

## SHARP INTEGRAL BOUNDS FOR WIGNER DISTRIBUTIONS

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ABSTRACT. The cross-Wigner distribution  $W(f, g)$  of two functions or temperate distributions  $f, g$  is a fundamental tool in quantum mechanics and in signal analysis. Usually, in applications in time-frequency analysis  $f$  and  $g$  belong to some modulation space and it is important to know which modulation spaces  $W(f, g)$  belongs to. Although several particular sufficient conditions have been appeared in this connection, the general problem remains open. In the present paper we solve completely this issue by providing the full range of modulation spaces in which the continuity of the cross-Wigner distribution  $W(f, g)$  holds, as a function of  $f, g$ . The case of weighted modulation spaces is also considered. The consequences of our results are manifold: new bounds for the short-time Fourier transform and the ambiguity function, boundedness results for pseudodifferential (in particular, localization) operators and properties of the Cohen class.

## 1. INTRODUCTION

The (cross-)Wigner distribution was first introduced in physics to account for quantum corrections to classical statistical mechanics in 1932 by Wigner [50] and in 1948 it was proposed in signal analysis by Ville [49]. This is why the Wigner distribution is also called Wigner-Ville distribution. Nowadays it can be considered one of the most important time-frequency representations, second only to the spectrogram, and it is one of the most commonly used quasiprobability distribution in quantum mechanics [21, 28].

Given two functions  $f_1, f_2 \in L^2(\mathbb{R}^d)$ , the *cross-Wigner distribution*  $W(f_1, f_1)$  is defined to be

$$(1) \quad W(f_1, f_2)(x, \xi) = \int f_1\left(x + \frac{t}{2}\right) \overline{f_2\left(x - \frac{t}{2}\right)} e^{-2\pi i \xi t} dt.$$

The quadratic expression  $Wf = W(f, f)$  is called the Wigner distribution of  $f$ .

An important issue related to such a distribution is the continuity of the map  $(f_1, f_2) \mapsto W(f_1, f_2)$  in the relevant Banach spaces. The basic result in this connection is the easily verified equality

$$\|W(f_1, f_2)\|_{L^2(\mathbb{R}^{2d})} = \|f_1\|_{L^2(\mathbb{R}^d)} \|f_2\|_{L^2(\mathbb{R}^d)}.$$

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Beside  $L^2$ , the time-frequency concentration of signals is often measured by the so-called modulation space norm  $M_m^{p,q}$ ,  $1 \leq p, q \leq \infty$ , for a suitable weight function  $m$  (cf. [22, 23, 28] and Section 2 below). In short, these spaces are defined as follows. For a fixed non-zero  $g \in \mathcal{S}(\mathbb{R}^d)$ , the short-time Fourier transform (STFT) of  $f \in \mathcal{S}'(\mathbb{R}^d)$  with respect to the window  $g$  is given by

$$(2) \quad V_g f(x, \xi) = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i \xi t} dt.$$

Then the space  $M_m^{p,q}(\mathbb{R}^d)$  is defined by

$$M_m^{p,q}(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : V_g f \in L_m^{p,q}(\mathbb{R}^{2d})\}$$

endowed with the obvious norm. Here  $L_m^{p,q}(\mathbb{R}^{2d})$  are mixed-norm weighted Lebesgue spaces in  $\mathbb{R}^{2d}$ ; see Section 2 below for precise definitions.

Both the STFT  $V_g f$  and the cross-Wigner distribution  $W(f, g)$  are defined on many pairs of Banach spaces. For example, they both map  $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$  into  $L^2(\mathbb{R}^{2d})$  and  $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$  into  $\mathcal{S}(\mathbb{R}^{2d})$  and can be extended to a map from  $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)$  into  $\mathcal{S}'(\mathbb{R}^{2d})$ .

In this paper we will mainly work with the polynomial weights

$$(3) \quad v_s(z) = \langle z \rangle^s = (1 + |z|^2)^{\frac{s}{2}}, \quad z \in \mathbb{R}^{2d}, \quad s \in \mathbb{R}.$$

For  $w = (z, \zeta) \in \mathbb{R}^{4d}$ , we write  $(1 \otimes v_s)(w) = v_s(\zeta)$ . Now, the problem addressed in this paper is to provide the full range of exponents  $p_1, p_2, q_1, q_2, p, q \in [1, \infty]$  such that

$$\|W(f_1, f_2)\|_{M_{1 \otimes v_s}^{p,q}} \lesssim \|f_1\|_{M_{v_s}^{p_1, q_1}} \|f_2\|_{M_{v_s}^{p_2, q_2}}.$$

These estimates were proved in [48, Theorem 4.2] (cf. also [47, Theorem 4.1] for modulation spaces without weights) under the conditions

$$p \leq p_i, q_i \leq q, \quad i = 1, 2$$

and

$$(4) \quad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{p} + \frac{1}{q}.$$

However, it is not clear whether these conditions are necessary as well.

Our main result shows that the sufficient conditions can be widened and such extension is sharp.

**Theorem 1.1.** *Assume  $p_i, q_i, p, q \in [1, \infty]$ ,  $s \in \mathbb{R}$ , such that*

$$(5) \quad p_i, q_i \leq q, \quad i = 1, 2$$

and that

$$(6) \quad \frac{1}{p_1} + \frac{1}{p_2} \geq \frac{1}{p} + \frac{1}{q}, \quad \frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{p} + \frac{1}{q}.$$

Then, if  $f_1 \in M_{v_{|s|}}^{p_1, q_1}(\mathbb{R}^d)$  and  $f_2 \in M_{v_s}^{p_2, q_2}(\mathbb{R}^d)$  we have  $W(f_1, f_2) \in M_{1 \otimes v_s}^{p, q}(\mathbb{R}^{2d})$ , and

$$(7) \quad \|W(f_1, f_2)\|_{M_{1 \otimes v_s}^{p, q}} \lesssim \|f_1\|_{M_{v_{|s|}}^{p_1, q_1}} \|f_2\|_{M_{v_s}^{p_2, q_2}}.$$

Viceversa, assume that there exists a constant  $C > 0$  such that

$$(8) \quad \|W(f_1, f_2)\|_{M^{p, q}} \leq C \|f_1\|_{M^{p_1, q_1}} \|f_2\|_{M^{p_2, q_2}}, \quad \forall f_1, f_2 \in \mathcal{S}(\mathbb{R}^{2d}).$$

Then (5) and (6) must hold.

The remarkable fact of this result, in our opinion, is that the conditions (5) and (6) turn out to be necessary too.

The consequences of this are manifold. First, in the framework of signal analysis and time-frequency representations, we obtain new estimates for the short-time Fourier transform  $V_{f_1} f_2$  and the ambiguity function  $A(f_1, f_2)$  (see Section 2 for definitions). In particular, we recapture the sharp Lieb's bounds in [36, Theorem 1]

$$\|A(f_1, f_2)\|_{L^q} \lesssim \|f_1\|_{L^2} \|f_2\|_{L^2},$$

valid for  $q \geq 2$ ; we also refer to [14, 15] for related estimates for the short-time Fourier transform and [11] for the strictly related Born-Jordan distribution.

Secondly, we easily provide new boundedness results for pseudodifferential operators (in particular, localization operators) with symbols in modulation spaces. Let us mention that the study of pseudodifferential operators in the context of modulation spaces has been pursued by many authors. The earliest works are due to Sjöstrand [42] and Tachizawa [45]. In the former work pseudodifferential operators with symbols in the modulation space  $M^{\infty, 1}$  (also called Sjöstrand's class) were investigated. Later, sufficient and some necessary boundedness conditions were investigated by Gröchenig and Heil [29, 30] and Labate [34, 35]. Since the year 2003 until today the contributions on this topic are so multiplied that there is hard to mention them all. Let us just recall some of them [2, 4, 18, 19, 31, 33, 38, 39, 43, 44, 47, 48].

Every continuous operator from  $\mathcal{S}(\mathbb{R}^d)$  to  $\mathcal{S}'(\mathbb{R}^d)$  can be represented as a pseudodifferential operator in the Weyl form  $L_\sigma$  and the connection with the cross-Wigner distribution is provided by

$$(9) \quad \langle L_\sigma f, g \rangle = \langle \sigma, W(g, f) \rangle, \quad f, g \in \mathcal{S}(\mathbb{R}^d).$$

By using this formula we can translate the boundedness results for the cross-Wigner distribution in Theorem 1.1 to boundedness results for Weyl operators.

Pseudodifferential operators of great interest in signal analysis are the so-called localization operators  $A_a^{\varphi_1, \varphi_2}$  (see Section 5), which can be represented as Weyl operators as follows (cf. [6, 13, 46])

$$(10) \quad A_a^{\varphi_1, \varphi_2} = L_{a * W(\varphi_2, \varphi_1)}$$

so that the Weyl symbol of the localization operator  $A_a^{\varphi_1, \varphi_2}$  is given by

$$(11) \quad \sigma = a * W(\varphi_2, \varphi_1).$$

Using this representation of localization operators as Weyl operators and using Theorem 1.1 we are able to obtain new boundedness results for localization operators, see Theorem 5.2 in Section 5 below.

Finally, another application of Theorem 1.1 is the investigation of the time-frequency properties of the Cohen class, introduced by Cohen in [8]. This class consists of elements of the type

$$(12) \quad M(f, f) = W(f, f) * \sigma$$

where  $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$  is called the Cohen kernel. When  $\sigma = \delta$ , then  $M(f, f) = W(f, f)$  and we come back to the Wigner distribution. For other choices of kernels we recapture the Born-Jordan distribution [10, 11, 12] or the  $\tau$ -Wigner distributions  $W_\tau(f, f)$  [7, Proposition 5.6]. In this framework we have the following result.

**Theorem 1.2.** *Assume  $s \geq 0$ ,  $p_1, q_1, p, q \in [1, \infty]$  such that*

$$(13) \quad 2 \min\left\{\frac{1}{p_1}, \frac{1}{q_1}\right\} \geq \frac{1}{p} + \frac{1}{q}.$$

*Consider a Cohen kernel  $\sigma \in M^{1, \infty}(\mathbb{R}^{2d})$ . If  $f \in M_{v_s}^{p_1, q_1}(\mathbb{R}^d)$ , then the Cohen distribution  $M(f, f)$  is in  $M_{1 \otimes v_s}^{p, q}(\mathbb{R}^{2d})$ , with*

$$(14) \quad \|M(f, f)\|_{M_{1 \otimes v_s}^{p, q}(\mathbb{R}^{2d})} \lesssim \|\sigma\|_{M^{1, \infty}(\mathbb{R}^{2d})} \|f\|_{M_{v_s}^{p_1, q_1}(\mathbb{R}^d)}^2.$$

In particular, the  $\tau$ -kernels and the Born-Jordan kernels enjoy such a property, cf. Section 6.

For the sake of clarity our results have been presented only for the polynomial weights  $v_s$ , but we remark that more general weights can also be considered, see the following Remark 3.2, (i).

Further developments of this research could involve the study of boundedness for bilinear/multilinear pseudodifferential operators and localization operators. This requires an extension of Theorem 1.1 to more general Wigner/Rihaczek distributions and STFT, see e.g., [3, 16, 27] and the recent contribution [37]. We leave this study to a subsequent paper.

In short, the paper is organized as follows. Section 2 is devoted to some preliminary results from time-frequency analysis and in particular to the computation of the STFT of a generalized Gaussian and its modulation norm. In Section 3 we prove Theorem 1.1. In Section 4 we show the continuity properties of pseudodifferential (and in particular localization) operators on modulation spaces. In Section 5 we present a time-frequency analysis of the Cohen class.

**Notation.** We define  $t^2 = t \cdot t$ , for  $t \in \mathbb{R}^d$ , and  $xy = x \cdot y$  is the scalar product on  $\mathbb{R}^d$ . The Schwartz class is denoted by  $\mathcal{S}(\mathbb{R}^d)$ , the space of tempered distributions by  $\mathcal{S}'(\mathbb{R}^d)$ . We use the brackets  $\langle f, g \rangle$  to denote the extension to  $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$  of the inner product  $\langle f, g \rangle = \int f(t)\overline{g(t)}dt$  on  $L^2(\mathbb{R}^d)$ . The Fourier transform is normalized to be  $\hat{f}(\xi) = \mathcal{F}f(\xi) = \int f(t)e^{-2\pi it\xi}dt$ , the involution  $g^*$  is  $g^*(t) = \overline{g(-t)}$ . The operators of translation and modulation are defined by  $T_x f(t) = f(t-x)$  and  $M_\xi f(t) = e^{2\pi it\xi} f(t)$ ,  $x, \xi \in \mathbb{R}^d$ .

## 2. PRELIMINARIES

**2.1. Modulation spaces.** The modulation and Wiener amalgam space norms are a measure of the joint time-frequency distribution of  $f \in \mathcal{S}'$ . For their basic properties we refer to the original literature [22, 23, 24] and the textbooks [21, 28].

For the description of the decay properties of a function/distribution, weight functions on the time-frequency plane are employed. We denote by  $v$  a continuous, positive, even, submultiplicative weight function (in short, a submultiplicative weight), i.e.,  $v(0) = 1$ ,  $v(z) = v(-z)$ , and  $v(z_1 + z_2) \leq v(z_1)v(z_2)$ , for all  $z, z_1, z_2 \in \mathbb{R}^{2d}$ . A positive, even weight function  $m$  on  $\mathbb{R}^{2d}$  is called  $v$ -moderate if  $m(z_1 + z_2) \leq Cv(z_1)m(z_2)$  for all  $z_1, z_2 \in \mathbb{R}^{2d}$ . Observe that  $v_s$  is a  $v_{|s|}$ -moderate weight, for every  $s \in \mathbb{R}$ . Given a non-zero window  $g \in \mathcal{S}(\mathbb{R}^d)$ , a  $v$ -moderate weight function  $m$  on  $\mathbb{R}^{2d}$ ,  $1 \leq p, q \leq \infty$ , the modulation space  $M_m^{p,q}(\mathbb{R}^d)$  consists of all tempered distributions  $f \in \mathcal{S}'(\mathbb{R}^d)$  such that the STFT  $V_g f$  (defined in (2)) is in  $L_m^{p,q}(\mathbb{R}^{2d})$  (weighted mixed-norm spaces), with norm

$$\|f\|_{M_m^{p,q}} = \|V_g f\|_{L_m^{p,q}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x, \xi)|^p m(x, \xi)^p dx \right)^{q/p} d\xi \right)^{1/q}.$$

(Obvious modifications occur when  $p = \infty$  or  $q = \infty$ ). If  $p = q$ , we write  $M_m^p$  instead of  $M_m^{p,p}$ , and if  $m(z) \equiv 1$  on  $\mathbb{R}^{2d}$ , then we write  $M^{p,q}$  and  $M^p$  for  $M_m^{p,q}$  and  $M_m^{p,p}$ . Then  $M_m^{p,q}(\mathbb{R}^d)$  is a Banach space whose definition is independent of the choice of the window  $g$ , in the sense that different nonzero window functions yield equivalent norms. The modulation space  $M^{\infty,1}$  is also called Sjöstrand's class [42].

We now recall the definition of the Wiener amalgam spaces that are image of the modulation spaces under the Fourier transform. For any even weigh functions  $u, w$  on  $\mathbb{R}^d$ , the Wiener amalgam spaces  $W(\mathcal{F}L_u^p, L_w^q)(\mathbb{R}^d)$  are given by the distributions  $f \in \mathcal{S}'(\mathbb{R}^d)$  such that

$$\|f\|_{W(\mathcal{F}L_u^p, L_w^q)(\mathbb{R}^d)} := \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x, \xi)|^p u^p(\xi) d\xi \right)^{q/p} w^q(x) dx \right)^{1/q} < \infty$$

(with natural changes for  $p = \infty$  or  $q = \infty$ ). Using Parseval identity we can write the so-called fundamental identity of time-frequency analysis  $V_g f(x, \xi) =$

$e^{-2\pi i x \xi} V_{\hat{g}} \hat{f}(\xi, -x)$ , hence  $|V_g f(x, \xi)| = |V_{\hat{g}} \hat{f}(\xi, -x)|$  so that (recall  $u(x) = u(-x)$ )

$$(15) \quad \|f\|_{M_{u \otimes w}^{p,q}} = \|\hat{f}\|_{W(\mathcal{F}L_u^p, L_w^q)}.$$

This proves that these Wiener amalgam spaces are the image under Fourier transform of modulation spaces:

$$(16) \quad \mathcal{F}(M_{u \otimes w}^{p,q}) = W(\mathcal{F}L_u^p, L_w^q).$$

In the sequel we will need the inclusion relations for modulation spaces. Assume  $m_1, m_2 \in \mathcal{M}_v(\mathbb{R}^{2d})$ , then

$$(17) \quad \begin{aligned} \mathcal{S}(\mathbb{R}^d) &\subseteq M_{m_1}^{p_1, q_1}(\mathbb{R}^d) \subseteq M_{m_2}^{p_2, q_2}(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R}^d), \\ \text{if } p_1 &\leq p_2, \quad q_1 \leq q_2, \quad m_2 \lesssim m_1. \end{aligned}$$

Moreover, we will often apply convolution relations for modulation spaces [13, Proposition 2.1] for the  $v_s$  weight functions as follows.

**Proposition 2.1.** *Let  $\nu(\xi) > 0$  be an arbitrary weight function on  $\mathbb{R}^d$ ,  $s \in \mathbb{R}$ , and  $1 \leq p, q, u, v, t \leq \infty$ . If*

$$\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{u}, \quad \text{and} \quad \frac{1}{t} + \frac{1}{t'} = \frac{1}{v},$$

then

$$(18) \quad M_{1 \otimes \nu}^{p,t}(\mathbb{R}^d) * M_{1 \otimes v_{|s|}^{\nu-1}}^{q,t'}(\mathbb{R}^d) \hookrightarrow M_{v_s}^{u,v}(\mathbb{R}^d)$$

with norm inequality  $\|f * h\|_{M_{v_s}^{u,v}} \lesssim \|f\|_{M_{1 \otimes \nu}^{p,t}} \|h\|_{M_{1 \otimes v_{|s|}^{\nu-1}}^{q,t'}}$ .

**2.2. Time-frequency tools.** To prove our main result, we will need to compute the STFT of the cross-Wigner distribution, proved in [28, Lemma 14.5.1]:

**Lemma 2.1.** *Fix a nonzero  $g \in \mathcal{S}(\mathbb{R}^d)$  and let  $\Phi = W(g, g) \in \mathcal{S}(\mathbb{R}^{2d})$ . Then the STFT of  $W(f_1, f_2)$  with respect to the window  $\Phi$  is given by*

$$(19) \quad V_{\Phi}(W(f_1, f_2))(z, \zeta) = e^{-2\pi i z_2 \zeta_2} \overline{V_g f_2(z_1 + \frac{\zeta_2}{2}, z_2 - \frac{\zeta_1}{2})} V_g f_1(z_1 - \frac{\zeta_2}{2}, z_2 + \frac{\zeta_1}{2}).$$

**Lemma 2.2.** *Consider the Gaussian function  $\varphi(x) = e^{-\pi x^2}$  and its rescaled version  $\varphi_{\lambda}(x) = e^{-\pi \lambda x^2}$ ,  $\lambda > 0$ . Then the cross-Wigner distribution is the following Gaussian function*

$$(20) \quad W(\varphi, \varphi_{\lambda})(x, \xi) = \frac{2^d}{(1 + \lambda)^{\frac{d}{2}}} e^{-a_{\lambda} \pi x^2} e^{-b_{\lambda} \pi \xi^2} e^{2\pi i c_{\lambda} x \xi}$$

with

$$(21) \quad a_{\lambda} = \frac{4\lambda}{1 + \lambda} \quad b_{\lambda} = \frac{4}{1 + \lambda} \quad c_{\lambda} = \frac{2(1 - \lambda)}{1 + \lambda}.$$

*Proof.* The proof is obtained by an easy computation. In particular, we will make the change of variables  $t = \frac{2s}{\sqrt{1+\lambda}} - 2\frac{1-\lambda}{\sqrt{1+\lambda}}x$ , so that  $dt = \frac{2^d}{(1+\lambda)^{d/2}}ds$ . In details,

$$\begin{aligned}
W(\varphi, \varphi_\lambda)(x, \xi) &= \int_{\mathbb{R}^d} e^{-\pi\left(x+\frac{t}{2}\right)^2 - \pi\lambda\left(x-\frac{t}{2}\right)^2} e^{-2\pi it\xi} dt \\
&= e^{-\pi(1+\lambda)x^2} \int_{\mathbb{R}^d} e^{-\frac{\pi}{4}[(1+\lambda)t^2 + 4(1-\lambda)xt]} e^{-2\pi it\xi} dt \\
&= e^{-\pi(1+\lambda)x^2} \int_{\mathbb{R}^d} e^{-\frac{\pi}{4}\left[\sqrt{1+\lambda}t + 2\frac{1-\lambda}{\sqrt{1+\lambda}}x\right]^2} e^{\pi\frac{(1-\lambda)^2}{1+\lambda}x^2} e^{-2\pi it\xi} dt \\
&= e^{-\pi\left[(1+\lambda) - \frac{(1-\lambda)^2}{1+\lambda}\right]x^2} \int_{\mathbb{R}^d} e^{-\frac{\pi}{4}\left[\sqrt{1+\lambda}t + 2\frac{1-\lambda}{\sqrt{1+\lambda}}x\right]^2} e^{-2\pi it\xi} dt \\
&= e^{-\pi\frac{4\lambda}{1+\lambda}x^2} \int_{\mathbb{R}^d} e^{-\pi s^2} e^{-2\pi i\left(\frac{2s}{\sqrt{1+\lambda}} - \frac{2(1-\lambda)}{1+\lambda}x\right)\xi} \frac{2^d}{(1+\lambda)^{d/2}} ds \\
&= \frac{2^d}{(1+\lambda)^{d/2}} e^{-\pi\frac{4\lambda}{1+\lambda}x^2} e^{4\pi i\frac{1-\lambda}{1+\lambda}x\xi} \int_{\mathbb{R}^d} e^{-\pi s^2} e^{-2\pi is\left(\frac{2\xi}{\sqrt{1+\lambda}}\right)} ds \\
&= \frac{2^d}{(1+\lambda)^{d/2}} e^{-\pi\frac{4\lambda}{1+\lambda}x^2} e^{-\pi\frac{4}{1+\lambda}\xi^2} e^{4\pi i\frac{1-\lambda}{1+\lambda}x\xi},
\end{aligned}$$

as desired.  $\square$

Hence the Wigner distribution above is a generalized Gaussian. Our goal will be to compute the modulation norm of this Wigner distribution. The first step is the calculation of the STFT of a generalized Gaussian.

**Proposition 2.2.** *Given  $a, b, c > 0$ , consider the generalized Gaussian function*

$$(22) \quad f(x, \xi) = e^{-\pi ax^2} e^{-\pi b\xi^2} e^{2\pi icx\xi}, \quad (x, \xi) \in \mathbb{R}^{2d}.$$

For  $\Phi(x, \xi) = e^{-\pi(x^2 + \xi^2)}$ ,  $z = (z_1, z_2)$ ,  $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$ , we obtain

$$\begin{aligned}
V_\Phi f(z, \zeta) &= \frac{1}{[(a+1)(b+1) + c^2]^{\frac{d}{2}}} e^{-\pi \frac{[a(b+1)+c^2]z_1^2 + [(a+1)b+c^2]z_2^2 + (b+1)\zeta_1^2 + (a+1)\zeta_2^2 - 2c(z_1\zeta_2 + z_2\zeta_1)}{(a+1)(b+1)+c^2}} \\
(23) \quad &\times e^{-\frac{2\pi i}{a+1} \left[ z_1\zeta_1 + (cz_1 - (a+1)\zeta_2) \frac{c\zeta_1 + (a+1)z_2}{(a+1)(b+1)+c^2} \right]}.
\end{aligned}$$

*Proof.* We write

$$\begin{aligned}
V_{\Phi}f(z, \zeta) &= \int_{\mathbb{R}^{2d}} e^{-\pi ax^2 - \pi b\xi^2 + 2\pi icx\xi} e^{-2\pi i(\zeta_1 x + \zeta_2 \xi)} e^{-\pi[(x-z_1)^2 + (\xi-z_2)^2]} dx d\xi \\
&= e^{-\pi(z_1^2 + z_2^2)} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} e^{-\pi[(a+1)x^2 - 2xz_1]} e^{-2\pi i x(\zeta_1 - c\xi)} dx \right) \\
&\quad \times e^{-\pi[(b+1)\xi^2 - 2\xi z_2]} e^{-2\pi i \zeta_2 \xi} d\xi \\
&= e^{-\pi\left(1 - \frac{1}{a+1}\right)z_1^2 - \pi z_2^2} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} e^{-\pi\left[\sqrt{a+1}x - \frac{z_1}{\sqrt{a+1}}\right]^2} e^{-2\pi i x(\zeta_1 - c\xi)} dx \right) \\
&\quad \times e^{-\pi[(b+1)\xi^2 - 2\xi z_2]} e^{-2\pi i \zeta_2 \xi} d\xi.
\end{aligned}$$

With the change of variables  $\sqrt{a+1}x - \frac{z_1}{\sqrt{a+1}} = t$ ,  $dx = \frac{dt}{(a+1)^{d/2}}$ , we obtain

$$\begin{aligned}
V_{\Phi}f(z, \zeta) &= \frac{1}{(a+1)^{d/2}} e^{-\pi\frac{a}{a+1}z_1^2 - \pi z_2^2} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} e^{-\pi t^2} e^{-2\pi i\left(\frac{t}{\sqrt{a+1}} + \frac{z_1}{a+1}\right)(\zeta_1 - c\xi)} dt \right) \\
&\quad \times e^{-\pi[(b+1)\xi^2 - 2\xi z_2]} e^{-2\pi i \zeta_2 \xi} d\xi \\
&= \frac{1}{(a+1)^{d/2}} e^{-\pi\frac{a}{a+1}z_1^2 - \pi z_2^2 - 2\pi i\frac{z_1 \zeta_1}{a+1}} \int_{\mathbb{R}^d} e^{-\pi\frac{(\zeta_1 - c\xi)^2}{a+1}} e^{2\pi i\frac{cz_1}{a+1}\xi} \\
&\quad \times e^{-\pi[(b+1)\xi^2 - 2\xi z_2]} e^{-2\pi i \zeta_2 \xi} d\xi \\
&= \frac{1}{(a+1)^{d/2}} e^{-\pi\frac{a}{a+1}z_1^2 - \pi z_2^2 - 2\pi i\frac{z_1 \zeta_1}{a+1} - \pi\frac{\zeta_1^2}{a+1}} \\
(24) \quad &\quad \times \int_{\mathbb{R}^d} e^{-\pi\left[(b+1)\xi^2 + \frac{c^2}{a+1}\xi^2 - 2\frac{c\zeta_1}{a+1}\xi - 2\xi z_2\right]} e^{2\pi i\left(\frac{cz_1}{a+1} - \zeta_2\right)\xi} d\xi.
\end{aligned}$$

The last integral can be computed as follows:

$$\begin{aligned}
I &:= \int_{\mathbb{R}^d} e^{-\pi\left[\frac{(a+1)(b+1)+c^2}{a+1}\xi^2 - 2\frac{c\zeta_1+z_2(a+1)}{a+1}\xi\right]} e^{2\pi i\left(\frac{cz_1}{a+1} - \zeta_2\right)\xi} d\xi \\
&= \int_{\mathbb{R}^d} e^{-\frac{\pi}{a+1}\{[(a+1)(b+1)+c^2]\xi^2 - 2[c\zeta_1+z_2(a+1)]\xi\}} e^{2\pi i\left(\frac{cz_1}{a+1} - \zeta_2\right)\xi} d\xi \\
&= \int_{\mathbb{R}^d} e^{-\frac{\pi}{a+1}\left[\sqrt{(a+1)(b+1)+c^2}\xi - \frac{c\zeta_1+z_2(a+1)}{\sqrt{(a+1)(b+1)+c^2}}\right]^2} e^{\frac{\pi}{a+1}\frac{[c\zeta_1+z_2(a+1)]^2}{(a+1)(b+1)+c^2}} e^{2\pi i\left(\frac{cz_1}{a+1} - \zeta_2\right)\xi} d\xi \\
&= e^{\frac{\pi}{a+1}\frac{[c^2\zeta_1^2+(a+1)^2]}{(a+1)(b+1)+c^2}} \int_{\mathbb{R}^d} e^{-\frac{\pi}{a+1}\left[\sqrt{(a+1)(b+1)+c^2}\xi - \frac{c\zeta_1+z_2(a+1)}{\sqrt{(a+1)(b+1)+c^2}}\right]^2} e^{2\pi i\left(\frac{cz_1}{a+1} - \zeta_2\right)\xi} d\xi.
\end{aligned}$$



Making the change of variables  $t = \frac{\sqrt{(a+1)(b+1)+c^2}\xi - \frac{c\zeta_1+z_2(a+1)}{\sqrt{(a+1)(b+1)+c^2}}}{\sqrt{a+1}}$ , so that  $d\xi = \left[\frac{a+1}{(a+1)(b+1)+c^2}\right]^{d/2} dt$ , we can write

$$\begin{aligned} I &= \frac{(a+1)^{\frac{d}{2}}}{[(a+1)(b+1)+c^2]^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\pi t^2} e^{2\pi i \left(\frac{c z_1}{a+1} - \zeta_2\right)} \left[ \frac{\sqrt{a+1}}{\sqrt{(a+1)(b+1)+c^2}} t - \frac{c\zeta_1+(a+1)z_2}{(a+1)(b+1)+c^2} \right] dt \\ &= \frac{(a+1)^{\frac{d}{2}}}{[(a+1)(b+1)+c^2]^{\frac{d}{2}}} e^{\frac{\pi}{a+1} \frac{(a+1)^2 z_2^2 + c^2 \zeta_1^2 + 2c(a+1)z_2 \zeta_1}{(a+1)(b+1)+c^2}} e^{-2\pi i \left(\frac{c z_1}{a+1} - \zeta_2\right) \frac{c\zeta_1+(a+1)z_2}{(a+1)(b+1)+c^2}} \\ &\quad \times e^{-\pi \left[ \frac{a+1}{(a+1)(b+1)+c^2} \left(\frac{c z_1}{a+1} - \zeta_2\right)^2 \right]}. \end{aligned}$$

The result then follows by substituting the value of the integral  $I$  in (24).  $\square$

**Corollary 2.3.** *Consider the generalized Gaussian  $f$  defined in (22) and the window function  $\Phi(x, \xi) = e^{-\pi(x^2+\xi^2)}$ . Then, for every  $1 \leq p, q \leq \infty$ , we have*

$$\|f\|_{M^{p,q}} \asymp \|V_{\Phi} f\|_{L^{p,q}} \asymp \frac{[(a+1)(b+1)+c^2]^{\frac{d}{p}+\frac{d}{q}-\frac{d}{2}} [(c^2+ab+a)(c^2+ab+b)]^{\frac{d}{2q}-\frac{d}{2p}}}{[b^2(a+1)+b(c^2+a+1)]^{\frac{d}{2q}} [a^2(b+1)+a(c^2+b+1)]^{\frac{d}{2q}}}.$$

The cases  $p = \infty$  or  $q = \infty$  can be obtained by using the rule  $1/\infty = 0$  in formula (25).

*Proof.* By Proposition 2.2, we can write

$$|V_{\Phi} f(z, \zeta)| = \frac{1}{[(a+1)(b+1)+c^2]^{\frac{d}{2}}} e^{-\pi \frac{[a(b+1)+c^2]z_1^2 + [(a+1)b+c^2]z_2^2 + (b+1)\zeta_1^2 + (a+1)\zeta_2^2 - 2c(z_1\zeta_2+z_2\zeta_1)}{(a+1)(b+1)+c^2}}.$$

It remains to compute the mixed  $L^{p,q}$ -norm of the previous function. We treat the cases  $1 \leq p, q < \infty$ . The cases either  $p = \infty$  or  $q = \infty$  are obtained with obvious modifications.

For simplicity, we set

$$\alpha = \frac{c^2 + a(b+1)}{(a+1)(b+1)+c^2}, \quad \beta = \frac{c^2 + (a+1)b}{(a+1)(b+1)+c^2}, \quad \gamma = \frac{(b+1)}{(a+1)(b+1)+c^2},$$

$$\delta = \frac{(a+1)}{(a+1)(b+1)+c^2}, \quad \sigma = \frac{c}{(a+1)(b+1)+c^2}.$$

Hence

$$(28) \quad \frac{\|V_{\Phi} f\|_{L^{p,q}}}{[(a+1)(b+1)+c^2]^{-\frac{d}{2p}}} = \left( \int_{\mathbb{R}^{2d}} I^{\frac{q}{p}} e^{-\pi q(\gamma\zeta_1^2 + \delta\zeta_2^2)} d\zeta_1 d\zeta_2 \right)^{\frac{1}{q}} =: A,$$

where  $I := \int_{\mathbb{R}^{2d}} e^{-\pi p \alpha z_1^2 - \pi p \beta z_2^2} e^{2\pi p \sigma (z_1 \zeta_2 + z_2 \zeta_1)} dz_1 dz_2$ . Now straightforward computations and change of variables yield

$$\begin{aligned} I &= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} e^{-\pi p (\alpha z_1^2 - 2\sigma \zeta_2 z_1)} dz_1 \right) e^{-\pi \beta z_2^2 + 2\pi p \sigma z_2 \zeta_1} dz_2 \\ &= e^{\pi p \frac{\sigma^2}{\alpha} \zeta_2^2} e^{\pi p \frac{\sigma^2}{\beta} \zeta_1^2} \int_{\mathbb{R}^d} e^{-\pi p (\sqrt{\alpha} z_1 - \frac{\sigma}{\sqrt{\alpha}} \zeta_2)^2} dz_1 \int_{\mathbb{R}^d} e^{-\pi p (\sqrt{\beta} z_2 - \frac{\sigma}{\sqrt{\beta}} \zeta_1)^2} dz_2 \\ &= p^{-\frac{d}{2}} \alpha^{-\frac{d}{2}} p^{-\frac{d}{2}} \beta^{-\frac{d}{2}} e^{\pi p \frac{\sigma^2}{\alpha} \zeta_2^2} e^{\pi p \frac{\sigma^2}{\beta} \zeta_1^2}. \end{aligned}$$

Substituting the value of the integral  $I$  in (28), we obtain

$$\begin{aligned} A &= p^{-\frac{d}{p}} \alpha^{-\frac{d}{2p}} \beta^{-\frac{d}{2p}} \left( \int_{\mathbb{R}^d} e^{\pi q \frac{\sigma^2}{\beta} \zeta_1^2 - \pi q \gamma \zeta_1^2} d\zeta_1 \int_{\mathbb{R}^d} e^{\pi q \frac{\sigma^2}{\alpha} \zeta_2^2 - \pi q \delta \zeta_2^2} d\zeta_2 \right)^{\frac{1}{q}} \\ &= p^{-\frac{d}{p}} \alpha^{-\frac{d}{2p}} \beta^{-\frac{d}{2p}} \left( \int_{\mathbb{R}^d} e^{-\pi q (\gamma - \frac{\sigma^2}{\beta}) \zeta_1^2} d\zeta_1 \int_{\mathbb{R}^d} e^{-\pi q (\delta - \frac{\sigma^2}{\alpha}) \zeta_2^2} d\zeta_2 \right)^{\frac{1}{q}} \\ &= p^{-\frac{d}{p}} \alpha^{-\frac{d}{2p}} \beta^{-\frac{d}{2p}} q^{-\frac{d}{q}} \left( \gamma - \frac{\sigma^2}{\beta} \right)^{-\frac{d}{2q}} \left( \delta - \frac{\sigma^2}{\alpha} \right)^{-\frac{d}{2q}}. \end{aligned}$$

Finally, the goal is attained by substituting in  $A$  the values of the parameters  $\alpha, \beta, \gamma, \delta, \sigma$  in (26) and (27) and observing that

$$\|f\|_{M^{p,q}} \asymp \|V_{\Phi} f\|_{L^{p,q}} = A[(a+1)(b+1) + c^2]^{-\frac{d}{2}}.$$

This concludes the proof.  $\square$

We have now all the tools to compute the modulation norm of the (cross-)Wigner distribution  $W(\varphi, \varphi_{\lambda})$  in (20). Precisely, setting in formula (25) the values  $a = a_{\lambda}$ ,  $b = b_{\lambda}$ ,  $c = c_{\lambda}$ , where  $a_{\lambda}, b_{\lambda}, c_{\lambda}$  are defined in (21), and making easy simplifications we attain the following result.

**Corollary 2.4.** *For  $\lambda > 0$  consider the (cross-)Wigner distribution  $W(\varphi, \varphi_{\lambda})$  defined in (20) (cf. Lemma 2.2). Then*

$$(29) \quad \|W(\varphi, \varphi_{\lambda})\|_{M^{p,q}} \asymp \frac{[(2\lambda+1)(\lambda+2)]^{\frac{d}{2q} - \frac{d}{2p}}}{\lambda^{\frac{d}{2q}} (1+\lambda)^{\frac{d}{2} - \frac{d}{p}}}.$$

The cases  $p = \infty$  or  $q = \infty$  can be obtained by using the rule  $1/\infty = 0$  in formula (29).

### 3. MAIN RESULT

In this Section we prove Theorem 1.1. We will focus separately on the sufficient and necessary part in the statement.

**Theorem 3.1 (Sufficient Conditions).** *If  $p_1, q_1, p_2, q_2, p, q \in [1, \infty]$  are indices which satisfy (5) and (6),  $s \in \mathbb{R}$ ,  $f_1 \in M_{v_{|s|}}^{p_1, q_1}(\mathbb{R}^d)$  and  $f_2 \in M_{v_s}^{p_2, q_2}(\mathbb{R}^d)$ , then  $W(f_1, f_2) \in M_{1 \otimes v_s}^{p, q}(\mathbb{R}^{2d})$ , and the estimate (7) holds true.*

*Proof.* We first study the case  $p, q < \infty$ . Let  $g \in \mathcal{S}(\mathbb{R}^d)$  and set  $\Phi = W(g, g) \in \mathcal{S}(\mathbb{R}^{2d})$ . If  $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$ , we write  $\tilde{\zeta} = (\zeta_2, -\zeta_1)$ . Then, from Lemma 2.1,

$$(30) \quad |V_\Phi(W(f_1, f_2))(z, \zeta)| = |V_g f_2(z + \frac{\tilde{\zeta}}{2})| |V_g f_1(z - \frac{\tilde{\zeta}}{2})|.$$

Consequently,

$$\|W(f_1, f_2)\|_{M_{1 \otimes v_s}^{p, q}} \asymp \left( \int_{\mathbb{R}^{2d}} \left( \int_{\mathbb{R}^{2d}} |V_g f_2(z + \frac{\tilde{\zeta}}{2})|^p |V_g f_1(z - \frac{\tilde{\zeta}}{2})|^p dz \right)^{\frac{q}{p}} \langle \zeta \rangle^{sq} d\zeta \right)^{1/q}.$$

After the change of variables  $z \mapsto z - \tilde{\zeta}/2$ , the integral over  $z$  becomes the convolution  $(|V_g f_2|^p * |(V_g f_1)^*|^p)(\tilde{\zeta})$ , and observing that  $(1 \otimes v_s)(z, \zeta) = \langle \zeta \rangle^s = v_s(\zeta) = v_s(\tilde{\zeta})$ , we obtain

$$\begin{aligned} \|W(f_1, f_2)\|_{M_{1 \otimes v_s}^{p, q}} &\asymp \left( \iint_{\mathbb{R}^{2d}} (|V_g f_2|^p * |(V_g f_1)^*|^p)^{\frac{q}{p}}(\tilde{\zeta}) v_s(\tilde{\zeta})^q d\zeta \right)^{1/p} \\ &= \| |V_g f_2|^p * |(V_g f_1)^*|^p \|_{L_{v_{ps}}^{\frac{q}{p}}}. \end{aligned}$$

Hence

$$(31) \quad \|W(f_1, f_2)\|_{M_{1 \otimes v_s}^{p, q}}^p \asymp \| |V_g f_2|^p * |(V_g f_1)^*|^p \|_{L_{v_{ps}}^{\frac{q}{p}}}.$$

Case  $p \leq q < \infty$ .

**Step 1.** Here we prove the desired result in the case  $p \leq p_i, q_i, i = 1, 2$ .

Suppose first that (4) are satisfied (and hence  $p_i, q_i \leq q, i = 1, 2$ ). Since  $q/p \geq 1$ , we can apply Young's Inequality for mixed-normed spaces (cf. [1], see also [26]) and majorize (31) as follows

$$\begin{aligned} \|W(f_1, f_2)\|_{M_{1 \otimes v_s}^{p, q}}^p &\lesssim \| |V_g f_2|^p \|_{L_{v_{p|s|}}^{r_2, s_2}} \| |(V_g f_1)^*|^p \|_{L_{v_{ps}}^{r_1, s_1}} \\ &= \| |V_g f_1|^p \|_{L_{v_{p|s|}}^{r_1, s_1}} \| |V_g f_2|^p \|_{L_{v_{ps}}^{r_2, s_2}} \\ &= \| |V_g f_1|^p \|_{L_{v_{|s|}}^{pr_1, ps_1}} \| |V_g f_2|^p \|_{L_{v_s}^{pr_2, ps_2}}, \end{aligned}$$

for every  $1 \leq r_1, r_2, s_1, s_2 \leq \infty$  such that

$$(32) \quad \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{s_1} + \frac{1}{s_2} = 1 + \frac{p}{q}.$$

Choosing  $r_i = p_i/p \geq 1, s_i = q_i/p \geq 1, i = 1, 2$ , the indices' relation (32) becomes (4) and we obtain

$$\|W(f_1, f_2)\|_{M_{1 \otimes v_s}^{p, q}} \lesssim \|V_g f_1\|_{L_{v_{|s|}}^{p_1, q_1}} \|V_g f_2\|_{L_{v_s}^{p_2, q_2}} \asymp \|f_1\|_{M_{v_{|s|}}^{p_1, q_1}} \|f_2\|_{M_{v_s}^{p_2, q_2}}.$$

Now, still assume  $p \leq p_i, q_i, i = 1, 2$  but

$$\frac{1}{p_1} + \frac{1}{p_2} \geq \frac{1}{p} + \frac{1}{q}, \quad \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{p} + \frac{1}{q},$$

(hence  $p_i, q_i \leq q, i = 1, 2$ ). We set  $u_1 = tp_1$ , and look for  $t \geq 1$  (hence  $u_1 \geq p_1$ ) such that

$$\frac{1}{u_1} + \frac{1}{p_2} = \frac{1}{p} + \frac{1}{q}$$

that gives

$$0 < \frac{1}{t} = \frac{p_1}{p} + \frac{p_1}{q} - \frac{p_1}{p_2} \leq 1$$

because  $p_1(1/p + 1/q) - p_1/p_2 \leq p_1(1/p_1 + 1/p_2) - p_1/p_2 = 1$  whereas the lower bound of the previous estimate follows by  $1/(tp_1) = 1/p + 1/q - 1/p_2 > 0$  since  $p \leq p_2$ . Hence the previous part of the proof gives

$$\begin{aligned} \|W(f_1, f_2)\|_{M_{1 \otimes v_s}^{p,q}} &\lesssim \|f_1\|_{M_{v_s}^{u_1, q_1}} \|f_2\|_{M_{v_s}^{p_2, q_2}} \\ &\lesssim \|f_1\|_{M_{v_s}^{p_1, q_1}} \|f_2\|_{M_{v_s}^{p_2, q_2}}. \end{aligned}$$

where the last inequality follows by inclusion relations for modulation spaces  $M_{v_s}^{p_1, q_1}(\mathbb{R}^d) \subseteq M_{v_s}^{u_1, q_1}(\mathbb{R}^d)$  for  $p_1 \leq u_1$ .

The general case

$$\frac{1}{p_1} + \frac{1}{p_2} \geq \frac{1}{p} + \frac{1}{q}, \quad \frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{p} + \frac{1}{q},$$

can be treated analogously.

**Step 2.** Assume now that  $p_i, q_i \leq q, i = 1, 2$ , and satisfy relation (6). If at least one out of the indices  $p_1, p_2$  is less than  $p$ , assume for instance  $p_1 \leq p$ , whereas  $p \leq q_1, q_2$ , then we proceed as follows. We choose  $u_1 = p, u_2 = q$ , and deduce by the results in Step 1 (with  $p_1 = u_1$  and  $p_2 = u_2$ ) that

$$\|W(f_1, f_2)\|_{M_{1 \otimes v_s}^{p,q}} \lesssim \|f_1\|_{M_{v_s}^{u_1, q_1}} \|f_2\|_{M_{v_s}^{u_2, q_2}} \lesssim \|f_1\|_{M_{v_s}^{p_1, q_1}} \|f_2\|_{M_{v_s}^{p_2, q_2}}$$

where the last inequality follows by inclusion relations for modulation spaces, since  $p_1 \leq u_1 = p$  and  $p_2 \leq u_2 = q$ .

Similarly we argue when at least one out of the indices  $q_1, q_2$  is less than  $p$  and  $p \leq p_1, p_2$  or when at least one out of the indices  $q_1, q_2$  is less than  $p$  and at least one out of the indices  $p_1, p_2$  is less than  $p$ . The remaining case  $p \leq p_i, q_i \leq q$  is treated in Step 1.

*Case  $p < q = \infty$ .* The argument are similar to the case  $p \leq q < \infty$ .

Case  $p = q = \infty$ . We use (30) and the submultiplicative property of the weight  $v_s$ ,

$$\begin{aligned}
\|W(f_1, f_2)\|_{M_{1 \otimes v_s}^\infty} &= \sup_{z, \zeta \in \mathbb{R}^{2d}} |V_g f_2(z + \frac{\tilde{\zeta}}{2})| |V_g f_1(z - \frac{\tilde{\zeta}}{2})| v_s(\zeta) \\
&= \sup_{z, \zeta \in \mathbb{R}^{2d}} \|V_g f_2(z)\| |(V_g f_1)^*(z - \zeta)| v_s(\zeta) \\
&= \sup_{z, \zeta \in \mathbb{R}^{2d}} \|V_g f_2(z)\| |(V_g f_1)^*(z - \tilde{\zeta})| v_s(\tilde{\zeta}) \\
&\leq \sup_{z \in \mathbb{R}^{2d}} (\|V_g f_1 v_{|s}\|_\infty |V_g f_2(z) v_s(z)|) = \|V_g f_1 v_{|s}\|_\infty \|V_g f_2 v_s\|_\infty \\
&\asymp \|f\|_{M_{v_{|s}}^\infty} \|g\|_{M_{v_s}^\infty} \leq \|f\|_{M_{v_{|s}}^{p_1, q_1}} \|f\|_{M_{v_s}^{p_2, q_2}},
\end{aligned}$$

for every  $1 \leq p_i, q_i \leq \infty$ ,  $i = 1, 2$ . Notice that in this case conditions (5) and (8) are trivially satisfied.

Case  $p > q$ . Using the inclusion relations for modulation spaces, we majorize

$$\|W(f_1, f_2)\|_{M_{1 \otimes v_s}^{p, q}} \lesssim \|W(f_1, f_2)\|_{M_{1 \otimes v_s}^{q, q}} \lesssim \|f_1\|_{M_{v_{|s}}^{p_1, q_1}} \|f_2\|_{M_{v_s}^{p_2, q_2}}$$

for every  $1 \leq p_i, q_i \leq q$ ,  $i = 1, 2$ . Here we have applied the case  $p \leq q$  with  $p = q$ . Notice that in this case condition (8) is trivially satisfied, since from  $p_i, q_i \leq q$  we infer  $1/p_1 + 1/p_2 \geq 1/q + 1/q$ ,  $1/q_1 + 1/q_2 \geq 1/q + 1/q$ . This concludes the proof.  $\square$

**Remark 3.2.** (i) *The result of Theorem 3.1 can be extended to more general weights. In particular, it holds for polynomial weights satisfying relation (4.10) in [48]. Hence our result extends Toft's result [48, Theorem 4.2] (cf. also [47, Theorem 4.1] for modulation spaces without weights). Other examples of suitable weights are given by sub-exponential weights of the type  $v(z) = e^{\alpha|z|^\beta}$  for  $\alpha > 0$  and  $0 < \beta < 1$ .*

(ii) *The particular case  $p = 1$ ,  $1 \leq q \leq \infty$ ,  $p_1 = q_1 = 1$ ,  $p_2 = q_2 = q$ ,  $s \geq 0$ , was already proved in [13, Prop. 2.2].*

(iii) *For  $p_i = q_i = p = q = 2$ ,  $i = 1, 2$ , we obtain the following continuity result for the cross-Wigner distribution acting between Shubin spaces and Sobolev spaces:*

*For  $s \geq 0$ ,  $f_1, f_2 \in Q_s(\mathbb{R}^d)$  (cf. Shubin's book [41]), the cross-Wigner distribution  $W(f_1, f_2)$  is in  $H^s(\mathbb{R}^{2d})$  with*

$$\|W(f_1, f_2)\|_{H^s(\mathbb{R}^{2d})} \lesssim \|f_1\|_{Q_s(\mathbb{R}^d)} \|f_2\|_{Q_s(\mathbb{R}^d)}.$$

(iv) *Continuity properties of the cross-Wigner distribution on modulation spaces with different weight functions can be easily inferred using the techniques of Theorem 3.1 and the Young type inequalities for weighted spaces shown by Johansson et al. in [32, Theorem 2.2].*

The estimate in (7) can be slightly improved if  $s \geq 0$ . Precisely, we have the following result.

**Theorem 3.3.** *If  $p_1, q_1, p_2, q_2, p, q \in [1, \infty]$  are indices which satisfy (5) and (6),  $s \geq 0$ ,  $f_1 \in M_{v_s}^{p_1, q_1}(\mathbb{R}^d)$  and  $f_2 \in M_{v_s}^{p_2, q_2}(\mathbb{R}^d)$ , then  $W(f_1, f_2) \in M_{1 \otimes v_s}^{p, q}(\mathbb{R}^{2d})$ , with*

$$\|W(f_1, f_2)\|_{M_{1 \otimes v_s}^{p, q}} \lesssim \|f_1\|_{M^{p_1, q_1}} \|f_2\|_{M^{p_2, q_2}} + \|f_1\|_{M_{v_s}^{p_1, q_1}} \|f_2\|_{M^{p_2, q_2}}.$$

*Proof.* The proof is similar to that of Theorem 3.1, but in this case from the estimate (31) we proceed by using

$$v_s(z) \lesssim v_s(z - w) + v_s(w), \quad s \geq 0$$

(with  $sp$  in place of  $s$ ) instead of  $v_s(z) \lesssim v_{|s|}(z - w)v_s(w)$ .  $\square$

If in particular we consider the Wigner distribution  $W(f, f)$ , then Theorem 3.3 can be rephrased as follows.

**Corollary 3.4.** *Assume  $s \geq 0$ ,  $p_1, q_1, p, q \in [1, \infty]$  such that*

$$2 \min\left\{\frac{1}{p_1}, \frac{1}{q_1}\right\} \geq \frac{1}{p} + \frac{1}{q}.$$

*If  $f \in M_{v_s}^{p_1, q_1}(\mathbb{R}^d)$ , then the Wigner distribution  $W(f, f)$  is in  $M_{1 \otimes v_s}^{p, q}(\mathbb{R}^{2d})$ , with*

$$(33) \quad \|W(f, f)\|_{M_{1 \otimes v_s}^{p, q}(\mathbb{R}^{2d})} \lesssim \|f\|_{M^{p_1, q_1}(\mathbb{R}^d)} \|f\|_{M_{v_s}^{p_1, q_1}(\mathbb{R}^d)}.$$

We now prove the sharpness of Theorem 3.1 (and Corollary 3.4) in the unweighted case  $s = 0$ .

**Theorem 3.5 (Necessary Conditions).** *Consider  $p_1, p_2, q_1, q_2, p, q \in [1, \infty]$ . Assume that there exists a constant  $C > 0$  such that*

$$(34) \quad \|W(f_1, f_2)\|_{M^{p, q}} \leq C \|f_1\|_{M^{p_1, q_1}} \|f_2\|_{M^{p_2, q_2}}, \quad \forall f_1, f_2 \in \mathcal{S}(\mathbb{R}^{2d}),$$

*then (5) and (6) must hold.*

*Proof.* Let us first demonstrate the necessity of (6). We consider the dilated Gaussians  $\varphi_\lambda(x) = \varphi(\sqrt{\lambda}x)$ , with  $\varphi(x) = e^{-\pi x^2}$ .

An easy computation (see also [28, formula (4.20)] (6)) shows that

$$(35) \quad W(\varphi_\lambda, \varphi_\lambda)(x, \xi) = 2^{\frac{d}{2}} \lambda^{-\frac{d}{2}} \varphi_{2\lambda}(x) \varphi_{2/\lambda}(\xi).$$

Now (see [15, Lemma 3.2], [47, Lemma 1.8])

$$\|\varphi_\lambda\|_{M^{r, s}} \asymp \lambda^{-\frac{d}{2r}} (\lambda + 1)^{-\frac{d}{2}(1 - \frac{1}{q} - \frac{1}{p})}$$

and observe that

$$\|W(\varphi_\lambda, \varphi_\lambda)\|_{M^{p, q}(\mathbb{R}^{2d})} = 2^{\frac{d}{2}} \lambda^{-\frac{d}{2}} \|\varphi_{2\lambda}\|_{M^{p, q}(\mathbb{R}^d)} \|\varphi_{2/\lambda}\|_{M^{p, q}(\mathbb{R}^d)}.$$

The assumption (8) in this case becomes

$$\lambda^{-\frac{d}{2}} (\lambda + 1)^{-\frac{d}{2}(1 - \frac{1}{q} - \frac{1}{p})} (\lambda^{-1} + 1)^{-\frac{d}{2}(1 - \frac{1}{q} - \frac{1}{p})} \lesssim \lambda^{-\frac{d}{2p_1}} (1 + \lambda)^{-\frac{d}{2}(1 - \frac{1}{q_1} - \frac{1}{p_1})} \lambda^{-\frac{d}{2p_2}} (1 + \lambda)^{-\frac{d}{2}(1 - \frac{1}{q_2} - \frac{1}{p_2})}$$

and letting  $\lambda \rightarrow +\infty$  we obtain

$$\frac{1}{p} + \frac{1}{q} \leq \frac{1}{q_1} + \frac{1}{q_2}$$

whereas for  $\lambda \rightarrow 0^+$

$$\frac{1}{p} + \frac{1}{q} \leq \frac{1}{p_1} + \frac{1}{p_2},$$

so that (6) must hold.

It remains to prove the sharpness of (5). We first show the conditions  $p_2, q_2 \leq q$ . We test (34) on the (cross-)Wigner distribution  $W(\varphi, \varphi_\lambda)$  defined in (20), that is

$$\|W(\varphi, \varphi_\lambda)\|_{M^{p,q}(\mathbb{R}^{2d})} \lesssim \|\varphi\|_{M^{p_1,q_1}(\mathbb{R}^d)} \|\varphi_\lambda\|_{M^{p_2,q_2}(\mathbb{R}^d)}.$$

Using Corollary 2.4 the previous estimate can be rephrased as

$$\frac{[(2\lambda + 1)(\lambda + 2)]^{\frac{d}{2q} - \frac{d}{2p}}}{\lambda^{\frac{d}{2q}}(1 + \lambda)^{\frac{d}{2} - \frac{d}{p}}} \lesssim \lambda^{-\frac{d}{2p_2}} \lambda^{-\frac{d}{2}(1 - \frac{1}{q_2} - \frac{1}{p_2})}, \quad \forall \lambda > 0.$$

Letting  $\lambda \rightarrow +\infty$  we attain

$$q_2 \leq q$$

whereas for  $\lambda \rightarrow 0^+$

$$p_2 \leq q.$$

The conditions  $p_1, q_1 \leq q$  then follows by using the cross-Wigner property

$$W(\varphi_\lambda, \varphi)(x, \xi) = \overline{W(\varphi, \varphi_\lambda)(x, \xi)},$$

so that

$$\|W(\varphi_\lambda, \varphi)\|_{M^{p,q}(\mathbb{R}^{2d})} = \|\overline{W(\varphi, \varphi_\lambda)}\|_{M^{p,q}(\mathbb{R}^{2d})} = \|W(\varphi, \varphi_\lambda)\|_{M^{p,q}(\mathbb{R}^{2d})}$$

and applying the same argument as before.  $\square$

#### 4. CONTINUITY RESULTS FOR THE SHORT-TIME FOURIER TRANSFORM AND THE AMBIGUITY DISTRIBUTION

This optimal bounds in Theorem 1.1 for the Wigner distribution can be translated in optimal new estimates for other time-frequency representations such that the STFT or the ambiguity function. Precisely, given  $f, g \in L^2(\mathbb{R}^d)$ , we recall the definition of the (cross-)ambiguity function

$$(36) \quad A(f_1, f_2)(x, \xi) = \int_{\mathbb{R}^d} e^{-2\pi i t \xi} f_1\left(t + \frac{x}{2}\right) \overline{f_2\left(t - \frac{x}{2}\right)} dt.$$

It is well-known that the Wigner distribution is the symplectic Fourier transform of the ambiguity function, see e.g., [21]. In other words, cf. [28, Lemma 4.3.4],

$$(37) \quad W(f_1, f_2)(x, \xi) = \mathcal{FUA}(f_1, f_2)(x, \xi), \quad f_1, f_2 \in L^2(\mathbb{R}^d),$$

where the operator  $\mathcal{U}$  is the rotation  $\mathcal{U}F(x, \xi) = F(\xi, -x)$  of a function  $F$  on  $\mathbb{R}^{2d}$ . We need the following norm equivalence.

**Lemma 4.1.** *For  $s \in \mathbb{R}$ ,  $1 \leq p, q \leq \infty$ , the following equivalence holds*

$$\|W(f_1, f_2)\|_{M_{1 \otimes v_s}^{p,q}} = \|A(f_1, f_2)\|_{W(\mathcal{F}L^p, L_{v_s}^q)} \asymp \|V_{f_2} f_1\|_{W(\mathcal{F}L^p, L_{v_s}^q)}.$$

*Proof.* Let us observe that the weight  $v_s$ ,  $s \in \mathbb{R}$ , is symmetric in each coordinate:

$$v_s(x, \xi) = v_s(x, -\xi) = v_s(-x, \xi) = v_s(-x, -\xi).$$

Using (37), the connection between modulation and Wiener amalgam spaces (15) and the symmetry of the weights  $v_s$  we can write

$$\begin{aligned} \|W(f_1, f_2)\|_{M_{1 \otimes v_s}^{p,q}} &= \|\mathcal{F}\mathcal{U}A(f_1, f_2)\|_{M_{1 \otimes v_s}^{p,q}} = \|\mathcal{U}A(f_1, f_2)\|_{W(\mathcal{F}L^p, L_{v_s}^q)} \\ &= \|A(f_1, f_2)\|_{W(\mathcal{F}L^p, L_{v_s}^q)}. \end{aligned}$$

Now a simple change of variables in (36) let us write

$$A(f_1, f_2)(x, \xi) = e^{\pi i x \xi} V_{f_2} f_1(x, \xi).$$

It was proved in [11, Proposition 3.2] that the function  $F(x, \xi) = e^{\pi i x \xi}$  is in the Wiener amalgam space  $W(\mathcal{F}L^1, L^\infty)$ . This means that, by the product properties for Wiener amalgam spaces, for every  $s \in \mathbb{R}$ ,

$$\|A(f_1, f_2)\|_{W(\mathcal{F}L^p, L_{v_s}^q)} \lesssim \|F\|_{W(\mathcal{F}L^1, L^\infty)} \|V_{f_2} f_1\|_{W(\mathcal{F}L^p, L_{v_s}^q)}$$

and since  $\bar{F}(x, \xi) = e^{-\pi i x \xi} \in W(\mathcal{F}L^1, L^\infty)$  as well, with  $\|\bar{F}\|_{W(\mathcal{F}L^1, L^\infty)} = \|F\|_{W(\mathcal{F}L^1, L^\infty)}$ , we can analogously write

$$\|V_{f_2} f_1\|_{W(\mathcal{F}L^p, L_{v_s}^q)} \lesssim \|F\|_{W(\mathcal{F}L^1, L^\infty)} \|A(f_1, f_2)\|_{W(\mathcal{F}L^p, L_{v_s}^q)}.$$

This proves the desired result.  $\square$

This observations, together with the Wigner property  $W(f_1, f_2)(x, \xi) = \overline{W(f_2, f_1)}$  let us translate Theorem 1.1 in terms of STFT acting from modulation spaces to Wiener amalgam spaces. Notice that the following two corollaries also hold for the ambiguity function  $A(f_1, f_2)$  in place of the STFT  $V_{f_1} f_2$ .

**Corollary 4.1.** *Consider  $s \in \mathbb{R}$  and assume that  $p_1, p_2, q_1, q_2, p, q \in [1, \infty]$  satisfy conditions (5) and (6). Then if  $f_1 \in M_{v_s}^{p_1, q_1}(\mathbb{R}^d)$  and  $f_2 \in M_{v_s}^{p_2, q_2}(\mathbb{R}^d)$ , we have  $V_{f_1} f_2 \in W(\mathcal{F}L^p, L_{v_s}^q)(\mathbb{R}^{2d})$  with*

$$(38) \quad \|V_{f_1} f_2\|_{W(\mathcal{F}L^p, L_{v_s}^q)(\mathbb{R}^{2d})} \lesssim \|f_1\|_{M_{v_s}^{p_1, q_1}(\mathbb{R}^d)} \|f_2\|_{M_{v_s}^{p_2, q_2}(\mathbb{R}^d)}.$$

*Viceversa, assume that there exists a constant  $C > 0$  such that*

$$(39) \quad \|V_{f_1} f_2\|_{W(\mathcal{F}L^p, L^q)(\mathbb{R}^{2d})} \leq C \|f_1\|_{M^{p_1, q_1}(\mathbb{R}^d)} \|f_2\|_{M^{p_2, q_2}(\mathbb{R}^d)}, \quad \forall f_1, f_2 \in \mathcal{S}(\mathbb{R}^{2d}).$$

*Then (5) and (6) must hold.*



The previous result has many special and interesting cases. Let us just give a flavour of the main important ones. For  $p_i = q_i$ ,  $i = 1, 2$ , we obtain the following result.

**Corollary 4.2.** *Assume that  $p_1, p_2, p, q \in [1, \infty]$  satisfy*

$$(40) \quad p_1, p_2 \leq q, \quad \frac{1}{p_1} + \frac{1}{p_2} \geq \frac{1}{p} + \frac{1}{q}.$$

*Then, for  $s \in \mathbb{R}$ , if  $f_1 \in M_{v_{|s|}}^{p_1}(\mathbb{R}^d)$  and  $f_2 \in M_{v_s}^{p_2}(\mathbb{R}^d)$  we have  $V_{f_1} f_2 \in W(\mathcal{F}L^p, L_{v_s}^q)(\mathbb{R}^{2d})$  with*

$$(41) \quad \|V_{f_1} f_2\|_{W(\mathcal{F}L^p, L_{v_s}^q)(\mathbb{R}^{2d})} \lesssim \|f_1\|_{M_{v_{|s|}}^{p_1}(\mathbb{R}^d)} \|f_2\|_{M_{v_s}^{p_2}(\mathbb{R}^d)}.$$

*Viceversa, assume that there exists a constant  $C > 0$  such that*

$$(42) \quad \|V_{f_1} f_2\|_{W(\mathcal{F}L^p, L^q)(\mathbb{R}^{2d})} \leq C \|f_1\|_{M^{p_1}(\mathbb{R}^d)} \|f_2\|_{M^{p_2}(\mathbb{R}^d)}, \quad \forall f_1, f_2 \in \mathcal{S}(\mathbb{R}^{2d}).$$

*Then (40) must hold.*

**Remark 4.3.** *The previous result holds also for the cross-Wigner distribution if we replace  $\|V_{f_1} f_2\|_{W(\mathcal{F}L^p, L_{v_s}^q)(\mathbb{R}^{2d})}$  by  $\|W(f_1, f_2)\|_{M_{1 \otimes v_s}^{p, q}(\mathbb{R}^{2d})}$*

If we choose  $s = 0$ ,  $p = q'$  and  $q \geq 2$  in the previous result, we can refine some Lieb's integral bounds for the ambiguity function showed in [36]. Namely, we obtain in particular the following sufficient conditions for boundedness.

**Corollary 4.4.** *Assume  $q \geq 2$ ,  $p_1, p_2, q_1, q_2 \leq q$  such that*

$$\frac{1}{p_1} + \frac{1}{p_2} \geq 1 \quad \frac{1}{q_1} + \frac{1}{q_2} \geq 1.$$

*If  $f_i \in M^{p_i, q_i}(\mathbb{R}^d)$ ,  $i = 1, 2$ , then the ambiguity function satisfy  $A(f_1, f_2) \in L^q(\mathbb{R}^{2d})$ , with*

$$\|A(f_1, f_2)\|_{L^q(\mathbb{R}^{2d})} \lesssim \|f_1\|_{M^{p_1, q_1}(\mathbb{R}^d)} \|f_2\|_{M^{p_2, q_2}(\mathbb{R}^d)}.$$

*Proof.* Since  $\mathcal{F}L^{q'} \subseteq L^q$  for  $q \geq 2$ , the inclusion relations for Wiener amalgam spaces give  $W(\mathcal{F}L^{q'}, L^q)(\mathbb{R}^{2d}) \subseteq W(L^q, L^q)(\mathbb{R}^{2d}) = L^q(\mathbb{R}^{2d})$ . The result then follows by Corollary 4.1.  $\square$

Observe that for  $p_i = q_i = 2$ ,  $i = 1, 2$ ,  $M^{p_i, q_i}(\mathbb{R}^d) = L^2(\mathbb{R}^d)$  and we recapture Lieb's bound, see [36, Theorem 1]. We also refer to [14, 15] for related estimates for the short-time Fourier transform.

## 5. PSEUDODIFFERENTIAL OPERATORS

In this section we apply Theorem 1.1 to the study of pseudodifferential operators on modulation spaces. The key tool is the weak definition of a Weyl operator  $L_\sigma$  by means of a duality pairing between the symbol  $\sigma$  and the cross-Wigner distribution  $W(g, f)$  as shown in (9).

The sharpest result concerning boundedness of pseudodifferential operators on (un-weighted) modulation spaces was proved by one of us with Tabacco and Wahlberg in [17, Theorem 1.1]. Such result covers previous sufficient boundedness conditions proved by Toft in [47, Theorem 4.3] and necessary boundedness conditions exhibited in our previous work [15, Proposition 5.3]. Our result in this framework extends [17, Theorem 1.1] to weighted modulation spaces, thus widening the sufficient boundedness conditions presented by Toft in [48, Theorem 4.3]. Using Theorem 1.1 the proof of the following result is decidedly simple.

**Theorem 5.1.** *Assume  $s \geq 0$ ,  $p_i, q_i, p, q \in [1, \infty]$ ,  $i = 1, 2$ , are such that*

$$(43) \quad \min\left\{\frac{1}{p_1} + \frac{1}{p_2}, \frac{1}{q_1} + \frac{1}{q_2}\right\} \geq \frac{1}{p'} + \frac{1}{q'}.$$

and

$$(44) \quad q \leq \min\{p'_1, q'_1, p_2, q_2\}.$$

Then the pseudodifferential operator  $T$ , from  $\mathcal{S}(\mathbb{R}^d)$  to  $\mathcal{S}'(\mathbb{R}^d)$ , having symbol  $\sigma \in M_{1 \otimes v_s}^{p,q}(\mathbb{R}^{2d})$ , extends uniquely to a bounded operator from  $M_{v_s}^{p_1, q_1}(\mathbb{R}^d)$  to  $M_{v_s}^{p_2, q_2}(\mathbb{R}^d)$ , with the estimate

$$(45) \quad \|Tf\|_{M_{v_s}^{p_2, q_2}} \lesssim \|\sigma\|_{M_{1 \otimes v_s}^{p,q}} \|f\|_{M_{v_s}^{p_1, q_1}}.$$

Vice-versa, if (45) holds for  $s = 0$  and for every  $f \in \mathcal{S}(\mathbb{R}^d)$ ,  $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ , then (43) and (44) must be satisfied.

*Proof.* Assume  $\sigma \in M_{1 \otimes v_s}^{p,q}(\mathbb{R}^{2d})$ ,  $f \in M_{v_s}^{p_1, q_1}(\mathbb{R}^d)$  such that (43) and (44) are satisfied. For  $g \in M_{v_{-s}}^{p'_2, q'_2}(\mathbb{R}^d)$  Theorem 1.1 says that the cross-Wigner distribution is in  $M_{1 \otimes v_{-s}}^{p', q'}(\mathbb{R}^{2d})$ , provided that  $p_1, q_1, p'_2, q'_2 \leq q'$  and

$$\min\{1/p_1 + 1/p'_2, 1/q_1 + 1/q'_2\} \geq 1/p' + 1/q',$$

that are conditions (44) and (43), respectively. Thereby there exists a constant  $C > 0$  such that

$$\begin{aligned} |\langle \sigma, W(g, f) \rangle| &\leq \|a\|_{M_{1 \otimes v_s}^{p,q}(\mathbb{R}^{2d})} \|W(g, f)\|_{M_{1 \otimes v_{-s}}^{p', q'}(\mathbb{R}^{2d})} \\ &\leq C \|f\|_{M_{v_s}^{p_1, q_1}(\mathbb{R}^d)} \|g\|_{M_{v_{-s}}^{p'_2, q'_2}(\mathbb{R}^d)}. \end{aligned}$$

Since  $|\langle L_\sigma f, g \rangle| = |\langle \sigma, W(g, f) \rangle|$ , this concludes the proof of the sufficient conditions. The necessary conditions are proved in [17, Theorem 1.1].  $\square$

We now present sharp boundedness results for localization operators.

Let us mention that, since their introduction by Daubechies [20] as a mathematical tool to localize a signal in the time-frequency plane, they have been investigated by many authors in the field of signal analysis, see [6, 13, 25, 40, 47, 51, 46] and references therein. Localization operators with Gaussian windows are well-known in quantum mechanics, under the name of anti-Wick operators [5, 41].

A localization operator  $A_a^{\varphi_1, \varphi_2}$  with symbol  $a$  and windows  $\varphi_1, \varphi_2$  is defined as

$$(46) \quad A_a^{\varphi_1, \varphi_2} f(t) = \int_{\mathbb{R}^{2d}} a(x, \xi) V_{\varphi_1} f(x, \xi) M_{\xi} T_x \varphi_2(t) dx d\xi.$$

In signal analysis the meaning is as follows: first, analyse the signal  $f$  by taking the STFT  $V_{\varphi_1} f$ , then localize  $f$  by multiplying with the symbol  $a$  (if in particular  $a = \chi_{\Omega}$ , for some compact set  $\Omega \subseteq \mathbb{R}^{2d}$ , it is considered only the part of  $f$  that lives on the set  $\Omega$  in the time-frequency plane), then reconstruct the signal by superposition of time-frequency shifts with respect to the window  $\varphi_2$ . If  $\varphi_1(t) = \varphi_2(t) = e^{-\pi t^2}$ , then  $A_a = A_a^{\varphi_1, \varphi_2}$  is the classical Anti-Wick operator and the mapping  $a \rightarrow A_a^{\varphi_1, \varphi_2}$  is interpreted as a quantization rule [5, 41, 51].

Rewriting a localization operator  $A_a^{\varphi_1, \varphi_2}$  as a Weyl operator, cf. (10), we can investigate boundedness properties for localization operators as boundedness conditions for Weyl operators having symbols  $a * W(\varphi_2, \varphi_1)$ , see (11). Again it comes into play Theorem 1.1.

**Theorem 5.2.** *Assume  $s \geq 0$ , the indices  $p_i, q_i, p, q \in [1, \infty]$ ,  $i = 1, 2$ , fulfil the relations (43) and (44). Consider  $r \in [1, 2]$ . If  $a \in M_{1/v-s}^{p,q}(\mathbb{R}^{2d})$  and  $\varphi_1, \varphi_2 \in M_{v_2s}^r(\mathbb{R}^d)$ , then the localization operator  $A_a^{\varphi_1, \varphi_2}$  is continuous from  $M_{v_s}^{p_1, q_1}(\mathbb{R}^d)$  to  $M_{v_s}^{p_2, q_2}(\mathbb{R}^d)$  with*

$$\|A_a^{\varphi_1, \varphi_2}\|_{op} \lesssim \|a\|_{M_{1/v-s}^{p,q}} \|\varphi_1\|_{M_{2s}^r} \|\varphi_2\|_{M_{v_2s}^r}.$$

*Proof.* Using Theorem 1.1 for  $\varphi_1, \varphi_2 \in M_{v_2s}^r(\mathbb{R}^d)$  we obtain that  $W(\varphi_2, \varphi_1) \in M_{1 \otimes v_2s}^{1, \infty}$ , for every  $r \in [1, 2]$ . Now the convolution relations in Proposition 2.1, in the form  $M_{1 \otimes v-s}^{p,q} * M_{1 \otimes v_2s}^{1, \infty} \subseteq M_{1 \otimes v_s}^{p,q}$ , yield that the Weyl symbol  $\sigma = a * W(\varphi_2, \varphi_1)$  belongs to  $M_{1 \otimes v_s}^{p,q}$ . The result now follows from Theorem 5.1.  $\square$

**Remark 5.3.** (i) *The previous result extends Theorem 3.2 in [13] and Theorem 4.11 in [48] for this particular choice of weights. We observe that further extensions of Theorem 5.2 can be considered by using more general polynomial weights satisfying condition (4.17) in [48].*

(ii) *Using the same techniques as in the proof Theorem 5.2 one can study conditions on symbols and window functions such that the operator  $A_a^{\varphi_1, \varphi_2}$  is in the Schatten class  $S^p$ , cf., e.g. [13, Theorem 3.4].*

## 6. FURTHER APPLICATIONS: THE COHEN CLASS

The Cohen class (12) was introduced by Cohen in [8] essentially to circumvent the problem of the lack of positivity of the Wigner distribution. Many different kinds of kernels were proposed, in particular we recall for  $\tau \in [0, 1] \setminus \{1/2\}$  the  $\tau$ -kernels

$$\sigma_\tau(x, \xi) = \frac{2^d}{|2\tau - 1|^d} e^{2\pi i \frac{2}{2\tau - 1} x \xi},$$

which provide the  $\tau$ -Wigner distributions  $W_\tau(f, f)$  [7, Proposition 5.6]:

$$W_\tau(f, f) = W(f, f) * \sigma_\tau.$$

We recall that such distributions can be used in the definition of the  $\tau$ -pseudodifferential operators, see e.g., [7, 47]. Another important kernel is the Cohen kernel  $\Theta_\sigma$ , which yields the Born-Jordan distribution [10, 11, 12], given by [12, Prop. 3.4]

$$\Theta_\sigma(\zeta_1, \zeta_2) = \begin{cases} -2 \operatorname{Ci}(4\pi|\zeta_1\zeta_2|), & (\zeta_1, \zeta_2) \in \mathbb{R}^2, d = 1 \\ \mathcal{F}(\chi_{\{|s|\geq 2\}}|s|^{d-2})(\zeta_1, \zeta_2), & (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}, d \geq 2, \end{cases}$$

where  $\operatorname{Ci}(t)$  is the cosine integral function. It was shown in [12, Sec. 4] that  $\sigma_\tau$ ,  $\tau \in [0, 1] \setminus \{1/2\}$ , and  $\Theta_\sigma$  belong to the modulation space  $M^{1,\infty}(\mathbb{R}^{2d})$ . Inspired by this result, Theorem 1.2 shows continuity properties for elements of the Cohen class having kernels in the modulation space  $M^{1,\infty}(\mathbb{R}^{2d})$ . Let us prove Theorem 1.2. The main ingredient will be Theorem 1.1.

*Proof of Theorem 1.2.* If  $f \in M_{v_s}^{p_1, q_1}(\mathbb{R}^d)$ , with  $p_1, q_1$  satisfying (13), Theorem 1.1 gives that the Wigner distribution is in the corresponding  $M_{1 \otimes v_s}^{p, q}(\mathbb{R}^{2d})$ . Then the result follows by the inclusion relation  $M_{1 \otimes v_s}^{p, q} * M^{1, \infty} \subseteq M_{1 \otimes v_s}^{p, q}$ ,  $s \geq 0$  (see Prop. 2.1).  $\square$

Observe that the indices' assumptions of Theorem 1.2 coincide with those of Corollary 3.4. Hence the continuity properties on modulation spaces of these Cohen kernels coincide with those of the Wigner distribution. In other words, the time-frequency properties of these Cohen distributions resemble those of the Wigner distribution.

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## REFERENCES

- [1] A. Benedek and R. Panzone. The space  $L^p$  with mixed-norm. *Duke Math. J.*, 28:301–324, 1961.
- [2] A. Bényi, K. Gröchenig, K.A. Okoudjou and L.G. Rogers. Unimodular Fourier multipliers for modulation spaces. *J. Funct. Anal.*, 246(2): 366–384, 2007.
- [3] Á. Bényi and K. A. Okoudjou. Bilinear pseudodifferential operators on modulation spaces. *J. Fourier Anal. Appl.*, 100 (3):301–313, 2004.
- [4] A. Bényi and K.A. Okoudjou. Time-frequency estimates for pseudodifferential operators. *Contemporary Math.*, Amer. Math. Soc., 428:13–22, 2007.
- [5] F. A. Berezin. Wick and anti-Wick symbols of operators. *Mat. Sb. (N.S.)*, 86(128):578–610, 1971.
- [6] P. Boggiatto, E. Cordero, and K. Gröchenig. Generalized Anti-Wick operators with symbols in distributional Sobolev spaces. *Integral Equations and Operator Theory*, 48(4):427–442, 2004.
- [7] P. Boggiatto, G. De Donno and A. Oliaro. Time-frequency representations of Wigner type and pseudo-differential operators. *Trans. Amer. Math. Soc.*, 362(9):4955–4981, 2010.
- [8] L. Cohen. Generalized phase-space distribution functions. *J. Math. Phys.*, 7:781–786, 1966.
- [9] L. Cohen. *Time Frequency Analysis: Theory and Applications*. Prentice Hall, 1995.
- [10] E. Cordero, M. de Gosson and F. Nicola. On the Invertibility of Born-Jordan Quantization. *J. Math. Pures Appl.*, 105(4):537–557, 2016.
- [11] E. Cordero, M. de Gosson and F. Nicola. On the reduction of the interferences in the Born-Jordan distribution. *Appl. Comput. Harmon. Anal.*, In Press. arXiv:1601.03719
- [12] E. Cordero, M. de Gosson and F. Nicola. Time-frequency Analysis of Born-Jordan Pseudo-differential Operators. *Submitted*. arXiv:1601.05303.
- [13] E. Cordero and K. Gröchenig. Time-frequency analysis of localization operators. *J. Funct. Anal.*, 205 (1):107–131, 2003.
- [14] E. Cordero and F. Nicola. Sharp continuity results for the short-time Fourier transform and for localization operators. *Monatsh. Math.*, 162:251–276, 2011.
- [15] E. Cordero and F. Nicola. Pseudodifferential operators on  $L^p$ , Wiener amalgam and modulation spaces. *Int. Math. Res. Notices*, 10:1860–1893, 2010.
- [16] E. Cordero and K. Okoudjou. Multilinear localization operators. *J. Math. Anal. Appl.*, 325 (2):1103–1116, 2007.
- [17] E. Cordero, A. Tabacco and P. Wahlberg. Schrödinger-type propagators, pseudodifferential operators and modulation spaces. *J. London Math. Soc.*, 88(2):375–395, 2013.
- [18] E. Cordero, J. Toft and P. Wahlberg. Sharp results for the Weyl product on modulation spaces. *J. Funct. Anal.*, 267(8):3016–3057, 2014.
- [19] W. Czaja. Boundedness of pseudodifferential operators on modulation spaces. *J. Math. Anal. Appl.*, 284(1):389–396, 2003.
- [20] I. Daubechies. Time-frequency localization operators: a geometric phase space approach. *IEEE Trans. Inform. Theory*, 34(4):605–612, 1988.
- [21] M. de Gosson. *Symplectic methods in Harmonic Analysis and in Mathematical Physics*, Birkhäuser, 2011.
- [22] H. G. Feichtinger. Banach convolution algebras of Wiener’s type, In *Proc. Conf. “Function, Series, Operators”*, Budapest August 1980, Colloq. Math. Soc. János Bolyai, 35, 509–524, North-Holland, Amsterdam, 1983.

- [23] H. G. Feichtinger. Banach spaces of distributions of Wiener's type and interpolation. In *Proc. Conf. Functional Analysis and Approximation, Oberwolfach August 1980*, Internat. Ser. Numer. Math., 69:153–165. Birkhäuser, Boston, 1981.
- [24] H. G. Feichtinger. Generalized amalgams, with applications to Fourier transform. *Canad. J. Math.*, 42(3):395–409, 1990.
- [25] H. G. Feichtinger and K. Nowak. A First Survey of Gabor Multipliers. In H. G. Feichtinger and T. Strohmer, editors, *Advances in Gabor Analysis*. Birkhäuser, Boston, 2002.
- [26] Y. V. Galperin. Young's Convolution Inequalities for weighted mixed (quasi-) norm spaces. *J. Ineq. Spec. Funct.*, 5(1):1–12, 2014.
- [27] L. Grafakos and R. H. Torres. Multilinear Calderón-Zygmund theory, *Adv. Math.*, 165(1):124–164, 2002.
- [28] K. Gröchenig. *Foundations of Time-Frequency Analysis*. Birkhäuser, Boston, 2001.
- [29] K. Gröchenig and C. Heil. Modulation spaces and pseudodifferential operators. *Integral Equations Operator Theory*, 34:439–457, 1999.
- [30] K. Gröchenig and C. Heil. Counterexamples for boundedness of pseudodifferential operators. *Osaka J. Math.*, 41:681–691, 2004.
- [31] K. Gröchenig and T. Strohmer. Pseudodifferential operators on locally compact abelian groups and Sjöstrand's symbol class. *J. Reine Angew. Math.*, 613:121–146, 2007.
- [32] K. Johansson, S. Pilipović, N. Teofanov and J. Toft. Sharp convolution and multiplication estimates in weighted spaces, *Analysis and Applications*, 13(5):457–480, 2015.
- [33] M. Kobayashi and M. Sugimoto. The inclusion relation between Sobolev and modulation spaces. *J. Funct. Anal.*, 260(11):3189–3208, 2011.
- [34] D. Labate. Time-frequency analysis of pseudodifferential operators. *Monatshefte Math.*, 133:143–156, 2001.
- [35] D. Labate. Pseudodifferential operators on modulation spaces. *J. Math. Anal. Appl.*, 262:242–255, 2001.
- [36] L. H. Lieb. Integral bounds for radar ambiguity functions and Wigner distributions. *J. Math. Phys.*, 31, No.3, 1990.
- [37] S. Molahajloo, K. A. Okoudjou and G. E. Pfander. Boundedness of Multilinear Pseudodifferential Operators on Modulation Spaces. *J. Fourier Anal. Appl.*, 1–35, 2016. DOI:10.1007/s00041-016-9461-2
- [38] S. Pilipović, N. Teofanov and J. Toft. Micro-local analysis in Fourier-Lebesgue and modulation spaces. Part II, *J. Pseudo-Differ. Oper. Appl.*, 1:341–376, 2010.
- [39] S. Pilipović, N. Teofanov and J. Toft. Micro-Local Analysis in Fourier-Lebesgue and modulation Spaces. Part I, *J. Fourier Anal. Appl.*, 17:374–407, 2011.
- [40] J. Ramanathan and P. Topiwala. Time-frequency localization via the Weyl correspondence. *SIAM J. Math. Anal.*, 24(5):1378–1393, 1993.
- [41] M. A. Shubin. *Pseudodifferential Operators and Spectral Theory*. Springer-Verlag, Berlin, second edition, 2001. Translated from the 1978 Russian original by Stig I. Andersson.
- [42] J. Sjöstrand. An algebra of pseudodifferential operators. *Math. Res. Lett.*, 1(2):185–192, 1994.
- [43] T. Strohmer, Pseudodifferential operators and Banach algebras in mobile communications. *Appl. Comput. Harmon. Anal.*, 20(2):237–249, 2006.
- [44] M. Sugimoto and N. Tomita. Boundedness properties of pseudo-differential operators and Calderón-Zygmund operators on modulation spaces. *J. Fourier Anal. Appl.*, 14(1):124–143, 2008.
- [45] K. Tachizawa. The boundedness of pseudo-differential operators on modulation spaces. *Math. Nachr.*, 168:263–277, 1994.

- [46] N. Teofanov. Continuity and Schatten von Neumann Properties for Localization Operators on Modulation Spaces. *Mediterr. J. Math.* 13:745–758, 2016.
- [47] J. Toft. Continuity properties for modulation spaces, with applications to pseudo-differential calculus. I. *J. Funct. Anal.*, 207(2):399–429, 2004.
- [48] J. Toft. Continuity properties for modulation spaces, with applications to pseudo-differential calculus. II. *Ann. Global Anal. Geom.*, 26(1):73–106, 2004.
- [49] J. Ville. Théorie et Applications de la Notion de Signal Analytique, Câbles et Transmission, 2:61–74, 1948.
- [50] E. Wigner. On the Quantum Correction for Thermodynamic Equilibrium. *Physical Review* 40:749–759, 1932.
- [51] M. W. Wong. *Wavelets Transforms and Localization Operators*, volume 136 of Operator Theory Advances and Applications. Birkhauser, 2002.

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