Maslov, Chern-Weil and Mean Curvature

Tommaso Pacini*

September 28, 2018

Abstract

We provide an integral formula for the Maslov index of a pair (E, F) over a surface Σ, where $E \to \Sigma$ is a complex vector bundle and $F \subset E_{|\partial\Sigma}$ is a totally real subbundle. As in Chern-Weil theory, this formula is written in terms of the curvature of E plus a boundary contribution.

When (E, F) is obtained via an immersion of $(\Sigma, \partial\Sigma)$ into a pair (M, L) where M is Kähler and L is totally real, the formula allows us to control the Maslov index in terms of the geometry of (M, L). We exhibit natural conditions on (M, L) which lead to bounds and monotonicity results.

1 Introduction

The goal of this paper is to investigate and generalize a rather surprising relationship between two apparently unrelated quantities: the Maslov index of a surface with boundary, and the mean curvature of its boundary constraint.

Some forms of this relationship are already known, cf. *e.g.* [10] concerning Lagrangian boundary data in $\mathbb{C}^n$ or [3] concerning surfaces Σ immersed in a Kähler-Einstein manifold M whose boundaries $\partial\Sigma$ lie on a minimal Lagrangian submanifold $L \subset M$.

These geometric assumptions on (M, L) are, however, extremely strong. The Maslov index is well-defined even with respect to non-integrable complex structures on M and totally real boundary data, while mean curvature can be defined with respect to any metric. It is thus interesting to understand if such a relationship exists in a more general context.

Our tool for studying this issue is a very general integral formula for the Maslov index of an abstract "bundle pair" (E, F) over a surface with boundary $(\Sigma, \partial\Sigma)$, cf. Equation (2). Standard definitions of this index are either topological or axiomatic. The integrand in our formula involves data arising from a connection on E. It thus provides a third, geometric, definition in the spirit of Chern-Weil theory.

When the pair is determined by an immersion of $(\Sigma, \partial\Sigma)$ into (M, L) we can rephrase this formula in terms of geometric data on (M, L), cf. Equation (3).

*University of Torino, Italy, tommaso.pacini@unito.it

This formula holds under very general assumptions on (M, L). In the special case where M is Kähler and L is totally real we show that the boundary contribution can be expressed in terms of the "J-mean curvature" of L, defined in terms of a perturbed volume functional specifically tailored towards totally real submanifolds, introduced in [1]. Finally, when M is Kähler and L is Lagrangian we recover the classical expression of the boundary contribution in terms of the standard mean curvature of L. Going in the opposite direction, we thus find that the "root cause" of the appearance of mean curvature is the same as that which generates the whole body of differential geometry of totally real submanifolds developed in [7], [6], [8].

Applications of our formulae are discussed in Section 5. We show that they provide a unifying point of view on several results in the literature, also extending them to the more general context of non-constant curvature, totally real boundary data or non-integrable J.

Acknowledgements. My interest in this topic was largely triggered by stimulating conversations with Claude LeBrun, Jason Lotay and Roberta Maccheroni.

Integral formulae for certain Maslov indices also appear for example in [4], [11] and [2]. These papers however mostly focus on pairs determined by immersions of Σ into a Kähler manifold M, and only consider Lagrangian boundary data. Only the formula in [2] allows for abstract bundle pairs, but it relies on a simplifying hypothesis which completely eliminates the (Lagrangian) boundary contribution. Our own formula includes all these cases, but is much more general. In summary, results concerning surfaces immersed in M Kähler and with boundary on L Lagrangian should probably be considered classical. Beyond this case, we are not aware of serious overlaps between this paper and the available literature.

2 Preliminaries

Induced curvatures. Let E be a k-dimensional complex bundle over any manifold N, possibly with boundary. We will typically denote sections of E by σ, sections of E^* by α and tangent vectors or vector fields on N by X.

Choose a connection ∇ on E, thus $\nabla : \Lambda^0(E) \to \Lambda^1(E)$. As usual, we can extend ∇ to all tensor bundles associated to E via the Leibniz rule and by imposing that the connections commute with contractions. We will denote these induced connections using the same symbol ∇. In particular,

- The induced connection on E^*, equivalently on $\overline{E}$, satisfies

$$(\nabla_X \alpha)\sigma = X(\alpha(\sigma)) - \alpha(\nabla_X \sigma).$$

- The induced connection on $\Lambda^k E := E \wedge \cdots \wedge E$ satisfies

$$\nabla_X(\sigma_1 \wedge \cdots \wedge \sigma_k) = ((\nabla_X \sigma_1) \wedge \cdots \wedge \sigma_k) + \cdots + (\sigma_1 \wedge \cdots \wedge (\nabla_X \sigma_k)).$$

Let $R \in \Lambda^2(N) \otimes \mathrm{gl}(E)$ denote the curvature of ∇, defined by

$$R(X,Y)\sigma := \nabla_X \nabla_Y \sigma - \nabla_Y \nabla_X \sigma - \nabla_{[X,Y]}\sigma.$$

One can use the analogous formula to calculate the curvatures of the induced connections. In particular,

- The curvature on E^* is $-R^*$, where $R^* \in \Lambda^2(N) \otimes \mathrm{gl}(E^*)$ is the dual of R.
- The curvature on $\Lambda^k E$ is $\mathrm{tr}(R) \in \Lambda^2(N) \otimes \mathrm{gl}(\Lambda^k E)$.

Using the identifications $\mathrm{gl}(\Lambda^k E) \simeq \mathbb{C} \simeq \mathrm{gl}(\Lambda^k E^*)$ we conclude that the curvature on $\Lambda^k E^*$ is $-\mathrm{tr}(R^*) = -\mathrm{tr}(R) \in \Lambda^2(N)$.

Any other connection on E is of the form $\widetilde{\nabla} = \nabla + A$, for some $A \in \Lambda^1(N) \otimes \mathrm{gl}(E)$. The corresponding curvature is $\tilde{\mathrm{R}} = R + dA + A \wedge A$. On the induced bundles one obtains

- On E^*, the connections are related by $\widetilde{\nabla} = \nabla - A^*$ where $A^* \in \Lambda^1(N) \otimes \mathrm{gl}(E^*)$ is the dual of A. The corresponding curvature is $-R^* - d(A^*) + A^* \wedge A^*$.
- On $\Lambda^k E$, $\widetilde{\nabla} = \nabla + \mathrm{tr}(A)$, where $\mathrm{tr}(A) \in \Lambda^1(N) \otimes \mathrm{gl}(\Lambda^k E) \simeq \Lambda^1(N)$. The corresponding curvature is $\mathrm{tr}(R) + d(\mathrm{tr}(A))$ because $\mathrm{tr}(A) \wedge \mathrm{tr}(A) = 0$.
- On $\Lambda^k E^*$, $\widetilde{\nabla} = \nabla - \mathrm{tr}(A^*) = \nabla - \mathrm{tr}(A)$. The corresponding curvature is $-\mathrm{tr}(R) - d(\mathrm{tr}(A))$.

Recall from Chern-Weil theory that $\mathrm{tr}(R) \in \Lambda^2(N)$ is closed. It follows from the above that it is independent of ∇ up to an exact form $d(\mathrm{tr}(A))$. Choosing ∇ to be unitary with respect to some choice of Hermitian metric h on E we obtain $\mathrm{tr}(R) \in i\Lambda^2(N, \mathbb{R})$.

The Maslov index. Let $(\Sigma, \partial\Sigma)$ be a compact surface with (possibly empty) boundary. Let E be a k-dimensional complex vector bundle over Σ and $F \subset E$ a k-dimensional totally real subbundle of $E_{|\partial\Sigma}$, defined over $\partial\Sigma$.

- When $\partial\Sigma \neq \emptyset$ (thus a collection of circles) it is well-known that E is trivial, though not in a unique way. Given a choice of trivialization, one can measure the twisting (winding number) of F around $\partial\Sigma$ with respect to the trivialization of E, obtaining an integer. This is a topological definition of the *Maslov index* of the pair (E, F). One can check that it is independent of the choice of trivialization. This index can alternatively be characterized axiomatically in terms of its behaviour under natural "connect sum" operations on (E, F) and $(\Sigma, \partial\Sigma)$: we refer to [9] Appendix C.3 for details.
- When $\partial\Sigma = \emptyset$ (thus F is not defined) the Maslov index coincides with twice the first Chern number $c_1(E) \cdot \Sigma$ of E.

The Maslov index appears in many different contexts in geometry and analysis. In particular, when Σ is a Riemann surface, it appears in the generalized Riemann-Roch theorem for surfaces with boundary, cf. [9] Appendix C.1: it then replaces the term $c_1(E)\cdot\Sigma$ (*i.e.* the degree of E) which appears in the classical Riemann-Roch theorem concerning holomorphic line bundles E over closed surfaces. In both cases, an important geometric application of this theorem concerns the deformation theory (moduli spaces) of Riemann surfaces immersed in complex manifolds (M, J), where J is not necessarily integrable. In this case one chooses E to be $TM_{|\Sigma}$ (alternatively, to prevent reparametrizations, E may be the complex normal bundle over Σ). When Σ has boundary, the theory is well-defined as long as $\partial\Sigma$ is constrained to lie on a given totally real submanifold $L \subset M$: the tangent space of L then defines the subbundle F. Bounds on the Maslov index give information on the dimension of the moduli space.

Monotonicity. Deformation theory plays an important role in Symplectic Geometry. In this case we are given a symplectic manifold (M,ω) and we choose a (not necessarily integrable) compatible complex structure J. In this setting there is an important subclass of totally real manifolds: *Lagrangian submanifolds*, defined by the condition $\omega_{|TL} \equiv 0$. When $\partial\Sigma$ lies on a Lagrangian submanifold, it is interesting to compare the Maslov index with the quantity $\int_\Sigma \omega$: the Lagrangian submanifold L is *monotone* if these quantities are proportional (with a fixed constant), for any such Σ. Monotonicity implies a very strong control on the Maslov index, thus on the deformation theory of surfaces with boundary on L: this has important consequences in Floer theory.

3 A Chern-Weil formula for the Maslov index

To simplify we restrict our attention to the oriented case. Specifically, let $(\Sigma, \partial\Sigma)$ be an oriented compact surface with boundary and (E, F) be a bundle pair as in Section 2, with $\dim_{\mathbb{C}}(E) = k$ and F orientable.

Fix ∇ on E. Choosing alternatively $N := \Sigma$ or $N := \partial\Sigma$ we can consider as in Section 2 the induced connections on both $\Lambda^k E \to \Sigma$ and on $\Lambda^k E \to \partial\Sigma$. In both cases the bundle is trivial, though not uniquely: different choices are parametrized by maps $N \to \mathbb{S}^1$.

In the latter case, however, the subbundle F provides a canonical choice (up to homotopy) of a trivialization, as follows. Choose a hermitian metric h on E and an orientation of F. For any point $x \in \partial\Sigma$, define

$$\sigma_J := \frac{\sigma_1 \wedge \cdots \wedge \sigma_k}{|\sigma_1 \wedge \cdots \wedge \sigma_k|_h}, \tag{1}$$

where $\sigma_1 \dots \sigma_k$ is any positively-oriented (real) basis of the fiber F_x. It is clear that this definition is independent of the choice of basis. Choosing a different metric $\tilde{h}$ we obtain a section $\tilde{\sigma}_J = f\sigma_J$, for some $f : \partial\Sigma \to \mathbb{R}^+$, proving that σ_J is well-defined up to homotopy.

The connection on $\Lambda^k E \to \partial\Sigma$ is thus completely defined by the *connection 1-form* $\theta \in \Lambda^1(M)$ defined by the identity

$$\nabla \sigma_J = \theta \otimes \sigma_J$$

Changing metric gives $\nabla\tilde{\sigma}_J = \tilde{\theta}\otimes\tilde{\sigma}$, where $\tilde{\theta} = \theta + df$. Changing the orientation on F gives $-\sigma_J$, thus the same connection 1-form.

The main formula. We combine the curvature of $\Lambda^k E$ on Σ with the connection 1-form on $\partial\Sigma$ to define the number

$$\mu(E,F) := \frac{i}{\pi}\left(\int_\Sigma \operatorname{tr}(R) - \int_{\partial\Sigma}\theta\right). \tag{2}$$

This is independent of θ, *i.e.* of h thus of σ_J, because $\int_{\partial\Sigma}\tilde{\theta} = \int_{\partial\Sigma}\theta$. It is also independent of the choice of ∇ because, using $\nabla + A$, the induced curvature on Σ changes to $\operatorname{tr}(R) + d(\operatorname{tr}(A))$ while the connection 1-form becomes $\theta + \operatorname{tr}(A)$: the additional terms then cancel via Stokes' theorem. Choosing ∇ unitary with respect to h we find $\mu(E,F) \in \mathbb{R}$.

Notice that the pair (E,F) induces a pair $(\Lambda^k E, \Lambda^k F)$ on $(\Sigma, \partial\Sigma)$. It is clear from the definition that $\mu(E,F) = \mu(\Lambda^k E, \Lambda^k F)$. Furthermore $\mu(\overline{E},F) = -\mu(E,F)$ because the induced connection on $\overline{E} \simeq E^*$ has the opposite sign. Finally, $\mu(E,F)$ changes sign if we change the orientation of Σ.

Example 3.1 Let Δ denote the closed unit disk in $\mathbb{C}$. Let $(\Sigma,\partial\Sigma) := (\Delta, \mathbb{S}^1)$, let $E := \mathbb{C}$ be the trivial bundle and $F := T\mathbb{S}^1$ be the tangent bundle to the boundary. Choose the trivial (flat) connection and the standard metric h. Then $\sigma_J = \partial\psi$, the standard unit vector field along $\mathbb{S}^1$, and $\nabla\sigma_J = id\psi \otimes \sigma_J$ so the connection 1-form is $\theta = id\psi$. It follows that $\mu(E,F) = 2$.

Remark When $\partial\Sigma = \emptyset$, (2) coincides with the standard Chern-Weil formula for $2c_1(E)\cdot\Sigma$.

Theorem 3.2 *The number $\mu(E,F)$ defined in (2) coincides with the Maslov index of (E,F). In particular it is integer-valued.*

Proof: It is simple to check that $\mu(E,F)$ satisfies the axioms listed in [9] Appendix C.3. Example 3.1 verifies, for example, the normalization axiom. □

4 The formula for immersed surfaces

Let (M,J) be a real $2n$-dimensional manifold endowed with a (not necessarily integrable) complex structure J. Let $L \subset M$ be an oriented immersed n-dimensional totally real submanifold. In this section we assume that Σ is immersed in M in such a way that $\partial\Sigma$ is immersed in L. We then obtain canonical data $E := TM_{|\Sigma}$, the pullback tangent bundle, and $F := TL_{|\partial\Sigma}$, the totally real subbundle defined by the tangent bundle of L. Notice that $\Lambda^n E$ coincides with the anti-canonical bundle of M, *i.e.* the dual of K_M.

We are interested in computing $\mu_L(\Sigma, \partial\Sigma) := \mu(E, F)$. Fix a Hermitian metric h on M and a unitary connection ∇. In [7] it is shown that $K_{M|L}$ is trivial and admits a canonical section Ω_J. The corresponding connection 1-form is an element of $i\Lambda^1(L, \mathbb{R})$: writing it as $i\xi_J$ we obtain a real-valued 1-form ξ_J on L called the *Maslov 1-form*. Restricted to $\partial\Sigma$, Ω_J is the dual of the section σ_J defined in (1) so the corresponding connection 1-forms differ by a sign.

Let P denote the 2-form on M defined, at any point $x \in M$, by

$$P(X, Y) := \omega(R(X, Y)e_i, e_i),$$

where R is the curvature of ∇ and $e_1 \dots e_{2n}$ is an orthonormal basis of T_xM. One can check that $P = 2i\,\mathrm{tr}(R)$, so general theory ensures that $d\xi_J = \frac{1}{2}P_{|TL}$.

It follows that, in this context, we can re-write the Maslov index in terms of the geometry of (M, L):

$$\mu_L(\Sigma, \partial\Sigma) = \frac{1}{2\pi}\int_\Sigma P - \frac{1}{\pi}\int_{\partial\Sigma} \xi_J. \tag{3}$$

Corollary 4.1 *If Σ_1 and Σ_2 belong to the same homology class in $H_2(M, L)$ then $\mu_L(\Sigma_1, \partial\Sigma_1) = \mu_L(\Sigma_2, \partial\Sigma_2)$.*

Proof: This is a standard fact which can be proved using the topological definition of μ_L. Alternatively, let T be a 3-dimensional submanifold in M such that $\partial T = \Sigma_1 - \Sigma_2 - \Sigma$, with $\Sigma \subseteq L$ and $\partial\Sigma = \partial\Sigma_1 - \partial\Sigma_2$. Then $\frac{1}{2}\int_\Sigma P = \int_\Sigma d\xi_J$ so the result follows from Equation (3) and Stokes' theorem. $\square$

The Maslov indices of such surfaces can thus be collected into a single relative cohomology class $\mu_L \in H^2(M, L)$, known as the *Maslov class* of L.

When M is Kähler one can check that $P(X, Y) = 2\rho(X, Y)$, where the latter is the standard *Ricci 2-form* $\rho(X, Y) := \mathrm{Ric}(JX, Y)$. Furthermore, for any totally real submanifold we prove in [7] the fundamental relationships

$$\xi_J = \omega(H_J, \cdot)_{|TL}, \quad d\xi_J = \rho_{|TL}, \tag{4}$$

where H_J is the negative gradient of the *J-volume functional* defined by integrating Ω_J over L. If L is Lagrangian the J-volume coincides with the standard Riemannian volume and H_J coincides with the standard mean curvature vector field H of L.

Corollary 4.2 *Assume (M, J, ω) is Kähler with Ricci 2-form ρ. The Maslov index of $(\Sigma, \partial\Sigma) \subset (M, L)$ has the following integral representation:*

- *For L totally real with J-mean curvature H_J,*

$$\mu_L(\Sigma, \partial\Sigma) = \frac{1}{\pi}\left(\int_\Sigma \rho - \int_{\partial\Sigma} \omega(H_J, \cdot)\right).$$

- *For L Lagrangian with mean curvature H,*

$$\mu_L(\Sigma, \partial\Sigma) = \frac{1}{\pi}\left(\int_\Sigma \rho - \int_{\partial\Sigma} \omega(H, \cdot)\right).$$

The Lagrangian case already appears, in implicit form, in [3]. Notice that in this case all terms in the Chern-Weil formula (2) for the Maslov index become classical geometric quantities.

In order to better understand the quantity H_J, let us recall the standard definition: a submanifold L is *minimal* if it is a critical point of the standard Riemannian volume, *i.e.* $H = 0$. Analogously, we say that a totally real submanifold L is *J-minimal* if it is a critical point of the J-volume functional, *i.e.* $H_J = 0$; equivalently, $\xi_J = 0$ or Ω_J is parallel.

Equation (4) shows that any minimal Lagrangian must automatically satisfy the additional condition $\rho_{|TL} \equiv 0$. Coupled with the Lagrangian condition $\omega_{|TL} \equiv 0$, the resulting system typically does not admit solutions unless M is Kähler-Einstein: in this case the two conditions coincide. The notion of J-minimal submanifolds provides an interesting extension of the minimal Lagrangian condition to the more general setting of Kähler manifolds: they are again defined variationally and Equation (4) again shows that they satify the condition $\rho_{|TL} \equiv 0$, but by definition they are only totally real, not necessarily Lagrangian. In [8] we show that J-minimal submanifolds also have several analytic properties in common with minimal Lagrangians, and provide examples.

Remark When M is not Kähler, torsion terms appear in gradient of the J-volume functional and in Equation (4). We refer to [7] for details.

5 Applications

We now list a few applications of the above formulae.

Comparison with the Gauss-Bonnet theorem. Recall the standard Gauss-Bonnet theorem: given a smooth domain U in an orientable surface M with metric g,

$$\int_U K \,\mathrm{vol}_g + \int_{\partial U} k\, ds = 2\pi\, \chi(U),$$

where K is the Gaussian curvature of M and k is the geodesic curvature of the boundary which, using an arclength parametrization $\gamma(s)$ of ∂U and normal vector field $n(s)$, is defined by the identity $\nabla\dot\gamma = k\, ds \otimes n(s)$.

For dimensional reasons such M is automatically Kähler and ∂U is trivially Lagrangian. Let us choose $(\Sigma, \partial\Sigma) = (\overline{U}, \partial U)$. Then $\sigma_J = \dot\gamma$ and $n = J\dot\gamma$ so $\theta = ik\, ds$. This shows that, up to a factor π, (2) coincides with the left-hand side of the Gauss-Bonnet formula. We conclude that $\mu_L(\Sigma, \partial\Sigma) = 2\chi(U)$, generalizing Example 3.1.

Vanishing conditions. Natural assumptions on (M, L) ensure the vanishing of one or both terms in the formula for μ_L:

- Assume $P \equiv 0$. This happens for example when M is Kähler Ricci-flat, *e.g.* when M is $\mathbb{C}^n$ or, more generally, Calabi-Yau. Then μ_L depends only on the boundary contribution.

- Assume L is such that Ω_J is parallel: this is similar to the "orthogonality" condition appearing in [2], but in our context it is geometrically motivated by the notion of J-minimal submanifolds. In this case the boundary contribution vanishes and the Maslov index is completely determined by P.

As an application of the first vanishing statement we generalize a result of [4] from Lagrangian to totally real boundary data.

Corollary 5.1 *Assume M is Kähler Ricci-flat. Then $\int_{\partial\Sigma} \omega(H_J, \cdot)$ is a multiple of π, for any totally real L and any $(\Sigma, \partial\Sigma) \subset (M, L)$.*

For example, when $M = \mathbb{C}^n$ we can produce a 1-parameter family of totally real submanifolds tL simply by rescaling. One can check that $H_J(t) = \frac{1}{t} H_J$, so the boundary contribution for $t\Sigma$ is constant. On the other hand Maslov indices are integral, thus also independent of t: this is consistent with Corollary 5.1.

The following result uses the second vanishing condition.

Corollary 5.2 *Let M be Kähler. Assume the Ricci 2-form ρ has a semi-definite sign (e.g., $\rho \leq 0$). Then, for any minimal Lagrangian L, $\mu_L(\Sigma, \partial\Sigma)$ has that same sign for any complex Σ (e.g., $\mu_L \leq 0$).*

More generally this holds for any J-minimal submanifold.

The same statement holds also when J is non-integrable, as long as one replaces ρ with P and L satisfies $\nabla \Omega_J \equiv 0$.

There are also interesting situations where both terms vanish, leading to Maslov-zero submanifolds. For example, (4) shows that ξ_J is closed when M is Kähler Ricci-flat. We thus obtain the following result.

Corollary 5.3 *Assume M is Kähler Ricci-flat and that L is either (i) minimal Lagrangian, or (ii) totally real and J-minimal, or (iii) totally real with first Betti number $b^1(L) = 0$. Then L is Maslov-zero.*

Situation (i) includes the case of special Lagrangian submanifolds in Calabi-Yau manifolds. We thus obtain a new proof of the well-known fact that special Lagrangians are Maslov-zero: the standard proof relies on the topological definition of the Maslov index in terms of the twisting of $\Lambda^n TL$ with respect to the parallel section of K_M determined by the Calabi-Yau condition.

Monotonicity of minimal Lagrangian submanifolds. As mentioned in Section 2, in the symplectic context it is interesting to compare the Maslov index with $\int_\Sigma \omega$, the *symplectic area*. This is best understood from the cohomological point of view introduced at the beginning of Section 5. Indeed, for Lagrangian boundary data it is simple to check that the symplectic areas of two surfaces Σ_1, Σ_2 coincide if they belong to the same relative homology class. Symplectic area can thus also be viewed as a relative cohomology class α_L, and our problem can be phrased in terms of comparing μ_L to α_L in $H^2(M,L)$.

Consider the long exact sequence

$$\cdots \to H^1(L) \xrightarrow{\delta} H^2(M,L) \xrightarrow{j} H^2(M) \to \dots \tag{5}$$

The fact that the Maslov index is twice the first Chern number when Σ has empty boundary shows that the image of μ_L in $H^2(M)$ is $2c_1(M)$. The image of α_L in $H^2(M)$ is $[\omega]$.

Assume M is Kähler-Einstein, *i.e.* $\rho = c\,\omega$ for some $c \in \mathbb{R}$. Then $2\pi c_1(M) = c[\omega]$ so $j(\pi\mu_L - c\,\alpha_L) = 0 \in H^2(M)$. Exactness implies that there must exist some $\beta \in H^1(L)$ such that $\delta(\beta) = \pi\mu_L - c\,\alpha_L$. We can find β using our integral formula, as follows.

Corollary 4.2 shows that

$$\pi\,\mu_L(\Sigma, \partial\Sigma) - c\int_\Sigma \omega = -\int_{\partial\Sigma} \omega(H,\cdot).$$

Equation (4) and the Kähler-Einstein hypothesis imply that, for L Lagrangian, $\omega(H,\cdot)$ is a closed 1-form on L so it defines a cohomology class in $H^1(L)$. We thus obtain the main result of [3].

Corollary 5.4 *Assume M is Kähler-Einstein with $\rho = c\,\omega$ and L is Lagrangian. Then*

$$\pi\mu_L - c\,\alpha_L = -[\omega(H,\cdot)].$$

In particular, any minimal Lagrangian L is monotone.

One should compare this with Corollary 5.2 which produces a definite sign for the Maslov index only for complex curves, but with less stringent hypotheses.

Monotonicity of J-minimal submanifolds. As discussed in Section 4, minimal Lagrangians make sense mostly in the context of Kähler-Einstein manifolds. In [8] it is shown that, in the more general context of Kähler manifolds with (positive or negative) definite Ricci curvature, J-minimal submanifolds provide an interesting substitute.

In this context ρ defines a second symplectic form on M and $\pm\rho(\cdot, J, \cdot)$ is a Riemannian metric. We will say that a submanifold is *ρ-Lagrangian* if $\rho_{|TL} \equiv 0$. Any such L is totally real. All standard results in Symplectic Geometry apply to ρ and to ρ-Lagrangians, *e.g.* the energy of holomorphic curves with boundary on ρ-Lagrangians is topologically determined by the homology class, thus bounded.

Given any surface Σ with boundary on a ρ-Lagrangian L, the analogue of the symplectic area is the quantity

$$\alpha_L(\Sigma, \partial\Sigma) := \int_\Sigma \rho. \tag{6}$$

As usual this defines a class $\alpha_L \in H^2(M, L)$. We will say that L is *ρ-monotone* if μ_L is proportional to α_L.

Our main reason for interest in this notion is the fact that any J-minimal submanifold is ρ-Lagrangian: this is immediate from the definitions and from Equation (4). In the long exact sequence (5), α_L has image $2\pi c_1(M) \in H^2(M)$ so $j(\pi\mu_L - \alpha_L) = 0$. We then obtain the precise description of a class $\beta \in H^1(L)$, as above. This leads to the following result which further reinforces the analogies between J-minimal submanifolds and minimal Lagrangians.

Corollary 5.5 *Assume M is Kähler with (positive or negative) definite Ricci curvature. Then any J-minimal submanifold is ρ-monotone.*

Monotonicity of solitons. Recall that an immersed Lagrangian submanifold $\iota : L \to \mathbb{C}^n$ is a *self-similar soliton* if, for some $c \in \mathbb{R}$ and all $x \in L$, $H(x) = c\,\iota(x)^\perp$. In this case, under mean curvature flow and up to reparametrization, ι evolves simply by rescaling: $\iota(t) = \rho(t) \cdot \iota$. Thus $H(t, x) = \frac{1}{\rho(t)} H(x)$ and each $\iota(t)$ is a soliton, with $c(t) = \frac{1}{\rho^2(t)}$.

Using the Lagrangian hypothesis the soliton equation can be re-written as an equation of 1-forms on L:

$$\iota^*\omega(H, \cdot) = c\,\iota^*\omega(\iota, \cdot) = c\,\iota^* g(J\iota, \cdot).$$

Setting $\lambda := \frac{1}{2}(xdy - ydx)$ and $\iota = (x, y)$ we find that the soliton equation is equivalent to $\iota^*\omega(H, \cdot) = 2c\,\iota^*\lambda$. Now notice that $\omega = d\lambda$. Corollary 4.2 then leads to the following fact, cf. [5].

Corollary 5.6 *Let L be a Lagrangian soliton in $\mathbb{C}^n$. Then $\mu_L = -\frac{2c}{\pi}\alpha_L$, so L is monotone.*

Notice that μ_L is integral and α_L rescales by $\rho^2(t)$: this is consistent with the rescaling of $c(t)$.

References

[1] Vincent Borrelli, *Maslov form and J-volume of totally real immersions*, J. Geom. Phys. **25** (1998), no. 3-4, 271–290.

[2] Cheol-Hyun Cho and Hyung-Seok Shin, *Chern-Weil Maslov index and its orbifold analogue*, Asian J. Math. **20** (2016), no. 1, 1–19.

[3] Kai Cieliebak and Edward Goldstein, *A note on the mean curvature, Maslov class and symplectic area of Lagrangian immersions*, J. Symplectic Geom. **2** (2004), no. 2, 261–266.

[4] Pierre Dazord, *Sur la géométrie des sous-fibrés et des feuilletages lagrangiens*, Ann. Sci. École Norm. Sup. (4) **14** (1981), no. 4, 465–480 (1982).

[5] K. Groh, M. Schwarz, K. Smoczyk, and K. Zehmisch, *Mean curvature flow of monotone Lagrangian submanifolds*, Math. Z. **257** (2007), no. 2, 295–327.

[6] Jason D. Lotay and Tommaso Pacini, *Complexified diffeomorphism groups, totally real submanifolds and Kähler-Einstein geometry*, to appear in Trans. AMS.

[7] ———, *From Lagrangian to totally real geometry: coupled flows and calibrations*, to appear in Comm. in Analysis and Geometry.

[8] ———, *From minimal Lagrangian to J-minimal submanifolds: persistence and uniqueness*, to appear in Bollettino UMI.

[9] Dusa McDuff and Dietmar Salamon, *J-holomorphic curves and symplectic topology*, American Mathematical Society Colloquium Publications, vol. 52, American Mathematical Society, Providence, RI, 2004.

[10] Jean-Marie Morvan, *Classe de Maslov d'une immersion lagrangienne et minimalité*, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), no. 13, 633–636. MR 625362

[11] Hajime Ono, *Integral formula of Maslov index and its applications*, Japan. J. Math. (N.S.) **30** (2004), no. 2, 413–421.