THE BLOW-UP OF $\mathbb{P}^4$ AT 8 POINTS AND ITS FANO MODEL, VIA VECTOR BUNDLES ON A DEL PEZZO SURFACE

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1. Introduction

Let $S = \mathrm{Bl}_{q_1,\dots,q_8}\mathbb{P}^2$ and $X = \mathrm{Bl}_{p_1,\dots,p_8}\mathbb{P}^4$ be the blow-ups respectively of $\mathbb{P}^2$ and $\mathbb{P}^4$ at 8 general points. There is a classical connection between these two varieties due to projective association, or Gale duality, which gives a bijection between sets of 8 general points in $\mathbb{P}^2$ and in $\mathbb{P}^4$, up to projective equivalence (see 2.18). In this framework, a beautiful relation among S and X has been established by Mukai, using moduli of sheaves on S, as follows.

Theorem 1.1 ([Muk05], §2)**.** *If $\{q_1,\dots,q_8\} \subset \mathbb{P}^2$ and $\{p_1,\dots,p_8\} \subset \mathbb{P}^4$ are associated sets of points, then X is isomorphic to the moduli space of rank 2 torsion free sheaves F on S, with $c_1(F) = -K_S$ and $c_2(F) = 2$, semistable with respect to $-K_S + 2h$, where $h \in \mathrm{Pic}(S)$ is the pull-back of $\mathcal{O}_{\mathbb{P}^2}(1)$ under the blow-up map $S \to \mathbb{P}^2$.*

This result is the starting point of this paper, which has three main subjects:

A. the moduli spaces $M_{S,L}$ of rank 2 torsion free sheaves F on a (smooth) degree one del Pezzo surface S, with $c_1(F) = -K_S$ and $c_2(F) = 2$, semistable (in the sense of Gieseker-Maruyama) with respect to $L \in \mathrm{Pic}(S)$ ample;
B. the smooth Fano 4-fold $Y := M_{S,-K_S}$;
C. the geometry of $X = \mathrm{Bl}_{p_1,\dots,p_8}\mathbb{P}^4$.

Mukai's proof of Th. 1.1 is based on the study of the birational geometry of $M_{S,L}$ in terms of the variation of the stability condition given by L. In this paper we resume and expand Mukai's study of these moduli spaces, proving that the birational geometry of $M_{S,L}$ is completely governed by the variation of stability conditions. Then we apply

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this to study the Fano 4-fold Y and the blow-up X of $\mathbb{P}^4$ at 8 general points. Let us describe in more detail these three points, and present our main results.

A. Moduli of vector bundles on a degree 1 del Pezzo surface. To describe the moduli spaces $M_{S,L}$, we introduce two convex rational polyhedral cones

$$\Pi \subset \mathcal{E} \subset \mathrm{Nef}(S) \subset H^2(S,\mathbb{R}).$$

See §2, in particular 2.3 and 2.10, for the explicit definitions; the cone Π has been introduced in [Muk05, p. 8].

Let us first state some general properties of $M_{S,L}$. In the following proposition some statements are standard; the new part is the characterization of the polarizations L for which the moduli space is non-empty. See Cor. 3.5, 3.26, 3.27, and 3.28, and Rem. 3.4.

Proposition 1.2. *Let $L \in \mathrm{Pic}(S)$ ample. The moduli space $M_{S,L}$ is non-empty if and only if $L \in \mathcal{E}$, and in this case $M_{S,L}$ is a smooth, projective, rational 4-fold. Every sheaf parametrized by $M_{S,L}$ is locally free and stable.*

Let us consider now the birational geometry of the moduli spaces $M_{S,L}$. The relation between variation of polarization via wall-crossings, and birational geometry of moduli spaces of sheaves on surfaces, is classical and has been intensively studied, see for instance [FQ95, EG95, MW97], and for the case of del Pezzo surfaces, [CMR99, Góm00] and the more recent [BMW14]. In our setting, this relation can be made completely explicit. Generalising [Muk05, Lemma 3], we determine all the (finitely many) walls for slope semistability (Cor. 3.11), and introduce the stability fan $\mathrm{ST}(S)$ in $H^2(S,\mathbb{R})$, supported on the cone $\mathcal{E}$, determined by these walls (see Def. 3.12). When the polarization L varies in the interior of a cone of maximal dimension of the stability fan, the stability condition determined by L is constant, and so is $M_{S,L}$. When L moves to a different cone in the stability fan, the moduli space $M_{S,L}$ undergoes a simple birational transformation. These results are presented in §3, and are mostly based on Mukai's work [Muk05]; see 3.1 for a more detailed overview.

The moduli spaces $M_{S,L}$ are Mori dream spaces (see 5.12 and references therein for the notions of Mori dream space and of Mori chamber decomposition). This follows from Th. 1.1 and Castravet and Tevelev's result [CT06, Th. 1.3], and also from the log Fano property of $M_{S,L}$, see Cor. 4.21. Thus the stability fan $\mathrm{ST}(S)$ in $H^2(S,\mathbb{R})$ has a counterpart in $H^2(M_{S,L},\mathbb{R})$, the fan $\mathrm{MCD}(M_{S,L})$ given by the Mori chamber decomposition, defined via birational geometry.

In §4, using the classical construction of determinant line bundles on the moduli space $M_{S,L}$, we define a group homomorphism

$$\rho\colon \mathrm{Pic}(S) \longrightarrow \mathrm{Pic}(M_{S,L})$$

and study its properties, see 4.1 for a more detailed overview. The determinant map ρ provides the bridge between stability chambers in $H^2(S,\mathbb{R})$ and cones of divisors in $H^2(M_{S,L},\mathbb{R})$: in §5 we show the following.

Theorem 1.3 (see 5.12). *Let $L \in \mathrm{Pic}(S)$ ample, $L \in \Pi$. The map $\rho\colon H^2(S,\mathbb{R}) \to H^2(M_{S,L},\mathbb{R})$ is an isomorphism, $\rho(\mathcal{E})$ is the cone of effective divisors $\mathrm{Eff}(M_{S,L})$, and $\rho(\Pi)$ is the cone of movable divisors $\mathrm{Mov}(M_{S,L})$. Moreover, ρ yields an isomorphism between the stability fan $\mathrm{ST}(S)$ in $H^2(S,\mathbb{R})$, and the fan $\mathrm{MCD}(M_{S,L})$ in $H^2(M_{S,L},\mathbb{R})$ given by the Mori chamber decomposition.*

This result relies on the classical positivity properties of the determinant line bundle and on Th. 1.1.

We recall that a pseudo-isomorphism is a birational map which is an isomorphism in codimension one, and similarly we define a pseudo-automorphism. When the polarization L varies in the cone Π, we get finitely many pseudo-isomorphic moduli spaces $M_{S,L}$, related by sequences of flips. We show that in fact, when $L \in \Pi$, the moduli space $M_{S,L}$ determines the surface S; we addressed this question in analogy with Bayer and Macrì's result [BM14, Cor. 1.3] on moduli of sheaves on K3 surfaces.

Theorem 1.4 (see 6.14). *Let S_1 and S_2 be del Pezzo surfaces of degree one, and $L_i \in \mathrm{Pic}(S_i)$ ample line bundles with $L_i \in \Pi_i \subset H^2(S_i, \mathbb{R})$, for $i = 1, 2$. Then $S_1 \cong S_2$ if and only if M_{S_1,L_1} and M_{S_2,L_2} are pseudo-isomorphic.*

Note that the assumption that $L_i \in \Pi_i$ here is essential, because every degree one del Pezzo surface S has a polarization L_0 such that $M_{S,L_0} \cong \mathbb{P}^4$ (see Prop. 3.20).

We also describe the group of pseudo-automorphisms of $M_{S,L}$, when $L \in \Pi$.

Theorem 1.5 (see 6.15). *Let $L \in \mathrm{Pic}(S)$ ample, $L \in \Pi$. Then the group of pseudo-automorphisms of $M_{S,L}$ is isomorphic to the automorphism group $\mathrm{Aut}(S)$ of S, where $f \in \mathrm{Aut}(S)$ acts on $M_{S,L}$ as $[F] \mapsto [(f^{-1})^*F]$.*

B. Geometry of the Fano model Y. The anticanonical class $-K_S$ in $H^2(S, \mathbb{R})$ belongs to the cone Π, and lies in the interior of a cone of the stability fan. It follows again from the classical properties of the determinant line bundle that for the polarization $L = -K_S$, the moduli space $M_{S,L}$ is Fano. More precisely, we have the following.

Proposition 1.6 (Prop. 4.20 and 6.1, Lemma 6.24). *The moduli space $Y := M_{S,-K_S}$ is a smooth, rational Fano 4-fold with index one and $b_2(Y) = 9$, $b_3(Y) = 0$, $h^{2,2}(Y) = b_4(Y) = 45$, $(-K_Y)^4 = 13$, $h^0(Y, -K_Y) = 6$, $h^0(Y, T_Y) = 0$, and $h^1(Y, T_Y) = 8$.*

Let us notice that, except products of del Pezzo surfaces, there are very few known examples of Fano 4-folds with $b_2 \geq 7$. In particular, to the authors' knowledge, the family of Fano 4-folds Y is the only known example of Fano 4-fold with $b_2 \geq 9$ which is not a product of surfaces. It is a very interesting family, whose construction and study was one of the motivations for this work.

By Th. 1.3, the determinant map $\rho\colon H^2(S, \mathbb{R}) \to H^2(Y, \mathbb{R})$ is an isomorphism and yields a completely explicitly description of the relevant cones of divisors $\mathrm{Eff}(Y)$, $\mathrm{Mov}(Y)$, and $\mathrm{Nef}(Y)$, and more generally of the Mori chamber decomposition of $\mathrm{Eff}(Y)$. We give here a statement on the cone of effective curves, and refer the reader to §6 for the descriptions of the other relevant cones.

Proposition 1.7 (see 6.3). *The cone of effective curves $\mathrm{NE}(Y)$ is isomorphic to the cone of effective curves $\mathrm{NE}(S)$ of S, and it has 240 extremal rays. Each extremal ray yields a small contraction[1] with exceptional locus a smooth rational surface.*

More generally, every contraction $\varphi\colon Y \to Z$ with $\dim Z > 0$ is birational with $\mathrm{codim}\,\mathrm{Exc}(\varphi) \geq 2$, *and* $\mathrm{Nef}(Y) \cap \partial\,\mathrm{Mov}(Y) = \{0\}$.

We also show that S and Y determine each other, and we determine $\mathrm{Aut}(Y)$.

Theorem 1.8 (see 6.14). *Let S_1 and S_2 be del Pezzo surfaces of degree 1, and set $Y_i := M_{S_i,-K_{S_i}}$ for $i = 1, 2$. Then $S_1 \cong S_2$ if and only if $Y_1 \cong Y_2$.*

[1]A *contraction* is a surjective map with connected fibers $\varphi\colon Y \to Z$, where Z is normal and projective.

Theorem 1.9 (see 6.15)**.** *The map $\psi\colon \operatorname{Aut}(S)\to\operatorname{Aut}(Y)$ given by $\psi(f)[F]=[(f^{-1})^*F]$, for $f\in\operatorname{Aut}(S)$ and $[F]\in Y$, is a group isomorphism. In particular $\operatorname{Aut}(Y)$ is finite, and if S is general, then $\operatorname{Aut}(Y)=\{\mathrm{Id}_Y,\iota_Y\}$, where $\iota_Y\colon Y\to Y$ is induced by the Bertini involution of S.*

The description of the automorphism group of Y, and of its action on $H^2(Y,\mathbb{R})$, is also used to show that Y is *fibre-like*, namely that it can appear as a fiber of a Mori fiber space, see 6.21.

Finally, motivated by the low values of $h^0(Y,-K_Y)$ and $(-K_Y)^4$ (see Prop. 1.6), and also by the analogy with degree one del Pezzo surfaces, in §7 we study the base loci of the anticanonical and bianticanonical linear systems of Y, and prove the following.

Theorem 1.10 (see 7.1 and 7.12)**.** *The linear system $|-K_Y|$ has a base locus of positive dimension, while the linear system $|-2K_Y|$ is base point free.*

C. The blow-up X of $\mathbb{P}^4$ at 8 general points. As we already recalled, association gives a bijection between (general) sets of 8 points in $\mathbb{P}^2$ and in $\mathbb{P}^4$. This gives a natural correspondence between pairs (S,h), where S is a del Pezzo surface of degree one, and $h\in\operatorname{Pic}(S)$ defines a birational map $S\to\mathbb{P}^2$, and blow-ups X of $\mathbb{P}^4$ at 8 general points (see 2.21). The interplay between S, X, and Y is the key point of this paper. This also yields new results on the blow-up X of $\mathbb{P}^4$ at 8 general points, which are mostly treated in §8; let us give an overview.

First of all we describe explicitly the relation among X and Y: the Fano 4-fold Y is obtained from X by flipping the transforms of the lines in $\mathbb{P}^4$ through 2 blown-up points, and of the rational normal quartics through 7 blown-up points (Lemma 5.18).

In particular, Th. 1.10 on the anticanonical and bianticanonical linear systems on Y is proved using the birational map $X \dashrightarrow Y$ and studying the corresponding linear systems in X. We show that the base locus of $|-K_X|$ contains the transform R of a smooth rational quintic curve in $\mathbb{P}^4$ through the 8 blown-up points, and that the transform of R in Y is contained in the base locus of $|-K_Y|$ (Cor. 7.6 and Lemma 7.7, see also Rem. 7.8).

We also have the following direct consequence of Th. 1.1 and 1.4.

Corollary 1.11. *Let $q_1^i,\dots,q_8^i\in\mathbb{P}^2$ be such that $S_i:=\mathrm{Bl}_{q_1^i,\dots,q_8^i}\mathbb{P}^2$ is a del Pezzo surface, for $i=1,2$. Let $p_1^i,\dots,p_8^i\in\mathbb{P}^4$ be the associated points to $q_1^i,\dots,q_8^i\in\mathbb{P}^2$, and set $X_i:=\mathrm{Bl}_{p_1^i,\dots,p_8^i}\mathbb{P}^4$, for $i=1,2$. Then $S_1\cong S_2$ if and only if X_1 and X_2 are pseudo-isomorphic.*

The previous results also give a description of the group of pseudo-automorphisms of X; we show (Prop. 8.9) that X has a unique non-trivial pseudo-automorphism ι_X, that we call the *Bertini involution* of X. Via the blow-up map $X\to\mathbb{P}^4$, this also defines a birational involution $\iota_{\mathbb{P}^4}\colon\mathbb{P}^4\dashrightarrow\mathbb{P}^4$.

The birational maps ι_X and $\iota_{\mathbb{P}^4}$ are in fact classically known, as they can be defined via the Cremona action of the Weyl group $W(E_8)$ on sets of 8 points in $\mathbb{P}^4$, see Dolgachev and Ortland [DO88, Ch. VI, §4, and p. 131] and Du Val [DV81, p. 199 and p. 201]. With the standard notation for divisors in X (see 2.22), we have

$$\iota_X^*H=49H-30\big(\sum_i E_i\big)$$

(see [DV81, (11) on p. 199]), thus $\iota_{\mathbb{P}^4}$ is defined by the linear system $V \subset |\mathcal{O}_{\mathbb{P}^4}(49)|$ of hypersurfaces having multiplicity at least 30 at $p_1, \dots, p_8$. As noted in [DV81, p. 201] and [DO88, p. 131], the classical definitions of ι_X and $\iota_{\mathbb{P}^4}$ do not give a geometrical description of these maps. Using the interpretation of X as a moduli space of vector bundles on S, we give a factorization of these maps as smooth blow-ups and blow-downs, see Prop. 8.9 and Cor. 8.10.

Finally, as a direct application of Th. 1.1 and 1.3, we describe the fixed[2] divisors of X in terms of conics in S. It has been shown by Castravet and Tevelev [CT06, Th. 2.7] that $\mathrm{Eff}(X)$ is generated by the classes of fixed divisors, which form an orbit under the action of the Weyl group $W(E_8)$ on $H^2(X,\mathbb{Z})$ (see 5.9). We get the following.

Proposition 1.12 (see 8.1)**.** *Let X be the blow-up of $\mathbb{P}^4$ at 8 general points. Then the cone of effective divisors $\mathrm{Eff}(X)$ is generated by the classes of 2160 fixed divisors, which are in bijection with the classes of conics in a del Pezzo surface S of degree 1. With the standard notation for divisors in S and X (see 2.2 and 2.22), if $C \sim dh - \sum_i m_i e_i$ is such a conic, then the corresponding fixed divisor E_C has class:*

$$E_C \sim \frac{1}{2}\Big(\sum_i m_i - d\Big)\Big(H - \sum_i E_i\Big) + \sum_i m_i E_i.$$

We also give generators for the semigroup of integral effective divisors of X (Lemma 8.3), by applying a result from [CT06].

We conclude by mentioning that our study of the blow-up X and of its Fano model Y is analogous to the study in [AC17] of the Fano model of the blow-up of $\mathbb{P}^n$ in $n+3$ general points, for n even.

We work over the field of complex numbers.

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2. Preliminaries on del Pezzo surfaces of degree 1 and association

Let S be a del Pezzo surface of degree 1; we always assume that S is smooth. In this section we collect the properties of S that are needed in the sequel, fix the relevant notation and terminology, and introduce the convex rational polyhedral cones

$$\mathcal{N} \subset \Pi \subset \mathcal{E} \subset \mathrm{Nef}(S) \subset H^2(S,\mathbb{R})$$

that play a crucial role for the rest of the paper. We also recall the relation via association among S and the blow-up X of $\mathbb{P}^4$ in 8 points.

We refer the reader to [Dol12, Ch. 8] for the classical properties of S. In particular, we recall that $H^2(S,\mathbb{Z})$ has a lattice structure given by the intersection form, and

[2] A prime divisor E is fixed if it is a fixed component of the linear system $|mE|$ for every $m \geq 1$.

the sublattice $K_S^\perp$ is an E_8-lattice; we denote by $W_S \cong W(E_8)$ its Weyl group of automorphisms.

With a slight abuse of notation, we will often write $C \in H^2(S,\mathbb{R})$ for the class of a curve $C \subset S$, and similarly for divisors in higher dimensional varieties.

2.1. The cones $\mathrm{NE}(S)$ **and** $\mathrm{Nef}(S)$**.** The cone of effective curves $\mathrm{NE}(S) \subset H^2(S,\mathbb{R})$ is generated by the classes of the 240 (-1)-curves, on which W_S acts transitively.

A **conic** C on S is a smooth rational curve such that $-K_S \cdot C = 2$ and $C^2 = 0$; every such conic yields a conic bundle $S \to \mathbb{P}^1$ having C as a fiber. There are 2160 conics in $H^2(S,\mathbb{Z})$, on which W_S acts transitively [Dol12, §8.2.5].

We will repeatedly use the explicit description of (-1)-curves and conics of S once a birational map $S \to \mathbb{P}^2$ is fixed; see [Dol12, Prop. 8.2.19, §8.2.6, §8.8.1].

The dual cone of $\mathrm{NE}(S)$ is the cone of nef divisors $\mathrm{Nef}(S)$, which has two types of generators. The first are the conics, which lie on the boundary of $\mathrm{NE}(S)$, and correspond to conic bundles $S \to \mathbb{P}^1$. The second type of generators are big and correspond to birational maps $\sigma\colon S \to \mathbb{P}^2$, which realise S as the blow-up of $\mathbb{P}^2$ in 8 distinct points; the corresponding generator of $\mathrm{Nef}(S)$ is $h := \sigma^*\mathcal{O}_{\mathbb{P}^2}(1)$; we call such h a **cubic**. There are 17280 cubics in $H^2(S,\mathbb{Z})$, which form an orbit under the action of W_S (see [Dol12, §8.2.5, 8.2.6, 8.8.1]). Summing-up we have:

$$\mathrm{Nef}(S) = \langle C, h \,|\, C \text{ a conic and } h \text{ a cubic}\rangle \subset H^2(S,\mathbb{R}),$$

where $\langle v_1,\dots,v_r\rangle$ denotes the convex cone generated by $v_1,\dots,v_r$ in a real vector space.

2.2. Notation for $S \to \mathbb{P}^2$**.** Given a cubic h, we use the following notation:

- $\sigma\colon S \to \mathbb{P}^2$ is the birational map defined by h
- $q_1,\dots,q_8 \in \mathbb{P}^2$ are the points blown-up by σ
- $e_i \subset S$ is the exceptional curve over q_i, for $i = 1\dots,8$
- $e := e_1 + \cdots + e_8$, so that $-K_S = 3h - e$
- $C_i \subset S$ is the transform of a general line through q_i, so that $C_i \sim h - e_i$, for $i = 1,\dots,8$
- $\ell_{ij} \subset S$ is the transform of the line $\overline{q_iq_j} \subset \mathbb{P}^2$, so that $\ell_{ij} \sim h - e_i - e_j$, for $1 \le i < j \le 8$.

2.3. The cone $\mathcal{E}$**.** We are interested in the subcone of $\mathrm{Nef}(S)$ generated by the conics:

$$\mathcal{E} := \langle C \,|\, C \text{ a conic}\rangle \subset H^2(S,\mathbb{R}).$$

Since $\mathcal{E} \subsetneq \mathrm{Nef}(S)$, dually we have $\mathcal{E}^\vee \supsetneq \mathrm{NE}(S)$, for the dual cone $\mathcal{E}^\vee$ of $\mathcal{E}$. We have:

$$\mathcal{E}^\vee = \langle \ell, 2h + K_S \,|\, \ell \text{ a } (-1)\text{-curve and } h \text{ a cubic}\rangle. \tag{2.4}$$

Indeed, given a cubic h, $(2h+K_S)^\perp \cap \mathcal{E}$ is a simplicial facet[3] of $\mathcal{E}$, generated by the conics C_i for $i = 1,\dots,8$ (notation as in 2.2). On the other hand, given a (-1)-curve ℓ, $\ell^\perp \cap \mathcal{E}$ is a non-simplicial facet of $\mathcal{E}$, generated by the 126 conics disjoint from ℓ.

It follows from (2.4) that the cone $\mathcal{E}$ can equivalently be described as:

$$\mathcal{E} = \{L \in \mathrm{Nef}(S) \,|\, L \cdot (2h + K_S) \ge 0 \text{ for every cubic } h\}. \tag{2.5}$$

Remark 2.6. Every $L \in \mathrm{Pic}(S)$ contained in the interior of $\mathcal{E}$ is ample. If $L \in \mathrm{Pic}(S)$ is contained in the boundary of $\mathcal{E}$, then L is ample if and only if L is in the relative interior of a facet $(2h+K_S)^\perp \cap \mathcal{E}$, where h is a cubic.

[3] A facet is a face of codimension one.

2.7. The cone $\mathcal{N}$. We set:

$$\mathcal{N} = \{L \in H^2(S,\mathbb{R}) \,|\, L \cdot (2\ell + K_S) \geq 0 \text{ for every } (-1)\text{-curve } \ell\},$$

equivalently $\mathcal{N}$ is defined via its dual cone:

$$\mathcal{N}^\vee = \langle 2\ell + K_S \,|\, \ell \text{ a } (-1)\text{-curve}\rangle.$$

The cone $\mathcal{N}^\vee$ has 240 extremal rays, and is isomorphic to $\mathrm{NE}(S)$ via the automorphism of $H^2(S,\mathbb{R})$ given by $\gamma \mapsto \gamma + (\gamma \cdot K_S)K_S$. This is a self-adjoint map and coincides with its transpose. Dually, $\mathrm{Nef}(S)$ is isomorphic to $\mathcal{N}$ via the same linear map. This gives a description of the generators of $\mathcal{N}$:

$$\mathcal{N} = \langle -2K_S + C, -3K_S + h \,|\, C \text{ a conic and } h \text{ a cubic}\rangle. \tag{2.8}$$

Lemma 2.9. *The cone $\mathcal{N}^\vee$ contains all (-1)-curves; equivalently, every $L \in \mathcal{N}$ is nef.*

Proof. Let ℓ be a (-1)-curve, and h a cubic such that $\ell = e_1$ (notation as in 2.2). Let ℓ' be the (-1)-curve such that $\ell' \sim 3h - 2e_2 - e_3 - \cdots - e_8$. Then $\ell \in \mathcal{N}^\vee$ because:

$$\ell = e_1 \sim e_2 + \ell' + K_S = \frac{1}{2}\left(2e_2 + K_S + 2\ell' + K_S\right). \qquad \blacksquare$$

2.10. The cone Π. We will also consider the following cone, defined in [Muk05, p. 8]:

$$\Pi = \{L \in \mathrm{Nef}(S) \,|\, L \cdot (2C + K_S) \geq 0 \text{ for every conic } C\},$$

equivalently Π is defined via its dual cone:

$$\Pi^\vee = \langle \ell, 2C + K_S \,|\, \ell \text{ is a } (-1)\text{-curve and } C \text{ is a conic}\rangle.$$

Lemma 2.11. *We have: $\mathcal{N} \subset \Pi \subset \mathcal{E} \subset \mathrm{Nef}(S)$.*

Proof. We show the dual inclusions $\mathcal{N}^\vee \supset \Pi^\vee \supset \mathcal{E}^\vee$. It is easy to see that for every cubic h, $2h + K_S \in \Pi^\vee$, hence $\mathcal{E}^\vee \subset \Pi^\vee$ by (2.4). Similarly, given a conic C, it is easy to see that $2C + K_S \in \mathcal{N}^\vee$, so that $\Pi^\vee \subset \mathcal{N}^\vee$ by Lemma 2.9. $\blacksquare$

In 6.7 we will show that the one-dimensional faces of Π contained in the interior of $\mathcal{E}$ are generated by $-K_S + 3h$, where h is a cubic.

2.12. The Bertini involution. Let $\iota_S\colon S \to S$ be the Bertini involution (see [Dol12, §8.8.2]); for S general, ι_S is the unique non-trivial automorphism of S. The pull-back ι_S^* acts on $\mathrm{Pic}(S)$ (and on $H^2(S,\mathbb{R})$) by fixing K_S and acting as -1 on $K_S^\perp$. This yields:

$$\iota_S^*\gamma = 2(\gamma \cdot K_S)K_S - \gamma \quad \text{for every } \gamma \in H^2(S,\mathbb{R}). \tag{2.13}$$

2.14. Other preliminary elementary properties of S.

Remark 2.15. Let ℓ, ℓ' be (-1)-curves in S.
(*a*) If $\ell \cdot \ell' = 0$, then $\ell \cap \ell' = \emptyset$, and there exists a cubic h such that $h \cdot \ell = h \cdot \ell' = 0$.
(*b*) If $\ell \cdot \ell' = 2$, then there exist (notation as in 2.2):
- a cubic h such that $\ell = \ell_{12} \sim h - e_1 - e_2$ and $\ell' \sim 2h - e_4 - \cdots - e_8$
- a cubic h' such that $\ell = e_1'$ and $\ell' \sim 3h' - 2e_1' - e_2' - \cdots - e_7'$.

(*c*) If $\ell \cdot \ell' \geq 3$, then $\ell \cdot \ell' = 3$ and $\ell' = \iota_S^*\ell \sim -2K_S - \ell$.

Lemma 2.16. *Let h, h' be cubics in S. Then $h \cdot h' \leq 17$, and equality holds if and only if $h' = \iota_S^* h \sim -6K_S - h$.*

Proof. Consider the cubic h; notation as in 2.2. We have $h' \sim mh - \sum_i a_i e_i$ where $m, a_i \in \mathbb{Z}$ and $m = h \cdot h'$. Since $(h')^2 = 1$ and $-K_S \cdot h' = 3$, we get

$$\sum a_i^2 = m^2 - 1 \quad \text{and} \quad \sum a_i = 3(m-1).$$

Now the inequality $(\sum_i a_i)^2 \leq 8 \sum_i a_i^2$ (given by Cauchy-Schwarz applied to $(a_1, \dots, a_8)$ and $(1, \dots, 1)$) yields $(m-1)(m-17) \leq 0$, hence $m \leq 17$. If $m = 17$, then $a_1 = \cdots = a_8 = 6$, thus $h' \sim 17h - 6(e_1 + \cdots + e_8) = -6K_S - h = \iota_S^* h$ (see (2.13)). ■

Remark 2.17. Let $L \in \operatorname{Pic}(S)$ be nef and such that $-K_S \cdot L = 2$. Then L is one of the following classes: $\{-2K_S, C, -K_S + \ell \,|\, C \text{ a conic}, \ell \text{ a } (-1)\text{-curve}\}$.

Indeed by vanishing and Riemann-Roch we have $h^0(S, L) > 0$. The semigroup of effective divisors of S is generated by (-1)-curves and $-K_S$ [BP04, Cor. 3.3], and since $-K_S \cdot L = 2$, then L is either $-2K_S$, $-K_S + \ell$, or $\ell + \ell'$. In this last case, since L is nef, we must have $\ell \cdot \ell' > 0$. If $\ell \cdot \ell' = 1$, then $\ell + \ell'$ is a conic, and if $\ell \cdot \ell' \geq 3$, then $\ell + \ell' \sim -2K_S$ (see Rem. 2.15(c)). Finally, if $\ell \cdot \ell' = 2$, then by Rem. 2.15(b) there exists a cubic h such that $\ell + \ell' \sim 3h - e_1 - e_2 - e_4 - \cdots - e_8 \sim -K_S + e_3$ (notation as in 2.2).

2.18. Association. We refer the reader to [DO88, EP00] for the definition and main properties of association, or Gale duality; here we just give a brief outline.

Consider the natural action of $\operatorname{Aut}(\mathbb{P}^2)$ on $(\mathbb{P}^2)^8$, and similarly of $\operatorname{Aut}(\mathbb{P}^4)$ on $(\mathbb{P}^4)^8$. In both cases, every semistable element is also stable [DO88, Ch. II, Cor. on p. 25]. Let us consider the GIT quotients $P_2^8 := ((\mathbb{P}^2)^8)^s / \operatorname{Aut}(\mathbb{P}^2)$ and $P_4^8 := ((\mathbb{P}^4)^8)^s / \operatorname{Aut}(\mathbb{P}^4)$.

Association is an algebraic construction which yields an isomorphism $a\colon P_2^8 \cong P_4^8$ [DO88, Ch. III, Cor. on p. 36]. In particular, to every stable ordered set of 8 points in $\mathbb{P}^2$, we associate a stable ordered set of 8 points in $\mathbb{P}^4$, unique up to projective equivalence, and viceversa. Moreover, the same bijection can be given for non-ordered sets of points [DO88, Ch. III, §1]. We also need the following.

Lemma 2.19. *Let $q_1, \dots, q_8 \in \mathbb{P}^2$ be in general linear position, and let $p_1, \dots, p_8 \in \mathbb{P}^4$ be the associated points. Then $p_1, \dots, p_8$ are in general linear position.*

Proof. Let A be a 3×8 matrix with columns the coordinates of the points q_i's, and similarly let B be a 5×8 matrix containing the coordinates of the points p_j's; by the definition of association we have $A \cdot B^t = 0$. Since $q_1, \dots, q_8 \in \mathbb{P}^2$ are in general linear position, every maximal minor of A is non-zero. For $I \subset \{1, \dots, 8\}$ with $|I| = 3$, let $a_I \in \mathbb{C}$ be the minor of A given by the columns in I, and b_I the minor of B given by the columns *not* in I.

Let $I, J \subset \{1, \dots, 8\}$ be such that $|I| = |J| = 3$ and $|I \cap J| = 2$. It is shown in [DO88, Ch. III, Lemma 1] that $a_I b_J + a_J b_I = 0$, thus $b_I = 0$ if and only if $b_J = 0$. Since by construction B has maximal rank, this shows that every maximal minor of B is non-zero, hence the points $p_1, \dots, p_8$ are in general linear position. ■

Remark 2.20. Let $q_1, \dots, q_8 \in \mathbb{P}^2$ be such that the blow-up of $\mathbb{P}^2$ at $q_1, \dots, q_8$ is a smooth del Pezzo surface (see [Dol12, Prop. 8.1.25]). In particular the q_i's are in general linear position, and hence stable [DO88, Ch. II, Th. 1]. This yields an open subset $U_{\mathrm{dP}} \subset P_2^8$. If $(p_1, \dots, p_8) \in a(U_{\mathrm{dP}})$, then $p_1, \dots, p_8 \in \mathbb{P}^4$ are in general linear position by Lemma 2.19.

2.21. Degree one del Pezzo surfaces and blow-ups of $\mathbb{P}^4$ in 8 points. Let S be a del Pezzo surface of degree 1, and h a cubic in S. We associate to (S,h) a blow-up X of $\mathbb{P}^4$ in 8 points in general linear position, as follows.

Let $q_1,\dots,q_8 \in \mathbb{P}^2$ be the points blown-up under the birational morphism $S \to \mathbb{P}^2$ defined by h, and let $p_1,\dots,p_8 \in \mathbb{P}^4$ be the associated points to $q_1,\dots,q_8 \in \mathbb{P}^2$ (which are in general linear position by Rem. 2.20). Then we set

$$X = X_h = X_{(S,h)} := \mathrm{Bl}_{p_1,\dots,p_8}\,\mathbb{P}^4.$$

We will always assume that $q_1,\dots,q_8 \in \mathbb{P}^2$ and $p_1,\dots,p_8 \in \mathbb{P}^4$ are associated as *ordered* sets of point.

Conversely, let X be a blow-up of $\mathbb{P}^4$ in 8 general points. Differently from the case of surfaces, the blow-up map $X \to \mathbb{P}^4$ is unique, see [DO88, p. 64]. Thus X determines $p_1,\dots,p_8 \in \mathbb{P}^4$ up to projective equivalence, which in turn determine $q_1,\dots,q_8 \in \mathbb{P}^2$ up to projective equivalence, and hence a pair (S,h), such that $X \cong X_{(S,h)}$. The pair (S,h) is unique up to isomorphism, therefore S is determined up to isomorphism, and h is determined up to the action of $\mathrm{Aut}(S)$ on cubics; in particular $X_{(S,h)} = X_{(S,\iota_S^* h)}$.

2.22. Notation for the blow-up X of $\mathbb{P}^4$ at 8 points. Let $p_1,\dots,p_8 \in \mathbb{P}^4$ be points in general linear position, and set $X := \mathrm{Bl}_{p_1,\dots,p_8}\,\mathbb{P}^4$. We use the following notation:

- $E_i \subset X$ is the exceptional divisor over $p_i \in \mathbb{P}^4$, for $i = 1,\dots,8$
- $H \in \mathrm{Pic}(X)$ is the pull-back of $\mathcal{O}_{\mathbb{P}^4}(1)$
- $L_{ij} \subset X$ is the transform of the line $\overline{p_i p_j} \subset \mathbb{P}^4$, for $1 \le i < j \le 8$
- $e_i \subset E_i$ is a line, for $i = 1,\dots,8$
- $h \subset X$ is the transform of a general line in $\mathbb{P}^4$
- $\gamma_i \subset \mathbb{P}^4$ is the rational normal quartic through $p_1,\dots,\check{p}_i,\dots,p_8$, for $i = 1,\dots,8$ (γ_i exists and is unique, see for instance [Har92, p. 14])
- $\Gamma_i \subset X$ is the transform of $\gamma_i \subset \mathbb{P}^4$, for $i = 1,\dots,8$.

The notation $h, e_1,\dots,e_8$ is standard and will be used both in S and in X (see 2.2); it will be clear from the context whether we are referring to classes in S or in X.

3. Moduli of rank 2 vector bundles on S with $c_1 = -K_S$ and $c_2 = 2$: non-emptyness, walls, special loci

3.1. Let S be a del Pezzo surface of degree 1, and $L \in \mathrm{Pic}(S)$ an ample line bundle. Following [Muk05], in this section we introduce the moduli space $M_{S,L}$ of rank 2 torsion free sheaves F on S, with $c_1(F) = -K_S$ and $c_2(F) = 2$, semistable with respect to L. We resume and expand the study made in [Muk05] of this moduli space.

More precisely, we determine explicitly all the walls for slope semistability, and introduce the stability fan $\mathrm{ST}(S)$ in $H^2(S,\mathbb{R})$ determined by these walls. We describe the birational transformation occurring in $M_{S,L}$ when the polarization L crosses a wall, and describe $M_{S,L}$ when L belongs to some wall. Finally, for every cubic h, we construct a chamber $\mathcal{C}_h$ such that for $L \in \mathcal{C}_h$, $M_{S,L} \cong \mathbb{P}^4$. Using these results, we show Prop. 1.2.

Many results in this section are due to Mukai [Muk05]. Our new contributions are: the determination of the cone $\mathcal{E}$ of the polarizations for which the moduli space is non-empty (Cor. 3.26), the completion of the description of the walls for slope semistability (Prop. 3.8 and Cor. 3.11), the description of the moduli space when the polarization is not in a chamber (Lemma 3.22), and the description of the exceptional locus of the morphism $\gamma\colon M_L \to M_L^{\mu}$ to the moduli space of slope semistable sheaves (Lemma 3.30).

3.2. The moduli space $M_{S,L} = M_L$. Let $L \in \mathrm{Pic}(S)$ be ample, and F a rank 2 torsion free sheaf on S. By "stable" and "semistable" we mean stable or semistable in the sense of Gieseker-Maruyama; we will use μ-stable or μ-semistable for slope stability. We refer the reader to Huybrechts and Lehn's book [HL10, §1.2] for these notions, and recall that:

$$\mu\text{-stable} \Rightarrow \text{ stable } \Rightarrow \text{ semistable } \Rightarrow \mu\text{-semistable.}$$

In particular, for a fixed polarization L, either the 4 notions of stability and semistability above coincide, or there exists a strictly μ-semistable sheaf.

Definition 3.3. Given $L \in \mathrm{Pic}(S)$ ample, $M_{S,L}$ is the moduli space of torsion-free sheaves F of rank 2 on S, with $c_1(F) = -K_S$ and $c_2(F) = 2$, semistable with respect to L. When the surface S is fixed, we will often write M_L for $M_{S,L}$.

Remark 3.4. Let $L \in \mathrm{Pic}(S)$ be ample, and let F be a rank 2 torsion-free sheaf with $c_1(F) = -K_S$ and $c_2(F) = 2$. Then either F is stable, or F is not semistable. Indeed by Riemann-Roch we have $\chi(S, F) = 1$.

The following is a standard application of Rem. 3.4, see for instance [BMW14, 3.1] and [HL10, Th. 4.5.4 and p. 115].

Corollary 3.5. *Let $L \in \mathrm{Pic}(S)$ be ample. If the moduli space M_L is non-empty, then it is smooth, projective, of pure dimension* 4*.*

We will see in Cor. 3.27 that M_L is always irreducible; this is already known, see [CMR99, Prop. 3.11] and [Góm00, Th. III].

Recall that in the ample cone of S, the polarizations L for which there exists a strictly μ-semistable sheaf belong to a (locally finite) set of rational hyperplanes (**walls**), which yield a chamber decomposition of the ample cone (where a **chamber** is a connected component of the complement of the walls in the ample cone). For any chamber $\mathcal{C}$ we have:

(1) for $L \in \mathcal{C}$, every μ-semistable sheaf is also stable and μ-stable;
(2) the stability condition is the same for every $L \in \mathcal{C}$, and for $L \in \mathcal{C}$ the moduli spaces M_L are all equal, hence they only depend on the chamber $\mathcal{C}$.

We will sometimes denote by $M_{\mathcal{C}}$ or $M_{S,\mathcal{C}}$ the moduli space M_L for $L \in \mathcal{C}$.

First of all, Mukai gives a necessary condition on the polarization L for the existence of μ-semistable sheaves with respect to L.

Lemma 3.6 ([Muk05], p. 9)**.** *Let $L \in \mathrm{Pic}(S)$ be ample. If there exists a μ-semistable torsion-free sheaf F of rank* 2 *with $c_1(F) = -K_S$ and $c_2(F) = 2$, then $L \cdot (2h + K_S) \geq 0$ for every cubic h, namely: $L \in \mathcal{E}$ (see* (2.5)*).*

3.7. Walls and special extensions. [Muk05, Lemma 3] describes all the walls that intersect the cone Π; we generalise it and describe every wall.

Proposition 3.8. *Let $L \in \mathrm{Pic}(S)$ be ample, and suppose that there exists a strictly μ-semistable torsion-free sheaf F of rank* 2 *with $c_1(F) = -K_S$ and $c_2(F) = 2$. Then we have the following.*

(*a*) *There exists a divisor D such that $L \cdot (2D + K_S) = 0$, and either D or $-K_S - D$ is linearly equivalent to a (-1)-curve, a conic, or a cubic;*

(b) F is locally free and there is an exact sequence:

$$0 \longrightarrow \mathcal{O}_S(D) \longrightarrow F \longrightarrow \mathcal{O}_S(-K_S-D) \longrightarrow 0. \tag{3.9}$$

(c) If D is effective, or if the extension is split, then F is not stable.
If $-K_S-D$ is effective, and the extension is not split, then F is stable.

For the proof of Prop. 3.8 we need the following elementary result.

Lemma 3.10. *Let $L \in \operatorname{Pic}(S)$ be ample, $L \in \mathcal{E}$, and let B be an effective divisor on S such that:*

$$B \cdot (K_S + B) = -2, \quad h^0(S, K_S + B) = 0, \quad \textit{and} \quad L \cdot (2B + K_S) = 0.$$

Then B is linearly equivalent to either a (-1)-curve, or a conic, or a cubic.

Proof. We can write $B \sim P+N$ where P and N are integral divisors, P is nef, $P \cdot N = 0$, and $N = \sum_{i=1}^r m_i\ell_i$ with $m_i \in \mathbb{Z}_{>0}$ and the ℓ_i's pairwise disjoint (-1)-curves (see [BPS17, Ex. 3.3]).

If $P = 0$, we have $B = m_1\ell_1 + \cdots + m_r\ell_r$ and $-2 = B \cdot (K_S+B) = -\sum_{i=1}^r m_i(m_i+1)$, hence $r = 1$, $m_1 = 1$, and B is a (-1)-curve.

Suppose that B has Iitaka dimension 1, and let $S \to \mathbb{P}^1$ be the conic bundle given by P, so that $P = m_0C$ where C is a general fiber and $m_0 \geq 1$. Then we have

$$-2 = (m_0C + N) \cdot (K_S + m_0C + N) = -\Big(2m_0 + \sum_{i=1}^r m_i(m_i+1)\Big),$$

which yields $m_0 = 1$ and $r = 0$, namely B is linearly equivalent to a conic.

Finally, suppose that B is big. We have $L \cdot (2h + K_S) \geq 0$ for every cubic h, because $L \in \mathcal{E}$ (see (2.5)). Since $L \cdot (2B + K_S) = 0$, if there exists some cubic h such that $B - h$ is effective, then $B = h$.

Since P is nef and big, by vanishing we have $h^i(S, K_S+P) = 0$ for $i = 1, 2$. Using the assumptions and Riemann-Roch, one gets $P \cdot (K_S+P) = -2$ and $N \cdot (K_S+N) = 0$. This yields that $N = 0$ and $B \sim P$ is nef and big, and moreover using again Riemann-Roch and vanishing one gets $h^0(S, B) = 2+B^2$. If $\sigma\colon S \to S'$ is the birational map given by B, we have $B = \sigma^*B'$ where B' is ample and again $h^0(S', B') = 2 + (B')^2$, namely the pair (S', B') has Δ-genus zero (see [BS95, §3.1]). These pairs have been classified by Fujita, see for instance [BS95, Prop. 3.1.2], and the possibilities are: $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, a))$ with $a \geq 1$, and $(\mathbb{F}_1, L)$ where L is ample and $L_{|F} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ for a fiber F of the $\mathbb{P}^1$-bundle on $\mathbb{F}_1$. In each case one can find a cubic h on S such that $B - h$ is effective, so that $B = h$. ■

Proof of Prop. 3.8. Since F is not μ-stable with respect to L, there is an exact sequence

$$0 \longrightarrow \mathcal{O}_S(D) \otimes \mathcal{I}_{Z_1} \longrightarrow F \longrightarrow \mathcal{O}_S(-K_S-D) \otimes \mathcal{I}_{Z_2} \longrightarrow 0$$

where Z_1 and Z_2 are 0-dimensional subschemes of S (see [HL10, Ex. 1.1.16]) and D is a divisor on S with $L \cdot D = \frac{1}{2}L \cdot (-K_S)$, namely $L \cdot (2D + K_S) = 0$. Using that $c_2(F) = 2$ we get $D \cdot (K_S + D) = l(Z_1) + l(Z_2) - 2$, where $l(Z_i)$ is the length of Z_i. Since both L and $-K_S$ are ample, we have $L \cdot (-K_S) > 0$, and hence $L \cdot D > 0$ and $L \cdot (D + K_S) < 0$. This yields $h^0(S, K_S + D) = 0$ and $h^2(S, K_S + D) = h^0(S, -D) = 0$, hence $\chi(S, \mathcal{O}_S(K_S+D)) = -h^1(S, K_S+D) \leq 0$. On the other hand by Riemann-Roch:

$$\chi(S, \mathcal{O}_S(K_S + D)) = 1 + \frac{1}{2}D \cdot (K_S + D) = \frac{1}{2}\big(l(Z_1) + l(Z_2)\big) \geq 0,$$

which yields $Z_1 = Z_2 = \emptyset$, $D \cdot (K_S + D) = -2$, and the exact sequence:

$$0 \longrightarrow \mathcal{O}_S(D) \longrightarrow F \longrightarrow \mathcal{O}_S(-K_S - D) \longrightarrow 0;$$

in particular F is locally free, and we have (b).

We have $L \in \mathcal{E}$ by Lemma 3.6. By [Muk05, Lemma 2] the sheaf F has a non-zero global section; thus by the sequence above, either $\mathcal{O}_S(D)$ or $\mathcal{O}_S(-K_S - D)$ must have a non-zero global section. Now if D is effective, then D is either a (-1)-curve, or a conic, or a cubic, by Lemma 3.10. Similarly, if $-K_S - D$ is effective, then it is either a (-1)-curve, or a conic, or a cubic, again by Lemma 3.10. This shows (a).

For (c), notice that if ℓ is a (-1)-curve, C a conic, and h a cubic, we have:

$$\chi(S, F) = 1, \quad \chi(S, \mathcal{O}_S(\ell)) = 1, \quad \chi(S, \mathcal{O}_S(C)) = 2, \quad \chi(S, \mathcal{O}_S(h)) = 3.$$

If D is linearly equivalent to ℓ, C, or h, then $\chi(S, \mathcal{O}_S(D)) > \frac{1}{2}\chi(S, F) = \frac{1}{2}$, thus F is not stable.

Suppose that $-K_S - D$ is linearly equivalent to ℓ, C, or h. If the extension is split, then we can exchange D with $-K_S - D$, so that again F is not stable. If instead the extension is not split, then $\mathcal{O}_S(D)$ is the unique locally free rank 1 subsheaf G of F with $\mu_L(G) = \mu_L(F)$, where μ_L denotes the slope with respect to L (see [Fri98, Ch. 4, Prop. 21]). Since $\chi(S, \mathcal{O}_S(D)) = \chi(S, F) - \chi(S, \mathcal{O}_S(-K_S - D)) \in \{0, -1, -2\}$, we deduce that F is stable. ■

Corollary 3.11. *Let $L \in \operatorname{Pic}(S)$ be ample, and suppose that there exists a strictly μ-semistable torsion-free sheaf F of rank 2 with $c_1(F) = -K_S$ and $c_2(F) = 2$. Then L belongs to a wall $(2\ell + K_S)^\perp$, $(2C + K_S)^\perp$, or $(2h + K_S)^\perp$, where ℓ is a (-1)-curve, C is a conic, and h is a cubic.*

Definition 3.12 (the stability fan)**.** The hyperplanes $(2h + K_S)^\perp$, $(2\ell + K_S)^\perp$, and $(2C + K_S)^\perp$ define a fan $\mathrm{ST}(S)$ in $H^2(S, \mathbb{R})$, supported on the cone $\mathcal{E}$, that we call the **stability fan**. The cones of maximal dimension of the fan are the closures $\overline{\mathcal{C}}$, where $\mathcal{C} \subset \mathcal{E}$ is a chamber. The cone Π is a union of cones of the fan, and the cone $\mathcal{N}$ belongs to the fan (see 2.10 and 2.7). Notice that the hyperplanes $(2h + K_S)^\perp$ cut the boundary of the cone $\mathcal{E}$, so that they do not separate different chambers for which the moduli space is non-empty (see Cor. 3.6).

Remark 3.13. The anticanonical class $-K_S$ does not belong to any wall and lies in the interior of the cone $\mathcal{N}$ (in particular $-K_S \in \Pi$ and $-K_S \in \mathcal{E}$), so that $\mathcal{N}$ is the closure of the chamber containing $-K_S$.

We now study more in detail the sheaves arising from extensions as in (3.9).

Remark 3.14. Given a (-1)-curve ℓ, a conic C, and a cubic h, we have:

$$\begin{aligned} &h^1(S, K_S + 2\ell) = 2, && h^1(S, K_S + 2C) = 1, && h^1(S, K_S + 2h) = 0, \\ &h^1(S, -K_S - 2\ell) = 3, && h^1(S, -K_S - 2C) = 4, && h^1(S, -K_S - 2h) = 5. \end{aligned}$$

Lemma 3.15. *Let F be a sheaf on S sitting in an extension*

$$0 \longrightarrow \mathcal{O}_S(D) \longrightarrow F \xrightarrow{\varphi} \mathcal{O}_S(-K_S - D) \longrightarrow 0$$

where D or $-K_S - D$ is either a (-1)-curve, or a conic, or a cubic. We have:

(a) F is locally free with rank 2, $c_1(F) = -K_S$, and $c_2(F) = 2$;

(b) φ is unique up to non-zero scalar multiplication;

(c) *either the extension is split, or F is determined, up to isomorphism, by an element of*

$$\mathbb{P}\big(\operatorname{Ext}^1(\mathcal{O}_S(-K_S-D),\mathcal{O}_S(D))\big)=\mathbb{P}\big(H^1(S,\mathcal{O}_S(K_S+2D))\big)$$

(here $\mathbb{P}$ stands for the classical projectivization, namely that of 1*-dimensional linear subspaces);*

(d) *if D is a cubic, then the extension is split.*

Proof. Statement (a) is straightforward. For (b), let $\varphi'\colon F\to\mathcal{O}_S(-K_S-D)$ be another surjective map. Then the restriction $\varphi'_{|\ker\varphi}\colon\mathcal{O}_S(D)\to\mathcal{O}_S(-K_S-D)$ must be trivial, because in all six possible cases for D, $-K_S-2D$ cannot be effective. Similarly, $\varphi_{|\ker\varphi'}=0$, thus $\ker\varphi=\ker\varphi'$ and hence $\varphi'=\lambda\varphi$. Statement (c) follows from (b), and (d) holds because $h^1(S,K_S+2h)=0$ (see Rem. 3.14). ■

Definition 3.16 (special loci in the moduli spaces M_L). Given a (-1)-curve ℓ and a conic C in S, we set:

$$P_\ell=\mathbb{P}(\operatorname{Ext}^1(\mathcal{O}_S(\ell),\mathcal{O}_S(-K_S-\ell)))\cong\mathbb{P}^2,\qquad Z_\ell=\mathbb{P}(\operatorname{Ext}^1(\mathcal{O}_S(-K_S-\ell),\mathcal{O}_S(\ell)))\cong\mathbb{P}^1,$$
$$E_C=\mathbb{P}(\operatorname{Ext}^1(\mathcal{O}_S(C),\mathcal{O}_S(-K_S-C)))\cong\mathbb{P}^3,$$

and we denote by F_C the unique sheaf on S sitting in a non-split extension:

$$0\longrightarrow\mathcal{O}_S(C)\longrightarrow F_C\longrightarrow\mathcal{O}_S(-K_S-C)\longrightarrow 0.$$

These loci are crucial in the description of the birational transformation occurring in M_L when L crosses a wall; following Mukai [Muk05, p. 8], we describe this in detail.

3.17. Crossing the wall $(2\ell+K_S)^\perp$. Let us fix a (-1)-curve ℓ. Let $\mathcal{C}$ be a chamber lying in the halfspace

$$(K_S+2\ell)^{>0}:=\{\gamma\in H^2(S,\mathbb{R})\,|\,\gamma\cdot(K_S+2\ell)>0\},$$

and such that $\overline{\mathcal{C}}\cap(2\ell+K_S)^\perp$ intersects the ample cone. Then for any non-trivial extension

$$0\longrightarrow\mathcal{O}_S(-K_S-\ell)\longrightarrow F\longrightarrow\mathcal{O}_S(\ell)\longrightarrow 0,$$

F is stable with respect to $L\in\mathcal{C}$ [Qin93, Ch. II, Th. 1.2.3]. By Lemma 3.15(c), these sheaves F are parametrized by $P_\ell\cong\mathbb{P}^2$, and we have $P_\ell\subset M_L$ [Qin93, Ch. II, Cor. 1.2.4].

Similarly, let $\mathcal{C}'$ be a chamber lying in the halfspace $(K_S+2\ell)^{<0}$, and such that $\overline{\mathcal{C}'}\cap(2\ell+K_S)^\perp$ intersects the ample cone. Then for any non-trivial extension

$$0\longrightarrow\mathcal{O}_S(\ell)\longrightarrow F'\longrightarrow\mathcal{O}_S(-K_S-\ell)\longrightarrow 0,$$

F' is stable with respect to $L'\in\mathcal{C}'$; these sheaves F are parametrized by $Z_\ell\cong\mathbb{P}^1$, and we have $Z_\ell\subset M_{L'}$.

Suppose moreover that $\overline{\mathcal{C}}$ and $\overline{\mathcal{C}'}$ share a common facet (which must lie on the hyperplane $(K_S+2\ell)^\perp$). Then the moduli spaces M_L and $M_{L'}$ are birational and isomorphic in codimension 1 under the natural map $[F]\mapsto[F]$. More precisely, the normal bundles of P_ℓ and Z_ℓ are $\mathcal{N}_{P_\ell/M_L}\cong\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$ and $\mathcal{N}_{Z_\ell/M_{L'}}\cong\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3}$ [FQ95, Prop. 3.6 and Lemma 3.2(ii)], the birational map $M_L\dashrightarrow M_{L'}$ is a K-negative flip, and factors as $M_L\leftarrow\widehat{M}\to M_{L'}$, where $\widehat{M}\to M_L$ is the blow-up of P_ℓ, and $\widehat{M}\to M_{L'}$ is the blow-up of Z_ℓ (see [Qin93, Th. 2] and [FQ95, Th. 3.9]).

3.18. Crossing the wall $(2C+K_S)^\perp$. Let us fix a conic C. Let $\mathcal{C}$ be a chamber lying in the halfspace $(K_S+2C)^{>0}$, such that $\overline{\mathcal{C}} \cap (2C+K_S)^\perp$ intersects the ample cone. Then for any non-trivial extension

$$0 \longrightarrow \mathcal{O}_S(-K_S-C) \longrightarrow F \longrightarrow \mathcal{O}_S(C) \longrightarrow 0,$$

F is stable with respect to $L \in \mathcal{C}$ [Qin93, Ch. II, Th. 1.2.3]. By Lemma 3.15(c), these sheaves F are parametrized by $E_C \cong \mathbb{P}^3$, and we have $E_C \subset M_L$ [Qin93, Ch. II, Cor. 1.2.4].

Similarly, let $\mathcal{C}'$ be a chamber lying in the halfspace $(K_S+2C)^{<0}$, and such that $\overline{\mathcal{C}'} \cap (2C+K_S)^\perp$ intersects the ample cone. Then for any non-trivial extension

$$0 \longrightarrow \mathcal{O}_S(C) \longrightarrow F' \longrightarrow \mathcal{O}_S(-K_S-C) \longrightarrow 0,$$

F' is stable with respect to $L' \in \mathcal{C}'$; then $F' \cong F_C$, which yields a point $[F_C] \in M_{L'}$.

Suppose moreover that $\overline{\mathcal{C}}$ and $\overline{\mathcal{C}'}$ share a common facet (which must lie on the hyperplane $(K_S+2C)^\perp$). Then the natural map $[F] \mapsto [F]$ yields a birational morphism $M_L \to M_{L'}$. More precisely, the normal bundle of E_C is $\mathcal{N}_{E_C/M_L} \cong \mathcal{O}_{\mathbb{P}^3}(-1)$ [FQ95, Prop. 3.6], and $M_L \to M_{L'}$ is just the blow-up of the smooth point $[F_C]$, with exceptional divisor E_C (see [Qin93, Th. 2] and [FQ95, Th. 3.9]).

3.19. The moduli space for the outer chamber $\mathcal{C}_h$. Let us fix a cubic h. Recall from 2.3 that the wall $(2h+K_S)^\perp$ cuts the facet $\tau_h = \langle C_1, \dots, C_8 \rangle$ of $\mathcal{E}$ (notation as in 2.2). We have $C_1 + \dots + C_8 \sim -K_S + 5h$, so $-K_S+5h$ belongs to the relative interior of τ_h.

All the walls different from $(2h+K_S)^\perp$ cut the cone τ_h along a proper face. Thus there is a unique chamber $\mathcal{C}_h \subset \mathcal{E}$ such that $\overline{\mathcal{C}}_h$ intersects the relative interior of τ_h; in particular τ_h is a facet of $\overline{\mathcal{C}}_h$. We have $\overline{\mathcal{C}}_h = \langle C_1, \dots, C_8, -K_S+3h \rangle$ (see Lemma 5.2 and the following discussion).

Proposition 3.20 ([Muk05], p. 9). *For every $[F] \in M_{\mathcal{C}_h}$ the sheaf F is locally free, and $M_{\mathcal{C}_h} \cong \mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_S(h), \mathcal{O}_S(-K_S-h))) = \mathbb{P}^4$.*

3.21. The moduli space M_L when L belongs to some wall.

Lemma 3.22. *Let $L \in \mathrm{Pic}(S)$ be ample, $L \in \mathcal{E}$. We have the following.*

(a) *There exists a unique chamber $\mathcal{C}$ such that $L \in \overline{\mathcal{C}}$ and $\mathcal{C} \subset (K_S+2D)^{>0}$ for every (-1)-curve, conic, or cubic D with $L \cdot (2D+K_S) = 0$.*

(b) *We have $M_L = M_{\mathcal{C}}$, and $L \in \Pi$ if and only if $\mathcal{C} \subset \Pi$.*

(c) *If L belongs to the boundary of $\mathcal{E}$, then L belongs to a unique wall $(2h+K_S)^\perp$ where h is a cubic; L is in the relative interior of the facet $\tau_h = (2h+K_S)^\perp \cap \mathcal{E}$, $\mathcal{C} = \mathcal{C}_h$, and $M_L = M_{\mathcal{C}_h} \cong \mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_S(h), \mathcal{O}_S(-K_S-h))) = \mathbb{P}^4$.*

(d) *Suppose that L is in the interior of $\mathcal{E}$, and that it is contained in the walls $(2\ell_i+K_S)^\perp$ and $(2C_j+K_S)^\perp$, ℓ_i a (-1)-curve and C_j a conic, for $i = 1, \dots, r$ and $j = 1, \dots, s$, with $r \geq 0$ and $s \geq 0$. Then M_L contains the special loci $P_{\ell_i} \cong \mathbb{P}^2$ and $E_{C_j} \cong \mathbb{P}^3$ for all i, j, and these loci are pairwise disjoint in M_L.*

Proof. The anticanonical class $-K_S$ belongs to $\mathcal{E}$ and is not contained in any wall (see Rem. 3.13). Thus we can choose $m \gg 0$ such that the class $L' := -K_S + mL$ is contained in a chamber, and such that no wall contains a convex combination of L and L' different from L. Then the chamber $\mathcal{C}$ containing L' satisfies (a), because for every D as in (a) we have $L' \cdot (K_S+2D) = -K_S \cdot (K_S+2D) > 0$.

It follows from standard arguments (see for instance [Fri98, Ch. 4, Prop. 22]; the argument for torsion free sheaves is the same) that:

$$L\text{-}\mu\text{-stable} \;\Rightarrow\; \mathcal{C}\text{-stable} \;\Rightarrow\; L\text{-}\mu\text{-semistable}. \tag{3.23}$$

Suppose that L is on the boundary of $\mathcal{E}$. Since L is ample, by Rem. 2.6 L is contained in the relative interior of a facet $\tau_h = (2h + K_S)^{\perp} \cap \mathcal{E}$, where h is a cubic. By 3.19 we see that $(2h + K_S)^{\perp}$ is the unique wall containing L, and that $\mathcal{C} = \mathcal{C}_h$.

We have described $\mathcal{C}_h$-stable sheaves in Prop. 3.20, they concide with non-split extensions

$$0 \longrightarrow \mathcal{O}_S(-K_S - h) \longrightarrow F \longrightarrow \mathcal{O}_S(h) \longrightarrow 0.$$

In particular such sheaves are all L-stable by Prop. 3.8(c).

Conversely, by Prop. 3.8 and Lemma 3.15(d), the strictly L-μ-semistable sheaves are the sheaves appearing in an extension as above, and the split case $\mathcal{O}_S(-K_S-h)\oplus\mathcal{O}_S(h)$; this last one is not L-stable.

Together with (3.23), this shows that L-stability coincides with $\mathcal{C}_h$-stability, and we get (c) by Prop. 3.20. Notice also that there are no L-μ-stable sheaves.

Suppose now that L is not on the boundary of $\mathcal{E}$, as in (d). By Prop. 3.8, the set of strictly L-μ-semistable sheaves is given by:

(1) the sheaves in non-split extensions $0 \longrightarrow \mathcal{O}_S(-K_S - D) \longrightarrow F \longrightarrow \mathcal{O}_S(D) \longrightarrow 0$
(2) the sheaves in extensions $0 \longrightarrow \mathcal{O}_S(D) \longrightarrow F \longrightarrow \mathcal{O}_S(-K_S - D) \longrightarrow 0$

where $D \in \{\ell_1, \dots, \ell_r, C_1, \dots, C_s\}$. The sheaves in (1) are L-stable by Prop. 3.8(c), and $\mathcal{C}$-stable as explained in 3.17 and 3.18. On the other hand, the sheaves in (2) are neither L-stable (again by Prop. 3.8(c)) nor $\mathcal{C}$-stable (because $L' \cdot (K_S + 2D) > 0$).

Together with (3.23), this shows that L-stability coincides with $\mathcal{C}$-stability, and we get (b). The sheaves in (1) yield the loci $P_{\ell_1}, \dots, P_{\ell_r}, E_{C_1}, \dots, E_{C_s}$ in M_L. Finally, given a strictly L-μ-semistable sheaf F as in (1), the subsheaf $\mathcal{O}_S(-K_S - D)$ is unique (see [Fri98, Ch. 4, Prop. 21]), so these loci are pairwise disjoint, and we get (d). ■

3.24. Properties of M_L.

Corollary 3.25. *Let $L_1, L_2 \in \mathrm{Pic}(S)$ be ample, $L_1, L_2 \in \mathcal{E}$. Then there exist non-empty dense open subsets $U_1 \subseteq M_{S,L_1}$ and $U_2 \subseteq M_{S,L_2}$ parametrizing the same objects; this yields a natural birational map $\varphi\colon M_{S,L_1} \dashrightarrow M_{S,L_2}$ given by $\varphi([F]) = [F]$.*

Suppose that $L_1 \in \Pi$. Then φ is a contracting[4] *birational map, and φ is a pseudo-isomorphism if and only if $L_2 \in \Pi$.*

Proof. By Lemma 3.22 we can assume that L_1 and L_2 do not belong to any wall. Then the first statement follows from the explicit description of wall crossings in 3.17 and 3.18. If $L_1 \in \Pi$, then φ is a compositions of flips and blow-downs, so it is contracting. Moreover, φ is a pseudo-isomorphism if and only if it can be factored as a sequence of flips, if and only if $L_2 \in \Pi$. See [Muk05, p. 9]. ■

The following are straightforward consequences of Cor. 3.6, Cor. 3.25 and Prop. 3.20.

Corollary 3.26. *Let $L \in \mathrm{Pic}(S)$ be ample. The following are equivalent:*

(i) *there exists a μ-semistable torsion-free sheaf F of rank 2 with $c_1(F) = -K_S$ and $c_2(F) = 2$;*

[4] A birational map $\varphi\colon X_1 \dashrightarrow X_2$ is *contracting* if there exist open subsets $U_1 \subseteq X_1$ and $U_2 \subseteq X_2$ such that φ yields an isomorphism between U_1 and U_2, and $\operatorname{codim}(X_2 \smallsetminus U_2) \geq 2$.

(ii) $M_L \neq \emptyset$;
(iii) $L \in \mathcal{E}$, *namely* $L \cdot (2h + K_S) \geq 0$ *for every cubic* h.

Corollary 3.27. *Let* $L \in \mathcal{E}$ *be an ample line bundle. Then* M_L *is irreducible.*

Corollary 3.28. *Let* $L \in \mathcal{E}$ *be an ample line bundle, and* F *a* μ*-semistable torsion-free sheaf* F *of rank* 2 *with* $c_1(F) = -K_S$ *and* $c_2(F) = 2$. *Then* F *is locally free.*

Proof. This holds for $L \in \mathcal{C}_h$ by Prop. 3.20. By the explicit description of wall crossings given in 3.17 and 3.18, the statement stays true whenever L is in a chamber. Finally if L does not belong to a chamber, the statement follows from Lemma 3.22(b). ■

3.29. The moduli space of μ-semistable sheaves. We consider now the moduli space M_L^μ of μ-semistable rank 2 torsion-free sheaves F on S, with $c_1(F) = -K_S$ and $c_2(F) = 2$, see [HL10, §8.2] and references therein. We recall that by Cor. 3.28, every such sheaf F is locally free.

By [HL10, Def. and Th. 8.2.8] M_L^μ is a projective scheme (which we consider with its reduced scheme structure), endowed with a natural morphism $\gamma\colon M_L \to M_L^\mu$, with the following properties:

(a) the points of M_L^μ are in bijection with equivalence classes of μ-semistable rank 2 locally free sheaves F on S, with $c_1(F) = -K_S$ and $c_2(F) = 2$;

(b) by [HL10, Def. 8.2.10 and Th. 8.2.11], if F_1 and F_2 are not isomorphic, then they are equivalent if and only if they are strictly μ-semistable, and in the exact sequences given by Prop. 3.8:
$$0 \longrightarrow \mathcal{O}_S(D_i) \longrightarrow F_i \longrightarrow \mathcal{O}_S(-K_S - D_i) \to 0 \qquad i = 1, 2$$
we have either $\mathcal{O}_S(D_1) \cong \mathcal{O}_S(D_2)$ or $\mathcal{O}_S(D_1) \cong \mathcal{O}_S(-K_S - D_2)$;

(c) for $[F] \in M_L$, $\gamma([F])$ is the class of F in M_L^μ, and γ is an isomorphism on the open subsets of μ-stable, locally free sheaves [HL10, Cor. 8.2.16].

Thus if the polarization L is in a chamber, it follows from (c) and Cor. 3.28 that γ is an isomorphism and the two moduli spaces coincide. In general we have the following.

Lemma 3.30. *Let* $L \in \mathrm{Pic}(S)$ *be ample,* $L \in \mathcal{E}$.

If L *belongs to the boundary of* $\mathcal{E}$, *then* M_L^μ *is a point.*

Suppose that L *is not on the boundary of* $\mathcal{E}$, *and that it is contained in the walls* $(2\ell_i + K_S)^\perp$ *and* $(2C_j + K_S)^\perp$, ℓ_i *a* (-1)*-curve and* C_j *a conic, for* $i = 1, \dots, r$ *and* $j = 1, \dots, s$, *with* $r \geq 0$ *and* $s \geq 0$. *Then* γ *is birational,* $\mathrm{Exc}(\gamma) = P_{\ell_1} \cup \cdots \cup P_{\ell_r} \cup E_{C_1} \cup \cdots \cup E_{C_s}$, *and the image of* $\mathrm{Exc}(\gamma)$ *is given by* $r + s$ *distinct points.*

Proof. The statement follows from (b) above and from Lemma 3.22 (and its proof). ■

4. The determinant map

4.1. In this section we recall the classical construction of determinant line bundles on the moduli space M_L, and use it to define a group homomorphism $\rho\colon \mathrm{Pic}(S) \to \mathrm{Pic}(M_L)$, that we call the determinant map. We study the properties of ρ, together with its real extension $\rho\colon H^2(S,\mathbb{R}) \to H^2(M_L,\mathbb{R})$ and its dual map $\zeta := \rho^t\colon \mathcal{N}_1(M_L) \longrightarrow \mathcal{N}_1(S) = H^2(S,\mathbb{R})$, where $\mathcal{N}_1(M_L)$ is the vector space of real 1-cycles in M_L up to numerical equivalence, and similarly for S. Using the classical positivity properties of the determinant line bundle, we determine the images via ζ of the lines in the exceptional loci in M_L (Cor. 4.16 and 4.17). We also compare the maps ρ for different L's (Lemma

4.12), and show that ρ is equivariant for the action of the group of automorphisms of S which fix the polarization L (Prop. 4.23). Finally we deduce that the moduli space M_{-K_S} is a Fano variety (Prop. 4.20), and more generally that M_L is always a Mori dream space (Cor. 4.21).

4.2. Determinant line bundles. We refer the reader to [HL10, Ch. 8] and references therein, in particular Le Potier [LP92] and Li [Li93], for the construction and properties of determinant line bundles; let us give a brief outline.

Let $L \in \mathrm{Pic}(S)$ be ample, $L \in \mathcal{E}$. To simplify the notation, set $M := M_L$. Let $K(S)$ be the Grothendieck group of coherent sheaves on S, and similarly for the moduli space M and the product $S \times M$; since S and M are smooth projective varieties, their Grothendieck groups of coherent sheaves are naturally isomorphic to the Grothendieck groups of locally free sheaves. Moreover, being S a rational surface, we have a group isomorphism

$$(\mathrm{rk}, c_1, \chi)\colon K(S) \longrightarrow \mathbb{Z} \oplus \mathrm{Pic}(S) \oplus \mathbb{Z}. \tag{4.3}$$

Under this isomorphism, the class $[\mathcal{O}_{pt}]$ corresponds to $(0, \mathcal{O}_S, 1)$, in particular it does not depend on the point.

Let $\mathfrak{f} \in K(S)$ be the class of a rank 2 torsion-free sheaf F on S with $c_1(F) = -K_S$ and $c_2(F) = 2$; for every $\mathfrak{c} \in K(S)$ we have

$$\chi(\mathfrak{f} \otimes \mathfrak{c}) = 2\chi(\mathfrak{c}) - K_S \cdot c_1(\mathfrak{c}) - \mathrm{rk}\,\mathfrak{c}.$$

We are interested in the subgroup $\mathfrak{f}^\perp = \{\mathfrak{c} \in K(S) \,|\, \chi(\mathfrak{f} \otimes \mathfrak{c}) = 0\}$. Notice that since $\chi(\mathfrak{f} \otimes [\mathcal{O}_{pt}]) = 2$, for every $\mathfrak{c} \in K(S)$ we have $2\mathfrak{c} - \chi(\mathfrak{f} \otimes \mathfrak{c})[\mathcal{O}_{pt}] \in \mathfrak{f}^\perp$.

By Rem. 3.4 and Cor. 3.28, for every $[F] \in M$ the sheaf F is locally free and stable, thus there exists a universal vector bundle $\mathcal{U}$ over $S \times M$, which is unique up to twists with pull-backs of line bundles on M. We denote by $\pi_S\colon S \times M \to S$ and $\pi_M\colon S \times M \to M$ the two projections. One defines a group homomorphism $\lambda_{\mathcal{U}}\colon K(S) \to \mathrm{Pic}(M)$ [HL10, Def. 8.1.1] as the composition:

$$\lambda_{\mathcal{U}}\colon K(S) \xrightarrow{\pi_S^*} K(S \times M) \xrightarrow{\otimes[\mathcal{U}]} K(S \times M) \xrightarrow{(\pi_M)_!} K(M) \xrightarrow{\det} \mathrm{Pic}(M),$$

where we recall that $(\pi_M)_! = \sum_i (-1)^i R^i(\pi_M)_*$, thus

$$\lambda_{\mathcal{U}}(\mathfrak{c}) = \det\big((\pi_M)_!(\pi_S^*\mathfrak{c} \otimes [\mathcal{U}])\big). \tag{4.4}$$

Given a line bundle $N \in \mathrm{Pic}(M)$, the vector bundle $\mathcal{U} \otimes \pi_M^* N$ is another universal family which yields another group homomorphism $\lambda_{\mathcal{U}\otimes\pi_M^*N}\colon K(S) \to \mathrm{Pic}(M)$. We have:

$$\lambda_{\mathcal{U}\otimes\pi_M^*N}(\mathfrak{c}) = \lambda_{\mathcal{U}}(\mathfrak{c}) \otimes [N^{\otimes\chi(\mathfrak{f}\otimes\mathfrak{c})}] \quad \text{for every } \mathfrak{c} \in K(S).$$

In order to have an intrinsic map, one restricts to the subgroup $\mathfrak{f}^\perp$, and set:

$$\lambda := (\lambda_{\mathcal{U}})_{|\mathfrak{f}^\perp}\colon \mathfrak{f}^\perp \longrightarrow \mathrm{Pic}(M).$$

Theorem 4.5 ([HL10], Th. 8.2.8)**.** *Let $H \in |L|$ be a curve, and set $\mathfrak{h} := [\mathcal{O}_H] \in K(S)$. Consider the class*

$$u_1 = -2\mathfrak{h} + \chi(\mathfrak{f} \otimes \mathfrak{h})[\mathcal{O}_{pt}] \in \mathfrak{f}^\perp \subset K(S),$$

and set $\mathcal{L}_1 := \lambda(u_1) \in \mathrm{Pic}(M_L)$ [HL10, Def. 8.1.9]*. Then $\mathcal{L}_1$ is semiample, and some positive multiple of $\mathcal{L}_1$ defines the map $\gamma\colon M_L \to M_L^\mu$ (see* (3.29)*).*

4.6. The map ρ. We define a map $\mathfrak{u}\colon \operatorname{Pic}(S) \to \mathfrak{f}^\perp \subset K(S)$, and use it to define a map $\rho := \lambda \circ \mathfrak{u}\colon \operatorname{Pic}(S) \to \operatorname{Pic}(M)$, as follows. We set, for every $P \in \operatorname{Pic}(S)$:

$$\mathfrak{u}(P) := 2\big([P^{\otimes(-1)}] - [\mathcal{O}_S]\big) - \chi\Big(\mathfrak{f} \otimes \big([P^{\otimes(-1)}] - [\mathcal{O}_S]\big)\Big)[\mathcal{O}_{pt}] \in \mathfrak{f}^\perp \subset K(S), \tag{4.7}$$

where $\chi(\mathfrak{f} \otimes ([P^{\otimes(-1)}] - [\mathcal{O}_S])) = P \cdot (P + 2K_S)$; one can check that

$$(\operatorname{rk}\mathfrak{u}(P), c_1(\mathfrak{u}(P)), \chi(\mathfrak{u}(P))) = (0, P^{\otimes(-2)}, -K_S \cdot P). \tag{4.8}$$

Since the map (4.3) is an isomorphism, (4.8) implies that $\mathfrak{u}$ is a group homomorphism. The following is the key property of $\mathfrak{u}$; it will be used in the proof of Cor. 4.11.

Remark 4.9. Let $P \in \operatorname{Pic}(S)$ be effective, $H \in |P|$ a curve, and set $\mathfrak{h} := [\mathcal{O}_H] \in K(S)$. The exact sequence

$$0 \longrightarrow \mathcal{O}_S(-H) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_H \longrightarrow 0$$

yields $\mathfrak{h} = [\mathcal{O}_S] - [P^{\otimes(-1)}]$ in $K(S)$, hence $\mathfrak{u}(P) = -2\mathfrak{h} + \chi(\mathfrak{f} \otimes \mathfrak{h})[\mathcal{O}_{pt}]$.

Finally we define $\rho := \lambda \circ \mathfrak{u}\colon \operatorname{Pic}(S) \to \operatorname{Pic}(M)$, namely $\rho(P) = \lambda(\mathfrak{u}(P))$ for every $P \in \operatorname{Pic}(S)$ (this is in fact twice the map defined in [Li93, (1.8)]).

Remark 4.10. The maps λ and ρ depend only on the moduli space M, so that ρ is the same for every polarization L which yields the same stability condition and hence the same moduli space (like in the situation of Lemma 3.22).

Whenever we need to highlight the dependence on the polarization, we will write $\rho = \rho_L\colon \operatorname{Pic}(S) \to \operatorname{Pic}(M_L)$ or $\rho = \rho_{\mathcal{C}}\colon \operatorname{Pic}(S) \to \operatorname{Pic}(M_{\mathcal{C}})$ if $\mathcal{C}$ is a chamber.

Corollary 4.11. *Let $L \in \operatorname{Pic}(S)$ be ample, $L \in \mathcal{E}$, and consider $\rho(L) \in \operatorname{Pic}(M_L)$. Then $\rho(L) = \mathcal{L}_1$ (notation as in Th. 4.5), and the following hold:*

(*a*) *$\rho(L)$ is semiample, and some positive multiple of $\rho(L)$ defines the map $\gamma\colon M_L \to M_L^\mu$ (see* (3.29)*);*
(*b*) *$\rho(L)$ is big if L lies in the interior of the cone $\mathcal{E}$, while $\rho(L) = 0$ if L lies on the boundary of $\mathcal{E}$;*
(*c*) *$\rho(L)$ is ample if and only if L belongs to a chamber.*
(*d*) *Suppose that L is in the interior of $\mathcal{E}$, and that it is contained in the walls $(2\ell_i+K_S)^\perp$ and $(2C_j + K_S)^\perp$, ℓ_i a (-1)-curve and C_j a conic, for $i = 1, \dots, r$ and $j = 1, \dots, s$, with $r \geq 0$ and $s \geq 0$. If $\Gamma \subset M_L$ is an irreducible curve, then $\rho(L) \cdot \Gamma = 0$ if and only if $\Gamma \subset P_{\ell_1} \cup \cdots \cup P_{\ell_r} \cup E_{C_1} \cup \cdots \cup E_{C_s}$.*

Proof. Rem. 4.9 and Th. 4.5 yield $\rho(L) = \mathcal{L}_1$, and (*a*). Then the remaining statements follow from Lemma 3.30. ∎

In particular, if $\mathcal{C} \subset \mathcal{E}$ is a chamber, it follows from (*c*) above that $\rho(L)$ is ample on $M_{\mathcal{C}}$ for every $L \in \mathcal{C}$.

Lemma 4.12. *Let $L, L' \in \mathcal{E}$ be ample line bundles, and let $\varphi\colon M_L \dashrightarrow M_{L'}$ be the natural birational map (see Cor. 3.25). Let $U \subseteq M_L$ and $U' \subseteq M_{L'}$ be the open subsets over which φ is an isomorphism. Then for every $P \in \operatorname{Pic}(S)$ we have*

$$\rho_L(P)_{|U} \cong \varphi^*\big(\rho_{L'}(P)_{|U'}\big).$$

If moreover L and L' belong to Π, then we have a commutative diagram:

$$\begin{tikzcd} & \mathrm{Pic}(S) \arrow[dl, "\rho_L"'] \arrow[dr, "\rho_{L'}"] & \\ \mathrm{Pic}(M_L) & & \mathrm{Pic}(M_{L'}). \arrow[ll, "\varphi^*"'] \end{tikzcd}$$

Proof. For simplicity set $M := M_L$ and $M' := M_{L'}$. We denote by $\pi_{A\times B,A}$ the projection $A \times B \to A$.

We can choose universal vector bundles $\mathcal{U}$ on $S \times M$ and $\mathcal{U}'$ on $S \times M'$ such that $\mathcal{U}_{|S\times U} = (\mathrm{Id}_S \times\varphi)^*\mathcal{U}'_{|S\times U'}$. Let F be a locally free sheaf on S. We have

$$\big((\pi_{S\times M,S})^*F\otimes\mathcal{U}\big)_{|S\times U} = (\pi_{S\times U,S})^*F\otimes\mathcal{U}_{|S\times U} = (\mathrm{Id}_S\times\varphi)^*\big((\pi_{S\times U',S})^*F\otimes\mathcal{U}'_{|S\times U'}\big)$$

and hence by base change:

$$\begin{aligned} \big(R^i(\pi_{S\times M,M})_*((\pi_{S\times M,S})^*F\otimes\mathcal{U})\big)_{|U} &= R^i(\pi_{S\times U,U})_*\big((\pi_{S\times M,S})^*F\otimes\mathcal{U}\big)_{|S\times U} \\ &= R^i(\pi_{S\times U,U})_*(\mathrm{Id}_S\times\varphi)^*\big((\pi_{S\times U',S})^*F\otimes\mathcal{U}'_{|S\times U'}\big) \\ \cong \varphi^*R^i(\pi_{S\times U',U'})_*\big((\pi_{S\times U',S})^*F\otimes\mathcal{U}'_{|S\times U'}\big) &= \varphi^*\big(R^i(\pi_{S\times M',M'})_*((\pi_{S\times M',S})^*F\otimes\mathcal{U}')\big)_{|U'}. \end{aligned}$$

Taking the determinant commutes with restricting to an open subset, therefore we get:

$$\big(\det(R^i(\pi_{S\times M,M})_*((\pi_{S\times M,S})^*F\otimes\mathcal{U})\big)_{|U} \cong \varphi^*\det\big(R^i(\pi_{S\times M,M})_*((\pi_{S\times M,S})^*F\otimes\mathcal{U}')\big)_{|U'},$$

and hence

$$\lambda_{\mathcal{U}}([F])_{|U} \cong \varphi^*\big(\lambda_{\mathcal{U}'}([F])_{|U'}\big),$$

which yields the first statement. Now if L and L' belong to Π, then by Cor. 3.25 the complements of U and U' both have codimension > 1, and we deduce that $\lambda_{\mathcal{U}}([F]) = \varphi^*(\lambda_{\mathcal{U}'}([F]))$ in $\mathrm{Pic}(M)$, which concludes the proof. ∎

Let $\mathcal{C} \subset \Pi$ be a chamber, and $L \in \overline{\mathcal{C}}$. Cor. 4.11 and Lemma 4.12 yield the following positivity property of $\rho(L)$ on $M_{\mathcal{C}}$.

Lemma 4.13. *Let $\mathcal{C} \subset \Pi$ be a chamber and $L \in \mathrm{Pic}(S)$ ample, $L \in \overline{\mathcal{C}}$. Suppose that L is contained in the walls $(2\ell_i + K_S)^\perp$ and $(2C_j + K_S)^\perp$, ℓ_i a (-1)-curve and C_j a conic, for $i = 1,\dots,r$ and $j = 1,\dots,s$, with $r \geq 0$ and $s \geq 0$. Suppose also that $\mathcal{C}$ is contained in $(2\ell_i + K_S)^{>0}$ for $i = 1,\dots,h$, and in $(2\ell_i + K_S)^{<0}$ for $i = h+1,\dots,r$, with $h \in \{0,\dots,r\}$.*

Then $M_{\mathcal{C}}$ contains the loci $P_{\ell_1},\dots,P_{\ell_h},Z_{\ell_{h+1}},\dots,Z_{\ell_r},E_{C_1}\dots,E_{C_s}$. Moreover $\rho(L) \in \mathrm{Pic}(M_{\mathcal{C}})$ is nef and big, and if $\Gamma \subset M_{\mathcal{C}}$ is an irreducible curve, then $\rho(L)\cdot\Gamma = 0$ if and only if $\Gamma \subset P_{\ell_1}\cup\cdots\cup P_{\ell_h}\cup Z_{\ell_{h+1}}\cup\cdots\cup Z_{\ell_r}\cup E_{C_1}\cup\cdots\cup E_{C_s}$.

Proof. Notice first of all that since $\mathcal{C} \subset \Pi$, we have $\mathcal{C} \subset (2C_j + K_S)^{>0}$ for $j = 1,\dots,s$, so the first statement follows from 3.17 and 3.18.

Let $\mathcal{C}'$ be the chamber such that $L \in \overline{\mathcal{C}'}$ and $\mathcal{C}' \subset (2\ell_i + K_S)^{>0}$, $\mathcal{C}' \subset (2C_j + K_S)^{>0}$ for every $i = 1,\dots,r$ and $j = 1,\dots,s$, as in Lemma 3.22(a). We consider both determinant maps

$$\rho_{\mathcal{C}}\colon \mathrm{Pic}(S) \longrightarrow \mathrm{Pic}(M_{\mathcal{C}}) \quad\text{and}\quad \rho_{\mathcal{C}'}\colon \mathrm{Pic}(S) \longrightarrow \mathrm{Pic}(M_{\mathcal{C}'}).$$

We have $M_L = M_{\mathcal{C}'}$ by Lemma 3.22(b). Thus by Cor. 4.11 (see also Rem. 4.10), $\rho_{\mathcal{C}'}(L) \in \mathrm{Pic}(M_{\mathcal{C}'})$ is nef and big, and has intersection zero precisely with curves contained in the loci P_{ℓ_i} and E_{C_j}, for $i = 1,\dots,r$ and $j = 1,\dots,s$.

Going from $\mathcal{C}'$ to $\mathcal{C}$, we cross the walls $(K_S+2\ell_i)^\perp$ for $i=h+1,\dots,r$; correspondingly (see 3.17) the natural birational map $\varphi\colon M_{\mathcal{C}'} \dashrightarrow M_{\mathcal{C}}$ is an isomorphism in codimension one and flips $P_{\ell_i}\cong\mathbb{P}^2$ to $Z_{\ell_i}\cong\mathbb{P}^1$, for every $i=h+1,\dots,r$.

Notice that in $M_{\mathcal{C}'}$ the loci $P_{\ell_1},\dots,P_{\ell_r},E_{C_1},\dots,E_{C_s}$ are pairwise disjoint (see Lemma 3.22(d)), so that $P_{\ell_1},\dots,P_{\ell_h},E_{C_1},\dots,E_{C_s}$ are contained in the open subset where φ is an isomorphism. Therefore $(\varphi^*)^{-1}(\rho_{\mathcal{C}'}(L))\in\operatorname{Pic}(M_{\mathcal{C}})$ is nef, and has intersection zero only with curves contained in $P_{\ell_1},\dots,P_{\ell_h},Z_{\ell_{h+1}},\dots,Z_{\ell_r},E_{C_1}\dots,E_{C_s}$. On the other hand $(\varphi^*)^{-1}(\rho_{\mathcal{C}'}(L))=\rho_{\mathcal{C}}(L)$ by Lemma 4.12, so we get the statement. ■

Lemma 4.14. *Let $\mathcal{C}\subset\Pi$ be a chamber. Then $\rho(-K_S)=-K_{M_{\mathcal{C}}}\in\operatorname{Pic}(M_{\mathcal{C}})$.*

Proof. Consider the moduli space $Y:=M_{-K_S}$, and the associated map $\rho_{-K_S}\colon \operatorname{Pic}(S)\to\operatorname{Pic}(Y)$. By Cor. 4.11 and [HL10, Th. 8.3.3] we have $\rho_{-K_S}(-K_S)=-K_Y$. On the other hand, since $-K_S\in\Pi$ and $\mathcal{C}\subset\Pi$, there is a pseudo-isomorphism $\varphi\colon M_{\mathcal{C}}\dashrightarrow Y$ (see Cor. 3.25), and $\varphi^*(-K_Y)=-K_{M_{\mathcal{C}}}$. The statement follows from Lemma 4.12. ■

4.15. The map ζ. Let us consider now the transpose map of ρ:

$$\zeta:=\rho^t\colon \mathcal{N}_1(M_{\mathcal{C}})\longrightarrow\mathcal{N}_1(S)=H^2(S,\mathbb{R}).$$

It follows from Lemmas 4.13 and 4.14 that we can determine the images, via ζ, of the lines in the exceptional loci of $M_{\mathcal{C}}$.

Corollary 4.16. *Let $\mathcal{C}\subset\Pi$ be a chamber such that $\overline{\mathcal{C}}$ has a facet on the wall $(2\ell+K_S)^\perp$, where ℓ is a (-1)-curve. Let $\Gamma_\ell\subset M_{\mathcal{C}}$ be an irreducible curve defined as follows:*

- *if $\mathcal{C}\subset(2\ell+K_S)^{>0}$, then Γ_ℓ is a line in $P_\ell\cong\mathbb{P}^2\subset M_{\mathcal{C}}$;*
- *if $\mathcal{C}\subset(2\ell+K_S)^{<0}$, then $\Gamma_\ell=Z_\ell\subset M_{\mathcal{C}}$.*

Then $\zeta(\Gamma_\ell)=\begin{cases}2\ell+K_S & \text{if } \mathcal{C}\subset(2\ell+K_S)^{>0};\\ -2\ell-K_S & \text{if } \mathcal{C}\subset(2\ell+K_S)^{<0}.\end{cases}$

Proof. If $\mathcal{C}\subset(2\ell+K_S)^{>0}$ we have $\mathcal{N}_{P_\ell/M_{\mathcal{C}}}\cong\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$ (see 3.17), thus $-K_{M_{\mathcal{C}}}\cdot\Gamma_\ell=1$. Similarly, if $\mathcal{C}\subset(2\ell+K_S)^{<0}$, we get $-K_{M_{\mathcal{C}}}\cdot\Gamma_\ell=-1$.

Let $L\in\operatorname{Pic}(S)$ be in the relative interior of the facet $\overline{\mathcal{C}}\cap(2\ell+K_S)^\perp$. The boundary of Π intersects the facet $\overline{\mathcal{C}}\cap(2\ell+K_S)^\perp$ along proper faces, hence L is in the interior of Π, and it is ample by Rem. 2.6. Consider $\rho(L)\in\operatorname{Pic}(M_{\mathcal{C}})$. Then by Lemma 4.13, $\rho(L)\cdot\Gamma_\ell=0$. Since the linear span of the facet is the hyperplane $(2\ell+K_S)^\perp$, this yields

$$\rho((2\ell+K_S)^\perp)\subseteq\Gamma_\ell^\perp\subset H^2(M_{\mathcal{C}},\mathbb{R}),$$

and dually $\zeta(\Gamma_\ell)=a(2\ell+K_S)$ for some $a\in\mathbb{R}$.

On the other hand we have $\rho(-K_S)=-K_{M_{\mathcal{C}}}$ by Lemma 4.14, so that

$$-K_{M_{\mathcal{C}}}\cdot\Gamma_\ell=\rho(-K_S)\cdot\Gamma_\ell=-K_S\cdot\zeta(\Gamma_\ell)=a(-K_S)\cdot(2\ell+K_S)=a.$$

This yields the statement. ■

With a similar proof, one shows the following.

Corollary 4.17. *Let $\mathcal{C}\subset\Pi$ be a chamber such that $\overline{\mathcal{C}}$ has a facet on the wall $(2C+K_S)^\perp$, where $C\subset S$ is a conic. Let $\Gamma_C\subset M_{\mathcal{C}}$ be a line in $E_C\cong\mathbb{P}^3\subset M_{\mathcal{C}}$. Then $\zeta(\Gamma_C)=2C+K_S$.*

Remark 4.18. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two chambers contained in Π, and consider the maps $\zeta_{\mathcal{C}_1}\colon \mathcal{N}_1(M_{\mathcal{C}_1}) \to H^2(S,\mathbb{R})$ and $\zeta_{\mathcal{C}_2}\colon \mathcal{N}_1(M_{\mathcal{C}_2}) \to H^2(S,\mathbb{R})$, with the obvious notation. Let $\varphi\colon M_{\mathcal{C}_1} \dashrightarrow M_{\mathcal{C}_2}$ be the natural birational map (see Cor. 3.25), $\Gamma \subset M_{\mathcal{C}_1}$ an irreducible curve contained in the open subset where φ is an isomorphism, and $\Gamma' := \varphi(\Gamma)$. Then $\zeta_{\mathcal{C}_1}(\Gamma) = \zeta_{\mathcal{C}_2}(\Gamma') \in H^2(S,\mathbb{R})$. Indeed if $L \in \mathrm{Pic}(S)$, using Lemma 4.12, we have:

$$\zeta_{\mathcal{C}_2}(\Gamma')\cdot L = \Gamma'\cdot \rho_{\mathcal{C}_2}(L) = \Gamma\cdot\varphi^*\rho_{\mathcal{C}_2}(L) = \Gamma\cdot\rho_{\mathcal{C}_1}(L) = \zeta_{\mathcal{C}_1}(\Gamma)\cdot L.$$

4.19. M_L is a Mori dream space. Mori dream spaces are projective varieties with an especially nice behaviour with respect to birational geometry and the Minimal Model Program (see [HK00] for more details). Fano varieties, and more generally log Fano varieties, are Mori dream spaces. We recall that a smooth projective variety M is log Fano if there exists an effective $\mathbb{Q}$-divisor Δ on M such that $-(K_M+\Delta)$ is ample, and the pair (M,Δ) is klt. We refer the reader to [KM98, Def. 2.34] for the notion of klt pair; since M is smooth, this is a condition on the singularities of Δ, which is automatically satisfied when $\Delta = 0$.

The following is another important consequence of Lemma 4.14; see [BMW14, Prop. 3.3] for a related result.

Proposition 4.20 (the Fano model Y)**.** *The moduli space $Y := M_{-K_S}$ is a smooth Fano 4-fold; we have $M_L = Y$ for every ample line bundle $L \in \mathcal{N}$.*

Proof. The moduli space Y is a smooth projective 4-fold by Cor. 3.5 and 3.27. Moreover $-K_Y = \rho(-K_S)$ by Lemma 4.14, so it is ample by Cor. 4.11(c), and Y is Fano. The second statement follows from Rem. 3.13 and Lemma 3.22. ∎

Corollary 4.21. *For every $L \in \mathrm{Pic}(S)$ ample, $L \in \mathcal{E}$, the moduli space M_L is log Fano and a Mori dream space.*

Via the relation with the blow-up of $\mathbb{P}^4$ in 8 general points (Th. 1.1), the Corollary above is already known by [CT06, Th. 1.3]; see also [AM16, Th. 1.3].

Proof. By Prop. 4.20, the moduli space $Y = M_{-K_S}$ is Fano. Let now $L \in \mathrm{Pic}(S)$ be ample, $L \in \mathcal{E}$. Consider the natural birational map $\varphi\colon Y \dashrightarrow M_L$ (see Cor. 3.25). Since $-K_S \in \Pi$, the map φ is contracting. Thus M_L is log Fano by [PS09, Lemma 2.8]. Finally, log Fano varieties are Mori dream spaces by [BCHM10, Cor. 1.3.2]. ∎

4.22. Automorphisms. Let $\mathrm{Aut}(S)_L$ be the subgroup of $\mathrm{Aut}(S)$ of automorphisms fixing L in $\mathrm{Pic}(S)$. There is a natural homomorphism $\psi\colon \mathrm{Aut}(S)_L \to \mathrm{Aut}(M_L)$ defined as follows:

$$\psi(f)([F]) = [(f^{-1})^*F] \qquad \text{for every } f \in \mathrm{Aut}(S)_L \text{ and } [F] \in M_L.$$

Proposition 4.23. *The determinant map $\rho\colon \mathrm{Pic}(S) \to \mathrm{Pic}(M_L)$ is equivariant for the action of $\mathrm{Aut}(S)_L$, namely for every $f \in \mathrm{Aut}(S)_L$ and $P \in \mathrm{Pic}(S)$ we have:*

$$\rho(f^*P) = \psi(f)^*(\rho(P)) \in \mathrm{Pic}(M_L).$$

Proof. For simplicity set $M := M_L$. Let $f \in \mathrm{Aut}(S)_L$, $g := \psi(f)$ the induced automorphism of M, and $h := (f,g)$ acting diagonally on $S \times M$.

Let $\mathcal{U}$ be a universal vector bundle over M. Let us check that $h^*\mathcal{U}$ is again a universal family. To this end, we compute the restriction over $S \times [F]$, for $[F] \in M$:

$$(h^*\mathcal{U})_{|S\times[F]} = (h_{|S\times[F]})^*(\mathcal{U}_{|S\times g([F])}) \cong f^*(f^{-1})^*F \cong F.$$

We have a commutative diagram

$$\begin{tikzcd} S \times M \arrow[r, "h"] \arrow[d, "\pi_M"] & S \times M \arrow[d, "\pi_M"] \\ M \arrow[r, "g"] & M, \end{tikzcd}$$

where the horizontal maps are isomorphism, hence for every locally free sheaf G on $S \times M$ and every $i \geq 0$ we have

$$R^i(\pi_M)_*(h^*G) \cong g^*(R^i(\pi_M)_*G),$$

thus in $\mathrm{Pic}(M)$:

$$\det\big((\pi_M)_![h^*G]\big) \cong g^*\big(\det((\pi_M)_![G])\big).$$

Therefore given a locally free sheaf V on S we have, by (4.4):

$$\begin{aligned} \lambda_{h^*\mathcal{U}}([f^*V]) &= \det\big((\pi_M)_![\pi_S^* f^*V \otimes h^*\mathcal{U}]\big) \cong \det\big((\pi_M)_![h^*(\pi_S^*V \otimes \mathcal{U})]\big) \\ &\cong g^*\big(\det((\pi_M)_![\pi_S^*V \otimes \mathcal{U}])\big) = g^*\lambda_{\mathcal{U}}([V]). \end{aligned}$$

The pullback via f of vector bundles on S induces a natural group automorphism $f^*\colon K(S) \to K(S)$, which preserves the subgroup $\mathfrak{f}^\perp$. Thus the equality above yields $\lambda(f^*\mathfrak{c}) = g^*\lambda(\mathfrak{c})$ for every $\mathfrak{c} \in \mathfrak{f}^\perp$. To conclude, we remark that $\mathfrak{u}(f^*P) = f^*(\mathfrak{u}(P))$ for every $P \in \mathrm{Pic}(S)$ (see (4.7)). ■

5. The relation with the blow-up X of $\mathbb{P}^4$ at 8 points – identification of the stability fan with the Mori chamber decomposition

5.1. Polarizations in the linear span of h and $-K_S$. In this preliminary subsection, for a fixed cubic h, we describe the chambers that intersect the plane in $H^2(S, \mathbb{R})$ spanned by h and $-K_S$ (see [Muk05, p. 9]). This will be needed to describe the birational maps that relate the associated moduli spaces. Notation as in 2.2; set

$$h' := \iota_S^* h = 17h - 6e = -6K_S - h,$$

so that h' is another cubic (see Lemma 2.16). Since both h and h' are nef and non-ample, the cone $\mathrm{Nef}(S)$ intersects the plane in $H^2(S, \mathbb{R})$ spanned by h and $-K_S$ in the cone $\langle h, h' \rangle$. For every $t \in (0,1) \cap \mathbb{Q}$ consider the ample class:

$$L_t := (1-t)h + th' = -6tK_S + (1-2t)h.$$

Lemma 5.2. *We have:*

$$L_t \in \mathcal{E} \iff t \in \left[\frac{1}{32}, \frac{31}{32}\right], \qquad L_t \in \Pi \iff t \in \left[\frac{1}{20}, \frac{19}{20}\right], \qquad L_t \in \mathcal{N} \iff t \in \left[\frac{1}{4}, \frac{3}{4}\right],$$

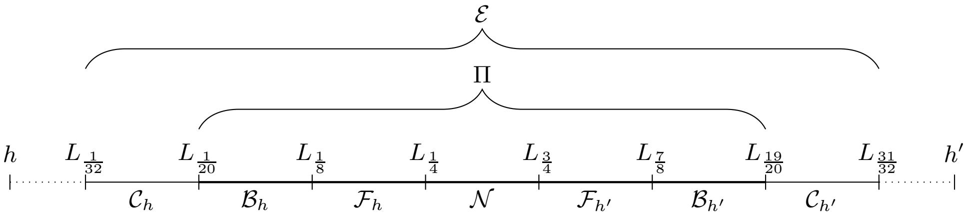
and L_t belongs to a wall if and only if $t \in \left\{\frac{1}{32}, \frac{1}{20}, \frac{1}{8}, \frac{1}{4}, \frac{3}{4}, \frac{7}{8}, \frac{19}{20}, \frac{31}{32}\right\}$. More precisely:

- $L_{\frac{1}{32}} = \frac{3}{16}(-K_S + 5h) \in (2h + K_S)^\perp$
- $L_{\frac{1}{20}} = \frac{3}{10}(-K_S + 3h) \in (2C_i + K_S)^\perp$, $i = 1, \ldots, 8$
- $L_{\frac{1}{8}} = \frac{3}{4}(-K_S + h) \in (2\ell_{ij} + K_S)^\perp$, $1 \leq i < j \leq 8$
- $L_{\frac{1}{4}} = \frac{1}{2}(-3K_S + h) \in (2e_i + K_S)^\perp$, $i = 1, \ldots, 8$
- $L_{\frac{3}{4}} \in (2\ell + K_S)^\perp$ *when ℓ is a (-1)-curve such that $\ell \sim 6h - 2e - e_i$, for $i = 1, \ldots, 8$*
- $L_{\frac{7}{8}} \in (2\ell + K_S)^\perp$ *when ℓ is a (-1)-curve such that $\ell \sim 5h - 2e + e_i + e_j$, with $1 \leq i < j \leq 8$*

- $L_{\frac{19}{20}} \in (2C'_i + K_S)^{\perp}$, *$C'_i$ the conic such that $C'_i \sim 11h - 4e + e_i$, for $i = 1, \ldots, 8$*
- $L_{\frac{31}{32}} \in (2h' + K_S)^{\perp}$

and no other wall contains some L_t.

FIGURE 5.3. Chambers intersecting the plane spanned by h and $-K_S$



We introduce some notation for the 7 chambers containing some L_t (see Fig. 5.3). First we notice that $L_{\frac{1}{32}} = \frac{3}{16}(-K_S + 5h)$ is in the relative interior of the facet τ_h of $\mathcal{E}$ cut by $(2h+K_S)^{\perp}$ (see 3.19), so that L_t belongs to the outer chamber $\mathcal{C}_h$ for $t \in (\frac{1}{32}, \frac{1}{20})$. In this chamber the moduli space is isomorphic to $\mathbb{P}^4$, as shown in Prop. 3.20.

Notation 5.4 (the chambers $\mathcal{B}_h$ and $\mathcal{F}_h$)**.** Given a cubic h, we denote by $\mathcal{B}_h$ the chamber containing $-K_S + 2h = \frac{7}{3}L_{\frac{1}{14}}$, and by $\mathcal{F}_h$ the chamber containing $-2K_S + h = \frac{5}{3}L_{\frac{1}{5}}$ (see Fig. 5.3).

Notice that $\mathcal{B}_h \subset \Pi$. It is not difficult to see that $-K_S + 3h$ and $-K_S + h$ generate one-dimensional faces of $\overline{\mathcal{B}}_h$, and that the hyperplanes $(2C_i + K_S)^{\perp}$ and $(2\ell_{jk} + K_S)^{\perp}$ intersect $\overline{\mathcal{B}}_h$ along a facet, for every $i = 1, \ldots, 8$ and $1 \le j < k \le 8$.

Proof of Lemma 5.2. We use Cor. 3.11. If $\tilde{h}$ is a cubic, and $m := h \cdot \tilde{h}$, we have $L_t \cdot (2\tilde{h} + K_S) = 4(9-m)t + 2m - 3$. Using Lemma 2.16 and (2.5), we get the statement for $\mathcal{E}$ and the walls $(2\tilde{h} + K_S)^{\perp}$. The computation is completely analogous for Π and the walls $(2C + K_S)^{\perp}$, and for $\mathcal{N}$ and the walls $(2\ell + K_S)^{\perp}$. ∎

5.5. The blow-up X of $\mathbb{P}^4$ in 8 points. Let h be a cubic in S, and recall from 2.21 that we associate to (S, h) a blow-up $X = X_{(S,h)}$ of $\mathbb{P}^4$ in 8 points in general linear position. The following is a refined version of Th. 1.1; notation as in 2.2 and 2.22.

Theorem 5.6 ([Muk05], p. 9)**.** *Let S be a del Pezzo surface of degree 1 and h a cubic in S. Then there is an isomorphism $f\colon M_{S,\mathcal{B}_h} \to X_{(S,h)}$, and moreover:*

$$f(E_{C_i}) = E_i \quad \text{and} \quad f(Z_{\ell_{jk}}) = L_{jk} \quad \text{for every } i = 1, \ldots, 8 \text{ and } 1 \le j < k \le 8.$$

We can see from Fig. 5.3 and Lemma 5.2 that, to go from the chamber $\mathcal{C}_h$ to $\mathcal{B}_h$, one has to cross the walls $(2C_i + K_S)^{\perp}$ for $i = 1, \ldots, 8$; this gives the blow-up map $M_{\mathcal{B}_h} \to M_{\mathcal{C}_h} \cong \mathbb{P}^4$, see Prop. 3.20 and 3.18.

Corollary 5.7. *Let X be a blow-up of $\mathbb{P}^4$ at 8 general points. Then there exists a smooth del Pezzo surface S of degree one, and a cubic h on S, such that $X \cong M_{S,\mathcal{B}_h}$.*

From now on we will identify $M_{\mathcal{B}_h}$ and X via the isomorphism f.

Proposition 5.8. *The maps $\rho\colon H^2(S,\mathbb{R}) \to H^2(X,\mathbb{R})$ and $\zeta\colon \mathcal{N}_1(X) \to H^2(S,\mathbb{R})$ are isomorphisms of real vector spaces, and we have (notation as in 2.2 and 2.22):*

$$\rho(h) = \sum_{j=1}^{8} E_j - H, \qquad \rho(e_i) = -2E_i + \sum_{j=1}^{8} E_j - H \quad \text{for } i = 1,\dots,8,$$
$$\zeta(h) = e - h = 2h + K_S, \qquad \zeta(e_i) = -2e_i + e - h \quad \text{for } i = 1,\dots,8.$$

Proof. The cone $\overline{\mathcal{B}}_h$ is contained in Π, intersects the walls $(2C_i+K_S)^\perp$ and $(2\ell_{ij}+K_S)^\perp$ along a facet, and $\mathcal{B}_h \subset (2\ell_{ij} + K_S)^{<0}$ (see 5.4). Thus by Th. 5.6 and Cor. 4.17 and 4.16 we have:

$$\zeta(e_i) = 2C_i+K_S = -2e_i+e-h, \quad \zeta(h-e_i-e_j) = \zeta(L_{ij}) = -2\ell_{ij}-K_S = h-e+2e_i+2e_j$$

which easily yields the statement. ∎

5.9. Relating the intersection product in S to Dolgachev's pairing in X. We recall that $H^2(X,\mathbb{Z})$ has a natural pairing, related to the action of the symmetric group S_8 and of the standard Cremona map, see [Dol83, DO88] and also [Muk01]. This pairing is defined by imposing that $H, E_1, \dots, E_8$ is an orthogonal basis, $H^2 = 3$, and $E_i^2 = -1$ for $i = 1,\dots,8$ (notation as in 2.22). The sublattice $K_X^\perp$ is an E_8-lattice; we denote by $W_X \cong W(E_8)$ its Weyl group of automorphisms.

Using Prop. 5.8, we see that $\rho^{-1}\colon H^2(X,\mathbb{R}) \to H^2(S,\mathbb{R})$ coincides with $\frac{1}{2}$ the linear map φ defined by Mukai in [Muk05, p. 7]. In particular, we get the following.

Lemma 5.10 ([Muk05], p. 7)**.** *Set $\tilde\rho := \frac{1}{2}\rho\colon H^2(S,\mathbb{R}) \to H^2(X,\mathbb{R})$. We have $\tilde\rho(K_S^\perp \cap H^2(S,\mathbb{Z})) = K_X^\perp \cap H^2(X,\mathbb{Z})$, and the restriction $\tilde\rho\colon K_S^\perp \cap H^2(S,\mathbb{Z}) \to K_X^\perp \cap H^2(X,\mathbb{Z})$ is an isometry. Moreover there is a group isomorphism $\phi\colon W_S \to W_X$ such that ρ and $\tilde\rho$ are equivariant with respect to ϕ.*

Remark 5.11. The relation between integral points in $H^2(S,\mathbb{R})$ and $H^2(X,\mathbb{R})$ via $\tilde\rho$ is the following:

$$\tilde\rho^{-1}(H^2(X,\mathbb{Z})) = \{L \in H^2(S,\mathbb{Z}) \,|\, K_S \cdot L \text{ is even}\}.$$

Indeed, we have $\tilde\rho(K_S) = \frac{1}{2}K_X$ by Lemma 4.14. Let $L \in H^2(S,\mathbb{R})$ and set $m := K_S \cdot L$; we have $L - mK_S \in K_S^\perp$ and $\tilde\rho(L) = \tilde\rho(L - mK_S) + \frac{1}{2}mK_X$.

If L is integral and m is even, then $\tilde\rho(L - mK_S) \in H^2(X,\mathbb{Z})$ by Lemma 5.10, so $\tilde\rho(L) \in H^2(X,\mathbb{Z})$.

Conversely, suppose that $\tilde\rho(L)$ is integral. It is not difficult to check that K_X and $K_X^\perp \cap H^2(X,\mathbb{Z})$ generate $H^2(X,\mathbb{Z})$ as a group, and since $\tilde\rho(L - mK_S) \in K_X^\perp$, we see that m must be an even integer and $\tilde\rho(L - mK_S)$ must be integral. By Lemma 5.10, $L - mK_S$ is integral, and hence L is integral.

5.12. Cones of divisors and chamber decompositions. We recall that if M is a Mori dream space, then in $\mathcal{N}^1(M)$ the convex cones $\operatorname{Eff}(M)$, $\operatorname{Mov}(M)$, and $\operatorname{Nef}(M)$ (respectively of effective, movable, and nef divisors) are all closed and polyhedral. Moreover there is a fan $\operatorname{MCD}(M)$, supported on $\operatorname{Eff}(M)$, called the **Mori chamber decomposition** (see [HK00, Oka16]); the cones of maximal dimension of the fan are in bijection with contracting birational maps $\varphi\colon M \dashrightarrow M'$ (up to isomorphism of the target), where M' is projective, normal and $\mathbb{Q}$-factorial. The cone corresponding to φ is $\varphi^*\operatorname{Nef}(M') + \langle E_1,\dots,E_r\rangle$, where $E_1,\dots,E_r \subset M$ are the exceptional prime divisors of

φ. In particular, $\varphi\colon M \dashrightarrow M'$ is a pseudo-isomorphism if and only if the corresponding cone is contained in $\mathrm{Mov}(M)$.

Let $\mathcal{C} \subset \Pi$ be a chamber. After Cor. 4.21, we know that $M_{\mathcal{C}}$ is a Mori dream space. Applying the previous results, we relate the Mori chamber decomposition $\mathrm{MCD}(M_{\mathcal{C}})$ to the stability fan $\mathrm{ST}(S)$ in S (see 3.12), via the determinant map $\rho\colon H^2(S,\mathbb{R}) \to H^2(M_{\mathcal{C}},\mathbb{R})$; this is our main result in this section.

Theorem 5.13. *Let $\mathcal{C}$ be a chamber contained in Π. We have the following:*

(*a*) *the determinant map $\rho\colon H^2(S,\mathbb{R}) \to H^2(M_{\mathcal{C}},\mathbb{R})$ is an isomorphism;*
(*b*) *$\rho(\overline{\mathcal{C}}) = \mathrm{Nef}(M_{\mathcal{C}})$, $\rho(\Pi) = \mathrm{Mov}(M_{\mathcal{C}})$, and $\rho(\mathcal{E}) = \mathrm{Eff}(M_{\mathcal{C}})$;*
(*c*) *ρ yields an isomorphism between the stability fan $\mathrm{ST}(S)$ in $H^2(S,\mathbb{R})$ (see 3.12), and the fan $\mathrm{MCD}(M_{\mathcal{C}})$ in $H^2(M_{\mathcal{C}},\mathbb{R})$ given by the Mori chamber decomposition.*

Before proving Th. 5.13, we need a preliminary property.

Definition 5.14 (the divisor E_C)**.** Let C be a conic and $\mathcal{C} \subset \Pi$ a chamber. We generalise Def. 3.16 and define a fixed prime divisor $E_C \subset M_{\mathcal{C}}$, as follows.

If $\overline{\mathcal{C}} \cap (2C+K_S)^{\perp}$ is a facet of $\overline{\mathcal{C}}$ and intersects the ample cone of S, then by 3.18 $M_{\mathcal{C}}$ contains the divisor $E_C \cong \mathbb{P}^3$, which is the exceptional divisor of a blow-up of a point.

In general, we choose a chamber $\mathcal{C}' \subset \Pi$ such that $\overline{\mathcal{C}'} \cap (2C+K_S)^{\perp}$ is a facet of $\overline{\mathcal{C}'}$ and intersects the ample cone of S, so that $M_{\mathcal{C}'}$ contains E_C as an exceptional divisor. Let $\varphi\colon M_{\mathcal{C}} \dashrightarrow M_{\mathcal{C}'}$ be the natural pseudo-isomorphism (see Cor. 3.25). Then we still denote by $E_C \subset M_{\mathcal{C}}$ the strict transform of E_C under φ.

Lemma 5.15. *In the setting of Def. 5.14, we have $\rho(C) = 2E_C$ in $H^2(M_{\mathcal{C}},\mathbb{R})$.*

Proof. There exists a cubic h such that $C = C_1$ (notation as in 2.2). Consider the chamber $\mathcal{B}_h \subset \Pi$ and $\rho_{\mathcal{B}_h}\colon H^2(S,\mathbb{R}) \to H^2(M_{\mathcal{B}_h},\mathbb{R})$. By Prop. 5.8 and Th. 5.6 we have $\rho_{\mathcal{B}_h}(C) = \rho_{\mathcal{B}_h}(h-e_1) = 2E_1 = 2E_C$, and this yields $\rho_{\mathcal{C}}(C) = 2E_C$ by Lemma 4.12. ∎

Proof of Th. 5.13. We first show (*a*) and that $\rho(\mathcal{E}) = \mathrm{Eff}(M_{\mathcal{C}})$. Let h be a cubic, and consider the chamber $\mathcal{B}_h \subset \Pi$. By Cor. 3.25 there is a pseudo-isomorphism $\varphi\colon M_{\mathcal{C}} \dashrightarrow M_{\mathcal{B}_h}$, and $\varphi^*\colon H^2(M_{\mathcal{B}_h},\mathbb{R}) \to H^2(M_{\mathcal{C}},\mathbb{R})$ is an isomorphism which preserves the effective cone. Thus, by Lemma 4.12, it is enough to prove the statements for the chamber $\mathcal{B}_h$. Now $\rho_{\mathcal{B}_h}$ is isomorphism by Prop. 5.8, so we have (*a*).

Let $X = X_{(S,h)}$. The cone $\mathrm{Eff}(X)$ is generated by the orbit $W_X \cdot E_1$, see [CT06, Th. 2.7]. On the other hand $\mathcal{E}$ is generated by conics, namely by the orbit $W_S \cdot C_1$ (notation as in 2.2). Since $\rho_{\mathcal{B}_h}(C_1) = 2E_1$ by Lemma 5.15, we have $\rho_{\mathcal{B}_h}(\mathcal{E}) = \mathrm{Eff}(X)$ by Lemma 5.10. Thus $\rho_{\mathcal{C}}(\mathcal{E}) = \mathrm{Eff}(M_{\mathcal{C}})$.

Let now $\mathcal{C}' \subset \mathcal{E}$ be a chamber and $\varphi'\colon M_{\mathcal{C}} \dashrightarrow M_{\mathcal{C}'}$ the natural birational map (see Cor. 3.25). Since $\mathcal{C} \subset \Pi$, the map φ' is contracting, so it determines a Mori chamber in $H^2(M_{\mathcal{C}},\mathbb{R})$. Let us show that $\rho_{\mathcal{C}}(\mathcal{C}')$ is contained in this Mori chamber, namely that:

$$\rho_{\mathcal{C}}(\mathcal{C}') \subseteq (\varphi')^*\,\mathrm{Nef}(M_{\mathcal{C}'}) + \langle E_1,\dots,E_r\rangle \quad \text{in } H^2(M_{\mathcal{C}},\mathbb{R}), \tag{5.16}$$

where $E_1,\dots,E_r \subset M_{\mathcal{C}}$ are the exceptional prime divisors of φ'.

Let $L \in \mathrm{Pic}(S)$, $L \in \mathcal{C}'$. By Cor. 4.11(*c*), $\rho_{\mathcal{C}'}(L)$ is ample on $M_{\mathcal{C}'}$, so $(\varphi')^*(\rho_{\mathcal{C}'}(L)) \in (\varphi')^*\,\mathrm{Nef}(M_{\mathcal{C}'})$. On the other hand, if $U \subseteq M_{\mathcal{C}}$ is the open subset where φ' is an isomorphism, we have $\rho_{\mathcal{C}}(L)_{|U} \cong (\varphi')^*(\rho_{\mathcal{C}'}(L))_{|U}$ by Lemma 4.12, and hence $\rho_{\mathcal{C}}(L) \cong (\varphi')^*(\rho_{\mathcal{C}'}(L)) + \sum_{i=1}^r a_iE_i$, where $a_i \in \mathbb{Z}$. We show that $a_i > 0$ for every $i = 1,\dots,r$, which yields (5.16).

Let $L_0 \in \mathcal{C}$, and consider the segment in $H^2(S,\mathbb{R})$ joining L_0 and L. By varying the polarization along the segment, we factor $\varphi' \colon M_{\mathcal{C}} \dashrightarrow M_{\mathcal{C}'}$ in flips and blow-downs as described in 3.17 and 3.18:

$$\begin{tikzcd} M_{\mathcal{C}} = M_{\mathcal{C}_0} \arrow[r, dashed, "\sigma_1"'] \arrow[rrrr, dashed, bend left, "\varphi'"] & M_{\mathcal{C}_1} \arrow[r, dashed] & \cdots\cdots \arrow[r, dashed] & M_{\mathcal{C}_{t-1}} \arrow[r, dashed, "\sigma_t"'] & M_{\mathcal{C}_t} = M_{\mathcal{C}'} \end{tikzcd}$$

where $t \geq r$ and the sequence contains precisely r blow-downs $\sigma_{i_1}, \dots, \sigma_{i_r}$, with exceptional divisors (the transforms of) $E_1, \dots, E_r$.

When $\sigma_i \colon M_{\mathcal{C}_{i-1}} \dashrightarrow M_{\mathcal{C}_i}$ is a flip, we have $\sigma_i^*(\rho_{\mathcal{C}_i}(L)) = \rho_{\mathcal{C}_{i-1}}(L)$ by Lemma 4.12.

Consider a blow-up $\sigma_{i_j} \colon M_{\mathcal{C}_{i_j-1}} \to M_{\mathcal{C}_{i_j}}$. Again by Lemma 4.12 we have $\rho_{\mathcal{C}_{i_j-1}}(L) = \sigma_{i_j}^* \rho_{\mathcal{C}_{i_j}}(L) + b_j E_j$, with $b_j \in \mathbb{Z}$. Let $\Gamma_j \subset E_j \cong \mathbb{P}^3 \subset M_{\mathcal{C}_{i_j-1}}$ be a line, and let $C_j \subset S$ be the conic such that σ_{i_j} corresponds to crossing the wall $(2C_j + K_S)^{\perp}$; by construction $L \cdot (2C_j + K_S) < 0$. Then by Cor. 4.17 and the projection formula we have

$$L \cdot (2C_j + K_S) = L \cdot \zeta_{\mathcal{C}_{i_j-1}}(\Gamma_j) = \rho_{\mathcal{C}_{i_j-1}}(L) \cdot \Gamma_j = \left(\sigma_{i_j}^* \rho_{\mathcal{C}_{i_j}}(L) + b_j E_j\right) \cdot \Gamma_j = -b_j,$$

thus $b_j > 0$. This shows that $a_i > 0$ for $i = 1, \dots, r$.

Let us show that different maximal cones in $\mathrm{ST}(S)$ yield different Mori chambers. Let $\mathcal{C}'' \subset \mathcal{E}$ be another chamber, and $\varphi'' \colon M_{\mathcal{C}} \dashrightarrow M_{\mathcal{C}''}$ the associated contracting birational map. If φ' and φ'' have the same Mori chamber, then there exists an isomorphism $\chi \colon M_{\mathcal{C}'} \to M_{\mathcal{C}''}$ such that $\varphi'' = \chi \circ \varphi'$. Then, by Cor. 3.25, there exists an open subset $U' \subseteq M_{\mathcal{C}'}$ such that $\mathrm{codim}(M_{\mathcal{C}'} \smallsetminus U') \geq 2$, and $\chi([F]) = [F]$ for every $[F] \in U'$. By comparing two universal families for $M_{\mathcal{C}'}$ and $M_{\mathcal{C}''}$, we see that $\chi([F]) = [F]$ for every $[F] \in M_{\mathcal{C}'}$. Therefore $\mathcal{C}'$ and $\mathcal{C}''$ yield the same stability condition, and hence $\mathcal{C}' = \mathcal{C}''$.

Thus the image of a chamber $\mathcal{C} \subset \mathcal{E}$ is contained in a unique Mori chamber. Then, using (5.16) and $\rho_{\mathcal{C}}(\mathcal{E}) = \mathrm{Eff}(M_{\mathcal{C}})$, it is not difficult to see that

$$\rho_{\mathcal{C}}(\overline{\mathcal{C}'}) = (\varphi')^* \operatorname{Nef} M_{\mathcal{C}'} + \langle E_1, \dots, E_r \rangle$$

for every chamber $\mathcal{C}' \subset \mathcal{E}$, and hence (c). For $\mathcal{C}' = \mathcal{C}$ we get $\rho_{\mathcal{C}}(\overline{\mathcal{C}}) = \mathrm{Nef}(M_{\mathcal{C}})$. Finally, by Cor. 3.25 $\mathcal{C}' \subset \Pi$ if and only if φ' is a pseudo-isomorphism, if and only if $\rho_{\mathcal{C}}(\mathcal{C}') \subset \mathrm{Mov}(M_{\mathcal{C}})$, so we get (b). ■

Proof of Th. 1.3. It is a direct consequence of Lemma 3.22 and Th. 5.13. ■

5.17. From the blow-up X to the Fano model Y. Let h be a cubic, and let us go back to the chambers intersecting the plane spanned by h and $-K_S$, described in 5.1. We have a diagram:

$$\begin{tikzcd} X \cong M_{\mathcal{B}_h} \arrow[r, dashed] \arrow[rr, dashed, bend left, "\xi"] & M_{\mathcal{F}_h} \arrow[r, dashed] & Y = M_{-K_S}, \end{tikzcd}$$

where the birational maps are the natural ones (see Cor. 3.25), and we denote by $\xi \colon X \dashrightarrow Y$ the composition $X \stackrel{f^{-1}}{\to} M_{\mathcal{B}_h} \dashrightarrow Y$. We will occasionally write $\xi_h \colon X_h \dashrightarrow Y$, when we need to stress that X_h and ξ_h depend on the chosen cubic h (while Y does not). Notation as in 2.22.

Lemma 5.18. *The birational map $\xi \colon X \dashrightarrow Y$ is the composition of* 36 *(K-positive) flips: first the flips of L_{ij} for $1 \leq i < j \leq 8$, and then the flips of Γ_k for $k = 1, \dots, 8$.*

There is a commutative diagram:

$$\begin{tikzcd} & \widehat{X} \arrow[dl] \arrow[dr] & \\ X \arrow[rr, dashed, "\xi"] & & Y \end{tikzcd}$$

where $\widehat{X} \to X$ is the blow-up of the curves L_{ij} and Γ_k, with every exceptional divisor isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$ with normal bundle $\mathcal{O}(-1,-1)$, and $\widehat{X} \to Y$ is the blow-up of 36 *pairwise disjoint smooth rational surfaces.*

Proof. Firstly, to go from the chamber $\mathcal{B}_h$ to $\mathcal{F}_h$, we have to cross the $\binom{8}{2} = 28$ walls $(2\ell_{ij} + K_S)^\perp$ (see Fig. 5.3 and Lemma 5.2). Moreover, by Th. 5.6, the loci $Z_{\ell_{ij}}$ correspond to the curves $L_{ij} \subset X$. Therefore the map $X \dashrightarrow M_{\mathcal{F}_h}$ is the composition of 28 flips, each replacing L_{ij} with $P_{\ell_{ij}} \cong \mathbb{P}^2$.

Secondly, to go from the chamber $\mathcal{F}_h$ to $\mathcal{N}$, we have to cross the 8 walls $(2e_k + K_S)^\perp$ (see again Fig. 5.3 and Lemma 5.2). Thus the second map $M_{\mathcal{F}_h} \dashrightarrow Y$ is a composition of 8 flips, each replacing $Z_{e_k} \cong \mathbb{P}^1$ with $P_{e_k} \cong \mathbb{P}^2$, for $k = 1, \ldots, 8$.

Let $k \in \{1, \ldots, 8\}$. We claim that $Z_{e_k} \subset M_{\mathcal{F}_h}$ is the transform of $\Gamma_k \subset X$. Indeed, consider the maps $\zeta_{\mathcal{F}_h} \colon \mathcal{N}_1(M_{\mathcal{F}_h}) \to H^2(S, \mathbb{R})$ and $\zeta_{\mathcal{B}_h} \colon \mathcal{N}_1(X) \to H^2(S, \mathbb{R})$. The curve Γ_k is contained in the open subset where the birational map $X \dashrightarrow M_{\mathcal{F}_h}$ is an isomorphism, and if we denote by $\Gamma'_k \subset M_{\mathcal{F}_h}$ the transform of Γ_k, Rem. 4.18 and Prop. 5.8 yield that

$$\zeta_{\mathcal{F}_h}(\Gamma'_k) = \zeta_{\mathcal{B}_h}(\Gamma_k) = \zeta_{\mathcal{B}_h}(4h - e + e_k) = -2e_k - K_S.$$

On the other hand, $\overline{\mathcal{F}}_h$ intersects $(2e_k + K_S)^\perp$ along a wall, and by Cor. 4.16 we also have $\zeta_{\mathcal{F}_h}(Z_{e_k}) = -2e_k - K_S$. Since $\zeta_{\mathcal{F}_h}$ is an isomorphism by Th. 5.13(a), we deduce that Γ'_k and Z_{e_k} are numerically equivalent; the class of Z_{e_k} generates an extremal ray of $\mathrm{NE}(M_{\mathcal{F}_h})$ whose locus is just Z_{e_k}, hence $\Gamma'_k = Z_{e_k}$.

Finally, the factorization of ξ as a sequence of smooth blow-ups and blow-downs follows from the explicit description of the flips in 3.17. ■

Corollary 5.19. *Let X be the blow-up of $\mathbb{P}^4$ at 8 general points. If $C \subset X$ is an irreducible curve with $-K_X \cdot C \leq 0$, then either $C = L_{ij}$ or $C = \Gamma_k$ for some $1 \leq i < j \leq 8$, $k = 1, \ldots, 8$.*

Proof. This follows from Lemma 5.18 and [Cas17, Lemma 2.8(2)]. ■

6. Geometry of the Fano model Y

Let S be a del Pezzo surface of degree one; in this section we study the Fano 4-fold $Y = M_{S,-K_S}$ (see Prop. 4.20).

Proposition 6.1 (numerical invariants)**.** *We have $b_2(Y) = 9$, $b_3(Y) = 0$, $h^{2,2}(Y) = b_4(Y) = 45$, $(-K_Y)^4 = 13$, and $h^0(Y, -K_Y) = 6$. Moreover Y has index one.*

Proof. Let h be a cubic in S, and let us consider $X = X_{(S,h)}$ (see 2.21) and the birational map $\xi \colon X \dashrightarrow Y$ (see 5.17). Since X is a blow-up of $\mathbb{P}^4$ in 8 points, one computes that $b_2(X) = b_4(X) = h^{2,2}(X) = 9$, $b_3(X) = 0$, and $(-K_X)^4 = 625 - 8 \cdot 81 = -23$. By the explicit description of ξ as a sequence of smooth blow-ups given in Lemma 5.18, this yields the Betti and Hodge numbers of Y (see for instance [Voi02, Th. 7.31]). Moreover $(-K_Y)^4 = (-K_X)^4 + 36 = 13$, and $h^0(Y, -K_Y) = h^0(\mathbb{P}^4, -K_{\mathbb{P}^4}) - 15 \cdot 8 = 6$ (see for instance [Cas17, Cor. 3.10 and Prop. 3.3]). Finally, it is not difficult to see that Y contains curves of anticanonical degree 1, for instance a line in a smooth rational surface in the indeterminacy locus of $\xi^{-1}\colon Y \dashrightarrow X$. Therefore Y has index one. ■

We are now going to describe the relevant cones of curves and divisors in $\mathcal{N}_1(Y)$ and $H^2(Y,\mathbb{R})$, using the following direct consequence of Th. 5.13.

Corollary 6.2. *The determinant map $\rho\colon H^2(S,\mathbb{R}) \to H^2(Y,\mathbb{R})$ yields an isomorphism between:*

$$\mathcal{N} \subset \Pi \subset \mathcal{E} \subset H^2(S,\mathbb{R}) \quad \textit{and} \quad \mathrm{Nef}(Y) \subset \mathrm{Mov}(Y) \subset \mathrm{Eff}(Y) \subset H^2(Y,\mathbb{R}).$$

Dually, the map $\zeta\colon \mathcal{N}_1(Y) \to H^2(S,\mathbb{R})$ yields an isomorphism between:

$$\mathrm{Mov}_1(Y) \subset \mathrm{NE}(Y) \subset \mathcal{N}_1(Y) \quad \textit{and} \quad \mathcal{E}^\vee \subset \mathcal{N}^\vee \subset H^2(S,\mathbb{R}).$$

Here $\mathrm{Mov}_1(Y) = \mathrm{Eff}(Y)^\vee$ is the convex cone generated by classes of curves moving in a family of curves covering Y (see [Laz04, §11.4.C and references therein]).

In the following subsections we give a geometric description of the extremal rays and the facets of these cones in terms of special divisors and curves in Y, using the explicit descriptions given in §2 of the cones $\mathcal{N}$, Π, $\mathcal{E}$ and their duals.

6.3. The cone of effective curves and the nef cone. The cone $\mathrm{NE}(Y)$ has 240 extremal rays, and is isomorphic to $\mathrm{NE}(S)$ (see 2.7). If ℓ is a (-1)-curve, the corresponding extremal ray of $\mathrm{NE}(Y)$ is generated by the class of a line Γ_ℓ in $P_\ell \cong \mathbb{P}^2 \subset Y$ (see Cor. 4.16). The corresponding elementary contraction is a small contraction, sending P_ℓ to a point. For completeness let us state here the following lemma on the relative positions of the special surfaces P_ℓ in Y; it will be proved in §8. Recall that if $\ell, \ell' \subset S$ are (-1)-curves, then $\ell \cdot \ell' \leq 3$, with equality if and only if $\ell' = \iota_S^*\ell$, see Rem. 2.15(c).

Lemma 6.4. *Let $\ell, \ell' \subset S$ be distinct (-1)-curves. If $\ell \cdot \ell' \leq 1$, then $P_\ell \cap P_{\ell'} = \emptyset$ in Y. If $\ell \cdot \ell' = 2$, then P_ℓ and $P_{\ell'}$ intersect transversally in one point in Y.*

Suppose that S is general. If $\ell \cdot \ell' = 3$, then P_ℓ and $P_{\ell'}$ intersect transversally in 3 *points in Y.*

The cone $\mathrm{Nef}(Y)$ is isomorphic to $\mathrm{Nef}(S)$. It has $19440 = 2160 + 17280$ extremal rays, one for each conic C and cubic h of S; the corresponding generators are $\rho(-2K_S + C)$ and $\rho(-3K_S + h)$ (see (2.8)).

Recall that extremal rays of $\mathrm{Nef}(Y)$ correspond to contractions $f\colon Y \to Z$ with $\rho_Z = 1$. Let us describe these contractions using Lemma 3.22(d) and Cor. 4.11.

Given a cubic h, the line bundle $-3K_S + h$ is contained in the walls $(2e_i + K_S)^\perp$, for $i = 1, \dots, 8$ (notation as in 2.2; see Lemma 5.2). Thus the contraction given by $\rho(-3K_S + h)$ is birational, small, and has exceptional locus $P_{e_1} \cup \cdots \cup P_{e_8}$, where the P_{e_i}'s are pairwise disjoint. Correspondingly, the classes of the curves $\Gamma_{e_1}, \dots, \Gamma_{e_8}$ span a simplicial facet of $\mathrm{NE}(Y)$.

Given a conic C, the line bundle $-2K_S + C$ is contained in 14 walls $(2\ell + K_S)^\perp$, where ℓ is a (-1)-curve such that $C \cdot \ell = 0$, namely ℓ is a component of a reducible conic linearly equivalent to C. Thus the contraction given by $\rho(-2K_S + C)$ is birational, small, and has exceptional locus the disjoint union of 14 P_ℓ's (all contained in the divisor E_C, see 6.5). This yields a non-simplicial facet of $\mathrm{NE}(Y)$.

This shows Prop. 1.7 from the Introduction.

6.5. The cone of effective divisors. The cone $\mathrm{Eff}(Y)$ has 2160 extremal rays, each generated by a fixed divisor E_C, where $C \subset S$ is a conic. Each such divisor comes, up to pseudo-isomorphism, from the blow-up of a smooth point.

The divisor $E_C \subset Y$ is smooth and is isomorphic to the blow-up of $\mathbb{P}^3$ in 14 points (in a special position). This can be seen by choosing a cubic h such that $C = C_1 \sim h - e_1$ (notation as in 2.2), so that E_C is the transform of the exceptional divisor $E_1 \cong \mathbb{P}^3 \subset X_h = X$ under $\xi\colon X \dashrightarrow Y$ (see Th. 5.6). By the explicit description of the map ξ given in Lemma 5.18, we see that E_1 is blown-up in the 14 points of intersection with $L_{12}, \dots, L_{18}, \Gamma_2, \dots, \Gamma_8$.

Recall that for every (-1)-curve ℓ, $\Gamma_\ell \subset P_\ell \cong \mathbb{P}^2 \subset Y$ generates an extremal ray of $\mathrm{NE}(Y)$. By Lemmas 5.15 and 4.16 we have

$$E_C \cdot \Gamma_\ell = \frac{1}{2}\rho(C) \cdot \Gamma_\ell = \frac{1}{2}C \cdot \zeta(\Gamma_\ell) = \frac{1}{2}C \cdot (2\ell + K_S) = C \cdot \ell - 1, \tag{6.6}$$

and there are 14 special loci P_ℓ (given by the (-1)-curves ℓ with $C \cdot \ell = 0$) contained in E_C; these are in E_C the exceptional divisors of the blow-up $E_C \to \mathbb{P}^3$.

6.7. The cone of movable divisors and the divisors $H_{Y,h}$. The cone $\mathrm{Mov}(Y)$ is isomorphic to the cone $\Pi \subset H^2(S, \mathbb{R})$ via ρ, and it has two types of facets, cut by $\zeta^{-1}(\ell)^\perp$ and $\zeta^{-1}(2C + K_S)^\perp$ for every (-1)-curve ℓ and conic C in S (see 2.10). The class $\zeta^{-1}(2\ell)$ is the class of a moving curve on Y, we will describe it in 6.11. The class $\zeta^{-1}(2C + K_S)$ is the class of the transform $\Gamma_C \subset E_C$ of a general line in $\mathbb{P}^3$ under the blow-up $E_C \to \mathbb{P}^3$, see Cor. 4.17.

Definition 6.8 (the map η_h)**.** Let h be a cubic in S, and consider the outer chamber $\mathcal{C}_h$ introduced in 3.19; by Prop. 3.20, we have $M_{\mathcal{C}_h} \cong \mathbb{P}^4$. Thus the natural contracting birational map $Y = M_{-K_S} \dashrightarrow M_{\mathcal{C}_h}$ (see Cor. 3.25) yields a map $\eta_h\colon Y \dashrightarrow \mathbb{P}^4$. By varying the polarization from $-K_S$ to $\mathcal{C}_h$ along the plane spanned by $-K_S$ and h (see Fig. 5.3), we factor η_h as

$$\begin{tikzcd} Y \arrow[r, dashed, "\xi_h^{-1}"'] \arrow[rr, dashed, bend left, "\eta_h"] & X_h \arrow[r] & \mathbb{P}^4 \end{tikzcd}$$

where ξ_h is described in Lemma 5.18 and $X_h \to \mathbb{P}^4$ is the blow-up of 8 points. It follows from Lemma 5.18 (and its proof) that the indeterminacy locus of η_h is the union of the surfaces P_{e_i} and $P_{\ell_{jk}}$ for $i = 1, \dots, 8$ and $1 \le j < k \le 8$, and these surfaces are pairwise disjoint.

Proposition 6.9. *Let h be a cubic, and set $H_{Y,h} := \frac{1}{2}\rho(-K_S + 3h) \in H^2(Y, \mathbb{R})$. Then $H_{Y,h} \in \mathrm{Pic}(Y)$ and is a movable class. Its complete linear system defines the contracting birational map $\eta_h\colon Y \dashrightarrow \mathbb{P}^4$, with exceptional divisors $E_{C_1}, \dots, E_{C_8}$. The images $\eta_h(E_{C_1}), \dots, \eta_h(E_{C_8})$ are 8 points in $\mathbb{P}^4$, associated to the points $q_1, \dots, q_8 \in \mathbb{P}^2$ (notation as in 2.2).*

Proof. After Prop. 5.8 we have $\rho_{\mathcal{B}_h}(-K_S + 3h) = 2H \in \mathrm{Pic}(X_h)$ (notation as in 2.22), so Lemma 4.12 and the definition of η_h yield

$$\rho_{-K_S}(-K_S + 3h) = (\xi_h^{-1})^*(\rho_{\mathcal{B}_h}(-K_S + 3h)) = (\xi_h^{-1})^*(2H) = \eta_h^*\mathcal{O}_{\mathbb{P}^4}(2).$$

Thus $H_{Y,h} = \eta_h^*\mathcal{O}_{\mathbb{P}^4}(1)$, and the rest of the statement follows from Th. 5.6. ■

Lemma 6.10. *The divisor $H_{Y,h}$ generates an extremal ray of* $\mathrm{Mov}(Y)$*, contained in the interior of* $\mathrm{Eff}(Y)$*.*

Conversely, let τ be an extremal ray of $\mathrm{Mov}(Y)$ *lying in the interior of* $\mathrm{Eff}(Y)$*. Then there exists a cubic $h' \subset S$ such that $H_{Y,h'} \in \tau$.*

Proof. The divisor $-K_S + 3h$ belongs to Π, and it lies on the facets $\Pi \cap (2C_i + K_S)^{\perp}$ of Π, for $i = 1, \dots, 8$ (notation as in 2.2). Since the classes $2C_i + K_S$, for $i = 1, \dots, 8$, are linearly independent, we see that $-K_S + 3h$ generates an extremal ray of Π, and via the isomorphism ρ we see that $H_{Y,h}$ generates an extremal ray of $\mathrm{Mov}(Y)$. Moreover $H_{Y,h}$ is big by Prop. 6.9.

For the second statement, a large enough integral divisor $D \in \tau$ defines a contracting birational map $f\colon Y \dashrightarrow Y'$, where Y' is $\mathbb{Q}$-factorial with $\rho_{Y'} = 1$. The prime exceptional divisors of f generate a simplicial facet of $\mathrm{Eff}(Y)$. By Th. 5.13(b) we have $\mathrm{Eff}(Y) \cong \mathcal{E}$, and every simplicial facet of $\mathcal{E}$ has the form $(2h' + K_S)^{\perp} \cap \mathcal{E}$ for a cubic h', see 2.3. The corresponding facet of $\mathrm{Eff}(Y)$ is generated by $E_{C'_1}, \dots, E_{C'_8}$, thus $f\colon Y \dashrightarrow Y'$ and $\eta_{h'}\colon Y \dashrightarrow \mathbb{P}^4$ have the same exceptional divisors. This means that the composition $f \circ \eta_{h'}^{-1}\colon \mathbb{P}^4 \dashrightarrow Y'$ is a pseudo-isomorphism, and hence an isomorphism. Therefore $H_{Y,h'} \in \tau$. ■

Let us notice that the birational map $\eta_h\colon Y \dashrightarrow \mathbb{P}^4$ allows to reconstruct the surface S from Y: indeed, by Prop. 6.9, it determines the points $q_1, \dots, q_8 \in \mathbb{P}^2$ blown-up by $S \to \mathbb{P}^2$. This is the key point for the proofs of Theorems 1.4, 1.5, 1.8, and 1.9.

6.11. The cone of moving curves. The cone $\mathrm{Mov}_1(Y)$ is isomorphic, via ζ^{-1}, to $\mathcal{E}^{\vee} \subset H^2(S, \mathbb{R})$. Thus it has $17520 = 17280 + 240$ extremal rays, generated by $\zeta^{-1}(2h + K_S)$ and $\zeta^{-1}(\ell)$ for every cubic h and (-1)-curve ℓ on S (see (2.4)). Let us describe some families of curves whose classes generate $\mathrm{Mov}_1(Y)$.

Let h be a cubic, and consider the birational map $\eta_h\colon Y \dashrightarrow \mathbb{P}^4$. It follows from Prop. 5.8 and Rem. 4.18 that $\zeta^{-1}(2h + K_S) \in \mathcal{N}_1(Y)$ is the class of the transform under η_h of a general line in $\mathbb{P}^4$. The corresponding facet of $\mathrm{Eff}(Y)$ is simplicial, generated by the exceptional divisors $E_{C_1}, \dots, E_{C_8}$ of η_h.

In order to describe the extremal ray generated by $\zeta^{-1}(\ell)$, we need the following.

Remark 6.12. Let X be the blow-up of $\mathbb{P}^4$ at 8 points $p_1, \dots, p_8$ in general linear position. Fix $i \in \{1, \dots, 8\}$ and let $\mathcal{P}_i \subset \mathbb{P}^3$ be the image of the set $\{p_1, \dots, \check{p}_i, \dots, p_8\}$ under the projection $\pi_{p_i}\colon \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ from p_i. Let T be the blow-up of $\mathbb{P}^3$ at the 7 points in $\mathcal{P}_i$. There is a pseudo-isomorphism $X \dashrightarrow X_i$ and a $\mathbb{P}^1$-bundle $X_i \to T$ extending π_{p_i} (see [Muk05, Ex. 1] and [AC17, Rem. 4.8]). Let $\pi_i\colon X \dashrightarrow T$ be the composite map. Then the general fiber of π_i is the transform in X of a general line in $\mathbb{P}^4$ through p_i, so it has class $h - e_i \in \mathcal{N}_1(X)$.

Remark 6.13. In the situation of Rem. 6.12, $-K_T$ is nef and big [BL12, Prop. 2.9], and $-K_T = 2Q$, $Q \in \mathrm{Pic}(T)$. Using vanishing and Riemann-Roch one computes that $h^0(T, Q) = 3$.

Set $D := \pi_i^* Q \in \mathrm{Pic}(X)$. Then $h^0(X, D) = 3$, and by Prop. 5.8 we have $D \sim 2H - \sum_{j=1}^{8} E_j - E_i = \frac{1}{2}\rho(-K_S + e_i)$.

Let now $\ell \subset S$ be a (-1)-curve. Choose a cubic h such that $\ell = e_1$ (notation as in 2.2), and consider $X = X_h$ and the $\mathbb{P}^1$-bundle $X_1 \to T$ described in Rem. 6.12. Set $Y_\ell := X_1$ and $T_\ell := T$. Then there is a pseudo-isomorphism $Y \dashrightarrow Y_\ell$ and a $\mathbb{P}^1$-bundle

$Y_\ell \to T_\ell$, and we claim that the general fiber f of the composite map $Y \dashrightarrow T_\ell$ has class $2\zeta^{-1}(\ell) \in \mathcal{N}_1(Y)$. Indeed by Prop. 5.8 we have $\zeta_{\mathcal{B}_h}(h-e_1) = 2e_1 \in H^2(S,\mathbb{R})$, and Rem. 4.18 yields $\zeta_{-K_S}(f) = \zeta_{\mathcal{B}_h}(h-e_1) = 2e_1 = 2\ell$. See [Muk05, p. 9-10] for a modular description of the map $Y \dashrightarrow T_\ell$. The facet of $\mathrm{Eff}(Y)$ cut by $\zeta^{-1}(\ell)^\perp$ is generated by the 126 E_C's such that C is a conic disjoint from ℓ.

In particular, we notice that Y has 240 distinct dominating families of rational curves of anticanonical degree 2.

6.14. Torelli type results.

Proof of Th. 1.8. One implication is clear. For the other, let $f\colon Y_1 \to Y_2$ be an isomorphism, and let h_2 be a cubic on S_2.

Consider the divisor class H_{Y_2,h_2} on Y_2. By Lemma 6.10, $f^*H_{Y_2,h_2}$ generates an extremal ray of $\mathrm{Mov}(Y_1)$, lying in the interior of $\mathrm{Eff}(Y_1)$. Again by Lemma 6.10, there exists a cubic h_1 on S_1 such that H_{Y_1,h_1} is a positive multiple of $f^*H_{Y_2,h_2}$. Therefore we have a commutative diagram:

$$\begin{tikzcd} Y_1 \arrow[r, "f"] \arrow[d, dashed, "\eta_{h_1}"'] & Y_2 \arrow[d, dashed, "\eta_{h_2}"] \\ \mathbb{P}^4 \arrow[r, "f'"] & \mathbb{P}^4 \end{tikzcd}$$

where f' is a projective transformation.

For $i = 1,2$ let $p_1^i,\dots,p_8^i \in \mathbb{P}^4$ be the images of the exceptional divisors of η_{h_i}, $\sigma_i\colon S_i \to \mathbb{P}^2$ the map induced by $h_i \in \mathrm{Pic}(S_i)$, and $q_1^i,\dots,q_8^i \in \mathbb{P}^2$ the points blown-up by σ_i. Then $p_1^i,\dots,p_8^i \in \mathbb{P}^4$ and $q_1^i,\dots,q_8^i \in \mathbb{P}^2$ are associated point sets by Prop. 6.9, and $p_1^1,\dots,p_8^1$ are projectively equivalent to $p_1^2,\dots,p_8^2$ by the diagram above. We conclude that $q_1^1,\dots,q_8^1$ and $q_1^2,\dots,q_8^2$ are projectively equivalent, and hence that $S_1 \cong S_2$. ■

Proof of Th. 1.4. If $S_1 \cong S_2$, then M_{S_1,L_1} and M_{S_2,L_2} are pseudo-isomorphic by Cor. 3.25.

Conversely, suppose that M_{S_1,L_1} and M_{S_2,L_2} are pseudo-isomorphic, and set $Y_i := M_{S_i,-K_{S_i}}$ for $i=1,2$. Then, again by Cor. 3.25, there is a pseudo-isomorphism $f\colon Y_1 \dashrightarrow Y_2$. Since Y_1 and Y_2 are Fano, f must be an isomorphism (because $f^*(-K_{Y_2}) = -K_{Y_1}$), hence $S_1 \cong S_2$ by Th. 1.8. ■

6.15. Automorphisms and pseudo-automorphisms.

Proof of Th. 1.9. We have a natural group homomorphism $\mathrm{Aut}(S) \to \mathrm{Aut}(H^2(S,\mathbb{R}))$, given by $f \mapsto (f^{-1})^*$, and similarly for Y. Moreover, the isomorphism $\rho\colon H^2(S,\mathbb{R}) \to H^2(Y,\mathbb{R})$ induces an isomorphism $\mathrm{Aut}(H^2(S,\mathbb{R})) \to \mathrm{Aut}(H^2(Y,\mathbb{R}))$, given by $\varphi \mapsto \rho\circ\varphi\circ\rho^{-1}$. These maps are related by the following diagram, which is commutative by Prop. 4.23:

$$\begin{tikzcd} \mathrm{Aut}(S) \arrow[r] \arrow[d, "\psi"'] & \mathrm{Aut}(H^2(S,\mathbb{R})) \arrow[d, "\wr"] \\ \mathrm{Aut}(Y) \arrow[r] & \mathrm{Aut}(H^2(Y,\mathbb{R})) \end{tikzcd} \tag{6.16}$$

The map $\mathrm{Aut}(S) \to \mathrm{Aut}(H^2(S,\mathbb{R}))$ is injective [Dol12, Prop. 8.2.39], thus ψ is injective. To show that ψ is also surjective, let $g\colon Y \to Y$ be an automorphism.

The first step is to show that, up to multiply g for an element in the image of ψ, we can assume that $g^*\colon H^2(Y,\mathbb{R}) \to H^2(Y,\mathbb{R})$ fixes the ray $\mathbb{R}_{\geq 0}H_{Y,h}$ for some cubic h of S. As in the proof of Th. 1.8 (see 6.14), we find two cubics h, h' on S such that $g^*H_{Y,h}$ is a multiple of $H_{Y,h'}$, and a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, dashed, "\eta_{h'}"'] & Y \arrow[d, dashed, "\eta_h"] \\ \mathbb{P}^4 \arrow[r, "g'"] & \mathbb{P}^4 \end{tikzcd} \tag{6.17}$$

where g' is a projective transformation. Moreover, if $\sigma\colon S \to \mathbb{P}^2$ and $\sigma'\colon S \to \mathbb{P}^2$ are the morphisms induced respectively by h and h', there is a projective transformation $f'\colon \mathbb{P}^2 \to \mathbb{P}^2$ sending the the points blown-up by σ' to the points blown-up by σ.

By the uniqueness of the blow-up, there exists an automorphism $f\colon S \to S$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\sigma'"'] & S \arrow[d, "\sigma"] \\ \mathbb{P}^2 \arrow[r, "f'"] & \mathbb{P}^2 \end{tikzcd} \tag{6.18}$$

and $f^*h = h'$. Now by Prop. 4.23 we get

$$\psi(f)^*H_{Y,h} = \psi(f)^*\Big(\frac{1}{2}\rho(-K_S + 3h)\Big) = \frac{1}{2}\rho\big(f^*(-K_S + 3h)\big) = \frac{1}{2}\rho(-K_S + 3h') = H_{Y,h'},$$

hence $(g \circ \psi(f^{-1}))^*H_{Y,h}$ is a positive multiple of $H_{Y,h}$.

We can now assume that g^* fixes the ray $\mathbb{R}_{\geq 0}H_{Y,h}$, so that $h = h'$ in (6.17) and $\sigma = \sigma'$ in (6.18). Since g^* also induces an automorphism of $H^2(Y,\mathbb{Z}) \subset H^2(Y,\mathbb{R})$, we actually have $g^*H_{Y,h} = H_{Y,h}$.

Let $E_{C_1}, \dots, E_{C_8} \subset Y$ be the exceptional divisors of η_h, $p_1, \dots, p_8 \in \mathbb{P}^4$ their images, and $q_1, \dots, q_8 \in \mathbb{P}^2$ the (ordered) associated points, namely the points blown-up by σ.

The projective transformation $g'\colon \mathbb{P}^4 \to \mathbb{P}^4$ fixes the set $\{p_1, \dots, p_8\}$, hence permutes the points p_i; let us call τ this permutation, so that $g^*E_{C_i} = E_{C_{\tau(i)}}$ for $i = 1, \dots, 8$.

We note that the map $\mathrm{Aut}(Y) \to \mathrm{Aut}(H^2(Y,\mathbb{R}))$ is injective. Indeed, suppose that $g^* = \mathrm{Id}_{H^2(Y,\mathbb{R})}$. Then τ is the identity, and g' fixes p_i for every $i = 1, \dots, 8$. Since $p_1, \dots, p_8$ are in general linear position (see Rem. 2.20), we get $g' = \mathrm{Id}_{\mathbb{P}^4}$ and hence $g = \mathrm{Id}_Y$ by (6.17).

We carry on with the proof that $g \in \mathrm{Im}(\psi)$. Since $p_1, \dots, p_8$ and $p_{\tau(1)}, \dots, p_{\tau(8)}$ are projectively equivalent, and $p_{\tau(1)}, \dots, p_{\tau(8)}$ (as an ordered set of points) is associated to $q_{\tau(1)}, \dots, q_{\tau(8)}$ [DO88, Ch. III, §1], we conclude that also $q_1, \dots, q_8$ and $q_{\tau(1)}, \dots, q_{\tau(8)}$ are projectively equivalent. Let us call k' the projective transformation of $\mathbb{P}^2$ which maps q_i in $q_{\tau(i)}$ for $i = 1, \dots, 8$. This induces an automorphism k of S. We claim that $\psi(k) = g$; by what precedes, it is enough to show that $\psi(k)^* = g^*$.

Notice that $H_{Y,h}, E_{C_1}, \dots, E_{C_8}$ is a basis of $H^2(Y,\mathbb{R})$, and $g^*E_{C_i} = E_{C_{\tau(i)}}$ for $i = 1, \dots, 8$. On the other hand $k^*h = h$, $k^*K_S = K_S$, and $k^*e_i = e_{\tau(i)}$ for $i = 1, \dots, 8$, hence $k^*C_i = C_{\tau(i)}$ for $i = 1, \dots, 8$. This easily implies, using Prop. 4.23 and the map $\rho\colon H^2(S,\mathbb{R}) \to H^2(Y,\mathbb{R})$, that $\psi(k)^*H_{Y,h} = H_{Y,h}$ and $\psi(k)^*E_{C_i} = E_{C_{\tau(i)}}$ for $i = 1, \dots, 8$, and finally that $\psi(k)^* = g^*$.

Therefore ψ is an isomorphism; in particular $\mathrm{Aut}(Y)$ is finite, see [Dol12, §8.8.4]. ■

Proof of Th. 1.5. By Cor. 3.25, there is a pseudo-isomorphism $\varphi\colon M_{S,L} \dashrightarrow Y := M_{S,-K_S}$, which induces an isomorphism between the group of pseudo-automorphisms of $M_{S,L}$ and that of Y. On the other hand, being Y Fano, every pseudo-automorphism of Y is an automorphism. Thus the statement follows from Th. 1.9. ■

Given a chamber $\mathcal{C} \subset \Pi$, it is not difficult to see that under the isomorphism given by Th. 1.5, $\mathrm{Aut}(M_{S,\mathcal{C}}) \cong \{f \in \mathrm{Aut}(S) \mid f^*\mathcal{C} = \mathcal{C}\}$. In particular, when S is general, $\mathrm{Aut}(M_{S,\mathcal{C}}) = \{\mathrm{Id}\}$ unless $M_{S,\mathcal{C}} = Y$, because the Bertini involution ι_S fixes only the central chamber $\mathcal{N}$ (see 2.12).

Definition 6.19 (the Bertini involution in Y)**.** The Bertini involution ι_S of S induces an involution $\iota_Y = \psi(\iota_S)$ of Y, which we still call the Bertini involution; explicitly $\iota_Y\colon Y \to Y$ is given by $\iota_Y([F]) = [\iota_S^* F]$. By Prop. 4.23, we have a commutative diagram:

(6.20)
$$\begin{array}{ccc} H^2(S,\mathbb{R}) & \xrightarrow{\iota_S^*} & H^2(S,\mathbb{R}) \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} \\ H^2(Y,\mathbb{R}) & \xrightarrow{\iota_Y^*} & H^2(Y,\mathbb{R}). \end{array}$$

6.21. Fibre-likeness. A Fano variety is *fibre-like* if it can appear as a fiber of a Mori fiber space; this notion has been introduced and studied in [CFST16]. Every Fano variety with $b_2 = 1$ is fibre-like, while fibre-likeness becomes a rather strong condition on Fano varieties with $b_2 > 1$.

In the case of the Fano 4-fold Y, our analysis of the automorphisms yields that the invariant part of $H^2(Y,\mathbb{R})$ by the action of the Bertini involution ι_Y is $\mathbb{R}K_Y$ (see (6.20) and 2.12). By [CFST16, Th. 1.2], this implies the following.

Proposition 6.22. *The Fano* 4*-fold Y is fibre-like.*

According to the authors' knowledge, this is the first explicit example of higher-dimensional non-toric smooth Fano variety with this property, which is not a product of lower dimensional varieties. The symmetries of numerical cones of Fano varieties with high Picard rank were one of our original motivations for this work.

6.23. Deformations.

Lemma 6.24. *We have $h^0(Y,T_Y) = 0$ and $h^1(Y,T_Y) = 8$.*

Thus $Y = M_{S,-K_S}$ varies in an 8-dimensional family, like S. For the proof, we need the following general formula.

Lemma 6.25. *Let Z be a smooth Fano* 4*-fold. Then*

$$h^0(Z,T_Z) - h^1(Z,T_Z) = 27 - 5h^0(-K_Z) + K_Z^4 + 3b_2(Z) - h^{1,2}(Z) - h^{2,2}(Z) + 3h^{1,3}(Z).$$

Proof. Since Z is Fano, by Nakano vanishing we have $h^i(T_Z) = 0$ for $i \geq 2$, so by Riemann-Roch

$$h^0(T_Z) - h^1(T_Z) = \chi(T_Z) = \frac{1}{12}\left(2K^4 - 5K^2 \cdot c_2 - 5K \cdot c_3 - 2\chi_{top}\right) + 4.$$

Riemann-Roch for $\mathcal{O}_Z(-K_Z)$ gives $K^2 \cdot c_2 = 2(6h^0(-K) - K^4 - 6)$, and Riemann-Roch for Ω^1_Z gives $K \cdot c_3 = 2(2h^{1,2} - 4b_2 + h^{2,2} - 4h^{1,3} - 22)$; this yields the statement. ■

Proof of Lemma 6.24. By Lemma 6.25 and Prop. 6.1 we have $h^0(T_Y) - h^1(T_Y) = -8$. On the other hand, $\mathrm{Aut}(Y)$ is finite by Th. 1.9, hence $h^0(T_Y) = 0$, and $h^1(T_Y) = 8$. ■

6.26. Other models. Let us mention two other interesting projective 4-folds that are pseudo-isomorphic to Y.

The first is the blow-up W of $(\mathbb{P}^1)^4$ in 5 general points. There exists a pseudo-isomorphism $W \dashrightarrow X$, where X is a blow-up pf $\mathbb{P}^4$ at 8 general points, see [Muk04, Remark at the end of §1]. Thus by Cor. 5.7 and Th. 5.13 there exist a del Pezzo surface S of degree 1, and a chamber $\mathcal{C} \subset \Pi \subset H^2(S,\mathbb{R})$, such that $W \cong M_{S,\mathcal{C}}$, and W is pseudo-isomorphic to $Y = M_{S,-K_S}$.

For the second, let G be the variety of lines contained in a smooth complete intersection of two quadric hypersurfaces in $\mathbb{P}^6$. Then G is a smooth Fano 4-fold with $b_2(G) = 8$, and G is pseudo-isomorphic to a blow-up of $\mathbb{P}^4$ in 7 general points (see [AC17] and references therein). Let $\mathrm{Bl}_p G$ be the blow-up of G at a general point. As for W above, there exist a del Pezzo surface S of degree 1, and a chamber $\mathcal{C}' \subset \Pi \subset H^2(S,\mathbb{R})$, such that $\mathrm{Bl}_p G \cong M_{S,\mathcal{C}'}$, and $\mathrm{Bl}_p G$ is pseudo-isomorphic to $Y = M_{S,-K_S}$. There is also a chamber $\mathcal{C}'' \subset \mathcal{E} \subset H^2(S,\mathbb{R})$ such that $G \cong M_{S,\mathcal{C}''}$, however $\mathcal{C}'' \not\subset \Pi$.

7. Anticanonical and bianticanonical linear systems

7.1. The anticanonical linear system. Let S be a degree one del Pezzo surface, and $Y = M_{S,-K_S}$ the associated Fano 4-fold. In this subsection we show the first part of Th. 1.10, namely that the linear system $|-K_Y|$ has a base locus of positive dimension.

It is enough to prove this statement when $Y = M_{S,-K_S}$ is general, *i.e.* when S is a general del Pezzo surface of degree 1; we will assume this throughout the subsection. Let us also fix for the whole subsection a cubic $h \subset S$, the corresponding blow-up $X = X_h$ of $\mathbb{P}^4$ at 8 points, and the birational map $\xi\colon X \dashrightarrow Y$ (see 5.17). We keep the notation as in 2.22. Notice that since S is general, X is a blow-up of $\mathbb{P}^4$ at 8 general points.

We analyse the base locus of $|-K_X|$. This contains the curves L_{ij} for $1 \le i < j \le 8$ and Γ_k for $k = 1,\dots,8$, because they have negative intersection with $-K_X$ (see Cor. 5.19). We will show (Cor. 7.6 and Lemma 7.7) that $\mathrm{Bs}|-K_X|$ also contains the transform R of a smooth rational quintic curve $R_4 \subset \mathbb{P}^4$ through $p_1,\dots,p_8$.

Let us recall that an elliptic normal quintic in $\mathbb{P}^4$ is a smooth curve of genus one, degree 5, not contained in a hyperplane.

Lemma 7.2 ([RS00],[Dol04])**.** *Let $p_1,\dots,p_8 \in \mathbb{P}^4$ be general points. Then there is a pencil of elliptic normal quintics in $\mathbb{P}^4$ through $p_1,\dots,p_8$, which sweeps out a cubic scroll $W \subset \mathbb{P}^4$.*

Let moreover $q_1,\dots,q_8 \in \mathbb{P}^2$ be the associated points to $p_1,\dots,p_8 \in \mathbb{P}^4$. Then there is a birational map $\alpha\colon W \to \mathbb{P}^2$ such that $\alpha(p_i) = q_i$ for $i = 1,\dots,8$, α sends the pencil of elliptic normal quintics to the pencil of plane cubics through $q_1,\dots,q_8$, and α is the blow-up of the ninth base point $q_0 \in \mathbb{P}^2$ of the pencil of plane cubics.

Proof. The first statement is [RS00, Prop. 5.2]. Let $B \subset W$ be an elliptic normal quintic through $p_1,\dots,p_8$, and $\Lambda \subset \mathbb{P}^4$ a general hyperplane. By [Dol04, 2.4], the complete linear system $|p_1 + \cdots + p_8 - \Lambda_{|B}|$ on B yields a map $B \to \mathbb{P}^2$, embedding B as a plane cubic, and sending p_i to q_i for $i = 1,\dots,8$.

Recall that W is isomorphic to the blow-up of $\mathbb{P}^2$ at a point. Let $e \subset W$ be the (-1)-curve, and $f \subset W$ a fiber of the $\mathbb{P}^1$-bundle on W. Then the blow-up $\alpha\colon W \to \mathbb{P}^2$ is the map associated to the complete linear system $|e+f|$ on W. In W we have $\Lambda_{|W} \sim e+2f$, $B \sim 2e+3f = -K_W$, and $B^2 = 8$, so that if B' is another quintic of the pencil, $B'_{|B} = p_1+\cdots+p_8$. Thus on B we have $p_1+\cdots+p_8-\Lambda_{|B} \sim (B'-\Lambda)_{|B} \sim (e+f)_{|B}$. Moreover it is not difficult to see that the restriction map $H^0(W, \mathcal{O}_W(e+f)) \to H^0(B, \mathcal{O}_B((e+f)_{|B}))$ is an isomorphism; this yields the statement. ■

Let $W' \subset X$ be the transform of the cubic scroll $W \subset \mathbb{P}^4$. We have a diagram:

(7.3)
$$\begin{tikzcd} & W' \subset X \arrow[dl, "\eta"'] \arrow[dd, "\alpha'"] \arrow[dr] & \\ W \subset \mathbb{P}^4 \arrow[dr, "\alpha"'] & & S \arrow[dl, "\sigma"] \\ & \mathbb{P}^2 & \end{tikzcd}$$

where $\eta\colon W' \to W$ is the blow-up of $p_1, \ldots, p_8$, so the composition $\alpha' := \alpha \circ \eta\colon W' \to \mathbb{P}^2$ is the blow-up of $q_0, \ldots, q_8$. Thus W' is isomorphic to the blow-up of S in the base point of $|-K_S|$, and there is an elliptic fibration $\pi\colon W' \to \mathbb{P}^1$, where the smooth fibers are the transforms of the elliptic normal quintics through $p_1, \ldots, p_8$ in $\mathbb{P}^4$.

Lemma 7.4. *The surface $W' \subset X$ is disjoint from L_{ij} for $1 \leq i < j \leq 8$ and from Γ_k for $k = 1, \ldots, 8$, and W' is contained in the open subset where $\xi\colon X \dashrightarrow Y$ is an isomorphism.*

Proof. Consider the rational normal quartic $\gamma_1 \subset \mathbb{P}^4$ through $p_2, \ldots, p_8$, so that $\Gamma_1 \subset X$ is the transform of γ_1. To show that W' is disjoint from Γ_1, we show that $W \cap \gamma_1 = \{p_2, \ldots, p_8\}$ and that the intersection is transverse.

Let $V \subset \mathbb{P}^4$ be the cone over γ_1 with vertex p_8. Then the 0-cycle given by the intersection of V and W has degree 9 and contains $p_2+\cdots+p_7+3p_8$, so it is $p_2+\cdots+p_7+3p_8$. Thus set-theoretically $W \cap V = \{p_2, \ldots, p_8\}$, and the intersection is transverse at $p_2, \ldots, p_7$.

This shows that set-theoretically $W \cap \gamma_1 = \{p_2, \ldots, p_8\}$, and that the intersection is transverse at $p_2, \ldots, p_7$. By considering the cone over γ_1 with vertex p_7, we see that the intersection is transverse also at p_8. Thus $W' \cap \Gamma_1 = \emptyset$, and similarly $W' \cap \Gamma_k = \emptyset$ for $k = 1, \ldots, 8$.

To show that $W' \cap L_{ij} = \emptyset$ for every $1 \leq i < j \leq 8$, one proceeds in a similarly way, by considering in $\mathbb{P}^4$ the intersection of W with a plane through 3 points among $p_1, \ldots, p_8$, and showing that W intersects the line $\overline{p_i p_j}$ only in p_i, p_j, and that the intersection is transverse.

The last statement follows from Lemma 5.18. ■

Lemma 7.5. *We have $(-K_X)_{|W'} = \mathcal{O}_{W'}(R+2F)$ and $R = \mathrm{Bs}\,|(-K_X)_{|W'}|$, where $F \subset W'$ is a fiber of the elliptic fibration, and $R \subset W'$ is a (-1)-curve and a section of the elliptic fibration.*

Proof. We have $-K_X = 5H - 3\sum_{i=1}^8 E_i$, and $E_{i|W'}$ is a (-1)-curve in W' for $i = 1, \ldots, 8$. Thus $((-K_X)_{|W'})^2 = 25(H_{|W'})^2 + 9\sum_i (E_{i|W'})^2 = 75 - 72 = 3$.

Let F be a smooth fiber of the elliptic fibration $\pi\colon W' \to \mathbb{P}^1$. Since F is the transform of an elliptic normal quintic in $\mathbb{P}^4$ through $p_1, \dots, p_8$, we have

$$-K_X \cdot F = \Big(5H - 3\sum_{i=1}^{8} E_i\Big) \cdot F = 25 - 24 = 1.$$

Since $-K_{W'} \sim F$, by Riemann-Roch we get $\chi(W', (-K_X)_{|W'}) = 3$.

Notice that $(-K_X)_{|W'}$ has positive intersection with every curve in W'. Indeed, by Lemma 5.19, there are finitely many irreducible curves in X having non-positive intersection with $-K_X$, and by Lemma 7.4 these curves are disjoint from W'.

By Nakai's criterion, $(-K_X)_{|W'}$ is ample on W', and $-K_{W'}$ is nef, so by Kodaira vanishing we have $h^i(W', (-K_X)_{|W'}) = h^i(W', K_{W'} - K_{W'} + (-K_X)_{|W'}) = 0$ for $i = 1, 2$, and $h^0(W', (-K_X)_{|W'}) = 3$. In particular the linear system $|(-K_X)_{|W'}|$ is non-empty.

We have $F \in |\pi^*\mathcal{O}_{\mathbb{P}^1}(1)|$, and every fiber of π is integral. Thus for every irreducible curve $C \subset W'$ we have $F \cdot C \geq 0$, and $F \cdot C = 0$ if and only if $C \sim F$ and C is a fiber of the elliptic fibration.

Let $D \in |(-K_X)_{|W'}|$. Then $1 = -K_X \cdot F = D \cdot F$ (where the first intersection is in X, and the second in W'), thus we must have $D = R + \sum_i m_i F_i$ where R is an irreducible curve such that $R \cdot F = 1$, and F_i are fibers of the elliptic fibration. In particular $(-K_X)_{|W'} \sim R + mF$, where $m = \sum_i m_i$.

Since R is a section of π, we have $R \cong \mathbb{P}^1$; moreover $-K_{W'} \cdot R = F \cdot R = 1$, hence R is a (-1)-curve. Since $((-K_X)_{|W'})^2 = 3$, we get $m = 2$, hence $D = R + F_1 + F_2$ (with possibly $F_1 = F_2$).

Now if $D' \in |(-K_X)_{|W'}|$ is another divisor, we have $D' = R' + F_1' + F_2'$ as above. Since all fibers of the elliptic fibration are linearly equivalent, we get $R \sim R'$ and hence $R = R'$, because R is a (-1)-curve. Thus $R = \operatorname{Bs}|(-K_X)_{|W'}|$. ■

Corollary 7.6. *The base locus of $|-K_X|$ contains the smooth rational curve R, and the base locus of $|-K_Y|$ contains the smooth rational curve $\xi(R)$.*

Proof. It follows from Lemma 7.5 that $\operatorname{Bs}|-K_X| \supseteq \operatorname{Bs}|(-K_X)_{|W'}| = R$. Moreover, by Lemma 7.4, R is contained in the open subset where the pseudo-isomorphism $\xi\colon X \dashrightarrow Y$ is an isomorphism, thus $\xi(R)$ is contained in the base locus of $|-K_Y|$. ■

We describe the images of the curve $R \subset X$ in $\mathbb{P}^4$ and in $\mathbb{P}^2$ in the following lemma, whose proof is not difficult and is left to the reader.

Lemma 7.7. *Let $R_4 \subset \mathbb{P}^4$ and $R_2 \subset \mathbb{P}^2$ be the images of R under $\eta\colon W' \subset X \to W \subset \mathbb{P}^4$ and $\alpha'\colon W' \to \mathbb{P}^2$ respectively (see* (7.3)*).*

Then R_4 is a smooth rational quintic curve through $p_1, \dots, p_8$, and R_2 is a rational plane quartic containing $q_1, \dots, q_8$ and having a triple point in q_0.

Remark 7.8 (communicated to the authors by Daniele Faenzi and John Christian Ottem)**.** Faenzi and Ottem have computed with Macaulay2 the dimension and the degree of the base locus of the linear system of quintics in $\mathbb{P}^4$ having multiplicity at least 3 at 8 general points; it turns out that this base locus has dimension 1 and degree 65. On the other hand the base locus contains the 28 lines $\overline{p_i p_j}$, the 8 quartics γ_i, and the quintic R_4, whose degrees sum to 65. This shows that the base locus of $|-K_Y|$ is given by the smooth rational curve $\xi(R)$, possibly union some zero-dimensional components.

Remark 7.9. The quartic $R_2 \subset \mathbb{P}^2$ is classically known and described in [Cob19, p. 252] and [Moo43, (6)]; it has the following geometrical description. Consider the pencil of plane cubics through $q_0, \dots, q_8$, and let C_λ be an element of the pencil. Consider the (projective) tangent line $T_{q_0}C_\lambda$ of C_λ at q_0, and let p_λ be the third point of intersection among C_λ and $T_{q_0}C_\lambda$. Then R_2 is the locus of the points p_λ when λ varies.

The point p_λ is related to the Bertini involution $\iota_{\mathbb{P}^2}$ defined by the pencil and q_0, because if $x \in C_\lambda$ is a general point, then $\iota_{\mathbb{P}^2}(x)$ is the third point of intersection of the line $\overline{p_\lambda x}$ with C_λ.

It is not difficult to see that, if F_1 and F_2 are the equations of two cubics of the pencil, and $L_i := \sum_{j=0}^2 \frac{\partial F_i}{\partial x_i}(q_0)x_j$ is the equation of the tangent line at q_0 of the cubic defined by F_i, for $i = 1, 2$, then R_2 has equation $F_1L_2 - F_2L_1 = 0$.

Remark 7.10. The curves R and F (notation as in Lemma 7.5) satisfy $-K_X \cdot R = -K_X \cdot F = 1$ and $E_i \cdot R = E_i \cdot F = 1$ for every $i = 1, \dots, 8$, so $R \equiv F$ in X, and since W' is contained in the domain of ξ, we also have $\xi(R) \equiv \xi(F)$ in Y. Under the map $\zeta \colon \mathcal{N}_1(X) \to H^2(S, \mathbb{R})$, we have $\zeta(R) = -K_S$ (see Prop. 5.8).

Remark 7.11. It is shown in [DP15, Th. 3.1] that a nef divisor in X is always base point free, and an ample divisor in X is always very ample. It is interesting to note that these properties are not preserved from X to Y.

7.12. The bianticanonical linear system. Let S be a degree one del Pezzo surface, and $Y = M_{S,-K_S}$ the associated Fano 4-fold. In this subsection we show the second part of Th. 1.10, namely that the linear system $|-2K_Y|$ is base point free.

Let us consider a fixed divisor $E_C \subset Y$, where $C \subset S$ is a conic. Then $E_C + \iota_Y^* E_C \in |-2K_Y|$ (see Du Val [DV81, p. 201]). Indeed we have $E_C = \frac{1}{2}\rho(C)$ by Lemma 5.15, and $\iota_S^* C \sim -4K_S - C$ (see (2.13)), so using (6.20) we get

$$\iota_Y^* E_C = \iota_Y^*\Big(\frac{1}{2}\rho(C)\Big) = \rho\Big(\frac{1}{2}\iota_S^* C\Big) = \rho\Big(-2K_S - \frac{1}{2}C\Big) = -2K_Y - E_C.$$

We are going to use the divisors $E_C + \iota_Y^* E_C$ to prove the statement. First we need the following intermediate result.

Lemma 7.13. *Let $h \subset S$ be a cubic; notation as in 2.2. Then* $\mathrm{Bs}\,|-2K_Y| \subseteq \bigcup_{i=1}^8 (P_{e_i} \cup P_{\iota_S^* e_i})$.

Proof. Since $E_{C_i} + E_{\iota_S^* C_i} \in |-2K_Y|$ for $i = 1, \dots, 8$, we have

$$\mathrm{Bs}\,|-2K_Y| \subseteq \bigcap_{i=1}^{8}(E_{C_i} \cup E_{\iota_S^* C_i}) = \bigcup_{\{1,\dots,8\} = I \sqcup J} \Big(\bigcap_{i\in I} E_{C_i} \cap \bigcap_{j \in J} E_{\iota_S^* C_j}\Big). \tag{7.14}$$

Let us consider $X = X_h$ and the birational map $\xi \colon X \dashrightarrow Y$. Recall from Lemma 5.18 that ξ flips the curves L_{ij} for $1 \le i < j \le 8$ and Γ_i for $i = 1, \dots, 8$ (notation as in 2.22); moreover $E_{C_i} \subset Y$ is the transform of the exceptional divisor $E_i \subset X$, by Th. 5.6.

In X the divisors $E_1, \dots, E_8$ are pairwise disjoint. Fix a partition $\{1, \dots, 8\} = I \sqcup J$ with the cardinality of I at least 3. Then the flipping curves in X which intersect every divisor E_i with $i \in I$ are Γ_j for $j \in J$, thus after the flips we have

$$\bigcap_{i \in I} E_{C_i} = \bigcup_{j \in J} P_{e_j} \subset Y.$$

Similarly, by considering the cubic ι_S^*h instead of h, we get

$$\bigcap_{i\in I} E_{\iota_S^* C_i} = \bigcup_{j\in J} P_{\iota_S^* e_j} \subset Y.$$

Thus for every partition $\{1,\dots,8\} = I \sqcup J$ we have

$$\bigcap_{i\in I} E_{C_i} \cap \bigcap_{j\in J} E_{\iota_S^* C_j} \subseteq \bigcup_{i=1}^{8} (P_{e_i} \cup P_{\iota_S^* e_i}).$$

which together with (7.14) yields the statement. ■

We are ready to show that $|-2K_Y|$ is base point free. First of all recall that if ℓ, ℓ' are (-1)-curves in S, we have $\iota_S^*\ell' \sim -2K_S - \ell'$ (see (2.13)) and hence $\ell \cdot \iota_S^*\ell' = 2 - \ell \cdot \ell'$.

It follows from Lemma 7.13 that $\mathrm{Bs}|-2K_Y|$ is contained in the union in Y of the loci P_ℓ, where ℓ is a (-1)-curve in S. We fix a (-1)-curve ℓ, and we show that $P_\ell \cap \mathrm{Bs}|-2K_Y| = \emptyset$; this gives the statement.

Let us choose a cubic h such that $\ell = 2h - e_1 - \cdots - e_5$. We have

$$\ell \cdot e_i = \ell \cdot \iota_S^* e_i = 1 \quad \text{for } i = 1,\dots,5, \quad \text{and} \quad \ell \cdot e_j = 0, \;\; \ell \cdot \iota_S^* e_j = 2 \quad \text{for } j = 6,7,8.$$

Therefore, by Lemma 6.4, we have $P_\ell \cap P_{e_i} = \emptyset$ for every $i = 1,\dots,8$, $P_\ell \cap P_{\iota_S^* e_i} = \emptyset$ for $i = 1,\dots,5$, and $P_\ell \cap P_{\iota_S^* e_i}$ is a point y_i for $i = 6,7,8$. By Lemma 7.13, we get $P_\ell \cap \mathrm{Bs}|-2K_Y| \subseteq \{y_6, y_7, y_8\}$.

Notice also that $\iota_S^* e_6 \cdot \iota_S^* e_7 = e_6 \cdot e_7 = 0$, hence $P_{\iota_S^* e_6} \cap P_{\iota_S^* e_7} = \emptyset$, in particular $y_6 \neq y_7$. Similarly one sees that the points y_6, y_7, y_8 are distinct.

Now let us consider the (-1)-curves ℓ and $\iota_S^* e_6$. Since $\ell \cdot \iota_S^* e_6 = 2$, there exists a different cubic h' of S such that $\iota_S^* e_6 = e_1'$ and $\ell \sim 3h' - 2e_1' - e_2' - \cdots - e_7'$ (see Rem. 2.15(b)). We have

$$\ell \cdot e_i' = \ell \cdot \iota_S^* e_i' = 1 \quad \text{for } i = 2,\dots,7, \qquad \ell \cdot \iota_S^* e_1' = \ell \cdot e_8' = 0, \quad \text{and} \quad \ell \cdot e_1' = \ell \cdot \iota_S^* e_8' = 2.$$

Thus, by Lemma 6.4, P_ℓ is disjoint from $P_{e_2'}, \dots, P_{e_8'}, P_{\iota_S^* e_1'}, \dots, P_{\iota_S^* e_7'}$, and intersects $P_{e_1'} = P_{\iota_S^* e_6}$ in y_6 and $P_{\iota_S^* e_8'}$ in a point y'. Again using Lemma 7.13, we conclude that $P_\ell \cap \mathrm{Bs}|-2K_Y| \subseteq \{y_6, y'\}$.

Finally, let us notice that $K_S + \ell \sim e_8' - e_1'$, hence $e_8' \sim K_S + \ell + \iota_S^* e_6 = \ell - K_S - e_6$. Therefore $\iota_S^* e_8' \cdot \iota_S^* e_7 = e_8' \cdot e_7 = (\ell - K_S - e_6) \cdot e_7 = 1$, and by Lemma 6.4 we have $P_{\iota_S^* e_8'} \cap P_{\iota_S^* e_7} = \emptyset$, in particular $y' \neq y_7$. Similarly we see that $y' \neq y_8$, and we conclude that $y_7, y_8 \notin \mathrm{Bs}|-2K_Y|$. Now repeating the argument by replacing $\iota_S^* e_6$ with $\iota_S^* e_7$, we conclude that $y_6 \notin \mathrm{Bs}|-2K_Y|$ and hence that $P_\ell \cap \mathrm{Bs}|-2K_Y| = \emptyset$.

7.15. Open question. Describe the fixed locus of ι_Y, the quotient Y/ι_Y, and the action of ι_Y on $|-K_Y|$ and $|-2K_Y|$.

8. Geometry of the blow-up X of $\mathbb{P}^4$ in 8 points

Let X be the blow-up of $\mathbb{P}^4$ at 8 general points $p_1, \dots, p_8$. In this section we apply our previous results to study the geometry of X.

8.1. Cones of divisors and fixed divisors. By Cor. 5.7, there are a degree one del Pezzo surface S and a cubic h on S such that $X \cong M_{S,\mathcal{B}_h}$. The determinant map $\rho\colon H^2(S,\mathbb{R}) \to H^2(X,\mathbb{R})$ is, in this case, a completely explicit linear isomorphism (see Prop. 5.8), which allows to describe the relevant cones of divisors in $H^2(X,\mathbb{R})$, after Th. 5.13. In particular, it is possible to write explicitly equations for $\mathrm{Mov}(X)$ in terms of the coefficients of a divisor $dH - \sum_i m_i E_i$; one gets one equation for each (-1)-curve and conic on S, corresponding to the generators of $\Pi^\vee$ (see 2.10). The same can be done, in principle, for $\mathrm{Eff}(X)$; however the generators of $\mathcal{E}^\vee$ given by cubics (see (2.4)) give a very large number of equations.

Concerning the generators of the effective cone, it follows from Th. 5.13 and Lemma 5.15 that they are given by the fixed divisors $E_C = \frac{1}{2}\rho(C)$, where C is a conic in S. Using Prop. 5.8, one computes that if $C \sim dh - \sum_i m_i e_i$, then the corresponding fixed divisor E_C has class:

$$E_C \sim \frac{1}{2}\big(\sum_i m_i - d\big)\big(H - \sum_i E_i\big) + \sum_i m_i E_i. \tag{8.2}$$

This proves Prop. 1.12; see also [LP17] for related results.

Let us give the first examples of fixed divisors in X. If $d = 1$, one gets the E_i's; otherwise, if $d \geq 2$, $E_C \subset X$ is the transform of a hypersurface $D_C \subset \mathbb{P}^4$ of degree $\frac{1}{2}(\sum_i m_i - d)$. For $d = 2$, D_C is a hyperplane through 4 blown-up points, and for $d = 3$ a quadric cone through 7 blown-up points, with vertex a line through two of the points.

When $d = 4$, there are two types of conics: $C_1 \sim 4h - e - 2e_i$, with $i \in \{1, \ldots, 8\}$, and $C_2 \sim 4h - \sum_{i \in I} e_i - e + e_k$, with $I \subset \{1, \ldots, 8\}$, $|I| = 3$, and $k \notin I$. We have $E_{C_1} \sim 3H - 2\sum_{j \neq i} E_j$, so that D_{C_1} is the secant variety of the rational normal quartic γ_i. To describe D_{C_2}, let $\pi_{p_k}\colon \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ be the projection from p_k, and set $p_i' := \pi_{p_k}(p_i)$ for $i \neq k$. Then D_{C_2} is the cone, with vertex p_k, over a Cayley nodal cubic surface in $\mathbb{P}^3$, containing the 7 points p_i' for $i \neq k$, and having the 4 nodes in p_j' for $j \in \{1, \ldots, 8\} \smallsetminus (I \cup \{k\})$.

More generally, whenever the conic C is such that $m_i = 0$ for some $i \in \{1, \ldots, 8\}$, D_C is a cone with vertex p_i, indeed it follows from (8.2) that D_C has in p_i a singular point of multiplicity equal to its degree.

Let ℓ be a (-1)-curve, and set $D_\ell := \frac{1}{2}\rho(-K_S + \ell) \in H^2(X,\mathbb{R})$; by Rem. 6.13, D_ℓ is the class of an integral divisor in X, and $h^0(X, D_\ell) = 3$.

Lemma 8.3. *The semigroup* $\mathrm{Eff}(X)_{sg} := \{L \in H^2(X,\mathbb{Z}) \,|\, h^0(X,L) > 0\}$ *is generated by the* 2401 *classes* $-K_X$, E_C, *and* D_ℓ, *where* C *is a conic and* ℓ *is a* (-1)*-curve.*

Proof. By [CT06, Th. 2.7] the semigroup $\mathrm{Eff}(X)_{\mathrm{sg}}$ is generated by its elements L such that $-K_X \cdot L = 3$ with respect to Dolgachev's pairing in $H^2(X,\mathbb{Z})$, see 5.9. Let L be such an element; in particular $L \in \mathrm{Eff}(X)$.

Recall the isomorphism $\tilde{\rho} = \frac{1}{2}\rho\colon H^2(S,\mathbb{R}) \to H^2(X,\mathbb{R})$ defined in Lemma 5.10, and consider $L' := \tilde{\rho}^{-1}(L) \in H^2(S,\mathbb{R})$. We have $L' \in H^2(S,\mathbb{Z})$ by Rem. 5.11, and $L' \in \mathcal{E}$ by Th. 5.13(b), so that L' is nef (see 2.3). Finally, using Lemma 5.10, it is not difficult to see that $-K_S \cdot L' = \frac{2}{3}(-K_X \cdot L) = 2$.

Therefore, by Rem. 2.17, L' is one of the classes $-2K_S, C, -K_S + \ell$, where C is a conic and ℓ is a (-1)-curve, and hence L is one of the classes $-K_X = \tilde{\rho}(-2K_S)$, $E_C = \tilde{\rho}(C)$, and $D_\ell = \tilde{\rho}(-K_S + \ell)$. Conversely, all these classes are effective and $-K_X \cdot (-K_X) = -K_X \cdot E_C = -K_X \cdot D_\ell = 3$, so they are all generators for $\mathrm{Eff}(X)_{\mathrm{sg}}$. ■

8.4. Special surfaces. Let S be a del Pezzo surface of degree one, and $Y = M_{S,-K_S}$ the associated Fano 4-fold. Consider a cubic $h \subset S$ and the map $\eta_h \colon Y \dashrightarrow \mathbb{P}^4$ (see 6.8); we have a factorization:

$$\begin{tikzcd} Y \arrow[r, dashed, "\xi_h^{-1}"'] \arrow[rr, dashed, bend left, "\eta_h"] & X_h \arrow[r] & \mathbb{P}^4 \end{tikzcd}$$

and the indeterminacy locus of η_h is the union of the surfaces P_ℓ for the (-1)-curves $\ell \subset S$ such that $h \cdot \ell \leq 1$.

Notation 8.5. For every (-1)-curve ℓ with $h \cdot \ell \geq 2$, we set $V_{h,\ell} := \overline{\eta_h(P_\ell)} \subset \mathbb{P}^4$.

We denote by $\widetilde{P}_\ell \subset X_h$ the transform of $P_\ell \subset Y$ under $\xi_h \colon X_h \dashrightarrow Y$, so that $V_{h,\ell} \subset \mathbb{P}^4$ is the image of $\widetilde{P}_\ell \subset X_h$ under $X_h \to \mathbb{P}^4$.

We denote by $\widetilde{\Gamma}_\ell \subset \widetilde{P}_\ell \subset X_h$ the transform of a general line $\Gamma_\ell \subset P_\ell \subset Y$.

Let $p_1, \dots, p_8 \in \mathbb{P}^4$ be the images of the exceptional divisors of $\eta_h \colon Y \dashrightarrow \mathbb{P}^4$. Together with the curves $\overline{p_i p_j}$ for $1 \leq i < j \leq 8$ and γ_i for $i = 1, \dots, 8$ (notation as in 2.22), and with the images in $\mathbb{P}^4$ of the fixed divisors in X_h described in (8.2), the surfaces $V_{h,\ell} \subset \mathbb{P}^4$ appear naturally in the base loci of linear systems in $\mathbb{P}^4$ with assigned multiplicities at $p_1, \dots, p_8$. We describe the degree and singularities of the special surfaces $V_{h,\ell}$ in Th. 8.6 below.

Let us first recall that an isolated surface singularity is of type $\frac{1}{3}(1,1)$ if it is analytically isomorphic to the vertex of the cone over a cubic; a normal surface singularity is of type $\frac{1}{3}(1,1)$ if and only if its minimal resolution has exceptional divisor a smooth rational curve E with $E^2 = -3$.

Theorem 8.6. *Let h be a cubic and ℓ a (-1)-curve with $h \cdot \ell \geq 2$.*

(a) *If $h \cdot \ell = 2$, we have $\ell \sim 2h - \sum_{j \notin I} e_j$ with $I \subset \{1, \dots, 8\}$, $|I| = 3$. Then $V_{h,\ell}$ is the plane through p_i for $i \in I$.*

(b) *If $h \cdot \ell = 3$, we have $\ell \sim 3h - e - e_i + e_j$ with $i, j \in \{1, \dots, 8\}$, $i \neq j$. Then $V_{h,\ell}$ is the cone over γ_i with vertex p_j.*

(c) *If $h \cdot \ell = 4$, we have $\ell \sim 4h - e - \sum_{i \in I} e_i$ with $I \subset \{1, \dots, 8\}$, $|I| = 3$. Then $V_{h,\ell}$ is a normal surface of degree 6 containing $p_1, \dots, p_8$, $\operatorname{Sing}(V_{h,\ell}) = \{p_j\}_{j \notin I}$, and $V_{h,\ell}$ has a singularity of type $\frac{1}{3}(1,1)$ in p_j for every $j \notin I$.*

Suppose that S is general.

(d) *If $h \cdot \ell = 5$, we have $\ell \sim 5h - 2e + e_i + e_j$ with $i, j \in \{1, \dots, 8\}$, $i < j$. Then $V_{h,\ell}$ is a surface of degree 10 with $\operatorname{Sing}(V_{h,\ell}) = \{p_k\}_{k \neq i,j} \cup \overline{p_i p_j}$, $V_{h,\ell}$ has a singularity of type $\frac{1}{3}(1,1)$ in p_k for every $k \neq i, j$, and $V_{h,\ell}$ has multiplicity 3 at the general point of the line $\overline{p_i p_j}$.*

(e) *If $h \cdot \ell = 6$, we have $\ell \sim 6h - 2e - e_i$ with $i \in \{1, \dots, 8\}$. Then $V_{h,\ell}$ is a surface of degree 15 with $\operatorname{Sing}(V_{h,\ell}) = \{p_i\} \cup \gamma_i$, $V_{h,\ell}$ has a singularity of type $\frac{1}{3}(1,1)$ in p_i, and $V_{h,\ell}$ has multiplicity 3 at the general point of γ_i.*

To prove Th. 8.6, we first determine the numerical class of $\widetilde{\Gamma}_\ell \subset X_h$ in Lemma 8.7. We use this Lemma to show Th. 8.6 (a) and (b), and this is used to prove Lemma 6.4 on the relative positions of the surfaces P_ℓ in Y. Finally we use Lemma 6.4 to prove the rest of Th. 8.6.

Lemma 8.7. *Let h be a cubic, and $\ell \sim dh - \sum_i m_i e_i$ a (-1)-curve with $d \geq 2$. Then*

$$\widetilde{\Gamma}_\ell \sim \Big(6d - 5 - \sum_i m_i\Big)h - \sum_i (d - m_i - 1)e_i \quad \text{in } \mathcal{N}_1(X_h).$$

Proof. Since $d = h \cdot \ell \geq 2$, $P_\ell \subset Y$ is not contained in the indeterminacy locus of $\xi_h^{-1} \colon Y \dashrightarrow X_h$. Therefore P_ℓ can intersect the indeterminacy locus of ξ_h^{-1} at most in finitely many points (see for instance [Cas17, Rem. 2.9]), and Γ_ℓ is contained in the open subset of X_h where ξ_h^{-1} is an isomorphism. By Rem. 4.18 and Cor. 4.16 we have

$$\begin{aligned}\zeta_{\mathcal{B}_h}(\widetilde{\Gamma}_\ell) = \zeta_{-K_S}(\Gamma_\ell) = 2\ell + K_S &\sim (2d-3)h - \sum_i (2m_i - 1)e_i \\ &= \zeta_{\mathcal{B}_h}\Big(\big(6d - 5 - \sum_i m_i\big)h - \sum_i (d - m_i - 1)e_i\Big),\end{aligned}$$

where the last equality follows from Prop. 5.8. Since $\zeta_{\mathcal{B}_h}$ is an isomorphism by Th. 5.13(a), we get the statement. ■

Proof of Th. 8.6 (a) and (b). If $\ell \sim 2h - \sum_{j \notin I} e_j$, it follows from Lemma 8.7 that $\widetilde{\Gamma}_\ell \subset X_h$ is the transform of a conic in $\mathbb{P}^4$ passing through p_i for $i \in I$, which gives (a).

For (b), set for simplicity $i = 1$ and $j = 8$; then Lemma 8.7 yields $\widetilde{\Gamma}_\ell \sim 5h - e_2 - \cdots - e_7 - 2e_8$. Let $B \subset \mathbb{P}^4$ be the image of $\widetilde{\Gamma}_\ell \subset X_h$, and let $\pi_{p_8} \colon \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ be the projection from p_8. Then both γ_1 and B have image, under π_{p_8}, the rational normal cubic through $\pi_{p_8}(p_2), \ldots, \pi_{p_8}(p_7)$ in $\mathbb{P}^3$, and the cone over γ_1 with vertex p_8 is (the closure of) the inverse image of this cubic. Thus B is contained in the cone, and $V_{h,\ell}$ coincides with the cone. ■

Proof of Lemma 6.4. Suppose that $\ell \cdot \ell' = 0$. Then there exists a cubic h such that $h \cdot \ell = h \cdot \ell' = 0$ (see Rem. 2.15(a)). The line bundle $L_0 = -3K_S + h$ is ample on S, lies on the boundary of the cone $\mathcal{N}$, and is contained in both the walls $(2\ell + K_S)^\perp$ and $(2\ell' + K_S)^\perp$. By Lemma 3.22(d), the surfaces P_ℓ and $P_{\ell'}$ are disjoint in $Y = M_{\mathcal{N}}$.

Suppose that $\ell \cdot \ell' = 1$. Then $\ell + \ell'$ is linearly equivalent to a conic C. Similarly as before, $L_1 = -2K_S + C$ is ample on S, lies on the boundary of $\mathcal{N}$, and is contained in the walls $(2\ell + K_S)^\perp$ and $(2\ell' + K_S)^\perp$, so $P_\ell \cap P_{\ell'} = \emptyset$ by Lemma 3.22(d).

Assume that $\ell \cdot \ell' = 2$. Then there exists a cubic h' such that $\ell = \ell'_{12} \sim h' - e'_1 - e'_2$ and $\ell' \sim 2h' - e'_4 - \cdots - e'_8$ (see Rem. 2.15(b)); let us consider the birational map $\xi_{h'}^{-1} \colon Y \dashrightarrow X_{h'}$. Then $P_\ell = P_{\ell'_{12}} \subset Y$ is contained in the indeterminacy locus of $\xi_{h'}^{-1}$ (by Lemma 5.18), while $P_{\ell'} \subset Y$ is the transform of the plane $\Lambda = V_{h',\ell'}$ through the points[5] $p_1, p_2, p_3 \in \mathbb{P}^4$ (by Th. 8.6(a)). Moreover it follows from the explicit factorization of $\xi_{h'}$ given in Lemma 5.18 that the induced birational map $\Lambda \dashrightarrow P_{\ell'}$ is a Cremona map centered in p_1, p_2, p_3. The corresponding 3 points in $P_{\ell'}$ are the intersection points with $P_{\ell'_{12}}$, $P_{\ell'_{13}}$, $P_{\ell'_{23}}$, and the intersection is transverse.

Finally suppose that S is general and that $\ell \cdot \ell' = 3$. Recall from Rem. 2.15(c) that $\ell' \sim -2K_S - \ell$. Let us choose a cubic h'' such that $\ell \sim 3h'' - e'' - e''_1 + e''_2$, and hence $\ell' \sim 3h'' - e'' - e''_2 + e''_1$.

By Th. 8.6(b), the surfaces $V_{h'',\ell}$ and $V_{h'',\ell'}$ in $\mathbb{P}^4$ are, respectively: the cone over γ_1 with vertex p_2, and the cone over γ_2 with vertex p_1. For a general choice of $p_1, \ldots, p_8$,

[5] Note that the points p_i here depend on h', as they are the images of the exceptional divisors of $\eta_{h'} \colon Y \dashrightarrow \mathbb{P}^4$. For simplicity we still denote them by $p_1, \ldots, p_8$, and similarly in the sequel of the proof.

$V_{h'',\ell}$ and $V_{h'',\ell'}$ are general cones over two general cubics contained in a hyperplane $H \subset \mathbb{P}^4$, and they intersect transversally at 9 points, including $p_3, \dots, p_8$. Thus the transforms $\widetilde{P}_\ell$ and $\widetilde{P}_{\ell'}$ of $V_{h'',\ell}$ and $V_{h'',\ell'}$ respectively in $X_{h''}$ intersect transversally in 3 points x_1, x_2, x_3.

It is not difficult to check that $\widetilde{P}_\ell$ intersects the indeterminacy locus of $\xi_{h''}\colon X_{h''} \dashrightarrow Y$ in $L_{23}, \dots, L_{28}, \Gamma_1$, and similarly $\widetilde{P}_{\ell'}$ intersects the indeterminacy locus of $\xi_{h''}$ in $L_{13}, \dots, L_{18}, \Gamma_2$. The curves L_{ij} and Γ_a are pairwise disjoint, so we conclude that $x_1, x_2, x_3 \in X_{h''}$ are contained in the open subset where $\xi_{h''}\colon X_{h''} \dashrightarrow Y$ is an isomorphism. Hence P_ℓ and $P_{\ell'}$ intersect transversally in 3 points $\xi_{h''}(x_1), \xi_{h''}(x_2), \xi_{h''}(x_3)$. ∎

Proof of Th. 8.6 (c), (d), and (e). We show (c); set for simplicity $I = \{1, 2, 3\}$. By Lemma 6.4, P_ℓ meets the indeterminacy locus of $\eta_h\colon Y \dashrightarrow \mathbb{P}^4$ in 13 isolated points:

$$x_i := P_{e_i} \cap P_\ell \ \text{ for } i = 1, 2, 3 \quad \text{and} \quad y_{ab} := P_{\ell_{ab}} \cap P_\ell \ \text{ for } 4 \leq a < b \leq 8$$

(recall that the components of the indeterminacy locus of η_h are pairwise disjoint in Y, see 6.8). By the description of the map ξ_h as a sequence of smooth blow-ups in Lemma 5.18, we have a diagram:

$$\begin{tikzcd}
 & \widetilde{P}_\ell \subset X_h \arrow[dl] \arrow[dr] & \\
V_{h,\ell} \subset \mathbb{P}^4 & & P_\ell \subset Y
\end{tikzcd}$$

where $\widetilde{P}_\ell \to P_\ell$ is the blow-up of $\mathbb{P}^2$ in the points x_i and y_{ab}, with exceptional curves Γ_i and L_{ab}, for $i = 1, 2, 3$ and $4 \leq a < b \leq 8$. The second morphism $\widetilde{P}_\ell \to V_{h,\ell}$ is the restriction of $X_h \to \mathbb{P}^4$, thus it is induced by $H_{|\widetilde{P}_\ell}$ and contracts the curve $(E_i)_{|\widetilde{P}_\ell}$ to p_i for $i = 1, \dots, 8$. In particular, we see that $V_{h,\ell} \smallsetminus \{p_1, \dots, p_8\}$ is smooth.

Recall that $\widetilde{\Gamma}_\ell \subset \widetilde{P}_\ell$ is the transform of a general line in $P_\ell \cong \mathbb{P}^2$, and $H \cdot \widetilde{\Gamma}_\ell = 8$ by Lemma 8.7. Since Γ_i and L_{ab} are the transforms respectively of a quartic and a line in $\mathbb{P}^4$, in X_h we have $H \cdot \Gamma_i = 4$ and $H \cdot L_{ab} = 1$, and in $\widetilde{P}_\ell$ we have

$$H_{|\widetilde{P}_\ell} \sim 8\widetilde{\Gamma}_\ell - 4\sum_{i=1}^{3} \Gamma_i - \sum_{4 \leq a < b \leq 8} L_{ab}.$$

Hence the degree of $V_{h,\ell}$ in $\mathbb{P}^4$ is $(H_{|\widetilde{P}_\ell})^2 = 6$.

Let $i \in \{1, \dots, 8\}$. In X_h the divisor E_i intersects Γ_j for $j \neq i$ and L_{ib} for $b \neq i$, while it is disjoint from Γ_i and from L_{ab} for $a, b \neq i$. Thus in Y its transform, that we still denote by E_i, contains P_{e_j} for $j \neq i$ and $P_{\ell_{ib}}$ for $b \neq i$, while it is disjoint from P_{e_i} and from $P_{\ell_{ab}}$ for $a, b \neq i$. By (6.6) we also have, in Y:

$$E_i \cdot \Gamma_\ell = C_i \cdot \ell - 1 = \begin{cases} 1 & \text{for } i = 1, 2, 3 \\ 2 & \text{for } i = 4, \dots, 8, \end{cases}$$

so $(E_i)_{|P_\ell}$ is a line in $P_\ell \cong \mathbb{P}^2$ for $i = 1, 2, 3$, and a conic for $i = 4, \dots, 8$.

Since E_1 contains P_{e_2} and P_{e_3}, it contains both x_2 and x_3; on the other hand these points are distinct, thus $(E_1)_{|P_\ell} = \overline{x_2 x_3}$, and E_1 does not contain other points of P_ℓ blown-up in $\widetilde{P}_\ell$. Similarly for E_2 and E_3.

Since E_4 contains $P_{e_1}, P_{e_2}, P_{e_3}$, and $P_{\ell_{4b}}$ for $b = 5, \dots, 8$, $(E_4)_{|P_\ell}$ is a conic containing the 7 points $x_1, x_2, x_3, y_{45}, y_{46}, y_{47}, y_{48}$. Notice that the divisors $E_1, \dots, E_8$ are pairwise disjoint in X_h, so $(E_4)_{|P_\ell}$ and $(E_i)_{|P_\ell}$ for $i=1,2,3$ can intersect only in the points x_1, x_2, x_3; this implies that $(E_4)_{|P_\ell}$ is a smooth conic, and similarly for $(E_i)_{|P_\ell}$ when $i=5,\dots,8$.

Since for $i=1,2,3$ $(E_i)_{|P_\ell}$ is a line containing two points blown-up in $\widetilde{P}_\ell \to P_\ell$, its transform $(E_i)_{|\widetilde{P}_\ell}$ is a (-1)-curve in the surface $\widetilde{P}_\ell$. This shows that around p_i, the map $\widetilde{P}_\ell \to V_{h,\ell}$ factors as the contraction of $(E_i)_{|\widetilde{P}_\ell}$ to a smooth point, followed by the normalization of $V_{h,\ell}$ at p_i. On the other hand, the scheme-theoretical fiber of p_i under the map $\widetilde{P}_\ell \to V_{h,\ell}$ is $(E_i)_{|\widetilde{P}_\ell}$, hence it is reduced; this shows that $p_i \in V_{h,\ell}$ is normal and hence smooth.

For $i=4,\dots,8$ $(E_i)_{|P_\ell}$ is a smooth conic containing 7 points blown-up in $\widetilde{P}_\ell \to P_\ell$. Thus its transform $(E_i)_{|\widetilde{P}_\ell}$ is a smooth rational curve with self-intersection $4-7=-3$ in the surface $\widetilde{P}_\ell \cong \mathrm{Bl}_{13\text{pts}}\,\mathbb{P}^2$. Similarly as before, this yields the statement on p_i for $i=4,\dots,8$.

We prove (d); set for simplicity $i=7$ and $j=8$. By Lemmas 5.18 and 6.4, P_ℓ meets the indeterminacy locus of $\eta_h\colon Y \dashrightarrow \mathbb{P}^4$ in 21 isolated points:

$$x_i := P_{e_i} \cap P_\ell \text{ for } i=1,\dots,6, \quad y_{ab} := P_{\ell_{ab}} \cap P_\ell \text{ for } a \le 6, b \ge 7, \quad \{z^1, z^2, z^3\} := P_{\ell_{78}} \cap P_\ell$$

(again, the components of the indeterminacy locus of η_h are pairwise disjoint in Y). By the description of the map ξ_h in Lemma 5.18, we have a diagram:

$$\begin{tikzcd} \widetilde{P}_\ell \subset X_h \arrow[d] & \widehat{P}_\ell \subset \widehat{X}_h \arrow[l] \arrow[d] \\ V_{h,\ell} \subset \mathbb{P}^4 & P_\ell \subset Y \end{tikzcd}$$

where $\widehat{P}_\ell \to P_\ell$ is the blow-up of $\mathbb{P}^2$ in the 21 points x_i, y_{ab}, z^j, with exceptional curves $\widehat{\Gamma}_i$, $\widehat{L}_{ab}$, and $\widehat{L}^j_{78}$ respectively, and $\widehat{P}_\ell \to \widetilde{P}_\ell$ is an isomorphism outside the curves $\widehat{L}^j_{78}$, while it glues the three curves $\widehat{L}^1_{78}, \widehat{L}^2_{78}, \widehat{L}^3_{78}$ onto $L_{78} \subset X_h$. Finally, the morphism $\widetilde{P}_\ell \to V_{h,\ell}$ is the restriction of $X_h \to \mathbb{P}^4$ and contracts the curve $(E_i)_{|\widetilde{P}_\ell}$ to p_i for $i=1,\dots,8$. In particular, we see that $V_{h,\ell} \smallsetminus (\{p_1,\dots,p_6\} \cup \overline{p_7p_8})$ is smooth, and that $V_{h,\ell}$ is singular along the line $\overline{p_7p_8}$, with a point of multiplicity 3 at the general point of the line.

Let $\widehat{H} \in \mathrm{Pic}(\widehat{P}_\ell)$ be the pull-back of $H_{|\widetilde{P}_\ell}$, and $\widehat{\Gamma}_\ell \subset \widehat{P}_\ell$ the transform of a general line in $P_\ell \cong \mathbb{P}^2$. In X_h we have $H \cdot L_{ab} = 1$ for every $a<b$, $H \cdot \Gamma_i = 4$, and $H \cdot \widetilde{\Gamma}_\ell = 11$ by Lemma 8.7. Using the projection formula, in $\widehat{P}_\ell$ we get

$$\widehat{H} \sim 11\widehat{\Gamma}_\ell - 4\sum_{i=1}^{6} \widehat{\Gamma}_i - \sum_{a \le 6, b \ge 7} \widehat{L}_{ab} - \sum_{j=1}^{3} \widehat{L}^j_{78}$$

and hence the degree of $V_{h,\ell}$ in $\mathbb{P}^4$ is $\widehat{H}^2 = 10$.

Let $i \in \{1,\dots,6\}$. Similarly to the proof of case (c), we see that in Y $(E_i)_{|P_\ell}$ is a smooth conic containing the 7 points $x_1,\dots,\check{x}_i\dots,x_6,y_{i7},y_{i8}$ and no other point blown-up in $\widehat{P}_\ell$. Thus the transform of $(E_i)_{|P_\ell}$ in $\widehat{P}_\ell$ is a smooth rational curve with self-intersection $4-7=-3$, which yields the statement on p_i.

The proof of (e) is very similar. ■

8.8. The Bertini involution in X and in $\mathbb{P}^4$.

Proposition 8.9. *Let X be a blow-up of $\mathbb{P}^4$ at 8 general points. Then X has a unique non-trivial pseudo-automorphism $\iota_X\colon X \dashrightarrow X$. It can be factored as follows:*

$$\begin{tikzcd} X \arrow[r, dashed, "\xi"'] \arrow[rrr, dashed, bend left, "\iota_X"] & Y & \widehat{X}_2 \arrow[l] \arrow[r] & X \end{tikzcd},$$

where

- *$\xi\colon X \dashrightarrow Y$ is described in Lemma 5.18;*
- *$\widehat{X}_2 \to Y$ is the blow-up of 36 pairwise disjoint smooth rational surfaces in Y, given by the transforms in Y of 8 surfaces of degree 10 and 28 surfaces of degree 15 in $\mathbb{P}^4$, all containing $p_1, \dots, p_8$; every exceptional divisor is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$ with normal bundle $\mathcal{O}(-1,-1)$;*
- *$\widehat{X}_2 \to X$ contracts the exceptional divisors to pairwise disjoint smooth rational curves.*

Proof. Let (S,h) be as in Cor. 5.7, so that $X \cong M_{\mathcal{B}_h}$. It follows from Th. 1.5 that X has a unique non-trivial pseudo-automorphism ι_X. Under the isomorphism with $M_{\mathcal{B}_h}$, ι_X is the map induced by the Bertini involution ι_S of S as follows:

$$M_{\mathcal{B}_h} \dashrightarrow M_{\mathcal{B}_h}, \quad [F] \mapsto [\iota_S^* F].$$

This is nothing but the natural birational map $M_{\mathcal{B}_h} \dashrightarrow M_{\mathcal{B}_{\iota_S^* h}}$ (given by $[F] \mapsto [F]$, see Cor. 3.25) composed with the natural isomorphism $M_{\mathcal{B}_{\iota_S^* h}} \cong M_{\mathcal{B}_h}$ induced by ι_S (note that $\iota_S^* \mathcal{B}_h = \mathcal{B}_{\iota_S^* h}$). In particular, we can factor ι_X as a sequence of flips by varying the polarization from $\mathcal{B}_h$ to $\mathcal{B}_{\iota_S^* h}$ along the plane spanned by h and $-K_S$ (see Fig. 5.3), and similarly for $\mathbb{P}^4$:

$$\begin{tikzcd} X = X_h \arrow[r, dashed, "\xi_h"'] \arrow[rr, dashed, bend left, "\iota_X"] & Y \arrow[r, dashed, "\xi^{-1}_{\iota_S^* h}"'] & X_{\iota_S^* h} \cong X \end{tikzcd} \qquad \begin{tikzcd} \mathbb{P}^4 \arrow[r, dashed, "\eta_h^{-1}"'] \arrow[rr, dashed, bend left, "\iota_{\mathbb{P}^4}"] & Y \arrow[r, dashed, "\eta_{\iota_S^* h}"'] & \mathbb{P}^4 \end{tikzcd}$$

The factorization of ξ_h is described in Lemma 5.18, so let us consider the second part $Y \dashrightarrow X$. To go from the chamber $\mathcal{N}$ to the chamber $\mathcal{F}_{\iota_S^* h}$, we have to cross the 8 walls $(2\ell_i + K_S)^\perp$, where $\ell_i \sim 6h - 2e - e_i$, for $i = 1, \dots, 8$ (see Fig. 5.3 and Lemma 5.2). Thus the map $Y \dashrightarrow M_{\mathcal{F}_{\iota_S^* h}}$ is the composition of 8 flips, each replacing $P_{\ell_i} \cong \mathbb{P}^2$ with a smooth rational curve. Moreover $P_{\ell_i} \subset Y$ is the transform of the surface $V_{h,\ell_i} \subset \mathbb{P}^4$, described in Th. 8.6($e$).

Secondly, to go from the chamber $\mathcal{F}_{\iota_S^* h}$ to the chamber $\mathcal{B}_{\iota_S^* h}$, we have to cross the 28 walls $(2\ell'_{ij} + K_S)^\perp$, where $\ell'_{ij} \sim 5h - 2e + e_i + e_j$, for $1 \le i < j \le 8$ (see Fig. 5.3 and Lemma 5.2). Thus the map $M_{\mathcal{F}_{\iota_S^* h}} \dashrightarrow X$ is the composition of 28 flips, each replacing $P_{\ell'_{ij}} \cong \mathbb{P}^2$ with a smooth rational curve. Moreover $P_{\ell'_{ij}} \subset Y$ is the transform of the surface $V_{h,\ell'_{ij}} \subset \mathbb{P}^4$, described in Th. 8.6($d$). ■

Corollary 8.10. *Let $p_1, \dots, p_8 \in \mathbb{P}^4$ be general points, and let $V \subset |\mathcal{O}_{\mathbb{P}^4}(49)|$ be the linear system of hypersurfaces having multiplicity at least 30 at $p_1, \dots, p_8$. Then $\dim V = 4$, and V defines a birational involution $\iota_{\mathbb{P}^4}\colon \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$. The base locus of V has dimension 2, and it is the union of 36 irreducible rational surfaces, 28 of degree 10, and 8 of degree 15. The birational map $\iota_{\mathbb{P}^4}$ contracts 8 irreducible rational hypersurfaces of degree 10.*

Proof. Most of the statement is a direct consequence of Prop. 8.9. The divisors contracted by $\iota_{\mathbb{P}^4}$ are the transforms of $E_{\iota_S^* C_i} \subset X$, for $i = 1, \dots, 8$. We have $C_i \sim h - e_i$, $\iota_S^* C_i \sim -4K_S - C_i \sim 11h - 4e + e_i$ (see (2.13)), and thus $E_{\iota_S^* C_i} \sim 10H - 6\sum_j E_j + E_i$ after Prop. 1.12. ■

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