1}{2}} \|u\|^{q-2} \|h\| \\
&\leq c_0^{q-2\beta} a(R) \left(R^{\frac{4\alpha-(2N-4-\gamma_0)(q-2)}{2}} \int_{B_R} V(|x|)|u|^2\,dx \right)^{\frac{1}{2}} \|u\|^{q-2} \|h\| \\
&\leq c_0^{q-2\beta} a(R) R^{\frac{4\alpha-(2N-4-\gamma_0)(q-2)}{4}} \|u\|^{q-1} \|h\|,
\end{aligned}$$

because $4\alpha - (2N-4-\gamma_0)(q-2) = 4\alpha + 4N - 2(\gamma_0+2) - (2N-4-\gamma_0)q > 0$. ☐

We can now give the proofs of Theorems 2.8 and 2.9. For convenience, define three functions $\alpha_1 = \alpha_1(\beta,\gamma)$, $\alpha_2 = \alpha_2(\beta)$ and $\alpha_3 = \alpha_3(\beta,\gamma)$ as follows:

$$\alpha_1(\beta,\gamma) := -(1-\beta)\gamma, \quad \alpha_2(\beta) := -(1-\beta)N, \quad \alpha_3(\beta,\gamma) := -\frac{N+(1-2\beta)\gamma}{2}. \tag{5.4}$$

Proof of Theorem 2.8. For brevity, define

$$\Lambda_\infty := \operatorname*{ess\,sup}_{r>R_2} \frac{K(r)}{r^{\alpha_\infty} V(r)^{\beta_\infty}} \quad \text{and} \quad \lambda_\infty := \operatorname*{ess\,inf}_{r>R_2} r^{\gamma_\infty} V(r).$$

Take $u \in H^2_{V,r}$ and $h \in H^2_V$ such that $\|u\| = \|h\| = 1$. Let $R \geq R_2$. Hereafter C denotes any positive constant independent from u, h and R. For all $\xi \geq 0$ we have

$$\operatorname*{ess\,sup}_{r>R} \frac{K(r)}{r^{\alpha_\infty+\xi\gamma_\infty} V(r)^{\beta_\infty+\xi}} \leq \operatorname*{ess\,sup}_{r>R_2} \frac{K(r)}{r^{\alpha_\infty} V(r)^{\beta_\infty} (r^{\gamma_\infty} V(r))^\xi} \leq \frac{\Lambda_\infty}{\lambda_\infty^\xi} \leq +\infty. \tag{5.5}$$

We now distinguish several cases. In each of them we will choose a suitable $\xi \geq 0$ and we will apply Lemma 4.1 with $\Omega = B_R^c$, $\alpha = \alpha_\infty + \xi\gamma_\infty$, $\beta = \beta_\infty + \xi$, $m = c_\infty \lambda_\infty^{-1/4} \|u\| = c_\infty \lambda_\infty^{-1/4}$, $\nu = \frac{2(N-2)-\gamma_\infty}{4}$ and

$$\Lambda = \operatorname*{ess\,sup}_{r>R} \frac{K(r)}{r^{\alpha_\infty+\xi\gamma_\infty} V(r)^{\beta_\infty+\xi}}.$$

In each case we will get

$$\int_{B_R^c} K(|x|)|u|^{q_2-1}|h|\,dx \leq CR^\delta$$

for some $\delta < 0$, independent from R, whence the thesis follows. Recalling the definitions (5.4), we set $\alpha_1 = \alpha_1(\beta_\infty,\gamma_\infty)$, $\alpha_2 = \alpha_2(\beta_\infty)$ and $\alpha_3 = \alpha_3(\beta_\infty,\gamma_\infty)$ for brevity.

• Case $\alpha_\infty \geq \alpha_1$. We take $\xi = 1-\beta_\infty$ and apply Lemma 4.1 with $\beta = \beta_\infty + \xi = 1$ and $\alpha = \alpha_\infty + \xi\gamma_\infty = \alpha_\infty + (1-\beta_\infty)\gamma_\infty$. We get

$$\begin{aligned}\int_{B_R^c} K(|x|)|u|^{q_2-1}|h|\,dx &\leq C\left(\int_{B_R^c} |x|^{2\alpha-2\nu(q_2-2)} V(|x|)|u|^2\,dx\right)^{\frac{1}{2}} \\ &\leq C\left(R^{2\alpha-2\nu(q_2-2)} \int_{B_R^c} V(|x|)|u|^2\,dx\right)^{\frac{1}{2}} \leq CR^{\alpha-\nu(q_2-2)},\end{aligned}$$

because

$$\begin{aligned}\alpha - \nu(q_2-2) &= \alpha_\infty + (1-\beta_\infty)\gamma_\infty - \frac{2(N-2)-\gamma_\infty}{4}(q_2-2) \\ &= \frac{2\alpha_\infty + \gamma_\infty - 2\beta_\infty\gamma_\infty + 2N - 4}{2} - \frac{2(N-2)-\gamma_\infty}{4} q_2 \\ &= \frac{2(N-2)-\gamma_\infty}{4}\left(2\frac{2\alpha_\infty + (1-2\beta_\infty)\gamma_\infty + 2(N-2)}{2(N-2)-\gamma_\infty} - q_2\right) \\ &= \frac{2(N-2)-\gamma_\infty}{4}(q_{**} - q_2) < 0.\end{aligned}$$

• Case $\max\{\alpha_2,\alpha_3\} < \alpha_\infty < \alpha_1$. We take $\xi = \frac{\alpha_\infty+(1-\beta_\infty)N}{N-\gamma_\infty}$ and apply Lemma 4.1 with $\beta = \beta_\infty + \xi$ and $\alpha = \alpha_\infty + \xi\gamma_\infty$. Notice that, if $\alpha_3 < \alpha_\infty < \alpha_1$, then

$$\beta = \beta_\infty + \xi = \frac{\alpha_\infty - \gamma_\infty\beta_\infty + N}{N-\gamma_\infty} \in \left(\frac{1}{2}, 1\right).$$

On the other hand, if $\alpha_\infty = \alpha_2$ ($= \max\{\alpha_2,\alpha_3\}$ when $\frac{1}{2} < \beta_\infty < 1$), then $\xi = 0$ and

$$\beta = \beta_\infty \in \left(\frac{1}{2}, 1\right).$$

We obtain

$$\int_{B_R^c} K(|x|)|u|^{q_2-1}|h|\,dx \le C\left(\int_{B_R^c} |x|^{\frac{\alpha-\nu(q_2-2\beta)}{1-\beta}}\,dx\right)^{1-\beta} \le C\left(R^{\frac{\alpha-\nu(q_2-2\beta)}{1-\beta}+N}\right)^{1-\beta},$$

because

$$\frac{\alpha-\nu(q_2-2\beta)}{1-\beta} + N = \frac{\nu}{1-\beta}\left(2\frac{\alpha_\infty - \beta_\infty\gamma_\infty + N}{N-\gamma_\infty} - q_2\right) = \frac{\nu}{1-\beta}(q_* - q_2) < 0.$$

• Case $\alpha_\infty \le 0 = \alpha_2$ ($= \max\{\alpha_2,\alpha_3\}$) and $\beta_\infty = 1$. We take $\xi = 0$ and apply Lemma 4.1 with $\beta = \beta_\infty + \xi = 1$ and $\alpha = \alpha_\infty + \xi\gamma_\infty = \alpha_\infty$. We get

$$\int_{B_R^c} K(|x|)|u|^{q_2-1}|h|\,dx \le C\left(\int_{B_R^c} |x|^{2\alpha_\infty-2\nu(q_2-2)}V(|x|)|u|^2\,dx\right)^{\frac{1}{2}} \le CR^{\alpha_\infty-\nu(q_2-2)},$$

because $\alpha_\infty - \nu(q_2-2) \le -\nu(q_2-2) < 0$.

• Case $\alpha_\infty \le \alpha_2$ ($= \max\{\alpha_2,\alpha_3\}$) and $\frac{1}{2} < \beta_\infty < 1$. We take $\xi = 0$ and apply Lemma 4.1 with $\beta = \beta_\infty \in \left(\frac{1}{2}, 1\right)$ and $\alpha = \alpha_\infty + \xi\gamma_\infty = \alpha_\infty$. We get

$$\int_{B_R^c} K(|x|)|u|^{q_2-1}|h|\,dx \le C\left(\int_{B_R^c} |x|^{\frac{\alpha_\infty-\nu(q_2-2\beta_\infty)}{1-\beta_\infty}}\,dx\right)^{1-\beta_\infty} \le C\left(R^{\frac{\alpha_\infty-\nu(q_2-2\beta_\infty)}{1-\beta_\infty}+N}\right)^{1-\beta_\infty},$$

because

$$\frac{\alpha_\infty-\nu(q_2-2\beta_\infty)}{1-\beta_\infty} + N = \frac{\alpha_\infty + (1-\beta_\infty)N - \nu(q_2-2\beta_\infty)}{1-\beta_\infty} = \frac{\alpha_\infty - \alpha_2 - \nu(q_2-2\beta_\infty)}{1-\beta_\infty} < 0.$$

• Case $\alpha_\infty \le \alpha_3$ ($= \max\{\alpha_2,\alpha_3\}$) and $\beta_\infty \le \frac{1}{2}$. We take $\xi = \frac{1-2\beta_\infty}{2} \ge 0$ and apply Lemma 4.1 with $\beta = \beta_\infty + \xi = \frac{1}{2}$ and $\alpha = \alpha_\infty + \xi\gamma_\infty = \alpha_\infty$. We get

$$\int_{B_R^c} K(|x|)|u|^{q_2-1}|h|\,dx \le C\left(\int_{B_R^c} |x|^{2\alpha-2\nu(q_2-1)}\,dx\right)^{\frac{1}{2}} \le CR^{\alpha-\nu(q_2-1)+\frac{N}{2}}$$

because $\alpha - \nu(q_2-1) + \frac{N}{2} = \alpha_\infty + \frac{1-2\beta_\infty}{2}\gamma_\infty + \frac{N}{2} - \nu(q_2-1) = \alpha_\infty - \alpha_3 - \nu(q_2-1) < 0.$ $\square$

Proof of Theorem 2.9. Define

$$\Lambda_0 := \operatorname*{ess\,sup}_{r\in(0,R_1)} \frac{K(r)}{r^{\alpha_0}V(r)^{\beta_0}} \quad \text{and} \quad \lambda_0 := \operatorname*{ess\,inf}_{r\in(0,R_1)} r^{\gamma_0}V(r).$$

If $\gamma_0 = 4$ the thesis derives from Theorem 2.5, so we assume $\gamma > 4$. To prove the result, we want to find a function $b(R) > 0$ such that $b(R) \to 0$ when $R \to 0^+$ and

$$\int_{B_R} K(|x|)|u|^{q_1-1}|h|\,dx \le b(R)\|u\|^{q_1-1}\|h\|, \;\; \forall u \in H^2_{V,r}, \; \forall h \in H^2_V.$$

So we fix $0 < R \le R_1$. Then

$$\lambda(R) := \operatorname*{ess\,inf}_{r\in(0,R)} r^{\gamma_0}V(r) \ge \lambda_0 > 0 \tag{5.6}$$

and for all $\xi \ge 0$ we have

$$\Lambda_{\alpha_0+\xi\gamma_0,\beta_0+\xi} := \operatorname*{ess\,sup}_{r\in(0,R)} \frac{K(r)}{r^{\alpha_0+\xi\gamma_0}V(r)^{\beta_0+\xi}} \le \operatorname*{ess\,sup}_{r\in(0,R_1)} \frac{K(r)}{r^{\alpha_0}V(r)^{\beta_0}(r^{\gamma_0}V(r))^{\xi}} \le \frac{\Lambda_0}{\lambda_0^{\xi}} < +\infty. \tag{5.7}$$

We now consider several cases.

• Case $4 < \gamma_0 < N$. In this case $(\alpha_0, q_1) \in \mathcal{A}_{\beta_0,\gamma_0}$ implies $\alpha_0 > \max\{\alpha_2, \alpha_3\}$ and

$$\max\{1, 2\beta_0\} < q_1 < \min\left\{2\frac{\alpha_0 - \gamma_0\beta_0 + N}{N - \gamma_0}, 2\frac{2\alpha_0 + (1-2\beta_0)\gamma_0 + 2N - 4}{2N - 4 - \gamma_0}\right\}.$$

Hence, we can find $\xi \ge 0$, independent from R, u and h, such that $\alpha = \alpha_0 + \xi\gamma_0$ and $\beta = \beta_0 + \xi$ satisfy

$$\frac{1}{2} \le \beta \le 1 \quad \text{and} \quad 2\beta < q_1 < \frac{4\alpha + 4N - 2(\gamma_0+4)\beta}{2N - 4 - \gamma_0}. \tag{5.8}$$

Recalling (5.6) and (5.7), we can apply Lemma 5.4 (with $q = q_1$), whence $\forall u \in H^2_{V,r}$ and $\forall h \in H^2_V$ we get

$$\int_{B_R} K(|x|)|u|^{q_1-1}|h|\,dx \le c_0^{q_1-2\beta} a(R) R^{\frac{4\alpha+4N-2(\gamma_0+4)\beta-(2N-4-\gamma_0)q_1}{4}} \|u\|^{q_1-1}\|h\|.$$

This implies the thesis because $R^{4\alpha+4N-2(\gamma_0+4)\beta-(2N-4-\gamma_0)q_1} \to 0$ as $R \to 0^+$ and

$$a(R) = \Lambda_{\alpha_0+\xi\gamma_0,\beta_0+\xi}(R)\left(\frac{1}{\sqrt{\lambda(R)}} + \frac{R^{\frac{\gamma_0-4}{2}}}{\lambda(R)}\right)^{\frac{q_1-2\beta}{2}} \le \frac{\Lambda_0}{\lambda_0^{\xi}}\left(\frac{1}{\sqrt{\lambda_0}} + \frac{R_1^{\frac{\gamma_0-4}{2}}}{\lambda_0}\right)^{\frac{q_1-2\beta}{2}}.$$

• Case $N \le \gamma_0 < 2N - 4$. Again, $(\alpha_0, q_1) \in \mathcal{A}_{\beta_0,\gamma_0}$ implies that we can find $\xi \ge 0$ such that $\alpha = \alpha_0 + \xi\gamma_0$ and $\beta = \beta_0 + \xi$ satisfy (5.8). We get the thesis applying Lemma 5.4.

• Case $\gamma_0 = 2N - 4$. In this case, from $(\alpha_0, q_1) \in \mathcal{A}_{\beta_0,\gamma_0}$ we infer that there exists $\xi \ge 0$ such that $\alpha = \alpha_0 + \xi\gamma_0$ and $\beta = \beta_0 + \xi$ satisfy

$$\frac{1}{2} \le \beta \le 1, \quad q_1 > 2\beta \quad \text{and} \quad 0 < 2\alpha + 2N - (\gamma_0 + 4)\beta.$$

As in the previous cases, the thesis follows from Lemma 5.4.

• Case $\gamma_0 > 2N-4$. In this case, the hypothesis $(\alpha_0, q_1) \in \mathcal{A}_{\beta_0,\gamma_0}$ implies that we can find $\xi \geq 0$ such that $\alpha = \alpha_0 + \xi\gamma_0$ and $\beta = \beta_0 + \xi$ satisfy

$$\frac{1}{2} \leq \beta \leq 1 \quad \text{and} \quad q_1 > \max\left\{2\beta, 2\frac{2\alpha + 2N - (\gamma_0+4)\beta}{2N-4-\gamma_0}\right\}.$$

Again, the thesis follows from Lemma 5.4. □

6 Application to the bilaplacian equation

In this section we state our existence results for Eq. (1.1), which are Theorems 6.2 and 6.3 below (see also Remark 6.5). We let $N \geq 5$ and assume that V, K and Q satisfy $(\mathbf{V})$, $(\mathbf{K})$ with $s > \frac{2N}{N+4}$ (cf. Lemma 7.1 below) and the following hypothesis:

$(\mathbf{Q})$ $Q : \mathbb{R}_+ \to [0, +\infty)$ is a measurable function such that the linear funtional $h \mapsto \int_{\mathbb{R}^N} Q(|x|)\, h\, dx$ is continuous on H^2_V.

We also assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the following condition, where q_1, q_2 will be specified later:

(f_{q_1,q_2}) $\exists M > 0$ such that $|f(t)| \leq M \min\left\{t^{q_1-1}, t^{q_2-1}\right\}$ for all $t \geq 0$.

Remark 6.1. 1. Assumption $(\mathbf{Q})$ is quite abstract, but it is easy to find explicit conditions on Q ensuring it. For example, by Rellich inequality, if $Q \in L^2(\mathbb{R}_+, r^{N+3}dr)$ then one has

$$\left|\int_{\mathbb{R}^N} Q(|x|)\, h\, dx\right| \leq \left(\int_{\mathbb{R}^N} Q(|x|)^2 |x|^4\, dx\right)^{1/2} \left(\int_{\mathbb{R}^N} \frac{|h|^2}{|x|^4} dx\right)^{1/2} \leq (\text{const.})\, \|h\|, \quad \forall h \in H^2_V.$$

In a similar way, $(\mathbf{Q})$ holds true if $Q \in L^{2N/(N+4)}(\mathbb{R}_+, r^{N-1}dr)$ (by Sobolev inequality) or $V^{-1/2}Q \in L^2(\mathbb{R}_+, r^{N-1}dr)$ (by definition of H^2_V). Other similar conditions ensuring the same result can be obtained by the interpolation Hardy-Sobolev inequalities of [15, 16] (see also [11]).

2. Assumption (f_{q_1,q_2}) implies $|f(t)| \leq M\, t^{q-1}$ for all $t \geq 0$ and $q \in [q_1, q_2]$, whence it is more stringent than a single-power growth assumption if $q_1 \neq q_2$. On the other hand we will never require $q_1 \neq q_2$, so that our results will also concern single-power nonlinearities as long as we can take $q_1 = q_2$ in (f_{q_1,q_2}).

We are interested in finding *radial weak solutions* of Eq. (1.1), i.e., functions $u \in H^2_{V,r}$ such that

$$\int_{\mathbb{R}^N} \triangle u \cdot \triangle h\, dx + \int_{\mathbb{R}^N} V(|x|)\, uh\, dx = \int_{\mathbb{R}^N} K(|x|)\, f(u)\, h\, dx + \int_{\mathbb{R}^N} Q(|x|)\, h\, dx \quad \text{for all } h \in H^2_V. \tag{6.1}$$

Our existence results are the following.

Theorem 6.2. *Assume $Q = 0$ and assume that there exist $q_1, q_2 > 2$ such that (f_{q_1,q_2}) and $\left(\mathcal{R}''_{q_1,q_2}\right)$ hold. Assume furthermore that f satisfies:*

(f_1) $\exists \theta > 2$ *such that* $0 \leq \theta F(t) \leq f(t)\, t$ *for all* $t \geq 0$;

(f_2) $\exists t_0 > 0$ *such that* $F(t_0) > 0$.

If $K(|\cdot|) \in L^1(\mathbb{R}^N)$*, we can replace assumptions* (f_1)-(f_2) *with the following one:*

(f_3) $\exists \theta > 2$ *and* $\exists t_0 > 0$ *such that* $0 < \theta F(t) \leq f(t)\, t$ *for all* $t \geq t_0$.

Then Eq. (1.1) *has a nonzero nonnegative radial weak solution.*

Theorem 6.3. *Assume that there exist* $q_1, q_2 \in (1,2)$ *such that* (f_{q_1,q_2}) *and* $\left(\mathcal{R}''_{q_1,q_2}\right)$ *hold. Assume furthermore that either* $Q \neq 0$ *(meaning that* Q *does not vanish almost everywhere), or* $Q = 0$ *and* f *satisfies the following condition:*

(f_4) $\exists \theta < 2$ *and* $\exists t_0, m > 0$ *such that* $F(t) \geq m t^\theta$ *for all* $0 \leq t \leq t_0$.

If $Q \neq 0$*, we also allow the case* $\max\{q_1, q_2\} = 2 > \min\{q_1, q_2\} > 1$*. Then Eq.* (1.1) *has a nonzero nonnegative radial weak solution.*

The above existence results will be proved in Section 7 and can be generalized and complemented by other results in different and quite standard ways (see Remark 6.5 below). They rely on assumption $\left(\mathcal{R}''_{q_1,q_2}\right)$, which is rather abstract but, as already discussed in Section 2, it can be granted in concrete cases through Theorems 2.5-2.9, which ensure $\left(\mathcal{R}''_{q_1,q_2}\right)$ for suitable ranges of exponents q_1 and q_2 by explicit conditions on the potentials. Some basic examples of nonlinearities satisfying (f_{q_1,q_2}) and the other assumptions of our results can be found in [4, Example 4.11].

Remark 6.4.

1. In Theorem 6.2, the information $K(|\cdot|) \in L^1(\mathbb{R}^N)$ actually allows weaker hypotheses on the nonlinearity, as assumptions (f_1) and (f_2) imply (f_3).
2. In Theorem 6.3, the case $\max\{q_1, q_2\} = 2 > \min\{q_1, q_2\} > 1$ cannot be considered if (f_4) holds, as (f_4) and (f_{q_1,q_2}) imply $\max\{q_1, q_2\} \leq \theta < 2$.

Remark 6.5.

1. Theorems 6.2 and 6.3 can be easily adapted to the case of Eq. (1.1) with a general right hand term $g(|x|, u)$ (see [3, Section 3] and [4, Section 4]). Moreover, they can be complemented with multiplicity results by standard variational techniques (see again [3, Section 3] and [4, Section 4]).
2. Theorems 6.2 and 6.3 can be used to derive existence results for Eq. (1.1) with Dirichlet boundary conditions in bounded balls or exterior radial domains, by suitably modifying the potentials V and K in order to reduce the Dirichlet problem to the problem in $\mathbb{R}^N$. In this cases, a single-power growth condition on the nonlinearity is sufficient and, respectively, only assumptions on the potentials near the origin or at infinity are needed. We leave the details to the interested reader, which we refer to [3, Section 5] for similar results and related arguments.
3. Using some ideas of [7], we think that our compactness and existence results can be easily extended to the case of inhomogeneous bilaplacian equations of the form

$$\Delta^2 u + V(|x|) u^{p-1} = K(|x|) f(u) + Q(|x|) \quad \text{in } \mathbb{R}^N$$

with $1 < p < N$, $p \neq 2$.

7 Proof of Theorems 6.2 and 6.3

In this section we apply the compactness results of Section 2 to prove the existence results of Section 6. Let $N \geq 5$ and assume that V, K and Q satisfy $(\mathbf{V})$, $(\mathbf{K})$ and $(\mathbf{Q})$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and set $F(t) := \int_0^t f(s)\, ds$.

The weak solutions of Eq. (1.1) are (at least formally) the critical points of the functional

$$I(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} K(|x|)\, F(u)\, dx - \int_{\mathbb{R}^N} Q(|x|)\, u\, dx. \tag{7.1}$$

As a matter of fact, by the continuous embedding of Theorem 2.4 and the results of [1] about Nemytskiĭ operators on the sum of Lebesgue spaces, (7.1) defines a C^1 functional on $H^2_{V,r}$ provided that (f_{q_1,q_2}) and $\left(\mathcal{S}'_{q_1,q_2}\right)$ hold for some $q_1, q_2 > 1$. In this case, the Fréchet derivative of I at any $u \in H^2_{V,r}$ is given by

$$I'(u)\, h = \int_{\mathbb{R}^N} (\triangle u \cdot \triangle h + V(|x|)\, uh)\, dx - \int_{\mathbb{R}^N} (K(|x|)\, f(u) + Q(|x|))\, h\, dx\,, \quad \forall h \in H^2_{V,r}\,, \tag{7.2}$$

but I does not need to be well defined on the whole space H^2_V, and therefore the classical Palais' Principle of Symmetric Criticality [10] does not actually ensure that the critical points of $I : H^2_{V,r} \to \mathbb{R}$ are weak solutions of Eq. (1.1). This is the aim of our first lemma, which relies on the following stronger version of condition $\left(\mathcal{S}'_{q_1,q_2}\right)$:

$\left(\mathcal{R}'_{q_1,q_2}\right)$ $\exists R_1, R_2 > 0$ such that $\mathcal{R}_0(q_1, R_1) < \infty$ and $\mathcal{R}_\infty(q_2, R_2) < \infty$.

Lemma 7.1. *Assume $s > \frac{2N}{N+4}$ in condition $(\mathbf{K})$ and assume that there exist $q_1, q_2 > 1$ such that (f_{q_1,q_2}) and $\left(\mathcal{R}'_{q_1,q_2}\right)$ hold. Then every critical point of $I : H^2_{V,r} \to \mathbb{R}$ is a weak solution to Eq.* (1.1).

Proof. Let $u \in H^2_{V,r}$ and assume $R_1 < R_2$ in $\left(\mathcal{R}'_{q_1,q_2}\right)$, which is not restrictive by the monotonicity of $\mathcal{R}_0$ and $\mathcal{R}_\infty$. By Lemma 3.1, there exists a constant $C > 0$ (also dependent on u) such that for all $h \in H^2_V$ we have

$$\int_{B_{R_2} \setminus B_{R_1}} K(|x|)\, |u|^{q_1 - 1} |h|\, dx \leq C \|h\|$$

and therefore, by (f_{q_1,q_2}), we get

$$\begin{aligned}
\int_{\mathbb{R}^N} K(|x|)\, |f(u)|\, |h|\, dx &\leq M \int_{\mathbb{R}^N} K(|x|) \min\{|u|^{q_1-1}, |u|^{q_2-1}\}\, |h|\, dx \\
&\leq M \left(\int_{B_{R_1}} K(|x|)\, |u|^{q_1-1} |h|\, dx + \int_{B^c_{R_2}} K(|x|)\, |u|^{q_2-1} |h|\, dx + C\|h\| \right) \\
&\leq M \left(\|u\|^{q_1-1} \mathcal{R}_0(q_1, R_1) + \|u\|^{q_2-1} \mathcal{R}_\infty(q_2, R_2) + C \right) \|h\|\,.
\end{aligned}$$

Together with assumption $(\mathbf{Q})$, this gives that the linear operator

$$T(u)\, h := \int_{\mathbb{R}^N} (\triangle u \cdot \triangle h + V(|x|)\, uh)\, dx - \int_{\mathbb{R}^N} (K(|x|)\, f(u) + Q(|x|))\, h\, dx$$

is well defined and continuous on H^2_V. Hence, by Riesz representation theorem, there exists a unique $\tilde{u} \in H^2_V$ such that $T(u)\, h = (\tilde{u}, h)$ for all $h \in H^2_V$, where $(\cdot, \cdot)$ is the scalar product defined in (2.4). By means of obvious changes of variables it is easy to infer that $\tilde{u} \in H^2_{V,r}$, so that $T(u) = 0$ on $H^2_{V,r}$ implies $\tilde{u} = 0$ and hence (6.1). □

Hereafter, we will assume that the hypotheses of Theorems 6.2 and 6.3 also include the following assumptions respectively: $f(t) = 0$ for $t < 0$ in Theorem 6.2, and f is odd in Theorem 6.3. This can be done without restriction, since Theorems 6.2 and 6.3 concern nonnegative solutions and all their assumptions still hold true if we replace $f(t)$ respectively with $f(t)\chi_{\mathbb{R}_+}(t)$ and $f(t)\chi_{\mathbb{R}_+}(t) - f(|t|)\chi_{\mathbb{R}_-}(t)$ ($\chi_{\mathbb{R}_\pm}$ denotes the characteristic function of $\mathbb{R}_\pm$).

With such additional assumptions, (f_{q_1,q_2}) implies that there exists $\tilde{M} > 0$ such that

$$|F(t)| \leq \tilde{M} \min\{|t|^{q_1}, |t|^{q_2}\} \quad \text{for all } t \in \mathbb{R}. \tag{7.3}$$

Lemma 7.2. *Let L_0 be the norm of the linear continuous functional $h \in H_V^2 \mapsto \int_{\mathbb{R}^N} Q(|x|)\, h\, dx$. If (f_{q_1,q_2}) and $\left(\mathcal{S}'_{q_1,q_2}\right)$ hold for some $q_1, q_2 > 1$, then there exist two constants $c_1, c_2 > 0$ such that*

$$I(u) \geq \frac{1}{2}\|u\|^2 - c_1\|u\|^{q_1} - c_2\|u\|^{q_2} - L_0\|u\| \qquad \text{for all } u \in H_{V,r}^2. \tag{7.4}$$

If $\left(\mathcal{S}''_{q_1,q_2}\right)$ also holds, then $\forall \varepsilon > 0$ there exist two constants $c_1(\varepsilon), c_2(\varepsilon) > 0$ such that (7.4) holds both with $c_1 = \varepsilon$, $c_2 = c_2(\varepsilon)$ and with $c_1 = c_1(\varepsilon)$, $c_2 = \varepsilon$.

Proof. Let $i \in \{1, 2\}$ and assume $R_1 < R_2$ in $\left(\mathcal{S}'_{q_1,q_2}\right)$, which is not restrictive by the monotonicity of $\mathcal{S}_0$ and $\mathcal{S}_\infty$. By Lemma 3.1 and the continuous embedding $H_{V,r}^2 \hookrightarrow L_{\mathrm{loc}}^{q_i}(\mathbb{R}^N)$, there exists a constant $c_{R_1,R_2}^{(i)} > 0$ such that for all $u \in H_{V,r}^2$ we have

$$\int_{B_{R_2} \setminus B_{R_1}} K(|x|)|u|^{q_i}\, dx \leq c_{R_1,R_2}^{(i)}\|u\|^{q_i}$$

and therefore, by (7.3),

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} (K(|x|)F(u) + Q(|x|)u)\, dx\right| \\
&\leq \int_{\mathbb{R}^N} K(|x|)|F(u)|\, dx + \left|\int_{\mathbb{R}^N} Q(|x|)u\, dx\right| \leq \tilde{M}\int_{\mathbb{R}^N} K(|x|)\min\{|u|^{q_1}, |u|^{q_2}\}\, dx + L_0\|u\| \\
&\leq \tilde{M}\left(\int_{B_{R_1}} K(|x|)|u|^{q_1}\, dx + \int_{B_{R_2}^c} K(|x|)|u|^{q_2}\, dx + \int_{B_{R_2}\setminus B_{R_1}} K(|x|)|u|^{q_i}\, dx\right) + L_0\|u\| \\
&\leq \tilde{M}\left(\|u\|^{q_1}\mathcal{S}_0(q_1, R_1) + \|u\|^{q_2}\mathcal{S}_\infty(q_2, R_2) + c_{R_1,R_2}^{(i)}\|u\|^{q_i}\right) + L_0\|u\| \qquad (7.5) \\
&= c_1\|u\|^{q_1} + c_2\|u\|^{q_2} + L_0\|u\|,
\end{aligned}$$

with obvious definition of the constants c_1 and c_2, independent of u. This proves (7.4). If $\left(\mathcal{S}''_{q_1,q_2}\right)$ also holds, then $\forall \varepsilon > 0$ we can fix $R_{1,\varepsilon} < R_{2,\varepsilon}$ such that $\tilde{M}\mathcal{S}_0(q_1, R_{1,\varepsilon}) < \varepsilon$ and $\tilde{M}\mathcal{S}_\infty(q_2, R_{2,\varepsilon}) < \varepsilon$, so that inequality (7.5) becomes

$$\left|\int_{\mathbb{R}^N} (K(|x|)F(u) + Q(|x|)u)\, dx\right| \leq \varepsilon\|u\|^{q_1} + \varepsilon\|u\|^{q_2} + c_{R_{1,\varepsilon},R_{2,\varepsilon}}^{(i)}\|u\|^{q_i} + L_0\|u\|.$$

The conclusion thus ensues by taking $i = 1$ and $c_1(\varepsilon) = \varepsilon + c_{R_{1,\varepsilon},R_{2,\varepsilon}}^{(1)}$, or $i = 2$ and $c_2(\varepsilon) = \varepsilon + c_{R_{1,\varepsilon},R_{2,\varepsilon}}^{(2)}$. □

Lemma 7.3. *Under the assumptions of Theorem 6.2, the functional $I : H_{V,r}^2 \to \mathbb{R}$ satisfies the Palais-Smale condition.*

Proof. By (f_1) or (f_3) together with the additional assumption $f(t) = 0$ for $t < 0$, we have that either (f_1) holds for all $t \in \mathbb{R}$, or $K(|\cdot|) \in L^1(\mathbb{R}^N)$ and f satisfies

$$\theta F(t) \leq f(t)\, t \quad \text{for all } |t| \geq t_0. \tag{7.6}$$

Let $\{u_n\}$ be a sequence in $H^2_{V,r}$ such that $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0$ in the dual space of $H^2_{V,r}$. Then

$$\frac{1}{2}\|u_n\|^2 - \int_{\mathbb{R}^N} K(|x|)\, F(u_n)\, dx = I(u_n) + \int_{\mathbb{R}^N} Q(|x|)\, u_n dx = O(1) + O(1)\|u_n\|$$

and

$$\|u_n\|^2 - \int_{\mathbb{R}^N} K(|x|)\, f(u_n)\, u_n dx = I'(u_n)\, u_n + \int_{\mathbb{R}^N} Q(|x|)\, u_n dx = o(1)\|u_n\| + O(1)\|u_n\|.$$

If f satisfies (f_1), we get

$$\begin{aligned} \frac{1}{2}\|u_n\|^2 + O(1) + O(1)\|u_n\| &= \int_{\mathbb{R}^N} K(|x|)\, F(u_n)\, dx \\ &\leq \frac{1}{\theta}\int_{\mathbb{R}^N} K(|x|)\, f(u_n)\, u_n dx = \frac{1}{\theta}\|u_n\|^2 + o(1)\|u_n\| + O(1)\|u_n\|, \end{aligned}$$

which implies that $\{\|u_n\|\}$ is bounded since $\theta > 2$. If $K(|\cdot|) \in L^1(\mathbb{R}^N)$ and f satisfies (7.6), we have

$$\begin{aligned} \int_{\{|u_n| \geq t_0\}} K(|x|)\, f(u_n)\, u_n dx &\leq \int_{\mathbb{R}^N} K(|x|)\, f(u_n)\, u_n dx + \int_{\{|u_n| < t_0\}} K(|x|)\, |f(u_n)\, u_n|\, dx \\ &\leq \int_{\mathbb{R}^N} K(|x|)\, f(u_n)\, u_n dx + M\int_{\{|u_n| < t_0\}} K(|x|) \min\{|u_n|^{q_1}, |u_n|^{q_2}\}\, dx \\ &\leq \int_{\mathbb{R}^N} K(|x|)\, f(u_n)\, u_n dx + M \min\{t_0^{q_1}, t_0^{q_2}\}\, \|K\|_{L^1(\mathbb{R}^N)}, \end{aligned}$$

and then, using (7.3), we get

$$\begin{aligned} &\frac{1}{2}\|u_n\|^2 + O(1) + O(1)\|u_n\| \\ &= \int_{\{|u_n| < t_0\}} K(|x|)\, F(u_n)\, dx + \int_{\{|u_n| \geq t_0\}} K(|x|)\, F(u_n)\, dx \\ &\leq \tilde{M}\int_{\{|u_n| < t_0\}} K(|x|) \min\{|u_n|^{q_1}, |u_n|^{q_2}\}\, dx + \frac{1}{\theta}\int_{\{|u_n| \geq t_0\}} K(|x|)\, f(u_n)\, u_n dx \\ &\leq \tilde{M} \min\{t_0^{q_1}, t_0^{q_2}\}\, \|K\|_{L^1(\mathbb{R}^N)} + \frac{1}{\theta}\int_{\mathbb{R}^N} K(|x|)\, f(u_n)\, u_n dx + \frac{M}{\theta}\min\{t_0^{q_1}, t_0^{q_2}\}\, \|K\|_{L^1(\mathbb{R}^N)} \\ &= \left(\tilde{M} + \frac{M}{\theta}\right) \min\{t_0^{q_1}, t_0^{q_2}\}\, \|K\|_{L^1(\mathbb{R}^N)} + \frac{1}{\theta}\|u_n\|^2 + o(1)\|u_n\| + O(1)\|u_n\|. \end{aligned}$$

This yields again that $\{\|u_n\|\}$ is bounded. Now, thanks to assumption $(\mathbf{Q})$ and since the embedding $H^2_{V,r} \hookrightarrow L^{q_1}_K + L^{q_2}_K$ is compact by Theorem 2.4 and the functional $u \mapsto \int_{\mathbb{R}^N} K(|x|)\, F(u)\, dx$ is C^1 on $L^{q_1}_K + L^{q_2}_K$ by [1, Proposition 3.8], it is a standard exercise to conclude that $\{u_n\}$ has a strongly convergent subsequence in $H^2_{V,r}$. $\square$

Proof of Theorem 6.2. We mean to apply the Mountain-Pass Theorem. To this end, from (7.4) of Lemma 7.2, where $L_0 = 0$ and $q_1, q_2 > 2$, we readily infer that $\exists \rho > 0$ such that

$$\inf_{u \in H^2_{V,r},\, \|u\| = \rho} I(u) > 0 = I(0). \tag{7.7}$$

Now we check that $\exists \bar{u} \in W_r$ such that $\|\bar{u}\| > \rho$ and $I(\bar{u}) < 0$. To this end, from (f_3) (which holds in any case, as (f_1) and (f_2) imply (f_3)), we deduce that $F(t) \geq t_0^{-\theta} F(t_0)\, t^\theta$ for all $t \geq t_0$. Then, taking into account assumption $(\mathbf{V})$, we fix a nonnegative function $u_0 \in C_c^\infty(\mathbb{R}^N) \cap H^2_{V,r}$ such that the set $\{x \in \mathbb{R}^N : u_0(x) \geq t_0\}$ has positive Lebesgue measure. We now distinguish the case of assumptions (f_1) and (f_2) from the case with $K(|\cdot|) \in L^1(\mathbb{R}^N)$. In the first one, for every $\lambda > 1$ we have

$$\begin{aligned}\int_{\mathbb{R}^N} K(|x|)\, F(\lambda u_0)\, dx &\geq \int_{\{\lambda u_0 \geq t_0\}} K(|x|)\, F(\lambda u_0)\, dx \geq \lambda^\theta F(t_0) \int_{\{\lambda u_0 \geq t_0\}} t_0^{-\theta} u_0^\theta dx \\ &\geq \lambda^\theta F(t_0) \int_{\{u_0 \geq t_0\}} t_0^{-\theta} u_0^\theta dx \geq \lambda^\theta F(t_0) \int_{\{u_0 \geq t_0\}} dx > 0\end{aligned}$$

and therefore, since $\theta > 2$, we get

$$\lim_{\lambda \to +\infty} I(\lambda u_0) \leq \lim_{\lambda \to +\infty} \left(\frac{\lambda^2}{2} \|u_0\|^2 - \lambda^\theta F(t_0) \int_{\{u_0 \geq t_0\}} dx - \lambda \int_{\mathbb{R}^N} Q(|x|)\, u_0 dx \right) = -\infty.$$

If $K(|\cdot|) \in L^1(\mathbb{R}^N)$, we observe that (7.3) implies $F(t) \geq -\tilde{M} \min\{t_0^{q_1}, t_0^{q_2}\}$ for all $0 \leq t \leq t_0$, so that, arguing as above about the integral over $\{\lambda u_0 \geq t_0\}$, for every $\lambda > 1$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} K(|x|)\, F(\lambda u_0)\, dx &= \int_{\{\lambda u_0 < t_0\}} K(|x|)\, F(\lambda u_0)\, dx + \int_{\{\lambda u_0 \geq t_0\}} K(|x|)\, F(\lambda u_0)\, dx \\ &\geq -\tilde{M} \min\{t_0^{q_1}, t_0^{q_2}\} \int_{\{\lambda u_0 < t_0\}} K(|x|)\, dx + \lambda^\theta F(t_0) \int_{\{u_0 \geq t_0\}} dx,\end{aligned}$$

which implies

$$I(\lambda u_0) \leq \frac{\lambda^2}{2} \|u_0\|^2 + \tilde{M} \min\{t_0^{q_1}, t_0^{q_2}\} \|K\|_{L^1(\mathbb{R}^N)} - \lambda^\theta F(t_0) \int_{\{u_0 \geq t_0\}} dx - \lambda \int_{\mathbb{R}^N} Q(|x|)\, u_0 dx \to -\infty$$

as $\lambda \to +\infty$. So, in any case, we can take $\bar{u} = \lambda u_0$ with λ sufficiently large. As a conclusion, taking into account Lemma 7.3, the Mountain-Pass Theorem provides the existence of a nonzero critical point $u \in H^2_{V,r}$ for I, which is a weak solutions to Eq. (1.1) by Lemma 7.1. Since the additional assumption $f(t) = 0$ for $t < 0$ implies $I'(u)\, u_- = -\|u_-\|^2$ (where $u_- \in H^2_{V,r}$ is the negative part of u), one has $u_- = 0$ and thus u is nonnegative.

Lemma 7.4. *Under the assumptions of Theorem 6.3, the functional $I : H^2_{V,r} \to \mathbb{R}$ is bounded from below and coercive. In particular, if $Q = 0$ and f satisfies (f_4), then*

$$\inf_{v \in H^2_{V,r}} I(v) < 0. \tag{7.8}$$

Proof. I is bounded below and coercive on $H^2_{V,r}$ thanks to Lemma 7.2. Indeed, the result readily follows from (7.4) if $q_1, q_2 \in (1,2)$, while, if $\max\{q_1,q_2\} = 2 > \min\{q_1,q_2\} > 1$, we fix $\varepsilon < 1/2$ and use the second part of the lemma in order to get

$$I(u) \geq \left(\frac{1}{2} - \varepsilon\right) \|u\|^2 - c(\varepsilon)\|u\|^{\min\{q_1,q_2\}} - L_0\|u\| \qquad \text{for all } u \in H^2_{V,r},$$

which yields again the conclusion. In order to prove (7.8) if $Q = 0$ and (f_4) holds, we use assumption $(\mathbf{V})$ to fix a nonzero function $u_0 \in C_c^\infty(\mathbb{R}^N) \cap H^2_{V,r}$ such that $0 \leq u_0 \leq t_0$. Then, by (f_4), for every $0 < \lambda < 1$ we get that $\lambda u_0 \in H^2_{V,r}$ satisfies

$$I(\lambda u_0) = \frac{1}{2}\|\lambda u_0\|^2 - \int_{\mathbb{R}^N} K(|x|) F(\lambda u_0)\, dx \leq \frac{\lambda^2}{2}\|u_0\|^2 - \lambda^\theta m \int_{\mathbb{R}^N} K(|x|) u_0^\theta dx.$$

Since $\theta < 2$, this implies $I(\lambda u_0) < 0$ for λ sufficiently small and therefore (7.8) ensues. □

Proof of Theorem 6.3. Recall Lemma 7.4 and let $\{v_n\}$ be any minimizing sequence for $\mu := \inf_{v \in H^2_{V,r}} I(v) \in \mathbb{R}$. As F is even and Q is nonnegative, we have

$$\begin{aligned} I(|v_n|) &= \frac{1}{2}\|v_n\|^2 - \int_{\mathbb{R}^N} K(|x|) F(|v_n|)\, dx - \int_{\mathbb{R}^N} Q(|x|)|v_n|\, dx \\ &= \frac{1}{2}\|v_n\|^2 - \int_{\mathbb{R}^N} K(|x|) F(v_n)\, dx - \int_{\{v_n \geq 0\}} Q(|x|) v_n dx + \int_{\{v_n < 0\}} Q(|x|) v_n dx \\ &= I(v_n) + 2\int_{\{v_n<0\}} Q(|x|) v_n dx \leq I(v_n), \end{aligned} \tag{7.9}$$

so that $|v_n| \in H^2_{V,r}$ is still a minimizing sequence. Hence we can assume $v_n \geq 0$. Since $\{v_n\}$ is bounded in $H^2_{V,r}$ by Lemma 7.4 and the embedding $H^2_{V,r} \hookrightarrow L^{q_1}_K + L^{q_2}_K$ is compact by assumption $\left(\mathcal{S}''_{q_1,q_2}\right)$ and Theorem 2.4, we can assume that there exists $u \in H^2_{V,r}$ such that (up to a subsequence) $v_n \rightharpoonup u$ in $H^2_{V,r}$ and $v_n \to u$ in $L^{q_1}_K + L^{q_2}_K$ and almost everywhere. Hence u is nonnegative and, thanks to $(\mathbf{Q})$ and the continuity of the functional $v \mapsto \int_{\mathbb{R}^N} K(|x|) F(v)\, dx$ on $L^{q_1}_K + L^{q_2}_K$ (which follows from (f_{q_1,q_2}) and [1, Proposition 3.8]), we have

$$\int_{\mathbb{R}^N} K(|x|) F(v_n)\, dx + \int_{\mathbb{R}^N} Q(|x|) v_n dx \to \int_{\mathbb{R}^N} K(|x|) F(u)\, dx + \int_{\mathbb{R}^N} Q(|x|)\, u\, dx.$$

By the weak lower semi-continuity of the norm, this implies

$$I(u) \leq \lim_{n\to\infty}\left(\frac{1}{2}\|v_n\|^2 - \int_{\mathbb{R}^N} K(|x|) F(v_n)\, dx + \int_{\mathbb{R}^N} Q(|x|) v_n dx\right) = \mu$$

and thus $I(u) = \mu$. So u is a critical point for I and thus a weak solutions to Eq. (1.1) by Lemma 7.1. It remains to show that $u \neq 0$. This is obvious if $Q = 0$ and f satisfies (f_4), since $\mu < 0$ by Lemma 7.4. If $Q \neq 0$, assume by contradiction that $u = 0$. From (7.2) we get $\int_{\mathbb{R}^N} Q(|x|)\, h\, dx = 0$ for all $h \in H^2_{V,r}$ and therefore $Q = 0$, which is a contradiction.

References

[1] M. Badiale, L. Pisani, S. Rolando, *Sum of weighted Lebesgue spaces and nonlinear elliptic equations*, NoDEA Nonlinear Differ. Equ. Appl., **18** (2011), 369405

[2] M. Badiale, M. Guida, S. Rolando, *Compactness and existence results in weighted Sobolev spaces of radial functions. Part I: compactness*, Calc. Var. **54** (2015), 1061-1090

[3] M. Badiale, M. Guida, S. Rolando, *Compactness and existence results in weighted Sobolev spaces of radial functions. Part II: Existence*, NoDEA Nonlinear Differ. Equ. Appl., **23** (2016), 61-95

[4] M. Badiale, M. Guida, S. Rolando, *Compactness and existence results for the p-Laplace equation*, J. Math. Anal. Appl.**451** (2017), 345-370

[5] W.D. Bastos, O.H. Miyagaki, R.S. Vieira, *Solution to biharmonic equation with vanishing potential*, Illinois J. Math. **57** (2013), 839854.

[6] P.C. Carrio, R. Demarque, O.H. Miyagaki, *Nonlinear Biharmonic Problems With Singular Potentials*, doi: 10.3934/cpaa.2014.13.2141, Communications On Pure And Applied Analysis, **13** (2014), 21412154

[7] R. Demarque, O.H. Miyagaki, *Radial solutions of inhomogeneous fourth order elliptic equations and weighted Sobolev embeddings*, Adv. Nonlinear Anal. **4** (2015), 135151

[8] F. Gazzola, H.Ch. Grunau, G. Sweers, *Polyharmonic Boundary Value Problems*, Springer, Berlin-Heidelberg, 2010

[9] S. Liang, J. Zhang, Y. Luo, *Existence of solutions for a class of biharmonic equations with critical nonlinearity in* $\mathbb{R}^N$ Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Math. RACSAM **110** (2016), no. 2, 681693

[10] R.S. Palais: The principle of symmetric criticality. Commun. Math. Phys. **69** (1979), 19-30 .

[11] T. Passalacqua, B. Ruf: Hardy-Sobolev inequalities for the biharmonic operator with remainder terms. J. Fixed Point Theory Appl. **15** (2014), 405-431

[12] E.S. Noussair, C.A. Swanson, J. Yang, *Transcritical Biharmonic Equations in* $\mathbb{R}^N$, Funkcialaj Ekvacioj, **35** (1992), 533543

[13] J. Sun, J. Chu, T.F. Wu, *Existence and multiplicity of nontrivial solutions for some biharmonic equations with p-Laplacian*, J. Differential Equations **262** (2017), 945977

[14] J. Sun, T.F. Wu, *Existence of nontrivial solutions for a biharmonic equation with p-Laplacian and singular sign-changing potential*, Appl. Math. Lett. **66** (2017), 6167

[15] Y.X. Yao, Y.T. Shen, Z.H. Chen, *Biharmonic equation and an improved Hardy inequality*, Acta Math. Appl. Sin. Engl. Ser. **20** (2004), 433440.

[16] Y. Yao, R. Wang, Y. Shen, *Nontrivial solution for a class of semilinear biharmonic equation involving critical exponents*, Acta Math. Sci. Ser. B Engl. Ed. **27** (2007), 509514.