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Remarks on the geometry of surfaces in the four-dimensional Mobius sphere

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The geometry of surfaces in the four-dimensional Möbius space

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Dedicated to the memory of a friend: Sergio Console

Abstract

We study the conformal geometry of surfaces immersed in the four-dimensional conformal sphere Q_4, viewed as a homogeneous space under the action of the Möbius group. We introduce the classes of $\pm$ isotropic surfaces and characterize them as those whose conformal Gauss map is antiholomorphic or holomorphic. We then relate these surfaces to Willmore surfaces and prove some interesting vanishing results and some bounds on the Euler characteristic of the surfaces. Finally, we characterize $-$ isotropic surfaces through an Enneper-Weierstrass-type parametrization.

Mathematical subject classification: 53A30, 14M15, 32L05

1 Introduction

In recent years, the study of the geometry of submanifolds of the conformal sphere has considerably flourished. The interest in the subject has various motivations spanning from it being a natural extension of the theory of curves and surfaces in the Euclidean space, to its connections with the theory of integrable systems and general relativity. In particular, the theory of Willmore surfaces has seen a great development in many directions. Among the numerous books and papers on this subject, [6] is undoubtedly worth mentioning, and we refer the reader to the references therein for a complete and updated bibliography on the subject.
Of all the different possible approaches that have been employed to deal with these topics, we chose Cartan's method of the moving frame because of its flexibility and intuitiveness and because, when dealing with homogeneous spaces, it seems to us to be the fittest.
This paper studies the geometry of surfaces in the conformal 4-sphere Q_4 and it is organized as follows. After a basic introduction on the generalities of the frame reduction procedure, needed to fix the notation, in Section 3 we introduce the conformal Grassmannian of 2-planes in $\mathbb{R}^6$ and its Kahler-Lorentzian structure. We also provide a holomorphic embedding of this

Grassmann manifold into a quadric in the complex projective space.
In Section 4 we define the conformal Gauss map of a surface in Q_4 and, inspired by [13], we identify a special class of Willmore surfaces, called isotropic surfaces, that we characterize as those surfaces whose conformal Gauss map is holomorphic or antiholomorphic (in what follows, for the sake brevity, we will write "$-$ holomorphic" to mean antiholomorphic and "$+$ holomorphic" instead of holomorphic). This result is stated in Theorem 4.4 and mirrors the well known characterization of Willmore surfaces as those with harmonic conformal Gauss map. This and other concepts and results studied here have been introduced in the study of minimal surfaces in the Riemannian 4-sphere and even in oriented Riemannian 4-manifolds. An interesting paper in this direction, besides the aforementioned [13], is [4].
We then employ some classical techniques such as Cauchy-Riemann inequalities and Carleman-type estimates that, combined with classical index theorems for vector fields and, more generally, for sections of suitable vector bundles, allow us to deduce an upper bound on the Euler characteristic of a compact, non isotropic surface. This result is stated in Theorem 4.8.
In Section 5 we consider the notion of S-Willmore surface, first introduced by Ejiri in [9]. There, the author proved that, in the riemannian setting an S-Willmore surface is a Willmore surface; this holds true also in our setting, as proved in Proposition 5.2. We also prove some vanishing and holomorphicity results that have nice topological consequences.
In the last part of the paper, we show that, roughly speaking, $-$ isotropic surfaces in the conformal 4-sphere are characterized by their conformal Gauss map: in Theorem 6.1 and Theorem 6.4 we establish a bijection between certain $-$ isotropic, weakly conformal branched immersions of a fixed Riemann surface in Q_4 and holomorphic maps, valued in the conformal Grassmannian, that are solutions of a suitable Pfaffian system.

2 The conformal sphere and its submanifolds

Consider $\mathbb{S}^n$ and $\mathbb{R}^n$ with their standard metrics of constant curvatures, and let $\sigma : \mathbb{S}^n \backslash \{N\} \to \mathbb{R}^n$ be the stereographic projection, where $N = (1, 0, \dots, 0) \in \mathbb{R}^{n+1}$ is the north pole. It is well known that σ is a conformal diffeomorphism. If $n \geq 3$, by Liouville's theorem ([8], pp.138-141; [12], pp.52-53, [18], pp. 289-290), every conformal diffeomorphism of $\mathbb{S}^n$ is of the form $\sigma^{-1} \circ g \circ \sigma$, where g is a composition of Euclidean similarities of $\mathbb{R}^n$ with possibly the inversion $\mathbb{R}^n \backslash \{0\} \ni x \mapsto x/|x|^2$. The assertion holds even for $n = 2$, although a proof of this fact relies, for instance, on standard compact Riemann surfaces theory since Liouville's theorem is false for $\mathbb{C}$. We observe that the group of conformal diffeomorphisms of the sphere, $\mathrm{Conf}(\mathbb{S}^2)$, can also be identified with the fractional linear transformations of $\mathbb{C}$, either holomorphic or anti-holomorphic. From now on, we let $n \geq 2$ and we fix the index convention $1 \leq A, B, C \leq n$. We denote by Q_n the Darboux hyperquadric

$$Q_n = \left\{ (x^0 : x^A : x^{n+1}) \mid \sum_A (x^A)^2 - 2x^0 x^{n+1} = 0 \right\} \subset \mathbb{P}^{n+1}(\mathbb{R}).$$

The Dirac-Weyl embedding $\chi : \mathbb{R}^n \to Q_n$ is defined by

$$\chi : x \longmapsto \left(1 : x : \frac{1}{2}|x|^2 \right)$$

and it extends to a diffeomorphism $\chi \circ \sigma : \mathbb{S}^n \to Q_n$ by setting $\chi \circ \sigma(N) = (0 : 0 : 1)$. The advantage of such a representation for the sphere is that every conformal diffeomorphism of $\mathbb{S}^n$ acts as a linear transformation on the homogeneous coordinates of Q_n, so that $\mathrm{Conf}(\mathbb{S}^n)$ can be viewed as the projectivized of the linear subgroup of $GL(n+2)$ preserving the quadratic form that defines the Darboux hyperquadric.
Endow $\mathbb{R}^{n+2}$ with the Lorentzian metric $\langle \, , \rangle$ represented, with respect to the standard basis $\{\eta_0, \eta_A, \eta_{n+1}\}$, by the matrix

$$S = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_n & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tag{1}$$

and let L^+ be the positive light cone, that is, $L^+ = \{v = {}^t(v^0, v^A, v^{n+1}) \in \mathbb{R}^{n+2} : {}^t v S v = 0, \; v^0 + v^{n+1} > 0\}$. Note that L^+ projectivizes to Q_n and that $\eta_0, \eta_{n+1} \in L^+$. Moreover, there is a bijection between $\mathrm{Conf}(\mathbb{S}^n)$ and the Lorentz group of $\langle \, , \rangle$ preserving the positive light cone (usually called the orthochronous Lorentz group). This gives a Lie group structure to the conformal group $\mathrm{Conf}(\mathbb{S}^n)$, which can be proved to be unique when the action of $\mathrm{Conf}(\mathbb{S}^n)$ on $\mathbb{S}^n$ is required to be smooth (see [14], pp. 95-98). In particular, the identity component of the Lorentz group is called the **Möbius group**, $\mathrm{Möb}(n)$, and coincides with the subgroup of the orientation preserving elements of $\mathrm{Conf}(\mathbb{S}^n)$. The transitivity of the action of $\mathrm{Möb}(n)$ on the n-sphere gives Q_n a homogeneous space structure, allowing us to identify it with the space of left cosets $\mathrm{Möb}(n)/\mathrm{Möb}(n)_0$, where $\mathrm{Möb}(n)_0$ is the isotropy subgroup of $[\eta_0] \in Q_n$:

$$\mathrm{Möb}(n)_0 = \left\{ \begin{pmatrix} r^{-1} & {}^t x A & \frac{1}{2} r |x|^2 \\ 0 & A & rx \\ 0 & 0 & r \end{pmatrix} \middle| \begin{array}{l} r > 0, \; x \in \mathbb{R}^n, \\ A \in SO(n) \end{array} \right\}. \tag{2}$$

It follows that the principal bundle projection $\pi : \mathrm{Möb}(n) \to Q_n$ associates to a matrix $G = (g_0 | g_A | g_{n+1})$ the point $[G\eta_0] = [g_0] \in Q_n$. From now on, we shall use the Einstein summation convention. Let $\mathfrak{m\ddot{o}b}(n)$ denote the Lie algebra of Möb(n); the Maurer-Cartan form Φ of Möb(n) is the $\mathfrak{m\ddot{o}b}(n)$-valued 1-form

$$\Phi = \begin{pmatrix} \Phi^0_0 & \Phi^0_B & 0 \\ \Phi^A_0 & \Phi^A_B & \Phi^A_{n+1} \\ 0 & \Phi^{n+1}_B & \Phi^{n+1}_{n+1} \end{pmatrix},$$

with the symmetry relations

$$\Phi^{n+1}_{n+1} = -\Phi^0_0, \quad \Phi^A_B = -\Phi^B_A, \quad \Phi^A_{n+1} = \Phi^0_A, \quad \Phi^{n+1}_B = \Phi^B_0$$

and satisfying the structure equation $d\Phi + \Phi \wedge \Phi = 0$, which component-wise reads

$$\begin{cases} d\Phi^0_0 &= -\Phi^0_A \wedge \Phi^A_0; \\ d\Phi^A_0 &= -\Phi^A_0 \wedge \Phi^0_0 - \Phi^A_B \wedge \Phi^B_0; \\ d\Phi^0_A &= -\Phi^0_0 \wedge \Phi^0_A - \Phi^0_B \wedge \Phi^B_A; \\ d\Phi^A_B &= -\Phi^A_0 \wedge \Phi^0_B - \Phi^A_C \wedge \Phi^C_B - \Phi^0_A \wedge \Phi^B_0. \end{cases} \tag{3}$$

Through a local section $s : U \subset Q_n \to$ Möb(n), Φ pulls back to a flat Cartan connection $\psi = s^*\Phi = s^{-1}ds$. In particular, the set $\{\psi^A_0\}$ gives a local basis for the cotangent bundle of Q_n. Under a change of section $\widetilde{s} = sK$, where $K : U \subset Q_n \to$ Möb$(n)_0$, the change of gauge becomes

$$\widetilde{\psi} = \widetilde{s}^{-1} d\widetilde{s} = K^{-1}\psi K + K^{-1}dK. \tag{4}$$

By the expression of Möb$(n)_0$ in (2), we have in particular

$$(\widetilde{\psi}^A_0) = r^{-1}\, {}^tA(\psi^A_0), \tag{5}$$

where (ψ^A_0) stands for the column vector whose A-th component is ψ^A_0. It follows that

$$\widetilde{\psi}^A_0 \otimes \widetilde{\psi}^A_0 = r^{-2}\psi^A_0 \otimes \psi^A_0, \qquad \widetilde{\psi}^1_0 \wedge \ldots \wedge \widetilde{\psi}^n_0 = r^{-n}\psi^1_0 \wedge \ldots \wedge \psi^n_0,$$

which implies that

$$\Big\{ \big(U, \psi^A_0 \otimes \psi^A_0\big) \ : \ U \subset Q_n \text{ domain of a local section } s : U \to \text{Möb}(n) \Big\}$$

defines a conformal structure on Q_n, that is, a collection of locally defined metrics varying conformally on the intersection of their domains of definition, together with an orientation (locally defined by $\psi^1_0 \wedge \ldots \wedge \psi^n_0$), both preserved by Möb(n). It is easy to prove that, with this conformal structure, $\chi \circ \sigma : \mathbb{S}^n \to Q_n$ is a conformal diffeomorphism. This gives sense to the whole construction.

Let now M be an m-dimensional, oriented manifold. We fix the index ranges

$$1 \leq i, j, \ldots \leq m, \qquad m+1 \leq \alpha, \beta, \ldots \leq n.$$

Let $f : M \to Q_n$ be an immersion. A **zeroth order frame field along** f is a smooth map e defined on an open set $U \subseteq M$ with values in Möb(n) such that $\pi \circ e = f_{|U}$. From now on, dealing with frames along f, we will omit specifying their domains of definition when no possible confusion will arise. We set

$$\phi = e^* \Phi$$

and observe that under a change of frames $\widetilde{e} = eK$, $\widetilde{\phi} = \widetilde{e}^* \Phi$ expresses in terms of ϕ as in (4). As a consequence, at any point $p \in M$ we can choose a zeroth order frame such that

$$\phi_0^\alpha = 0. \tag{6}$$

The isotropy subgroup at this point is given by

$$\text{Möb}(n)_1 = \left\{ \left(\begin{array}{cccc} r^{-1} & {}^t x A & {}^t y B & \frac{1}{2} r\big(|x|^2 + |y|^2\big) \\ 0 & A & 0 & rx \\ 0 & 0 & B & ry \\ 0 & 0 & 0 & r \end{array} \right) \middle| \begin{array}{l} r \in \mathbb{R}^+, A \in SO(m), \\ B \in SO(n-m), \\ x \in \mathbb{R}^m, y \in \mathbb{R}^{n-m} \end{array} \right\}. \tag{7}$$

and since it is independent of p, smooth zeroth order frame fields such that (6) holds can be chosen in an appropriate neighborhood of each point of M by general theory, see [18].
A zeroth order frame field e such that (6) holds on its domain of definition is called **first order frame**. Any two such frame fields are related by $\widetilde{e} = eK$, where now K takes values in Möb$(n)_1$.
It can be easily verified that, with respect to first order frames, the quadratic form $ds^2 = \sum_i \phi_0^i \otimes \phi_0^i$ and the volume form $dV = \phi_0^1 \wedge \ldots \wedge \phi_0^m$ define a conformal structure on M and, with respect to these natural structures, f becomes a conformal immersion.
Differentiating (6) and using the structure equations of Möb(n) and Cartan's lemma, we find that there exist (locally defined) functions h_{ij}^α such that

$$\phi_i^\alpha = h_{ij}^\alpha \phi_0^j, \qquad h_{ij}^\alpha = h_{ji}^\alpha. \tag{8}$$

We use (4) and (7) to obtain that, under a change of first order frame fields

$$\widetilde{h}_{ij}^\alpha = r B_\alpha^\beta A_j^l (A_i^k h_{lk}^\beta - A_i^l y^\beta). \tag{9}$$

Taking the trace of (9) with respect to i and j we obtain

$$\widetilde{h}_{ii}^\alpha = r B_\alpha^\beta (h_{kk}^\beta - m y^\beta).$$

The next step is therefore to consider at any point $p \in M$ a first order frame such that

$$h^\alpha_{kk} = 0. \tag{10}$$

The isotropy subgroup is given by

$$\text{Möb}(n)_D = \left\{ \begin{pmatrix} r^{-1} & {}^t x A & 0 & \frac{1}{2} r|x|^2 \\ 0 & A & 0 & rx \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & r \end{pmatrix} \middle| \begin{array}{l} A \in SO(m), \\ B \in SO(n-m), \\ r \in \mathbb{R}^+, x \in \mathbb{R}^m \end{array} \right\}, \tag{11}$$

which is again independent of the point p considered, so that first order frames with the above property can be smoothly chosen in an appropriate neighborhood of any point. We define a **Darboux frame field along** f as a first order frame field for which (10) holds.
Any two Darboux frame fields are related again by $\widetilde{e} = eK$ where now K is a smooth function taking values in $\text{Möb}(n)_D$.

We observe that for Darboux frames (9) becomes

$$\widetilde{h}^\alpha_{ij} = rB^\beta_\alpha A^l_j A^k_i h^\beta_{kl}. \tag{12}$$

For further details on the generality of the frame reduction procedure, we refer the reader to [17], [19], [18].
Differentiating (8), using the structure equations and Cartan's lemma, with respect to a Darboux frame e we have

$$dh^\alpha_{ij} - h^\alpha_{ik}\phi^k_j - h^\alpha_{kj}\phi^k_i + h^\beta_{ij}\phi^\alpha_\beta + h^\alpha_{ij}\phi^0_0 + \delta_{ij}\phi^0_\alpha = h^\alpha_{ijk}\phi^k_0, \tag{13}$$

for some (locally defined) functions h^α_{ijk} symmetric in the lower indices.
Taking the trace of (13) with respect to i and j and using (10) we obtain

$$\phi^0_\alpha = p^\alpha_k \phi^k_0 \tag{14}$$

where we have set

$$p^\alpha_k = \frac{1}{m} h^\alpha_{iik}. \tag{15}$$

We say that a point $p \in M$ is an **umbilical point** if and only if for some (hence any) Darboux frame

$$h^\alpha_{ij} = 0 \qquad \text{at } p.$$

A totally umbilical submanifold is actually an m-dimensional sphere, as stated in the following proposition.

Proposition 2.1. *Let $f : M \to Q_n$ be an immersion, M oriented, $m = \dim M \geq 2$, for which $h^\alpha_{ij} \equiv 0$ at every point. Then, there exists $Q_m \subset Q_n$ such that $f(M) \subseteq Q_m$. Furthermore, if M is compact, f is a diffeomorphism onto Q_m.*

The proof relies on a standard technique in the method of the moving frame and therefore we omit it.
The form (11) of the isotropy subgroup $\text{Möb}(n)_D$ of Darboux frames along f suggests the following considerations: let us consider the matrix of 1-forms Ψ defined by

$$\Psi = \begin{pmatrix} \phi_0^0 & \phi_i^0 & 0 \\ \phi_0^i & \phi_j^i & \phi_i^0 \\ 0 & \phi_0^i & -\phi_0^0 \end{pmatrix}. \tag{16}$$

We can clearly think of Ψ as taking values in the Lie algebra of $\text{Möb}(m)$.
Under a change of Darboux frames $\widetilde{e} = eK$, where K takes values in $\text{Möb}(n)_D$, we have

$$\widetilde{\Psi} = \bar{K}^{-1}\Psi\bar{K} + \bar{K}^{-1}d\bar{K},$$

with

$$\bar{K} = \begin{pmatrix} r^{-1} & {}^t xA & \frac{r}{2}|x|^2 \\ 0 & A & rx \\ 0 & 0 & r \end{pmatrix},$$

$x \in \mathbb{R}^m$, $A \in SO(m)$, $r \in \mathbb{R}^+$.
We therefore conclude that Ψ defines a Cartan connection on M.
Moreover, we can define a suitable vector bundle N over M whose role should parallel that of the normal bundle of an isometric immersion into a Riemannian manifold. In order to do this, with respect to any Darboux frame, we define the fiber N_p to be the $(n-m)$-dimensional vector space generated by $\{e_\alpha\}$. Because of (11), it is trivial to see that the bundle N is well defined and on it there is a naturally defined inner product $(\ ,\ )$ for which $\{e_\alpha\}$ is an orthonormal basis at p. With respect to this inner product we define a metric connection

$$D : \Gamma(N) \to \Gamma(T^*M \otimes N)$$

by setting

$$De_\alpha = \phi_\alpha^\beta \otimes e_\beta.$$

The curvature forms Λ_β^α are defined via the structure equations

$$d\phi_\beta^\alpha = -\phi_\gamma^\alpha \wedge \phi_\beta^\gamma + \Lambda_\beta^\alpha.$$

Using the structure equations of the group $\text{Möb}(n)$ and (8), setting

$${}^\perp\tau_{\beta ij}^\alpha = h_{ki}^\alpha h_{kj}^\beta - h_{kj}^\alpha h_{ki}^\beta, \tag{17}$$

we obtain

$$\Lambda_\beta^\alpha = \frac{1}{2}{}^\perp\tau_{\beta ij}^\alpha \phi_0^i \wedge \phi_0^j.$$

Observe that we have the symmetry relations

$${}^\perp\tau_{\beta ij}^\alpha = -{}^\perp\tau_{\beta ji}^\alpha = -{}^\perp\tau_{\alpha ij}^\beta$$

Moreover, with respect to Darboux frames $\widetilde{e}$, e

$$^{\perp}\widetilde{\tau}^{\alpha}_{\beta ij} = r^2 B^{\gamma}_{\alpha} B^{\rho}_{\beta} A^t_i A^v_j {}^{\perp}\tau^{\gamma}_{\rho tv}$$

It follows that we can define a tensor $^{\perp}\tau$ by locally setting

$$^{\perp}\tau = {}^{\perp}\tau^{\alpha}_{\beta ij}\phi^i_0 \otimes \phi^j_0 \otimes e_\alpha \otimes e_\beta.$$

We will call $^{\perp}\tau$ the normal curvature tensor.

3 The conformal Grassmannian

Set $s = n - m \geq 1$ and let $\{\varepsilon_0, \dots, \varepsilon_m, \varepsilon_{m+1}, \dots, \varepsilon_n, \varepsilon_{n+1}\}$ be the standard basis of $\mathbb{R}^{n+2}$. Fix as an origin in the Grassmann manifold of oriented s-planes in $\mathbb{R}^{n+2}$, $G_s(\mathbb{R}^{n+2})$, the point $O = [\varepsilon_{m+1}, \dots, \varepsilon_n]$ and consider the orbit $\mathcal{Q}_s(\mathbb{R}^{n+2})$ of the point O under the left action (by matrix multiplication) of the group Möb(n) onto $G_s(\mathbb{R}^{n+2})$. Then the isotropy subgroup of the action on the orbit at the point O is given by

$$H_0 = \left\{ \begin{pmatrix} a & {}^t z & 0 & b \\ x & A & 0 & y \\ 0 & 0 & B & 0 \\ c & {}^t w & 0 & d \end{pmatrix} \middle| \begin{array}{c} \begin{pmatrix} a & {}^t z & b \\ x & A & y \\ c & {}^t w & d \end{pmatrix} \in \text{Möb}(m), \\ B \in SO(s) \end{array} \right\} \subseteq \text{Möb}(n). \tag{18}$$

Note that, since $H_0 \subseteq$ Möb(n), $z, w, x, y, a, b, c, d, A$ cannot be chosen arbitrarily but have to satisfy certain compatibility relations between them that will be essential in determining that certain quantities are globally well defined.

Thus $\mathcal{Q}_s(\mathbb{R}^{n+2})$ is identified with the homogeneus space Möb$(n)/H_0$ with the canonical projection

$$\widehat{\pi} : \text{Möb}(n) \to \mathcal{Q}_s(\mathbb{R}^{n+2})$$

given by

$$\widehat{\pi} : P \mapsto [P_{m+1}, \dots, P_n] \tag{19}$$

where P_0, P_A, P_{n+1} are the columns of the matrix P.

On their common domain of definition, two local sections of the bundle $\widehat{\pi} : \text{Möb}(n) \to \mathcal{Q}_s(\mathbb{R}^{n+2})$ are related by $\widetilde{s} = sK$ where K is a function taking values in H_0. Considering the components Φ^0_α, Φ^i_α, Φ^α_0 of the Maurer-Cartan form of Möb(n) and setting $\varphi = s^*\Phi$, we find that their pull-backs under the sections s, $\widetilde{s}$ are related by the following transformation laws:

$$\begin{cases} \widetilde{\varphi}^0_\alpha = d\,\varphi^0_\beta B^\beta_\alpha - y^i \varphi^i_\beta B^\beta_\alpha + b\varphi^\beta_0 B^\beta_\alpha \\ \widetilde{\varphi}^i_\alpha = -w^i \varphi^0_\beta B^\beta_\alpha + A^k_i \varphi^k_\beta B^\beta_\alpha - z^i \varphi^\beta_0 B^\beta_\alpha \\ \widetilde{\varphi}^\alpha_0 = c\,\varphi^0_\beta B^\beta_\alpha - x^k \varphi^k_\beta B^\beta_\alpha + a\varphi^\beta_0 B^\beta_\alpha \end{cases} \tag{20}$$

where the meaning of $d, c, a, b, y, x, w, z, A, B$ is given in (18). From (20) and the relations defining the group Möb(n), it is not hard to deduce that the quadratic form dl^2 of signature $(s, s(m+1))$ given by

$$dl^2 = -\varphi^0_\alpha \otimes \varphi^\alpha_0 - \varphi^\alpha_0 \otimes \varphi^0_\alpha + \sum_{i,\alpha} \varphi^i_\alpha \otimes \varphi^i_\alpha \tag{21}$$

is well defined on $\mathcal{Q}_s\big(\mathbb{R}^{n+2}\big)$ and determines a pseudo-metric on it. In particular the forms φ^0_α, φ^α_0, φ^i_α constitute a local (non orthonormal) coframe on $\mathcal{Q}_s\big(\mathbb{R}^{n+2}\big)$ whose dimension is $s(m+2)$. It is convenient to set

$$\theta^{0,\alpha} = \varphi^\alpha_0, \qquad \theta^{\alpha,0} = \varphi^0_\alpha, \qquad \theta^{\alpha,i} = \varphi^i_\alpha \tag{22}$$

and to order the pairs $(\alpha, 0)$, (α, i), $(0, \alpha)$ as

$$\begin{aligned}
&(\gamma, 0) < (\beta, i) < (0, \alpha) \qquad \forall \alpha, \beta, \gamma, i\\
&(0, \beta) < (0, \alpha) \qquad \text{iff } \beta < \alpha\\
&(\beta, j) < (\alpha, i) \qquad \text{iff } \beta < \alpha \text{ or } \beta = \alpha \text{ and } j < i\\
&(\beta, 0) < (\alpha, 0) \qquad \text{iff } \beta < \alpha.
\end{aligned} \tag{23}$$

Thus, representing with the symbols $\widetilde{A}, \widetilde{B}, \dots$ the $s(m+2)$ indices $(\alpha, 0)$, (α, i), $(0, \alpha)$, we can write dl^2 as

$$dl^2 = g_{\widetilde{A}\widetilde{B}} \theta^{\widetilde{A}} \otimes \theta^{\widetilde{B}} \tag{24}$$

with

$$\big(g_{\widetilde{A}\widetilde{B}}\big) = \begin{pmatrix} 0 & 0 & -I_s \\ 0 & I_{sm} & 0 \\ -I_s & 0 & 0 \end{pmatrix} \qquad s = n - m. \tag{25}$$

The Levi-Civita connection forms $\theta^{\widetilde{A}}_{\widetilde{B}}$ with respect to the previous coframe are therefore characterized by the equations

$$\begin{cases} d\theta^{\widetilde{A}} = -\theta^{\widetilde{A}}_{\widetilde{B}} \wedge \theta^{\widetilde{B}} \\ g_{\widetilde{A}\widetilde{C}} \theta^{\widetilde{C}}_{\widetilde{B}} + g_{\widetilde{B}\widetilde{C}} \theta^{\widetilde{C}}_{\widetilde{A}} = 0. \end{cases} \tag{26}$$

This allows us to determine the connection forms by simply taking exterior derivatives of (22) and using the structure equations of the group Möb(n). We obtain

$$\begin{cases}
\theta^{\alpha,0}_{\beta,0} = \delta^\alpha_\beta \varphi^0_0 + \varphi^\alpha_\beta, & \theta^{\alpha,0}_{\beta,i} = \delta^\alpha_\beta \varphi^0_i, & \theta^{\alpha,0}_{0,\beta} = 0 \\
\theta^{\alpha,i}_{\beta,0} = \delta^\alpha_\beta \varphi^i_0, & \theta^{\alpha,i}_{\beta,k} = \delta^\alpha_\beta \varphi^i_k + \delta^i_k \varphi^\alpha_\beta, & \theta^{\alpha,i}_{0,\beta} = \delta^\alpha_\beta \varphi^0_i \\
\theta^{0,\alpha}_{\beta,0} = 0, & \theta^{0,\alpha}_{\beta,i} = \delta^\alpha_\beta \varphi^i_0, & \theta^{0,\alpha}_{0,\beta} = \varphi^\alpha_\beta - \delta^\alpha_\beta \varphi^0_0
\end{cases} \tag{27}$$

and, by a simple computation, one checks the validity of the skew-symmetry relations given by the second of (26).

It is worth considering the special case $s=2$, that is $m=n-2$. Indeed, starting from the $2n$ independent forms φ^0_α, φ^i_α, φ^α_0 we can construct the n independent forms over $\mathbb{C}$

$$\zeta^0 = \varphi^0_{n-1} + i\varphi^0_n, \qquad \zeta^k = \varphi^k_{n-1} + i\varphi^k_n, \qquad \zeta^{n-1} = \varphi^{n-1}_0 + i\varphi^n_0. \tag{28}$$

Using the structure equations, it is immediate to verify that their differentials belong to the ideal they generate, showing that $\mathcal{Q}_2\big(\mathbb{R}^{n+2}\big)$ is a complex manifold, in fact complex Lorentzian. Indeed the complex structure J induced by the forms (28) is determined by

$$\zeta^0(X+iJX) = \zeta^k(X+iJX) = \zeta^{n-1}(X+iJX) = 0 \qquad \forall X \in T\mathcal{Q}_2\big(\mathbb{R}^{n+2}\big),$$

that is

$$\varphi^0_{n-1}(X) = \varphi^0_n(JX) \qquad \varphi^k_{n-1}(X) = \varphi^k_n(JX) \qquad \varphi^{n-1}_0(X) = \varphi^n_0(JX).$$

It is therefore trivial to verify that the metric dl^2 is Hermitian-Lorentzian:

$$\begin{aligned} dl^2(JX,JY) = &- \varphi^0_{n-1}(JX)\varphi^{n-1}_0(JY) - \varphi^0_n(JX)\varphi^n_0(JY) + \\ &- \varphi^{n-1}_0(JX)\varphi^0_{n-1}(JY) - \varphi^n_0(JX)\varphi^0_n(JY) + \\ &+ \varphi^i_{n-1}(JX)\varphi^i_{n-1}(JY) + \varphi^i_n(JX)\varphi^i_n(JY) = \\ =& \, dl^2(X,Y). \end{aligned}$$

We verify that $\mathcal{Q}_2\big(\mathbb{R}^{n+2}\big)$ is Kähler by showing that the differential of the Kähler form

$$\mathcal{K}(X,Y) = dl^2(JX,Y)$$

vanishes identically. This is a simple exercise using (28) and the Maurer-Cartan structure equations. Indeed we have that

$$\begin{aligned} \mathcal{K} = &- \varphi^0_{n-1}\wedge\varphi^n_0 - \varphi^{n-1}_0\wedge\varphi^0_n + \varphi^i_{n-1}\wedge\varphi^i_n = \\ =& \frac{i}{2}\Big(-\zeta^0\wedge\overline{\zeta^{n-1}} - \zeta^{n-1}\wedge\overline{\zeta^0} + \zeta^k\wedge\overline{\zeta^k}\Big), \end{aligned} \tag{29}$$

therefore $d\mathcal{K}=0$.
Finally we describe the complex projective structure of the conformal Grassmannian.

Proposition 3.1. *There is a holomorphic embedding of the conformal Grassmannian $\mathcal{Q}_2\big(\mathbb{R}^{n+2}\big)$ into the hyperquadric of $\mathbb{P}^{n+1}_{\mathbb{C}}$ whose homogeneous equation is*

$$-2x^0x^{n+1} + \sum_{A=1}^{n}(x^A)^2 = 0. \tag{30}$$

Proof. There is a natural injection of $\mathcal{Q}_2\big(\mathbb{R}^{n+2}\big)$ in $\mathbb{P}^{n+1}_{\mathbb{C}}$ defined as follows. Let $[G\varepsilon_{n-1}, G\varepsilon_n]$, with $G \in \text{Möb}(n)$, be a 2-plane of $\mathcal{Q}_2\big(\mathbb{R}^{n+2}\big)$. The map sending $[G\varepsilon_{n-1}, G\varepsilon_n]$ to the projectivization of the complex, non-zero vector $G(\varepsilon_{n-1} + i\varepsilon_n)$ is well defined and injective, and thus provides a complex projective representation for the whole conformal Grassmannian of 2-planes in $\mathbb{R}^{n+2}$.

Indeed, let $[G\varepsilon_{n-1}, G\varepsilon_n]$ and $[G'\varepsilon_{n-1}, G'\varepsilon_n]$ be two representatives for the same 2-plane in $\mathcal{Q}_2\big(\mathbb{R}^{n+2}\big)$, then G and G' must differ by an element of the isotropy subgroup H_0, namely $G' = GH$ for some $H \in H_0$. But H has an expression as in (18), with $B \in SO(2)$, that is

$$B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

for some $\theta \in \mathbb{R}$, so we have

$$\begin{aligned} G'(\varepsilon_{n-1} + i\varepsilon_n) =& GH(\varepsilon_{n-1} + i\varepsilon_n) = \\ =& G(\cos\theta\varepsilon_{n-1} + \sin\theta\varepsilon_n - i\sin\theta\varepsilon_{n-1} + i\cos\theta\varepsilon_n) = \\ =& e^{-i\theta}G(\varepsilon_{n-1} + i\varepsilon_n) \end{aligned}$$

which projects to the same complex projective class as $G(\varepsilon_{n-1}+i\varepsilon_n)$. As for injectivity, if $G(\varepsilon_{n-1}+i\varepsilon_n)$ and $G'(\varepsilon_{n-1}+i\varepsilon_n)$ project to the same projective class, then there exists $\rho > 0$ and $\theta \in \mathbb{R}$ such that

$$G'(\varepsilon_{n-1} + i\varepsilon_n) = \rho e^{i\theta}G(\varepsilon_{n-1} + i\varepsilon_n) = \rho GH(\varepsilon_{n-1} + i\varepsilon_n),$$

where

$$H = \begin{pmatrix} I_{n-1} & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

clearly belongs to H_0. So $[G\varepsilon_{n-1}, G\varepsilon_n]$ and $[G'\varepsilon_{n-1}, G'\varepsilon_n]$ are in fact the same 2-plane in $\mathcal{Q}_2\big(\mathbb{R}^{n+2}\big)$.

We will show that, as a matter of fact, $\mathcal{Q}_2\big(\mathbb{R}^{n+2}\big)$ can be identified with an open submanifold of the projective quadric of homogeneous equation (30). As we have explained above, the image in $\mathbb{P}^{n+1}_{\mathbb{C}}$ of a 2-plane of $\mathcal{Q}_2\big(\mathbb{R}^{n+2}\big)$ is the projective class of a complex vector of the form $G(\varepsilon_{n-1} + i\varepsilon_n)$, for some $G \in \text{Möb}(n)$. Now, the vector $\varepsilon_{n-1} + i\varepsilon_n$ trivially satisfies equation (30), and therefore lies in the quadric. Note that the quadric (30) is represented by the matrix

$$S = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_n & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

introduced in (1) and, since $G \in \text{Möb}(n)$,

$$
\begin{aligned}
{}^t[G(\varepsilon_{n-1} + i\varepsilon_n)]S[G(\varepsilon_{n-1} + i\varepsilon_n)] &= {}^t(\varepsilon_{n-1} + i\varepsilon_n)^t GSG(\varepsilon_{n-1} + i\varepsilon_n) = \\
&= {}^t(\varepsilon_{n-1} + i\varepsilon_n)S(\varepsilon_{n-1} + i\varepsilon_n) = 0.
\end{aligned}
$$

Therefore $G(\varepsilon_{n-1} + i\varepsilon_n)$ lies in the quadric (30).
However, the conformal Grassmannian does not cover the whole quadric. Indeed the points of the quadric coming from a 2-plane in $\mathcal{Q}_2(\mathbb{R}^{n+2})$ are those that have a representative $v + iw \in \mathbb{C}^{n+2}$ such that, with respect to the Lorentzian product in $\mathbb{R}^{n+2}$, $\|v\|^2 = \|w\|^2 > 0$. This leaves out the projective classes represented by vectors $v + iw$ where v and w are isotropic and non zero. All such vectors lie in the quadric but cannot be obtained from ε_{n-1} or ε_n through a matrix of $\text{Möb}(n)$, because such matrices preserve the Lorentzian norm defined through the matrix S. □

4 The geometry of surfaces in Q_4

Let $f : M \to Q_4$ be an oriented immersed surface. Assume that M has been given the structure of a Riemann surface starting from an assigned metric g and assume that f is conformal in the sense that the conformal structure that it induces on M coincides with that of M as a Riemann surface.
We let $e : U \subset M \to \text{Möb}(n)$ be a local first order frame along f, so that, according to (6),

$$\phi^\alpha_0 = 0 \qquad 3 \le \alpha \le 4$$

and the isotropy subgroup is given by (7). Then

$$\phi^\alpha_i = h^\alpha_{ij}\phi^j_0, \qquad h^\alpha_{ij} = h^\alpha_{ji} \qquad 1 \le i, j \le 2 \tag{31}$$

and we have the transformation laws (9).

Starting from first order frames, we are now going to introduce a number of geometric invariants. We let L^α denote the **Hopf transform** of the symmetric matrix (h^α_{ij}), that is

$$L^\alpha = \frac{1}{2}(h^\alpha_{11} - h^\alpha_{22}) - ih^\alpha_{12}. \tag{32}$$

Setting

$$A = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

and using (9), we compute

$$\widetilde{h}^\alpha_{11} = rB^\beta_\alpha\Big(\cos^2 t\, h^\beta_{11} + 2\cos t \sin t\, h^\beta_{12} + \sin^2 t\, h^\beta_{22} - y^\beta\Big),$$

$$\widetilde{h}^\alpha_{22} = rB^\beta_\alpha\Big(\sin^2 t\, h^\beta_{11} - 2\sin t \cos t\, h^\beta_{12} + \cos^2 t\, h^\beta_{22} - y^\beta\Big),$$

$$\widetilde{h}^{\alpha}_{12} = rB^{\beta}_{\alpha}\Big(-\sin t\cos t(h^{\beta}_{11} - h^{\beta}_{22}) + (-\sin^2 t + \cos^2 t)h^{\beta}_{12}\Big).$$

From the above formulae, we can deduce the one expressing the transformation of L^{α} under a change of first order frames, that is

$$\widetilde{L}^{\alpha} = re^{2it}B^{\beta}_{\alpha}L^{\beta}. \tag{33}$$

Therefore, setting

$$B = \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix},$$

we obtain that

$$\widetilde{L}^3 \pm i\widetilde{L}^4 = re^{2it}e^{\mp is}\big(L^3 \pm iL^4\big). \tag{34}$$

Using this, we see that the real, locally defined 2-forms

$$\omega_{\pm} = \left|L^3 \pm iL^4\right|^2 \phi^1_0 \wedge \phi^2_0, \tag{35}$$

are in fact globally defined and smooth. We will say that $f : M \to Q_4$ is $+$ **or** $-$ **isotropic** respectively if $\omega_+ \equiv 0$ or $\omega_- \equiv 0$.
Note that, when f is at the same time $+$ and $-$ isotropic, then

$$h^{\alpha}_{12} = 0, \qquad h^{\alpha}_{11} = h^{\alpha}_{22}.$$

Thus, passing to a Darboux frame, $h^{\alpha}_{ij} = 0$ for every α, i, j, and $f(M) \subseteq Q_2 \subset Q_4$ according to Proposition 2.1.

We underline the fact that the forms $\omega_{\pm}$ are invariant with respect to first order frames.

It is also easy to see, using (9), that the 2-form

$$\mathrm{w} = \frac{1}{4}\left\{\sum_{\alpha}(h^{\alpha}_{11} - h^{\alpha}_{22})^2 + 4(h^{\alpha}_{12})^2\right\}\phi^1_0 \wedge \phi^2_0 = \big(|L^3|^2 + |L^4|^2\big)\phi^1_0 \wedge \phi^2_0 \tag{36}$$

is globally defined. In particular the form η is globally defined, which satisfies

$$\mathrm{w} = \omega_{\pm} \mp \eta. \tag{37}$$

We now identify η. A simple computation, using the definitions of w and $\omega_{\pm}$ yields

$$\eta = -i\Big(L^3\overline{L^4} - L^4\overline{L^3}\Big)\phi^1_0 \wedge \phi^2_0. \tag{38}$$

Expressing it in terms of the h^{α}_{ij}'s we obtain

$$-i\Big(L^3\overline{L^4} - L^4\overline{L^3}\Big) = h^3_{11}h^4_{12} - h^3_{22}h^4_{12} - h^3_{12}h^4_{11} + h^3_{12}h^4_{22}.$$

If we specialise to a Darboux frame e along f, since $h^{\alpha}_{11} + h^{\alpha}_{22} = 0$ we obtain

$$-i\Big(L^3\overline{L^4} - L^4\overline{L^3}\Big) = 2(h^3_{11}h^4_{12} - h^3_{12}h^4_{11}).$$

We go back to the normal bundle N introduced in Section 2. The curvature K_N of this bundle is now given by

$$\Lambda_4^3 = \frac{1}{2}{}^{\perp}\tau_{4ij}^3 \phi_0^i \wedge \phi_0^j = K_N \phi_0^1 \wedge \phi_0^2$$

and using (17) we deduce that

$$K_N = -i\left(L^3\overline{L}^4 - L^4\overline{L}^3\right) \tag{39}$$

or, in other words

$$d\phi_4^3 = K_N \phi_0^1 \wedge \phi_0^2 = \eta. \tag{40}$$

Using (37), (40) and the generalized Gauss-Bonnet theorem, having set

$$W(f) = \int_M \mathrm{w} \tag{41}$$

in the case of M compact, we obtain

Theorem 4.1. *Let $f : M \to Q_4$ be an immersion of a compact orientable surface; then*

$$W(f) = \int_M \omega_\pm \mp 2\pi\chi(N) \tag{42}$$

where $\chi(N)$ is the Euler number of the bundle N introduced above.

The functional $W(f)$ defined in (41) for M compact or, more generally on compact domains of M, is called the **Willmore functional**.

Corollary 4.2. *Let $f : M \to Q_4$ be an immersion of a compact orientable surface. Then*

$$\int_M \omega_\pm \geq \pm 2\pi\chi(N)$$

equality holding if and only if $f(M) = Q_2 \subset Q_4$.

Proof. An easy computation shows that

$$\mathrm{w} = \frac{1}{2}\sum_{i,j,\alpha}(h_{ij}^\alpha)^2 \phi_0^1 \wedge \phi_0^2,$$

so, clearly, $W(f) \geq 0$ and $W(f) = 0$ if and only if $f(M) = Q_2 \subset Q_4$ by Proposition 2.1. □

Remark 4.3. *Suppose that M is compact and orientable;* (42) *implies that, if M is either $+$ or $-$ isotropic, then the values of $W(f)$ are "quantized".*

Our next goal is to give a geometric interpretation to $+$ and $-$ isotropic immersions. Towards this aim we introduce the conformal Gauss map. We let $\mathcal{Q}_2\big(\mathbb{R}^6\big)$ be the conformal Grassmannian of 2-planes introduced in Section 3. As we have seen, $\mathcal{Q}_2\big(\mathbb{R}^6\big)$ has the structure of a Kähler-Lorentzian manifold with a local basis of $(1,0)$-type forms given by

$$\zeta^0 = \varsigma^*\Phi^0_3 + i\varsigma^*\Phi^0_4, \qquad \zeta^k = \varsigma^*\Phi^k_3 + i\varsigma^*\Phi^k_4, \qquad \zeta^3 = \varsigma^*\Phi^3_0 + i\varsigma^*\Phi^4_0, \tag{43}$$

where ς is any local section of $\widehat{\pi}$.
Let $f : M \to Q_4$ be an immersed oriented surface and let e be a (local) Darboux frame along f. The **conformal Gauss map** $\gamma_f : M \to \mathcal{Q}_2\big(\mathbb{R}^6\big)$ is defined by setting

$$\gamma_f : p \mapsto [e_3, e_4]_p$$

where with $[e_3, e_4]_p$ we denote the oriented 2-plane generated by the vectors e_3, e_4 at the point p.
We observe that, under a change of Darboux frames, γ_f is in fact globally well defined, and the orientation of the 2-plane $[e_3, e_4]$ is also preserved.
We set, in a Darboux frame e,

$$k^\alpha = \frac{1}{2}(p^\alpha_1 - ip^\alpha_2), \tag{44}$$

where p^α_k was defined in (15).

Theorem 4.4. *Let $f : M \to Q_4$ be an immersed oriented Riemann surface. Then f is $\pm$ isotropic if and only if $\gamma_f : M \to \mathcal{Q}_2\big(\mathbb{R}^6\big)$ is $\mp$ holomorphic.*

Proof. We recall that, given a Riemann surface M, a map $f : M \to \mathcal{Q}_2\big(\mathbb{R}^6\big)$ is respectively $\pm$ holomorphic (that is, holomorphic or antiholomorphic) if the pull-back of the forms ζ^0, ζ^k, ζ^3 in (43) is respectively of type $(1,0)$ or $(0,1)$.
We begin by observing that if e is any Darboux frame along f, then the following diagram is commutative.

$$\begin{tikzcd}
 & \text{Möb}(4) \arrow[dl, "\widehat{\pi}"'] \arrow[dr, "\pi"] & \\
\mathcal{Q}_2\big(\mathbb{R}^6\big) & & Q_4 \\
 & M \arrow[ul, "\gamma_f"] \arrow[uu, "e"] \arrow[ur, "f"'] &
\end{tikzcd}$$

This fact enables us to compute in a simple way $\gamma_f^*\zeta^0$, $\gamma_f^*\zeta^k$, $\gamma_f^*\zeta^3$. Indeed, setting

$$\theta^{0,\alpha} = \varsigma^*\Phi^\alpha_0, \qquad \theta^{\alpha,0} = \varsigma^*\Phi^0_\alpha, \qquad \theta^{\alpha,i} = \varsigma^*\Phi^i_\alpha \tag{45}$$

and using (15), (44) and (31) we have:

$$\begin{cases} \gamma_f^* \theta^{\alpha,0} = p_k^\alpha \phi_0^k \\ \gamma_f^* \theta^{\alpha,i} = -h_{ik}^\alpha \phi_0^k \\ \gamma_f^* \theta^{0,\alpha} = 0. \end{cases} \tag{46}$$

In order to see this, we observe that

$$\gamma_f^* \varsigma^* \Phi = (\widehat{\pi} \circ e)^* \varsigma^* \Phi = e^* (\varsigma \circ \widehat{\pi})^* \Phi.$$

And since $\widehat{\pi} \circ (\varsigma \circ \widehat{\pi}) = \widehat{\pi}$, then for every g in the inverse image through $\widehat{\pi}$ of the domain of definition of ς, it holds

$$\varsigma(\widehat{\pi}(g)) = g\widetilde{K}(g),$$

where $\widetilde{K}$ is an H_0-valued function. Therefore

$$(\varsigma \circ \widehat{\pi})^* \Phi_g = \widetilde{K}(g)^{-1} g^{-1} dg \widetilde{K}(g) + \widetilde{K}(g)^{-1} d\widetilde{K}_g,$$

and since $\widetilde{K}(g)^{-1} d\widetilde{K}_g$ has values in the Lie algebra of H_0, we deduce that

$$\begin{aligned} (\varsigma \circ \widehat{\pi})^* \Phi^\alpha_{0\,g_0} &= \Big(\widetilde{K}(g_0)^{-1} g_0^{-1} dg_{g_0} \widetilde{K}(g_0)\Big)^\alpha_0 \\ (\varsigma \circ \widehat{\pi})^* \Phi^0_{\alpha g_0} &= \Big(\widetilde{K}(g_0)^{-1} g_0^{-1} dg_{g_0} \widetilde{K}(g_0)\Big)^0_\alpha \\ (\varsigma \circ \widehat{\pi})^* \Phi^\alpha_{i\,g_0} &= \Big(\widetilde{K}(g_0)^{-1} g_0^{-1} dg_{g_0} \widetilde{K}(g_0)\Big)^\alpha_i. \end{aligned}$$

If for a fixed $\widetilde{g}$ we replace the section ς with the section $\varsigma\widetilde{K}(\widetilde{g})^{-1}$ obtained multiplying ς by a constant matrix, we will have defined a new section $\widetilde{\varsigma}$ which satisfies, at the point $\widetilde{g}$ (and in general only there), the equality $\widetilde{\varsigma}(\widehat{\pi}(\widetilde{g})) = \widetilde{g}$, and therefore

$$\begin{aligned} (\widetilde{\varsigma} \circ \widehat{\pi})^* \Phi^\alpha_{0\,\widetilde{g}} &= \big(\widetilde{g}^{-1} dg_{\widetilde{g}}\big)^\alpha_0 = \Phi^\alpha_{0\,\widetilde{g}} \\ (\widetilde{\varsigma} \circ \widehat{\pi})^* \Phi^0_{\alpha\widetilde{g}} &= \big(\widetilde{g}^{-1} dg_{\widetilde{g}}\big)^0_\alpha = \Phi^0_{\alpha\widetilde{g}} \\ (\widetilde{\varsigma} \circ \widehat{\pi})^* \Phi^\alpha_{i\,\widetilde{g}} &= \big(\widetilde{g}^{-1} dg_{\widetilde{g}}\big)^\alpha_i = \Phi^\alpha_{i\,\widetilde{g}}. \end{aligned}$$

Now let us fix $p_0 \in M$ and set $\widetilde{g} = e(p_0)$. Given a section ς defined in a neighborhood of $\gamma_f(p_0)$, and possibly replacing it with the section $\varsigma\widetilde{K}(e(p_0))^{-1}$, which we shall still call ς, we have at the point p_0

$$\varsigma(\widehat{\pi}(e(p_0))) = e(p_0),$$

and thus

$$\begin{aligned} \big(\gamma_f^* \varsigma^* \Phi^\alpha_0\big)_{p_0} &= (e^*(\varsigma \circ \widehat{\pi})^* \Phi^\alpha_0)_{p_0} = (e^* \Phi^\alpha_0)_{p_0} = \phi^\alpha_{0\,p_0} = 0 \\ \big(\gamma_f^* \varsigma^* \Phi^0_\alpha\big)_{p_0} &= \phi^0_{\alpha p_0} = p^\alpha_k(p_0) \phi^k_{0 p_0} \\ \big(\gamma_f^* \varsigma^* \Phi^\alpha_i\big)_{p_0} &= \phi^\alpha_{i\,p_0} = h^\alpha_{ik}(p_0) \phi^k_{0 p_0}. \end{aligned}$$

Hence, setting $\varphi = \phi_0^1 + i\phi_0^2$ and observing that, if α_k, β_k are real-valued functions, one has

$$(\alpha_k + i\beta_k)\phi_0^k = \left\{\frac{\alpha_1+\beta_2}{2} + i\frac{\beta_1-\alpha_2}{2}\right\}\varphi + \left\{\frac{\alpha_1-\beta_2}{2} + i\frac{\beta_1+\alpha_2}{2}\right\}\overline{\varphi}, \quad (47)$$

we get, with the aid of (32), at the point p_0,

$$\begin{aligned}
\gamma_f^*\zeta^0 &= \big(k^3+ik^4\big)\varphi + \Big(\overline{k^3-ik^4}\Big)\overline{\varphi}\\
\gamma_f^*\zeta^1 &= -\frac{1}{2}\big(L^3+iL^4\big)\varphi - \frac{1}{2}\Big(\overline{L^3-iL^4}\Big)\overline{\varphi}\\
\gamma_f^*\zeta^2 &= -\frac{i}{2}\big(L^3+iL^4\big)\varphi + \frac{i}{2}\Big(\overline{L^3-iL^4}\Big)\overline{\varphi}\\
\gamma_f^*\zeta^3 &= 0.
\end{aligned}$$

It is therefore clear, using (35), that if γ_f is $\mp$ holomorphic, then f is $\pm$ isotropic. To prove the converse, we need to show that

$$L^3 \pm iL^4 = 0 \qquad \text{implies} \qquad k^3 \pm ik^4 = 0. \quad (48)$$

Towards this aim we differentiate the first of (48) and we use (13) to perform the computations. Note that, since we are using Darboux frames along f,

$$L^\alpha = h_{11}^\alpha - ih_{12}^\alpha$$

and we can compute

$$\begin{aligned}
d\big(L^3 \pm iL^4\big) =&(h_{11k}^3 - p_k^3 \pm h_{12k}^4)\phi_0^k \pm i(h_{11k}^4 - p_k^4 \mp h_{12k}^3)\phi_0^k +\\
&+ \big(L^3 \pm iL^4\big)\big[i(2\phi_1^2 \mp \phi_3^4) - \phi_0^0\big],
\end{aligned}$$

Now, using (47), some further computation yields

$$d\big(L^3 \pm iL^4\big) = (L^3 \pm iL^4)\big[i(2\phi_1^2 \mp \phi_3^4) - \phi_0^0\big] + (\zeta^3 \pm i\zeta^4)\varphi + (k^3 \pm ik^4)\overline{\varphi} \quad (49)$$

where, for ease of notation, we have set

$$\zeta^\alpha = \overline{k^\alpha} - i(h_{112}^\alpha - ih_{122}^\alpha). \quad (50)$$

Now, if the first of (48) holds, then in particular the coefficient of $\bar{\varphi}$ in (49) must vanish, which is the claim. $\square$

Let us now further analyze the quantities k^α defined in (44). It is not hard to show that under a change of Darboux frames

$$\widetilde{k}^3 \pm i\widetilde{k}^4 = r^2 e^{it} e^{\mp is}\left\{k^3 \pm ik^4 + \frac{1}{2}(x^1+ix^2)(L^3 \pm iL^4)\right\}. \quad (51)$$

For $p > 2$, consider the condition

$$\exists \gamma \in L^p_{\mathrm{loc}}(M) \quad \text{such that} \quad \left|k^3 \pm ik^4\right| \leq \gamma \left|L^3 \pm iL^4\right| \qquad \text{a.e.} \tag{52}$$

Of course we have to check that this condition actually makes sense, since the quantities involved strongly depend on the choice of the Darboux frame. To this end we use (51) and (34) and observe that if condition (52) holds for some Darboux frame, then for any other Darboux frame we can estimate

$$\begin{aligned}\left|\widetilde{k}^3 \pm i\widetilde{k}^4\right| &= r^2 \left|k^3 \pm ik^4 + \frac{1}{2}(x^1 + ix^2)(L^3 \pm iL^4)\right| \leq \\ &\leq r^2 \left(\gamma + \frac{1}{2}\left|x^1 + ix^2\right|\right)\left|L^3 \pm iL^4\right| = r\left(\gamma + \frac{1}{2}\left|x^1 + ix^2\right|\right)\left|\widetilde{L}^3 \pm i\widetilde{L}^4\right|.\end{aligned}$$

Therefore condition (52) still holds, provided we replace γ with another suitable function in $L^p_{\mathrm{loc}}(M)$. We recall the following result by Eschenburg and Tribuzy (see [10]).

Lemma 4.5. *Let $U \subset \mathbb{C}$ be an open domain containing 0 and $f : U \to \mathbb{C}^n$ a smooth function satisfying the Cauchy-Riemann condition*

$$\left|\frac{\partial f}{\partial \bar{z}}\right| \leq \gamma |f| \tag{53}$$

for some L^p-function γ with $p > 2$. Then, in a neighborhood of the origin, either $f \equiv 0$ or

$$f(z) = z^k f_0(z)$$

for some nonnegative integer k and a continuous function f_0 such that $f_0(0) \neq 0$.

We also recall that, if $E \to M$ is a complex vector bundle over a Riemann surface M, a smooth section s of E is said to be of **analytic type** if it either vanishes identically or, near any zero p, we have

$$s = z^k s_0$$

for some positive integer k and some continuous section s_0 with $s_0(p) \neq 0$, where z is any holomorphic chart centered at p. Sections of analytic type, and particularly functions of analityc type, are quite useful in many different settings, and have therefore been studied thoroughly (see e.g. [2]).
Cauchy-Riemann conditions have many applications in this context, starting with the following

Proposition 4.6. *Let $f : M \to Q_4$ be an immersion satisfying (52). Then either γ_f is $\pm$ holomorphic or the set $\mathcal{I}_\mp$ of $\mp$ isotropic points of M is discrete.*

Proof. Keeping in mind the expression (49) for the differential of the functions $L^3 \pm iL^4$, we use (52) in order to apply Lemma 4.5 to these functions. The claim then follows readily. $\square$

Let us now consider the canonical projection $p : \mathbb{R}^6 \setminus \{0\} \to \mathbb{P}^5_{\mathbb{R}}$, sending x to its projective class $[x]$. Given two Darboux frames e and $\widetilde{e}$ along $f : M \to Q_4$, we have

$$p_{*\widetilde{e}_0}\widetilde{e}_\alpha = rB^\beta_\alpha p_{*e_0}e_\beta.$$

Indeed, since $p(\lambda x) = p(x)$ for every $\lambda \in \mathbb{R}^*$ and for every $x \in \mathbb{R}^6 \setminus \{0\}$, then $p_{*\lambda x}\lambda_{*x}v = p_{*x}v$, that is $p_{*\lambda x}\lambda v = p_{*x}v$. Therefore

$$p_{*\widetilde{e}_0}\widetilde{e}_\alpha = p_{*r^{-1}e_0}\widetilde{e}_\alpha = p_{*e_0}(r\widetilde{e}_\alpha) = rB^\beta_\alpha p_{*e_0}e_\beta.$$

Hence, setting $E_\alpha = p_{*e_0}e_\alpha$, we get

$$\widetilde{E}_\alpha = rB^\beta_\alpha E_\beta. \tag{54}$$

It follows that the bundle P over M locally spanned by E_3, E_4 is globally well defined. Let P_c be its complexification and $P_c = P_c^{(1,0)} \oplus P_c^{(0,1)}$ the splitting of P_c into $(1,0)$ and $(0,1)$ parts, locally spanned by $E_3 - iE_4$ and $E_3 + iE_4$ respectively. Observe that under a change of Darboux frames, by virtue of (54) we have

$$\widetilde{E}_3 \pm i\widetilde{E}_4 = re^{\mp is}(E_3 \pm iE_4). \tag{55}$$

On the other hand, if $\varphi = \phi^1_0 + i\phi^2_0$ is the form that gives M its complex structure, we have that

$$\widetilde{\varphi} = r^{-1}e^{-it}\varphi. \tag{56}$$

From (34), (55) and (56) we conclude that

$$\mu_\mp = \big(L^3 \mp iL^4\big)(E_3 \pm iE_4) \otimes \varphi \otimes \varphi$$

are sections of the bundles

$$P_c^{(0,1)} \otimes T^*M^{(1,0)} \otimes T^*M^{(1,0)} \quad \text{and} \quad P_c^{(1,0)} \otimes T^*M^{(1,0)} \otimes T^*M^{(1,0)}$$

respectively, which are globally defined on M. Under assumption (52) we can deduce that these sections either vanish identically or have isolated zeros with positive integer multiplicities. Indeed, since φ is a holomorphic section of $T^*M^{(1,0)}$, then

$$D_{\frac{\partial}{\partial \bar z}}\mu_\mp = d(L^3 \mp iL^4)\left(\frac{\partial}{\partial \bar z}\right)(E^3 \pm iE^4) \otimes \varphi^2 + (L^3 \mp iL^4)D_{\frac{\partial}{\partial \bar z}}(E^3 \pm iE^4) \otimes \varphi^2$$

and now, using (49), assumption (52), and the fact that $P_c^{(1,0)}$ and $P_c^{(0,1)}$ are line bundles, we have

$$\left\| D_{\frac{\partial}{\partial \bar z}}\mu_\mp \right\| \leq \gamma|L^3 \mp iL^4|\,\left\|E^3 \pm iE^4\right\| = \gamma\|\mu_\mp\|$$

for some $\gamma \in L^p_{\mathrm{loc}}(M)$. Thus the sections $\mu_\mp$ satisfy a Cauchy-Riemann type inequality; we can therefore apply Lemma 4.5 to their local trivializations and deduce that they are of analytic type.
Assume now M compact. By the Poincaré-Hopf index theorem (see, e.g. [10] and [11]) we have

Proposition 4.7. *Let M be a compact Riemann surface and L a complex line bundle over M. If $s \not\equiv 0$ is a section of L of analytic type, then the Euler number of L, $\chi(L)$, is equal to the sum of the orders of the zeros of s.*

By virtue of this result, assuming γ_f not $\pm$ holomorphic and letting $z(\mu_\mp)$ be the sum of the orders of the zeros of $\mu_\mp$, then using the properties of the Chern classes of line bundles we obtain

$$\begin{cases} z(\mu_-) = -2\chi(M) + \chi\big(P_c^{(0,1)}\big) = -2\chi(M) - \chi(P) \\ z(\mu_+) = -2\chi(M) + \chi\big(P_c^{(1,0)}\big) = -2\chi(M) + \chi(P). \end{cases}$$

We have therefore proved the following

Theorem 4.8. *Let $f : M \to Q_4$ be an immersed compact surface satisfying (52). Then either $\gamma_f : M \to \mathcal{Q}_2\big(\mathbb{R}^6\big)$ is $\pm$ holomorphic or*

$$2\chi(M) \leq -|\chi(P)|.$$

5 Willmore surfaces and S-Willmore surfaces

We recall that an immersion $f : M \to Q_n$ is a **Willmore surface** if it is a critical point of the Willmore functional, that is if, for any compact $K \subseteq M$ and any smooth variation $f_t : M \to Q_n$ with support in K, we have

$$\frac{d}{dt}\Big|_{t=0} W_K(f_t) = 0,$$

where

$$W_K(f) = \int_K \mathrm{w}. \tag{57}$$

It is a known fact that an immersed surface $f : M \to Q_n$ is Willmore if and only if its conformal Gauss map is harmonic (as was first proved in [15]). We shall see that this is still true with our representation of the conformal Grassmannian and our definition of the conformal Gauss map. To this end, we introduce the following geometric quantities. Consider the equality

$$\phi^0_\alpha = p^\alpha_k \phi^k_0 \tag{58}$$

(note that in what follows we can consider arbitrary dimension $m \geq 2$ and codimension n). Taking the exterior derivative of the above equation and using the Maurer-Cartan structure equations together with Cartan's lemma, we obtain

$$dp_i^\alpha - p_k^\alpha \phi_i^k + p_i^\beta \phi_\beta^\alpha + 2p_i^\alpha \phi_0^0 - h_{ki}^\alpha \phi_k^0 = p_{ik}^\alpha \phi_0^k \tag{59}$$

with

$$p_{ik}^\alpha = p_{ki}^\alpha. \tag{60}$$

With a simple but tedious computation, one verifies that under a change of Darboux frames we have

$$\begin{aligned}\widetilde{p_{ij}^\alpha} =& r^3 B_\alpha^\beta A_i^k A_j^t \Big(p_{kt}^\beta + x^l h_{lkt}^\beta - x^t x^l h_{lk}^\beta - x^k x^l h_{lt}^\beta - \frac{1}{2} x^l x^l h_{kt}^\beta - 2x^t p_k^\beta - 2x^k p_t^\beta \Big) + \\ &+ r^3 B_\alpha^\beta \delta_{ij} \Big(x^l x^t h_{lt}^\beta + x^l p_l^\beta \Big)\end{aligned} \tag{61}$$

so that, tracing with respect to i and j

$$\widetilde{p_{ii}^\alpha} = r^3 B_\alpha^\beta \Big(p_{tt}^\beta + (m-2) \Big(2x^l p_l^\beta + x^l x^t h_{lt}^\beta \Big) \Big) \tag{62}$$

showing that, when $m = 2$, the system of equations

$$p_{ii}^\alpha = 0 \tag{63}$$

is conformally invariant.
It was proved in [16] and [5] that condition (63) is equivalent to f being a Willmore surface. As a matter of fact, (63) is also equivalent to the harmonicity of the conformal Gauss map. This result is summarized in the following

Theorem 5.1. *Let $f : M \to Q_n$ be an immersed oriented Riemann surface with conformal Gauss map $\gamma_f : M \to \mathcal{Q}_{n-2}(\mathbb{R}^{n+2})$. Then f is a Willmore surface if and only if γ_f is harmonic.*

The proof of this result is achieved by direct computation of the tension field of the conformal Gauss map.
Let us now go back to surfaces in Q_4. In this context the concepts of harmonicity and $\pm$ holomorphicity of the conformal Gauss map both make sense, and since $\pm$ holomorphicity implies harmonicity, we find that $\pm$ isotropic surfaces in Q_4 are in particular Willmore surfaces.

In [9], Ejiri has introduced the notion of S-Willmore surface. In our setting, with respect to a Darboux frame along f, the notion corresponds to the two following conditions

$$\begin{aligned} &(a) \quad L^\alpha e_\alpha \not\parallel \overline{L^\alpha} e_\alpha \\ &(b) \quad k^\alpha e_\alpha // L^\alpha e_\alpha, \end{aligned} \tag{64}$$

whose conformal invariance is apparent once we recognize that, at $p \in M$, condition $(5.2a)$ is equivalent to

$$\begin{vmatrix} L^3 & L^4 \\ \overline{L^3} & \overline{L^4} \end{vmatrix} \neq 0 \qquad \text{that is} \qquad L^3 \overline{L^4} - \overline{L^3} L^4 \neq 0,$$

and, by (39), this translates to

$$K_N(p) \neq 0.$$

On the other hand, condition (5.2b) can be expressed as

$$k^3L^4 - k^4L^3 = 0.$$

The quantity on the left-hand side, under a change of Darboux frames, obeys the transformation law

$$\widetilde{k}^3\widetilde{L}^4 - \widetilde{k}^4\widetilde{L}^3 = r^3 e^{3it}(k^3L^4 - k^4L^3),$$

as can be readily seen using (33), thus the element of $\bigotimes^3 T^*M^{(1,0)}$

$$\alpha_1 = (k^3L^4 - k^4L^3)\varphi \otimes \varphi \otimes \varphi \tag{65}$$

is globally defined on M and condition (5.2b) is satisfied at $p \in M$ if and only if

$$\alpha_1(p) = 0.$$

Ejiri proved that, in the Riemannian setting, an S-Willmore surface is a Willmore surface. This can be easily checked in our setting, too.

Proposition 5.2. *Let $f : M \to Q_4$ be an S-Willmore surface, namely an immersed oriented Riemann surface such that $K_N \neq 0$ and $\alpha_1 = 0$. Then f is a Willmore surface.*

Proof. Suppose f is S-Willmore. In particular $k^3L^4 - k^4L^3 = 0$ on M. Differentiating the left-hand side and using the structure equations, we find

$$\begin{aligned} d(k^3L^4 - k^4L^3) = &- 3(k^3L^4 - k^4L^3)(\phi_0^0 + i\phi_2^1) + \frac{1}{2}(Q^3L^4 - Q^4L^3)\varphi + \\ &+ (k^3\zeta^4 - k^4\zeta^3)\varphi + \frac{1}{4}(p^3_{kk}L^4 - p^4_{kk}L^3)\overline{\varphi}, \end{aligned} \tag{66}$$

where

$$Q^\alpha = \frac{1}{2}(p^\alpha_{11} - p^\alpha_{22}) - ip^\alpha_{12}$$

and ζ^α has been defined in (50). This can be shown through a quite lengthy, but straightforward, computation, that we omit.
Now, setting $k^3L^4 - k^4L^3 = 0$ in (66), we can deduce that, in particular,

$$p^3_{kk}L^4 = p^4_{kk}L^3.$$

Assume by contradiction that f is not a Willmore surface, that is, either $p^3_{kk} \neq 0$ or $p^4_{kk} \neq 0$, say $p^3_{kk} \neq 0$. Then we have

$$iK_N = L^3\overline{L^4} - \overline{L^3}L^4 = \frac{p^4_{kk}}{p^3_{kk}}(L^3\overline{L^3} - \overline{L^3}L^3) = 0$$

which contradicts (5.2a). □

From the proof of Theorem 4.4, we have that γ_f is $\pm$ holomorphic if and only if $k^3 = \pm ik^4$ and $L^3 = \pm iL^4$, hence in this case we automatically have $\alpha_1 = 0$, so that

Proposition 5.3. *Let $f : M \to Q_4$ be a $\pm$ isotropic immersed surface. Then f is S-Willmore if and only if $K_N \neq 0$ on M.*

The next risult is another application of Lemma 4.5.

Proposition 5.4. *Let $f : M \to Q_4$ be an immersion without umbilical points and such that the set of $\pm$ isotropic points is not discrete. If f satisfies condition* (52)*, then f is S-Willmore.*

Proof. By Proposition 4.6, f must be $\pm$ isotropic. This implies $\alpha_1 = 0$ and

$$K_N = -i(L^3\overline{L^4} - \overline{L^3}L^4) = \mp 2\left|L^4\right|^2 = \mp 2\left|L^3\right|^2.$$

Therefore $K_N(p) = 0$ if and only if p is an umbilical point, and the result follows. $\square$

Observe that under a change of Darboux frames we have

$$\widetilde{p}^3_{kk}\widetilde{L}^4 - \widetilde{p}^4_{kk}\widetilde{L}^3 = r^3 e^{3it}\left(p^3_{kk}L^4 - p^4_{kk}L^3\right), \tag{67}$$

therefore, applying once more Lemma 4.5 we have the following

Theorem 5.5. *Let $f : M \to Q_4$ be an immersion such that*

$$\exists \gamma \in L^p_{\mathrm{loc}}(M) \quad \text{such that} \quad \left|p^3_{kk}L^4 - p^4_{kk}L^3\right| \leq \gamma\left|k^3L^4 - k^4L^3\right| \quad a.e. \tag{68}$$

for some $p > 2$. Then either $\alpha_1 \equiv 0$ or its zero set is discrete. In this latter case, for M compact we have

$$z(\alpha_1) = -3\chi(M),$$

where $z(\alpha_1)$ is the sum of the orders of the zeros of α_1.

Remark 5.6. *If M is a Willmore surface, condition* (68) *is automatically satisfied. Moreover, if M is a topological 2-sphere, then $\alpha_1 \equiv 0$.*

Proposition 5.7. *Let $f : M \to Q_4$ be a Willmore surface. Then α_1 is holomorphic.*

Proof. Let e be a Darboux frame along f and $g = \phi^1_0 \otimes \phi^1_0 + \phi^2_0 \otimes \phi^2_0$ be the local metric on M defined by e. There exists a local isothermal coordinate $z = x + iy$ on M such that $g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = g\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = r^2$; therefore $\widetilde{g} = r^{-2}g$ is a flat local metric, conformally related to g. Since $\left\{r^{-1}\frac{\partial}{\partial x}, r^{-1}\frac{\partial}{\partial y}\right\}$ is an orthonormal frame for g, we can consider the locally defined, $SO(2)$-valued function A given by $A_1^i = \phi_0^i\big(r^{-1}\frac{\partial}{\partial x}\big)$, $A_2^i = \phi_0^i\Big(r^{-1}\frac{\partial}{\partial y}\Big)$. If we set $\widetilde{e} = eK$, with K defined by

$$K = \begin{pmatrix} r^{-1} & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & r \end{pmatrix}$$

then $\widetilde{e}$ is a Darboux frame, since K is obviously $\text{Möb}(n)_D$-valued. Moreover, trivially $\widetilde{\phi}_0^1 \otimes \widetilde{\phi}_0^1 + \widetilde{\phi}_0^1 \otimes \widetilde{\phi}_0^2 = \widetilde{g}$ and the dual frame to the coframe $\Big\{\widetilde{\phi}_0^i\Big\}$ is just $\Big\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\Big\}$. Indeed, for instance,

$$\widetilde{\phi}_0^1\left(\frac{\partial}{\partial x}\right) = r^{-1}A_1^j\phi_0^j\left(\frac{\partial}{\partial x}\right) = \phi_0^j\left(r^{-1}\frac{\partial}{\partial x}\right)\phi_0^j\left(r^{-1}\frac{\partial}{\partial x}\right) = 1.$$

Now, from the structure equations, we have $d\widetilde{\varphi} = (\widetilde{\phi}_0^0 + i\widetilde{\phi}_2^1)\wedge\widetilde{\varphi}$ and, denoting $Z = \dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right)$ and $W = \overline{Z}$ we get

$$d\widetilde{\varphi}(W, Z) = d(\widetilde{\varphi}(W))(Z) - d(\widetilde{\varphi}(Z))(W) + \widetilde{\varphi}([Z, W]) = 0$$

since $\widetilde{\varphi}(W) = 0$, $\widetilde{\varphi}(Z) = 1$ and $[Z, W] = 0$. On the other hand

$$[(\widetilde{\phi}_0^0 + i\widetilde{\phi}_2^1) \wedge \widetilde{\varphi}](Z, W) = -(\widetilde{\phi}_0^0 + i\widetilde{\phi}_2^1)(W),$$

proving that $\widetilde{\phi}_0^0 + i\widetilde{\phi}_2^1$ is of type $(1,0)$, and hence can be expressed as $\widetilde{\phi}_0^0 + i\widetilde{\phi}_2^1 = \mu\widetilde{\varphi}$, for some locally defined complex valued function μ.
Now, with respect to $\widetilde{e}$, (65) is the expression of α_1 in a local holomorphic trivialization of the bundle $\bigotimes^3 T^*M^{(1,0)}$ so, in order to check if α_1 is holomorphic (that is, if $\bar{\partial}\alpha_1 = 0$) we only need to check that the differential of its coefficient in such trivialization, $k^3L^4 - k^4L^3$, is a local form of type $(1,0)$. But, assuming that f is Willmore, (66) (with respect to the frame $\widetilde{e}$) shows that this is exactly the case. □

6 Enneper-Weierstrass type representations for surfaces in Q_4

So far we have considered immersions of oriented surfaces in the conformal sphere Q_4 and we have associated to them certain maps with values in the conformal Grassmannian $\mathcal{Q}_2(\mathbb{R}^6)$, i.e. the conformal Gauss map. This map has some remarkable properties, for instance it is holomorphic if and only if the original immersion is $-$ isotropic. Now we are going to do the converse: starting from a holomorphic map γ with values in $\mathcal{Q}_2(\mathbb{R}^6)$ we want to see if, and under what conditions, it is possible to retrieve a Q_4-valued map whose conformal Gauss map is exactly the map γ.

First of all, let us observe that, given a $-$ isotropic immersion $f : M \to Q_4$, the conformal Gauss map γ_f is constant if and only if f is totally umbilical, namely $f(M) \subseteq Q_2$, or equivalently $W_K(f) = 0$ for any compact domain $K \subset M$.

Let M be a Riemann surface and $\gamma : M \to \mathcal{Q}_2(\mathbb{R}^6)$ a non constant holomorphic map. Let φ be a (local) $(1,0)$-form defining the complex structure on M and let $s : U \subset \mathcal{Q}_2(\mathbb{R}^6) \to \text{Möb}(4)$ be a local section of $\widehat{\pi}$. Then

$$\gamma^*\zeta^0 = \Lambda^0\varphi, \qquad \gamma^*\zeta^k = \Lambda^k\varphi, \qquad \gamma^*\zeta^3 = \Lambda^3\varphi, \tag{69}$$

where ζ^0, ζ^k and ζ^3 are defined as in (28) with respect to the section s. The vector Λ of components Λ^0, Λ^k, Λ^3 is of analytic type, i.e. it either vanishes identically or has isolated zeros. Indeed, let ω be such that $d\varphi = i\omega \wedge \varphi$; then, differentiating (69) and using (28) and the structure equations, we have

$$\begin{aligned} d(\gamma^*\zeta^0) &= d\Lambda^0 \wedge \varphi + \Lambda^0 d\varphi = \left(d\Lambda^0 + i\Lambda^0\omega\right) \wedge \varphi = \\ &= \gamma^* d\zeta^0 = -\gamma^* s^*(\Phi_0^0 + i\Phi_3^4) \wedge \Lambda^0\varphi - \gamma^* s^* \Phi_k^0 \wedge \Lambda^k\varphi. \end{aligned}$$

Hence

$$\Big(d\Lambda^0 + i\Lambda^0\omega + \Lambda^0\gamma^*s^*\left(\Phi_0^0 + i\Phi_3^4\right) + \Lambda^k\gamma^*s^*\Phi_k^0\Big) \wedge \varphi = 0$$

and similarly for $d(\gamma^*\zeta^k)$ and $d(\gamma^*\zeta^3)$, so that we obtain

$$\begin{cases} d\Lambda^0 = -i\Lambda^0\left(\omega + \gamma^*s^*\Phi_3^4 - i\gamma^*s^*\Phi_0^0\right) - \Lambda^k\gamma^*s^*\Phi_k^0 & \mod \varphi \\ d\Lambda^k = -i\Lambda^k\left(\omega + \gamma^*s^*\Phi_3^4\right) - \Lambda^j\gamma^*s^*\Phi_j^k - \Lambda^0\gamma^*s^*\Phi_0^k - \Lambda^3\gamma^*s^*\Phi_k^0 & \mod \varphi \\ d\Lambda^3 = -i\Lambda^3\left(\omega + \gamma^*s^*\Phi_3^4 + i\gamma^*s^*\Phi_0^0\right) - \Lambda^k\gamma^*s^*\Phi_0^k & \mod \varphi. \end{cases}$$

Thus $d\Lambda^a = \Psi_b^a\Lambda^b$ modulo φ, for some $\mathfrak{gl}(4,\mathbb{C})$-valued 1-form $\Psi = (\Psi_b^a)$, namely the vector Λ is a solution of the system

$$\frac{\partial\Lambda}{\partial\bar{z}} = \Psi\left(\frac{\partial}{\partial\bar{z}}\right)\Lambda$$

and, by Lemma 4.5 (but see also [7] for a direct proof of this case), the claim follows.

Since we assumed γ to be non constant, it follows that the zeros of Λ are isolated, and in a neighborhood of any zero, Λ factorizes as $\Lambda = z^t\widetilde{\Lambda}$, with $\widetilde{\Lambda} \neq 0$, z a local holomorphic chart centered at the zero and $t \in \mathbb{N}$.

Since $\mathcal{Q}_2(\mathbb{R}^6)$ can be identified with an open subset of a quadric in $\mathbb{P}^5_{\mathbb{C}}$, the map γ can be lifted to a smooth, $\mathbb{C}^6 \setminus \{0\}$-valued map $\{\gamma\} = e_3 + ie_4$, where $e = s \circ \gamma : U \subset M \to \text{Möb}(n)$ (note that e is not necessarily an immersion, because in general γ is not). Denoting $\phi = e^{-1}de$, we have

$$\begin{aligned} \Lambda^0\varphi &= \gamma^*\zeta^0 = e^*\Phi_3^0 + ie^*\Phi_4^0 = \phi_3^0 + i\phi_4^0, \\ \Lambda^k\varphi &= \gamma^*\zeta^k = \phi_3^k + i\phi_4^k, \\ \Lambda^3\varphi &= \gamma^*\zeta^3 = \phi_3^5 + i\phi_4^5, \end{aligned}$$

and since $de = e\phi$,

$$\begin{aligned} d\{\gamma\} &= i(e_3 + ie_4)\phi_4^3 + e_0(\phi_3^0 + i\phi_4^0) + e_k(\phi_3^k + i\phi_4^k) + e_5(\phi_3^5 + i\phi_4^5) = \\ &= i\{\gamma\}\phi_4^3 + (\Lambda^0 e_0 + \Lambda^k e_k + \Lambda^3 e_5)\varphi \end{aligned}$$

If $p : \mathbb{C}^6 \setminus \{0\} \to \mathbb{P}^5_{\mathbb{C}}$ is the canonical projection, then $\gamma = p \circ \{\gamma\}$ and

$$d\gamma_x = \gamma_{*x} = p_{*\{\gamma\}(x)}\{\gamma\}_{*x} = \varphi\, p_{*\{\gamma\}(x)}\Big(\Lambda^0 e_0 + \Lambda^k e_k + \Lambda^3 e_5\Big).$$

The complex tangent line to the curve $\gamma(M)$ at the point $\gamma(x)$ is therefore the vector space spanned over $\mathbb{C}$ by the non-zero vector $p_{*\{\gamma\}(x)}(\Lambda^0 e_0 + \Lambda^k e_k + \Lambda^3 e_5)$. This prompts us to define a new map, called the "derivative" of γ, $\gamma' : M \to \mathbb{P}^5_{\mathbb{C}}$ which associates to the point $x \in M$ the projectivization of the non-zero vector $\Lambda^0 e_0 + \Lambda^k e_k + \Lambda^3 e_5$. This map is trivially well defined and does not depend on the choice of the section s.
We will need to add the further assumption that γ' be valued in the quadric $\mathcal{Q}_2(\mathbb{R}^6)$; this happens if and only if the vector ${}^t(\Lambda^0, \Lambda^k, 0, 0, \Lambda^3)$ satisfies the equation

$$-2\Lambda^0\Lambda^3 + \Lambda^k\Lambda^k = 0.$$

Definition 6.1. *A map $\gamma : M \to \mathcal{Q}_2(\mathbb{R}^6)$ will be called a **totally isotropic holomorphic map** if it is holomorphic, non constant, and if γ' is valued in $\mathcal{Q}_2(\mathbb{R}^6)$.*

Let $\widetilde{s}$ be another local section of the bundle $\widehat{\pi} : \text{Möb}(4) \to \mathcal{Q}_2(\mathbb{R}^6)$, and $\widetilde{e} = \widetilde{s} \circ \gamma$. Then $\widetilde{e} = eK$ where K takes values in H_0 as defined in (18). At any point $p \in M$ we can therefore choose a section such that $\Lambda^3 = 0$, hence $\Lambda^0 = a$, $\Lambda^1 = \lambda$ and $\Lambda^2 = i\lambda$, for some $a, \lambda \in \mathbb{C}$. Since Λ is of analytic type, such sections can be locally smoothly chosen in a neighborhood of p. The frame e corresponding to such section will be called an **isotropic frame**, and the isotropy subgroup for such frames is exactly $\text{Möb}(n)_D$ as defined in (11). With this choice of frame, (69) rewrites as

$$\gamma^*\zeta^0 = a\varphi, \quad \gamma^*\zeta^1 = \lambda\varphi, \quad \gamma^*\zeta^2 = i\lambda\varphi, \quad \gamma^*\zeta^3 = 0. \tag{70}$$

We can associate, to any totally isotropic holomorphic map γ, a map $J_\gamma : M \to Q_4$ defined as follows. Let e be any isotropic frame along γ and set $J_\gamma = [e_0]$. In this way J_γ is well defined, because isotropic frames change by matrices in $\text{Möb}(n)_D$. Differentiating the second and third equalities of (70), we obtain

$$\begin{aligned} d(\gamma^*\zeta^1) =& -\phi_0^1 \wedge \gamma^*\zeta^0 - \phi_2^1 \wedge \gamma^*\zeta^2 - i\phi_3^4 \wedge \gamma^*\zeta^1 - \phi_1^0 \wedge \gamma^*\zeta^3 = \\ =& \big(-a\phi_0^1 - i\lambda\phi_2^1 - i\lambda\phi_3^4\big) \wedge \varphi, \\ d(\gamma^*\zeta^2) =& (-a\phi_0^2 - \lambda\phi_1^2 + \lambda\phi_3^4) \wedge \varphi, \end{aligned}$$

but on the other hand $\gamma^*\zeta^2 = i\gamma^*\zeta^1$, so we have

$$\left(-ia\phi_0^1 + \lambda\phi_2^1 + \lambda\phi_3^4\right) \wedge \varphi = \left(-a\phi_0^2 - \lambda\phi_1^2 + \lambda\phi_3^4\right) \wedge \varphi$$

that is, $ia\left(\phi_0^1 + i\phi_0^2\right) \wedge \varphi = 0$. Differentiating the last of (70) we get

$$0 = d(\gamma^*\zeta^3) = \left(-\lambda\phi_0^1 - i\lambda\phi_0^2\right) \wedge \varphi.$$

Therefore we have obtained

$$\begin{aligned} a\left(\phi_0^1 + i\phi_0^2\right) \wedge \varphi = 0 \\ \lambda\left(\phi_0^1 + i\phi_0^2\right) \wedge \varphi = 0 \end{aligned}$$

Since Λ is of analytic type, outside a discrete set (the set of zeros of a and λ), we must have

$$\phi_0^1 + i\phi_0^2 = \mu\varphi \tag{71}$$

for some locally defined complex function μ, whose vanishing is independent of the choice of the isotropic frame. Differentiating (71), we have

$$d\mu \wedge \varphi + i\mu\omega \wedge \varphi = d\phi_0^1 + id\phi_0^2 = \mu\phi_0^0 \wedge \varphi + i\mu\phi_2^1 \wedge \varphi,$$

that is

$$d\mu = -i\mu\left(\omega - \phi_2^1 + i\phi_0^0\right) \qquad \mathrm{mod}\ \varphi.$$

Therefore μ is of analytic type, and so it either vanishes identically or has isolated zeros.
Let us now consider an open set $U \subset M$ where μ is nonzero and let e be an isotropic frame along γ defined on U. Then e is trivially a zeroth order frame along J_γ, since $\pi \circ e = J_\gamma$. Moreover, it is a first order frame, since from (70)

$$0 = \gamma^*\zeta^3 = \phi_0^3 + i\phi_0^4,$$

so $\phi_0^\alpha = 0$. Also, J_γ is a conformal immersion on U, since the only points where J_γ is not an immersion are the zeros of μ. In the case of μ vanishing identically, then J_γ is constant. Indeed in this case not only $\phi_0^\alpha = 0$, but also $\phi_0^1 = \phi_0^2 = 0$. So

$$dJ_\gamma = p_* de_0 = p_*(e_0\phi_0^0 + e_A\phi_0^A) = \phi_0^A p_* e_A = 0$$

where $p : \mathbb{R}^6 \setminus \{0\} \to \mathbb{P}^5_{\mathbb{R}}$ is the canonical projection.
Thus, either J_γ is constant on M or it is a weakly conformal branched immersion. Assume to be in this latter case; we will prove that an isotropic frame e along γ is a Darboux frame along J_γ.
To this end we use (70) to deduce that

$$\gamma^*\zeta^2 = i\,\gamma^*\zeta^1. \tag{72}$$

Now we set, as usual, $\phi_i^\alpha = h_{ij}^\alpha \phi_0^j$, $h_{ij}^\alpha = h_{ji}^\alpha$, and observe that

$$\gamma^* \zeta^k = e^*(\Phi_3^k + i\Phi_4^k) = -\phi_k^3 - i\phi_k^4 = -(h_{kj}^3 + ih_{kj}^4)\phi_0^j$$

and equation (72) is equivalent to the following system

$$\begin{cases} h_{1j}^3 = h_{2j}^4 \\ h_{2j}^3 = -h_{1j}^4 \end{cases}$$

which gives

$$h_{11}^3 = h_{21}^4 = -h_{22}^3, \qquad h_{11}^4 = -h_{21}^3 = -h_{22}^4.$$

Moreover, it is trivial to see that, outside the branch points of J_γ, we have $\gamma_{J_\gamma} = \gamma$, and J_γ is $-$ isotropic, since γ_{J_γ} is holomorphic by assumption.

On the other hand, consider a weakly conformal branched immersion $f : M \to Q_4$ with the property that its Gauss map γ_f can be continuously extended to the branch points, and let e be any Darboux frame along f. If f is $-$ isotropic (outside the branch points), then γ_f is holomorphic, and in this case, with the notations of (69), we have

$$\Lambda^0 = k^3 + ik^4, \qquad \Lambda^1 = -\frac{1}{2}\big(L^3 + iL^4\big), \qquad \Lambda^2 = -\frac{i}{2}\big(L^3 + iL^4\big), \qquad \Lambda^3 = 0,$$

so that

$$-2\Lambda^0\Lambda^3 + \sum_k \Lambda^k\Lambda^k = 0$$

and γ_f is a totally isotropic map. Furthermore, $J_{\gamma_f} = f$.
We have therefore proved the following

Theorem 6.1. *Let M be a Riemann surface. There is a bijective correspondence between $-$ isotropic, non totally umbilical, weakly conformal branched immersions $f : M \to Q_4$, whose conformal Gauss map can be continuously extended at the branch points, and non constant, holomorphic, totally isotropic maps $\gamma : M \to \mathcal{Q}_2\big(\mathbb{R}^6\big)$ with non constant associated map J_γ. The bijection is realized via the conformal Gauss map.*

Using an appropriate Grassmann bundle, we can extend the previous result so as to include the totally umbilical surfaces.
Let us consider the product manifold $Q_4 \times \mathcal{Q}_2\big(\mathbb{R}^6\big)$ and define $\mathcal{Q}_2(Q_4)$ as the orbit of the point $([\eta_0], [\varepsilon_3, \varepsilon_4]) \in Q_4 \times \mathcal{Q}_2\big(\mathbb{R}^6\big)$ with respect to the natural left action (defined componentwise) of the group Möb(4). In other words

$$\mathcal{Q}_2(Q_4) = \{([\eta], [s_1, s_2]) \mid \eta = P\eta_0,\ s_1 = P\varepsilon_3,\ s_2 = P\varepsilon_4,\ P \in \text{Möb}(4)\}. \tag{73}$$

It is trivial to see that Möb(4) acts transitively on $\mathcal{Q}_2(Q_4)$, the action being given, for $P \in$ Möb(4) and $([\eta], [s_1, s_2]) \in \mathcal{Q}_2(Q_4)$, by

$$P([\eta], [s_1, s_2]) = ([P\eta], [Ps_1, Ps_2]).$$

Let us compute the isotropy subgroup of the point $([\eta_0],[\varepsilon_3,\varepsilon_4])$. If $P \in$ Möb(4) fixes the point $([\eta_0],[\varepsilon_3,\varepsilon_4])$, then in particular it must fix the first component, hence P must be an element of G_0, defined in (2), so it is bound to be of the form

$$P = \begin{pmatrix} r^{-1} & {}^t x A & \frac{1}{2} r|x|^2 \\ 0 & A & rx \\ 0 & 0 & r \end{pmatrix}.$$

But, for $P[\varepsilon_3]$ to belong to $[\varepsilon_3,\varepsilon_4]$, we must have $x^3 = 0$, $A^1_3 = A^2_3 = 0$ and analogously, imposing $P[\varepsilon_4] \in [\varepsilon_3,\varepsilon_4]$, we deduce $x^4 = 0$ and $A^1_4 = A^2_4 = 0$. Putting these conditions together we find that $P \in \text{Möb}(n)_D$. Since in turn any element of $\text{Möb}(n)_D$ fixes $([\eta_0],[\varepsilon_3,\varepsilon_4])$, we can conclude that the isotropy subgroup is exactly $\text{Möb}(n)_D$. Hence $\mathcal{Q}_2(Q_4) \simeq \text{Möb}(4)/\text{Möb}(n)_D$ is realized as a homogeneous space with projection

$$\bar{\pi} : \text{Möb}(4) \to \mathcal{Q}_2(Q_4)$$

given by

$$\bar{\pi} : P \mapsto ([P\eta_0],[P\varepsilon_3, P\varepsilon_4]),$$

that is, $\bar{\pi} = \pi \times \widehat{\pi}$. Also, we will denote by $\breve{\pi} : \mathcal{Q}_2(Q_4) \to Q_4$ the canonical projection

$$\breve{\pi} : ([\eta],[s_1,s_2]) \mapsto [\eta].$$

Observe that $\mathcal{Q}_2(Q_4)$ has a natural integrable complex structure defined as follows: let ξ be a local section of the bundle $\bar{\pi} : \text{Möb}(4) \to \mathcal{Q}_2(Q_4)$; then we declare the forms

$$\begin{aligned} \sigma^{-1} &= \xi^*\Phi^1_0 + i\xi^*\Phi^2_0, \\ \sigma^0 &= \xi^*\Phi^0_3 + i\xi^*\Phi^0_4, \\ \sigma^k &= \xi^*\Phi^k_3 + i\xi^*\Phi^k_4, \\ \sigma^3 &= \xi^*\Phi^3_0 + i\xi^*\Phi^4_0 \end{aligned} \tag{74}$$

a local basis of the space of the forms of type $(1,0)$ over $\mathcal{Q}_2(Q_4)$. In order to do this, first we need to check that the ideal they generate is differential. Setting, for the sake of simplicity, $\varphi = \xi^*\Phi$ and using the structure equations, we have

$$\begin{aligned} d\sigma^{-1} =& -\sigma^{-1} \wedge (\varphi^0_0 + i\varphi^1_2) - \varphi^1_3 \wedge \varphi^3_0 - \varphi^1_4 \wedge \varphi^4_0 - i\varphi^2_3 \wedge \varphi^3_0 - i\varphi^2_4 \wedge \varphi^4_0 = \\ =& -\sigma^{-1} \wedge (\varphi^0_0 + i\varphi^1_2) + i\sigma^1 \wedge \varphi^4_0 + i\sigma^2 \wedge \varphi^3_0 + \sigma^3 \wedge (\varphi^1_3 + \varphi^2_4) \end{aligned}$$

and likewise for the differentials of the other forms. Lastly, one can easily check that the space generated by these forms is well defined, i.e., it is independent of the choice of the section ξ.

Proposition 6.2. *The fibers of $\breve{\pi} : \mathcal{Q}_2(Q_4) \to Q_4$ are integral submanifolds of the (invariantly defined) Pfaffian system*

$$\begin{cases} \sigma^{-1} = 0 \\ \sigma^3 = 0. \end{cases} \tag{75}$$

Proof. Since $\mathcal{Q}_2(Q_4) \subset Q_4 \times \mathcal{Q}_2(\mathbb{R}^6)$, for $([\eta],[s_1,s_2]) \in \mathcal{Q}_2(Q_4)$, we have

$$T_{([\eta],[s_1,s_2])}\mathcal{Q}_2(Q_4) \subset T_{[\eta]}Q_4 \times T_{[s_1,s_2]}\mathcal{Q}_2(\mathbb{R}^6).$$

Thus, we can regard a tangent vector of $\mathcal{Q}_2(Q_4)$ as a pair (X,V) with $X \in T_{[\eta]}Q_4$ and $V \in T_{[s_1,s_2]}\mathcal{Q}_2(\mathbb{R}^6)$. Now $\breve{\pi}$ is the projection on the first component, so

$$\breve{\pi}_{*([\eta],[s_1,s_2])}(X,V) = X$$

and

$$\ker \breve{\pi}_{*([\eta],[s_1,s_2])} = \left\{(0,V) \in T_{([\eta],[s_1,s_2])}\mathcal{Q}_2(Q_4)\right\}$$

We want to prove that

$$\begin{cases} \sigma^{-1}_{([\eta],[s_1,s_2])}(0,V) = 0 \\ \sigma^{3}_{([\eta],[s_1,s_2])}(0,V) = 0, \end{cases}$$

or equivalently that, if ξ is a local section of $\bar{\pi} : \text{Möb}(4) \to \mathcal{Q}_2(Q_4)$, then

$$\xi^*\Phi^A_{0\,([\eta],[s_1,s_2])}(0,V) = 0.$$

To this end we set $g = \xi([\eta],[s_1,s_2])$ and compute

$$\begin{aligned} \xi^*\Phi^A_{0\,([\eta],[s_1,s_2])}(0,V) =& \Phi^A_{0\,g}(\xi_{*([\eta],[s_1,s_2])}(0,V)) = \left(\Phi_g(\xi_{*([\eta],[s_1,s_2])}(0,V))\right)^A_0 = \\ =& (g^{-1})^A_b\left(\xi_{*([\eta],[s_1,s_2])}(0,V)\right)^b_0, \end{aligned}$$

where in the last equality we used the definition of the Maurer-Cartan form for classical groups (see [1] or [3] for details):

$$\Phi_P(X) = P^{-1}X.$$

Now take $([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])$ in the domain of ξ, set $\widetilde{g} = \xi([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])$ and observe that

$$\bar{\pi}(\xi([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])) = \bar{\pi}(\widetilde{g}) = ([\widetilde{g}\eta_0],[\widetilde{g}\varepsilon_3,\widetilde{g}\varepsilon_4])$$

and, since $\bar{\pi} \circ \xi = id$,

$$([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2]) = (\bar{\pi} \circ \xi)([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2]) = ([\widetilde{g}\eta_0],[\widetilde{g}\varepsilon_3,\widetilde{g}\varepsilon_4]).$$

In particular we have that $[\widetilde{\eta}] = [\widetilde{g}\eta_0]$ and

$$[\widetilde{g}\eta_0] = [\widetilde{g}_0] = [(\xi([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2]))_0] = [\xi_0([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])],$$

that is, the projective class of the vector $\xi_0([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])$ coincides with that of $\widetilde{\eta}$. In other words, calling

$$p:\mathbb{R}^6\setminus\{0\}\to\mathbb{P}^5_{\mathbb{R}}$$

the canonical projection, we find that $p(\xi_0([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2]))=p(\widetilde{\eta})$. Hence $p\circ\xi_0=\check{\pi}$ and

$$(p\circ\xi_0)_{*([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])}(0,V)=\check{\pi}_{*([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])}(0,V)=0,$$

that is

$$p_{*\xi_0([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])}\xi_{0*([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])}(0,V)=0.$$

Thus $\xi_{0*([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])}(0,V)\in\ker p_{*\xi_0([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])}$, implying

$$\xi_{0*([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])}(0,V)=\lambda\xi_0([\widetilde{\eta}],[\widetilde{s}_1,\widetilde{s}_2])$$

for some $\lambda\in\mathbb{R}$. Therefore

$$\big(\xi_{*([\eta],[s_1,s_2])}(0,V)\big)^b_0=\lambda(\xi([\eta],[s_1,s_2]))^b_0=\lambda g^b_0.$$

So eventually,

$$\xi^*\Phi^A_{0\,([\eta],[s_1,s_2])}(0,V)=\lambda(g^{-1})^A_b g^b_0=\lambda\delta^A_0=0.$$

□

Let us consider the canonical projection $c:\mathcal{Q}_2(Q_4)\to\mathcal{Q}_2\big(\mathbb{R}^6\big)$ defined by

$$c([\eta],[s_1,s_2])=[s_1,s_2],$$

which makes the following diagram commutative

$$\begin{tikzcd} & \text{Möb}(4) \arrow[dl, "\bar{\pi}"'] \arrow[dr, "\widehat{\pi}"] & \\ \mathcal{Q}_2(Q_4) \arrow[rr, "c"] & & \mathcal{Q}_2\big(\mathbb{R}^6\big) \end{tikzcd}$$

that is, $\widehat{\pi}=c\circ\bar{\pi}$.

Proposition 6.3. *The map $c:\mathcal{Q}_2(Q_4)\to\mathcal{Q}_2\big(\mathbb{R}^6\big)$ defined above is holomorphic.*

Proof. Fix $p_0=([\eta],[s_1,s_2])\in\mathcal{Q}_2(Q_4)$ and consider ξ a local section of the bundle $\bar{\pi}:\text{Möb}(4)\to\mathcal{Q}_2(Q_4)$, defined on a neighborhood of p_0 and ς a local section of the bundle $\widehat{\pi}:\text{Möb}(4)\to\mathcal{Q}_2\big(\mathbb{R}^6\big)$ defined on a neighborhood of $[s_1,s_2]$. We have to show that $c^*\zeta^0$, $c^*\zeta^k$ and $c^*\zeta^3$, defined as in (28), are forms of type $(1,0)$.

Set $g_0 = \xi(p_0)$. As in the proof of Theorem 4.4, we can assume that the section ς satisfies $\varsigma(\widehat{\pi}(g_0)) = g_0$, and

$$(\varsigma \circ \widehat{\pi})^* (\Phi^0_\alpha)_{g_0} = (\Phi^0_\alpha)_{g_0}.$$

Then, observing that $c = \widehat{\pi} \circ \xi$, we have that

$$\left(c^* \zeta^0\right)_{p_0} = \left(\xi^* \widehat{\pi}^* \zeta^0\right)_{p_0} = \xi^* \left(\widehat{\pi}^* \varsigma^* (\Phi^0_3 + i\Phi^0_4)\right)_{g_0} = \xi^* \Phi^0_{3g_0} + i\xi^* \Phi^0_{4g_0} = \sigma^0_{p_0},$$

and analogously for $c^*\zeta^k$ and $c^*\zeta^3$. □

Definition 6.2. *Let $f : M \to Q_4$ be an immersed oriented surface. The **conformal Gauss lift** $\Gamma_f : M \to \mathcal{Q}_2(Q_4)$ is defined as*

$$\Gamma_f = f \times \gamma_f,$$

that is, given $p \in M$ and e any Darboux frame along f, defined on a neighborhood of p,

$$\Gamma_f = \bar{\pi} \circ e;$$

in other words,

$$\Gamma_f : p \mapsto ([e_0]_p, [e_3, e_4]_p).$$

We are now ready to state the generalization of Theorem 6.1.

Theorem 6.4. *Let M be a Riemann surface. There is a bijective correspondence between − isotropic, weakly conformal branched immersions $f : M \to Q_4$ whose conformal Gauss map can be continuously extended at the branch points, and holomorphic maps $\Gamma : M \to \mathcal{Q}_2(Q_4)$, solutions of the Pfaffian system*

$$\begin{cases} \sigma^3 = 0 \\ \sigma^2 - i\sigma^1 = 0 \end{cases}$$

but not of $\sigma^{-1} = 0$. The bijection is realized via the conformal Gauss lift Γ_f.

Proof. Let $f : M \to Q_4$ be as in the statement of the theorem. Then, in order to show that the conformal Gauss lift Γ_f is holomorphic, we proceed as for the conformal Gauss map γ_f in the proof of Theorem 4.4. Let us fix $p_0 \in M$ such that it is not a branch point for f and choose a Darboux frame e along f defined on a neighborhood U of p_0 and a section ξ of the bundle $\bar{\pi} : \text{Möb}(4) \to \mathcal{Q}_2(Q_4)$ defined in a neighborhood of $\Gamma_f(p_0)$. We set $e(p_0) = g_0$; then since $\bar{\pi} \circ (\xi \circ \bar{\pi}) = \bar{\pi}$, there must exist a function $K : \bar{\pi}^{-1}(U) \to \text{Möb}(n)_D$ such that, for every $g \in \bar{\pi}^{-1}(U)$

$$\xi(\bar{\pi}(g)) = gK(g)$$

and

$$(\xi \circ \bar{\pi})^* \Phi_g = K(g)^{-1} g^{-1} dg K(g) + K(g)^{-1} dK_g$$

In particular we have

$$
\begin{aligned}
(\xi\circ\bar\pi)^*\Phi^k_{0\,g} &= \big(K(g)^{-1}g^{-1}dgK(g)\big)^k_0\\
(\xi\circ\bar\pi)^*\Phi^0_{\alpha g} &= \big(K(g)^{-1}g^{-1}dgK(g)\big)^0_\alpha\\
(\xi\circ\bar\pi)^*\Phi^k_{\alpha g} &= \big(K(g)^{-1}g^{-1}dgK(g)\big)^k_\alpha\\
(\xi\circ\bar\pi)^*\Phi^\alpha_{0\,g} &= \big(K(g)^{-1}g^{-1}dgK(g)\big)^\alpha_0,
\end{aligned}
$$

because $K^{-1}dK$ is valued in the Lie algebra of the group $\text{Möb}(n)_D$. Replacing, if necessary, the section ξ with $\xi K(g_0)^{-1}$, we can assume that

$$\xi(\bar\pi(g_0)) = g_0$$

and hence

$$
\begin{aligned}
(\xi\circ\bar\pi)^*\Phi^k_{0\,g_0} &= \Phi^k_{0\,g_0}\\
(\xi\circ\bar\pi)^*\Phi^0_{\alpha g_0} &= \Phi^0_{\alpha g_0}\\
(\xi\circ\bar\pi)^*\Phi^k_{\alpha g_0} &= \Phi^k_{\alpha g_0}\\
(\xi\circ\bar\pi)^*\Phi^\alpha_{0\,g_0} &= \Phi^\alpha_{0\,g_0}.
\end{aligned}
$$

Therefore we can compute

$$\big(\Gamma_f^*\sigma^{-1}\big)_{p_0} = \big((\xi\circ\bar\pi\circ e)^*(\Phi^1_0 + i\Phi^2_0)\big)_{p_0} = \big(e^*(\Phi^1_0+i\Phi^2_0)\big)_{p_0} = \varphi_{p_0} \qquad (76)$$

and likewise for σ^k and σ^3. This proves the holomorphicity of Γ_f outside the set of branch points of f. But since f is continuous and by assumption γ_f can be continuously extended to the branch points, then $\Gamma_f = f\times\gamma_f$ is continuous on M, and therefore holomorphic.

The same computation also proves that Γ_f is a solution of the Pfaffian system $\sigma^3 = 0$, $\sigma^2 - i\sigma^1 = 0$, since it is easily verified that

$$
\begin{aligned}
\Gamma_f^*\sigma^3 &= 0,\\
\Gamma_f^*\sigma^1 &= -\frac{1}{2}\big(L^3 + iL^4\big)\varphi\\
\Gamma_f^*\sigma^2 &= -\frac{i}{2}\big(L^3 + iL^4\big)\varphi.
\end{aligned}
$$

Moreover, (76) assures that

$$\Gamma_f^*\sigma^{-1} \neq 0.$$

On the contrary, assume $\Gamma : M \to \mathcal{Q}_2(Q_4)$ is a holomorphic map such that $\Gamma^*\sigma^3 = 0$, $\Gamma^*\sigma^2 = i\Gamma^*\sigma^1$ and $\Gamma^*\sigma^{-1}\neq 0$ and define $f_\Gamma = \check\pi\circ\Gamma$. For any local section ξ of $\bar\pi$, the map $e = \xi\circ\Gamma$ is a local frame along f_Γ, since

$$\pi\circ e = \pi\circ\xi\circ\Gamma = \check\pi\circ\bar\pi\circ\xi\circ\Gamma = \check\pi\circ\Gamma = f_\Gamma.$$

Moreover, let φ be a local $(1,0)$-form defining the complex structure on M; then, since Γ is holomorphic, there must exist a smooth function $\mu \not\equiv 0$ such that

$$e^*(\Phi_0^1 + i\Phi_0^2) = \Gamma^*\sigma^{-1} = \mu\varphi.$$

As usual, we set $\phi = e^*\Phi$, so that the previous equality becomes $\phi_0^1 + i\phi_0^2 = \mu\varphi$. Differentiating this last equality and using the structure equation we can deduce that

$$d\mu = -i\mu\big(\omega - \phi_2^1 + i\phi_0^0\big) \qquad \bmod \varphi,$$

where ω is such that $d\varphi = i\omega \wedge \varphi$. Hence μ is of analytic type, and its zeros must be isolated and of finite order, proving that f_Γ is a weakly conformal branched immersion. In addition, since by assumption $\Gamma^*\sigma^3 = 0$, we know that e is a first order frame along f_Γ. We can prove that e is actually a Darboux frame along f_Γ using

$$\Gamma^*\sigma^2 = i\Gamma^*\sigma^1. \tag{77}$$

Indeed, setting as usual $\phi_i^\alpha = h_{ij}^\alpha\phi_0^j$, $h_{ij}^\alpha = h_{ji}^\alpha$,

$$\Gamma^*\sigma^k = e^*(\Phi_3^k + i\Phi_4^k) = -\phi_k^3 - i\phi_k^4 = -(h_{kj}^3 + ih_{kj}^4)\phi_0^j$$

and equation (77) becomes

$$\begin{cases} h_{1j}^3 = h_{2j}^4 \\ h_{2j}^3 = -h_{1j}^4 \end{cases}$$

which gives

$$h_{11}^3 = h_{21}^4 = -h_{22}^3, \qquad h_{11}^4 = -h_{21}^3 = -h_{22}^4.$$

Now since $e = \xi \circ \Gamma$ is a Darboux frame along f_Γ, it makes sense to consider its conformal Gauss map, defined as usual as

$$\gamma_{f_\Gamma} = [e_3, e_4] = \widehat{\pi} \circ e$$

outside the branch points of f_Γ. We want to prove that γ_{f_Γ} can be continuously extended at the branch points, and that the extension is holomorphic. To this end, we define $\gamma : M \to \mathcal{Q}_2\big(\mathbb{R}^6\big)$ as follows

$$\gamma = c \circ \Gamma \tag{78}$$

and observe that Proposition 6.3 implies that γ is holomorphic. By the commutativity of the following diagram

$$\begin{tikzcd}
 & & \text{Möb}(4) \arrow[dl, "\bar{\pi}"'] \arrow[dr, "\widehat{\pi}"] & \\
 & \mathcal{Q}_2(Q_4) \arrow[rr, "c"] \arrow[dl, "\check{\pi}"'] & & \mathcal{Q}_2\big(\mathbb{R}^6\big) \\
Q_4 & & M \arrow[ll, "f_\Gamma"] \arrow[ul, "\Gamma"'] \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

we have that, on the open set where γ_{f_Γ} is defined,

$$\gamma_{f_\Gamma} = \widehat{\pi} \circ e = \widehat{\pi} \circ \xi \circ \Gamma = c \circ \bar{\pi} \circ \xi \circ \Gamma = c \circ \Gamma = \gamma.$$

Therefore γ_{f_Γ} is holomorphic, hence f_Γ is $-$ isotropic. Lastly, we obviously have

$$\Gamma_{f_\Gamma} = \bar{\pi} \circ e = \bar{\pi} \circ \xi \circ \Gamma = \Gamma$$

and

$$f_{\Gamma_f} = \check{\pi} \circ \Gamma_f = \check{\pi} \circ \bar{\pi} \circ e = \pi \circ e = f,$$

so the claim is proved. □

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