ON THE AUTOMORPHISM GROUP OF A SYMPLECTIC HALF-FLAT 6-MANIFOLD

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ABSTRACT. We prove that the automorphism group of a compact 6-manifold M endowed with a symplectic half-flat SU(3)-structure has abelian Lie algebra with dimension bounded by $\min\{5, b_1(M)\}$. Moreover, we study the properties of the automorphism group action and we discuss relevant examples. In particular, we provide new complete examples on $T\mathbb{S}^3$ which are invariant under a cohomogeneity one action of SO(4).

1. Introduction

An SU(3)-structure on a six-dimensional smooth manifold M is the data of an almost Hermitian structure (g, J) with fundamental 2-form $\omega := g(J\cdot, \cdot)$ and a complex volume form $\Psi = \psi + i\,\widehat{\psi} \in \Omega^{3,0}(M)$ such that

$$\psi \wedge \widehat{\psi} = \frac{2}{3}\,\omega^3. \tag{1.1}$$

By [11], the whole data (g, J, Ψ) is completely determined by the real 2-form ω and the real 3-form ψ, provided that they satisfy suitable conditions (see §3 for more details).

An SU(3)-structure (ω, ψ) is said to be *symplectic half-flat* if both ω and ψ are closed. In this case, the intrinsic torsion can be identified with a unique real $(1,1)$-form σ which is primitive with respect to ω, i.e., $\sigma \wedge \omega^2 = 0$, and fulfills $d\widehat{\psi} = \sigma \wedge \omega$ (see e.g. [4]). This SU(3)-structure is *half-flat* according to [4, Def. 4.1], namely $d(\omega^2) = 0$ and $d\psi = 0$, and the corresponding almost complex structure J is integrable if and only if σ vanishes identically. When this happens, (M, ω, ψ) is a *Calabi-Yau* 3-fold. Otherwise, the symplectic half-flat structure is said to be *strict*.

In recent years, symplectic half-flat structures turned out to be of interest in supersymmetric string theory. For instance, in [10] the authors proved that supersymmetric flux vacua with constant intermediate SU(2)-structure [2] are related to the existence of special classes of half-flat structures on the internal 6-manifold. In particular, they showed that solutions of Type IIA SUSY equations always admit a symplectic half-flat structure. In [12], the definition of symplectic half-flat structures, which are called supersymmetric of Type IIA, is generalized in higher dimensions, and it is proved that semi-flat supersymmetric structures of Type IIA correspond to semi-flat supersymmetric structures of Type IIB via the SYZ and Fourier-Mukai transformations.

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In mathematical literature, symplectic half-flat structures were first introduced and studied in [6] and then in [8], while explicit examples were exhibited in [5, 7, 9, 16, 20]. Most of them consist of simply connected solvable Lie groups endowed with a left-invariant symplectic half-flat structure. Moreover, in [9] it was proved that every six-dimensional compact solvmanifold with an invariant symplectic half-flat structure also admits a solution of Type IIA SUSY equations.

Let M be a 6-manifold endowed with a strict symplectic half-flat structure (ω, ψ). In the present paper, we are interested in studying the properties of the automorphism group $\mathrm{Aut}(M, \omega, \psi) \coloneqq \{f \in \mathrm{Diff}(M) \mid f^*\omega = \omega,\ f^*\psi = \psi\}$, aiming at understanding how to construct non-trivial examples with high degree of symmetry.

In [16], we proved the non-existence of compact homogeneous examples and we classified all non-compact cases which are homogeneous under the action of a semisimple Lie group of automorphisms. Here, in Theorem 2.1 we show that the Lie algebra of $\mathrm{Aut}(M, \omega, \psi)$ is abelian with dimension bounded by $\min\{5, b_1(M)\}$ whenever M is compact. This allows to obtain a direct proof of the aforementioned non-existence result. In the same theorem, we also provide useful information on geometric properties of the $\mathrm{Aut}^0(M, \omega, \psi)$-action on the manifold, proving in particular that the automorphism group acts by cohomogeneity one only when M is diffeomorphic to a torus. Some relevant examples are then discussed in order to show that the automorphism group can be non-trivial and that the upper bound on its dimension can be actually attained.

As our previous result on non-compact homogeneous spaces suggests, the non-compact ambient might provide a natural setting where looking for new examples. In section 3, we obtain new complete examples of symplectic half-flat structures on the tangent bundle $T\mathbb{S}^3$ which are invariant under the natural cohomogeneity one action of $\mathrm{SO}(4)$. These include also the well-known Calabi-Yau example constructed by Stenzel [19].

2. The automorphism group

Let M be a six-dimensional manifold endowed with an $\mathrm{SU}(3)$-structure (ω, ψ). The *automorphism group* of (M, ω, ψ) consists of the diffeomorphisms of M preserving the $\mathrm{SU}(3)$-structure, namely

$$\mathrm{Aut}(M, \omega, \psi) \coloneqq \{f \in \mathrm{Diff}(M) \mid f^*\omega = \omega,\ f^*\psi = \psi\}\,.$$

Clearly, $\mathrm{Aut}(M, \omega, \psi)$ is a closed Lie subgroup of the isometry group $\mathrm{Iso}(M, g)$, as every automorphism preserves the Riemannian metric g induced by the pair (ω, ψ). The Lie algebra of the identity component $\mathrm{G} \coloneqq \mathrm{Aut}^0(M, \omega, \psi)$ is

$$\mathfrak{g} = \{X \in \mathfrak{X}(M) \mid \mathcal{L}_X\omega = 0,\ \mathcal{L}_X\psi = 0\}\,,$$

and every $X \in \mathfrak{g}$ is a Killing vector field for the metric g. Moreover, the Lie group $\mathrm{Aut}(M, \omega, \psi) \subset \mathrm{Iso}(M, g)$ is compact whenever M is compact.

If (M, ω, ψ) is a Calabi-Yau 3-fold, i.e., if ω, ψ and $\widehat{\psi}$ are all closed, then the Riemannian metric g is Ricci-flat and $\mathrm{Hol}(g) \subseteq \mathrm{SU}(3)$. When M is compact and the holonomy group is precisely $\mathrm{SU}(3)$, it follows from Bochner's Theorem that $\mathrm{Aut}(M, \omega, \psi)$ is finite.

We now focus on *strict symplectic half-flat* structures, namely SU(3)-structures (ω, ψ) such that

$$d\omega = 0, \quad d\psi = 0, \quad d\widehat{\psi} = \sigma \wedge \omega,$$

with $\sigma \in [\Omega_0^{1,1}(M)] \coloneqq \left\{\kappa \in \Omega^2(M) \mid J\kappa = \kappa,\ \kappa \wedge \omega^2 = 0\right\}$ not identically vanishing. Notice that the condition on σ is equivalent to requiring that the almost complex structure J induced by (ω, ψ) is non-integrable (cf. e.g. [4]). In this case, we can show the following result.

Theorem 2.1. *Let M be a compact six-dimensional manifold endowed with a strict symplectic half-flat structure (ω, ψ). Then, there exists an injective map*

$$\mathscr{F} : \mathfrak{g} \to \mathscr{H}^1(M), \quad X \mapsto \iota_X \omega,$$

where $\mathscr{H}^1(M)$ is the space of Δ_g-harmonic 1-forms. Consequently, the following properties hold:

1) $\dim(\mathfrak{g}) \leq b_1(M)$*;*
2) $\mathfrak{g}$ *is abelian with* $\dim(\mathfrak{g}) \leq 5$*;*
3) for every $p \in M$, the isotropy subalgebra $\mathfrak{g}_p$ has dimension $\dim(\mathfrak{g}_p) \leq 2$*. If* $\dim(\mathfrak{g}_p) = 2$ *for some p, then* $\mathrm{G}_p = \mathrm{G}$*;*
4) the G*-action is free when* $\dim(\mathfrak{g}) \geq 4$*. In particular, when* $\dim(\mathfrak{g}) = 5$ *the manifold M is diffeomorphic to* $\mathbb{T}^6$*.*

Before proving the theorem, we show a general lemma.

Lemma 2.2. *Let (ω, ψ) be an* SU(3)*-structure. Then, for every vector field X the following identity holds*

$$\iota_X \psi \wedge \psi = -2 * (\iota_X \omega),$$

where $*$ denotes the Hodge operator determined by the Riemannian metric g and the orientation $dV_g = \frac{1}{6}\omega^3$.

Proof. From the equation $\iota_X \Psi \wedge \Psi = 0$, which holds for every vector field X, we have

$$\iota_X \psi \wedge \psi = \iota_X \widehat{\psi} \wedge \widehat{\psi}, \quad \iota_X \psi \wedge \widehat{\psi} = -\iota_X \widehat{\psi} \wedge \psi.$$

Using the above identities and the relations $\iota_X \psi = \iota_{JX} \widehat{\psi}$, $\iota_{JX} \psi = -\iota_X \widehat{\psi}$, we get

$$\begin{aligned} \iota_X \psi \wedge \psi &= \iota_{JX} \widehat{\psi} \wedge \psi \\ &= \iota_{JX}(\widehat{\psi} \wedge \psi) + \widehat{\psi} \wedge \iota_{JX} \psi \\ &= \iota_{JX}(\widehat{\psi} \wedge \psi) - \widehat{\psi} \wedge \iota_X \widehat{\psi} \\ &= \iota_{JX}(\widehat{\psi} \wedge \psi) - \psi \wedge \iota_X \psi. \end{aligned}$$

Hence, $2\, \iota_X \psi \wedge \psi = \iota_{JX}(\widehat{\psi} \wedge \psi)$. Now, from condition (1.1) we know that $\widehat{\psi} \wedge \psi = -\frac{2}{3}\, \omega^3 = -4\, dV_g$. Thus,

$$\iota_X \psi \wedge \psi = -2\, \iota_{JX} dV_g = -2 * (JX)^\flat = -2 * (\iota_X \omega).$$

□

Proof of Theorem 2.1.
Let $X \in \mathfrak{g}$. Then, using the closedness of ω we have $0 = \mathcal{L}_X\omega = d(\iota_X\omega)$. Moreover, since $d\psi = 0$ and $\mathcal{L}_X\psi = 0$, then $d(\iota_X\psi \wedge \psi) = 0$ and Lemma 2.2 implies that $d * (\iota_X\omega) = 0$. Hence, the 1-form $\iota_X\omega$ is Δ_g-harmonic and $\mathscr{F}$ coincides with the injective map $Z \mapsto \iota_Z\omega$ restricted to $\mathfrak{g}$. From this 1) follows.

In order to prove 2), we begin recalling that every Killing field on a compact manifold preserves every harmonic form. Consequently, for all $X, Y \in \mathfrak{g}$ we have

$$0 = \mathcal{L}_Y(\iota_X\omega) = \iota_{[Y,X]}\omega + \iota_X(\mathcal{L}_Y\omega) = \iota_{[Y,X]}\omega.$$

Since the map $Z \mapsto \iota_Z\omega$ is injective, we obtain that $\mathfrak{g}$ is abelian. Now, G is compact abelian and it acts effectively on the compact manifold M. Therefore, the principal isotropy is trivial and $\dim(\mathfrak{g}) \leq 6$. When $\dim(\mathfrak{g}) = 6$, M can be identified with the 6-torus $\mathbb{T}^6$ endowed with a left-invariant metric, which is automatically flat. Hence, if (ω, ψ) is strict symplectic half-flat, then $\dim(\mathfrak{g}) \leq 5$.

As for 3), we fix a point p of M and we observe that the image of the isotropy representation $\rho : \mathrm{G}_p \to \mathrm{O}(6)$ is conjugate into SU(3). Since SU(3) has rank two and G_p is abelian, the dimension of $\mathfrak{g}_p$ is at most two. If $\dim(\mathfrak{g}_p) = 2$, then the image of ρ is conjugate to a maximal torus of SU(3) and its fixed point set in T_pM is trivial. As $T_p(\mathrm{G} \cdot p) \subseteq (T_pM)^{\mathrm{G}_p}$, the orbit $\mathrm{G} \cdot p$ is zero-dimensional, which implies that $\dim(\mathfrak{g}) = 2$.

Assertion 4) is equivalent to proving that G_p is trivial for every $p \in M$ whenever $\dim(\mathfrak{g}) \geq 4$. In this case, $\dim(\mathfrak{g}_p) \leq 1$ by 3), and therefore $\dim(\mathrm{G} \cdot p) \geq 3$. If G_p contains a non-trivial element h, then $\rho(h)$ fixes every vector in $T_p(\mathrm{G} \cdot p)$ and, consequently, its fixed point set in T_pM must be non-trivial of dimension at least three. On the other hand, a non-trivial element of SU(3) is easily seen to have a fixed point set of dimension at most two. This shows that $\mathrm{G}_p = \{1_{\mathrm{G}}\}$. The last assertion follows immediately from [14]. □

Point 2) in the above theorem gives a direct proof of a result obtained in [16].

Corollary 2.3. *There are no compact homogeneous 6-manifolds endowed with an invariant strict symplectic half-flat structure.*

It is worth observing here that the non-compact case is less restrictive. For instance, it is possible to exhibit non-compact examples which are homogeneous under the action of a *semisimple* Lie group of automorphisms (see e.g. [16]). Moreover, in the next section we shall construct non-compact examples of cohomogeneity one with respect to a semisimple Lie group of automorphisms.

The next example was given in [8]. It shows that G can be non-trivial, that the upper bound on its dimension given in 2) can be attained, and that 4) is only a sufficient condition.

Example 2.4. On $\mathbb{R}^6$ with standard coordinates $(x^1, \ldots, x^6)$ consider three smooth functions $a(x^1)$, $b(x^2)$, $c(x^3)$ in such a way that

$$\lambda_1 \coloneqq b(x^2) - c(x^3), \quad \lambda_2 \coloneqq c(x^3) - a(x^1), \quad \lambda_3 \coloneqq a(x^1) - b(x^2),$$

are $\mathbb{Z}^6$-periodic. Then, the following pair of $\mathbb{Z}^6$-invariant differential forms on $\mathbb{R}^6$ induces an SU(3)-structure on $\mathbb{T}^6 = \mathbb{R}^6/\mathbb{Z}^6$:

$$\omega = dx^{14} + dx^{25} + dx^{36}, \quad \psi = -e^{\lambda_3}\, dx^{126} + e^{\lambda_2}\, dx^{135} - e^{\lambda_1}\, dx^{234} + dx^{456},$$

where $dx^{ijk\cdots}$ is a shorthand for the wedge product $dx^i \wedge dx^j \wedge dx^k \wedge \cdots$. It is immediate to check that (ω, ψ) is strict symplectic half-flat whenever the functions λ_i are not all constant. The automorphism group of $(\mathbb{T}^6, \omega, \psi)$ is $\mathbb{T}^3$ when $a(x^1)\, b(x^2)\, c(x^3) \not\equiv 0$, while it becomes $\mathbb{T}^4$ ($\mathbb{T}^5$) when one (two) of them vanishes identically.

Finally, we observe that there exist examples where the upper bound on the dimension of $\mathfrak{g}$ given in 1) is more restrictive than the upper bound given in 2).

Example 2.5. In [5], the authors obtained the classification of six-dimensional nilpotent Lie algebras admitting symplectic half-flat structures. The only two non-abelian cases are described up to isomorphism by the following structure equations

$$(0,0,0,0,e^{12},e^{13}), \quad (0,0,0,e^{12},e^{13},e^{23}).$$

Denote by N the simply connected nilpotent Lie group corresponding to one of the above Lie algebras, and endow it with a left-invariant strict symplectic half-flat structure (ω, ψ). By [13], there exists a co-compact discrete subgroup $\Gamma \subset \mathrm{N}$ giving rise to a compact nilmanifold $\Gamma\backslash\mathrm{N}$. Moreover, the left-invariant pair (ω, ψ) on N passes to the quotient defining an SU(3)-structure of the same type on $\Gamma\backslash\mathrm{N}$. By [15], we have that $b_1(\Gamma\backslash\mathrm{N})$ is either four or three.

3. Non-compact cohomogeneity one examples

In this section, we construct complete examples of strict symplectic half-flat structures on a non-compact 6-manifold admitting a cohomogeneity one action of a semisimple Lie group of automorphisms. This points out the difference between the compact and the non-compact case, and together with the results in [16, §4.3] it suggests that the non-compact ambient provides a natural setting to obtain new examples.

From now on, we consider the natural cohomogeneity one action on $M = T\mathbb{S}^3 \cong \mathbb{S}^3 \times \mathbb{R}^3$ induced by the transitive SO(4)-action on $\mathbb{S}^3$. Then, we have

$$T\mathbb{S}^3 \cong \mathrm{SO}(4) \times_{\mathrm{SO}(3)} \mathbb{R}^3.$$

We refer the reader to [1, 14, 17, 18] for basic notions on cohomogeneity one isometric actions. Following the notation of [18], we consider the Lie algebra $\mathfrak{so}(4) \cong \mathfrak{su}(2) + \mathfrak{su}(2)$ and we fix the following basis of $\mathfrak{su}(2)$

$$H \coloneqq \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad E \coloneqq \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad V \coloneqq \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Let $\gamma : \mathbb{R} \to M$ be a normal geodesic such that $p \coloneqq \gamma(0) \in \mathbb{S}^3$ and $\gamma_t \coloneqq \gamma(t)$ is a regular point for all $t \neq 0$. The singular isotropy subalgebra is $\mathfrak{so}(4)_p = \mathfrak{su}(2)_{\mathrm{diag}}$, while the principal isotropy subalgebra $\mathfrak{k} \coloneqq \mathfrak{so}(4)_{\gamma_t}$, $t \neq 0$, is one-dimensional and spanned by (H,H). We consider the following basis of $\mathfrak{so}(4) \cong \mathfrak{su}(2) + \mathfrak{su}(2)$

$$E_1 \coloneqq (E,0), \quad V_1 \coloneqq (V,0), \quad E_2 \coloneqq (0,E), \quad V_2 \coloneqq (0,V),$$
$$U \coloneqq (H,H), \quad A \coloneqq (H,-H).$$

We let $\xi \coloneqq \frac{\partial}{\partial t}$, and for any $Z \in \mathfrak{so}(4)$ we denote by $\widehat{Z}$ the corresponding fundamental vector field on M. Then, a basis of $T_{\gamma_t}M$ for $t \neq 0$ is given by

$$(\xi, \widehat{A}, \widehat{E}_1, \widehat{V}_1, \widehat{E}_2, \widehat{V}_2)_{\gamma_t}.$$

We shall denote the dual coframe along γ_t by $(\xi^*, A^*, E_1^*, V_1^*, E_2^*, V_2^*)_{\gamma_t}$, where $\xi^* \coloneqq dt$.

Let $\mathrm{K} \subset \mathrm{SO}(4)$ be the principal isotropy subgroup corresponding to the Lie algebra $\mathfrak{k}$. The space of K-invariant 2-forms on $T_{\gamma_t}M$, $t \neq 0$, is spanned by

$$\omega_1 \coloneqq \xi^* \wedge A^*, \qquad \omega_2 \coloneqq E_1^* \wedge V_1^*, \qquad \omega_3 \coloneqq E_2^* \wedge V_2^*,$$

$$\omega_4 \coloneqq E_1^* \wedge E_2^* + V_1^* \wedge V_2^*, \qquad \omega_5 \coloneqq E_1^* \wedge V_2^* - V_1^* \wedge E_2^*.$$

These forms extend as SO(4)-invariant 2-forms on the regular part $M_0 \coloneqq \mathbb{S}^3 \times \mathbb{R}^+$. By [18], their differentials along γ_t are

$$\begin{aligned} d\omega_1|_{\gamma_t} &= \tfrac{1}{4}\,\xi^* \wedge (\omega_2 - \omega_3)\,, \quad & d\omega_2|_{\gamma_t} &= d\omega_3|_{\gamma_t} = 0\,, \\ d\omega_4|_{\gamma_t} &= -2\,A^* \wedge \omega_5, & d\omega_5|_{\gamma_t} &= 2\,A^* \wedge \omega_4. \end{aligned} \tag{3.1}$$

We now describe the general SO(4)-invariant symplectic 2-form ω on M. Along γ_t, $t \neq 0$, we have

$$\omega|_{\gamma_t} = \sum_{i=1}^{5} f_i(t)\,\omega_i,$$

for suitable smooth functions $f_i \in \mathcal{C}^\infty(\mathbb{R}^+)$. By [18, Prop. 6.1], the SO(4)-invariant 2-form ω on M_0 corresponding to $\omega|_{\gamma_t}$ admits a smooth extension to the whole M if and only if the functions f_i extend smoothly on $\mathbb{R}$ as follows:

i) f_1 and f_4 are even and f_2, f_3, f_5 are odd;
ii) $f_3'(0) = \frac{1}{2}\,f_1(0) + f_2'(0)$, $f_5'(0) = -\frac{1}{4}\,f_1(0) - f_2'(0)$, and $f_4(0) = 0$.

Moreover, $\omega|_p$ is non-degenerate if and only if $f_1(0) \neq 0$.

Using (3.1), we compute $d\omega$ and we see that ω is closed if and only if

$$f_4, f_5 \equiv 0, \quad \begin{cases} f_2' = -\frac{1}{4}\,f_1 \\ f_3' = \frac{1}{4}\,f_1 \end{cases}.$$

Combining this with the extendability conditions, we obtain that every SO(4)-invariant symplectic 2-form ω on M can be written as

$$\omega|_{\gamma_t} = f_1(t)\,\omega_1 + f_2(t)\,\omega_2 + f_3(t)\,\omega_3, \quad t \neq 0, \tag{3.2}$$

with $f_1 \in \mathcal{C}^\infty(\mathbb{R})$ even and nowhere vanishing, and

$$f_2(t) = -\frac{1}{4}\int_0^t f_1(s)\,ds = -f_3(t).$$

Notice that $\omega^3|_{\gamma_t} = -6 f_1 f_2^2\,\omega_1 \wedge \omega_2 \wedge \omega_3$ at every regular point of the geodesic γ_t. As f_1 is nowhere zero, we may assume that $f_1 < 0$, so that the volume form $\xi^* \wedge A^* \wedge E_1^* \wedge V_1^* \wedge E_2^* \wedge V_2^*$ defines the same orientation on $T_{\gamma_t}M$ as $\frac{1}{6}\,\omega^3|_{\gamma_t}$ for all $t \in \mathbb{R}^+$.

We now search for an SO(4)-invariant closed 3-form $\psi \in \Omega^3(M)^{\mathrm{SO}(4)}$ so that the pair (ω, ψ) defines an SO(4)-invariant symplectic half-flat structure on M. For the sake of simplicity, we make the following Ansatz

$$\psi = du, \;\; u \in \Omega^2(M)^{\mathrm{SO}(4)}.$$

As before, along γ_t, $t \neq 0$, we can write

$$u|_{\gamma_t} = \sum_{i=1}^{5} \phi_i(t)\, \omega_i, \tag{3.3}$$

for some smooth functions $\phi_i \in \mathcal{C}^\infty(\mathbb{R}^+)$ satisfying the same extendability conditions as the f_i's. Then, omitting the dependence on t for brevity, we have

$$\psi|_{\gamma_t} = \psi_2\, \xi^* \wedge \omega_2 + \psi_3\, \xi^* \wedge \omega_3 + \phi_4'\, \xi^* \wedge \omega_4 + \phi_5'\, \xi^* \wedge \omega_5 + 2\, A^* \wedge (\phi_5\, \omega_4 - \phi_4\, \omega_5), \tag{3.4}$$

where $\psi_2 \coloneqq \frac{1}{4}\,\phi_1 + \phi_2'$ and $\psi_3 \coloneqq \phi_3' - \frac{1}{4}\,\phi_1$.

By [11], the pair (ω, ψ) defines an SU(3)-structure if and only if the following conditions hold:

a) the compatibility condition $\omega \wedge \psi = 0$;
b) the stability condition $P(\psi) < 0$, P being the characteristic quartic polynomial defined on 3-forms (see below for the definition);
c) denoted by J the almost complex structure induced by (ω, ψ), then the complex volume form $\Psi \coloneqq \psi + i\,\widehat{\psi}$ with $\widehat{\psi} \coloneqq J\psi$ fulfills the normalization condition (1.1);
d) the symmetric bilinear form $g \coloneqq \omega(\cdot, J\cdot)$ is positive definite.

The compatibility condition a) along γ_t reads $f_2\psi_3 + f_3\psi_2 = 0$. Since $f_2 = -f_3 \neq 0$, this implies

$$\psi_2 = \psi_3. \tag{3.5}$$

Recall that at each point $q \in M$ the 3-form ψ gives rise to an endomorphism $S \in \mathrm{End}(T_qM)$ defined as follows for every $\theta \in T_q^*M$ and every $v \in T_qM$

$$\iota_v\psi \wedge \psi \wedge \theta = \theta(S(v))\frac{\omega^3}{6}.$$

The endomorphism S satisfies $S^2 = P(\psi)\mathrm{Id}$, and it gives rise to the almost complex structure $J \coloneqq \frac{1}{\sqrt{|P(\psi)|}}S$ when $P(\psi) < 0$.

From the expressions

$$\begin{aligned}
\iota_\xi\psi \wedge \psi|_{\gamma_t} &= 2\left(\psi_2^2 - (\phi_4')^2 - (\phi_5')^2\right)\, \xi^* \wedge \omega_2 \wedge \omega_3 - 4\left(\phi_4'\phi_5 - \phi_4\phi_5'\right)\, A^* \wedge \omega_2 \wedge \omega_3,\\
\iota_{\widehat{A}}\psi \wedge \psi|_{\gamma_t} &= 4\left(\phi_4\phi_5' - \phi_4'\phi_5\right)\, \xi^* \wedge \omega_2 \wedge \omega_3 - 8\left(\phi_4^2 + \phi_5^2\right)\, A^* \wedge \omega_2 \wedge \omega_3,
\end{aligned}$$

we see that the endomorphism $S \in \mathrm{End}(T_{\gamma_t}M)$ maps the subspace of $T_{\gamma_t}M$ spanned by ξ and $\widehat{A}|_{\gamma_t}$ into itself with associated matrix given by

$$-\frac{1}{f_1 f_2^2}\begin{pmatrix} 4\,(\phi_4'\phi_5 - \phi_4\phi_5') & 8\left(\phi_4^2 + \phi_5^2\right) \\ 2\left(\psi_2^2 - (\phi_4')^2 - (\phi_5')^2\right) & -4\,(\phi_4'\phi_5 - \phi_4\phi_5') \end{pmatrix}. \tag{3.6}$$

Since the curve γ_t must be a normal geodesic for the metric g induced by (ω, ψ), it follows that the tangent vector ξ is orthogonal to the orbit $\mathrm{SO}(4)\cdot\gamma_t$ at every regular point of γ_t. In particular, we have

$$0 = g(\xi, \widehat{A}) = \omega(\xi, J(\widehat{A})) = \frac{1}{\sqrt{|P(\psi)|}}\,\omega(\xi, S(\widehat{A})) = \frac{4}{f_2^2\,\sqrt{|P(\psi)|}}\,(\phi_4'\phi_5 - \phi_4\phi_5'),$$

from which we get

$$\phi_4'\phi_5 = \phi_4\phi_5'. \tag{3.7}$$

Using (3.5), (3.7) and the definition of $P(\psi)$, we obtain

$$P(\psi) = \frac{16}{f_1^2 f_2^4}\left(\phi_4^2+\phi_5^2\right)\left(\psi_2^2 - (\phi_4')^2 - (\phi_5')^2\right). \tag{3.8}$$

Consequently, the stability condition b) reads

$$\psi_2^2 - (\phi_4')^2 - (\phi_5')^2 < 0, \quad \phi_4^2+\phi_5^2 \neq 0, \tag{3.9}$$

for all $t \in \mathbb{R}^+$.

We now note that the vector field $J(\xi)$ is tangent to the SO(4)-orbits and it belongs to the space of K-fixed vectors in $T_{\gamma_t}(\mathrm{SO}(4)\cdot\gamma_t)^{\mathrm{K}}$, which is spanned by $\widehat{A}|_{\gamma_t}$. Since the geodesic γ_t has unit speed, we see that

$$1 = g(\xi,\xi) = \omega(\xi, J(\xi)) = -\frac{2}{f_2^2\sqrt{|P(\psi)|}}\left(\psi_2^2 - (\phi_4')^2 - (\phi_5')^2\right). \tag{3.10}$$

Using (3.8), the relation (3.10) implies that

$$4\left(\phi_4^2+\phi_5^2\right) = f_1^2\left((\phi_4')^2 + (\phi_5')^2 - \psi_2^2\right). \tag{3.11}$$

Let us now focus on c). From (3.6) and (3.11), we obtain $J(\xi) = \frac{1}{f_1}\hat{A}|_{\gamma_t}$. Using this and the identity $\widehat{\psi} = J\psi = -\psi(J\cdot,\cdot,\cdot)$, we have

$$\widehat{\psi}|_{\gamma_t} = \xi^* \wedge \left(2\,\frac{\phi_4}{f_1}\,\omega_5 - 2\,\frac{\phi_5}{f_1}\,\omega_4\right) + f_1\,A^* \wedge \left(\psi_2\,(\omega_2+\omega_3) + \phi_4'\,\omega_4 + \phi_5'\,\omega_5\right). \tag{3.12}$$

Now, the normalization condition $\psi \wedge \widehat{\psi} = \frac{2}{3}\,\omega^3$ gives

$$4(\phi_4^2+\phi_5^2) - f_1^2\,\left(\psi_2^2 - (\phi_4')^2 - (\phi_5')^2\right) = 2\,f_1^2 f_2^2.$$

Combining this with (3.11), we obtain

$$\phi_4^2+\phi_5^2 = \frac{1}{4}(f_1f_2)^2. \tag{3.13}$$

Note that (3.8), (3.11) and (3.13) imply

$$P(\psi) \equiv -4$$

along the geodesic γ_t. Thus, the stability of ψ holds also at $t=0$.

Going back to (3.7), we see that either $\phi_4 = \lambda\phi_5$ or $\phi_5 = \lambda\phi_4$ for some $\lambda \in \mathbb{R}\smallsetminus\{0\}$. Since ϕ_4 and ϕ_5 extend as an even and an odd function on $\mathbb{R}$, respectively, we see that either $\phi_4 \equiv 0$ or $\phi_5 \equiv 0$. As f_1f_2 is an odd function on $\mathbb{R}$, (3.13) implies that

$$\phi_4 \equiv 0, \quad \phi_5 = \pm\frac{1}{2}f_1f_2. \tag{3.14}$$

The matrix associated with the symmetric bilinear form $\omega(\cdot, J\cdot)$ along γ_t, $t \in \mathbb{R}^+$, is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & f_1^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\,\frac{\phi_5'\phi_5}{f_1f_2} & 0 & -2\,\frac{\psi_2\phi_5}{f_1f_2} & 0 \\ 0 & 0 & 0 & -2\,\frac{\phi_5'\phi_5}{f_1f_2} & 0 & -2\,\frac{\psi_2\phi_5}{f_1f_2} \\ 0 & 0 & -2\,\frac{\psi_2\phi_5}{f_1f_2} & 0 & -2\,\frac{\phi_5'\phi_5}{f_1f_2} & 0 \\ 0 & 0 & 0 & -2\,\frac{\psi_2\phi_5}{f_1f_2} & 0 & -2\,\frac{\phi_5'\phi_5}{f_1f_2} \end{pmatrix},$$

and condition d) can be written as

$$-2\,\frac{\phi_5'\phi_5}{f_1f_2} > 0, \quad \psi_2^2 < (\phi_5')^2.$$

The former condition is equivalent to $(f_2^2)'' > 0$, while the latter is satisfied whenever ψ is stable (cf. (3.9)).

Note that the metric g extends smoothly over the singular orbit $\mathbb{S}^3$ to a Hermitian symmetric bilinear form. The restriction of g on $T_p\mathbb{S}^3$ is positive definite as $g_p(\widehat{A}, \widehat{A}) = f_1^2(0) > 0$ and the orbit $\mathrm{SO}(4) \cdot p$ is isotropy irreducible. Moreover, $T_pM = T_p\mathbb{S}^3 \oplus J(T_p\mathbb{S}^3)$, and from this we see that g_p is positive definite.

Summing up, the existence of a complete $\mathrm{SO}(4)$-invariant symplectic half-flat structure (ω, ψ) on M is equivalent to the existence of a smooth function $f_1 \in \mathcal{C}^\infty(\mathbb{R})$ satisfying the following conditions:

1) f_1 is even and negative;
2) the function $f_2(t) := -\frac{1}{4}\int_0^t f_1(s)ds$ satisfies $(f_2^2)'' > 0$;
3) there exists an even smooth function $\psi_2 \in \mathcal{C}^\infty(\mathbb{R})$ satisfying $\psi_2^2 = [(f_2^2)'']^2 - f_2^2$.

Indeed, given f_1 we define the symplectic form ω on M as in (3.2), with $f_3 = -f_2$. As for ψ, we let $\psi_3 := \psi_2$, $\phi_4 := 0$, and $\phi_5 := \pm\frac{1}{2}f_1f_2$ in (3.4). Then, (3.11) and (3.13) imply $\psi_2^2 = (\phi_5')^2 - f_2^2$, and we can choose the sign in the definition of ϕ_5 so that the extendability condition $\phi_5'(0) = -\psi_2(0)$ given in ii) is satisfied. It is also easy to see that we may choose ϕ_1, ϕ_2, ϕ_3 so that $\psi_2 = \frac{1}{4}\,\phi_1 + \phi_2'$ and $\psi_3 = \phi_3' - \frac{1}{4}\,\phi_1$, and the corresponding u as in (3.3) extends to a global 2-form on M. The resulting 3-form ψ is then stable by condition 3) and (3.8). The stability condition together with the inequality in 2) implies that the induced bilinear form g is everywhere positive definite. Hence, we have proved the following result.

Proposition 3.1. *The existence of a complete* $\mathrm{SO}(4)$*-invariant symplectic half-flat structure* (ω, ψ) *on* $T\mathbb{S}^3 = \mathrm{SO}(4) \times_{\mathrm{SO}(3)} \mathbb{R}^3$ *with* $\psi \in d\Omega^2(M)$ *is equivalent to the existence of a smooth function* $f_1 \in \mathcal{C}^\infty(\mathbb{R})$ *satisfying conditions* 1), 2), 3).

Recall that the symplectic half-flat structure (ω, ψ) is strict if and only if the unique 2-form $\sigma \in [\Omega_0^{1,1}(M)]$ fulfilling $d\widehat{\psi} = \sigma \wedge \omega$ is not identically zero. Starting from (3.12), using (3.1) and the identity $dA^*|_{\gamma_t} = \frac{1}{4}(\omega_3 - \omega_2)$ (cf. [18, (3.27)]), we obtain

$$d\widehat{\psi}|_{\gamma_t} = \left((f_1\phi_5')' - 4\,\frac{\phi_5}{f_1} \right) \omega_1 \wedge \omega_5 + (f_1\psi_2)'\,\omega_1 \wedge (\omega_2 + \omega_3),$$

whence

$$\sigma|_{\gamma_t} = \frac{1}{f_1}\,(f_1\psi_2)'\,(\omega_2 + \omega_3) + \frac{1}{f_1}\left((f_1\phi_5')' - 4\,\frac{\phi_5}{f_1} \right) \omega_5.$$

By [3], we know that the scalar curvature of the metric g induced by a symplectic half-flat structure is given by $\mathrm{Scal}(g) = -\frac{1}{2}|\sigma|^2$. Hence, in our case we have

$$\mathrm{Scal}(g)|_{\gamma_t} = -\frac{1}{f_1^2 f_2^2}\left[((f_1\psi_2)')^2 - \left((f_1\phi_5')' - 4\,\frac{\phi_5}{f_1} \right)^2 \right] = -\left(\frac{(f_1\psi_2)'}{f_1\phi_5'} \right)^2,$$

where the second equality follows from the relations obtained so far.

We may construct plenty of complete $\mathrm{SO}(4)$-invariant strict symplectic half-flat structures on M by choosing a suitable f_1 as above. For instance, the function

$$f_1(t) \coloneqq -\cosh(t), \quad t \in \mathbb{R},$$

fits in with conditions 1), 2), 3). With this choice, the scalar curvature is

$$\mathrm{Scal}(g)|_{\gamma_t} = -\tanh^2(t)\frac{(6\cosh^2(t) - 5)^2}{4\cosh^4(t) - 8\cosh^2(t) + 5}.$$

This shows that the resulting symplectic half-flat structure is strict and non-homogeneous.

Note that the vanishing of σ is equivalent to the vanishing of $\mathrm{Scal}(g)$. Hence, setting $(f_1\psi_2)' = 0$, the resulting $\mathrm{SU}(3)$-structure (ω, ψ) is Calabi-Yau and the associated metric is the well-known Stenzel's Ricci-flat metric on $T\mathbb{S}^3$ (cf. [19]).

Finally, we remark that the scalar curvature always vanishes at $t = 0$. Indeed, $(f_1\psi_2)'(0) = 0$, as $f_1\psi_2$ is even, while $f_1(0)\phi_5'(0) \neq 0$. This implies that an $\mathrm{SO}(4)$-invariant symplectic half-flat structure (ω, ψ) with ψ exact has constant scalar curvature if and only if it is Calabi-Yau.

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