Radial nonlinear elliptic problems with singular or vanishing potentials

Marino Badiale*, Federica Zaccagni

Dipartimento di Matematica "Giuseppe Peano"
Università degli Studi di Torino, Via Carlo Alberto 10, 10123 Torino, Italy
e-mail: marino.badiale@unito.it

Abstract

In this paper we prove existence of radial solutions for the nonlinear elliptic problem

$$-\mathrm{div}(A(|x|)\nabla u) + V(|x|)u = K(|x|)f(u) \quad \text{in } \mathbb{R}^N,$$

with suitable hypotheses on the radial potentials A, V, K. We first get compact embeddings of radial weighted Sobolev spaces into sum of weighted Lebesgue spaces, and then we apply standard variational techniques to get existence results.

Keywords. Nonlinear elliptic equations, weighted Sobolev spaces, compact embeddings, unbounded or decaying potentials

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1 Introduction

In this paper we will study the following non linear elliptic equation

$$-\mathrm{div}(A(|x|)\nabla u) + V(|x|)u = K(|x|)f(u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous nonlinearity satisfying $f(0) = 0$ and $V \geq 0, A, K > 0$ are given radial potentials. When $A = 1$ the differential operator is the usual laplacian, and this kind of problems have been much studied in last years, with different sets of hypotheses on the nonlinearity f and the potentials V, K. Much work has been devoted in particular to problems in which such potentials can be vanishing or divergent at 0 and ∞, because this prevents the use of standard embeddings between Sobolev spaces of radial functions, and new embedding and compactness results must be proved (see

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for example [1], [2], [3], [4], [9], [10], [11], [12], [13], [14], [19], [20], [21], and the references therein). The case in which the potential A is not trivial has been studied in [22], [15], [18] for the p-laplacian equation, in [16] and [17] for bounded domains, and in [23] for exterior domains. The typical result obtained in these works says, roughly speaking, that given suitable asymptotic behavior at 0 and ∞ for the potentials, there is a suitable range of exponent q such that, if f behaves like the power t^{q-1}, then problem (1.1) has a radial solution.

In this paper we study problem (1.1) using the ideas introduced in [5], [6], [7]. The main novelty of this approach is that the nonlinearity f is not a pure power as before, but has different power-like behaviors at zero and infinity. The typical example is $f(t) = \min\{t^{q_1-1}, t^{q_2-1}\}$. Also, we do not introduce hypotheses on the asymptotic behavior of V, K, but on their ratio. The typical result is the existence of two intervals $\mathcal{I}_1, \mathcal{I}_2$ such that if $q_1 \in \mathcal{I}_1$ and $q_2 \in \mathcal{I}_2$, and f as above, then problem (1.1) has a radial solution. When $\mathcal{I}_1 \cap \mathcal{I}_2 \neq \emptyset$, it is possible to choose $q_1 = q_2 = q \in \mathcal{I}_1 \cap \mathcal{I}_2$, so that $f(t) = t^{q-1}$ and we get results similar to those already known in literature. The main technical device for our results is given by compact embeddings of Sobolev spaces of radial functions in sum of Lebesgue spaces. We refer to [8] for an introduction to sum of Lebesgue spaces and to the main results we shall use in this paper.

The paper is organized as follows: after the introduction, in section 2 we introduce the main function spaces we shall use, and prove some preliminary embedding results. In section 3 we introduce some sufficient conditions for compactness of the embeddings, and in section 4 we prove compactness. In section 5 we apply the previous results to get existence and multiplicity results for (1.1). Finally, in section 6 we give some concrete examples that, we hope, could help the reader to understand what is new in our results. Notice that the main hypotheses of our results are introduced at the beginning of section 2, while the main results for (1.1) are theorems 5.3 and 5.4.

Notations. We end this introductory section by collecting some notations used in the paper.

• For every $R > 0$, we set $B_R := \{x \in \mathbb{R}^N : |x| < R\}$.
• ω_N is the $(N-1)$-dimensional measure of the surface $\partial B_1 = \{x \in \mathbb{R}^N : |x| = 1\}$.
• For any subset $A \subseteq \mathbb{R}^N$, we denote $A^c := \mathbb{R}^N \setminus A$. If A is Lebesgue measurable, $|A|$ stands for its measure.
• By $\to$ and $\rightharpoonup$ we respectively mean *strong* and *weak* convergence.
• $\hookrightarrow$ denotes *continuous* embeddings.
• $C_c^\infty(\Omega)$ is the space of the infinitely differentiable real functions with compact support in the open set $\Omega \subseteq \mathbb{R}^N$; $C_{c,r}^\infty(\mathbb{R}^N)$ is the radial subspace of $C_c^\infty(\mathbb{R}^N)$. If B is a ball with center in 0, $C_{c,r}^\infty(B)$ is the radial subspace of $C_c^\infty(B)$.
• If $1 \leq p \leq \infty$ then $L^p(A)$ and $L^p_{\mathrm{loc}}(A)$ are the usual real Lebesgue spaces (for any mea-

surable set $A \subseteq \mathbb{R}^N$). If $\rho : A \to (0, +\infty)$ is a measurable function, then $L^p(A, \rho(z)\,dz)$ is the real Lebesgue space with respect to the measure $\rho(z)\,dz$ (dz stands for the Lebesgue measure on $\mathbb{R}^N$).

2 Hypotheses and pointwise estimates

Assume $N \geq 3$. Let V, K and A be three potentials satisfying the following hypothesis:

[A] $A : (0, +\infty) \to (0, +\infty)$ is a continuous function such that there exist real numbers $2 - N < a_0, a_\infty \leq 2$ and $c_0, c_\infty > 0$ satisfying:

$$c_0 \leq \liminf_{r \to 0^+} \frac{A(r)}{r^{a_0}} \leq \limsup_{r \to 0^+} \frac{A(r)}{r^{a_0}} < +\infty$$

$$c_\infty \leq \liminf_{r \to +\infty} \frac{A(r)}{r^{a_\infty}} \leq \limsup_{r \to +\infty} \frac{A(r)}{r^{a_\infty}} < +\infty.$$

[V] $V : (0, +\infty) \to [0, +\infty)$ belongs to $L^1_{loc}(0, +\infty)$.

[K] $K : (0, +\infty) \to (0, +\infty)$ belongs to $L^s_{loc}(0, +\infty)$ for some $s > \max\left\{ \frac{2N}{N - a_0 + 2}, \frac{2N}{N - a_\infty + 2} \right\}$.

For any $q > 1$, we define the weighted Lebesgue space $L^q_K = L^q(\mathbb{R}^N, K(|x|)dx)$ whose norm is $||u||_{L^q_K} = \left(\int_{\mathbb{R}^N} K(|x|)|u|^q dx \right)^{1/q}$.

Definition 2.1. For $q_1, q_2 > 1$ we define the sum space $\mathcal{L}_K = L^{q_1}_K + L^{q_2}_K$ as

$$\mathcal{L}_K = L^{q_1}_K + L^{q_2}_K = \left\{ u = u_1 + u_2 \;\middle|\; u_i \in L^{q_i}_K \right\},$$

with norm $||u||_{\mathcal{L}_K} = \inf \left\{ \max\{||u_1||_{L^{q_1}_K}, ||u_2||_{L^{q_2}_K}\} \;\middle|\; u = u_1 + u_2,\ u_i \in L^{q_i}_K \right\}$.

By $L^{q_1} + L^{q_2}$ we mean the sum space obtained when $K \equiv 1$, that is, when the $L^{q_i}_K$'s are the usual Lebesgue spaces. We refer to [8] for a treatment of such spaces.

We are now going to prove some pointwise estimates for functions in $C^\infty_{c,r}(\mathbb{R}^N)$, which are the starting point of our arguments. In all this paper, when dealing with a radial function u, we will often write, with a little abuse of notation, $u(x) = u(|x|) = u(r)$ for $|x| = r$.

Remark 2.2. It is easy to check that the hypothesis $[A]$ implies that, for each $R > 0$, there exist $C_0 = C_0(R) > 0$ and $C_\infty = C_\infty(R) > 0$ such that

$$A(|x|) \geq C_0 |x|^{a_0} \quad \text{for all } 0 < |x| \leq R, \tag{2.1}$$

$$A(|x|) \geq C_\infty |x|^{a_\infty} \quad \text{for all } |x| \geq R. \tag{2.2}$$

Lemma 2.3. *Assume the hypothesis $[A]$. Fix $R > 0$. Then there exists a constant $C = C(N, R, a_\infty) > 0$ such that, for each $u \in C^\infty_{c,r}(\mathbb{R}^N)$, there holds*

$$|u(x)| \leq C\, |x|^{-\frac{N+a_\infty-2}{2}} \left(\int_{B_R^c} A(|x|)\, |\nabla u|^2\, dx \right)^{1/2} \qquad for\ R \leq |x| < +\infty. \tag{2.3}$$

Proof. If $u \in C^\infty_{c,r}(\mathbb{R}^N)$ and $|x| = r \geq R$, we have

$$-\,u(r) = \int_r^\infty u'(s) ds. \tag{2.4}$$

Using the hypothesis $[A]$, we obtain

$$|u(r)| \leq \int_r^\infty |u'(s)| ds$$

$$= \int_r^\infty |u'(s)| s^{\frac{N+a_\infty-1}{2}} s^{-\frac{N+a_\infty-1}{2}} ds$$

$$\leq \left(\int_r^\infty |u'(s)|^2 s^{N-1} s^{a_\infty} ds \right)^{\frac{1}{2}} \left(\int_r^\infty s^{-(N+a_\infty-1)} ds \right)^{\frac{1}{2}}$$

$$= (\omega_N)^{-\frac{1}{2}} \left(\int_{B_r^c} |x|^{a_\infty} |\nabla u|^2 dx \right)^{\frac{1}{2}} \left(\int_r^\infty s^{-(N+a_\infty-1)} ds \right)^{\frac{1}{2}}$$

$$\leq (C_\infty(R))^{1/2} (\omega_N)^{-\frac{1}{2}} \left(\int_{B_r^c} A(|x|) |\nabla u|^2 dx \right)^{\frac{1}{2}} \left(\int_r^\infty s^{-(N+a_\infty-1)} ds \right)^{\frac{1}{2}}.$$

As

$$\int_r^\infty s^{-(N+a_\infty-1)} ds = \frac{r^{-(N+a_\infty-2)}}{N+a_\infty-2},$$

it follows

$$|u(r)| \leq C\, r^{-\frac{N+a_\infty-2}{2}} \left(\int_{B_R^c} A(|x|)\, |\nabla u|^2\, dx \right)^{1/2}$$

where $C = (C_\infty(R))^{1/2} (\omega_N)^{-\frac{1}{2}} \left(\frac{1}{N+a_\infty-2} \right)^{1/2} = C(N, R, a_\infty)$, and this is our thesis. □

Lemma 2.4. *Assume the hypothesis* $[A]$*. Fix* $R > 0$*. Then there exists a constant* $C = C(N, R, a_0) > 0$ *such that, for each* $u \in C^\infty_{c,r}(B_R)$*, there holds*

$$|u(x)| \leq C\,|x|^{-\frac{N+a_0-2}{2}} \left(\int_{B_R} A(|x|)\,|\nabla u|^2\,dx\right)^{1/2} \quad for\ 0 < |x| < R. \tag{2.5}$$

Proof. Let $u \in C^\infty_{c,r}(B_R)$ and take $|x| = r < R$. Since $u(R) = 0$, we have

$$-\,u(r) = u(R) - u(r) = \int_r^R u'(s)ds. \tag{2.6}$$

The same arguments of Lemma 2.3 yield

$$|u(r)| \leq \int_r^R |u'(s)|ds$$

$$\leq \left(\int_r^R |u'(s)|^2 s^{N-1} s^{a_0} ds\right)^{\frac{1}{2}} \left(\int_r^R s^{-(N+a_0-1)} ds\right)^{\frac{1}{2}}$$

$$\leq (\omega_N)^{-\frac{1}{2}} \left(\int_{B_R\setminus B_r} A(|x|)|\nabla u|^2 dx\right)^{\frac{1}{2}} \left(\int_r^R s^{-(N+a_0-1)} ds\right)^{\frac{1}{2}}$$

$$\leq (\omega_N)^{-\frac{1}{2}}\,(C_0(R))^{1/2} \left(\int_{B_R} A(|x|)\,|\nabla u|^2\,dx\right)^{1/2} \left(\frac{1}{N+a_0-2}\right)^{1/2} r^{-\frac{N+a_0-2}{2}}$$

that is

$$|u(r)| \leq C r^{-\frac{N+a_0-2}{2}} \left(\int_{B_R} A(|x|)\,|\nabla u|^2\,dx\right)^{1/2}$$

where $C = (\omega_N)^{-\frac{1}{2}}\,(C_0(R))^{1/2} \left(\frac{1}{N+a_0-2}\right)^{1/2} = C(N, R, a_0)$, which is the thesis. $\square$

We now introduce another function space.

Definition 2.5.

$$S_A = \left\{ u \in C^\infty_{c,r}(\mathbb{R}^N) \,\middle|\, \int_{\mathbb{R}^N} A(|x|)\,|\nabla u|^2\,dx < +\infty \right\}$$

S_A is a subspace of $C^\infty_{c,r}(\mathbb{R}^N)$. We define on S_A the norm $||u||_A = \left(\int_{\mathbb{R}^N} A(|x|)\,|\nabla u|^2\,dx\right)^{1/2}$.

Definition 2.6. Let $2-N<a_0,\ a_\infty\leq 2$. We define the following real numbers

$$p_0:=\frac{2N}{N+a_0-2},\quad p_\infty:=\frac{2N}{N+a_\infty-2}. \tag{2.7}$$

Remark 2.7. Notice that $p_0,p_\infty\geq 2$.

The main property of S_A is given by the following lemma.

Lemma 2.8. *Consider A satisfying the hypothesis $[A]$. The embedding*

$$S_A\hookrightarrow L^{p_0}(\mathbb{R}^N)+L^{p_\infty}(\mathbb{R}^N)$$

is continuous.

Proof. Let us fix $R>0$ and $C>0$ such that $A(|x|)\geq C|x|^{a_\infty}$ for $|x|\geq R$. Take $u\in S_A$. We want to estimate $\int_{B_R^c}|u|^{p_\infty}dx$. With an integration by parts, and using Lemma 2.3, we obtain

$$\int_{B_R^c}|u|^{p_\infty}dx=\omega_N\int_R^\infty r^{N-1}|u(r)|^{p_\infty}dr$$

$$\leq\frac{2\omega_N}{N+a_\infty-2}\int_R^\infty r^N|u(r)|^{p_\infty-1}|u'(r)|dr$$

$$\leq\frac{2\omega_N}{N+a_\infty-2}\left(\int_R^\infty r^{a_\infty}|u'(r)|^2r^{N-1}dr\right)^{\frac{1}{2}}\left(\int_R^\infty r^{(N-\frac{N-1+a_\infty}{2})2}|u(r)|^{2(p_\infty-1)}dr\right)^{\frac{1}{2}}$$

$$\leq C\frac{2\omega_N^{\frac{1}{2}}}{N+a_\infty-2}\left(\int_{B_R^c}A(|x|)|\nabla u|^2dx\right)^{\frac{1}{2}}\left(\int_R^\infty r^{N-1}|u(r)|^{p_\infty}r^{2-a_\infty}|u(r)|^{p_\infty-2}dr\right)^{\frac{1}{2}}$$

$$\leq C^{\frac{p_\infty-2}{2}}\frac{2\omega_N^{\frac{1}{2}}}{N+a_\infty-2}\left(\int_{B_R^c}A(|x|)|\nabla u|^2dx\right)^{\frac{p_\infty}{4}}\left(\int_R^\infty r^{N-1}|u(r)|^{p_\infty}dr\right)^{\frac{1}{2}}$$

$$\leq C\left(\int_{B_R^c}A(|x|)|\nabla u|^2dx\right)^{\frac{p_\infty}{4}}\left(\int_{B_R^c}|u|^{p_\infty}dx\right)^{\frac{1}{2}},$$

where $C=C(N,R,a_\infty)$ may change from line to line. From this we obtain

$$\left(\int_{B_R^c}|u|^{p_\infty}dx\right)^{\frac{1}{p_\infty}}\leq C\left(\int_{B_R^c}A(|x|)|\nabla u|^2dx\right)^{\frac{1}{2}}\leq C\|u\|_A. \tag{2.8}$$

Assume now that $C = C_0(R+1) > 0$ is such that $A(|x|) \geq C|x|^{a_0}$ for $0 < |x| = r \leq R+1$. We want to estimate the integral $\int_{B_R} |u|^{p_0} dx$. Let us define a radial cut-off function $\rho(x) \in C^\infty_{0,r}(\mathbb{R}^N)$ such that $\rho(x) \in [0,1]$ for all x and

$$\rho(x) = \rho(|x|) = \begin{cases} 1 & 0 < |x| \leq R \\ 0 & |x| \geq R + \frac{1}{2} \end{cases}$$

Of course $\rho u \in C^\infty_{c,r}(B_{R+1})$ and we can employ Lemma 2.4. With the same computations used in the previous case, we have

$$\int_{B_{R+1}} |\rho u|^{p_0} dx$$

$$\leq C \left(\int_{B_{R+1}} A(|x|)|\nabla(\rho u)|^2 dx \right)^{\frac{p_0}{4}} \left(\int_{B_{R+1}} |\rho u|^{p_0} dx \right)^{\frac{1}{2}},$$

where $C = C(N, R, a_0)$. From this we derive

$$\int_{B_{R+1}} |\rho u|^{p_0} dx \leq C \left(\int_{B_{R+1}} A(|x|)|\nabla(\rho u)|^2 dx \right)^{\frac{p_0}{2}}.$$

Using the continuity of A in the compact set $\overline{B_{R+1} \setminus B_R}$ and thanks to Lemma 2.3, we obtain

$$\int_{B_{R+1}} A(|x|)|\nabla(\rho u)|^2 dx$$

$$\leq C \int_{B_{R+1}} A(|x|)(|\nabla u|^2 \rho^2 + |\nabla \rho|^2 u^2) dx$$

$$\leq C \left(\int_{B_{R+1}} A(|x|)|\nabla u|^2 dx + \int_{B_{R+1} \setminus B_R} |\nabla \rho|^2 u^2 dx \right)$$

$$\leq C \left(\|u\|_A^2 + \|u\|_A^2 \int_{B_{R+1} \setminus B_R} |x|^{-(N+a_\infty-2)} dx \right)$$

$$\leq C\|u\|_A^2,$$

where $C = C(N, R, a_\infty)$. Thus, it holds

$$\int_{B_{R+1}} |\rho u|^{p_0} dx \leq C\|u\|_A^{p_0},$$

where $C = C(N, R, a_0, a_\infty)$. As a consequence we get

$$\int_{B_R} |u|^{p_0} dx = \int_{B_R} |\rho u|^{p_0} dx \leq \int_{B_{R+1}} |\rho u|^{p_0} dx \leq C\|u\|_A^{p_0},$$

hence

$$\left(\int_{B_R}|u|^{p_0}dx\right)^{\frac{1}{p_0}}\leq C\|u\|_A. \tag{2.9}$$

Now we define

$$\begin{cases}\bar{u}_1 := u\chi_{B_R} \\ \bar{u}_2 := u\chi_{B_R^c}\end{cases},$$

so we obtain

$$u=\bar{u}_1+\bar{u}_2$$

with $\bar{u}_1 \in L^{p_0}(\mathbb{R}^N)$ and $\bar{u}_2 \in L^{p_\infty}(\mathbb{R}^N)$, from which it follows

$$\|u\|_{L^{p_0}+L^{p_\infty}} \leq \|\bar{u}_1\|_{L^{p_0}(\mathbb{R}^N)}+\|\bar{u}_2\|_{L^{p_\infty}(\mathbb{R}^N)}$$

$$=\|u\|_{L^{p_0}(B_R)}+\|u\|_{L^{p_\infty}(B_R^c)}\leq C\|u\|_A.$$

This hold for any $u\in S_A$, with a constant $C=C(N,R,a_0,a_\infty)$. Thus, the embedding

$$S_A \hookrightarrow L^{p_0}(\mathbb{R}^N)+L^{p_\infty}(\mathbb{R}^N)$$

is continuous. □

We want now to introduce the completion of S_A with respect to $||\cdot||_A$.

Definition 2.9. D_A is the space of all $u\in L^{p_0}+L^{p_\infty}$ for which there is a sequence $\{u_n\}_n\subset S_A$ such that

- $u_n\to u$ in $L^{p_0}+L^{p_\infty}$.
- $\{u_n\}_n$ is a Cauchy sequence with respect to $||\cdot||_A$.

Of course, D_A is a linear subspace of $L^{p_0}+L^{p_\infty}$. From the previous results we deduce the following two lemmas, which say that D_A is the completion of S_A wth respect to $||\cdot||_A$. The arguments are essentially standard, so we will skip the details.

Lemma 2.10. *Assume* $[A]$*. Let* $u \in D_A$*. Then* u *has weak derivatives* $D_i u$ *in the open set* $\Omega = \mathbb{R}^N \backslash \{0\}$ *(*$i = 1, ..., N$*) and it holds* $D_i u \in L^2_{loc}(\Omega)$*.*
If $\{u_n\}_n$ *is the sequence in* S_A *given by the definition of* $u \in D_A$*, then*

$$\int_{\mathbb{R}^N} A(|x|)|\nabla u(x) - \nabla u_n|^2 \, dx \to 0.$$

In particular $\int_{\mathbb{R}^N} A(|x|)|\nabla u(x)| \, dx < +\infty$*, and* $||u||_A = \left(\int_{\mathbb{R}^N} A(|x|)|\nabla u(x)| \, dx\right)^{1/2}$ *is a norm on* D_A*.*

Proof. The proof is a simple exercise on weak derivatives, and we leave it to the reader.

□

Lemma 2.11. *Assume* $[A]$*. If we consider the space* D_A *endowed with the norm* $|| \cdot ||_A$ *we have*

$$D_A \hookrightarrow L^{p_0}(\mathbb{R}^N) + L^{p_\infty}(\mathbb{R}^N)$$

with continuous embedding.
Furthermore, D_A *endowed with the norm* $|| \cdot ||_A$ *is complete, so it is an Hilbert space.*

Proof. To prove continuity of embedding, let $||u||_{\mathcal{L}}$ be the norm of $u \in L^{p_0} + L^{p_\infty}$. For $u \in D_A$ let $\{u_n\}_n$ be the sequence in S_A given by the definition. By the previous lemma we have $||u_n||_{\mathcal{L}} \leq C \, ||u_n||_{D_A}$, for a suitable positive constant C. But $||u_n - u||_{\mathcal{L}} \to 0$ by hypothesis and $||u_n - u||_{D_A} \to 0$ by (i), so we get

$$||u||_{\mathcal{L}} \leq C \, ||u||_{D_A},$$

which is the thesis.
To prove completeness, let $\{v_n\}_n \subset D_A$ be a Cauchy sequence. From (ii), it is a Cauchy sequence in $L^{p_0} + L^{p_\infty}$, so $u_n \to u$ in $L^{p_0} + L^{p_\infty}$. By (i), for each n we can choose $v_n \in S_A$ such that $||u_n - v_n||_A \leq 1/n$. Then we have

$$||u - v_n||_{\mathcal{L}} \leq ||u - u_n||_{\mathcal{L}} + ||u_n - v_n||_{\mathcal{L}} \leq ||u - u_n||_{\mathcal{L}} + C||u_n - v_n||_{D_A} \to 0.$$

Also, it is

$$||v_m - v_n||_{D_A} \leq ||v_m - u_m||_{D_A} + ||u_n - u_m||_{D_A} + ||u_n - v_n||_{D_A} \leq \frac{1}{m} + ||u_n - u_m||_{D_A} + \frac{1}{n},$$

so also $\{v_n\}_n$ is a Cauchy sequence with respect to $|| \cdot ||_{D_A}$. By definition of D_A we have $u \in D_A$, and by (i) $||u - v_n||_A \to 0$. Now we get

$$||u - u_n||_A \leq ||u - v_n||_A + ||v_n - u_n||_A \to 0,$$

and the proof is complete.

□

We will also need the following corollary.

Corollary 2.12. *Consider A satisfying the hypothesis $[A]$. We define*

$$a := \max\{a_0, a_\infty\}, \quad p_* := \frac{2N}{N+a-2} = \min\{p_0, p_\infty\} \geq 2.$$

Then, for any finite measure set $E \subseteq \mathbb{R}^N$, the embedding

$$D_A \hookrightarrow L^{p_*}(E)$$

is continuous.

Proof. This derives from the previous lemma together with prop. 2.17 $ii)$ of [8]. □

We now define another function space.

Definition 2.13.

$$X = \left\{ u \in D_A \, : \, \int_{\mathbb{R}^N} V(|x|)\, |u|^2 \, dx < +\infty \right\}.$$

with norm

$$||u||^2 = ||u||_X^2 = \int_{\mathbb{R}^N} A(|x|)\, |\nabla u|^2 + \int_{\mathbb{R}^N} V(|x|)\, |u|^2 \, dx.$$

X is an Hilbert spaces with respect to the norm $|| \cdot ||_X$. We will write $(u|v)$ for the scalar product in X, that is

$$(u|v) = \int_{\mathbb{R}^N} A(|x|)\, \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^N} V(|x|)\, u\, v \, dx.$$

We look for weak solutions of equation (1.1) in the space X. This means that a solution (1.1) is a function $u \in X$ such that, for all $h \in X$, it holds

$$\int_{\mathbb{R}^N} A(|x|) \nabla u \nabla h \, dx + \int_{\mathbb{R}^N} V(|x|) u \, h dx - \int_{\mathbb{R}^N} K(|x|)\, f(u) h \, dx = 0.$$

We will obtain such weak solutions by standard variational methods, that is we will introduce (in section 5) a functional on X whose critical points are weak solutions. To get such critical points we need, as usual, some compactness properties for the functional, which we will derive from compactness of suitable embeddings. So in the following sections we will prove that the space X is compactly embedded in $L_K^{q_1} + L_K^{q_2}$, for suitable q_1, q_2. The following lemma is a step to obtain these compact embeddings.

Lemma 2.14. *Assume* $[A]$, $[V]$, $[K]$. *Let* $a = \max\{a_0, a_\infty\}$ *and take* $1 < q < \infty$, $0 < r < R$. *Then there exists a constant* $C = C(N, a_0, a_\infty, r, R, q, s) > 0$ *such that, for any* $u, h \in X$ *we have*

$$\frac{\int_{B_R\setminus B_r} K(|x|)|u|^{q-1}|h|dx}{C\|K(|\cdot|)\|_{L^s(B_R\setminus B_r)}} \leq \begin{cases} \left(\int_{B_R\setminus B_r} |u|^2 dx\right)^{\frac{q-1}{2}} \|h\| & if\ q \leq \tilde{q} \\ \left(\int_{B_R\setminus B_r} |u|^2 dx\right)^{\frac{\tilde{q}-1}{2}} \|u\|^{q-\tilde{q}}\|h\| & if\ q > \tilde{q} \end{cases}$$

where $\tilde{q} = 2\left(1 + \frac{1}{N} - \frac{1}{s}\right) - \frac{a}{N}$.

Proof. We denote with σ the conjugate exponent of p_*, i.e. $\sigma = \frac{2N}{N-a+2} = \max\left\{\frac{2N}{N-a_0+2}, \frac{2N}{N-a_\infty+2}\right\}$. Thanks to Hölder's inequality, initially applied with $p_* > 1$ and then with $\frac{s}{\sigma} > 1$, we obtain

$$\begin{aligned} &\int_{B_R\setminus B_r} K(|x|)|u|^{q-1}|h|dx \\ &\leq \left(\int_{B_R\setminus B_r} K(|x|)^\sigma |u|^{(q-1)\sigma} dx\right)^{\frac{1}{\sigma}} \left(\int_{B_R\setminus B_r} |h|^{p_*} dx\right)^{\frac{1}{p_*}} \\ &\leq \left(\left(\int_{B_R\setminus B_r} K(|x|)^s dx\right)^{\frac{\sigma}{s}} \left(\int_{B_R\setminus B_r} |u|^{(q-1)\sigma(\frac{s}{\sigma})'} dx\right)^{\frac{1}{(\frac{s}{\sigma})'}}\right)^{\frac{1}{\sigma}} C\|h\| \\ &\leq C\|K(|\cdot|)\|_{L^s(B_R\setminus B_r)}\|h\| \left(\int_{B_R\setminus B_r} |u|^{2\frac{q-1}{\tilde{q}-1}} dx\right)^{\frac{\tilde{q}-1}{2}}, \end{aligned}$$

where we use the following computations

$$\frac{1}{\left(\frac{s}{\sigma}\right)'} = 1 - \frac{\sigma}{s} = \frac{s(N-a+2) - 2N}{s(N-a+2)}$$

hence

$$\left(\frac{s}{\sigma}\right)' = \frac{s(N-a+2)}{s(N-a+2) - 2N}$$

so that

$$\sigma\left(\frac{s}{\sigma}\right)' = \frac{2N}{N-a+2}\frac{s(N-a+2)}{s(N-a+2)-2N} = 2\frac{sN}{sN+2s-2N-as} = \frac{2}{\tilde{q}-1}.$$

- If $q = \tilde{q}$, the proof is over.

- If $q < \tilde{q}$, we apply Hölder's inequality again, with conjugate exponents $\frac{\tilde{q}-1}{q-1} > 1$, $\frac{\tilde{q}-1}{\tilde{q}-q}$, obtaining

$$\int_{B_R\setminus B_r} K(|x|)|u|^{q-1}|h|dx$$

$$\leq C\,\|K(|\cdot|)\|_{L^s(B_R\setminus B_r)}\|h\|\left(|B_R\setminus B_r|^{\frac{\tilde q-q}{\tilde q-1}}\left(\int_{B_R\setminus B_r}|u|^2dx\right)^{\frac{q-1}{\tilde q-1}}\right)^{\frac{\tilde q-1}{2}}$$

$$= C\,\|K(|\cdot|)\|_{L_s(B_R\setminus B_r)}\|h\|\left(\int_{B_R\setminus B_r}|u|^2dx\right)^{\frac{q-1}{2}}.$$

- If $q>\tilde q$, we have $\frac{q-1}{\tilde q-1}>1$. Thanks to Lemma 2.3 we obtain

$$\int_{B_R\setminus B_r}K(|x|)|u|^{q-1}|h|dx$$

$$\leq C\|K(|\cdot|)\|_{L^s(B_R\setminus B_r)}\|h\|\left(\int_{B_R\setminus B_r}|u|^{2\frac{q-1}{\tilde q-1}-2}|u|^2dx\right)$$

$$\leq C\|K(|\cdot|)\|_{L^s(B_R\setminus B_r)}\|h\|\left(\left(\frac{C\|u\|}{r^{\frac{N+a_\infty-2}{2}}}\right)^{2\frac{q-\tilde q}{\tilde q-1}}\int_{B_R\setminus B_r}|u|^2dx\right)^{\frac{\tilde q-1}{2}}$$

$$\leq C\|K(|\cdot|)\|_{L^s(B_R\setminus B_r)}\|h\|\|u\|^{q-\tilde q}\left(\int_{B_R\setminus B_r}|u|^2dx\right)^{\frac{\tilde q-1}{2}}.$$

In all the previous computations, C may mean different positive constants, depending only on $N, a_0, a_\infty, r, R, q, s$. □

Lemma 2.15. *Consider A satisfying the hypothesis $[A]$. Let Ω be a smooth bounded open set such that $\overline{\Omega}\subset\mathbb{R}^N\setminus\{0\}$. Then the embedding*

$$D_A\hookrightarrow L^2(\Omega)$$

is continuous and compact.

Proof. The continuity of the embedding is obvious thanks to Corollary 2.12 and the fact that $p_*\geq 2$. We prove now the compactness of the embedding. Let $\{u_n\}_n$ be a bounded sequence in D_A. By continuity of the embedding we obtain

$$\|u_n\|_{L^2(\Omega)}\leq C.$$

Moreover as the function $A(x)$ is continuous and strictly positive in the compact set $\overline{\Omega}$, there holds

$$\int_\Omega|\nabla u_n|^2dx\leq C\int_\Omega A(|x|)|\nabla u_n|^2dx\leq C||u_n||^2_{D_A}\leq C.$$

Thus, $\{u_n\}_n$ is bounded also in the space $H^1(\Omega)$. Thanks to Rellich's Theorem, $\{u_n\}_n$ has a convergent subsequence in $L^2(\Omega)$, and this gives our thesis. ☐

3 Some results on embeddings

Following [5] we now introduce some new functions, whose study will help us in getting conditions for compactness.

Definition 3.1. For $q > 1$ and $R > 0$ define

$$\mathcal{S}_0(q,R) := \sup_{u\in X, \|u\|=1} \int_{B_R} K(|x|)|u|^q dx$$

$$\mathcal{S}_\infty(q,R) := \sup_{u\in X, \|u\|=1} \int_{B_R^c} K(|x|)|u|^q dx.$$

Theorem 3.2. *Assume $N \geq 3$ and $[A]$, $[V]$, $[K]$. Take $q_1, q_2 > 1$.*

1. *If*

$$\mathcal{S}_0(q_1,R_1) < \infty \ \text{ and } \ \mathcal{S}_\infty(q_2,R_2) < \infty \ \text{ for some } R_1, R_2 > 0, \tag{3.1}$$

then

$$X \hookrightarrow L_K^{q_1}(\mathbb{R}^N) + L_K^{q_2}(\mathbb{R}^N),$$

with continuous embedding.

2. *If*

$$\lim_{R\to 0^+} \mathcal{S}_0(q_1,R) = \lim_{R\to +\infty} \mathcal{S}_\infty(q_2,R) = 0, \tag{3.2}$$

then the embedding of X into $L_K^{q_1}(\mathbb{R}^N) + L_K^{q_2}(\mathbb{R}^N)$ is compact.

Proof. As to 1, we remark that $\mathcal{S}_0$ and $\mathcal{S}_\infty$ are monotone, so it is not restrictive to assume $R_1 < R_2$. Assume $u \in X, \ u \neq 0$, then

$$\int_{B_{R_1}} K(|x|)|u|^{q_1} dx = \|u\|^{q_1} \int_{B_{R_1}} K(|x|) \frac{|u|^{q_1}}{\|u\|^{q_1}} dx \leq \|u\|^{q_1} \mathcal{S}_0(q_1, R_1) \tag{3.3}$$

and, in the same way,

$$\int_{B_{R_2}^c} K(|x|)|u|^{q_2} dx \leq \|u\|^{q_2} \mathcal{S}_\infty(q_2, R_2). \tag{3.4}$$

Here $||u|| = ||u||_X$ is the norm in X.
Using lemma 2.14 (with $h = u$) and lemma 2.15 we obtain, for a suitable $C > 0$, independent from u,

$$\int_{B_{R_2}\setminus B_{R_1}} K(|x|)|u|^{q_1}dx \leq C\|u\|^{q_1}. \tag{3.5}$$

Hence $u \in L_K^{q_1}(B_{R_2}) \cap L_K^{q_2}(B_{R_2}^c)$ so, by proposition 2.3 in [8], $u \in L_K^{q_1} + L_K^{q_2}$. Moreover, if $u_n \to 0$ in X, then, thanks to (3.3), (3.4) and (3.5), we obtain

$$\int_{B_{R_2}} K(|x|)|u_n|^{q_1}dx + \int_{B_{R_2}^c} K(|x|)|u_n|^{q_2}dx \to 0 \quad \text{as } n \to \infty.$$

It follows that $u_n \to 0$ in $L_K^{q_1} + L_K^{q_2}$ (see proposition 2.7 in [8]).
As to 2, we assume hypothesis (3.2) and let $u_n \rightharpoonup 0$ in X. Then, $\{u_n\}$ is bounded in X. Thanks to (3.3) and (3.4) we get that, for a fixed $\varepsilon > 0$, it is possible to obtain R_ϵ and r_ϵ such that $R_\epsilon > r_\epsilon > 0$ and for all $n \in \mathbb{N}$,

$$\int_{B_{r_\epsilon}} K(|x|)|u_n|^{q_1}dx \leq \|u_n\|^{q_1}\mathcal{S}_0(q_1, r_\epsilon) \leq \sup_n \|u_n\|^{q_1}\, \mathcal{S}_0(q_1, r_\epsilon) < \frac{\epsilon}{3}$$

and

$$\int_{B_{R_\epsilon}^c} K(|x|)|u_n|^{q_2}dx \leq \sup_n \|u_n\|^{q_2}\, \mathcal{S}_\infty(q_1, R_\epsilon) < \frac{\epsilon}{3}.$$

Thanks to Lemma 2.14 and to the boundedness of $\{u_n\}_n$ in X, there exist two constants $C, \mu > 0$, independent from n, such tat

$$\int_{B_{R_\epsilon}\setminus B_{r_\epsilon}} K(|x|)|u_n|^{q_2}dx \leq C\left(\int_{B_{R_\epsilon}\setminus B_{r_\epsilon}} |u_n|^2 dx\right)^{\mu} \to 0 \quad \text{as } n \to \infty,$$

thanks to Lemma 2.15. For n big enough we have

$$\int_{B_{R_\epsilon}} K(|x|)|u_n|^{q_1}dx + \int_{B_{R_\epsilon}^c} K(|x|)|u_n|^{q_2}dx < \epsilon.$$

From this, thanks to proposition 2.7 in [8], we get $u_n \to 0$ in $L_K^{q_1} + L_K^{q_2}$. □

4 Compactness of embeddings

We start this section proving the following two lemmas, which give the most important technical steps for our compactness results. For future purposes, these two lemmas are stated in a form which is a little more general than needed in the present paper.

Lemma 4.1. *Let $N \geq 3$ and $R > 0$. Assume $[A]$, $[V]$, $[K]$. If we assume*

$$\Lambda := \operatorname*{ess\,sup}_{x \in B_R} \frac{K(|x|)}{|x|^\alpha V(|x|)^\beta} < +\infty \quad for\ some\ 0 \leq \beta \leq 1\ and\ \alpha \in \mathbb{R},$$

then, $\forall u, h \in X$ and $\forall q > \max\{1, 2\beta\}$ we have

$$\int_{B_R} K(|x|)|u|^{q-1}|h|dx$$

$$\leq \begin{cases} \Lambda||u||^{q-1}\, C\, \left(\int_{B_R} |x|^{\frac{\alpha-\nu(q-1)}{N-a_0+2(1-2\beta+a_0\beta)}2N} dx\right)^{\frac{N-a_0+2(1-2\beta+a_0\beta)}{2N}} \|h\| & 0 \leq \beta \leq \frac{1}{2} \\ \Lambda||u||^{q-1}\, C\, \left(\int_{B_R} |x|^{\frac{\alpha-\nu(q-2\beta)}{1-\beta}} dx\right)^{1-\beta} \|h\| & \frac{1}{2} < \beta < 1 \\ \Lambda||u||^{q-2}\, C\, \left(\int_{B_R} |x|^{2\alpha-2\nu(q-2)} V(|x|)|u|^2 dx\right)^{\frac{1}{2}} \|h\| & \beta = 1 \end{cases}$$

where $\nu := \frac{N+a_0-2}{2}$ and $C = C(N, R, a_0, a_\infty)$.

Proof. We study the various cases separately:

1. If $\beta = 0$, we apply Hölder's inequality with conjugate exponents $p_0 = \frac{2N}{N+a_0-2}$, $\frac{2N}{N-a_0+2}$. We apply also (2.9) and Lemma 2.4 and we get

$$\frac{1}{\Lambda}\int_{B_R} K(|x|)|u|^{q-1}|h|dx \leq \int_{B_R} |x|^\alpha |u|^{q-1}|h|dx$$

$$\leq \left(\int_{B_R} \left(|x|^\alpha |u|^{q-1}\right)^{\frac{2N}{N-a_0+2}} dx\right)^{\frac{N-a_0+2}{2N}} \left(\int_{B_R} |h|^{p_0} dx\right)^{\frac{1}{p_0}}$$

$$\leq ||u||^{q-1}\, C\, \left(\int_{B_R} |x|^{\frac{\alpha-\nu(q-1)}{N-a_0+2}2N} dx\right)^{\frac{N-a_0+2}{2N}} \|h\|.$$

2. If $0 < \beta < 1/2$ then it is possible to apply Hölder's inequality with conjugate exponents $\frac{1}{\beta}$, $\frac{1}{1-\beta}$:

$$\frac{1}{\Lambda}\int_{B_R} K(|x|)|u|^{q-1}|h|dx$$

$$\leq \int_{B_R} |x|^\alpha V(|x|)^\beta |u|^{q-1}|h|dx = \int_{B_R} |x|^\alpha |u|^{q-1}|h|^{1-2\beta} V(|x|)^\beta |h|^{2\beta} dx$$

$$\leq \left(\int_{B_R} \left(|x|^\alpha |u|^{q-1}|h|^{1-2\beta}\right)^{\frac{1}{1-\beta}} dx\right)^{1-\beta} \left(\int_{B_R} V(|x|)|h|^2 dx\right)^\beta$$

$$\leq \left(\int_{B_R} \left(|x|^{\alpha}|u|^{q-1}|h|^{1-2\beta}\right)^{\frac{1}{1-\beta}} dx\right)^{1-\beta} \|h\|^{2\beta}.$$

We apply again Hölder's inequality with exponent $\frac{1-\beta}{1-2\beta}\, p_0 > 1$. Its conjugate exponent is given through the formula

$$\frac{1}{\left(\frac{1-\beta}{1-2\beta}p_0\right)'} = 1 - \frac{1-2\beta}{p_0(1-\beta)} = \frac{p_0(1-\beta)-(1-2\beta)}{p_0(1-\beta)}$$

$$= \frac{\frac{2N}{N+a_0-2}(1-\beta)-(1-2\beta)}{\frac{2N}{N+a_0-2}(1-\beta)} = \frac{2N(1-\beta)-(N+a_0-2)(1-2\beta)}{2N(1-\beta)}$$

$$= \frac{N-a_0+2(1-2\beta+a_0\beta)}{2N(1-\beta)}.$$

We obtain that

$$\frac{1}{\Lambda}\int_{B_R} K(|x|)|u|^{q-1}|h|dx$$

$$\leq \left(\left(\int_{B_R} \left(|x|^{\frac{\alpha}{1-\beta}}|u|^{\frac{q-1}{1-\beta}}\right)^{\left(\frac{1-\beta}{1-2\beta}p_0\right)'} dx\right)^{\frac{1}{\left(\frac{1-\beta}{1-2\beta}p_0\right)'}} \left(\int_{B_R} |h|^{p_0}dx\right)^{\frac{1-2\beta}{(1-\beta)p_0}}\right)^{1-\beta} \|h\|^{2\beta}$$

$$\leq ||u||^{q-1} \left(\left(\int_{B_R} \left(|x|^{\frac{\alpha}{1-\beta}-\nu\frac{q-1}{1-\beta}}\right)^{\left(\frac{1-\beta}{1-2\beta}p_0\right)'} dx\right)^{\frac{1}{\left(\frac{1-\beta}{1-2\beta}p_0\right)'}} C\,\|h\|^{\frac{1-2\beta}{1-\beta}}\right)^{1-\beta} \|h\|^{2\beta}$$

$$= ||u||^{q-1}C\left(\int_{B_R} |x|^{\frac{\alpha-\nu(q-1)}{N-a_0+2(1-2\beta+a_0\beta)}2N}dx\right)^{\frac{N-a_0+2(1-2\beta+a_0\beta)}{2N}} \|h\|,$$

where we used (2.9) and Lemma 2.3.

3. If $\beta = \frac{1}{2}$, there follows

$$\frac{1}{\Lambda}\int_{B_R} K(|x|)|u|^{q-1}|h|dx \leq \int_{B_R} |x|^{\alpha}|u|^{q-1}V(|x|)^{\frac{1}{2}}|h|dx$$

$$\leq \left(\int_{B_R} |x|^{2\alpha}|u|^{2(q-1)}dx\right)^{\frac{1}{2}} \left(\int_{B_R} V(|x|)|h|^2dx\right)^{\frac{1}{2}}$$

$$\leq ||u||^{q-1}C\left(\int_{B_R} |x|^{2\alpha-2\nu(q-1)}dx\right)\|h\|.$$

4. If $\frac{1}{2} < \beta < 1$, then we can apply Hölder's inequality first with conjugate exponents 2, then $\frac{1}{2\beta-1}$, $\frac{1}{2(1-\beta)}$. We get

$$\frac{1}{\Lambda}\int_{B_R} K(|x|)|u|^{q-1}|h|dx$$

$$\leq \int_{B_R} |x|^{\alpha} V(|x|)^{\beta}|u|^{q-1}|h|dx \leq \int_{B_R} |x|^{\alpha} V(|x|)^{\frac{2\beta-1}{2}}|u|^{q-1}V(|x|)^{\frac{1}{2}}|h|dx$$

$$\leq \left(\int_{B_R} |x|^{2\alpha}V(|x|)^{2\beta-1}|u|^{2(q-1)}dx\right)^{\frac{1}{2}} \left(\int_{B_R} V(|x|)|h|^2 dx\right)^{\frac{1}{2}}$$

$$\leq \left(\int_{B_R} |x|^{2\alpha}|u|^{2(q-2\beta)}V(|x|)^{2\beta-1}|u|^{2(2\beta-1)}dx\right)^{\frac{1}{2}} \|h\|$$

$$\leq \left(\left(\int_{B_R} |x|^{\frac{\alpha}{1-\beta}}|u|^{\frac{q-2\beta}{1-\beta}}dx\right)^{2(1-\beta)} \left(\int_{B_R} V(|x|)|u|^2 dx\right)^{2\beta-1}\right)^{\frac{1}{2}} \|h\|$$

$$\leq C\,||u||^{q-2\beta} \left(\left(\int_{B_R} |x|^{\frac{\alpha}{1-\beta}-\nu\frac{q-2\beta}{1-\beta}}dx\right)^{2(1-\beta)} \left(\int_{B_R} V(|x|)|u|^2 dx\right)^{2\beta-1}\right)^{\frac{1}{2}} \|h\|$$

$$= C\,||u||^{q-2\beta} \left(\int_{B_R} |x|^{\frac{\alpha-\nu(q-2\beta)}{1-\beta}}dx\right)^{1-\beta} \left(\int_{B_R} V(|x|)|u|^2 dx\right)^{\frac{2\beta-1}{2}} \|h\|$$

$$\leq C \left(\int_{B_R} |x|^{\frac{\alpha-\nu(q-2\beta)}{1-\beta}}dx\right)^{1-\beta} \|u\|_A^{q-1}\|h\|.$$

5. If $\beta = 1$, then the hypothesis $q > \max\{1, 2\beta\}$ implies $q > 2$. Thus, we have

$$\frac{1}{\Lambda}\int_{B_R} K(|x|)|u|^{q-1}|h|dx \leq \int_{B_R} |x|^{\alpha}V(|x|)|u|^{q-1}|h|dx$$

$$\leq \int_{B_R} |x|^{\alpha}V(|x|)^{\frac{1}{2}}|u|^{q-1}V(|x|)^{\frac{1}{2}}|h|dx$$

$$\leq \left(\int_{B_R} |x|^{2\alpha}V(|x|)|u|^{2(q-1)}dx\right)^{\frac{1}{2}} \left(\int_{B_R} V(|x|)|h|^2 dx\right)^{\frac{1}{2}}$$

$$\leq \left(\int_{B_R} |x|^{2\alpha}|u|^{2(q-2)}V(|x|)|u|^2 dx\right)^{\frac{1}{2}} \|h\|$$

$$\leq C||u||^{q-2}\left(\int_{B_R}|x|^{2\alpha-2\nu(q-2)}V(|x|)|u|^2dx\right)^{\frac{1}{2}}\|h\|.$$

The proof is now concluded. □

Lemma 4.2. *Let $N \geq 3$ and $R > 0$. Assume $[A]$, $[V]$, $[K]$. Assume also that*

$$\Lambda := \operatorname*{ess\,sup}_{x\in B_R^c}\frac{K(|x|)}{|x|^\alpha V(|x|)^\beta} < +\infty \quad for\ some\ 0 \leq \beta \leq 1\ and\ \alpha \in \mathbb{R}.$$

Then $\forall u, h \in X$ and $\forall q > \max\{1, 2\beta\}$ we have

$$\int_{B_R^c} K(|x|)|u|^{q-1}|h|dx$$

$$\leq \begin{cases} \Lambda||u||^{q-1}\,C\,s_0^{1-2\beta}\left(\int_{B_R^c}|x|^{\frac{\alpha-\nu(q-1)}{N-a_\infty+2(1-2\beta+a_\infty\beta)}2N}dx\right)^{\frac{N-a_\infty+2(1-2\beta+a_\infty\beta)}{2N}}\|h\| & 0\leq\beta\leq\frac{1}{2}\\ \Lambda||u||^{q-1}\,C\,\left(\int_{B_R^c}|x|^{\frac{\alpha-\nu(q-2\beta)}{1-\beta}}dx\right)^{1-\beta}\|h\| & \frac{1}{2}<\beta<1\\ \Lambda||u||^{q-2}\,C\,\left(\int_{B_R^c}|x|^{2\alpha-2\nu(q-2)}V(|x|)|u|^2dx\right)^{\frac{1}{2}}\|h\| & \beta=1\end{cases}$$

where $\nu := \frac{N+a_\infty-2}{2}$ and $C = C(N, R, a_0, a_\infty)$.

The proof of Lemma 4.2 is the same as that of Lemma 4.1, and we will skip it.

Definition 4.3. For $\alpha \in \mathbb{R}$, $\beta \in [0, 1]$ and $a \in (2-N, 2]$ we define the functions $\alpha^*(a, \beta)$ and $q^*(a, \alpha, \beta)$ as follows:

$$\alpha^*(a,\beta) := \max\left\{2\beta-1-\frac{N}{2}-a\beta+\frac{a}{2}, -(1-\beta)N\right\}$$

$$= \begin{cases} 2\beta-1-\frac{N}{2}-a\beta+\frac{a}{2} & if\ 0\leq\beta\leq\frac{1}{2}\\ -(1-\beta)N & if\ \frac{1}{2}\leq\beta\leq 1\end{cases}$$

$$q^*(a,\alpha,\beta) := 2\,\frac{\alpha-2\beta+N+a\beta}{N+a-2}.$$

Theorem 4.4. *Let $N \geq 3$. Assume $[A]$, $[V]$, $[K]$. Assume also that there exists $R_1 > 0$ such that*

$$\operatorname*{ess\,sup}_{x \in B_{R_1}} \frac{K(|x|)}{|x|^{\alpha_0} V(|x|)^{\beta_0}} < +\infty \quad for\ some\ 0 \leq \beta_0 \leq 1\ and\ \alpha_0 > \alpha^*(a_0, \beta_0). \tag{4.1}$$

Then $\lim_{R \to 0^+} \mathcal{S}_0(q_1, R) = 0$ *for any* $q_1 \in \mathbb{R}$ *such that*

$$\max\{1, 2\beta_0\} < q_1 < q^*(a_0, \alpha_0, \beta_0). \tag{4.2}$$

Proof. Let $u, h \in X$ satisfy $\|u\| = \|h\| = 1$. Take R such that $0 < R < R_1$. Of course we have

$$\operatorname*{ess\,sup}_{x \in B_R} \frac{K(|x|)}{|x|^{\alpha_0} V(|x|)^{\beta_0}} \leq \operatorname*{ess\,sup}_{x \in B_{R_1}} \frac{K(|x|)}{|x|^{\alpha_0} V(|x|)^{\beta_0}} < +\infty,$$

so we can apply Lemma 4.1 with $\alpha = \alpha_0$ and $\beta = \beta_0$ and $h = u$.

1. If $0 \leq \beta_0 \leq \frac{1}{2}$ we obtain

$$\int_{B_R} K(|x|)|u|^{q_1} dx = \int_{B_R} K(|x|)|u|^{q_1 - 1} |u| \, dx$$

$$\leq C \left(R^{\frac{2\alpha_0 - 4\beta_0 + 2N + 2a_0\beta_0 - (N + a_0 - 2)q_1}{N - a_0 + 2(1 - 2\beta_0 + a_0\beta_0)} N} \right)^{\frac{N - a_0 + 2(1 - 2\beta_0 + a_0\beta_0)}{2N}},$$

since

$$\frac{\alpha_0 - \nu(q-1)}{N - a_0 + 2(1 - 2\beta_0 + a_0\beta_0)} 2N + N = \frac{2\alpha_0 - 4\beta_0 + 2N + 2a_0\beta_0 - (N + a_0 - 2)q_1}{N - a_0 + 2(1 - 2\beta_0 + a_0\beta_0)} N$$

and it is easy to check that

$$2\alpha_0 - 4\beta_0 + 2N + a_0\beta_0 - (N + a_0 - 2)q_1 = (N + a_0 - 2)(q^*(a_0, \alpha_0, \beta_0) - q_1) > 0,$$

$$N - a_0 + 2(1 - 2\beta_0 + a_0\beta_0) \geq N > 0.$$

2. If $\frac{1}{2} < \beta_0 < 1$ we get

$$\int_{B_R} K(|x|)|u|^{q_1} dx = \int_{B_R} K(|x|)|u|^{q_1 - 1} |u| \, dx \leq C \left(R^{\frac{2\alpha_0 - (N + a_0 - 2)(q_1 - 2\beta_0)}{2(1 - \beta_0)} + N} \right)^{1 - \beta_0},$$

because

$$\frac{\alpha_0 - \nu(q_1 - 2\beta_0)}{1-\beta_0} + N = \frac{\alpha_0 - \frac{N+a_0-2}{2}(q_1 - 2\beta_0)}{1-\beta_0} + N$$

$$= \frac{2\alpha_0 - (N+a_0-2)(q_1-2\beta_0)}{2(1-\beta_0)} + N = \frac{N+a_0-2}{2(1-\beta_0)}(q^*(a_0,\alpha_0,\beta_0) - q_1) > 0.$$

3. Finally, if $\beta_0 = 1$, it holds that

$$\int_{B_R} K(|x|)|u|^{q_1} dx = \int_{B_R} K(|x|)|u|^{q_1-1}\,|u|\,dx \le CR^{\frac{2\alpha_0-(N+a_0-2)(q_1-2)}{2}},$$

because

$$2\alpha_0 - 2\nu(q_1-2) = 2\alpha_0 - 2\frac{N+a_0-2}{2}(q_1-2)$$

$$= 2\alpha_0 - (N+a_0-2)(q_1-2) = (N+a_0-2)(q^*(a_0,\alpha_0,1) - q_1) > 0.$$

Hence, in any of the previous cases there exist a constant $\delta = \delta(N,a_0,\alpha_0,\beta_0,q_1) > 0$ such that

$$\mathcal{R}_0(q_1,R) \le CR^{\delta} \to 0 \quad \text{as } R \to 0,$$

from which our thesis follows. ☐

Theorem 4.5. *Let $N \ge 3$. Assume $[A]$, $[V]$, $[K]$. Assume also that there exists $R_2 > 0$ such that*

$$\operatorname*{ess\,sup}_{|x|>R_2} \frac{K(|x|)}{|x|^{\alpha_\infty} V(|x|)^{\beta_\infty}} < +\infty \quad \textit{for some } 0 \le \beta_\infty \le 1 \textit{ and } \alpha_\infty \in \mathbb{R}.$$

Then $\lim_{R\to+\infty} \mathcal{S}_\infty(q_2,R) = 0$ for each $q_2 \in \mathbb{R}$ such that

$$q_2 > \max\{1, 2\beta_\infty, q^*(a_\infty,\alpha_\infty,\beta_\infty)\}. \tag{4.3}$$

Proof. Let $u \in X$ satisfy $\|u\| = 1$. Consider $R \ge R_2$, of course we have

$$\operatorname*{ess\,sup}_{x\in B_R^c} \frac{K(|x|)}{|x|^{\alpha_\infty} V(|x|)^{\beta_\infty}} \le \operatorname*{ess\,sup}_{|x|>R_2} \frac{K(|x|)}{|x|^{\alpha_\infty} V(|x|)^{\beta_\infty}} < +\infty,$$

hence we can apply Lemma 4.2 with $\alpha = \alpha_\infty$ and $\beta = \beta_\infty$. The arguments are the same as in Theorem 4.4, so we will skip the details.

1. If $0 \leq \beta_\infty \leq \frac{1}{2}$, with similar considerations of those used for β_0, we find

$$\int_{B_R} K(|x|)|u|^{q_2} dx = \int_{B_R} K(|x|)|u|^{q_2-1}\,|u|\,dx$$

$$\leq C\left(R^{\frac{2\alpha_\infty - 4\beta_\infty + 2N + 2a_\infty\beta_\infty - (N+a_\infty-2)q_2}{N-a_\infty+2(1-2\beta_\infty+a_\infty\beta_\infty)}N}\right)^{\frac{N-a_\infty+2(1-2\beta_\infty+a_\infty\beta_\infty)}{2N}}$$

since

$$2\alpha_\infty - 4\beta_\infty + 2N + a_\infty\beta_\infty - (N+a_\infty-2)q_2 < 0,$$

$$N - a_\infty + 2(1-2\beta_\infty + a_\infty\beta_\infty) \geq N > 0.$$

2. On the other hand, if $\frac{1}{2} < \beta_\infty < 1$, we have

$$\int_{B_R} K(|x|)|u|^{q_2} dx = \int_{B_R} K(|x|)|u|^{q_2-1}\,|u|\,dx \leq C\left(R^{\frac{2\alpha_\infty - (N+a_\infty-2)(q_2-2\beta_\infty)}{2(1-\beta_\infty)}+N}\right)^{1-\beta_\infty},$$

as

$$\frac{2\alpha_\infty - (N+a_\infty-2)(q_2-2\beta_\infty)}{2(1-\beta_\infty)} + N < 0.$$

3. Finally, if $\beta_\infty = 1$, we obtain

$$\int_{B_R} K(|x|)|u|^{q_2} dx = \int_{B_R} K(|x|)|u|^{q_2-1}\,|u|\,dx \leq CR^{\frac{2\alpha_\infty - (N+a_\infty-2)(q_2-2)}{2}},$$

because

$$2\alpha_\infty - (N+a_\infty-2)(q_2-2) < 0.$$

In each of the previous cases, there exists $\delta = \delta(N, a, \alpha_\infty, \beta_\infty, q_2) > 0$ such that

$$\mathcal{S}_\infty(q_2, R) \leq CR^{-\delta} \to 0, \quad \text{as } R \to \infty,$$

from which our thesis follows. $\square$

From the previous theorems we easily derives our main compactness result.

Theorem 4.6. *Assume* $N \geq 3$. *Assume* $[A]$, $[V]$, $[K]$. *Moreover, assume the hypotheses of the two previous theorems, that is: there are* $R_1, R_2 > 0$, $\alpha_0, \alpha_\infty \in \mathbb{R}$, $\beta_0, \beta_\infty \in [0,1]$ *such that*

$$\operatorname{ess}\sup_{|\mathrm{x}|<\mathrm{R}_1} \frac{K(|x|)}{|x|^{\alpha_0} V(|x|)^{\beta_0}} < +\infty, \ \operatorname{ess}\sup_{|\mathrm{x}|\geq\mathrm{R}_2} \frac{K(|x|)}{|x|^{\alpha_\infty} V(|x|)^{\beta_\infty}} < +\infty.$$

Thus, for q_1 *and* q_2 *such that*

$$\begin{cases} q_1 \in \mathcal{I}_1 = (\max\{1, 2\beta_0\},\ q^*(a_0, \alpha_0, \beta_0)) \\ q_2 \in \mathcal{I}_2 = (\max\{1, 2\beta_\infty,\ q^*(a_\infty, \alpha_\infty, \beta_\infty)\}, +\infty) \end{cases}$$

the embedding

$$X \hookrightarrow L_K^{q_1}(\mathbb{R}^N) + L_K^{q_2}(\mathbb{R}^N).$$

is continuous and compact.

5 Applications: existence results

We now use these results on compact embeddings to obtain existence and multiplicity results for nonlinear elliptic equations. We will deal with equation (1.1), and we will assume hypotheses $[A], [V], [K]$. As to the nonlinearity f we will assume the following hypotheses.

(f_1) $f : \mathbb{R} \to \mathbb{R}$ is a continuous functions, and there are constants $q_1, q_2 > 2$ and $M > 0$ such that

$$|f(t)| \leq M \min\{|t|^{q_1-1}, |t|^{q_2-1}\}, \quad \text{for all } t \in \mathbb{R}$$

(f_2) Define $F(t) = \int_0^t f(s)\,ds$, then there is $\theta > 2$ such that $0 \leq \theta F(t) \leq f(t)t$ for all t. Furthermore there is $t_0 > 0$ such that $F(t_0) > 0$.

The simplest example of a function satisfying (f_1), (f_2) is given by

$$f(t) = \min\{t^{q_1-1}, t^{q_2-1}\}$$

if $t \geq 0$, and $f(t) = -f(-t)$ if $t \leq 0$ (or also $f(t) = 0$ if $t \leq 0$), with $q_1, q_2 > 2$. Notice that if $q_1 \neq q_2$ there is no pure power function, i.e. $f(t) = t^q$, satisfying (f_1). However we do not assume $q_1 \neq q_2$, so pure power functions are included in our results, when the hypotheses will allow to choose $q_1 = q_2$.
We define the functional $I : X \to \mathbb{R}$

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} A(|x|)\,|\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} V(|x|)\,u^2\,dx - \int_{\mathbb{R}^N} K(|x|)\,F(u)\,dx.$$

Theorem 5.1. *Assume the hypotheses of Theorem 4.6. Assume* $(f_1), (f_2)$ *with* $q_i \in \mathcal{I}_i$*, where the intervals* $\mathcal{I}_i$ *are given in Theorem 4.6. Then* I *is a* C^1 *functional on* X*, whose differential is given by*

$$I'(u)h = \int_{\mathbb{R}^N} A(|x|)\, \nabla u \nabla h \, dx + \int_{\mathbb{R}^N} V(|x|) u \, h \, dx - \int_{\mathbb{R}^N} K(|x|)\, f(u) h \, dx$$

for all $u, h \in X$.

Proof. We know that the embedding of X in $L_K^{q_1} + L_K^{q_2}$ is continuous. By the previous results and proposition 3.8 of [8] we also know that the functional

$$\Phi(u) = \int_{\mathbb{R}^N} K(|x|)\, F(u) dx$$

is of class C^1 on $L_K^{q_1} + L_K^{q_2}$, with differential given by

$$\Phi'(u)h = \int_{\mathbb{R}^N} K(|x|)\, f(u) h dx.$$

Obviously the quadratic part of I is C^1, with differential given by

$$h \to \int_{\mathbb{R}^N} A(|x|) \nabla u \nabla h dx + \int_{\mathbb{R}^N} V(|x|) f(u) h dx.$$

The thesis easily follows.

□

Theorem 5.2. *Assume the hypotheses of Theorem 4.6. Assume* $(f_1), (f_2)$ *with* $q_i \in \mathcal{I}_i$*, where the intervals* $\mathcal{I}_i$ *are given in Theorem 4.6. Then* $I : X \to \mathbb{R}$ *satisfies the Palais-Smale condition.*

Proof. Assume that $\{u_n\}_n$ is a sequence in X such that $I(u_n)$ is bounded and $I'(u_n) \to 0$ in X'. We have to prove that $\{u_n\}_n$ has a converging subsequence. For this, notice that from the hypotheses we derive, for a suitable positive constant C,

$$C + C||u_n|| \geq I(u_n) + \frac{1}{\theta} I'(u_n) u_n = \left(\frac{1}{2} - \frac{1}{\theta} \right) ||u_n||^2 + \int_{\mathbb{R}^N} \left(\frac{1}{\theta} f(u_n) u_n - F(u_n) \right) dx$$

$$\geq \left(\frac{1}{2} - \frac{1}{\theta} \right) ||u_n||^2,$$

and this implies that $\{u_n\}_n$ is bounded. So we can assume, up to a subsequence, $u_n \rightharpoonup u$ in X and $u_n \to u$ in $L_K^{q_1}(\mathbb{R}^N) + L_K^{q_2}(\mathbb{R}^N)$. Now we have

$$||u_n - u||^2 = (u_n|u_n - u) - (u|u_n - u) = I'(u_n)(u_n - u) + \Phi'(u_n)(u_n - u) - (u|u_n - u).$$

Of course $(u|u_n - u) \to 0$ because $u_n \rightharpoonup u$ in X. We also have that $I'(u_n) \to 0$ in X' while $u_n - u$ is bounded in X, so $I'(u_n)(u_n - u) \to 0$. Lastly, we know that Φ is C^1 in the space $L^{q_1}_K(\mathbb{R}^N) + L^{q_2}_K(\mathbb{R}^N)$, and $u_n \to u$ in that space, so $\Phi'(u_n)$ is bounded (as a sequence in the dual space) and $u_n - u \to 0$, so $\Phi'(u_n)(u_n - u) \to 0$. Hence we get $||u_n - u||^2 \to 0$, which is the thesis.

□

Theorem 5.3. *Assume the hypotheses of Theorem 4.6. Assume* $(f_1), (f_2)$ *with* $q_i \in \mathcal{I}_i$*, where the intervals* $\mathcal{I}_i$ *are given in Theorem 4.6. Then* $I : X \to \mathbb{R}$ *has a non negative and non trivial critical point.*

Proof. Firstly, to have non negative solution, we assume as usual $f(t) = 0$ for $t \leq 0$. To prove the theorem we apply the standard Mountain Pass Lemma. We have proven that I satisfies the Palais-Smale condition, so it is enough to prove that it has the usual Mountain Pass geometry, that is, we have to prove the following two conditions:

(i) There are $\rho, \alpha > 0$ such that $I(u) \geq \alpha$ for all $||u|| = \rho$.

(ii) There is $v \in X$ such that $||v|| \geq \rho$ and $I(u) \leq 0$.

As for (i), let us take $0 < R_1 < R_2$ such that

$$\mathcal{S}_0(q_1, R_1) < +\infty, \quad \mathcal{S}_\infty(q_2, R_2) < +\infty,$$

which is possible because $q_i \in \mathcal{I}_i$. Then, using the definition of $\mathcal{S}_0, \mathcal{S}_\infty$, lemma 2.14 and the embedding of X in $L^2(B_{R_2} \backslash B_{R_1})$, we get

$$\int_{B_{R_1}} K(|x|)|u|^{q_1} \leq c_1||u||^{q_1}, \int_{B^c_{R_2}} K(|x|)|u|^{q_2} \leq c_2||u||^{q_2}, \int_{B_{R_2} \backslash B_{R_1}} K(|x|)|u|^{q_1} \leq c_1||u||^{q_i}.$$

Hence

$$\Big| \int_{\mathbb{R}^N} K(|x|)F(u)\, dx \Big| \leq$$

$$\int_{B_{R_1}} K(|x|)F(u)\, dx + \int_{B_{R_2} \backslash B_{R_1}} K(|x|)F(u)\, dx + \int_{\mathbb{R}^N \backslash B_{R_2}} K(|x|)F(u)\, dx \leq$$

$$M \int_{B_{R_1}} K(|x|)|u|^{q_1}\, dx + c_1||u||^{q_1} + M \int_{\mathbb{R}^N \backslash B_{R_2}} K(|x|)|u|^{q_2}\, dx \leq$$

$$c_3||u||^{q_1} + c_4||u||^{q_2},$$

so that

$$I(u) \geq \frac{1}{2}||u||^2 - c_3||u||^{q_1} - c_4||u||^{q_2},$$

and (i) easily follows.
To get (ii), we start remarking that, from (f_2), there is $c > 0$ such that, for all $t \geq t_0$, $F(t) \geq c\,t^\theta$. The potential K is not zero a.e., and from this fact it is easy to deduce that there are $\delta > 0$ and a measurable subset $A_\delta \subset (\delta, 1/\delta)$ such that $|A_\delta| > \delta$ and $K(r) > \delta$ in A_δ. Now take a function $\varphi \in C^\infty(\mathbb{R})$ such that $0 \leq \varphi(r) \leq 1$ for all r, $\varphi(r) = 1$ for $r \in (\delta, 1/\delta)$, $\varphi(r) = 0$ for $r \leq \frac{1}{2\delta}$ and $r \geq 1 + 1/\delta$. Define now, for $x \in \mathbb{R}$, $\psi(x) = \varphi(|x|)$. As $\varphi \in C_c^\infty(\mathbb{R}\backslash\{0\})$, it is $\psi \in C_c^\infty(\mathbb{R}^N\backslash\{0\})$, and furthermore ψ is radial, so $\psi \in X$. Define $\Omega_\delta = \{x \in \mathbb{R}^N \mid |x| \in A_\delta\}$. Hence, if we take $\lambda > t_0$ we get

$$K(|x|)F(\lambda\psi(x)) \geq \delta\, c\lambda^\theta \quad \text{in } \Omega_\delta,$$

and $K(|x|)F(\lambda\psi(x)) \geq 0$ for all x, so that

$$\int_{\mathbb{R}^N} K(|x|)F(\lambda\psi(x)) \geq \int_{\Omega_\delta} K(|x|)F(\lambda\psi(x)) \geq c\delta\,\lambda^\theta|\Omega_\delta| = C_\delta\lambda^\theta,$$

where $C_\delta > 0$ depends only on δ and N. We then get

$$I(\lambda\psi) \leq \lambda^2||\psi||^2 - C_\delta\lambda^\theta,$$

so $I(\lambda\psi) \to -\infty$ as $\lambda \to +\infty$, and this gives the result.

□

As I satisfies the Palais-Smale condition, arguing as in the proof of Theorem 1.2 in [8], we also get a result of existence of infinity solutions.

Theorem 5.4. *Assume the hypotheses of Theorem 4.6. Assume $(f_1), (f_2)$ with $q_i \in \mathcal{I}_i$, where the intervals $\mathcal{I}_i$ are given in Theorem 4.6. Assume furthermore the following two assumptions*

(f_3) *There exists $m > 0$ such that $F(t) \geq m\,\min\{t^{q_1}, t^{q_2}\}$ for all $t > 0$.*

(f_4) *f is an odd function.*

Then $I : X \to \mathbb{R}$ has a sequence $\{u_n\}_n$ of critical points such that $I(u_n) \to +\infty$.

6 Examples

In this section we give some examples that could help to understand what is new (and what is not) in our results. We will make a comparison, in some concrete cases, between our results and those of [22]. In this paper the authors study a p-laplacian equation, so we compare their results with ours only in the case $p = 2$. Our problem is also linked to those studied in [15] and [18], but in the following examples we assume $A(r) = \min\{r^\alpha, r^\beta\}$, with $\alpha \neq \beta$, and this rules out the results of [15] and [18], in which it is $A(r) = r^\alpha$, for some $\alpha \in \mathbb{R}$. In [22], the authors define three functions q^*, q_*, q_{**} which depends on the asymptotic behavior of the potentials, and find existence of solutions for, say, $f(t) = t^{q_1-1} + t^{q_2-1}$ when $q_i \in (q_*, q^*)$ or when $q_i > \max\{q_*, q_{**}\}$ $(i = 1, 2)$.

1. Let us choose the functions A, V, K as follows:

$$A(r) = \min\{r^2, r^{3/2}\}; \quad V(r) = \min\left\{1, \frac{1}{r^{1/2}}\right\}; \quad K(r) = \max\{r^{1/2}, r^{3/2}\}.$$

It is simple to verify that in this case the results of [22] do not apply, because if we compute the functions q^* and q_* it happens $q_* = \frac{4N+6}{2N-1}$ and $q^* = \frac{2N+1}{N}$ (and q_{**} is not defined), so that $q^* < q_*$ while $q^* > q_*$ is a needed hypothesis. To apply our results, we can choose $\beta_0 = \beta_\infty = 0$, $\alpha_0 = 1/2$, $\alpha_\infty = 3/2$, $a_0 = 2$, $a_\infty = 3/2$. We then get $q^*(a_0, \alpha_0, \beta_0) = \frac{2N+1}{N} > 2$ and $q^*(a_\infty, \alpha_\infty, \beta_\infty) = \frac{4N+6}{2N-1}$. Hence, if we choose

$$2 < q_1 < \frac{2N+1}{N} < \frac{4N+6}{2N-1} < q_2,$$

and $f(t) = \min\{t^{q_1-1}, t^{q_2-1}\}$, we can apply our existence results. Notice that in this case $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$.

2. Assume $N \geq 6$ and choose the functions A, V, K as follows:

$$A(r) = \max\left\{\frac{1}{r^2}, \frac{1}{r^3}\right\}; \quad V(r) = \frac{1}{r^4}; \quad K(r) = \min\left\{1, \frac{1}{r^2}\right\}.$$

In this case, following [22], the computations give $q^* = \frac{2N}{N-5}$ and $q_* = \frac{2(N-2)}{N-4}$, and it is $q_* < q^*$ for $N \geq 3$, so in this case they get existence for $q_i \in \left(\frac{2(N-2)}{N-4}, \frac{2N}{N-5}\right)$. To apply our results, we can choose $\beta_0 = \beta_\infty = \alpha_0 = 0$, $\alpha_\infty = -2$, $a_0 = -3$ and $a_\infty = -2$. We then get $q^*(a_0, \alpha_0, \beta_0) = \frac{2N}{N-5}$ and $q^*(a_\infty, \alpha_\infty, \beta_\infty) = \frac{2(N-2)}{N-4}$, so $\mathcal{I}_1 \cap \mathcal{I}_2 = \left(\frac{2(N-2)}{N-4}, \frac{2N}{N-5}\right)$, the same interval. Hence, for pure power functions we obtain exactly the same result as in [22], while we can not treat functions like $f(t) = t^{q_1-1} + t^{q_2-1}$. On the other hand, we are free to choose $2 < q_1 < \frac{2(N-2)}{N-4} < q_2$ and $f(t) = \min\{t^{q_1-1}, t^{q_2-1}\}$ and such a function does not satisfy the hypotheses of [22], because it satisfies (f_2) with $\theta = q_1 < \frac{2(N-2)}{N-4}$, which is not allowed in [22].

3. Finally, assume again $N \geq 6$ and choose the functions A, V, K as follows:

$$A(r) = \max\left\{\frac{1}{r^2}, \frac{1}{r^3}\right\}; \quad V(r) = e^{2r}; \quad K(r) = e^r.$$

In this case the results of [22] do not apply because of the exponential growth of the potential K. We can choose $a_0 = -3$, $a_\infty = -2$, $\beta_0 = \alpha_0 = \alpha_\infty = 0$, $\beta_\infty = \frac{1}{2}$, and we get, as before, $q^*(a_0, \alpha_0, \beta_0) = \frac{2N}{N-5}$ and $q^*(a_\infty, \alpha_\infty, \beta_\infty) = \frac{2(N-2)}{N-4}$, so again we get existence of solution for functions like $f(t) = \min\{t^{q_1-1}, t^{q_2-1}\}$ and for the same range of exponents q_i as above. In particular we can choose $f(t) = t^{q-1}$ for $q \in \left(\frac{2(N-2)}{N-4}, \frac{2N}{N-5}\right)$.

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