Eigenvalue estimates for submanifolds of warped product spaces

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Abstract

In this paper, we give lower bounds for the fundamental tone of open sets in minimal submanifolds immersed into warped product spaces of type $N^n \times_f Q^q$, where $f \in C^\infty(N)$. This setting allows to deal, among others, with minimal submanifolds of pieces of cylinders, cones, spheres and pseudo-hyperbolic spaces, most of these examples being not covered by the previous literature. Applications also include the study of the essential spectrum of hyperbolic graphs over compact regions of the boundary at infinity.

1. *Introduction*

Let M be a connected Riemannian manifold, possibly incomplete, and let $\Delta = \operatorname{div} \circ \nabla$ be the Laplace-Beltrami operator acting on $C_c^\infty(M)$, the space of smooth functions with compact support. To study the spectrum of Δ, we shall fix a self-adjoint extension: hereafter, will always consider the Friedrichs extension of Δ, that is, the unique extension of $(\Delta, C_c^\infty(M))$ whose domain lies in that of the closure of the associated quadratic form

$$\mathcal{Q} \quad : \qquad \varphi \in C_c^\infty(M) \ \longmapsto \ \int_M |\nabla\varphi|^2.$$

We remark that, when M is geodesically complete, by a result in [**17**, **19**, **30**] Δ is essentially self-adjoint, that is, the Friedrichs extension is indeed the unique self-adjoint extension of $(\Delta, C_c^\infty(M))$. Denote with $\sigma(-\Delta)$ and $\sigma_{\mathrm{ess}}(-\Delta)$, respectively, the spectrum and the essential spectrum of $-\Delta$. Given an open subset $\Omega \subset M$, the fundamental tone of Ω, $\lambda^*(\Omega)$, is defined by

$$\lambda^*(\Omega) = \inf \sigma(-\Delta) = \inf \left\{ \frac{\int_\Omega |\nabla f|^2}{\int_\Omega f^2};\ f \in H_0^1(\Omega)\backslash\{0\} \right\}.$$

When Ω has compact closure and Lipschitz boundary, $\lambda^*(\Omega)$ coincides with the first eigenvalue $\lambda_1(\Omega)$ of Ω, with Dirichlet boundary data on $\partial\Omega$. Its associated eigenspace is 1-dimensional and spanned by any nontrivial solution u of

$$\begin{cases} \Delta u + \lambda_1(\Omega) u = 0 & \text{on } \ \Omega, \\ u = 0 \quad \text{on } \ \partial\Omega. \end{cases}$$

The relations between the fundamental tone of open sets of M and their geometric invariants has been the subject to an intensive research in the past 50 years. Among a huge literature, we limit ourselves to quote the classics [**5**, **6**, **15**] and references therein for a detailed picture. In particular, a great effort has been done to estimate the fundamental tone of minimal submanifolds of well-behaved ambient spaces (for instance, in [**8**, **9**, **14**, **16**, **18**]). In this paper, we move a step further by giving lower bounds for the fundamental tone of manifolds which are minimally immersed in ambient spaces $N^n \times_f Q^q$ carrying a warped product structure. As we shall see in the last section, the generality of our setting allows applications to submanifolds in pieces of cylinders, cones, tubes, spheres, also improving certain recent results in the literature ([**7**, **8**, **9**]). We remark that there have been an increasing interest in the study of minimal and constant mean curvature submanifolds in product spaces $N \times \mathbb{R}$, after the discovery of many beautiful examples such as those in [**24**, **25**], and this motivates a thorough investigation of the spectrum of such submanifolds.

To introduce our main result, Theorem 8 below, we shall need some preliminary material, and we therefore prefer to postpone the statement of Theorem 8 after some definitions and a brief overview of basic results. Section 2 also contains a key technical lemma, Lemma 3 below. The statement and proof of Theorem 8 will appear in Section 3, and the final Section 4 is devoted to applications.

2. *Preliminaries*

Isometric immersions

Let M and W be smooth Riemannian manifolds of dimension m and $n+q$ respectively and $\varphi\colon M \hookrightarrow W$ be an isometric immersion. Consider a smooth function $F : W \to \mathbb{R}$ and the composition $F \circ \varphi\colon M \to \mathbb{R}$. Identifying X with $d\varphi(X)$, the Hessian of $F \circ \varphi$ at $x \in M$ is given by

$$\mathrm{Hess}_M(F \circ \varphi)(x)\,(X,Y) = \mathrm{Hess}_W F(\varphi(x))\,(X,Y) + \langle \nabla F,\, \sigma(X,Y)\rangle_{\varphi(x)}, \tag{2·1}$$

where $\sigma(X,Y)$ is the second fundamental form of φ. Tracing (2·1) with respect to an orthonormal basis $\{e_1, \dots e_m\}$,

$$\begin{aligned}\Delta_M(F \circ \varphi)(x) &= \sum_{i=1}^{m} \left\{ \mathrm{Hess}_W F(\varphi(x))\,(e_i,e_i) + \langle \nabla F,\, \sum_{i=1}^{m} \sigma(e_i,e_i)\rangle \right\} \\ &= \sum_{i=1}^{m} \mathrm{Hess}_W F(\varphi(x))\,(e_i,e_i) + m\langle \nabla F, H\rangle,\end{aligned} \tag{2·2}$$

where $H = m^{-1}\mathrm{tr}(\sigma)$ is the normalized mean curvature vector. Formulae (2·1) and (2·2) are well known in the literature, see [**22**].

Models and Hessian comparisons

Hereafter, we denote with $\mathbb{R}_0^+ = [0, +\infty)$. Let $g \in C^2(\mathbb{R}_0^+)$ be positive in $(0, R_0)$, for some $0 < R_0 \le \infty$, and satisfying

$$g(0) = 0, \qquad g'(0) = 1.$$

The κ-dimensional model manifold $\mathbb{Q}_g^\kappa$ constructed from the function g is just the ball $B_R(o) \subseteq \mathbb{R}^\kappa$ with metric given, in polar geodesic coordinates centered at o, by

$$\mathrm{d}s_g^2 = \mathrm{d}r^2 + g(r)^2 \,\langle\,,\rangle_{\mathbb{S}^{\kappa-1}}\,,$$

where $\langle\,,\rangle_{\mathbb{S}^{\kappa-1}}$ is the standard metric on the unit $(\kappa-1)$-sphere. The radial sectional curvature and the Hessian of the distance function r on $\mathbb{Q}^\kappa_g$ are given by the expressions

$$K^{\mathrm{rad}} = -\frac{g''(r)}{g(r)}, \qquad \mathrm{Hess}\, r = \frac{g'(r)}{g(r)}\Big(\mathrm{d}s^2 - \mathrm{d}r \otimes \mathrm{d}r\Big).$$

From the first relation, we see that a model can, equivalently, be specified by prescribing its radial sectional curvature $G \in C^\infty(\mathbb{R}^+_0)$ and recovering g as the solution of

$$\begin{cases} g'' - Gg = 0, \\ g(0) = 0, \quad g'(0) = 1, \end{cases} \tag{2·3}$$

on the maximal interval $(0, R_0)$ where $g > 0$. For future use, we will denote with G_- the negative part of G, i.e. $G_- = \max\{0, -G\}$.

For the proof of our main results we will make use of the following version of the Hessian Comparison Theorem, see [**21**] and [**27**, Chapter 2].

Theorem 1. *Let Q^q be a complete Riemannian q-manifold. Fix a point $o \in Q$, denote by $\rho_{Q}(x)$ the Riemannian distance function from o and let $D_o = Q\backslash cut(o)$ be the domain of the normal geodesic coordinates centered at o. Given $G \in C^\infty(\mathbb{R}^+_0)$, let g be the solution of the Cauchy problem (2·3), and let $(0, R_0) \subseteq [0, +\infty)$ be the maximal interval where g is positive. If the radial sectional curvature of Q satisfies*

$$K^{\mathrm{rad}}_Q \le -G(\rho_Q) \qquad (\textit{respectively}, \quad K^{\mathrm{rad}}_Q \ge -G(\rho_Q)), \tag{2·4}$$

on $B(o, R_0)$, then

$$\mathrm{Hess}_{\,Q}\,\rho_Q \ge \frac{g'(\rho_Q)}{g(\rho_Q)}\Big(\langle\,,\rangle_Q - d\rho_Q \otimes d\rho_Q\Big) \qquad (\textit{respectively}, \ \le)$$

on $D_o \cap B(o, R_0)\backslash\{o\}$, in the sense of quadratic forms.

Eigenvalues and Eigenfunctions

The generalized version of Barta's Eigenvalue Theorem [**4**], proved in [**9**] will be important in the sequel.

Theorem 2. *Let Ω be an open set in a Riemannian manifold M and let $f \in C^2(\Omega)$, $f > 0$ on Ω. Then*

$$\lambda^*(\Omega) \ge \inf_\Omega \left(-\frac{\Delta f}{f}\right). \tag{2·5}$$

We recall that, given a model $\mathbb{Q}^\kappa_g$ with $g > 0$ on $(0, R_0)$, and given $R \in (0, R_0)$, the first eigenfunction v of the geodesic ball $B_g(R)$ centered at o is radial. This can be easily seen by proving that its spherical mean

$$\bar{v}(r) = \frac{1}{g(r)^{\kappa-1}} \int_{\partial B_g(R)} v$$

is still an eigenfunction associated to $\lambda_1(B_g(R))$ and using the fact that the space of first eigenfunctions has dimension 1. With a slight abuse of notation, we can thus identify the first eigenfunction $v \in C^\infty(B_g(R))$ of $B_g(R)$ with the solution $v\colon [0, R] \to \mathbb{R}$ of

$$\begin{cases} v'' + (\kappa-1)\dfrac{g'}{g}v' + \lambda_1(B_g(R))v = 0 & \text{on } (0, R), \\ v(0) = 1, \quad v'(0) = 0, \quad v(R) = 0, \quad v > 0 \text{ on } [0, R). \end{cases} \tag{2·6}$$

Note that, multiplying the ODE by $g^{\kappa-1}$, integrating and using the initial condition, one can easily argue that $v' < 0$ on $(0, R]$.

We will need the following technical lemma, which extends a result due to Bessa-Costa, see [**7**, Lemma 2.4].

Lemma 3. *Let $\mathbb{Q}_g^\kappa$ be a model manifold with radial sectional curvature $-G(r)$, and suppose that $g' > 0$ on $[0, R)$. Let $v \in C^2(B_g(R))$ be a first positive eigenfunction of $B_g(R) \subset \mathbb{Q}_g^\kappa$. If*

$$\lambda_1(B_g(R)) \geq \kappa \|G_-\|_{L^\infty([0,R])}. \tag{2·7}$$

Then the following inequality holds:

$$\kappa \frac{g'(t)}{g(t)} v'(t) + \lambda_1(B_g(R)) v(t) \leq 0, \qquad t \in (0, R]. \tag{2·8}$$

Proof. For simplicity of notation, we denote by $\lambda = \lambda_1(B_g(R))$. Multiplying (2·6) by $g^{\kappa-1}$ we deduce that $v(t)$ satisfies the following differential equation:

$$\begin{cases} (g^{\kappa-1}v')' + \lambda g^{\kappa-1} v = 0 & \text{on } (0, R), \\ v(0) = 1, \quad v'(0) = 0, \quad v(R) = 0, \quad v > 0 \text{ on } [0, R). \end{cases} \tag{2·9}$$

Our aim is to deduce (2·8) via some modified Sturm-type arguments. In order to do so, we search for a positive function μ solving

$$\kappa \mu'(t) \frac{g'(t)}{g(t)} + \lambda \mu(t) = 0 \ \text{ on } (0, R). \tag{2·10}$$

Integrating, we get that $\log \mu(t) = -\dfrac{\lambda}{\kappa} \displaystyle\int_0^t \frac{g(s)}{g'(s)} \mathrm{d}s$ is a solution, giving

$$\mu(t) = e^{\left(-\frac{\lambda}{\kappa} \int_0^t \frac{g(s)}{g'(s)} \mathrm{d}s\right)}.$$

The above expression is well defined since $g' > 0$ on $[0, R)$. Since $\mu'(t) = -\dfrac{\lambda}{\kappa} \dfrac{g(t)}{g'(t)} \mu(t)$ we deduce

$$\mu'(t)v(t) - v'(t)\mu(t) = -\frac{\lambda}{\kappa} \frac{g(t)}{g'(t)} e^{\left(-\frac{\lambda}{\kappa} \int_0^t \frac{g(s)}{g'(s)} \mathrm{d}s\right)} v(t) - v'(t) e^{\left(-\frac{\lambda}{\kappa} \int_0^t \frac{g(s)}{g'(s)} \mathrm{d}s\right)}$$

$$= -\frac{1}{\kappa} \frac{g(t)}{g'(t)} e^{\left(-\frac{\lambda}{\kappa} \int_0^t \frac{g(s)}{g'(s)} \mathrm{d}s\right)} \left(\kappa \frac{g'(t)}{g(t)} v'(t) + \lambda v(t)\right). \tag{2·11}$$

From (2·11) we see that $\kappa \dfrac{g'(t)}{g(t)} v'(t) + \lambda v(t) \leq 0$ on $(0, R)$ if and only if

$$\mu'(t)v(t) - v'(t)\mu(t) \geq 0 \qquad \text{on } (0, R),$$

and we are going to prove this last inequality.

Differentiating (2·10) and multiplying by $(1/\kappa)$ both sides of the equality, we have

$$\mu''(t) \frac{g'(t)}{g(t)} + \mu'(t) \left[G(t) - \left(\frac{g'(t)}{g(t)}\right)^2 + \frac{\lambda}{\kappa} \right] = 0,$$

that is,

$$\mu''(t) = -\mu'(t)\frac{g(t)}{g'(t)}\left[G(t) - \left(\frac{g'(t)}{g(t)}\right)^2 + \frac{\lambda}{\kappa}\right].$$

Since $\mu'(t)\dfrac{g(t)}{g'(t)} = -\dfrac{\lambda}{\kappa}\mu(t)\left(\dfrac{g(t)}{g'(t)}\right)^2$ we can rewrite $\mu''(t)$ in the following way:

$$\mu''(t) = \frac{\lambda}{\kappa}\mu(t)\left[G(t)\left(\frac{g(t)}{g'(t)}\right)^2 - 1 + \frac{\lambda}{\kappa}\left(\frac{g(t)}{g'(t)}\right)^2\right].$$

Multiplying the above equation by $g^{\kappa-1}(t)$, and then adding and subtracting the term $(\kappa-1)g^{\kappa-2}(t)g'(t)\mu'(t)$, we obtain

$$(g^{\kappa-1}\mu')'(t) = -\lambda g^{\kappa-1}(t)\mu(t)\left[-\frac{G(t)}{\kappa}\left(\frac{g(t)}{g'(t)}\right)^2 - \frac{\lambda}{\kappa^2}\left(\frac{g(t)}{g'(t)}\right)^2 + 1\right]. \qquad (2{\cdot}12)$$

Next, we multiply (2·12) by $v(t)$ and (2·9) by $-\mu(t)$, and we add them to get

$$(g^{\kappa-1}\mu')'(t)v(t) - (g^{\kappa-1}v')'(t)\mu(t) = \frac{\lambda}{\kappa}g^{\kappa-1}(t)\mu(t)v(t)\left(\frac{g(t)}{g'(t)}\right)^2\left[G(t) + \frac{\lambda}{\kappa}\right].$$

Integrating from 0 to t gives

$$g^{\kappa-1}\left(\mu'v - v'\mu\right)(t) = \int_0^t \frac{\lambda}{\kappa}g^{\kappa-1}(s)\left(\frac{g(s)}{g'(s)}\right)^2\left[G(s) + \frac{\lambda}{\kappa}\right]\mu(s)v(s)\mathrm{d}s. \qquad (2{\cdot}13)$$

Now, from (2·7) we deduce that

$$\frac{\lambda}{\kappa}g^{\kappa-1}(t)\left(\frac{g(t)}{g'(t)}\right)^2\left[G(t) + \frac{\lambda}{\kappa}\right]\mu(t)v(t) \geq 0,$$

whence $\mu'(t)v(t) - v'(t)\mu(t) \geq 0$ for $t \in (0, R)$, as claimed. $\square$

Remark 4. It is important to find conditions to ensure the lower bound (2·7). For instance, if $-G(r) = B^2$, where B is a positive constant, then the solution g_B of (2·3) is

$$g_B(r) = B^{-1}\sin(Br), \qquad \text{thus} \qquad g'_B > 0 \quad \text{on } [0, \pi/(2B)). \qquad (2{\cdot}14)$$

The function g_B yields the model manifold $\mathbb{Q}^\kappa_{g_B} = \mathbb{S}^\kappa(B^2)$, the κ-dimensional sphere of constant sectional curvature B^2 and diameter $\mathrm{diam}_{\mathbb{S}^\kappa(B^2)} = \pi/B$. Note that the first eigenvalue of the geodesic ball of $\mathbb{S}^\kappa(B^2)$ of radius $R = \pi/2B$ is $\lambda_1(B_{\mathbb{S}^\kappa(B^2)}(\pi/2B)) = \kappa B^2$ and $v(r) = \cos(Br)$ is its first eigenfunction.

When $-G(r) \leq B^2$ and $R \leq \pi/(2B)$, by Sturm's argument a solution g of (2·3) satisfies

$$\frac{g'}{g} \geq \frac{g'_B}{g_B} > 0 \quad \text{on } \left[0, \frac{\pi}{2B}\right).$$

By Cheng's Comparison Theorem (version proved by Bessa-Montenegro in [**10**]),

$$\lambda_1(B_g(R)) \geq \lambda_1(B_{g_B}(R)), \quad R \in \left[0, \frac{\pi}{2B}\right).$$

In order to get $\lambda_1(B_g(R)) \geq \kappa\|G_-\|_{L^\infty([0,R))}$ it is sufficient to have

$$\lambda_1(B_{g_B}(R)) = \lambda_1(B_{\mathbb{S}^\kappa(B^2)}(R)) \geq \kappa\|G_-\|_{L^\infty([0,R))}. \qquad (2{\cdot}15)$$

On the other hand, we can see $\kappa\|G_-\|_{L^\infty([0,R))}$ as a first eigenvalue of a ball of radius $\widetilde{R}$ in a κ-dimensional sphere of sectional curvature $\widetilde{B}^2$, i.e.

$$\kappa\|G_-\|_{L^\infty([0,R))} = \lambda_1(B_{\mathbb{S}^\kappa(\widetilde{B}^2)}(\widetilde{R})),$$

where $\widetilde{R} = \dfrac{\pi}{2\sqrt{\|G_-\|_{L^\infty([0,R))}}}$ and $\widetilde{B}^2 = \|G_-\|_{L^\infty([0,R))}$.

We then conclude that the inequality (2·15) holds whenever

$$R \leq \frac{\pi}{2\sqrt{\|G_-\|_{L^\infty([0,R))}}}.$$

Remark 5. We remark that if

$$t\int_t^\infty G_-(s)\mathrm{d}s \leq \frac{1}{4} \qquad \text{for every } t \in \mathbb{R}^+,$$

then both g and g' are strictly positive on $\mathbb{R}^+$. This criterion has been proved in [**13**, Prop. 1.21].

A preliminary computation.
From now on, we will consider the case when the ambient space is a warped product $W^{n+q} = N \times_f Q$ of two Riemannian manifolds $(N^n, \langle,\rangle_N)$ and $(Q^q, \langle,\rangle_Q)$, with the Riemannian metric on W given by

$$\langle\!\langle\, , \,\rangle\!\rangle = \langle\, , \,\rangle_N + f^2\langle\, , \,\rangle_Q$$

for some smooth positive function $f\colon N \to \mathbb{R}^+$. We fix the index convention

$$1 \leq j, k \leq n, \quad n+1 \leq \alpha, \beta \leq n+q.$$

For $(p,q) \in W$, we choose a chart (U,ψ) on N around p, with coordinate tangent basis $\{\partial_j\} = \{\partial/\partial\psi_j\}$, and a chart (V,ϕ) on Q around q, with basis $\{\partial_\alpha\} = \{\partial/\partial\phi_\alpha\}$. Then, with respect to the product chart $(U \times V, \psi \times \phi)$ around (p,q), the Hessian of F at (p,q) has components

$$\begin{cases} \operatorname{Hess}_W F(\partial_j, \partial_\kappa) = \operatorname{Hess}_N F(\partial_j, \partial_\kappa), \\ \operatorname{Hess}_W F(\partial_j, \partial_\alpha) = \partial_j\partial_\alpha F - \dfrac{1}{f}\partial_j f \partial_\alpha F, \\ \operatorname{Hess}_W F(\partial_\alpha, \partial_\beta) = \operatorname{Hess}_Q F(\partial_\alpha, \partial_\beta) + \dfrac{1}{f}\left\langle \nabla^N f, \nabla F \right\rangle_N \langle\!\langle \partial_\alpha, \partial_\beta \rangle\!\rangle, \end{cases} \tag{2·16}$$

where $\operatorname{Hess}_N F$ and $\operatorname{Hess}_Q F$ mean respectively $\operatorname{Hess}(F \circ i_N)$ and $\operatorname{Hess}(F \circ i_Q)$ and the inclusions are given by

$$\begin{aligned} i_N &\colon (N, \langle,\rangle_N) \to N \times_f \{q\} \subseteq N \times_f Q, \quad x \mapsto (x,q), \\ i_Q &\colon (Q, \langle,\rangle_Q) \to \{p\} \times_f Q \subseteq N \times_f Q, \quad y \mapsto (p,y). \end{aligned}$$

From (2·16) we observe that if $F(p,q) = f(p)\cdot h(q)$, where f is the warping function and $h : Q \to \mathbb{R}$ is a smooth function on Q, then $\operatorname{Hess}_W F$ has a block structure, that is

$$\operatorname{Hess}_W F(X,Z) = 0 \quad \forall X \in T_{(p,q)}\big(N \times_f \{q\}\big),\ Z \in T_{(p,q)}\big(\{p\} \times_f Q\big).$$

More precisely, we have the following result.

Lemma 6. *Let $F \in C^\infty(N \times_f Q)$ be given by $F(p,q) = f(p)\cdot h(q)$, where $h \in C^\infty(Q)$.*

Then

$$\begin{cases} \operatorname{Hess}_W F(X,Y) = h \operatorname{Hess}_N f(X,Y), \\ \operatorname{Hess}_W F(X,Z) = 0, \\ \operatorname{Hess}_W F(Z,W) = f \operatorname{Hess}_Q h(Z,W) + h \dfrac{\left|\nabla^N f\right|_N^2}{f} \langle\!\langle Z, W \rangle\!\rangle , \end{cases} \tag{2·17}$$

for every $X, Y \in T_{(p,q)}\big(N \times_f \{q\}\big)$ *and* $Z, W \in T_{(p,q)}\big(\{p\} \times_f Q\big)$.

3. *Main results*

Let $\varphi\colon M^m \to N^n \times_f Q^q$, $m > n$, be a minimal immersion. Hereafter, we shall require the following

ASSUMPTION 7. Suppose that the radial sectional curvature of Q satisfies

$$K_Q^{\mathrm{rad}} \leq -G(\rho_Q), \text{ where } G \in C^\infty(\mathbb{R}_0^+),$$

where $\rho_Q(x) = \mathrm{dist}_Q(o,x)$. We assume that the solution g of (2·3) is positive and $g' > 0$ on $[0,R)$, and that $B_Q(o,R) \subseteq Q \backslash \mathrm{cut}(o)$.

Let $v : \overline{B_g(R)} \to \mathbb{R}$ be the first eigenfunction of the ball $B_g(R) \subset \mathbb{Q}_g^{m-n}$. As remarked, up to normalizing and possibly changing sign $v > 0$ on $B_g(R)$, v is radial and solves

$$\begin{cases} v''(t) + (m-n-1)\dfrac{g'(t)}{g(t)} v'(t) + \lambda_1(B_g(R)) v(t) = 0, \quad t \in (0,R) \\ v(0) = 1, \quad v(R) = 0, \quad v > 0 \text{ on } [0,R), \quad v' < 0 \text{ on } (0,R]. \end{cases} \tag{3·1}$$

Observe that, when $m = n+1$, the equation simply becomes

$$v''(t) + \lambda_1(B_g(R)) v(t) = 0.$$

THEOREM 8. *Let* $\varphi : M^m \to N^n \times_f Q^q$ *be an* m*-dimensional submanifold minimally immersed into the warped product space* $(N^n \times_f Q^q, \langle\!\langle\, , \,\rangle\!\rangle)$, *where*

$$\langle\!\langle\, , \,\rangle\!\rangle = \langle\, , \,\rangle_N + f^2 \langle\, , \,\rangle_Q,$$

$0 < f \in C^\infty(N)$, *Q satisfies Assumption 7 and $m > n$. Suppose that the warping function f satisfies*

$$\operatorname{Hess}_N f(\cdot,\cdot) - \frac{\left|\nabla^N f\right|_N^2}{f} \langle , \rangle_N \leq 0. \tag{3·2}$$

Let $U \subseteq N$ be an open subset, and let $\Omega \subset \varphi^{-1}(U \times_f B_Q(o,R))$ be a connected component. Then, if R is such that

$$R \leq \frac{\pi}{2\sqrt{\|G_-\|_{L^\infty([0,R))}}} \tag{3·3}$$

the following estimate holds:

$$\lambda^*(\Omega) \geq \inf_{p \in U} \left(\frac{\lambda_1(B_g(R)) - m\,|\nabla^N f|_N^2(p)}{|f(p)|^2} \right), \tag{3·4}$$

where $B_g(R)$ is the geodesic ball of radius R in the model manifold $\mathbb{Q}_g^{m-n}$ or the interval $[-R,R]$ if $m = n+1$.

Proof. We start defining $F\colon U \times_f B_Q(o,R) \to \mathbb{R}$ by $F(p,q) = f(p)\cdot h(q)$, where $h \in C^\infty(B_Q(o,R))$ is given by $h(q) = (v\circ \rho_{_Q})(q)$ and $v \in C^\infty([0,R])$ is the solution of (3·1). By Theorem 2, we have that

$$\lambda^*(\Omega) \geq \inf_{\Omega}\left(-\frac{\Delta(F\circ\varphi)}{F\circ\varphi}\right). \tag{3·5}$$

We are going to give a lower bound for $-\Delta(F\circ\varphi)/(F\circ\varphi)$. Let $x\in\Omega$ and let $\{e_1,\ldots,e_m\}$ be an orthonormal basis for $T_x\Omega$. Let $\varphi(x) = (p(x),q(x))$, $t(x) = \rho_{_Q}(q(x))$ and denote by $P_{_N} : T_{_{(p,q)}}\big(N\times_f Q\big) \to T_{_{(p,q)}}\big(N\times_f\{q\}\big)$ and $P_{_Q} : T_{_{(p,q)}}\big(N\times_f Q\big) \to T_{_{(p,q)}}\big(\{p\}\times_f Q\big)$ the orthogonal projections onto the tangent spaces of the two fibers. Then, by (2·2) and the minimality of M, the Laplacian of $F\circ\varphi$ at x has the expression

$$\begin{aligned}\Delta\,(F\circ\varphi)(x) &= \sum_{i=1}^m \mathrm{Hess}_{\,W}F(\varphi(x))(e_i,e_i)\\ &= \sum_{i=1}^m\Big[\mathrm{Hess}_{\,W}F(\varphi(x))(P_{_N}e_i,P_{_N}e_i) + \mathrm{Hess}_{\,W}F(\varphi(x))(P_{_Q}e_i,P_{_Q}e_i)\Big]\end{aligned}$$

where $W = N\times_f Q$. Using Lemma 6, and writing $t = t(x)$ for the ease of notation, we deduce

$$\begin{aligned}\Delta\,(F\circ\varphi)(x) = v(t)&\sum_{i=1}^m \mathrm{Hess}_{\,N}f(P_{_N}e_i,P_{_N}e_i)(p) + f(p)\sum_{i=1}^m \mathrm{Hess}_{\,Q}v(t)(P_{_Q}e_i,P_{_Q}e_i)\\ &+ v(t)\frac{\big|\nabla^N f\big|_N^2}{f}(p)\sum_{i=1}^m \langle\!\langle P_{_Q}e_i,P_{_Q}e_i\rangle\!\rangle.\end{aligned} \tag{3·6}$$

Let $\{E_1,\ldots,E_n\}$ be an orthonormal basis for T_pN, and consider the tangent basis $\big\{\partial/\partial\rho_{_Q}, \{\partial/\partial\theta^\gamma\}_{\gamma=n+2}^{n+q}\big\}$, associated to normal coordinates at Q. Then the set $\{\xi_l\}_{l=1}^{n+q}$ given by

$$\xi_j = E_j \;\;\forall j=1,\ldots,n, \quad \xi_{n+1} = \frac{1}{f}\frac{\partial}{\partial\rho_{_Q}}, \quad \xi_\gamma = \frac{1}{f}\frac{\partial}{\partial\theta^\gamma} \;\;\forall\gamma = n+2,\ldots,n+q$$

is an orthonormal basis of $T_{(p,q)}\big(N\times_f Q\big)$. So, we can write e_i as a linear combination of vectors of this basis in the following way:

$$e_i = \sum_{j=1}^n a_i^j\cdot\xi_j + b_i\cdot\xi_{n+1} + \sum_{\gamma=n+2}^{n+q} c_i^\gamma\cdot\xi_\gamma,$$

for constants a_i^j, b_i, c_i^γ satisfying

$$\sum_{j=1}^n (a_i^j)^2 + b_i^2 + \sum_{\gamma=n+2}^{n+q}(c_i^\gamma)^2 = 1,\ \forall i=1,\ldots,m. \tag{3·7}$$

From

$$\nabla^Q v(t) = v'(t)\frac{\partial}{\partial\rho_{_Q}}, \quad \mathrm{Hess}_{\,Q}v(t) = v'(t)\mathrm{Hess}_{\,Q}\rho_{_Q} + v''(t)\mathrm{d}\rho_{_Q}\otimes\mathrm{d}\rho_{_Q},$$

we can rewrite (3·6) in the following way:

$$\begin{aligned}\Delta(F\circ\varphi)(x) =& v(t)\sum_{i=1}^m \mathrm{Hess}_N f(P_N e_i, P_N e_i)(p) + f(p)\sum_{i=1}^m \Big[P_Q e_i(v'(t))\left\langle \frac{\partial}{\partial\rho_Q}, P_Q e_i\right\rangle_Q \\ &+ v'(t)\mathrm{Hess}_Q\rho_Q(P_Q e_i, P_Q e_i)\Big] + v(t)\frac{\left|\nabla^N f\right|_N^2}{f}(p)\sum_{i=1}^m \langle\!\langle P_Q e_i, P_Q e_i\rangle\!\rangle \\ =& v(t)\sum_{i=1}^m\left(\mathrm{Hess}_N f(P_N e_i, P_N e_i) + \frac{\left|\nabla^N f\right|_N^2}{f}\Big(1-\langle\!\langle P_N e_i, P_N e_i\rangle\!\rangle\Big)\right)(p) \\ &+ \frac{1}{f(p)}\left(v''(t)\sum_{i=1}^m b_i^2 + v'(t)\sum_{i=1}^m\sum_{\gamma=n+2}^{n+q}(c_i^\gamma)^2\mathrm{Hess}_Q\rho_Q\left(\frac{\partial}{\partial\theta^\gamma},\frac{\partial}{\partial\theta^\gamma}\right)\right).\end{aligned}$$

Using (3·2) and the fact that v is positive we have

$$\begin{aligned}-\Delta(F\circ\varphi)(x) \geq& -\frac{1}{f(p)}\Big[mv(t)|\nabla^N f|_N^2(p) + v''(t)\sum_{i=1}^m b_i^2 \\ &+ v'(t)\sum_{i=1}^m\sum_{\gamma=n+2}^{n+q}(c_i^\gamma)^2\mathrm{Hess}_Q\rho_Q\left(\frac{\partial}{\partial\theta^\gamma},\frac{\partial}{\partial\theta^\gamma}\right)\Big].\end{aligned}$$

Since $v'(t)\leq 0$, we can apply the Hessian Comparison Theorem, to obtain

$$\begin{aligned}-\Delta(F\circ\varphi)(x) \geq& -\frac{1}{f(p)}\Big[mv(t)\left|\nabla^N f\right|_N^2(p) + v''(t)\sum_{i=1}^m b_i^2 \\ &+ v'(t)\frac{g'(t)}{g(t)}\sum_{i=1}^m\sum_{\gamma=n+2}^{n+q}(c_i^\gamma)^2\Big] \\ =& -\frac{1}{f(p)}\Big[v''(t)\sum_{i=1}^m b_i^2 + v'(t)\frac{g'(t)}{g(t)}\Big(m - \sum_{i=1}^m\sum_{j=1}^n (a_i^j)^2 - \sum_{i=1}^m b_i^2\Big) \\ &+ mv(t)\left|\nabla^N f\right|_N^2(p)\Big]\end{aligned}$$

where the last equality follows by an algebraic manipulation that uses (3·7) summed for $i=1,\ldots,m$. Now, by a simple rearranging,

$$\begin{aligned}-\Delta(F\circ\varphi)(x) \geq& -\frac{1}{f(p)}\Big[v''(t) + (m-n-1)v'(t)\frac{g'(t)}{g(t)} - v''(t)\left(1-\sum_{i=1}^m b_i^2\right) \\ &+ v'(t)\frac{g'(t)}{g(t)}\left(n-\sum_{i=1}^m\sum_{j=1}^n (a_i^j)^2 + 1 - \sum_{i=1}^m b_i^2\right) + mv(t)\left|\nabla^N f\right|_N^2(p)\Big].\end{aligned}$$

From (3·1) we get

$$\begin{aligned}-\Delta(F\circ\varphi)(x) \geq& \frac{v(t)}{f(p)}\left(\lambda_1(B_g(R)) - m\left|\nabla^N f\right|_N^2(p)\right) \\ &+ \frac{1}{f(p)}\left[v''(t)\left(1-\sum_{i=1}^m b_i^2\right) - v'(t)\frac{g'(t)}{g(t)}\left(n-\sum_{i=1}^m\sum_{j=1}^n (a_i^j)^2 + 1 - \sum_{i=1}^m b_i^2\right)\right].\end{aligned} \quad (3·8)$$

We claim that the last line of (3·8) is nonnegative, that is,

$$v''(t)\left(1-\sum_{i=1}^m b_i^2\right) - v'(t)\frac{g'(t)}{g(t)}\left(n - \sum_{i=1}^m\sum_{j=1}^n (a_i^j)^2 + 1 - \sum_{i=1}^m b_i^2\right) \geq 0. \quad (3·9)$$

To prove this, we substitute $v''(t) = -(m-n-1)v'(t)\dfrac{g'(t)}{g(t)} - \lambda_1(B_g(R))v(t)$ in (3·9) to get

$$
\begin{aligned}
v''(t)\left(1-\sum_{i=1}^m b_i^2\right) - v'(t)\frac{g'(t)}{g(t)}\left(n - \sum_{i=1}^m\sum_{j=1}^n (a_i^j)^2 + 1 - \sum_{i=1}^m b_i^2\right) &= \\
-\left((m-n)v'(t)\frac{g'(t)}{g(t)} + \lambda_1(B_g(R))v(t)\right)\left(1-\sum_{i=1}^m b_i^2\right)& \qquad (3\cdot10)\\
-v'(t)\frac{g'(t)}{g(t)}\left(n - \sum_{i=1}^m\sum_{j=1}^n (a_i^j)^2\right)&,
\end{aligned}
$$

so that (3·9) is equivalent to show that

$$
\begin{aligned}
-\left((m-n)v'(t)\frac{g'(t)}{g(t)} + \lambda_1(B_g(R))v(t)\right)\left(1-\sum_{i=1}^m b_i^2\right)& \qquad (3\cdot11)\\
-v'(t)\frac{g'(t)}{g(t)}\left(n - \sum_{i=1}^m\sum_{j=1}^n (a_i^j)^2\right) &\geq 0.
\end{aligned}
$$

Now, in our assumption (3·3), by Remark 4 it holds

$$\lambda_1(B_g(R)) \geq (m-n)\|G_-\|_{L^\infty([0,R])}.$$

Hence, applying Lemma 3 we infer that

$$(m-n)v'(t)\frac{g'(t)}{g(t)} + \lambda_1(B_g(R))v(t) \leq 0.$$

Moreover, it is clear that $\left(1 - \sum_{i=1}^m b_i^2\right) \geq 0$, and finally we observe the inequality

$$\sum_{i=1}^m\sum_{j=1}^n (a_i^j)^2 = \sum_{j=1}^n\left(\sum_{i=1}^m \langle\!\langle e_i, \xi_j\rangle\!\rangle\right) = \sum_{j=1}^n |P_{_M}\xi_j|^2 \leq \sum_{j=1}^n |\xi_j|^2 = \sum_{j=1}^n 1 = n,$$

where $P_{_M}$ is the projection on M.

Keeping in mind that $v' \leq 0$, this concludes the proof of the claimed (3·11). From (3·8) we have

$$-\frac{\Delta(F\circ\varphi)}{F\circ\varphi}(x) \geq \frac{1}{f^2(p)}\left(\lambda_1(B_g(R)) - m\left|\nabla^N f\right|_N^2(p)\right). \qquad (3\cdot12)$$

Therefore, by (3·5) we conclude the desired (3·4). □

Remark 9. In the case $N = \mathbb{R}$, we observe that the mean curvature function of the fibers $\{p\}\times_f Q$ is given by $\mathcal{H}(y) = f'(y)/f(y)$. Therefore, condition (3·2) is equivalent to $f\mathcal{H}' \leq 0$, that is, $\mathcal{H}' \leq 0$. There exists a large class of functions for which $\mathcal{H}' \leq 0$. For instance, $f(y) =$ constant, $f(y) = y$ and $f(y) = e^{cy}$, where $c \in \mathbb{R}$.

4. *Applications*

To show the generality of Theorem 8, we conclude this paper with a number of different examples, and we discuss the sharpness of the estimates produced.

4·1. *Balls*

In the limiting case when N is a point, computations hold with the obvious simplifications and we get the following result, which is basically known (see QUOTATION!!)

COROLLARY 10. *Let $\varphi\colon M^m \to Q^q$ be an m-dimensional submanifold minimally immersed into Q^q. Suppose that Q satisfies Assumption 7. Let $\Omega \subset \varphi^{-1}(B_Q(o,R))$ be a connected component with*

$$R \leq \frac{\pi}{2\sqrt{\|G_-\|_{L^\infty([0,R))}}}.$$

Then

$$\lambda^*(\Omega) \geq \lambda_1(B_g^m(R)).$$

Here $B_g^m(R)$ is a geodesic ball of radius R in an m-dimensional model manifold $\mathbb{Q}_g^m$.

For instance, a minimal submanifold $\varphi : M^m \to \mathbb{R}^q$ in a ball of radius R has fundamental tone greater or equal than that of the flat m-ball of radius R:

$$\lambda^*(M) \geq \lambda_1(B_R^m) = \left(\frac{c_m}{R}\right)^2,$$

where c_m is the first zero of the $J_{m/2-1}$-Bessel function. Similarly, the first eigenvalue of a minimal submanifold $\varphi : M^m \to \mathbb{S}_+^q$ contained in half of a unit sphere is greater or equal than that of the upper half m-sphere:

$$\lambda^*(M) \geq \lambda_1\big(B_{\mathbb{S}^m}(\pi/2)\big) = m.$$

We underline that, although the sphere is well studied, the first eigenvalue $\lambda_1(B_{\mathbb{S}^m}(r))$ for $r \notin \{\pi/2, \pi\}$ is still pretty much unknown. Estimates for spherical cups have been developed in [**1**, **28**, **29**] for dimension two, [**20**] for dimension three and [**2**, **3**, **12**] for all dimensions.

4·2. *Cylinders*

A very similar situation occurs when the ambient space is a cylinder $\mathbb{R}\times Q^q$. Considering $f = 1$ and $N = \mathbb{R}$ in Theorem 8 we obtain a generalized version of Theorem 1.1 of [**7**].

COROLLARY 11. *Let $\varphi\colon M^m \to \mathbb{R} \times Q^q$ be an m-dimensional submanifold minimally immersed into $\mathbb{R} \times Q^q$ with Q satisfying the Assumption 7. Let $\Omega \subset \varphi^{-1}(\mathbb{R} \times B_Q(o,R))$ be a connected component with*

$$R \leq \frac{\pi}{2\sqrt{\|G_-\|_{L^\infty([0,R))}}}.$$

Then

$$\lambda^*(\Omega) \geq \lambda_1(B_g^{m-1}(R)). \tag{4·1}$$

Here, $B_g^{m-1}(R)$ is a geodesic ball of radius R in an $(m-1)$-dimensional model manifold $\mathbb{Q}_g^{m-1}$.

Note that, differently from Corollary 10, here the comparison ball in estimate (4·1) has dimension $m-1$ and not m. In particular, when $Q^q = \mathbb{R}^q$ in the last corollary we get the following result in the Euclidean space proved by Bessa and Costa in [**7**].

COROLLARY 12. *Let $\varphi\colon M^m \to \mathbb{R}^{q+1}$ be an m-dimensional submanifold minimally* *immersed into* $\mathbb{R}^{q+1}$*. Let* $\Omega \subset \varphi^{-1}(\mathbb{R} \times B_{\mathbb{R}^q}(o,R))$ *be a connected component. Then*

$$\lambda^*(\Omega) \geq \lambda_1(B_{\mathbb{R}^{m-1}}(o,R)) = \left(\frac{c_{m-1}}{R}\right)^2. \tag{4·2}$$

Here c_{m-1} *is the first zero of the* $J_{(m-1)/2-1}$*-Bessel function.*

4·3. *Pseudo-hyperbolic and hyperbolic spaces*

The pseudo-hyperbolic spaces, introduced by Tashiro in [**31**], are warped products $\mathbb{R} \times_f Q^q$ with

$$(i)\quad f(y) = ae^{by}, \qquad \text{or} \qquad (ii)\quad f(y) = a\cosh(by),$$

for some constants $a, b > 0$. In the case (i), we observe that condition (3·2) is satisfied, as it shows

$$f'' - \frac{(f')^2}{f} = \begin{cases} 0 & \text{in case } (i), \\ ab^2/\cosh(by) > 0 & \text{in case } (ii). \end{cases}$$

We state the following corollary in the case $f(y) = e^{by}$.

COROLLARY 13. *Let* $\varphi : M^m \to \mathbb{R} \times_{e^{by}} Q^q$ *be an* m*-dimensional submanifold minimally immersed into* $\mathbb{R} \times_{e^{by}} Q^q$*, with* Q *satisfying Assumption 7. Let* $\Omega \subset \varphi^{-1}\big((\alpha,\beta) \times_{e^{by}} B_Q(o,R)\big)$ *be a connected component with*

$$R \leq \frac{\pi}{2\sqrt{\|G_-\|_{L^\infty([0,R))}}}\cdot$$

Then,

$$\lambda^*(\Omega) \geq \frac{\lambda_1(B_g^{m-1}(R))}{e^{2b\beta}} - mb^2. \tag{4·3}$$

Here $B_g^{m-1}(R)$ *is the geodesic ball of* $(m-1)$*-dimensional model space* $\mathbb{Q}_g^{m-1}$*.*

Foliating through horospheres, we can represent the hyperbolic space $\mathbb{H}^{q+1}$ as the warped product $\mathbb{R} \times_{e^y} \mathbb{R}^q$. By Corollary 13 we have the following eigenvalue estimate.

COROLLARY 14. *Let* $\varphi\colon M^m \to \mathbb{H}^{q+1}$ *be an* m*-dimensional submanifold minimally immersed into* $\mathbb{H}^{q+1}$*. Let* $\Omega \subset \varphi^{-1}((-\infty,\beta) \times_{e^y} B_{\mathbb{R}^q}(o,R))$ *be a connected component. Then*

$$\lambda^*(\Omega) \geq \frac{\lambda_1(B_{\mathbb{R}^{m-1}}(o,R))}{e^{2\beta}} - m = e^{-2\beta}\left(\frac{c_{m-1}}{R}\right)^2 - m, \tag{4·4}$$

where c_{m-1} *is the first zero of the* $J_{(m-1)/2-1}$*-Bessel function.*

4·4. *Cones*

A $(q+1)$-dimensional cone $\mathcal{C}^{q+1}(Q) \subseteq \mathbb{R}^m$ over an open subset $Q \subset \mathbb{S}^q$ can be seen as the warped product $\mathcal{C}^{q+1}(Q) = (0,+\infty) \times_f Q$ where $f(y) = y$. In order to match with Assumption 7 we shall suppose that $Q \subset B_{\mathbb{S}^q}(o,R)$ for some $R \leq \pi/2$. More generally, we can consider cones $\mathcal{C}^{q+1}(Q)$ over open subsets $Q \subset W$ of Riemannian manifolds W, with Q satisfying Assumption 7. We have the following result.

COROLLARY 15. *Let* $\varphi\colon M^m \to \mathcal{C}^{q+1}(Q)$ *be a* m*-dimensional submanifold minimally immersed into* $\mathcal{C}^{q+1}(Q)$ *with* Q *satisfying the Assumption 7. Let* $\Omega \subset \varphi^{-1}\big((0,a) \times_y$

$B_Q(o,R)\big)$ *be a connected component with*

$$R \le \frac{\pi}{2\sqrt{\|G_-\|_{L^\infty([0,R))}}}\cdot$$

Then,

$$\lambda^*(\Omega) \ge \frac{1}{a^2}\big(\lambda_1(B_g^{m-1}(R)) - m\big), \tag{4·5}$$

where $B_g^{m-1}(R)$ *is the geodesic ball of radius* R *in the model manifold* $\mathbb{Q}_g^{m-1}$.

Exploiting the warped product structure of the sphere $\mathbb{S}^{q+1} = (0,\pi)\times_{\sin y}\mathbb{S}^q$, our estimate can be applied for minimal submanifolds in sectors of $\mathbb{S}^{q+1}$ which are different from spherical cups, leading to the next

Corollary 16. *Let* $\varphi\colon M^m \to \mathbb{S}^{q+1} = (o,\pi)\times_{\sin y}\mathbb{S}^q$ *be an* m*-dimensional submanifold minimally immersed into* $\mathbb{S}^{q+1}$. *Let* $\Omega \subset \varphi^{-1}((o,r)\times_{\sin y} B_{\mathbb{S}^q}(\theta))$, $\theta < \pi/2$ *be a connected component. Then*

$$\lambda^*(\Omega) \ge \begin{cases} \dfrac{\lambda_1(B_{\mathbb{S}^{m-1}}(\theta)) - m}{(\sin r)^2} & \text{if } r \le \pi/2, \\[2ex] \lambda_1(B_{\mathbb{S}^{m-1}}(\theta)) - m & \text{if } r \ge \pi/2. \end{cases} \tag{4·6}$$

These estimates are effective when θ is small, that is, when the minimal submanifold is contained in a small slice of the sphere. In particular, in this case they improve on the estimates in Corollary 10 when both are applicable.

4·5. *Essential spectrum*

The ideas developed above can be applied to study the essential spectrum of $-\Delta$ of submanifolds properly immersed into the hyperbolic spaces with fairly weak bounds on the mean curvature vector. Via Persson formula ([**26**] and [**11**, Prop. 3.2]), one can express the bottom of the essential spectrum of $-\Delta$ as follows: for every exhaustion of M by relatively compact open sets $\{K_j\}$ with Lipschitz boundary,

$$\inf \sigma_{\mathrm{ess}}(-\Delta) = \lim_{j\to+\infty} \lambda^*(M\backslash K_j). \tag{4·7}$$

It therefore follows that $-\Delta$ has purely discrete spectrum if and only if

$$\lim_{j\to+\infty} \lambda^*(M\backslash K_j) = \infty.$$

Our next application regards the essential spectrum of graph hypersurfaces of $\mathbb{H}^{q+1}$ whose boundary lies in a relatively compact region of $\mathbb{H}^q_\infty$, the boundary at infinity of $\mathbb{H}^{q+1}$.

Corollary 17. *Consider the upper half-space model of the hyperbolic space* $\mathbb{H}^{q+1}$, $q \ge 2$, *with coordinates* $(x_0, x_1, \ldots, x_q) = (x_0, \bar{x})$ *and metric*

$$\langle\, ,\rangle = \frac{1}{x_0^2}\Big(\mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \ldots + \mathrm{d}x_q^2\Big),$$

and let $\mathbb{H}^q_\infty$ *be its boundary at infinity, with chart* $\bar{x}$. *Consider a hypersurface without boundary* $\varphi : M^q \to \mathbb{H}^{q+1}$ *that can be written as the graph of a function* u *over a relatively compact, open set* $W \subseteq \mathbb{H}^q_\infty$, *and denote with* $H(\bar{x})$ *its mean curvature. For* $z > 0$, *define*

$$H_z = \sup\Big\{\big|H(\bar{x})\big| \,:\, \bar{x} \in W,\, u(\bar{x}) = z\Big\}$$

If

$$\lim_{z\to 0} z^2 H_z = 0, \tag{4·8}$$

then M has purely discrete spectrum.

Proof. Setting $y = \log x_0$, we can rewrite the metric on $\mathbb{H}^{q+1}$ as the one of the warped product $\mathbb{R}\times_{e^y}\mathbb{R}^q$. In our assumptions, since M has no boundary and is a graph over W it holds $y(\varphi(x)) \to -\infty$ as x diverges in M^q. We identify the factor $\mathbb{R}^q$ in the warped product structure with $\mathbb{H}^q_\infty$ endowed with the Euclidean metric, we fix an origin $o \in \mathbb{H}^q_\infty$ and we let R be large enough that $W \subset B_{\mathbb{R}^q}(o,R)$. Let $\{z_j\} \downarrow 0^+$ be a chosen sequence, set $\beta_j = \log z_j \downarrow -\infty$ and define

$$K_j = \varphi^{-1}\big((\beta_j, +\infty)\times W\big), \qquad \Omega_j = M\backslash K_j.$$

In our assumptions, K_j is relatively compact for every j and $\{K_j\}$ is a smooth exhaustion of M. Consider a positive first eigenfunction v of the geodesic ball $B_{\mathbb{R}^{q-1}}(o,2R)$, with the normalization $\|v\|_{L^\infty} = 1$. Define $F\colon (-\infty,\beta_j)\times_{e^y} B_{\mathbb{R}^q}(o,2R) \to \mathbb{R}$ as

$$F(y,p) = e^y\cdot h(p),$$

where $h(p) = v(\rho_{_{\mathbb{R}^q}}(p))$. By Theorem 2 and formula (2·2),

$$\begin{aligned}\lambda^*(M\setminus K_j) &\geq \inf_{M\backslash K_j} \frac{-\Delta(F\circ\varphi)}{F\circ\varphi}\\ &= \inf_{M\backslash K_j} -\frac{1}{F\circ\varphi}\left[\sum_{i=1}^q \mathrm{Hess}_{\,\mathbb{H}^{q+1}} F(\varphi(x))\,(e_i,e_i) + q\langle\nabla F, H\rangle\right].\end{aligned}$$

The proof of Theorem 8, in particular inequality (3·12), shows that, for $x\in M\backslash K_j$,

$$-\frac{1}{F\circ\varphi}\sum_{i=1}^q \mathrm{Hess}_{\,\mathbb{H}^{q+1}} F(\varphi(x))\,(e_i,e_i) \geq \frac{\lambda_1(B_{\mathbb{R}^{q-1}}(o,2R))}{e^{2y(x)}} - q,$$

therefore, on $M\setminus K_j$,

$$-\frac{\Delta(F\circ\varphi)}{F\circ\varphi}(x) \geq \frac{\lambda_1(B_{\mathbb{R}^{q-1}}(o,2R))}{e^{2y(x)}} - q - q\,|H|\,\frac{|\nabla F|}{F}(\varphi(x)).$$

On the other hand, $\nabla F = F\,\nabla y + e^y\nabla h$ and thus $|\nabla F|/F \leq 1 + |\nabla h|/h$. Since $1\geq h > 0$ on $\overline{B_{\mathbb{R}^{q-1}}(o,R)}$, we infer that

$$\sup_{B_{\mathbb{R}^{q-1}}(o,R)} \frac{|\nabla F|}{F} \leq C(R),$$

where

$$C(R) = 1 + \sup_{B_{\mathbb{R}^{q-1}}(o,R)} \frac{|\nabla h|}{h} > 0.$$

From the above, we have

$$\lambda^*(M\setminus K_j) \geq \inf_{M\backslash K_j}\left[\frac{\lambda_1(B_{\mathbb{R}^{q-1}}(o,2R)) - qC(R)|H(x)|e^{2y(x)} - qe^{2y(x)}}{e^{2y(x)}}\right]. \tag{4·9}$$

In our assumptions, on $M\backslash K_j$,

$$|H(x)|e^{2y(x)} \leq H_{x_0(x)}e^{2y(x)} = H_{x_0(x)}\big[x_0(x)\big]^2.$$

By (4·8), this latter goes to zero uniformly for $x \in M\backslash K_j$ and divergent j. In particular, for each fixed $\varepsilon > 0$, there exists j_ε large such that, for $j \geq j_\varepsilon$, $|H(x)|e^{2y(x)} \leq \varepsilon$ on $M\backslash K_j$. It therefore follows that, for j large enough,

$$\lambda^*(M \setminus K_j) \geq \inf_{M\backslash K_j} \left[\frac{\lambda_1(B_{\mathbb{R}^{q-1}}(o, 2R)) - qC(R)\varepsilon - qx_0(x)^2}{x_0(x)^2}\right].$$

Choosing ε sufficiently small, letting $j \to +\infty$ and using that $x_0(x)^2 \leq e^{2\beta_j} \to 0^+$ for $x \in M\backslash K_j$ and divergent j, we deduce that $\lambda^*(M\backslash K_j) \to +\infty$, and the claim follows by Persson formula. □

To conclude, we consider the essential spectrum of submanifolds satisfying some strong non-properness assumption. This includes submanifolds with bounded image immersed in a complete manifold. We begin with recalling the following

Definition 18. *Let M, W be Riemannian manifolds and let $\varphi\colon M \to W$ be an isometric immersion. The limit set of φ, denoted by $\lim \varphi$, is a closed set defined as follows*

$$\lim \varphi = \big\{p \in W;\ \exists \{p_k\} \subset M,\ \mathrm{dist}_M(o, p_k) \to \infty \text{ and } \mathrm{dist}_W(p, \varphi(p_k)) \to 0\big\}.$$

Observe that:

- An isometric immersion $\varphi\colon M \to W$ is proper if and only if $\lim \varphi = \emptyset$.
- The closure of the set $\varphi^{-1}[W \setminus T_\epsilon(\lim \varphi)]$ may not be a compact subset of M. Here $T_\epsilon(\lim \varphi) = \{y \in W\colon \mathrm{dist}_W(y, \lim \varphi) < \epsilon\}$ is the ϵ-tubular neighborhood of $\lim \varphi$.

Definition 19. *An isometric immersion $\varphi\colon M \to W$ is strongly non-proper if for all $\epsilon > 0$ the closed subset $\varphi^{-1}(W \setminus T_\epsilon \lim \varphi)$ is compact in M.*

Remark 20. A strongly non-proper immersions is not necessarily bounded: for example, the graph immersion $\varphi : B_1(0)\backslash\{0\} \subset \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}$ given by

$$\varphi(x) = (w, z) = \left(x, \frac{1 - r(x)}{r(x)} \sin\big(r(x)(1 - r(x))\big)\right)$$

is strongly non-proper, and $\lim \varphi = \{w = 0\} \cup \{r(w) = 1,\ z = 0\}$.

Corollary 21. *Let $\varphi\colon M^m \to N^n \times_f Q^q$ be a strongly non-proper minimal submanifold. Suppose that Q satisfies Assumption 7. Assume in addition that the warping function f satisfies $\inf_N f > c_1 > 0$, $\sup_N |\nabla f| \leq c_2 < \infty$ and*

$$\mathrm{Hess}_N f(\cdot, \cdot) - \frac{\left|\nabla^N f\right|_N^2}{f} \langle , \rangle_N \leq 0.$$

Then, if $\lim \varphi \subset N \times_f \{o\}$, the spectrum of M is discrete.

Proof. Let $T_j(N) = N \times_f B_Q(o, 1/j)$, for j large enough that $B_Q(o, 2/j) \Subset M$ is a regular, convex ball. Let $K_j = \varphi^{-1}[(N \times_f Q) \setminus T_j(N)]$ be an exhaustion of M by relatively compact, open sets. Note that $\varphi(M \setminus K_j) \subset T_j(N)$. We now proceed as in the proof of Corollary 17. Define $F = f(p)v_j(\rho_Q(q))$, where v_j is the first eigenfunction of $B_g(2/j) \subset \mathbb{Q}_g^{m-n}$, normalized according to $\|v_j\|_{L^\infty} = 1$, and note that

$$\|\nabla \log F\|_{L^\infty(T_j(N))} \leq \|\nabla \log f\| + \|\nabla \log v_j\| \leq \frac{c_2}{c_1} + \|\nabla \log v_j\|.$$

By gradient estimates (see for instance, [**23**, Thm. 6.1].)

$$\|\nabla \log v_j\|_{L^\infty(T_j)} = \left\| \frac{v_j'}{v_j} \right\|_{L^\infty([0,j])} \leq C \cdot j,$$

for some absolute constant $C > 0$, and so $\|\nabla \log F\|_{L^\infty(T_j)} \leq Cj$. Using formula (4·9) and proceeding as in the proof of Corollary 17, we have that

$$\lambda^*(M \setminus K_j) \geq \inf_{p\in N} \left(\frac{\lambda_1(B_g(2/j)) - m\,|\nabla^N f|_N^2(p)}{|f(p)|^2} \right)$$

$$- m\|H\|_{L^\infty(M)}\|\nabla \log F\|_{L^\infty(T_j)}.$$

Since

$$\frac{\lambda_1(B_g(2/j)) - m\,|\nabla^N f|_N^2(p)}{|f(p)|^2} \geq \frac{\lambda_1(B_g(2/j)) - mc_2^2}{c_1^2},$$

we deduce

$$\lambda^*(M \setminus K_j) \geq \frac{\lambda_1(B_g(2/j)) - mc_2^2}{c_1^2} - m\|H\|_{L^\infty(M)}Cj.$$

Taking into account the standard asymptotic $\lambda_1(B_g(2/j)) \sim Cj^2$, for some $C > 0$, we conclude that

$$\lim_{j\to+\infty} \lambda^*(M \setminus K_j) = +\infty,$$

and the thesis follows by Persson formula. □

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