Quantum Holonomies and the Heisenberg Group

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Abstract

Quantum holonomies of closed paths on the torus $\mathbb{T}^2$ are interpreted as elements of the Heisenberg group H_1. Group composition in H_1 corresponds to path concatenation and the group commutator is a deformation of the relator of the fundamental group π_1 of $\mathbb{T}^2$, making explicit the signed area phases between quantum holonomies of homotopic paths. Inner automorphisms of H_1 adjust these signed areas, and the discrete symplectic transformations of H_1 generate the modular group of $\mathbb{T}^2$.

1 Introduction

Motivated by studies in 2+1 dimensional gravity [1, 2], in previous articles [3, 4, 5, 6] we used piecewise linear (PL) paths in $\mathbb{R}^2$ to represent closed paths on the torus $\mathbb{T}^2$. Their corresponding quantum holonomies were represented by quantum $SL(2,\mathbb{R})$ matrices which in general do not commute. The traces of holonomies corresponding to paths which intersect on $\mathbb{T}^2$ satisfy a linear algebra [3], a generalization of the Goldman bracket [7], where on the right hand side (RHS) there are traces of holonomies of paths which are rerouted in two different ways at the intersections.

Rerouted paths on $\mathbb{T}^2$ arising from intersections of closed paths naturally correspond to PL paths in $\mathbb{R}^2$. The Heisenberg group, a non–commutative Lie group, has wide applications in both mathematics and physics, and can be interpreted geometrically in terms of PL paths on $\mathbb{R}^2$ [8].

Our purpose here is to present an interesting and novel relationship between quantum holonomies that represent PL paths and elements of the 3–parameter Heisenberg group H_1, where $SL(2,\mathbb{R})$ matrices are replaced by elements of H_1. A new feature is that area phases between homotopic paths (which arise, indirectly, from a non–zero quantum curvature [4]) are now explicit. Group composition in H_1 and the group commutator are interpreted geometrically. Inner

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automorphisms of H_1 i.e. conjugation by a fixed element of H_1, adjust the signed area of a PL path (the geometric area between a PL path and a straight path between the same points) and discrete symplectic transformations generate the modular group of $\mathbb{T}^2$.

The plan of the paper is as follows: in Section 2 we review PL paths, their associated quantum holonomies, and the concept of signed area. In Section 3 we present the Heisenberg group H_1. In Section 4 we connect the two.

2 PL paths and quantum holonomies

2.1 Piecewise linear paths

Closed paths (loops) on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ are identified with PL paths on its covering space $\mathbb{R}^2$, where a PL path is between integer points $(m, n) \in \mathbb{Z}^2$. All integer points are identified, and correspond to the same point on $\mathbb{T}^2$. A path in $\mathbb{R}^2$ representing a closed path on $\mathbb{T}^2$ can therefore be replaced by any parallel path starting at a different integer point. Two paths on $\mathbb{T}^2$ are homotopic if and only if the corresponding paths in $\mathbb{R}^2$ have the same integer starting point and the same integer endpoint.

A natural subclass of paths in $\mathbb{R}^2$ are those straight paths denoted $p = (m, n)$ that start at the origin $(0, 0)$ and end at an integer point (m, n). They generalize the cycles γ_1, γ_2 (corresponding to the paths $(1, 0)$ and $(0, 1)$ respectively) which satisfy the relator of $\pi_1(\mathbb{T}^2)$

$$\gamma_1 \cdot \gamma_2 \cdot \gamma_1^{-1} \cdot \gamma_2^{-1} = \mathbb{I}. \tag{1}$$

2.2 Quantum holonomies and signed area

In previous articles [3, 6] a quantum $SL(2, \mathbb{R})$ matrix was assigned to any straight path (m, n)

$$\hat{U}_{(m,n)} = \exp \int_{(m,n)} \hat{A}. \tag{2}$$

This extends straightforwardly to any PL path between integer points: assign a quantum matrix to each linear segment of the path, as in (2) and multiply the matrices in the same order as the segments along the path. The general relation is

$$p \to \hat{U}_p = P \exp \int_p \hat{A}. \tag{3}$$

where P denotes path-ordering. In (2), (3), $\hat{A}$ is a constant connection with non-zero curvature [4], parametrized by two non-commuting parameters $\hat{r}_1$, $\hat{r}_2$. For example, for the cycles γ_i which satisfy (1), equation (3) is

$$\hat{U}_i = \exp \int_{\gamma_i} \hat{A} = \begin{pmatrix} e^{\hat{r}_i} & 0 \\ 0 & e^{-\hat{r}_i} \end{pmatrix} \quad i = 1, 2. \tag{4}$$

where $[\hat{r}_1, \hat{r}_2] = -i\hbar\frac{\sqrt{-\Lambda}}{4}$, and Λ is a (negative) cosmological constant [1, 2].

In [3] it was shown that the holonomies (2) satisfy the q–relations

$$\hat{U}_{(m,n)}\hat{U}_{(s,t)} = q^{mt-ns}\,\hat{U}_{(s,t)}\hat{U}_{(m,n)}, \quad \hat{U}_{(m,n)}\hat{U}_{(s,t)}\hat{U}_{(m,n)}^{-1}\hat{U}_{(s,t)}^{-1} = q^{mt-ns}\,\mathbb{I}_2, \quad (5)$$

with

$$q = \exp(-\frac{i\hbar\sqrt{-\Lambda}}{4}). \tag{6}$$

For $\hat{U}_1 = \hat{U}_{\gamma_1} = \hat{U}_{(1,0)}$, $\hat{U}_2 = \hat{U}_{\gamma_2} = \hat{U}_{(0,1)}$, the q–relations (5) are

$$\hat{U}_1\hat{U}_2 = q\,\hat{U}_2\hat{U}_1, \quad \hat{U}_1\hat{U}_2\hat{U}_1^{-1}\hat{U}_2^{-1} = q\,\mathbb{I}_2 \tag{7}$$

i.e. a deformation of the holonomy relator that follows from (1).

From (4) and the Baker-Campbell-Hausdorff identity

$$e^X e^Y = e^{X+Y} e^{\frac{[X,Y]}{2}} \tag{8}$$

(valid when $[X, Y]$ is a c–number) we also have a triangle equation

$$\hat{U}_{(m,n)}\hat{U}_{(s,t)} = q^{\frac{1}{2}(mt-ns)}\hat{U}_{(m+s,n+t)}. \tag{9}$$

Consider two homotopic paths on $\mathbb{T}^2$ represented by two PL paths p_1, p_2 on $\mathbb{R}^2$. It was shown in [3] that the following relationship holds for the respective quantum matrices:

$$\hat{U}_{p_1} = q^{S(p_1,p_2)}\hat{U}_{p_2}, \tag{10}$$

where $S(p_1, p_2)$ denotes the signed area enclosed between the paths p_1 and p_2, and q is the quantum phase (6). The signed area between two PL paths is defined as follows: for any finite region R enclosed by p_1 and p_2, if the boundary of R consists of oriented segments of p_1 and p_2^{-1} (the path p_2 followed in the opposite direction), and is globally oriented in the positive (anticlockwise), or negative (clockwise) sense, this gives a contribution of $+\text{area}(R)$, or $-\text{area}(R)$ respectively, to the signed area $S(p_1, p_2)$ (otherwise the contribution is zero).

3 The Heisenberg group H_1

The Heisenberg algebra h_1 is a 3–dimensional Lie algebra with generators X, Y, Z satisfying the commutation relations

$$[X, Y] = XY - YX = Z, \quad [X, Z] = [Y, Z] = 0. \tag{11}$$

The 3-parameter Heisenberg group H_1 has elements

$$(a, b, c) = \exp(aX + bY + cZ) \tag{12}$$

where a, b, c are canonical, or exponential coordinates, and X, Y, Z satisfy (11). If $a, b, c \in \mathbb{Z}$ then H_1 is the discrete Heisenberg group. Otherwise it is the continuous Heisenberg group. The third element c in (12) is a *phase*, and from (11) it commutes with all elements of H_1.

We list some selected properties [10]:

1. The identity in H_1 is $(0,0,0)$ and the inverse of (a,b,c) is $(-a,-b,-c)$.

2. The group composition is, from (8), (11) and (12)

$$(a,b,c) * (a',b',c') = (a+a', b+b', c+c' + \frac{1}{2}(ab' - a'b)). \qquad (13)$$

3. The center of H_1 consists of elements $\{(0,0,c)\}$, $c \in \mathbb{R}$. An example is the group commutator

$$(a,b,c) * (a',b',c') * (a,b,c)^{-1} * (a',b',c')^{-1} \qquad (14)$$

which from (13) is equal to $(0,0,ab' - a'b)$.

3.1 Automorphisms of H_1

The map $A : \; f \to A(f)$, with $f, \; A(f) \in H_1$ is an automorphism of H_1 if $A(f * g) = A(f) * A(g)$, where $*$ is group composition. The automorphisms of H_1 are [10]

- Inner i.e. conjugation by a fixed element of H_1. This adds a phase since from (13)

$$(x',y',z') \to (x,y,z)*(x',y',z')*(x,y,z))^{-1} = (x',y',z'+xy'-yx'). \qquad (15)$$

- Outer i.e.

  Dilations $(x,y,z) \to (rx, ry, r^2 z), \quad r > 0, \quad r \in \mathbb{R}.$

  Inversions $(x,y,z) \to (y,x,-z)$ (a reflection about the $x-y$ diagonal).

  Symplectic

$$\begin{pmatrix} x \\ y \end{pmatrix} \to A \begin{pmatrix} x \\ y \end{pmatrix} \quad A \in SL(2,\mathbb{R}) \qquad (16)$$

  and do not affect z i.e. there is no change of phase. Special cases are the modular transformations, $A \in SL(2,\mathbb{Z})$, whose generators S and T are

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad (17)$$

  and satisfy $S^2 = -\mathbb{I}, (ST)^3 = \mathbb{I}$.

  Other symplectic transformations are rotations by an angle θ in the $x-y$ plane

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \qquad (18)$$

4   Quantum holonomies as elements of H_1

Our notation is as follows: holonomies represented by quantum $SL(2, \mathbb{R})$ matrices are denoted $\hat{U}$. Elements of H_1 are (a, b, c) as in (12). Holonomies, denoted U or U_p, are assignments of an element of H_1 to a PL path p. One advantage of this assignment is that the phase c is now explicit *in the elements of H_1*, and geometrically is equal to the signed area in $\mathbb{R}^2$ between the path p and its chord (the straight line connecting the integer points $(0, 0)$ and (a, b)). See Figure 1. Group composition and the group commutator have natural geometrical interpretations. This construction is related to [9] where it was shown that, for a curve γ from the origin to (a, b) in $\mathbb{R}^2$, there is a unique lifted curve in H_1 which connects the identity $(0, 0, 0)$ to an element (a, b, c), where c is the signed area between γ and its chord.

Note that for PL paths we require that $a, b \in \mathbb{Z}$, but not necessarily c.
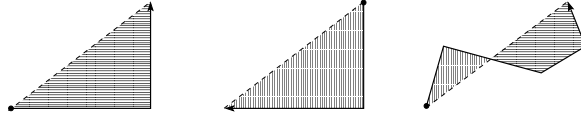

Figure 1: A PL path, its inverse, a general PL path, their chords (the dashed lines) and signed areas. Horizontal shading denotes positive areas and vertical shading denotes negative areas.

Quantum holonomies (3) are represented by elements of H_1, with properties

1. The identity in H_1 $(0, 0, 0)$ is the identity holonomy (representing the point $(0, 0)$, or the trivial path with zero signed area). The inverse of an element in H_1 is the inverse holonomy (and corresponds to the inverted PL path). See Figure 1.

2. Group composition (13) corresponds to path concatenation. To see this, consider the composition of two elements (a, b, c) and (a', b', c') in H_1.

   This concatenation is shown in the first two figures of Figure 2. The signed area between the resulting PL path and its chord $(a + a', b + b')$ is $c + c' + \frac{1}{2}(ab' - a'b)$. It is easily seen that $\frac{1}{2}(ab' - a'b)$ is the signed area between the two concatenated straight paths $((a, b)$, and the straight path from the point (a, b) to the point $(a + a', b + b'))$, and their chord (also $(a + a', b + b'))$, shown on the right of Figure 2.

   For straight paths $(c = c' = 0)$ group composition (13) corresponds to the triangle relation (9).

3. Homotopy and area phases. Consider two elements $U_{p_1} = (x, y, z)$ and $U_{p_2} = (x, y, z')$ of H_1. The group composition (13) of U_{p_1} and $U_{p_2}{}^{-1}$ is

$$(x, y, z) * (-x, -y, -z') = (0, 0, z - z') = e^{(z-z')Z} (0, 0, 0). \qquad (19)$$
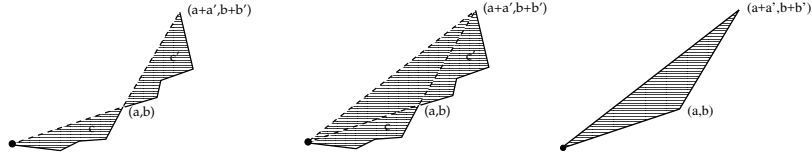
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Figure 2: Concatenation of two PL paths and two straight paths.

The RHS of equation (19) can be understood in terms of the signed area enclosed between two PL paths p_1 and p_2 with the same starting point $(0,0)$ and endpoint (x,y) but with different phases. In terms of holonomies

$$U_{p_1} = (x,y,z) = e^{zZ} \, U_{(x,y)}, \; U_{p_2^{-1}} = (-x,-y,-z') = e^{-z'Z} \, U_{(-x,-y)}$$

and using the triangle relation (9)

$$U_{p_1} \, U_{p_2^{-1}} = e^{(z-z')Z} \, U_{(0,0)} = (0,0,z-z') \qquad (20)$$

in agreement with (19). Further, from (20) we deduce that $U_{p_1} = e^{(z-z')Z} \, U_{p_2}$, i.e. the PL paths p_1 and p_2 are homotopic, with relative signed area $z - z'$.

An element (a,b,c) of H_1 corresponds to an equivalence class of homotopic paths which have zero relative area phase. Consider e.g. the chord $(a,b,0)$. Any homotopic zig-zag path with equal area above and below the chord has zero area phase with the chord. Therefore all such paths also have zero relative area phase, since signed area is additive [4] for composition of homotopies of paths.

4. The center of H_1 consists of elements $\{(0,0,c)\}$, $c \in \mathbb{R}$, interpreted as the holonomies U_p of closed paths p in $\mathbb{R}^2$, where c is the enclosed area. An example is the group commutator (14) which corresponds to the closed path $p = p_1 p_2 p_1^{-1} p_2^{-1}$, with $U_{p_1} = (a,b,c)$, $U_{p_2} = (a',b',c')$ shown in Figure 3. The enclosed area is $ab' - a'b$.
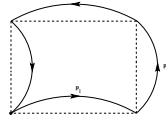

Figure 3: The closed path $p_1 p_2 p_1^{-1} p_2^{-1}$.

The group commutator (14) also describes the deformed relator (7).

5. It is natural to interpret signed area phases in terms of crossed modules and surface transport in higher gauge theory [11]. There is a crossed module given by the inclusion homomorphism $[H_1, H_1] \to H_1$, with trivial action of H_1 on $[H_1, H_1]$, where $[H_1, H_1]$ denotes the commutator subgroup

of H_1 (the subgroup generated by commutators). Here it is the subgroup of elements $\{(0, 0, c)\}$, $c \in \mathbb{R}$, i.e. the center of H_1.

6. Automorphisms.

   - Inner automorphisms - from (15), conjugation by an element of H_1 adds a phase. Geometrically, this can be interpreted as the action of a path p_1 on a path p_2, i.e. $p_2 \mapsto p_1 p_2 p_1^{-1}$ which adjusts the signed area between p_2 and its chord, as can be seen from Figure 3.
   - Outer automorphisms - the symplectic transformations in H_1 with $A \in SL(2, \mathbb{Z})$ (17) generate the modular group (with $q = e^Z$)

$$S : U_1 \to U_2, \quad U_2 \to U_1^{-1}$$
$$T : U_1 \to q^{-\frac{1}{2}} U_1 U_2, \quad U_2 \to U_2, \tag{21}$$

   a symmetry group of the deformed relator (7).

Final Remarks

We have found a novel relationship between quantum holonomies representing PL paths and elements of the Heisenberg group H_1. Group composition corresponds to path concatenation, the group commutator describes a deformation of the relator of the fundamental group π_1 of $\mathbb{T}^2$, and there are interpretations in terms of homotopy and area phases. Inner automorphisms of H_1 correspond to adjusting the signed area of a PL path, and discrete symplectic transformations generate the modular group.

The triangle relation (9) is reminiscent of a quantum bracket for traces of holonomies of straight paths derived in [3]

$$[\hat{T}(m, n), \hat{T}(s, t)] = \hat{T}(m, n)\hat{T}(s, t) - \hat{T}(s, t)\hat{T}(m, n)$$
$$= (q^{\frac{1}{2}(mt-ns)} - q^{-\frac{1}{2}(mt-ns)}) \, (\hat{T}(m + s, n + t) - \hat{T}(m - s, n - t)), \tag{22}$$

where $\hat{T}(m, n)$ is the trace of the holonomy $\hat{U}_{(m,n)}$. If $\hat{U}_{(m,n)} \in SL(2, \mathbb{R})$ this is

$$\hat{U}_{(m,n)} + \hat{U}_{(m,n)}^{-1} = \hat{U}_{(m,n)} + \hat{U}_{(-m,-n)} = tr \, \hat{U}_{(m,n)} \, \mathbb{I}_2. \tag{23}$$

The RHS of (22) contains two terms, corresponding to paths rerouted in two different ways at the intersections of the paths represented by $\hat{U}_{(m,n)}$ and $\hat{U}_{(s,t)}$. However, the RHS of (9) contains only one rerouting. To make contact with the bracket (22), it would be necessary to analyse the intersections of arbitrary PL paths, and to include inverses of holonomies, as for example, in (23).

Holonomies in 2+1 gravity [3, 4] were represented by quantum $SL(2, \mathbb{R})$ matrices (with q–number entries, composed by matrix multiplication). Representations of H_1 are by 3×3 matrices (with c-number entries and matrix multiplication) or by differential operators (with operator composition). The

correlation presented here is, however, independent of the representations of either.

The above issues, their applications and implications will be discussed elsewhere [12].

Acknowledgements

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